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FIXED POINT THEORY AND TRACE FOR BICATEGORIES

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Abstract. — The Lefschetz fixed point theorem follows easily from the identification of the Lefschetz number with the fixed point index. This identification is a consequence of the functoriality of the trace in symmetric monoidal categories.

There are refinements of the Lefschetz number and the fixed point index that give a converse to the Lefschetz fixed point theorem. An important part of this theorem is the identification of these different invariants.

We define a generalization of the trace in symmetric monoidal categories to a trace in bicategories with shadows. We show the invariants used in the converse of the Lefschetz fixed point theorem are examples of this trace and that the functoriality of the trace provides some of the necessary identifications. The methods used here do not use simplicial techniques and so generalize readily to other contexts.

***Résumé* (Théorie du point fixe et trace pour les bicatégories).** — Le théorème du point fixe de Lefschetz découle facilement de l'identification du nombre de Lefschetz avec l'indice de point fixe. Cette identification est une conséquence de la fonctorialité de la trace dans les catégories symétriques monoïdales.

Ce sont des raffinements du nombre de Lefschetz et de l'indice de point fixe qui fournissent la réciproque du théorème du point fixe de Lefschetz. Une partie importante de ce théorème est l'identification de ces invariants.

Nous définissons une généralisation de la trace dans les catégories symétriques monoïdales, en une trace dans les bicatégories avec ombres. Nous montrons que les invariants utilisés dans la réciproque du théorème du point fixe de Lefschetz sont des exemples de cette trace, et que la fonctorialité de la trace fournit certaines identifications nécessaires. Les méthodes présentées ici n'utilisent pas de technique simpliciale et peuvent donc être généralisées facilement dans d'autres contextes.

CONTENTS

INTRODUCTION

There are many approaches to determining when a continuous endomorphism of a topological space has a fixed point. One of the simplest is given by the Lefschetz fixed point theorem.

***Theorem A* (Lefschetz fixed point theorem).** — *Let M be a compact ENR and $f\colon M \to M$ be a continuous map. If f has no fixed points then the Lefschetz number of f is zero.*

The Lefschetz number of a map is defined using rational homology and so is relatively easy to compute. Further, if M is a simply connected closed smooth manifold of dimension at least three then a converse to the Lefschetz fixed point theorem also holds.

Theorem B. — *Let $f\colon M \to M$ be a continuous map of a simply connected closed smooth manifold of dimension at least three. Then the Lefschetz number of f is zero if and only if f is homotopic to a map with no fixed points.*

Note that we have replaced 'the map f has no fixed points' with 'the map f is homotopic to a map with no fixed points'. This change only reflects the fact that the Lefschetz number is defined using homology and so cannot distinguish between homotopic maps. In particular, the Lefschetz number cannot determine if a map has no fixed points, it can only determine if it is homotopic to a map with no fixed points.

Unfortunately, Theorem B does not hold if we remove the hypothesis that the space is simply connected. However, by sacrificing some of the computability we can refine the Lefschetz number to an invariant, called the Nielsen number, that detects if the map has fixed points.

Theorem C. — *Let $f\colon M \to M$ be a continuous map of a closed smooth manifold of dimension at least three. The Nielsen number of f, $N(f)$, is the minimum number of fixed points among all maps homotopic to f. In particular, $N(f)$ is zero if and only if f is homotopic to a map with no fixed points.*

The idea behind the Nielsen number is to incorporate information about the fundamental group into the invariant itself. This additional information corresponds to recording which fixed points can be eliminated by a homotopy of the original map.

The Nielsen number is not the most convenient description of this information for defining generalizations of this invariant to other categories and for proving results about relationships between the Nielsen number and basic topological constructions such as cofiber sequences or products. The invariant that retains the necessary information is called the Reidemeister trace. This invariant was defined by Wecken and Reidemeister in [**40, 45**]. It can be used to prove a theorem similar to Theorem C.

Theorem D. — *Let $f\colon M \to M$ be a continuous map of a closed smooth manifold of dimension at least three. The Reidemeister trace of f is zero if and only if f is homotopic to a map with no fixed points.*

Classically, all four of these results were proved using simplicial techniques. In [**11**], Dold and Puppe proposed an alternative approach. Their idea was to focus on the identification of the Lefschetz number, which is a global invariant, with a local invariant, the fixed point index. It is immediate from the definition that the fixed point index is zero for a map that has no fixed points or is homotopic to a map with no fixed points. Using this observation, the Lefschetz fixed point theorem is a consequence of the identification of the Lefschetz number with the index.

Dold and Puppe approached this identification by defining a more general construction that includes both of these invariants as special cases. Their construction is a 'trace' in any symmetric monoidal category. In some cases the trace is functorial. Dold and Puppe showed that the identification of the Lefschetz number with the index is an example of this functoriality.

In addition to giving an alternate proof of the Lefschetz fixed point theorem, Dold and Puppe's definition of trace can be used to describe generalizations of the fixed point index to other categories. If $f\colon X \to X$ and $p\colon X \to B$ are continuous maps such that $p \circ f = p$ we say that f is a fiberwise map. In [**8**], Dold defined an index for fiberwise maps and showed that the index is zero for a map that is fiberwise homotopic to a map with no fixed points. The fiberwise index is an example of the trace in symmetric monoidal categories.

It is possible to prove results for the trace in symmetric monoidal categories that can be applied to the special cases of the Lefschetz number and the index. For example, the Lefschetz number and the index are both additive on cofiber sequences. This follows from the additivity of the trace in (some) symmetric monoidal categories, see [**32**].

Unfortunately, the trace in symmetric monoidal categories cannot be used to describe the invariants of Theorems C and D. Invariants that include information about the fundamental group do not fit into a symmetric monoidal category. However, by replacing symmetric monoidal categories by appropriate bicategories and similarly modifying the definition of the trace we can accommodate these invariants.

Here we implement this philosophy. First we show that the Reidemeister trace is an example of a more general trace. This trace is defined here and is a trace in bicategories with some additional structure; these bicategories are called bicategories with shadows. Just as the Lefschetz number can be identified with the fixed point index, there is more than one description of the Reidemeister trace. There are generalizations of the fixed point index, defined by Reidemeister and Wecken, and of the Lefschetz number, defined by Husseini in **[19]**. Both of these invariants are examples of the trace in bicategories with shadows, and the functoriality of the trace can be used to identify them. There is also an invariant defined by Klein and Williams in **[25]** that can be identified with another example of the trace in a bicategory with shadows.

Next we show that this change in perspective gives definitions and proofs that generalize more easily than the classical approaches. One element of the classical invariants that causes problems for equivariant and fiberwise generalizations is the role played by a base point. Both classical definitions of the Reidemeister trace require that a base point be chosen, but a different choice of the base point does not change the invariant. Modified forms of the Reidemeister trace can be defined without a base point. We show that these invariants are also examples of trace in bicategories, and we use the formal structure of the trace to show that these unbased invariants can be identified with the classical invariants.

The second source of problems for generalizations is only obvious when trying to prove a converse to the Lefschetz fixed point theorem like Theorem D. In **[41]**, Scofield defined a generalization of the Nielsen number to fiberwise maps and gave an example that showed this invariant does not give a converse to the fiberwise Lefschetz fixed point theorem. More recently, Klein and Williams have defined a fiberwise invariant that does give a converse to the fiberwise Lefschetz fixed point theorem.

Theorem E. — *Let $M \to B$ be a fiber bundle with closed smooth manifold fibers F such that $\dim(F) - 3 \geq \dim(B)$. A fiberwise map $f\colon M \to M$ is fiberwise homotopic to a map with no fixed points if and only if the fiberwise Reidemeister trace of f is zero.*

There is another invariant, defined by Crabb and James in **[6]**, that can help to explain the discrepancy between Scofield's invariant and Klein and Williams' invariant. The invariant defined by Crabb and James is a derived form of the Reidemeister trace and so in the transition from a classical invariant to a fiberwise invariant it is sensitive to information that the other forms of the Reidemeister trace, like Scofield's invariant, miss. Crabb and James' invariant can be identified with the invariant defined by Klein and Williams. Crabb and James' invariant, in both its classical and fiberwise forms, is an example of the trace in bicategories with shadows.

More concretely, our goal is to convert Dold and Puppe's outline for proving Theorem A into an approach for proving Theorems D and E. Dold and Puppe's proof identified the Lefschetz number and the fixed point index and then used the observation that the index is zero for maps with no fixed points. Our first step is the same. We start by identifying the form of the Reidemeister trace defined by Husseini with Reidemeister and Wecken's form of the Reidemeister trace. Unfortunately, it is not obvious that Reidemeister and Wecken's form of the Reidemeister trace is zero only when the map is homotopic to a map with no fixed points. The next step in our proof is to identify Reidemeister and Wecken's form of the Reidemeister trace with Crabb and James' version. This invariant can then be identified with the invariant defined by Klein and Williams. Klein and Williams' proof in [25] then completes the proof of Theorem D.

To implement this plan we need to make connections between four different invariants. The first three invariants are examples of the trace in bicategories with shadows. Functoriality gives an identification of the Reidemeister trace defined by Husseini with the Reidemeister trace defined by Reidemeister and Wecken. Functoriality also shows that the Reidemeister trace defined by Reidemeister and Wecken is zero when the Reidemeister trace defined by Crabb and James is zero. The converse of this fact is not formal.

In the fiberwise setting of Theorem E not all of the steps in our proof of Theorem D make sense. Here we only have two invariants, the invariant defined by Klein and Williams and the fiberwise version of the invariant defined by Crabb and James. Klein and Williams' proof has an immediate fiberwise generalization and their fiberwise invariant can be identified with the fiberwise version of Crabb and James' invariant in complete analogy with the classical case.

We could interpret these proofs either as category theory with topological applications or as topological proofs that have a formal part. Here we will try to aim for the middle since, in reality, the category theory motivates the topology and the topology motivates the category theory. This balance is reflected in the structure of this paper. We start with some motivation from fixed point theory and category theory. Then we give reinterpretations of the fixed point theory that further suggests our definition of trace in a bicategory and the results that are entirely category theory. Motivated by these descriptions we define shadows and traces in bicategories. Using these formal results, we then give new proofs of some classical and fiberwise fixed point theory results. The last chapter returns to category theory and consists of further examples that are closely related to the topological examples given earlier.

In Chapters 1 and 2 we recall the elements of topological fixed point theory that will be the motivation for much of the later chapters. In Chapter 1 we define the Lefschetz number and the fixed point index. We also summarize Dold and Puppe's results on duality and trace in symmetric monoidal categories and their applications to fixed point theory. In Chapter 2 we focus on the converse to the Lefschetz fixed point theorem. We define the Nielsen number and the two versions of the Reidemeister trace defined by Husseini and Reidemeister and Wecken. We also describe Klein and Williams' proof of the converse to the Lefschetz fixed point theorem.

Chapter 3 serves as a transition between the classical fixed point theory of Chapter 2 and the definition of trace in a bicategory with shadows in Chapter 4. Here we give alternate descriptions of the versions of the Reidemeister trace defined by Wecken and Reidemeister and Crabb and James that suggest the definitions of Chapter 4. This chapter does not contain rigorous proofs, which are delayed to Chapters 5, 6, 7, and 8, but instead makes it clear that the Reidemeister trace has many features in common with trace in symmetric monoidal categories.

In Chapter 4 we define shadows in a bicategory and trace in a bicategory with shadows. We also prove some basic results about the trace and give some algebraic examples. In Chapters 5 and 6 we describe topological examples of duality and trace and show that the Reidemeister trace as defined by Reidemeister and Wecken and Crabb and James can be described using the trace in a bicategory with shadows. We also show that functoriality gives identifications of some of the forms of the Reidemeister trace. In Chapters 7 and 8 we show that many of the results from Chapters 5 and 6 carry over to fiberwise spaces.

Chapter 9 consists of examples of bicategories with shadows that either motivate or are motivated by the topological examples in Chapters 5, 6, 7, and 8. While the earlier chapters can be read without these examples, some of the results and constructions we use in Chapters 5, 6, 7, and 8 have more straightforward analogues in Chapter 9. In the earlier chapters we will indicate when there is a relevant section in Chapter 9. Chapter 9 can be read after Chapter 4.

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CHAPTER 1

A REVIEW OF FIXED POINT THEORY

This chapter and the next are primarily a review of the definitions and results from classical fixed point theory that motivate the remaining chapters. This chapter also contains an introduction to Dold and Puppe's definitions of duality and trace in symmetric monoidal categories.

The two invariants described in this chapter, the Lefschetz number and the fixed point index, are examples of trace in symmetric monoidal categories. Since the fixed point index is zero for maps that have no fixed points, the Lefschetz fixed point theorem follows from the identification of the Lefschetz number with the index. This identification is a consequence of the functoriality of the trace in symmetric monoidal categories.

Some references for the standard approach to topological fixed point theory include **[2, 14, 20, 21]**.

1.1. Classical fixed point theory

The Lefschetz fixed point theorem is a familiar result that relates a local, geometric invariant to a global, algebraic invariant. The algebraic invariant is the Lefschetz number.

Definition 1.1.1. — Let K be a field and C_* a finitely generated chain complex of vector spaces over K. If $f\colon C_* \to C_*$ is a map of chain complexes, the *Lefschetz number* of f, $L(f)$, is the alternating sum of the levelwise traces.

***Theorem 1.1.2* (Lefschetz Fixed Point Theorem).** — *Let M be a closed smooth manifold and $f\colon M \to M$ a continuous map. If the Lefschetz number of*

$$f_*\colon H_*(M;\mathbb{Q}) \to H_*(M;\mathbb{Q})$$

is nonzero then f has a fixed point.

Note that the Lefschetz number of the identity map is the Euler characteristic.

Since homology is a homotopy invariant, we could replace the conclusion of this theorem with "then all maps homotopic to f have a fixed point." Additionally, we can use the integers rather than the rational numbers as our coefficients. The Lefschetz number of $H_*(f;\mathbb{Z})$ is defined and is equal to the Lefschetz number of $H_*(f;\mathbb{Q})$.

To refine the Lefschetz fixed point theorem as described in the introduction we need another invariant, the fixed point index. There are many ways to define the index. We will use Dold's definition using homology and fundamental classes from **[7, 10]**.

For a generator $[S^n]$ of $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ and any pair $K \subset V \subset \mathbb{R}^n$, V open and K compact, there is a *fundamental class*

$$[S^n]_K \in H_n(V, V-K;\mathbb{Z})$$

around K. This class, $[S^n]_K$, is the image of $[S^n]$ under the map

$$H_n(S^n;\mathbb{Z}) \to H_n(S^n, S^n - K;\mathbb{Z}) \cong H_n(V, V-K;\mathbb{Z}).$$

Definition 1.1.3. — **[10**, VII.5] Let $V \subset \mathbb{R}^n$ be open and $f\colon V \to \mathbb{R}^n$ be continuous. Assume

$$F = \{x \in V | f(x) = x\}$$

is compact and let $[S^n]_F$ be the fundamental class of F. Then $I_f \in \mathbb{Z}$, the *fixed point index* of f, is defined by $I_f[S^n] = (\mathrm{id} - f)_*[S^n]_F$ where

$$(\mathrm{id} - f)\colon (V, V-F) \to (\mathbb{R}^n, \mathbb{R}^n - 0)$$

is defined by $(\mathrm{id} - f)(x) = x - f(x)$.

The index is additive. If there are open sets V_i such that $\bigcup V_i = V$ and $(F \cap V_i) \cap (F \cap V_j) = \varnothing$ for $i \neq j$, then $\sum I_{f|V_i} = I_f$. The index is local. If $F \subseteq W \subseteq V$ for some open set W, then $I_{f|W} = I_f$. The index is commutative. If $V \subset \mathbb{R}^n$, $V' \subset \mathbb{R}^m$ are open sets and $f\colon V \to \mathbb{R}^m$, $g\colon V' \to \mathbb{R}^n$ are continuous maps then

$$U = f^{-1}(V') \xrightarrow{gf} \mathbb{R}^n \qquad \text{and} \qquad U' = g^{-1}(V) \xrightarrow{fg} \mathbb{R}^m$$

have homeomorphic fixed point sets. If these sets are compact $I_{fg} = I_{gf}$.

If Y is any topological space and $U \subset Y$ is an open set which is also an ENR, then every map $f\colon U \to Y$ admits a factorization $f = \beta\alpha$ where

$$U \xrightarrow{\alpha} V \xrightarrow{\beta} Y$$

and V is open in some $\mathbb{R}^n$. If $F_f = \{y \in U | f(y) = y\}$ is compact then the fixed point index $I_{\alpha\beta}$ of $\alpha\beta\colon \beta^{-1}(U) \to V \subset \mathbb{R}^n$ is defined and is independent of the factorization $f = \beta\alpha$. This number is defined to be the index of f. In particular, the index of an endomorphism of an ENR is well defined.

Remark 1.1.4. — The index is an invariant of homotopy classes of maps, so homotopic maps have the same index. Additivity, localization, commutativity, homotopy invariance along with an additional axiom, normalization, characterize the index, see **[2**, IV]. The normalization axiom ensures that the index agrees with the Lefschetz number. Alternatively, in **[8**, 5.1], a characterization of the index is given using a variation of the homotopy invariance axiom and a normalization axiom.

***Theorem 1.1.5* (Lefschetz-Hopf).** — *If M is a closed smooth manifold and $f\colon M \to M$ is a continuous map then the index of f, I_f, equals the Lefschetz number of*

$$f_*\colon H_*(M;\mathbb{Z}) \to H_*(M;\mathbb{Z}).$$

The Lefschetz Fixed Point Theorem is a consequence of this theorem since if f has no fixed points the index is zero. There is a familiar proof of this theorem that uses simplicial homology, see [**1**, 9.6] or [**16**, 2.C]. We will describe an alternative, conceptual proof using duality in symmetric monoidal categories in the next two sections.

1.2. Duality and trace in symmetric monoidal categories

This section is a summary of the results of [**11**] that we will generalize. Other references for this section include [**28**, III.1] and [**33**]. We will define trace and duality for any symmetric monoidal category, but our focus will be on examples in the category of modules over a commutative ring R and the stable homotopy category.

Let $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$, unit object I, and symmetry isomorphism γ.

Definition 1.2.1. — We say that $A \in \mathrm{ob}\mathscr{C}$ is *dualizable* if there is a $B \in \mathrm{ob}\mathscr{C}$ and morphisms $\eta\colon I \to A \otimes B$, called *coevaluation*, and $\epsilon\colon B \otimes A \to I$, called *evaluation*, in $\mathscr{C}$ such that the following composites are the identity maps

$$A \cong I \otimes A \xrightarrow{\eta \otimes \mathrm{id}_A} A \otimes B \otimes A \xrightarrow{\mathrm{id}_A \otimes \epsilon} A \otimes I \cong A$$

$$B \cong B \otimes I \xrightarrow{\mathrm{id}_B \otimes \eta} B \otimes A \otimes B \xrightarrow{\epsilon \otimes \mathrm{id}_B} I \otimes B \cong B.$$

We call B the *dual* of A and we say (A, B) is a *dual pair*.

Note that any two duals of a dualizable object are isomorphic.

Let R be a commutative ring and Mod_R be the category of R-modules. Then Mod_R is a symmetric monoidal category using the usual tensor product over R. The ring R thought of as a module over itself is the unit.

The dual of a finitely generated free R-module M is $\mathrm{Hom}_R(M, R)$. This is also a finitely generated free R-module. If M has a basis $\{m_1, m_2, \ldots, m_n\}$ and dual basis $\{m_1', m_2', \ldots, m_n'\}$ the coevaluation and evaluation for the dual pair,

$$\eta\colon R \longrightarrow M \otimes_R \mathrm{Hom}_R(M, R) \qquad \text{and} \qquad \epsilon\colon \mathrm{Hom}_R(M, R) \otimes_R M \longrightarrow R,$$

are R-module homomorphisms given by $\epsilon(\phi, m) = \phi(m)$ and by linearly extending the map $\eta(1) = \sum_i m_i \otimes m_i'$.

If M is a finitely generated projective module it is also dualizable with dual $\mathrm{Hom}_R(M, R)$. The evaluation map is $\epsilon(\phi, m) = \phi(m)$. The dual basis theorem implies that there is a 'basis' $\{m_1, m_2, \ldots, m_n\}$ of M and dual 'basis' $\{m_1', m_2', \ldots, m_n'\}$ of $\mathrm{Hom}_R(M, R)$. The coevaluation map is given by linearly extending $\eta(1) = \sum_i m_i \otimes m_i'$.

Let Ch_R be the symmetric monoidal category of chain complexes of modules over a commutative ring R and chain maps. The dualizable chain complexes are the chain complexes that are projective in each degree and finitely generated. The dual of a finitely generated projective chain complex M is the chain complex $\mathrm{Hom}_R(M, R)$.

Theorem 1.2.2. — *Let A and B be objects in $\mathscr{C}$ and $\epsilon\colon B\otimes A \to I$ be a morphism in $\mathscr{C}$. Then the following are equivalent.*

(i) *B is the dual of A with evaluation ϵ.*

(ii) *The map $\epsilon/(-)\colon \mathscr{C}(C, D\otimes B) \to \mathscr{C}(C\otimes A, D)$ which sends $f\colon C\to D\otimes B$ to*

$$C\otimes A \xrightarrow{f\otimes \mathrm{id}} D\otimes B\otimes A \xrightarrow{\mathrm{id}\otimes\epsilon} D\otimes I \cong D$$

is a bijection for all objects $C, D \in \mathscr{C}$.

There is a similar characterization for a map $\eta\colon I\to A\otimes B$. If the category $\mathscr{C}$ is closed, there is another characterization of dualizable objects; see **[28]**.

Definition 1.2.3. — For a dualizable object A, the *trace* of $f\colon A\to A$ is the composite

$$I \xrightarrow{\eta} A\otimes B \xrightarrow{f\otimes \mathrm{id}} A\otimes B \xrightarrow{\gamma} B\otimes A \xrightarrow{\epsilon} I.$$

The trace is independent of the choice of B, η, and ϵ.

Let R be a commutative ring and M be a finitely generated projective R-module with 'basis' $\{m_1,\ldots,m_n\}$. The trace of a map of R-modules $f\colon M\to M$ is a map $R\to R$ and the image of 1 is

$$\sum_i m_i' f(m_i).$$

If R is a field the image of 1 is the usual trace of f regarded as a matrix. For a map $f\colon M\to M$ of chain complexes, the trace is also a map $R\to R$ and the image of 1 is

$$\sum_i (-1)^{\deg(m_i)} m_i'(f(m_i)),$$

the Lefschetz number of f. The sign comes from the sign in the symmetry isomorphism.

Let $\mathscr{C}$ and $\mathscr{C}'$ be symmetric monoidal categories. A *lax monoidal functor* consists of a functor

$$F\colon \mathscr{C}\to\mathscr{C}'$$

and natural transformations

$$\phi\colon FA\otimes FB \longrightarrow F(A\otimes B) \qquad \text{and} \qquad I' \longrightarrow FI$$

subject to the standard coherence conditions.

A lax monoidal functor is *symmetric* if the following diagram commutes.

$$\begin{array}{ccc} F(A)\otimes F(B) & \xrightarrow{\phi} & F(A\otimes B) \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle F(\gamma)} \\ F(B)\otimes F(A) & \xrightarrow{\phi} & F(B\otimes A) \end{array}$$

Lemma 1.2.4. — *If A is a dualizable object with dual DA, $F\colon \mathscr{C} \to \mathscr{C}'$ is a lax symmetric monoidal functor, and $\phi\colon FA \otimes FDA \to F(A \otimes DA)$ and $I' \to FI$ are isomorphisms, then FA and FDA are a dual pair with evaluation*

$$FDA \otimes FA \xrightarrow{\phi} F(DA \otimes A) \xrightarrow{F(\epsilon)} FI \cong I'$$

and coevaluation

$$I' \cong FI \xrightarrow{F(\eta)} F(A \otimes DA) \xrightarrow{\phi^{-1}} FA \otimes FDA.$$

This lemma follows from the definition of a dual pair. As an immediate consequence of this lemma we have the following corollary.

Corollary 1.2.5. — *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a lax symmetric monoidal functor and A be a dualizable object of $\mathscr{C}$ with dual DA such that $\phi\colon FA \otimes FDA \to F(A \otimes DA)$ and $I' \to FI$ are isomorphisms. Then*

$$\mathrm{trace}(Ff) = F(\mathrm{trace}(f))$$

for any endomorphism f of A.

The Künneth theorem implies that the homology functor satisfies the conditions of Corollary 1.2.5 if C_* is a finitely generated chain complex of projective modules, the images of the boundary maps are projective, and each $H_i(C_*)$ is projective. In particular, the rational singular or cellular chains on a compact manifold satisfy these conditions.

Remark 1.2.6. — There are also axiomatic descriptions of traces, see [**23**, **30**]. For symmetric monoidal categories where all objects are dualizable these two notions of trace coincide.

1.3. Duality and trace for topological spaces

Spanier-Whitehead duality for topological spaces is an example of duality in symmetric monoidal categories using the stable homotopy category. We will describe an equivalent definition that is more intuitive. For the perspective using the stable homotopy category see [**11**], [**28**, III], or [**34**, 15].

Definition 1.3.1. — [**28**, III.3.5] Let X be a compact, based topological space. A based space Y is *n-dual* to X if there are maps $\eta\colon S^n \to X \wedge Y$, called *coevaluation*, and $\epsilon\colon Y \wedge X \to S^n$, called *evaluation*, such that the following diagrams commute stably up to homotopy.

$$\begin{array}{ccc} S^n \wedge X & \xrightarrow{\eta \wedge \mathrm{id}} & (X \wedge Y) \wedge X \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \cong} \\ X \wedge S^n & \xleftarrow[\mathrm{id} \wedge \epsilon]{} & X \wedge (Y \wedge X) \end{array} \qquad \begin{array}{ccc} Y \wedge S^n & \xrightarrow{\mathrm{id} \wedge \eta} & Y \wedge (X \wedge Y) \\ {\scriptstyle (\sigma \wedge \mathrm{id})\gamma}\downarrow & & \downarrow{\scriptstyle \cong} \\ S^n \wedge Y & \xleftarrow[\epsilon \wedge \mathrm{id}]{} & (Y \wedge X) \wedge Y \end{array}$$

Here $\sigma\colon S^n \to S^n$ is a map of degree $(-1)^n$.

For closed smooth manifolds and compact ENR's we can explicitly describe dual pairs.

Theorem 1.3.2. — [**11**, 3.1][**28**, III.5.1]

(i) *Let $K \subset \mathbb{R}^n$ be a compact ENR. Then K_+ and the cone on the inclusion of $\mathbb{R}^n \setminus K$ into $\mathbb{R}^n$ are n-dual.*
(ii) *Let M be a closed smooth manifold embedded in $\mathbb{R}^n$. Then M_+ and the Thom space of the normal bundle ν of $M \to \mathbb{R}^n$ are n-dual.*

As usual, M_+ is M with a disjoint base point added. We will denote the Thom space of the embedding of M in $\mathbb{R}^n$ by $T\nu$.

Many of the explicit characterizations of dual pairs for topological spaces can be stated in this form. Compact ENR's are the natural generality, but closed smooth manifolds are the practical generality. Since the duality maps for manifolds are easier to describe, we will state most results in that form but there are also generalizations to compact ENR's.

The coevaluation map for the dual pair $(M_+, T\nu)$

$$S^n \to T\nu \to M_+ \wedge T\nu,$$

is the composite of the Pontryagin-Thom map for the normal bundle of the embedding $M \to S^n$ and the Thom diagonal. If $\sigma\colon M \to \nu$ is the zero section, the evaluation map

$$T\nu \wedge M_+ \to M_+ \wedge S^n \to S^n$$

is the composite of the Pontryagin-Thom map associated to a tubular neighborhood of

$$M \stackrel{\triangle}{\to} M \times M \stackrel{\sigma\times\mathrm{id}}{\to} \nu \times M$$

and the projection $M_+ \wedge S^n \to S^n$.

Let $\{-,-\}$ denote the stable maps of based spaces. From the characterizations of dual pairs in Theorem 1.2.2 and Theorem 1.3.2 we have the following corollary.

Corollary 1.3.3. — *If M is a closed smooth manifold embedded in $\mathbb{R}^n$ and Z and W are based spaces then*

$$\{Z \wedge M_+, W\} \cong \{S^n \wedge Z, W \wedge T\nu\}.$$

Definition 1.3.4. — Let X be a based space with n-dual Y and coevaluation and evaluation maps $\eta\colon S^n \to X \wedge Y$ and $\epsilon\colon Y \wedge X \to S^n$. If $f\colon X \to X$ is a continuous map, then the *trace* of f is the stable homotopy class of the map

$$S^n \xrightarrow{\eta} X \wedge Y \xrightarrow{f\wedge\mathrm{id}} X \wedge Y \xrightarrow{\gamma} Y \wedge X \xrightarrow{\epsilon} S^n.$$

By examining the explicit duality maps for a closed smooth manifold M (or a compact ENR) we can see that for $f\colon M \to M$,

$$\mathrm{index}(f) = \tilde{H}_*(\mathrm{trace}(f_+); \mathbb{Q}).$$

This is described in detail in [**7, 8, 11**].

Applying rational homology to the coevaluation and evaluation maps of the dual pair $(M_+, T\nu)$ we get a pair of maps

$$\eta\colon \mathbb{Q} \to \sum_i \tilde{H}_i(M_+; \mathbb{Q}) \otimes \tilde{H}_{n-i}(T\nu; \mathbb{Q})$$

and

$$\epsilon\colon \sum_i \tilde{H}_{n-i}(T\nu; \mathbb{Q}) \otimes \tilde{H}_i(M_+; \mathbb{Q}) \to \mathbb{Q}.$$

Since these maps come from the dual pair $(M_+, T\nu)$ they are coevaluation and evaluation maps that make $(\tilde{H}_i(M_+; \mathbb{Q}), \tilde{H}_{n-i}(T\nu; \mathbb{Q}))$ a dual pair. The trace of a map does not depend on the choice of dual pair, so the trace of $\tilde{H}_*(f_+; \mathbb{Q})$ with respect to this dual pair is the Lefschetz number. Theorem 1.1.5 follows from this observation:

$$\mathrm{index}(f) = \text{Lefschetz number}(f).$$

1.4. Duality and trace for fiberwise topological spaces

We can also define trace and duality in the symmetric monoidal category of ex-spaces over a fixed space B. The objects in this category are spaces E with maps $B \xrightarrow{\sigma} E \xrightarrow{p} B$ such that $p \circ \sigma$ is the identity map of B. The map σ is called the *section* and p is called the *projection*. The morphisms are maps $E \to E'$ that commute with the section and projection. The product is the internal smash product, $\wedge_B$. Given two ex-spaces X and Y over B, we can form the pullback along the maps to B, $X \times_B Y$, and the pushout along the maps from B, $X \vee_B Y$. The *internal smash product* is the pushout

$$\begin{array}{ccc} X \vee_B Y & \longrightarrow & X \times_B Y \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \wedge_B Y. \end{array}$$

Remark 1.4.1. — For all ex-spaces we require that the base space and total space are of the homotopy type of CW complexes, the projection map is a Hurewicz fibration and the section is a fiberwise cofibration. The category of these spaces and the fiberwise homotopy classes of maps is equivalent to the model theoretic homotopy category of ex-spaces defined in [**34**]. See [**34**, 9.1.2].

If we need to consider an ex-space that does not satisfy these conditions we will replace it with an equivalent space that does satisfy our requirements. We will not indicate the replacement in the notation.

Definition 1.4.2. — Let X be an ex-space over B. An ex-space Y is *n-dual* to X if there are fiberwise stable maps

$$\eta\colon S^n \times B \to X \wedge_B Y \qquad \text{and} \qquad \epsilon\colon Y \wedge_B X \to S^n \times B$$

such that

$$\begin{array}{ccc} S^n \wedge X & \xrightarrow{\eta\wedge \mathrm{id}} & (X \wedge_B Y) \wedge_B X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\cong} \\ X \wedge S^n & \xleftarrow[\mathrm{id}\wedge\epsilon]{} & X \wedge_B (Y \wedge_B X) \end{array} \qquad \begin{array}{ccc} Y \wedge S^n & \xrightarrow{\mathrm{id}\wedge \eta} & Y \wedge_B (X \wedge_B Y) \\ {\scriptstyle(\sigma\wedge\mathrm{id})\gamma}\downarrow & & \downarrow{\scriptstyle\cong} \\ S^n \wedge Y & \xleftarrow[\epsilon\wedge\mathrm{id}]{} & (Y \wedge_B X) \wedge_B Y \end{array}$$

commute up to stable fiberwise homotopy.

This definition is very similar to the definition of n-duality in the category of based topological spaces. Using parametrized spectra, this definition can be expressed as duality in a symmetric monoidal category, see **[11, 34]**. We also have explicit characterizations of dual pairs similar to those in Theorem 1.3.2.

Theorem 1.4.3. — [**6**, II.12.18][**11**, 6.1][**34**, 15.1.1]

(i) *Let L be an ENR_B over a paracompact space B such that $p\colon L \to B$ is proper. Then $L_+ = L \amalg B$ and the cone on the inclusion of $(B \times \mathbb{R}^n) \setminus L$ in $B \times \mathbb{R}^n$ are an n-dual pair.*

(ii) *Let N be an ex-space over B that satisfies the conditions of Remark 1.4.1. Then N is dualizable as an ex-space if and only if $p^{-1}(b)$ for $b \in B$ is dualizable in the sense of Definition 1.3.1.*

In particular, if $E \to B$ is a fiber bundle with closed smooth manifold fibers the ex-space E_+ is dualizable.

The definition of trace for a fiberwise map is identical to the definition of trace for a map of based spaces.

Definition 1.4.4. — Let X be an n-dualizable ex-space over B with dual Y and $f\colon X \to X$ be a fiberwise map over B. The *trace* of f is the fiberwise stable homotopy class of the composite

$$B \times S^n \xrightarrow{\eta} X \wedge_B Y \xrightarrow{f\wedge\mathrm{id}} X \wedge_B Y \xrightarrow{\gamma} Y \wedge_B X \xrightarrow{\epsilon} B \times S^n.$$

When X is a compact fiberwise ENR over a compactly generated paracompact base space, this is the fiberwise Dold index of f as defined in **[8]**.

Remark 1.4.5. — There is a category $\mathscr{F}$ with objects ex-spaces over B as before but whose morphisms are pairs of maps $f\colon E \to E$ and $\bar{f}\colon B \to B$ such that

$$\begin{array}{ccc} B & \xrightarrow{\bar{f}} & B \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ E & \xrightarrow{f} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\bar{f}]{} & B \end{array}$$

commutes. The category of ex-spaces is the subcategory of $\mathscr{F}$ with the same objects whose morphisms are the pairs $(f, \bar{f})$ where $\bar{f}$ is the identity.

Fixed point theory in $\mathscr{F}$ and in the category of ex-spaces are very different. We will not discuss fixed point theory in $\mathscr{F}$; some references for it include **[18, 26]**.

CHAPTER 2

THE CONVERSE TO THE LEFSCHETZ FIXED POINT THEOREM

The invariants described in the previous chapter only give a converse to the Lefschetz fixed point theorem under additional hypotheses. The crucial assumption is that the spaces are simply connected. We can relax this assumption by changing the invariant. The Nielsen number and various forms of the Reidemeister trace are choices for this refined invariant.

The Reidemeister trace, in any of its forms, is not an example of trace in symmetric monoidal categories. The reasons for this incompatibility will become clear as we define these invariants and compare different features of these invariants in this chapter and in the following chapters.

Despite the many differences between the Reidemeister trace and trace in symmetric monoidal categories it will also become clear that the Reidemeister trace can be described using a trace very much like the trace in a symmetric monoidal category. The structure suggested in this chapter and the next chapter is explicitly described in Chapters 4, 5, and 6.

2.1. The Nielsen number

To define the index we needed to assume that the fixed point set

$$F = \{x \in M | f(x) = x\}$$

is compact; now we will also assume that it is discrete. All invariants discussed here are invariants of homotopy classes of maps so choosing a homotopic representative doesn't change the invariant. If M is a manifold, transversality implies that any endomorphism of M is homotopic to a map with a discrete fixed point set.

Definition 2.1.1. — Two fixed points of $f\colon M \to M$, x and y, are in the same *fixed point class* if there is a lift of f to

$$\tilde{f}\colon \tilde{M} \to \tilde{M}$$

on the universal cover of M and lifts of x and y to $\tilde{x}$ and $\tilde{y}$ in $\tilde{M}$ such that $\tilde{f}(\tilde{x}) = \tilde{x}$ and $\tilde{f}(\tilde{y}) = \tilde{y}$.

There is an equivalent definition of fixed point classes using paths in M rather than the universal cover.

Lemma 2.1.2. — *Two fixed points x and y of $f\colon M \to M$ are in the same fixed point class if and only if there is a path α from x to y such that α is homotopic to $f(\alpha)$ with endpoints fixed.*

We can give another interpretation of this lemma in terms of the fundamental groupoid, ΠM, of M. Let Fix be the groupoid defined by the equalizer

$$\mathrm{Fix} \longrightarrow \Pi M \underset{f}{\overset{\mathrm{id}}{\rightrightarrows}} \Pi M.$$

The objects of Fix are the fixed points of f and the morphisms are the homotopy classes of paths with $[\alpha] = [f \circ \alpha]$. If we think of the fixed points as a discrete category, this category includes into the category Fix. The lemma shows that two fixed points are in the same fixed point class if and only if their images are in the same connected component of Fix. A fourth description of fixed point classes can be found in [**3**].

Since the fixed point set is assumed to be compact and discrete, it must be finite. Let $F_1, F_2, \ldots, F_k$ be the fixed point classes of f. This is also a finite set, $\cup F_i = F$, and $F_i \cap F_j = \varnothing$ if $i \neq j$.

Let $f\colon M \to M$ be a continuous map with a compact and discrete, and hence finite, fixed point set. For a fixed point class F_i, let V_i be an open set in M such that $F_i \subset V_i$ and $V_i \cap F_j$ is empty if $i \neq j$. Define the index of F_i, $i(F_i)$, to be the index of $f|V_i$. Since the index is local this is well defined.

Definition 2.1.3. — The *Nielsen number* of f, $N(f)$, is the number of fixed point classes with nonzero index.

***Theorem 2.1.4* (Theorem C).** — *The Nielsen number of a continuous endomorphism of a closed smooth manifold of dimension at least three is zero if and only if the map is homotopic to a map with no fixed points.*

The standard proof of this result uses simplicial techniques, see [**2**, VIII]. That proof shows that the theorem holds for simplicial complexes where the star of each vertex is connected. We will give a more conceptual proof in Chapter 6.

The dimension hypothesis is necessary. In [**22**], Jiang constructed an endomorphism for any two dimensional connected manifold with negative Euler characteristic that is not homotopic to a fixed point free map but whose Nielsen number is zero.

2.2. The geometric Reidemeister trace

Using the fixed point classes and the index of a continuous map $f\colon M \to M$ we can also define the geometric Reidemeister trace. This invariant contains all of the information in the Nielsen number so it can also be used to prove a converse to the Lefschetz Fixed Point Theorem.

Choose a base point $*$ in M, a path ζ from $*$ to $f(*)$, and a path γ_x from $*$ to x for each fixed point x of f. The map that takes a fixed point x to the homotopy class of $\gamma_x^{-1}f(\gamma_x)\zeta$ defines a function from the fixed points of f to the fundamental group of M.

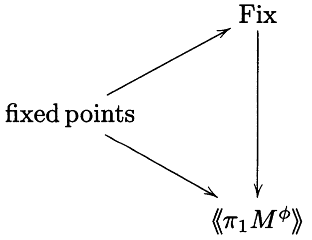
Definition 2.2.1. — Let π be a group and $\psi\colon \pi \to \pi$ a homomorphism. The set $\langle\!\langle \pi^\psi \rangle\!\rangle$ of *semiconjugacy classes* of π is the set π modulo the relation $\alpha \sim \beta\alpha\psi(\beta^{-1})$ for $\alpha, \beta \in \pi$.

Using the path ζ we can define a homomorphism $\phi\colon \pi_1 M \to \pi_1 M$ by $\phi(\alpha) = \zeta^{-1}f(\alpha)\zeta$. The map from the fixed points of f to $\pi_1 M$ descends to a well-defined injection from the fixed point classes of f to $\langle\!\langle \pi_1 M^\phi \rangle\!\rangle$ that is independent of all choices of paths.

The paths ζ and γ_x also define a map

$$\mathrm{Fix} \to \langle\!\langle \pi_1 M^\phi \rangle\!\rangle$$

by $x \mapsto \gamma_x^{-1}f(\gamma_x)\zeta$, and the diagram



commutes.

We let $\mathbb{Z}\langle\!\langle \pi_1 M^\phi \rangle\!\rangle$ denote the free abelian group on the set $\langle\!\langle \pi_1 M^\phi \rangle\!\rangle$. The injection above gives an identification of a fixed point class F_k with its image in $\langle\!\langle \pi_1 M^\phi \rangle\!\rangle$.

Definition 2.2.2. — The *geometric Reidemeister trace* of f, $R^{geo}(f)$, is

$$\sum_{\text{Fixed Point Classes } F_k} i(F_k) \cdot F_k \in \mathbb{Z}\langle\!\langle \pi_1(M)^\phi \rangle\!\rangle.$$

The index of f is the sum of the coefficients in the Reidemeister trace. The Nielsen number is the number of elements in $\langle\!\langle \pi_1(M)^\phi \rangle\!\rangle$ with nonzero coefficients in the Reidemeister trace.

Note that the geometric Reidemeister trace is zero if and only if the Nielsen number is zero.

***Theorem 2.2.3* (Theorem D).** — *If M is a closed smooth manifold of dimension at least three, then the geometric Reidemeister trace of an endomorphism f of M is zero if and only if f is homotopic to a map with no fixed points.*

2.3. The algebraic Reidemeister trace

We generalized the index to the geometric Reidemeister trace using the fixed point classes of an endomorphism. We can also use fixed point classes to generalize the Lefschetz number to the algebraic Reidemeister trace.

The algebraic Reidemeister trace is based on a generalization of the trace for linear transformations to a trace for homomorphisms of finitely generated projective modules. We will recall the definition of this generalized trace, called the Hattori-Stallings trace, first and then use it to define the algebraic Reidemeister trace.

Let R be a ring. A *trace function* is a function, T, from square matrices over R to an abelian group such that

(i) If $A, B \in \mathscr{M}_{p\times p}(R)$ then $T(A+B) = T(A) + T(B)$.
(ii) If $A \in \mathscr{M}_{p\times q}(R)$ and $B \in \mathscr{M}_{q\times p}(R)$ then $T(AB) = T(BA)$.

These condition imply $T(A) = \sum_i T(a_{ii})$ for $A = (a_{ij})$.

From a ring R we define an abelian group $\langle\langle R\rangle\rangle$ as the quotient of R by the subgroup generated by elements of the form

$$r_1 r_2 - r_2 r_1$$

for $r_1, r_2 \in R$.

Proposition 2.3.1. — [43] *The universal trace function, a trace function through which every trace function can be factored, is given on 1×1-matrices by the quotient map*

$$\mathscr{T}\colon R \to \langle\langle R\rangle\rangle.$$

This extends to $n\times n$ matrices by $\mathscr{T}(A) = \sum_i \mathscr{T}(a_{ii})$ for $A = (a_{ij})$.

Let M be a finitely generated free right R-module. Given any trace function T we can define a map $T\colon \mathrm{End}(M) \to \langle\langle R\rangle\rangle$. For each endomorphism ϕ choose a matrix A representing ϕ and define

$$T(\phi) := T(A).$$

By the second property of a trace function this is well defined.

We can also use a trace function T to define a map

$$T\colon \mathrm{End}(M) \to \langle\langle R\rangle\rangle$$

for a finitely generated projective right R-module M. Let N be a module such that $M \oplus N$ is a finitely generated free R-module. If ϕ is an endomorphism of M define an endomorphism of the free module $M\oplus N$ by $\phi \oplus 0$. Then $T(\phi)$ is defined to be $T(\phi \oplus 0)$. This is independent of all choices.

This description is given in terms of a 'basis' since it is defined using matrices. There is an equivalent definition of the universal trace function that does not require explicit use of the basis. For any right R-modules P and M there is a map

$$\nu\colon P\otimes_R \mathrm{Hom}_R(M,R) \longrightarrow \mathrm{Hom}_R(M,P) \tag{2.3.2}$$

defined by $\nu(p\otimes\phi)(m) = p\phi(m)$. If M is a finitely generated projective right R-module this map is an isomorphism.

Proposition 2.3.3. — *If M is a finitely generated projective right R-module, the universal trace function is the composite map*

$$\mathrm{Hom}_R(M,M) \xrightarrow{\nu^{-1}} M\otimes_R \mathrm{Hom}_R(M,R) \xrightarrow{\delta} \langle\!\langle R\rangle\!\rangle$$

where $\delta(m\otimes\phi) = \mathscr{T}(\phi(m))$.

The algebraic Reidemeister trace requires a generalization of the universal trace function. Before we can generalize the trace we need a more general target for a trace function.

Definition 2.3.4. — Let P be an R-R-bimodule. Then $\langle\!\langle P\rangle\!\rangle$ is the quotient of P, as an abelian group, by the subgroup generated by elements of the form $pr - rp$ for $r\in R$ and $p\in P$.

Let $\mathscr{T}\colon P\to\langle\!\langle P\rangle\!\rangle$ be the quotient map.

The generalization of the universal trace function is easier to describe when not explicitly using a basis, so we will generalize the description given in Proposition 2.3.3. Let R be a ring, P be an R-R-bimodule, and M be a finitely generated projective right R-module.

Definition 2.3.5. — The *Hattori-Stallings trace*, tr, of a map $f\colon M\to M\otimes_R P$ is the image of f under the composite

$$\mathrm{Hom}_R(M,M\otimes_R P) \xrightarrow{\nu^{-1}} (M\otimes_R P)\otimes_R \mathrm{Hom}_R(M,R) \xrightarrow{\delta} \langle\!\langle P\rangle\!\rangle$$

where $\delta(m\otimes p\otimes\phi) = \mathscr{T}(p\phi(m))$.

If C_* is a finitely generated chain complex of projective right R-modules, P is an R-R-bimodule, and $f\colon C_*\to C_*\otimes_R P$ is a map of chain complexes, the *Hattori-Stallings trace* of f is

$$\sum(-1)^i\mathrm{tr}(f_i),$$

the alternating sum of the levelwise traces. The sign enters since the evaluation map requires a transposition.

The example of the Hattori-Stallings trace we are most interested in is the algebraic Reidemeister trace. To define this invariant we must first fix some conventions. Composition of paths in a space X is given by $(\beta,\alpha)\mapsto\beta\alpha$ where α is a path from a to b, β is a path from b to c and $\beta\alpha$ is a path from a to c. This induces the group multiplication in $\pi_1 X$. If we think of $\tilde{X}$ as homotopy classes of paths in X that start at the base point $*$ there is an action of $\pi_1 X$ on $\tilde{X}$ from the right by $(\gamma, \alpha) \mapsto \gamma\alpha$.

Let X be a finite connected CW complex. Pick a base point $*$ in X. Then the cellular chain complex of $\tilde{X}$ is a finitely generated free right $\mathbb{Z}\pi_1(X,*)$-module. A continuous map $f\colon X \to X$ is not required to preserve a base point, and so we define an induced map $\tilde{f}$ on the universal cover by $\tilde{f}(\alpha) = f(\alpha)\zeta$ for some choice of path ζ from $*$ to $f(*)$. Define a group homomorphism

$$\phi\colon \pi_1(X,*) \to \pi_1(X,*)$$

by $\phi(\alpha) = \zeta^{-1} f(\alpha)\zeta$. The map $\tilde{f}$ is ϕ-equivariant in the sense that

$$\tilde{f}(\gamma\alpha) = \tilde{f}(\gamma)\phi(\alpha)$$

for $\alpha \in \pi_1(X,*)$ and $\gamma \in \tilde{X}$.

Let $\mathbb{Z}\pi_1(X,*)^\phi$ be the $\mathbb{Z}\pi_1(X,*) - \mathbb{Z}\pi_1(X,*)$-bimodule that is $\mathbb{Z}\pi_1(X,*)$ as an abelian group with the usual left action of $\pi_1(X,*)$ and the right action given by first applying ϕ and then using the group multiplication. Then $\tilde{f}$ defines a map

$$\tilde{f}_*\colon C_*\tilde{X} \to C_*\tilde{X} \otimes_{\mathbb{Z}\pi_1(X,*)} \mathbb{Z}\pi_1(X,*)^\phi$$

and this is a map of right $\mathbb{Z}\pi_1(X,*)$-modules.

Definition 2.3.6. — The *algebraic Reidemeister trace* of f, $R^{alg}(f)$, is the Hattori-Stallings trace of $\tilde{f}_*$.

Theorem 2.3.7. — *There is an isomorphism of abelian groups*

$$\mathbb{Z}\langle\!\langle \pi_1(X,*)^\phi \rangle\!\rangle \to \langle\!\langle \mathbb{Z}\pi_1(X,*)^\phi \rangle\!\rangle$$

and under this isomorphism

$$R^{geo}(f) = R^{alg}(f).$$

A proof of this theorem can be found in [**14**, 3.4] or [**19**, 1.13]. In Chapter 6 we will give a more conceptual proof of this result.

2.4. A proof of the converse to the Lefschetz fixed point theorem

In [**46**], Wecken showed that for some finite polyhedra the Nielsen number of an endomorphism is zero if and only if the map is homotopic to a map with no fixed points. Shi [**13**] later proved a refinement of Wecken's result. These proofs used simplicial techniques. Similar techniques can be used to prove an equivariant analogue of this result, but they are not as useful when trying to prove fiberwise results.

Here we will present the main ideas of an alternative proof due to Klein and Williams from [**25**]. This proof gives the converse to the Lefschetz fixed point theorem for manifolds of dimension at least three and has fiberwise and equivariant generalizations. The details of the fiberwise version are in Section 8.2. The missing details in this section can be recovered from that proof.

For their proof Klein and Williams translate fiberwise homotopy theory into equivariant homotopy theory using a loop group construction. They observe that their proof works equally well without this transformation and that it would be necessary to eliminate this transition to prove a converse to the fiberwise Lefschetz fixed point theorem. Here we present the main ideas of Klein and Williams' proof of the converse to the Lefschetz fixed point theorem using fiberwise homotopy theory.

Proposition 2.4.1. — **[12][25]**

(i) *Let X be a topological space and $f\colon X \to X$ a continuous map. Homotopies of f to a fixed point free map correspond to liftings which make the following diagram commute up to homotopy.*

$$\begin{array}{ccc} & & X \times X - \Delta \\ & \nearrow & \downarrow \\ X & \xrightarrow[\Gamma_f]{} & X \times X. \end{array}$$

Here Γ_f is the graph of f.

(ii) *For a continuous map $h\colon X \to Z$ let $r(h)\colon N(h) \to Z$ be a Hurewicz fibration such that*

$$\begin{array}{ccc} X & \longrightarrow & N(h) \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle r(h)} & \\ & Z & \end{array}$$

commutes and the map $X \to N(h)$ is a homotopy equivalence. There is a bijective correspondence between liftings up to homotopy in the diagram

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

and sections of the fibration $g^*N(h) \to Y$.

This proposition converts a fixed point question into a question about sections of fibrations. We will define a fixed point theory invariant by defining an invariant that detects sections of Hurewicz fibrations.

Let $p\colon E \to B$ be a Hurewicz fibration over a connected space B. The *unreduced fiberwise suspension* of E over B is the double mapping cylinder

$$S_B E := B \times \{0\} \cup_p E \times [0,1] \cup_p B \times \{1\}.$$

The map $p\colon E \to B$ induces a map $q\colon S_B E \to B$ which is also a fibration. [(1)] Let

$$\sigma_-, \sigma_+ \colon B \to S_B E$$

be the sections of $S_B E \to B$ given by the inclusions of $B \times \{0\}$ and $B \times \{1\}$ into $S_B E$.

Proposition 2.4.2. — [**25**, 3.1] *If $p\colon E \to B$ admits a section then σ_- and σ_+ are homotopic over B.*

Conversely, assume $p\colon E \to B$ is $(r+1)$-connected and B is homotopically a retract of a cell complex with cells in dimensions $\leq 2r+1$. If σ_- and σ_+ are homotopic over B, then p has a section.

From this point we will work in the category of ex-spaces, rather than spaces over B. This means that all spaces over B have a section and all maps respect the section. In particular, $S_B E$ is an ex-space with section σ_-.

Let S^0_B be the ex-space over B with total space two disjoint copies of B. The inclusion of B into one of the copies of B is the section. The projection map is the identity on each component. Under the assumptions in Proposition 2.4.2, a fiberwise version of the Freudenthal suspension theorem [**6**, 3.19] gives the following isomorphism

$$[S^0_B, S_B E]_B \cong \{S^0_B, S_B E\}_B.$$

The $\{-,-\}_B$ notation indicates fiberwise (sectioned) stable homotopy classes of fiberwise maps.

Definition 2.4.3. — [**25**, 3.4] The *stable cohomotopy Euler class* of p is the element of

$$\{S^0_B, S_B E\}_B$$

that corresponds to the map $\sigma_- \amalg \sigma_+$.

For a continuous map $f\colon M \to M$ there is an associated fibration

$$\Gamma^*_f(r(i))\colon \Gamma^*_f(N(i)) \to M$$

given by pulling the fibration associated to the inclusion

$$i\colon M \times M - \triangle \to M \times M$$

back along the graph of f. For this particular case we can give another description of the stable cohomotopy Euler class. Let $\Lambda^f M = \{\gamma \in M^I | f(\gamma(1)) = \gamma(0)\}$.

(1) If $\lambda\colon Np \to E^I$ is a lifting function for p with adjoint $\bar{\lambda}$ define

$$\bar{\chi}\colon Nq \times I \to S_B E$$

by $\bar{\chi}((e,t),\beta,s) = (\bar{\lambda}(e,\beta,s),t)$ for $(e,t) \in E \times (0,1)$ and $\bar{\chi}(b,\beta,s) = b \in B \times \{0\}$ for $b \in B \times \{0\}$ and similarly for $b \in B \times \{1\}$. Then the adjoint of $\bar{\chi}$,

$$\chi\colon Nq \to (S_B E)^I,$$

is a lifting function for q. See [**15, 44**] for similar results.

Proposition 2.4.4. — [**25**, 4.1, 5.1] *There is an isomorphism*

$$\{S^0_M, S_M(\Gamma^*_f(N(i)))\}_M \cong \{S^0, \Lambda^f M_+\}.$$

We will denote the image of $\sigma_- \amalg \sigma_+$ under this isomorphism by $R^{KW}(f)$.

If M is of dimension n then $\Gamma^*_f N(i) \to M$ is $(n-1)$-connected, see [**25**, 6.1,6.2]. If n is at least three then the corollary below follows from Proposition 2.4.2.

Corollary 2.4.5. — [**25**, 10.1] *If M is a closed smooth manifold of dimension at least 3, f is homotopic to a map with no fixed points if and only if $R^{KW}(f)$ is zero.*

CHAPTER 3

TOPOLOGICAL DUALITY AND FIXED POINT THEORY

In the previous chapters we recalled the definitions of some classical fixed point theory invariants. The definition of the algebraic Reidemeister trace is very similar to the trace in the symmetric monoidal category of modules over a commutative ring. From the definition we gave in the last chapter it is less clear that the geometric Reidemeister trace resembles the trace in a symmetric monoidal category.

In this chapter we will give another description of the geometric Reidemeister trace that will make the similarity with the trace in a symmetric monoidal category more clear. We will give definitions of duality and trace that resemble the definitions for a symmetric monoidal category and are also similar to the definition of the Hattori-Stallings trace for modules over a ring. We will show these constructions are examples of duality and trace in a bicategory with shadows in Chapters 4 and 5.

We will also describe an invariant, originally defined by Crabb and James, that can be identified with Klein and Williams' invariant. The definition of this invariant is similar to the geometric Reidemeister trace but it has two very significant differences. This invariant does not require a base point. We also replace homotopy classes of paths with path spaces.

In this chapter we are mostly interested in the impact of these changes on the invariants that we have already described, but the changes are even more important for fiberwise invariants. For a fiberwise space choosing a base point would correspond to choosing a section and sections do not always exist. So fiberwise invariants need to be unbased. The change to path spaces reflects the greater variety of invariants for fiberwise spaces. In the classical case, the invariant defined by Crabb and James is only zero when the geometric Reidemeister trace is zero. These invariants have fiberwise generalizations that do not share this property.

Since the techniques of Chapter 4 significantly simplify some of the proofs of the results in this chapter we will delay most of the proofs until Chapters 5 and 6.

3.1. Duality for spaces with group actions

In this section and the next section we will give an alternate definition of the geometric Reidemeister trace that is closer to the trace in a symmetric monoidal category. We first define duality for spaces with an action by a group π. The motivating example of a space with a group action is the action of the fundamental group of a manifold on the universal cover by deck transformations. In the next section we use this definition of dual pairs to define a trace. This trace is similar to the Hattori-Stallings trace.

There are two important observations about the duality defined in this section. First, this duality is more similar in perspective to the duality defined by Ranicki in [**39**, 3] than to duality in the symmetric monoidal category of G-spaces for a compact Lie group G. Second, despite the action of the group π, this duality is used to study classical invariants, not equivariant ones.

Let π be a discrete group. For a based right π-space X and a based left π-space Y, let $X \odot Y$ denote the based bar complex $B(X, \pi, Y)$.

The bar complex $B(X, \pi, Y)$ is the geometric realization of the simplicial based space with n simplices

$$X \wedge (\pi^n)_+ \wedge Y,$$

face maps

$$\begin{aligned} \partial_0(x, g_1, g_2, \ldots, g_n, y) &= (xg_1, g_2, \ldots, g_n, y) \\ \partial_i(x, g_1, g_2, \ldots, g_n, y) &= (x, g_1, \ldots, g_i g_{i+1}, \ldots, g_n, y) \text{ for } 0 < i < n \\ \partial_n(x, g_1, g_2, \ldots, g_n, y) &= (x, g_1, \ldots, g_{n-1}, g_n y) \end{aligned}$$

and degeneracy maps

$$s_i(x, g_1, g_2, \ldots, g_n, y) = (x, g_1, \ldots, g_i, e, g_{i+1}, \ldots g_n, y).$$

If Z has left and right actions by π then $B(X, \pi, B(Z, \pi, Y))$ is isomorphic to $B(B(X, \pi, Z), \pi, Y)$. Also, $B(X, \pi, \pi_+)$ and X are equivalent, but not necessarily isomorphic, as right π-spaces.

We think of $B(X, \pi, Y)$ as the homotopy coequalizer of the maps

$$X \wedge \pi_+ \wedge Y \rightrightarrows X \wedge Y$$

where the maps $X \wedge \pi_+ \wedge Y \to X \wedge Y$ are the action of π on X and π on Y. We will denote the actual coequalizer of these maps $X \wedge_\pi Y$.

We use a homotopy coequalizer to define $\odot$ so that the result has the correct homotopy type. Alternatively, we could make assumptions on the actions of π so that the bar resolution is equivalent to $X \wedge_\pi Y$. This will be the case in the examples we consider.

Lemma 3.1.1. — [**31**, 8.5] *If π acts principally on X and effectively on Y then $B(X, \pi, Y)$ is weakly equivalent to $X \wedge_\pi Y$.*

Let K be a based space (without an action by π). Then $\vee_\pi K$, the wedge product of copies of K indexed by the elements of π, has left and right actions of π given by permutations of the factors.

Definition 3.1.2. — We say a based right π-space X has *n-dual* Y if Y is a based left π-space, there is a map $\eta\colon S^n \to X \odot Y$ and a $\pi-\pi$ equivariant map $\epsilon\colon Y \wedge X \to \vee_\pi S^n$ such that the diagrams below commute up to equivariant homotopy after smashing with S^m for some $m \in \mathbb{N}$.

$$\begin{array}{ccc} S^n \wedge X & \xrightarrow{\eta\wedge \mathrm{id}} & (X\odot Y)\wedge X \\ & & \big\downarrow{\cong} \\ \big\downarrow{\gamma} & & X\odot (Y\wedge X) \\ & & \big\downarrow{\mathrm{id}\odot\epsilon} \\ X\wedge S^n & \xrightarrow[\cong]{} & X\odot(\vee_\pi S^n) \end{array} \qquad \begin{array}{ccc} Y \wedge S^n & \xrightarrow{\mathrm{id}\wedge \eta} & Y\wedge (X\odot Y) \\ & & \big\downarrow{\cong} \\ \big\downarrow{(\sigma\wedge\mathrm{id})\gamma} & & (Y\wedge X)\odot Y \\ & & \big\downarrow{\epsilon\odot\mathrm{id}} \\ S^n\wedge Y & \xrightarrow[\cong]{} & (\vee_\pi S^n)\odot Y \end{array}$$

The map $\sigma\colon S^n \to S^n$ is a map of degree $(-1)^n$. If Y is the dual of X we say that (X, Y) is a dual pair.

Not many π-spaces are dualizable in this sense. For example, let σ be a nontrivial subgroup of π. If the π space $(\pi/\sigma)_+$ was dualizable with dual Y there would be a π-π-equivariant map

$$\epsilon : Y \wedge (\pi/\sigma)_+ \to \vee_\pi S^n$$

for some integer n such that the diagrams in Definition 3.1.2 are satisfied. Since ϵ is a π-π-equivariant map its image must be the basepoint of $\vee_\pi S^n$.

Let M be a closed smooth manifold with universal cover $\tilde{M}$, quotient map $\pi\colon \tilde{M} \to M$, and normal bundle ν.

Lemma 3.1.3. — *For a closed smooth manifold M, $\tilde{M}_+$ is dualizable as a right $\pi_1 M$ space with dual $T\pi^*\nu$.*

Let S^ν be the fiberwise one point compactification of the normal bundle of M and $\tilde{M} \to \pi^* S^\nu$ be the inclusion as the points at infinity. Then $T\pi^*\nu$ is the pushout of the maps $\tilde{M} \to \pi^* S^\nu$ and $\tilde{M} \to *$. We use T since we want to suggest the Thom space in analogy with the duality described in Chapter 1. The space $T\pi^*\nu$ is a left $\pi_1 M$ space with action given by $\alpha \cdot (\gamma, v) = (\gamma\alpha^{-1}, v)$ for $\alpha \in \pi_1 M$ and $(\gamma, v) \in T\pi^*\nu$. Since the right action of $\pi_1 M$ on $\tilde{M}$ is free, $\tilde{M}_+ \odot T\pi^*\nu$ is equivalent to $\tilde{M} \wedge_{\pi_1 M} T\pi^*\nu$.

The coevaluation map for $\tilde{M}$ is the composite

$$S^n \xrightarrow{\chi} T\nu \longrightarrow \tilde{M}_+ \odot T\pi^*\nu.$$

The first map is the Pontryagin-Thom map for an embedding of M in $\mathbb{R}^n$. The second map is defined by

$$v \mapsto (\gamma_{\rho(v)}, \gamma_{\rho(v)}, v)$$

where $\rho\colon \nu \to M$ is the projection and $\gamma_{\rho(v)}$ is any lift of $\rho(v)$ to $\tilde{M}$. The second map is independent of the chosen lift since quotienting by the action of $\pi_1 M$ will identify any two different choices.

Before we define the evaluation map we need the following preliminary lemma.

Lemma 3.1.4. — [**6**, II.5.2] *Let K be an ENR. Then there is an open neighborhood W of the diagonal in $K \times K$ and a homotopy $H\colon W \times I \to K$ such that $H_0(x, y) = x$, $H_1(x, y) = y$ and $H_t(x, x) = x$ for all $(x, y) \in W$ and $t \in I$.*

We fix such a homotopy H and use it when defining all similar evaluation maps.

The evaluation map for $\tilde{M}_+$,

$$T\pi^*\nu \wedge \tilde{M}_+ \longrightarrow (\pi_1 M)_+ \wedge S^n = \vee_{\pi_1 M} S^n,$$

is defined by

$$(\gamma_m, v, \gamma_n) \mapsto (\gamma_m^{-1} H(n, m)\gamma_n, \epsilon(v, n))$$

where ϵ is the evaluation map from the dual pair $(M_+, T\nu)$. The element

$$\gamma_m^{-1} H(n, m)\gamma_n$$

of $\pi_1 M$ is the unique $g \in \pi_1 M$ such that $\gamma_m \cdot g$ is contained in a small neighborhood of γ_n.

If $\sigma\colon M \to T\nu$ is the zero section, points outside of a small neighborhood N of the image of $(\sigma \times \mathrm{id})\Delta$ are mapped by ϵ to the base point of S^n. If necessary we can shrink N so that for all $(v, m) \in N$, $(\rho(v), m) \in W$. If $H(n, m)$ is not defined then the triple (γ_n, γ_m, v) is mapped to the base point. The evaluation map is continuous and independent of choices. It is also compatible with the actions of $\pi_1 M$ on $T\pi^*\nu$ and $\tilde{M}_+$. We show that these maps make the required diagrams commute in the proof of Lemma 5.3.3.

3.2. The geometric Reidemeister trace as a trace

Before we can define the trace, we need to introduce a little more structure. This additional structure plays the role of the symmetry isomorphism in the definition of trace in a symmetric monoidal category and allows us to compare the target of the coevaluation with the source of the evaluation for dual pairs of π-spaces.

Definition 3.2.1. — Let Z be a based π-π space. The *shadow* of Z, $\langle\langle Z\rangle\rangle$, is the cyclic bar resolution $C(Z, \pi)$.

The cyclic bar resolution $C(Z, \pi)$ is the geometric realization of the simplicial based space with n simplices

$$(\pi^n)_+ \wedge Z,$$

face maps

$$\begin{aligned}
\partial_0(g_1,g_2,\dots,g_n,z) &= (g_1,\dots,g_{n-1},g_nz)\\
\partial_i(g_1,g_2,\dots,g_n,z) &= (g_1,\dots,g_{n-i}g_{n-i+1},\dots,g_n,z) \text{ for } 0<i<n\\
\partial_n(g_1,g_2,\dots,g_n,z) &= (g_2,\dots,g_n,zg_1),
\end{aligned}$$

and degeneracy maps

$$s_i(g_1,g_2,\dots,g_n,z) = (g_1,\dots,g_i,e,g_{i+1},\dots g_n,z).$$

We think of $\langle\!\langle Z\rangle\!\rangle$ as the homotopy coequalizer of the two actions of π on Z.

For a homomorphism $\phi\colon \pi\to\pi$ let π^ϕ be π as a set. On the left π acts on π^ϕ by multiplication and on the right π acts by applying ϕ and then acting by multiplication. There is a simplicial map from $C(\pi^\phi,\pi)$ to the constant simplicial set on the set of semiconjugacy classes of π with respect to ϕ given by

$$(g_1,g_2,\dots,g_n,h)\mapsto (g_1g_2\dots g_nh).$$

This map is an isomorphism on components since the images of $h_1,h_2\in\pi^\phi$ in the set of semiconjugacy classes coincide if and only if there is an element $(g,h)\in\pi\times\pi^\phi$ such that $gh=h_1$ and $h\phi(g)=h_2$. Note that $\pi_0(\langle\!\langle\pi^\phi\rangle\!\rangle)$ coincides with $\langle\!\langle\pi^\phi\rangle\!\rangle$ in Definition 2.2.1.

Taking the shadow of a π-π space is similar to applying the functor $\langle\!\langle-\rangle\!\rangle$ to an R-R-bimodule for a ring R. In both cases these solutions seem like an ad hoc resolution to a very small problem. In the next chapter we will show that structures very similar to these are very important in the definition of trace in a bicategory.

In this section we will define trace for π-maps

$$f\colon X\to X^\phi$$

where $\phi\colon\pi\to\pi$ is a homomorphism, X is a right π-space and X^ϕ is the space X with right action of π given by $x\cdot g=x\phi(g)$. There is a simplicial map from $B(X,\pi,\pi^\phi_+)$ to the constant simplicial set X^ϕ given by the action of π on X. This map is a simplicial homotopy equivalence with inverse given by

$$x\mapsto (x,e,\dots e)$$

and so $X\odot\pi^\phi_+$ is equivalent to X^ϕ as a right π-space.

Definition 3.2.2. — Let X be a right π-space with n-dual Y. Let $\phi\colon\pi\to\pi$ be a homomorphism. The *trace* of an equivariant map $f\colon X\to X^\phi$ is the stable homotopy class of the map

$$S^n\xrightarrow{\eta} X\odot Y\xrightarrow{f\odot \mathrm{id}} X^\phi\odot Y\cong (X\odot\pi^\phi_+)\odot Y\cong\langle\!\langle Y\wedge X\odot(\pi^\phi)_+\rangle\!\rangle\xrightarrow{\langle\!\langle\epsilon\odot\mathrm{id}\rangle\!\rangle}\vee_{\langle\!\langle\pi^\phi\rangle\!\rangle}S^n.$$

Let $f\colon M\to M$ be a continuous map of a closed smooth manifold. Let $\phi\colon\pi_1(M)\to\pi_1(M)$ be the induced map given by a choice of path ζ from the base point to its image under f. Define

$$\tilde f\colon\tilde M\to\tilde M$$

by $\tilde{f}(\gamma) = f(\gamma)\zeta$. Then $\tilde{f}$ satisfies

$$\tilde{f}(\gamma\alpha) = f(\gamma)f(\alpha)\zeta = f(\gamma)\zeta\zeta^{-1}f(\alpha)\zeta = \tilde{f}(\gamma)\phi(\alpha)$$

and defines an equivariant map

$$\tilde{f}\colon \tilde{M}_+ \to \tilde{M}_+^\phi.$$

Recall that the geometric Reidemeister trace of a map f is

$$\sum_{\text{Fixed Point Classes } F_k} i(F_k)\cdot F_k \in \mathbb{Z}\langle\langle \pi_1(M)^\phi\rangle\rangle$$

where $i(F_k)$ is the index of the fixed point class F_k.

Let $\pi_i^s(X)$ be the i^{th} stable homotopy group of X.

Proposition 3.2.3. — *There is an isomorphism $\pi_0^s(X) \to H_0(X)$. The image of the trace of $\tilde{f}$ under this isomorphism is the geometric Reidemeister trace of f.*

Proof. — The isomorphism $\pi_0^s(X) \to H_0(X)$ is the composite

$$\pi_0^s(X) \longleftarrow \pi_q(\Sigma^q X) \longrightarrow H_q(\Sigma^q X) \longleftarrow H_0(X)\ .$$

The first map is the inclusion. For sufficiently large q the Freudenthal suspension theorem implies this map is an isomorphism. The second map is the Hurewicz homomorphism. It is an isomorphism since $\pi_i(\Sigma^q X)$ is trivial for all i less than q. The last map is the suspension isomorphism.

By Lemma 3.1.3, $(\tilde{M}_+, T\pi^*\nu)$ is a dual pair. The trace of $\tilde{f}$ with respect to this dual pair is the composite

$$S^n \xrightarrow{\eta} T\nu \longrightarrow \tilde{M}_+ \odot T\pi^*\nu \xrightarrow{\tilde{f}\odot \mathrm{id}} \tilde{M}_+ \odot \pi_1 M^\phi \odot T\pi^*\nu \xrightarrow{\epsilon} \vee_{\langle\langle \pi_1 M^\phi\rangle\rangle} S^n$$

The trace of $\tilde{f}$ is a map into a wedge product, so specifying a point in S^n and an element of $\langle\langle \pi_1 M^\phi\rangle\rangle$ identifies the image of $v \in S^n$ under the trace of $\tilde{f}$. The image of $v \in S^n$ under the trace $\tilde{f}$ in S^n is $\epsilon(\eta(v), f\rho(\eta(v)))$. The image of $v \in S^n$ under the trace $\tilde{f}$ in $\langle\langle \pi_1 M^\phi\rangle\rangle$ is

$$\gamma_{\rho\eta(v)}^{-1} H(f\rho\eta(v), \rho\eta(v)) f(\gamma_{\rho\eta(v)})\zeta.$$

The map η is the Pontryagin-Thom map for an embedding $M \to \mathbb{R}^n$, ϵ is the evaluation for the dual pair $(M_+, T\nu)$, and H is as in Lemma 3.1.4. At a fixed point, the group element

$$\gamma_{\rho\eta(v)}^{-1} H(f\rho\eta(v), \rho\eta(v)) f(\gamma_{\rho\eta(v)})\zeta$$

is $\gamma_{\rho\eta(v)}^{-1} f(\gamma_{\rho\eta(v)})\zeta$, the image of the fixed point under the injection from the fixed point classes to the semiconjugacy classes described in Section 2.2. Since the index is local, the trace of $\tilde{f}$ restricted to a neighborhood of a fixed point is a map of degree equal to the index of the fixed point.

The trace of $\tilde{f}$ is an element of $\pi_0^s(\langle\!\langle \pi_1 M^\phi \rangle\!\rangle_+)$ and it is the image of an element in $\pi_n(\Sigma^n \langle\!\langle \pi_1 M^\phi \rangle\!\rangle_+)$ if M is n-dualizable. The image of this representative under the Hurewicz homomorphism is $(\mathrm{tr}(f))_*([S])$. The projection

$$H_n(\Sigma^n \langle\!\langle \pi_1 M^\phi \rangle\!\rangle_+) \cong \oplus H_n(S^n) \to H_n(S^n)$$

to the component corresponding to $\alpha \in \langle\!\langle \pi_1(M)^\phi \rangle\!\rangle$ takes $(\mathrm{tr}(f))_*([S])$ to the index of the fixed point class corresponding to α. □

The trace of $\tilde{f}$ and the other invariants we will define here are more naturally described as elements of stable homotopy groups rather than homology classes. We will think of them as homotopy classes of maps, and use the isomorphism $\pi_0^s(X) \cong H_0(X)$ when it is necessary to make a connection with homology. We will use this isomorphism to refer to the trace of $\tilde{f}$ as the geometric Reidemeister trace.

Remark 3.2.4. — By looking at the explicit description of the trace of $\tilde{f}$ for a map $f\colon M \to M$ we see that $R^{geo}(f)$ does not depend on the choice of the lift of f. Also note that $R^{geo}(f)$ is an invariant of the homotopy class of f since the trace of $\tilde{f}$ is an invariant of the homotopy class of $\tilde{f}$.

In the next chapter we will define duality and trace in a bicategory. In Chapter 5 we will show that the duality and trace defined in this section are examples of duality and trace in a bicategory.

3.3. Duality for spaces with path monoid actions

In the previous section we gave a description of the geometric Reidemeister trace. In this section we will describe an invariant that is similar to the geometric Reidemeister trace, and coincides with the invariant defined by Klein and Williams.

In Section 3.1 the motivating example was the action of the fundamental group of a topological space on the universal cover by deck transformations. In this section the motivating example is the 'action' of the free path space of a topological space on itself by composition. From this action we can define modules over the path space and then define duality for modules and trace for homomorphisms.

We will use this trace to define the homotopy Reidemeister trace which is a 'derived' form of the geometric Reidemeister trace. The homotopy Reidemeister trace is zero only when the geometric Reidemeister trace is zero.

In the previous section we worked with homotopy classes of paths. In this section we will work with free Moore paths. The free Moore paths of a space have composition, but composition is only defined for paths with compatible endpoints. This path space also satisfies unit conditions, but it has many units rather than just one. We use Moore paths rather than regular paths since the composition of Moore paths is strictly associative.

In the previous section all spaces had base points and we had to make adjustments since the maps were not based. In this section we will not use base points at all. This has several advantages. The base point is part of the construction of the geometric Reidemeister trace but the invariant does not depend on the choice of base point. As we mentioned before, fiberwise spaces don't always have sections, so in that case we will need unbased descriptions.

The free Moore path space of a space M is

$$\mathcal{P}M = \{(\gamma, u) \in \mathrm{Map}([0,\infty), M) \times [0,\infty) \,|\, \gamma(t) = \gamma(u) \text{ for all } t \geq u\}$$

This space is given the subspace topology from $\mathrm{Map}([0,\infty), M) \times [0,\infty)$. There are two maps $s, t\colon \mathcal{P}M \to M$, given by $s(\gamma, u) = \gamma(0)$ and $t(\gamma, u) = \gamma(u)$. Later we will think of $\mathcal{P}M$ as a category and so s and t denote source and target. Two paths $(\beta, v), (\alpha, u) \in PM$ can be composed if $\alpha(u) = \beta(0)$ and this composition defines a unital and associative product

$$\mathcal{P}M \times_M \mathcal{P}M = \{((\beta, v), (\alpha, u)) | \beta(0) = \alpha(u)\} \to \mathcal{P}M.$$

If we restrict to paths that start and end at some chosen point $* \in M$ the composition induces the group multiplication in the fundamental group.

Let X be an ex-space with section and projection maps $M \xrightarrow{\sigma} X \xrightarrow{p} M$ that satisfies the conditions of Remark 1.4.1. We define an ex-space over M, $\mathcal{P}M \boxtimes X$, by imposing additional identifications on the fiber product

$$\mathcal{P}M \times_M X = \{((\gamma, u), x) \in \mathcal{P}M \times X | \gamma(0) = p(x)\}.$$

We identify $((\gamma, u), x)$ and $((\gamma', u'), x')$ if x and x' are both in the image of the section and $\gamma(u) = \gamma'(u')$. The projection map

$$\mathcal{P}M \boxtimes X \to X$$

is $((\gamma, u), x) \mapsto \gamma(u)$. The section

$$M \to \mathcal{P}M \boxtimes X$$

is $m \mapsto ((c_m, 0), \sigma(m))$ where c_m is the constant path at m in M. A different description of this product is given in Chapter 5. The ex-space $X \boxtimes \mathcal{P}M$ is defined similarly.

Similarly, for two ex-spaces X and Y over M we define a based space $X \boxtimes Y$ by

$$\{(x, y) | p(x) = p'(y)\}/\sim$$

where (x, y) is identified with (x', y') if one of element of each pair is in the image of the section. The base point is the point of $X \boxtimes Y$ given by the equivalence relation.

Definition 3.3.1. — An ex-space X over M is a *right $\mathcal{P}M$-module* if there is a map over and under M

$$\kappa\colon X \boxtimes \mathcal{P}M \to X$$

that is associative and unital with respect to the product of $\mathcal{P}M$.

The definitions of a left module and a bimodule are similar. The space $\mathcal{P}M$ is a space over $M \times M$ via the map $t \times s$. With a disjoint section added $\mathcal{P}M$ is a ex-space over $M \times M$. This ex-space is written $(\mathcal{P}M, t \times s)_+$. Using composition of paths $(\mathcal{P}M, t \times s)_+$ is a $\mathcal{P}M$-$\mathcal{P}M$-bimodule. Using only one of the maps t or s we can think of $\mathcal{P}M$ as either a left or right $\mathcal{P}M$-module. For example, $(\mathcal{P}M, s)_+$ is a right $\mathcal{P}M$-module.

If X and Y are ex-spaces over M, the external smash product $Y \bar{\wedge} X$ is an ex-space over $M \times M$. If X_n is the fiber of X over n and Y_m is the fiber of Y over m, the fiber of $Y \bar{\wedge} X$ over (m, n) is $Y_m \wedge X_n$. If X is a right $\mathcal{P}M$-module and Y is a left $\mathcal{P}M$-module then $Y \bar{\wedge} X$ is a $\mathcal{P}M$-$\mathcal{P}M$-bimodule.

For a right $\mathcal{P}M$-module X and a left $\mathcal{P}M$-module Y, $X \odot Y$ is defined to be the bar resolution $B(X, \mathcal{P}M, Y)$. The product used to define the bar resolution is the $\boxtimes$ product defined above. This is analogous to the $\odot$ for two spaces with an action by a group π or to the tensor product of modules over a ring.

Definition 3.3.2. — We say that a right $\mathcal{P}M$-module X is *n-dualizable* if there is a left $\mathcal{P}M$-module Y, a continuous map $\eta\colon S^n \to X \odot Y$ and a map $\epsilon\colon Y \bar{\wedge} X \to S^n \bar{\wedge} (\mathcal{P}M, t \times s)_+$ of $\mathcal{P}M$-$\mathcal{P}M$-bimodules such that

$$\begin{array}{ccc} S^n \bar{\wedge} X & \xrightarrow{\eta \wedge \mathrm{id}} & (X \odot Y) \bar{\wedge} X \\ & & \downarrow \cong \\ \gamma \downarrow & & X \odot (Y \bar{\wedge} X) \\ & & \downarrow \mathrm{id} \odot \epsilon \\ X \bar{\wedge} S^n & \xrightarrow[\cong]{} & X \odot (S^n \bar{\wedge} (\mathcal{P}M, t \times s)_+) \end{array} \qquad \begin{array}{ccc} Y \bar{\wedge} S^n & \xrightarrow{\mathrm{id} \wedge \eta} & Y \bar{\wedge} (X \odot Y) \\ & & \downarrow \cong \\ (\sigma \wedge \mathrm{id})\gamma \downarrow & & ((Y \bar{\wedge} X) \odot Y) \\ & & \downarrow \epsilon \odot \mathrm{id} \\ S^n \bar{\wedge} Y & \xrightarrow[\cong]{} & (S^n \bar{\wedge} (\mathcal{P}M, t \times s)_+) \odot Y \end{array}$$

commute stably up to homotopy respecting the action by $\mathcal{P}M$.

Let $s^* S^\nu$ denote the pullback of S^ν along $s\colon \mathcal{P}M \to M$. Then $T_M s^* S^\nu$ is the quotient of $s^* S^\nu$ where we identify $((\gamma, u), v)$ and $((\gamma', u'), v')$ if v and v' are in the image of the section and $\gamma(u) = \gamma'(u')$. This is an ex-space over M with projection given by

$$((\gamma, u), v) \mapsto \gamma(u).$$

We use T_M to denote the quotient since $T_M s^* S^\nu$ is related to the Thom space. This is a left $\mathcal{P}M$-module

Lemma 3.3.3. — *Let M be a closed smooth manifold. Then $(\mathcal{P}M, s)_+$ is dualizable with dual $T_M s^* S^\nu$.*

We prove this lemma in Chapter 5 where it is part of Lemma 5.4.3.

The coevaluation map

$$S^n \to (\mathcal{P}M, s)_+ \odot T_M s^* S^\nu$$

is the composite of the Pontryagin-Thom map for the normal bundle of M with the map that takes an element v of the normal bundle to $(c_{\rho(v)}, c_{\rho(v)}, v)$. Here c_x is the constant path at x. The evaluation map is more difficult to describe. It is closely related to the evaluation map in Lemma 3.1.3 and it is defined in Lemma 5.4.3.

3.4. The homotopy Reidemeister trace as a trace

Before we define the trace we need to define the shadow of a space with two actions of $\mathcal{P}M$.

Definition 3.4.1. — Let Z be a $\mathcal{P}M$-$\mathcal{P}M$-bimodule. The *shadow* of $Z, \langle\langle Z\rangle\rangle$, is the cyclic bar resolution $C(Z, \mathcal{P}M)$.

For a map $f\colon M \to M$, $(\mathcal{P}^f M, t \times s)_+$ is the $\mathcal{P}M$-$\mathcal{P}M$-bimodule defined by

$$(\mathcal{P}^f M, t \times s) := \{(m, (\gamma, u)) \in M \times \mathcal{P}M | f(m) = \gamma(0)\}.$$

This is a space over $M \times M$ with projection map $(m, \gamma) \mapsto (\gamma(u), m)$. The actions of $\mathcal{P}M$ are given by composition of paths on the left and composing with f followed by composition of paths on the right. There is a map from the shadow of $(\mathcal{P}^f M, t \times s)_+$ to the space

$$\Lambda^f M = \{(\gamma, u) \in \mathcal{P}M | f(\gamma(u)) = \gamma(0)\}_+.$$

This is an isomorphism on components. See Section 6.2 for a description of this comparison.

For a map $f\colon M \to M$ there is an induced map

$$\tilde{f}\colon (\mathcal{P}M, s)_+ \to (\mathcal{P}^f M, s)_+$$

of right $\mathcal{P}M$-modules.

Definition 3.4.2. — If M is closed smooth manifold and $f\colon M \to M$ is a map, the *trace* of $\tilde{f}$ is the stable homotopy class of the map

$$S^n \longrightarrow (\mathcal{P}M, s)_+ \odot T_M s^* S^\nu \xrightarrow{\cong} \langle\langle T_M s^* S^\nu \bar{\wedge} (\mathcal{P}M, s)_+\rangle\rangle$$
$$\xrightarrow{\mathrm{id}\,\bar{\wedge}\tilde{f}} \langle\langle T_M s^* S^\nu \bar{\wedge} (\mathcal{P}^f M, s)_+\rangle\rangle \longrightarrow S^n \wedge \langle\langle (\mathcal{P}^f M, t \times s)_+\rangle\rangle$$

Definition 3.4.3. — The *homotopy Reidemeister trace* of f, $R^{\mathrm{htpy}}(f)$, is the trace of $\tilde{f}$.

The duality and trace defined in these two sections are also examples of the duality and trace in bicategories defined in Chapter 4. The bicategory used to define this duality is described in Chapter 5.

CHAPTER 4

WHY BICATEGORIES?

In the previous two chapters we have explained some of the reasons why Dold and Puppe's trace in symmetric monoidal categories cannot describe the Reidemeister trace. We have also explained why there should be some structure, similar to the trace in symmetric monoidal categories, that does describe the Reidemeister trace. In this chapter we will describe that structure.

To achieve the necessary additional generality we replace symmetric monoidal categories by bicategories. Bicategories have structure that is very similar to a symmetric monoidal category in the ways that are important for defining duality and trace. In particular, bicategories have 'tensor products', or composition, and units. Bimodules over a ring and spaces with group actions are examples of bicategories.

Without some additional structure bicategories are not similar enough to symmetric monoidal categories to have a trace. To define a trace we add shadows. The shadows are closely related to the bicategory composition and they play a role similar to that of the symmetry isomorphism in a symmetric monoidal category.

Some results that follow easily from the definitions of duality and trace significantly simplify proofs of results stated in Chapter 3. We include those results here and complete the proofs omitted from Chapter 3 in Chapters 5 and 6.

We omit most proofs in this section since they are diagram chases from the definitions. Additional information can be found in **[38]**.

4.1. Definitions

Bicategories can be thought of as monoidal categories with many objects. Instead of having objects and morphisms, bicategories have 0-cells, 1-cells, and 2-cells. Each 1-cell or 2-cell in a bicategory $\mathscr{B}$ has a source 0-cell and a target 0-cell. For two 0-cells A and B, the 1-cells and 2-cells with source A and target B form a category with objects the 1-cells and morphisms the 2-cells. This category is usually written $\mathscr{B}(A, B)$. In addition, for 0-cells A, B, and C there is a functor

$$\odot \colon \mathscr{B}(B, C) \times \mathscr{B}(A, B) \to \mathscr{B}(A, C)$$

that acts as 'composition' for 1-cells and 2-cells. For each 0-cell A there is a functor U_A from the category with one object and one morphism to $\mathscr{B}(A, A)$. Up to isomorphism 2-cells, the functors $\odot$ are associative and unital with respect to the functors U_A.

Definition 4.1.1. — [27, 1.0] A *bicategory* $\mathscr{B}$ consists of

(*i*) A collection ob$\mathscr{B}$.

(*ii*) Categories $\mathscr{B}(A, B)$ for each $A, B \in$ ob$\mathscr{B}$.

(*iii*) Functors

$$\odot\colon \mathscr{B}(B,C) \times \mathscr{B}(A,B) \to \mathscr{B}(A,C)$$
$$U_A\colon * \to \mathscr{B}(A,A)$$

for A, B and C in ob$\mathscr{B}$.

Here $*$ denotes the category with one object and one morphism. The functors $\odot$ are required to satisfy unit and associativity conditions up to natural isomorphism 2-cells.

Remark 4.1.2. — There are two choices for the convention used for the bicategory composition. We follow the convention used in [29], but the choice

$$\odot\colon \mathscr{B}(A,B) \times \mathscr{B}(B,C) \to \mathscr{B}(A,C)$$

is also used.

Some examples of bicategories include:

— The bicategory with 0-cells rings, 1-cells bimodules, and 2-cells homomorphisms.

— The bicategory with 0-cells categories, 1-cells functors, and 2-cells natural transformations.

— The bicategory Ex of ex-spaces with 0-cells spaces; and for two spaces A and B the category Ex(A, B) is the category of ex-spaces over $B \times A$. (1)

A monoidal category is a bicategory with a single 0-cell. The objects of the monoidal category are the 1-cells of the bicategory and the morphisms are the 2-cells. The monoidal product of the monoidal category is the $\odot$ in the bicategory and the unit object I is used to define the single functor U_I. Chapter 9 contains more examples of bicategories.

Definition 4.1.3. — A *lax functor* $F\colon \mathscr{B} \to \mathscr{B}'$ between bicategories consists of

(i) A function F from the 0-cells of $\mathscr{B}$ to the 0-cells of $\mathscr{B}'$;

(ii) Functors $F\colon \mathscr{B}(A,B) \to \mathscr{B}'(FA, FB)$ for all pairs of 0-cells A and B of $\mathscr{B}$;

(iii) Natural transformations

$$\phi_{X,Y}\colon F(X) \odot F(Y) \to F(X \odot Y)$$

for all 1-cells X and Y;

(iv) Natural transformations $\phi_A\colon U'_{F(A)} \to F(U_A)$ for each 0-cell A

that satisfy some coherence conditions.

(1) The name Ex is used to refer to a different, but related, category in [34].

This is analogous to a monoidal functor and its natural transformations

$$F(A) \otimes F(B) \to F(A \otimes B)$$

and $I' \to F(I)$. A *strong functor* of bicategories is a lax functor where the natural transformations $\phi_{X,Y}$ and ϕ_A are natural isomorphisms.

For more detailed definitions see [**27, 34**].

4.2. Rings, bimodules, and maps

Many of the important features of bicategories can be seen in the example of rings, bimodules, and homomorphisms. The 0-cells of this bicategory are rings, the 1-cells with source A and target B are the B-A-bimodules. The 2-cells between two B-A bimodules are the bimodule homomorphisms. The $\odot$ is the usual tensor product of modules over a ring. The functor U_A associated to a ring A is A regarded as an A-A-bimodule. This bicategory is denoted by Mod.

We used the Hattori-Stallings trace of an endomorphism of a finitely generated projective module to define the algebraic Reidemeister trace. In the definition we used the map

$$\nu\colon X \otimes_A P \otimes_A \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_A(X, X \otimes_A P)$$

which is an isomorphism for all right A-modules P and finitely generated projective right A-modules X. We can also define a map

$$\eta\colon \mathbb{Z} \to \mathrm{Hom}_A(X, X)$$

for any right A-module X by $\eta(n) = n \cdot \mathrm{id}_X$. If X is a finitely generated projective right A-module then the composite of η with ν^{-1} gives a map

$$\mathbb{Z} \to X \otimes_A \mathrm{Hom}_A(X, A).$$

In fact, the existence of a map

$$\mathbb{Z} \to X \otimes_A \mathrm{Hom}_A(X, A)$$

satisfying some additional conditions is equivalent to ν being an isomorphism.

Proposition 4.2.1. — *The following are equivalent for a right A-module X.*

(i) *X is a finitely generated projective right A-module.*

(ii) *The map*

$$\nu\colon P \otimes_A \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_A(X, P)$$

is an isomorphism for all A-A-bimodules P.

(iii) *There is a map $\eta\colon \mathbb{Z} \to X \otimes_A D_A X$, $D_A X := \mathrm{Hom}_A(X, A)$, such that*

$$\begin{array}{ccccc}
D_A X & \xrightarrow{\cong} & D_A X \otimes_{\mathbb{Z}} \mathbb{Z} & \xrightarrow{\mathrm{id} \otimes \eta} & D_A X \otimes_{\mathbb{Z}} (X \otimes_A D_A X) \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & & & \Big\downarrow{\scriptstyle \cong} \\
D_A X & \xrightarrow{\cong} & A \otimes_A D_A X & \xleftarrow[\mathrm{ev} \otimes \mathrm{id}]{} & (D_A X \otimes_{\mathbb{Z}} X) \otimes_A D_A X
\end{array}$$

and

$$\begin{array}{ccccc} X & \xrightarrow{\cong} & \mathbb{Z}\otimes_{\mathbb{Z}} X & \xrightarrow{\eta\otimes \mathrm{id}} & (X\otimes_A D_A X)\otimes_{\mathbb{Z}} X \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \cong} \\ X & \xrightarrow{\cong} & X\otimes_A A & \xleftarrow[\mathrm{id}\otimes \mathrm{ev}]{} & X\otimes_A (D_A X\otimes_{\mathbb{Z}} X) \end{array}$$

commute.

Using the map η and the functor $\langle\!\langle - \rangle\!\rangle$ from Definition 2.3.4 we can give another description of the Hattori-Stallings trace.

Proposition 4.2.2. — *If X is a finitely generated projective right A-module, P is an A-A-bimodule, and $f\colon X \to X\otimes_A P$ is a map of right A-modules, then the Hattori-Stallings trace of f is the image of 1 under the composite*

$$\begin{array}{ccccc} \mathbb{Z} & & & & \langle\!\langle P\rangle\!\rangle \\ \downarrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \langle\!\langle \mathrm{ev}\rangle\!\rangle} \\ X\otimes_A D_A(X) & \xrightarrow{f\otimes \mathrm{id}} & X\otimes_A P\otimes_A D_A(X) & \xrightarrow{\cong} & \langle\!\langle P\otimes_A D_A(X)\otimes_{\mathbb{Z}} X\rangle\!\rangle. \end{array}$$

In the rest of this chapter we generalize this description of the Hattori-Stallings trace to a trace in bicategories with shadows. We begin by recalling the generalization of Proposition 4.2.1 to a bicategory from **[34]**. Then we define shadows which generalize the functors $\langle\!\langle - \rangle\!\rangle$.

4.3. Duality

Duality in a bicategory is similar to duality in a symmetric monoidal category. Many of the differences are already seen in Proposition 4.2.1, which we generalize here. This section is based on Chapter 16 of **[34]** which contains additional details.

Definition 4.3.1. — A 1-cell $X\in \mathscr{B}(B,A)$ is *right dualizable* if there is a 1-cell $Y\in \mathscr{B}(A,B)$ and maps $\eta\colon U_A \to X\odot Y$, called the *coevaluation*, and $\epsilon\colon Y\odot X\to U_B$, called the *evaluation*, such that the following diagrams commute in $\mathscr{B}(B,A)$ and $\mathscr{B}(A,B)$, respectively.

$$\begin{array}{ccccc} X & \xleftarrow{\cong} & U_A\odot X & \xrightarrow{\eta\odot \mathrm{id}} & (X\odot Y)\odot X \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \cong} \\ X & \xleftarrow[\cong]{} & X\odot U_B & \xleftarrow[\mathrm{id}\odot \epsilon]{} & X\odot (Y\odot X) \end{array} \qquad \begin{array}{ccccc} Y & \xleftarrow{\cong} & Y\odot U_A & \xrightarrow{\mathrm{id}\odot \eta} & Y\odot (X\odot Y) \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \cong} \\ Y & \xleftarrow[\cong]{} & U_B\odot Y & \xleftarrow[\epsilon\odot \mathrm{id}]{} & (Y\odot X)\odot Y \end{array}$$

If X is right dualizable with dual Y we say that Y is the *right dual* of X and that (X,Y) is a *dual pair.* We also say that Y is *left dualizable* and that X is the *left dual* of Y.

Proposition 4.3.2. — *Let X be a 1-cell in $\mathscr{B}(B,A)$, Y be a 1-cell in $\mathscr{B}(A,B)$, and $\epsilon\colon Y \odot X \to U_B$ be a 2-cell in $\mathscr{B}(B,B)$. Then the following are equivalent.*

(i) *Y is the right dual of X with evaluation ϵ.*

(ii) *The map $\epsilon/(-)\colon \mathscr{B}(W, Z\odot Y) \to \mathscr{B}(W\odot X, Z)$ which takes a 2-cell $f\colon W \to Z\odot Y$ to the 2-cell*

$$W\odot X \xrightarrow{f\odot \mathrm{id}} Z\odot Y\odot X \xrightarrow{\mathrm{id}\odot\epsilon} Z\odot U_B \cong Z$$

is a bijection for all $W\in\mathscr{B}(A,C)$ and $Z\in\mathscr{B}(B,C)$.

There is a similar characterization using the coevaluation. If $\mathscr{B}$ is a closed bicategory there is also a characterization using the internal hom.

As in the symmetric monoidal case, any two right duals of a right dualizable object are isomorphic. If (X,Y) is a dual pair with coevaluation η and evaluation ϵ and (X,Y') is another dual pair with coevaluation η' and evaluation ϵ' then

$$Y\cong Y\odot U_A \xrightarrow{\mathrm{id}\odot\eta'} Y\odot X\odot Y' \xrightarrow{\epsilon\odot\mathrm{id}} U_B\odot Y'\cong Y'$$

is an isomorphism with inverse

$$Y'\cong Y'\odot U_A \xrightarrow{\mathrm{id}\odot\eta} Y'\odot X\odot Y \xrightarrow{\epsilon'\odot\mathrm{id}} U_B\odot Y\cong Y\ .$$

A dual pair in a symmetric monoidal category is also a dual pair in the corresponding bicategory with one 0-cell. However, in a bicategory an object that is right dualizable might not be left dualizable, and conversely.

The following results about composites of dual pairs follow immediately from the definition of a dual pair. While they are both very easy to prove they have significant consequences.

Theorem 4.3.3. — *Let X be a right dualizable 1-cell in $\mathscr{B}(B,A)$ with dual Y and W be a right dualizable 1-cell in $\mathscr{B}(C,B)$ with dual Z.*

Let (η,ϵ) be coevaluation and evaluation maps for the dual pair (X,Y), and let (ζ,ψ) be coevaluation and evaluation maps for the dual pair (W,Z). Then the composites

$$U_A \xrightarrow{\eta} X\odot Y \xrightarrow{\cong} X\odot U_B\odot Y \xrightarrow{\mathrm{id}\odot\zeta\odot\mathrm{id}} (X\odot W)\odot(Z\odot Y)$$

and

$$(Z\odot Y)\odot(X\odot W) \xrightarrow{\mathrm{id}\odot\epsilon\odot\mathrm{id}} Z\odot U_B\odot W \xrightarrow{\cong} Z\odot W \xrightarrow{\psi} U_C$$

are coevaluation and evaluation maps that exhibit $(X\odot W, Z\odot Y)$ as a dual pair of 1-cells.

Theorem 4.3.4. — *Let (X,Y) be a dual pair with evaluation $\epsilon\colon Y\odot X\to U_B$. Let Z be another 1-cell and suppose $(X\odot Z, V)$ is a dual pair. If ϵ is an isomorphism then $(Z, V\odot X)$ is a dual pair.*

Let (Z, W) be a dual pair with coevaluation $\chi\colon U_B \to Z \odot W$ and X be a 1-cell. If $(X \odot Z, V)$ is a dual pair and χ is an isomorphism, then $(X, Z \odot V)$ is a dual pair.

Strong functors of bicategories are compatible with dual pairs, but weaker hypotheses can also give compatibility between functors and dual pairs.

Proposition 4.3.5. — *Let (X,Y) be a dual pair in a bicategory $\mathscr{B}$, $X \in \mathscr{B}(B,A)$, and $F\colon \mathscr{B} \to \mathscr{B}'$ be a lax functor of bicategories such that $\phi_{X,Y}$ and ϕ_B are isomorphisms. Then (FX, FY) is a dual pair in $\mathscr{B}'$.*

If the coevaluation and evaluation maps for the dual pair (X,Y) are

$$\eta\colon U_A \to X \odot Y \quad \text{and} \quad \epsilon\colon Y \odot X \to U_B,$$

then the coevaluation and evaluation maps for the dual pair (FX, FY) are

$$U'_{FA} \xrightarrow{\phi_A} F(U_A) \xrightarrow{F(\eta)} F(X \odot Y) \xrightarrow{(\phi_{X,Y})^{-1}} FX \odot FY$$

and

$$FY \odot FX \xrightarrow{\phi_{Y,X}} F(Y \odot X) \xrightarrow{F(\epsilon)} F(U_B) \xrightarrow{(\phi_B)^{-1}} U'_{FB}.$$

Let Ch be the bicategory with 0-cells rings, 1-cells chain complexes of bimodules, and 2-cells maps of chain complexes. Let C be a chain complex of left R-modules and D be a chain complex of right R-modules. Suppose (D, C) is a dual pair. There is a map

$$H_*(D) \odot H_*(C) = H_*(D) \otimes_R H_*(C) \to H_*(D \otimes_R C).$$

The Künneth Theorem implies this map is an isomorphism if each C_i is a projective module, the boundaries of C_i are projective and the homology of C is projective in each degree. The natural transformations ϕ_R are the identity for all rings R and so when the hypotheses of the Künneth Theorem are satisfied Proposition 4.3.5 implies that the homology of a dualizable complex is dualizable.

4.4. Shadows

In symmetric monoidal categories the symmetry isomorphism

$$X \otimes Y \to Y \otimes X$$

provides a way to compare the target of the coevaluation with the source of the evaluation for a dual pair (X,Y). This is an important part of the definition of the trace. To define trace in a bicategory we will also need to be able to compare the target of coevaluation with the source of evaluation. For example, in the bicategory of rings, bimodules, and homomorphisms it is necessary to compare $X \otimes_A Y$ with $Y \otimes_B X$ for an B-A-bimodule X with dual Y.

The comparisons we need are not automatically part of the structure of a bicategory, as we can see in the bicategory Mod. In Mod we introduced the functors $\langle\!\langle - \rangle\!\rangle$ to define the Hattori-Stallings trace. In this section we describe how to generalize these functors to other bicategories.

Definition 4.4.1. — A *shadow* for a bicategory $\mathscr{B}$ consists of functors

$$\langle\!\langle - \rangle\!\rangle\colon \mathscr{B}(A,A) \to \mathbf{T}$$

for each object A of $\mathscr{B}$ and some fixed category $\mathbf{T}$, and a natural isomorphism

$$\theta\colon \langle\!\langle X \odot Y \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle Y \odot X \rangle\!\rangle$$

for all pairs of 1-cells $X \in \mathscr{B}(B,A)$ and $Y \in \mathscr{B}(A,B)$ such that the following diagrams commute whenever they make sense: (2)

$$\begin{array}{ccccc}
\langle\!\langle (X \odot Y) \odot Z \rangle\!\rangle & \xrightarrow{\theta} & \langle\!\langle Z \odot (X \odot Y) \rangle\!\rangle & \longrightarrow & \langle\!\langle (Z \odot X) \odot Y \rangle\!\rangle \\
\downarrow & & & & \uparrow \theta \\
\langle\!\langle X \odot (Y \odot Z) \rangle\!\rangle & \xrightarrow{\theta} & \langle\!\langle (Y \odot Z) \odot X \rangle\!\rangle & \longrightarrow & \langle\!\langle Y \odot (Z \odot X) \rangle\!\rangle
\end{array}$$

$$\begin{array}{ccccc}
\langle\!\langle Z \odot U_A \rangle\!\rangle & \xrightarrow{\theta} & \langle\!\langle U_A \odot Z \rangle\!\rangle & \xrightarrow{\theta} & \langle\!\langle Z \odot U_A \rangle\!\rangle \\
& \searrow & \downarrow & \swarrow & \\
& & \langle\!\langle Z \rangle\!\rangle & &
\end{array}$$

Shadows can be thought of as 'cyclic tensor products' since the natural isomorphisms will allow cyclic permutations of 1-cells and 2-cells.

As we noted before, from a symmetric monoidal category we can define a bicategory with a single 0-cell, 1-cells the objects of the category, and 2-cells the morphisms. The identity functor is a shadow for this bicategory and the isomorphism θ is the symmetry isomorphism.

The bicategory Ch is a bicategory with shadows. The shadows are given by applying the shadows of Mod levelwise. The isomorphism $\langle\!\langle X \odot Y \rangle\!\rangle \to \langle\!\langle Y \odot X \rangle\!\rangle$ is the usual exchange of elements and adds a sign determined by degree.

Since shadows are not automatically part of the structure of a bicategory, it is not surprising that additional hypotheses will be needed before a lax functor is considered compatible with shadows.

Definition 4.4.2. — Let $\mathscr{B}$ and $\mathscr{B}'$ be bicategories with shadows. A *lax shadow functor* is a lax functor $F\colon \mathscr{B} \to \mathscr{B}'$, a functor $F\colon \mathbf{T} \to \mathbf{T}'$, and a natural transformation

$$\psi_A\colon \langle\!\langle F(-) \rangle\!\rangle \to F\langle\!\langle - \rangle\!\rangle$$

(2) The following diagrams are due to Michael Shulman.

for each 0-cell A of $\mathscr{B}$ such that

$$\begin{array}{ccc}
\langle\!\langle FX \odot FY \rangle\!\rangle & \xrightarrow{\theta} & \langle\!\langle FY \odot FX \rangle\!\rangle \\
\downarrow{\scriptstyle \phi_{X,Y}} & & \downarrow{\scriptstyle \phi_{Y,X}} \\
\langle\!\langle F(X \odot Y) \rangle\!\rangle & & \langle\!\langle F(Y \odot X) \rangle\!\rangle \\
\downarrow{\scriptstyle \psi_A} & & \downarrow{\scriptstyle \psi_B} \\
F\langle\!\langle X \odot Y \rangle\!\rangle & \xrightarrow[\theta]{} & F\langle\!\langle Y \odot X \rangle\!\rangle
\end{array}$$

commutes for all 1-cells $X \in \mathscr{B}(B, A)$ and $Y \in \mathscr{B}(A, B)$.

Homology is a lax shadow functor. The lax functor is the identity on 0-cells and the usual homology functor on 1-cells and 2-cells. Define ψ_R by the coequalizer

$$\begin{array}{ccccccc}
R \otimes H_*(C_*) & \longrightarrow & H_*(R \otimes C_*) & \underset{H_*(\kappa\gamma)}{\overset{H_*(\kappa)}{\rightrightarrows}} & H_*(C_*) & \longrightarrow & \langle\!\langle H_*(C_*) \rangle\!\rangle \\
\downarrow & & & & \| & & \vdots\, {\scriptstyle \psi_R} \\
H_*(R \otimes C_*) & & \underset{H_*(\kappa\gamma)}{\overset{H_*(\kappa)}{\rightrightarrows}} & & H_*(C_*) & \longrightarrow & H_*(\langle\!\langle C_* \rangle\!\rangle).
\end{array}$$

4.5. Trace

Motivated by the definition of trace in a symmetric monoidal category and the Hattori-Stallings trace we can now use duality in a bicategory to define trace in a bicategory with shadows.

In this section $\mathscr{B}$ is a bicategory with shadows and $X \in \mathscr{B}(B, A)$ is a 1-cell with right dual $Y \in \mathscr{B}(A, B)$. Let $\eta\colon U_A \to X \odot Y$ and $\epsilon\colon Y \odot X \to U_B$ be the coevaluation and evaluation for the dual pair (X, Y).

Definition 4.5.1. — For 1-cells $P \in \mathscr{B}(B, B)$ and $Q \in \mathscr{B}(A, A)$ and a 2-cell

$$f\colon Q \odot X \to X \odot P$$

the *trace* of f is the composite

$$\begin{array}{c}
\langle\!\langle Q \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle Q \odot U_A \rangle\!\rangle \xrightarrow{\langle\!\langle \mathrm{id} \odot \eta \rangle\!\rangle} \langle\!\langle Q \odot X \odot Y \rangle\!\rangle \xrightarrow{\langle\!\langle f \odot \mathrm{id} \rangle\!\rangle} \langle\!\langle X \odot P \odot Y \rangle\!\rangle \cong \langle\!\langle Y \odot X \odot P \rangle\!\rangle \xrightarrow{\langle\!\langle \epsilon \odot \mathrm{id} \rangle\!\rangle} \langle\!\langle U_B \odot P \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle P \rangle\!\rangle.
\end{array}$$

In a symmetric monoidal category, trace was defined only for endomorphisms of dualizable objects. In a bicategory we add the 1-cells P and Q since we want to use this trace in fixed point theory applications. For these examples the maps we want to take the trace of are not endomorphisms of 1-cells. Rather, they are 2-cells of the form $X \to X \odot P$. This can be seen in the definition of the algebraic Reidemeister trace where $X = C_*(\tilde{M};\mathbb{Q})$ for some closed smooth manifold M and $P = \mathbb{Z}\pi_1(M)^\phi$. While Q is not needed in our applications, we add it to the definition for symmetry.

We define the trace of a 2-cell $g\colon Y \odot Q \to P \odot Y$ similarly.

The following lemmas describe basic properties of the trace. All of these lemmas are easy to prove.

Lemma 4.5.2. — *If (X, Y) and (X, Y') are dual pairs, the trace of f with respect to (X, Y) is equal to the trace of f with respect to (X, Y').*

Let f' be the composite

$$\begin{array}{ccccccc} Y\odot Q & & & & & & P\odot Y. \\ \downarrow\cong & & & & & & \uparrow\cong \\ Y\odot Q\odot U_A & \xrightarrow{\mathrm{id}\odot\mathrm{id}\odot\eta} & Y\odot Q\odot X\odot Y & \xrightarrow{\mathrm{id}\odot f\odot\mathrm{id}} & Y\odot X\odot P\odot Y & \xrightarrow{\epsilon\odot\mathrm{id}\odot\mathrm{id}} & U_B\odot P\odot Y \end{array}$$

The 2-cell f' is the *dual* of f.

Lemma 4.5.3. — *For f and f' as above,*

$$\mathrm{tr}(f) = \mathrm{tr}(f').$$

One of the defining properties of a trace function on matrices is commutativity. The trace in bicategories is also commutative.

Lemma 4.5.4. — *If X and Z are right dualizable 1-cells, $g\colon R \odot Z \to X \odot S$, $f\colon Q \odot X \to Z \odot P$ are 2-cells, and the composites*

$$Q\odot R\odot Z \xrightarrow{\mathrm{id}_Q\odot g} Q\odot X\odot S \xrightarrow{f\odot\mathrm{id}_S} Z\odot P\odot S$$
$$R\odot Q\odot X \xrightarrow{\mathrm{id}_R\odot f} R\odot Z\odot P \xrightarrow{g\odot\mathrm{id}_P} X\odot S\odot P$$

are defined, then

$$\mathrm{tr}((f\odot\mathrm{id}_S)(\mathrm{id}_Q\odot g)) = \mathrm{tr}((g\odot\mathrm{id}_P)(\mathrm{id}_R\odot f)).$$

The trace respects the $\odot$ structure.

Lemma 4.5.5. — *If X and Z are right dualizable 1-cells, $f\colon Q\odot X \to X$ and $g\colon Z \to Z\odot P$ are 2-cells, and*

$$g\odot f\colon Z\odot Q\odot X \to Z\odot P\odot X$$

is defined, then

$$\mathrm{tr}(g\odot f) = \mathrm{tr}(g)\mathrm{tr}(f).$$

Some of the dual pairs that we will consider later have much more structure than is required by the definitions. The additional structure gives more information about the traces.

Lemma 4.5.6. — (i) *Let (X, Y) be a dual pair such that the evaluation ϵ is an isomorphism. Let Z be another 1-cell such that $X \odot Z$ is dualizable. For a 2-cell $g \colon Q \odot Z \to Z \odot P$ let $g^\star$ be the composite*

$$\begin{array}{ccc} X \odot Q \odot Y \odot X \odot Z & \xrightarrow{\mathrm{id} \odot \mathrm{id} \odot \epsilon \odot \mathrm{id}} & X \odot Q \odot U_A \odot Z & & \\ & & \downarrow \cong & & \\ & & X \odot Q \odot Z & \xrightarrow{\mathrm{id} \odot g} & X \odot Z \odot P. \end{array}$$

Then

$$\langle\!\langle X \odot Q \odot Y \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle Y \odot X \odot Q \rangle\!\rangle \xrightarrow{\langle\!\langle \epsilon \odot \mathrm{id} \rangle\!\rangle} \langle\!\langle U_A \odot Q \rangle\!\rangle \cong \langle\!\langle Q \rangle\!\rangle \xrightarrow{\mathrm{tr}(g)} \langle\!\langle P \rangle\!\rangle$$

is the trace of $g^\star$.

(ii) *Let (Z, W) be a dual pair such that the coevaluation χ is an isomorphism and let X be another 1-cell such that $X \odot Z$ is dualizable. If $f \colon Q \odot X \to X \odot P$ is a 2-cell let $f^\star$ be the composite*

$$\begin{array}{ccc} Q \odot X \odot Z & \xrightarrow{f \odot \mathrm{id}} & X \odot P \odot Z & & \\ & & \downarrow \cong & & \\ & & X \odot U_A \odot P \odot Z & \xrightarrow{\mathrm{id} \odot \chi \odot \mathrm{id} \odot \mathrm{id}} & X \odot Z \odot W \odot P \odot Z. \end{array}$$

Then

$$\langle\!\langle Q \rangle\!\rangle \xrightarrow{\mathrm{tr}(f)} \langle\!\langle P \rangle\!\rangle \cong \langle\!\langle P \odot U_A \rangle\!\rangle \xrightarrow{\langle\!\langle \mathrm{id} \odot \chi \rangle\!\rangle} \langle\!\langle P \odot Z \odot W \rangle\!\rangle \xrightarrow{\cong} \langle\!\langle W \odot P \odot Z \rangle\!\rangle$$

is the trace of $f^\star$.

Strong symmetric monoidal functors preserve dual pairs and trace in a symmetric monoidal category, as do lax symmetric monoidal functors that satisfy some additional hypotheses. Strong functors of bicategories, and lax functors of bicategories where some of the coherence natural transformations are isomorphisms, preserve dual pairs in a bicategory. Strong functors of bicategories that are also shadow functors almost preserve the trace.

Proposition 4.5.7. — *Let F be a lax shadow functor and (X, Y) a dual pair such that*

$$\phi_{X,Y} \colon F(X) \odot F(Y) \to F(X \odot Y)$$

and

$$\phi_B \colon U'_{F(B)} \to F(U_B)$$

are isomorphisms. If $f \colon Q \odot X \to X \odot P$ is a 2-cell, $\phi_{Q,X}$ is an isomorphism and $\hat{f}$ is the composite

$$FQ \odot FX \xrightarrow{\phi_{Q,X}^{-1}} F(Q \odot X) \xrightarrow{F(f)} F(X \odot P) \xrightarrow{\phi_{X,P}} FX \odot FP$$

then the following diagram commutes.

$$\begin{array}{ccc}
\langle\!\langle FQ\rangle\!\rangle & \xrightarrow{\operatorname{tr}(\hat{f})} & \langle\!\langle FP\rangle\!\rangle \\
{\scriptstyle\psi_A}\downarrow & & \downarrow{\scriptstyle\psi_B} \\
F\langle\!\langle Q\rangle\!\rangle & \xrightarrow[F(\operatorname{tr}(f))]{} & F\langle\!\langle P\rangle\!\rangle
\end{array}$$

For the homology functor, the natural transformations ϕ_A are all the identity and so the conditions of Proposition 4.5.7 are all consequences of the Künneth Theorem.

Corollary 4.5.8. — *Let C be a finitely generated chain complex of projective right R-modules such that the boundaries and homology of C are projective in each degree. If $f\colon C \to C \otimes_R P$ is map of chain complexes then*

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\operatorname{tr}(\phi\circ H_*(f))} & \langle\!\langle H_*(P)\rangle\!\rangle \\
\Vert & & \downarrow{\scriptstyle\psi_A} \\
\mathbb{Z} & \xrightarrow[H_*(\operatorname{tr}(f))]{} & H_*(\langle\!\langle P\rangle\!\rangle)
\end{array}$$

commutes.

In particular, if M is a finite CW complex and $H_*(\tilde{M};\mathbb{Z})$ is projective as a right module over $\mathbb{Z}\pi_1 M$ then the algebraic Reidemeister trace computed using the chains on M is the same as the trace of the induced map on homology. Compare this observation with [**11**, 4.3.b] and [**19**, 1.4].

Remark 4.5.9. — The trace defined in this section generalizes the trace in symmetric monoidal categories defined in [**11**]. It also satisfies generalizations of several of the defining properties of the trace in [**23, 30**]; see [**38**]. In particular, the trace in this section is natural and, up to unit isomorphisms, the trace of a 2-cell

$$f\colon Q \odot U_A \to U_A \odot P$$

is $\langle\!\langle f\rangle\!\rangle$. Since the bicategorical trace is 'asymmetric' the remaining properties can't be generalized in all cases.

In [**38**] there is another definition of trace in bicategories. This approach uses an extension of the definition of the shadow to double categories. This definition does not use dualizability, but in case of dualizable 1-cells the definitions coincide.

CHAPTER 5

DUALITY FOR PARAMETRIZED MODULES

In this chapter we give several examples of the duality defined in the previous chapter. We will first describe the bicategory of ex-spaces and one particular example of duality in this bicategory.

From the bicategory of ex-spaces we can define a bicategory that is a topological analogue of the bicategory of rings, bimodules, and homomorphisms. After defining the bicategory, we give several examples of dual pairs. These examples are similar to those in Chapter 3, but now we use the formal results from Chapter 4 to simplify many proofs.

The results from Chapter 4 that do the most to simplify the proofs here are the results about composites of dual pairs. The first of these results shows that the composite of two dual pairs is a dual pair. This result, along with a particular dual pair for a closed smooth manifold, will produce the dual pairs we described in Chapter 3.

In the next chapter we use these dual pairs to show that several forms of the Reidemeister trace are examples of trace in bicategories and we use functoriality of the trace to relate these invariants.

5.1. Costenoble-Waner Duality

We first define the bicategory Ex of ex-spaces. The 0-cells of Ex are spaces. A 1-cell in $\mathsf{Ex}(A,B)$ is an ex-space X over $B\times A$, a space X with maps

$$B\times A\xrightarrow{\sigma}X\xrightarrow{p}B\times A$$

such that $p\circ\sigma=\mathrm{id}$. The 2-cells of Ex are maps of total spaces that commute with the section and projection. If Y is an ex-space over B we think of it as an object of $\mathsf{Ex}(B,*)$.

Recall from Remark 1.4.1 that for homotopical control we will usually consider parametrized spaces X over $A\times B$ where the map $X\to A\times B$ is a fibration and the map $A\times B\to X$ is a fiberwise cofibration. This assumption implies that maps in the homotopy category will correspond to fiberwise homotopy classes of maps. While this is a restrictive assumption, in many of the examples we are interested in this condition is satisfied. When this does not hold we choose an equivalent replacement that does satisfy these conditions. See [**34**, 9.1.2] for further details.

The external smash product $\bar{\wedge}$ of an ex-space X over A with an ex-space Y over B is a parametrized space over $A \times B$. The fiber of the external smash product over (a, b) is the fiber of X over a smashed with the fiber of Y over b.

If X is a parametrized space over $A \times B$ and Y is a parametrized space over $B \times C$ then we define $X \boxtimes Y$, a parametrized space over $A \times C$, as the pullback along $\triangle\colon B \to B \times B$ and then pushout along $r\colon B \to *$ of $X \bar{\wedge} Y$.

$$\begin{array}{ccccc}
A\times C & \xleftarrow{\mathrm{id}\times r\times \mathrm{id}} & A\times B\times C & \xrightarrow{\mathrm{id}\times\triangle\times\mathrm{id}} & A\times B\times B\times C\\
\downarrow & & \downarrow & & \downarrow\\
X\boxtimes Y & \longleftarrow & (\mathrm{id}\times\triangle\times\mathrm{id})^*(X\bar{\wedge}Y) & \longrightarrow & X\bar{\wedge}Y\\
\downarrow & & \downarrow & & \downarrow\\
A\times C & \xleftarrow[\mathrm{id}\times r\times \mathrm{id}]{} & A\times B\times C & \xrightarrow[\mathrm{id}\times\triangle\times\mathrm{id}]{} & A\times B\times B\times C
\end{array}$$

Following [**34**], we write this as $X \boxtimes Y = r_! \triangle^*(X \bar{\wedge} Y)$ where $(-)^*$ indicates pullback and $(-)_!$ indicates pushout. This is the bicategory composition in Ex. For more details on these definitions see Chapter 17 of [**34**]. With the assumption that the projection maps are fibrations and the sections are fiberwise cofibrations $X \boxtimes Y$ will have the correct homotopy type.

For each 0-cell B, $(B, \triangle)_+ \in \mathsf{Ex}(B, B)$ denotes the ex-space with projection map the diagonal map $\triangle\colon B \to B \times B$ and a disjoint section. This is the unit for $\boxtimes$ and so it will be denoted U_B. More generally, if $p\colon X \to B$ is a continuous map, then $(X, p)_+$ is the parametrized space with projection p and a disjoint section. We regard $(X, p)_+$ as an object of $\mathsf{Ex}(B, *)$.

Costenoble-Waner duality [**34**, Chapter 18] for parametrized spaces is an example of duality in a more sophisticated stable version of the bicategory Ex but it also has an interpretations in terms of n-duality in Ex.

Definition 5.1.1. — [**34**, 18.3.1] An ex-space X over B is *Costenoble-Waner n-dualizable* if there is an ex-space Y over B and maps

$$S^n \xrightarrow{\eta} X \boxtimes tY \qquad \text{and} \qquad tY \boxtimes X \xrightarrow{\epsilon} \triangle_! S^n_B$$

such that

$$\begin{array}{ccc}
S^n\boxtimes X & \xrightarrow{\eta\boxtimes\mathrm{id}} & (X\boxtimes tY)\boxtimes X\\
 & & \downarrow\cong\\
\gamma\downarrow & & X\boxtimes(tY\boxtimes X)\\
 & & \downarrow \mathrm{id}\boxtimes\epsilon\\
X\boxtimes S^n & \xrightarrow[\cong]{} & X\boxtimes\triangle_! S^n_B
\end{array}
\qquad
\begin{array}{ccc}
tY\boxtimes S^n & \xrightarrow{\mathrm{id}\boxtimes\eta} & tY\boxtimes(X\boxtimes tY)\\
 & & \downarrow\cong\\
(\sigma\boxtimes\mathrm{id})\gamma\downarrow & & (tY\boxtimes X)\boxtimes tY\\
 & & \downarrow \epsilon\boxtimes\mathrm{id}\\
S^n\boxtimes tY & \xrightarrow[\cong]{} & \triangle_! S^n_B\boxtimes tY
\end{array}$$

commute up to fiberwise homotopy.

Note that $\triangle_!(S^n_B) \in \mathsf{Ex}(B, B)$ is the pushout of $S^n \times B$ along the diagonal map of B. The t indicates that we are thinking of Y as an element of $\mathsf{Ex}(*, B)$ rather than $\mathsf{Ex}(B, *)$.

Theorem 5.1.2. — [**34**, 18.6.1] *Let M be a closed smooth manifold with an embedding in $\mathbb{R}^n$. Then (S^0_M, tS^ν) is a Costenoble-Waner n-dual pair.*

The ex-space $S^0_M \in \mathsf{Ex}(M, *)$ has total space two copies of M. The projection map is the identity map on both components. The ex-space $S^\nu \in \mathsf{Ex}(M, *)$ is the fiberwise one point compactification of the normal bundle of M. The section is the inclusion of M as the points added by the compactification. It is a space over M via the projection map $\rho\colon S^\nu \to M$.

The coevaluation map

$$\eta\colon S^n \to S^0_M \boxtimes tS^\nu \cong T\nu$$

is the Pontryagin-Thom map for the normal bundle of the embedding $M \to S^n$. The diagonal gives an inclusion of M into $\nu \times M$. Let e be an identification of a neighborhood V of M in $\nu \times M$ with the trivial bundle $\mathbb{R}^n \times M$. We can define a map

$$E\colon V \to \mathrm{Map}(I, M) \times (\mathbb{R}^n \times M)$$

by

$$E(v, m) = (H(\rho(v), m), e(v, m))$$

where $H(\rho(v), m)$ is a path from $\rho(v)$ to m as in Lemma 3.1.4.

The evaluation map ϵ is the composite of the Pontryagin-Thom map for the embedding $M \to \nu \times M$ with the map E. This is related to the evaluation map described for the dual pair in Lemma 3.1.3.

Since Costenoble-Waner dual pairs are examples of dual pairs in a bicategory there are other characterizations of Costenoble-Waner duals. Let $\{-, -\}$ denote stable homotopy classes of maps and $\{-, -\}_B$ denote fiberwise stable homotopy classes of maps over B.

Corollary 5.1.3. — *If X is Costenoble-Waner n-dualizable with dual Y,*

$$\{Z \boxtimes X, W\}_B \cong \{S^n \wedge Z, W \boxtimes tY\}$$

for $Z \in \mathsf{Ex}(*, *)$ and $W \in \mathsf{Ex}(B, *)$,

In particular, for a closed smooth manifold M

$$\{S^0_M, U\}_M \cong \{S^0, U \boxtimes tS^\nu\}$$

for $U \in \mathsf{Ex}(M, *)$.

For any space B the parametrized spaces $(B, \mathrm{id})_+ \in \mathsf{Ex}(B, *)$ and $t(B, \mathrm{id})_+ \in \mathsf{Ex}(*, B)$ form a dual pair. The coevaluation is the diagonal map

$$\triangle\colon B \to B \times B.$$

If $r\colon B \to *$ is the map to a point, the evaluation map is

$$r_+\colon B_+ \to S^0.$$

For a manifold M, the dual pairs $((M,\mathrm{id})_+, t(M,\mathrm{id})_+)$ and (S^0_M, tS^ν) can be composed to give the dual pair $(M_+, T\nu)$ described in Theorem 1.3.2.

5.2. A bicategory of bimodules over parametrized monoids

In this section we define the bicategory that describes the topological dual pairs we will use later. This is a bicategory of monoids, bimodules, and maps of bimodules and its construction is similar to the construction of the bicategory of rings, bimodules, and homomorphisms from the category of abelian groups.

The bicategory in this section is a special case of the bicategory described in Section 9.3 with one exception. In Section 9.3 we make frequent use of colimits. In this section we will use homotopy colimits. Here we are primarily interested in homotopical information and homotopy colimits will give the right homotopy types. Also, we must use homotopy colimits to be able to connect our invariants with classical invariants, especially the invariant defined by Klein and Williams.

Definition 5.2.1. — A *monoid* in Ex is a parametrized space $\mathscr{A} \in \mathsf{Ex}(A, A)$ with parametrized maps

$$\mu\colon \mathscr{A}\boxtimes\mathscr{A} \longrightarrow \mathscr{A} \qquad \text{and} \qquad \iota\colon U_A \longrightarrow \mathscr{A}$$

such that

$$\mathscr{A} \cong U_A\boxtimes\mathscr{A} \xrightarrow{\iota\boxtimes\mathrm{id}} \mathscr{A}\boxtimes\mathscr{A} \xrightarrow{\mu} \mathscr{A}$$

and

$$\mathscr{A} \cong \mathscr{A}\boxtimes U_A \xrightarrow{\mathrm{id}\boxtimes\iota} \mathscr{A}\boxtimes\mathscr{A} \xrightarrow{\mu} \mathscr{A}$$

are the identity and

$$\begin{array}{ccc} \mathscr{A}\boxtimes\mathscr{A}\boxtimes\mathscr{A} & \xrightarrow{\mu\boxtimes\mathrm{id}} & \mathscr{A}\boxtimes\mathscr{A} \\ {\scriptstyle \mathrm{id}\boxtimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathscr{A}\boxtimes\mathscr{A} & \xrightarrow[\mu]{} & \mathscr{A} \end{array}$$

commutes.

We think of μ as composition and ι as the unit.

Definition 5.2.2. — Let $\mathscr{A}$ and $\mathscr{B}$ be two monoids in Ex. An *$\mathscr{A}$-$\mathscr{B}$-bimodule* is an object $\mathscr{X} \in \mathsf{Ex}(B, A)$ and two parametrized maps

$$\kappa\colon \mathscr{A}\boxtimes\mathscr{X} \to \mathscr{X}$$

and

$$\kappa'\colon \mathscr{X}\boxtimes\mathscr{B} \to \mathscr{X}$$

that are unital and associative with respect to the monoid structure of $\mathscr{A}$ and $\mathscr{B}$. We also require that the actions κ and κ' commute.

A monoid $\mathscr{A}$ defines an $\mathscr{A}$-$\mathscr{A}$ bimodule with left and right actions given by the monoid multiplication μ. We will denote this bimodule by $U_{\mathscr{A}}$ since it is the unit in a bicategory.

A parametrized space $\mathscr{X}$ over A is trivially a bimodule. Thought of as a space over $* \times A$, $\mathscr{X}$ has a left action by U_* using the obvious isomorphism. It also has a right action by U_A using the unit isomorphism

$$\mathscr{X} \boxtimes U_A \to \mathscr{X}.$$

Definition 5.2.3. — Let $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{A}$-$\mathscr{B}$-bimodules. A *map of bimodules* is a parametrized map $f\colon \mathscr{X} \to \mathscr{Y}$ such that

$$\begin{array}{ccc} \mathscr{A} \boxtimes \mathscr{X} & \xrightarrow{\kappa} & \mathscr{X} \\ {\scriptstyle \mathrm{id} \boxtimes f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathscr{A} \boxtimes \mathscr{Y} & \xrightarrow[\kappa]{} & \mathscr{Y} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathscr{X} \boxtimes \mathscr{B} & \xrightarrow{\kappa'} & \mathscr{X} \\ {\scriptstyle f \boxtimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathscr{Y} \boxtimes \mathscr{B} & \xrightarrow[\kappa']{} & \mathscr{Y} \end{array}$$

commute.

Definition 5.2.4. — Let $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule and $\mathscr{Y}$ a $\mathscr{B}$-$\mathscr{C}$-bimodule. Then $\mathscr{X} \odot \mathscr{Y}$ is the bar resolution $B(\mathscr{X}, \mathscr{B}, \mathscr{Y})$. This is an $\mathscr{A}$-$\mathscr{C}$-bimodule.

The bar resolution $B(\mathscr{X}, \mathscr{B}, \mathscr{Y})$ is the geometric realization of the simplicial ex-space over $C \times A$ with n simplices

$$\mathscr{X} \boxtimes (\mathscr{B})^n \boxtimes \mathscr{Y},$$

face maps

$$\begin{aligned} \partial_0 &= \kappa' \boxtimes \mathrm{id}_{\mathscr{B}}^{n-1} \boxtimes \mathrm{id}_{\mathscr{Y}} \\ \partial_i &= \mathrm{id}_{\mathscr{X}} \boxtimes \mathrm{id}_{\mathscr{B}}^{i-1} \boxtimes \mu \boxtimes \mathrm{id}_{\mathscr{B}}^{n-i-1} \boxtimes \mathrm{id}_{\mathscr{Y}} \quad \text{for } 0 < i < n \\ \partial_n &= \mathrm{id}_{\mathscr{X}} \boxtimes \mathrm{id}_{\mathscr{B}}^{n-1} \boxtimes \kappa \end{aligned}$$

and degeneracy maps

$$s_i = \mathrm{id}_{\mathscr{X}} \boxtimes \mathrm{id}_{\mathscr{B}}^{i} \boxtimes \iota \boxtimes \mathrm{id}_{\mathscr{B}}^{n-i} \boxtimes \mathrm{id}_{\mathscr{Y}}.$$

We think of $\mathscr{X} \odot \mathscr{Y}$ as the homotopy coequalizer

$$\mathscr{X} \boxtimes \mathscr{B} \boxtimes \mathscr{Y} \underset{\mathrm{id} \boxtimes \kappa}{\overset{\kappa' \boxtimes \mathrm{id}}{\rightrightarrows}} \mathscr{X} \boxtimes \mathscr{Y} \longrightarrow \mathscr{X} \odot \mathscr{Y}$$

as an ex-space.

The bar resolution is associative up to isomorphism. To see this recall that the geometric realization is a tensor product of functors, see **[42]**. Then the comparison of $B(\mathscr{X}, \mathscr{B}, B(\mathscr{Y}, \mathscr{C}, \mathscr{Z}))$ with $B(B(\mathscr{X}, \mathscr{B}, \mathscr{Y}), \mathscr{C}, \mathscr{Z})$ is a comparison of coequalizers. The product $\mathscr{X} \odot \mathscr{B}$ is homotopy equivalent to $\mathscr{X}$ using a simplicial homotopy and the extra degeneracy in $\mathscr{B}$.

This defines a bicategory $\mathscr{M}_{\mathsf{Ex}}$ with 0-cells monoids, 1-cells bimodules, and 2-cells homotopy classes of maps of bimodules. The $\odot$ of Definition 5.2.4 is the bicategory composition. The unit associated to a monoid $\mathscr{A}$ is that monoid regarded as a $\mathscr{A}$-$\mathscr{A}$-bimodule.

5.3. Ranicki duality for parametrized bimodules

Since the bicategory $\mathscr{M}_{\mathsf{Ex}}$ is defined using spaces instead of spectra, the definition of duality has to be modified a little from the definition of duality in a bicategory. We imitate the definition of n-duality for parametrized spaces.

If $\mathscr{X}$ is a $\mathscr{A}$-$\mathscr{B}$-bimodule then $\mathscr{X}\bar{\wedge}S^n$ and $S^n\bar{\wedge}\mathscr{X}$ are also $\mathscr{A}$-$\mathscr{B}$-bimodules.

Definition 5.3.1. — Let $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule. Then $\mathscr{X}$ is *n-dualizable* if there is a $\mathscr{B}$-$\mathscr{A}$-bimodule $\mathscr{Y}$ and maps of bimodules

$$\eta\colon S^n\bar{\wedge}U_{\mathscr{A}} \longrightarrow \mathscr{X}\odot\mathscr{Y} \qquad \text{and} \qquad \epsilon\colon \mathscr{Y}\odot\mathscr{X} \longrightarrow S^n\bar{\wedge}U_{\mathscr{B}}$$

such that the following diagrams commute up to stable parametrized homotopy respecting the module structure.

$$\begin{array}{ccccc} S^n\bar{\wedge}\mathscr{X} & \xrightarrow{\cong} & (S^n\bar{\wedge}U_{\mathscr{A}})\odot\mathscr{X} & \xrightarrow{\eta\odot\mathrm{id}} & \mathscr{X}\odot\mathscr{Y}\odot\mathscr{X} \\ \downarrow{\scriptstyle\gamma} & & & & \downarrow{\scriptstyle\mathrm{id}\odot\epsilon} \\ \mathscr{X}\bar{\wedge}S^n & & \xleftarrow[\cong]{\qquad\qquad} & & \mathscr{X}\odot(S^n\bar{\wedge}U_{\mathscr{B}}) \end{array}$$

$$\begin{array}{ccccc} \mathscr{Y}\bar{\wedge}S^n & \xrightarrow{\cong} & \mathscr{Y}\odot(S^n\bar{\wedge}U_{\mathscr{A}}) & \xrightarrow{\mathrm{id}\odot\eta} & \mathscr{Y}\odot\mathscr{X}\odot\mathscr{Y} \\ \downarrow{\scriptstyle(\sigma\bar{\wedge}\mathrm{id})\gamma} & & & & \downarrow{\scriptstyle\epsilon\odot\mathrm{id}} \\ S^n\bar{\wedge}\mathscr{Y} & & \xleftarrow[\cong]{\qquad\qquad} & & (S^n\bar{\wedge}U_{\mathscr{B}})\odot\mathscr{Y}. \end{array}$$

As before, σ is a map of degree $(-1)^n$.

By neglect of structure any $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{X}$ defines an $\mathscr{A}$-U_B-bimodule denoted $L(\mathscr{X})$ and a U_A-$\mathscr{B}$-bimodule denoted $R(\mathscr{X})$.

Lemma 5.3.2. — *Let $\mathscr{A}$ be a monoid. Then $(R(U_{\mathscr{A}}), L(U_{\mathscr{A}}))$ is a dual pair. The coevaluation map*

$$U_A \to R(U_{\mathscr{A}})\odot L(U_{\mathscr{A}})$$

is the unit map. The evaluation map

$$L(U_{\mathscr{A}})\odot R(U_{\mathscr{A}}) \to U_{\mathscr{A}}$$

is the monoid multiplication.

The simplicial ex-space used to define $R(U_{\mathscr{A}}) \odot L(U_{\mathscr{A}})$ has an 'extra' degeneracy given by regarding an element of $R(U_{\mathscr{A}})$ or $L(U_{\mathscr{A}})$ as an element of $\mathscr{A}$. This means that $R(U_{\mathscr{A}}) \odot L(U_{\mathscr{A}})$ is equivalent to $N(U_{\mathscr{A}})$, the monoid $\mathscr{A}$ regarded as an U_A-U_A-bimodule.

Costenoble-Waner duality is an example of duality in the bicategory $\mathscr{M}_{\mathsf{Ex}}$. The Costenoble-Waner dual of an ex-space X over B is the dual of X as a U_*-U_B-bimodule in $\mathscr{M}_{\mathsf{Ex}}$.

Costenoble-Waner duality only uses monoids defined using unit isomorphisms. The Ranicki dual pair described in Section 3.1 requires less trivial monoids. With the discrete topology, $(\pi_1 M)_+ \in \mathsf{Ex}(*,*)$ is a monoid. The monoid multiplication

$$(\pi_1 M)_+ \boxtimes (\pi_1 M)_+ = (\pi_1 M \times \pi_1 M)_+ \to (\pi_1 M)_+$$

is the group multiplication and the unit $\iota\colon S^0 \to (\pi_1 M)_+$ is the inclusion of the identity element of $\pi_1 M$. The universal cover $\tilde{M}_+ \in \mathsf{Ex}(*,*)$ is a right $(\pi_1 M)_+$ module with the right action given by the usual action of $\pi_1 M$ on $\tilde{M}$. With this interpretation, a Ranicki dual for $\tilde{M}_+$ is a dual for the module $\tilde{M}_+$ in the bicategory $\mathscr{M}_{\mathsf{Ex}}$.

We can use the quotient map to regard $\tilde{M}$ as a space over M. In contrast with the convention above, we choose to regard $(\tilde{M},\pi)_+$ as an element of $\mathsf{Ex}(*,M)$. This choice is consistent with our later convention for path spaces since we think of $\tilde{M}$ as the homotopy classes of paths in M that start at a base point. Then $(\tilde{M},\pi)_+$ is a right $(\pi_1 M)_+$-module. Note that $\tilde{M}_+$ is equivalent to $S^0_M \odot (\tilde{M},\pi)_+$. Let $_*\tilde{M}$ denote the universal cover of M regarded as homotopy classes of paths ending at the base point rather than starting at the base point. Then $(_*\tilde{M},\pi)_+$ is a left $\pi_1 M$-module.

Lemma 5.3.3. — *For a closed smooth manifold M we have the following dual pairs.*
 (i) $((\tilde{M},\pi)_+, (_*\tilde{M},\pi)_+)$.
 (ii) $(\tilde{M}_+, T\pi^* S^\nu)$

Proof. — As before, $T\pi^*S^\nu$ is the pushout of the maps $\tilde{M} \to \pi^* t S^\nu$ and $\tilde{M} \to *$. This is equivalent to $(_*\tilde{M},\pi)_+ \odot S^\nu$. The ex-space

$$(\tilde{M},\pi)_+ \odot (_*\tilde{M},\pi)_+$$

is equivalent to the coequalizer of the two actions of $\pi_1 M$ on

$$(\tilde{M} \times \tilde{M}, \pi \times \pi)_+$$

since $\pi_1 M$ acts freely on $\tilde{M}$. Similarly,

$$\tilde{M}_+ \odot T\pi^* S^\nu$$

is equivalent to the coequalizer of the two actions of $\pi_1 M$ on

$$\tilde{M}_+ \wedge T\pi^* S^\nu.$$

The coevaluation map for $((\tilde{M},\pi)_+, (_*\tilde{M},\pi)_+)$

$$U_M \to (\tilde{M},\pi)_+ \odot (_*\tilde{M},\pi)_+ \simeq (\tilde{M} \times_{\pi_1 M} \tilde{M}, \pi \times \pi)_+$$

is given by $m \mapsto (\gamma_m, \gamma_m)$ for any lift γ_m of m. This map is well defined since the action of $\pi_1 M$ identifies all possible choices. The evaluation map

$$({}_*\tilde{M}, \pi)_+ \odot (\tilde{M}, \pi)_+ = \{(\alpha, \beta) \in \tilde{M} \times \tilde{M} | \pi(\alpha) = \pi(\beta)\}_+ \to U_{\pi_1 M}$$

is given by $(\alpha, \beta) \mapsto \alpha\beta$.

Note that the required diagrams for this dual pair commute strictly and without needing to stabilize. This is closely related to the observation above that monoids produce dual pairs.

The second dual pair is the composite of the dual pairs $((\tilde{M}, \pi)_+, ({}_*\tilde{M}, \pi)_+)$ and (S^0_M, tS^ν). The coevaluation map

$$S^n \xrightarrow{\eta} T\nu \longrightarrow \tilde{M}_+ \wedge_{\pi_1 M} T\pi^* S^\nu \simeq \tilde{M}_+ \odot T\pi^* S^\nu$$

is the composite of the coevaluation, η, for the dual pair (S^0_M, tS^ν) and the map

$$v \mapsto (\gamma_{p(v)}, \gamma_{p(v)}, v).$$

The evaluation map

$$T\pi^* S^\nu \odot \tilde{M}_+ \simeq T\pi^* S^\nu \wedge \tilde{M}_+ \longrightarrow (\pi_1 M)_+ \wedge S^n = \vee_{\pi_1 M} S^n$$

is given by $(\gamma, v, \alpha) \mapsto (\gamma H(\alpha(1), \gamma(0))\alpha, \epsilon(v, \alpha(1)))$ where ϵ is the evaluation map for the dual pair (S^0_M, tS^ν). The path $H(\alpha(1), \gamma(0))$ is defined in Lemma 3.1.4. $\square$

This lemma completes the proof of Lemma 3.1.3.

5.4. Moore loops and bicategories

In the previous section we defined duality in the bicategory of monoids and bimodules in parametrized spaces and gave examples of dual pairs. With the exception of the first dual pair in Lemma 5.3.3, the dual pair for the universal cover of a manifold regarded as a space over that manifold, we haven't used the flexibility the bicategory Ex offers. In this section we will begin to exploit this greater flexibility.

One undesirable aspect of using Ranicki duality to describe duality for universal covers was the need to choose a base point. There are two ways of dealing with this problem. The first is to verify that different choices of the base point give "the same" dual pairs. The second is to use all possible choices of base point. In other words, use objects like the fundamental groupoid rather than the fundamental group. The first approach is used in **[2, 21]** and the second in **[4]**. We will use the second approach here since it will also be useful when defining fiberwise dual pairs.

For our topological applications, the fundamental groupoid is not exactly the right object. First, we would rather have a space of objects and a space of morphisms rather than sets. Second, for fiberwise applications we would rather consider all paths than homotopy classes of paths. Instead of the fundamental groupoid we will consider a topologized version of the fundamental groupoid and the space of Moore paths.

Let M be a closed smooth manifold and ΠM the space of homotopy classes of paths in M with endpoints fixed. Topologize this space using the quotient topology from the usual compact open topology on $\mathrm{Map}(I, M)$. There is a Hurewicz fibration $t \times s\colon \Pi M \to M \times M$ given by $s(\gamma) = \gamma(0)$ and $t(\gamma) = \gamma(1)$. The fiber product

$$\Pi M \times_M \Pi M = \{(\gamma_2, \gamma_1) \in \Pi M \times \Pi M | \gamma_1(1) = \gamma_2(0)\}$$

is a space with a map to $M \times M$ given by $(\gamma_2, \gamma_1) \mapsto (\gamma_2(1), \gamma_1(0))$. Composition gives a strictly associative map $\mu\colon \Pi M \times_M \Pi M \to \Pi M$. This is a map over $M \times M$.

The inclusion ι of M into ΠM by constant paths is also a map over $M \times M$ if M is regarded as a space over $M \times M$ using the diagonal map

$$\triangle\colon M \to M \times M.$$

We regard $\Pi M \times_M M$ as a space over $M \times M$ by the map $(\gamma, m) \mapsto (\gamma(1), m)$. Then $\Pi M \times_M M$ is homeomorphic to ΠM as a space over $M \times M$. The map ι acts as the unit for μ in the sense that the following maps are the identity

$$\Pi M \cong \Pi M \times_M M \to \Pi M \times \Pi M \to \Pi M$$

$$\Pi M \cong M \times_M \Pi M \to \Pi M \times \Pi M \to \Pi M.$$

With a disjoint section, ΠM is a monoid in Ex.

In contrast with the previous sections, we will not use a different notation for a path space monoid and that monoid regarded as a bimodule.

Recall that for an $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{X}$, $R(\mathscr{X})$ is $\mathscr{X}$ regarded as a U_A-$\mathscr{B}$-bimodule, $L(\mathscr{X})$ is $\mathscr{X}$ regarded as a $\mathscr{A}$-U_B-bimodule, and $N(\mathscr{X})$ is $\mathscr{X}$ regarded as a U_A-U_B-bimodule.

The parametrized space $(\Pi M, s)_+$ has a right action of $(\Pi M, t \times s)_+$ by composition of paths. Recall that S^ν is the fiberwise one point compactification of the normal bundle of M. Then $T_M s^* S^\nu$ is defined to be $L(\Pi M, t \times s)_+ \odot tS^\nu$. This is an ex-space over M, and it has a left action by ΠM.

The dual pairs in Lemma 5.4.1 are the unbased versions of the dual pairs in Lemma 5.3.3. We make this comparison explicit in Lemma 5.4.2.

Lemma 5.4.1. — *For a closed smooth manifold M we have the following dual pairs.*

(i) $(R(\Pi M, t \times s)_+, L(\Pi M, t \times s)_+)$

(ii) $((\Pi M, s)_+, T_M s^* S^\nu)$

Proof. — As noted before $R(\Pi M, t \times s)_+ \odot L(\Pi M, t \times s)_+$ is equivalent to $N(\Pi M, t \times s)_+$.

The first dual pair is a dual pair arising from a monoid and so this dual pair follows from Lemma 5.3.2. The second dual pair is the composite of

$$(R(\Pi M, t \times s)_+, L(\Pi M, t \times s)_+)$$

with the dual pair (S^0_M, tS^ν).

In (i), the coevaluation map

$$U_M \to R(\Pi M, t \times s)_+ \odot L(\Pi M, t \times s)_+$$

is given by $m \mapsto (c_m, c_m)$ where c_m is the constant path at m. The evaluation map

$$L(\Pi M, t \times s)_+ \odot R(\Pi M, t \times s)_+ \to (\Pi M, t \times s)_+$$

is given by $(\alpha, \beta) \mapsto \alpha\beta$.

In (ii), note that

$$((\Pi M, s)_+, T_M s^* S^\nu) \simeq (S^0_M \odot R(\Pi M, t \times s)_+, L(\Pi M, t \times s)_+ \odot tS^\nu).$$

The coevaluation map

$$S^n \to T\nu \to (\Pi M, s)_+ \odot T_M s^* S^\nu$$

is given by $v \mapsto (c_{\rho(\eta(v))}, c_{\rho(\eta(v))}, \eta(v))$, where η is the Pontryagin-Thom map for the normal bundle of M. The evaluation map

$$T_M s^* S^\nu \odot (\Pi M, s)_+ \to S^n \bar{\wedge} L(\Pi M, t \times s)_+ \odot R(\Pi M, t \times s)_+ \to S^n \bar{\wedge} (\Pi M, t \times s)_+$$

is given by $(\alpha, v, \beta) \mapsto (\epsilon(v, \beta(1)), \alpha H(\beta(1), \alpha(0))\beta)$ where H is as in Lemma 3.1.4 and ϵ is the evaluation map for (S^0_M, tS^ν). □

Lemma 5.4.2. — *$\tilde{M}_+$ is dualizable as a $\pi_1 M$-space if and only if $(\Pi M, s)_+$ is dualizable as a $(\Pi M, t \times s)_+$-module.*

Proof. — This result follows from Theorems 4.3.3 and 4.3.4. Let x be a point in M and $(\Pi M_x, t)_+$ be the universal cover of M thought of as homotopy classes of paths in M that start at x. This has a right action of $\pi_1(M, x)$ and a left action of $(\Pi M, t \times s)_+$. This space is dualizable with dual $({}_x\Pi M, s)_+$, the universal cover thought of as homotopy classes of paths in M ending at x with a right action by $(\Pi M, t \times s)_+$ and a left action by $\pi_1(M, x)$. This dual pair satisfies the additional condition that the evaluation map is an isomorphism.

Then $(\Pi M, s)_+ \odot (\Pi M_x, t)_+$ is equivalent to $\tilde{M}_+$ regarded as a right $\pi_1(M, x)$-space. By Theorems 4.3.3 and 4.3.4, $\tilde{M}_+$ is dualizable as a $\pi_1 M$-space if and only if $(\Pi M, s)_+$ is dualizable as a $(\Pi M, t \times s)_+$-space. □

Recall that $\mathcal{P}M = \{(\gamma, u) \in M^{[0,\infty)} \times [0, \infty) | \gamma(t) = \gamma(u) \,\text{for}\, t \geq u\}$ is the space of free Moore paths in M. With a disjoint section, $\mathcal{P}M$ is a monoid over $M \times M$ in Ex. The parametrized space $(\mathcal{P}M, s)_+$ has a right action of $(\mathcal{P}M, t \times s)_+$ by composition of paths.

Lemma 5.4.3. — *For a closed smooth manifold M we have the following dual pairs.*

(i) $(R(\mathcal{P}M, t \times s)_+, L(\mathcal{P}M, t \times s)_+)$
(ii) $((\mathcal{P}M, s)_+, T_M s^* S^\nu)$

Proof. — $T_M s^* S^\nu$ is defined to be $L(\mathcal{P}M, t \times s)_+ \odot S^\nu$.

The first dual pair is a dual pair arising from a monoid as in Lemma 5.3.2. The second dual pair is the composite of $(R(\mathcal{P}M, t \times s)_+, L(\mathcal{P}M, t \times s)_+)$ with the dual pair (S^0_M, tS^ν).

In (i), the coevaluation map

$$U_M \to R(\mathscr{P}M, t\times s)_+ \odot L(\mathscr{P}M, t\times s)_+ \simeq (\mathscr{P}M, t\times s)_+$$

is the inclusion of M into $\mathscr{P}M$ as constant paths. The evaluation map

$$(\mathscr{P}M \times_M \mathscr{P}M, t\times s)_+ \simeq L(\mathscr{P}M, t\times s)_+ \odot R(\mathscr{P}M, t\times s)_+ \to (\mathscr{P}M, t\times s)_+$$

is given by composition of paths.

In (ii), note that

$$(S^0_M \odot R(\mathscr{P}M, t\times s)_+, L(\mathscr{P}M, t\times s)_+ \odot S^\nu) \simeq ((\mathscr{P}M, s)_+, T_M s^* S^\nu).$$

The coevaluation map

$$S^n \to S^0_M \boxtimes S^\nu \simeq T\nu \to (\mathscr{P}M, s)_+ \odot T_M s^* S^\nu$$

is given by $v \mapsto (c_{\rho\eta(v)}, c_{\rho\eta(v)}, \eta(v))$ where η is the Pontryagin-Thom for the embedding of M in $\mathbb{R}^n$. The evaluation map

$$\begin{array}{ccc} T_M s^* S^\nu \odot (\mathscr{P}M, s)_+ & \longrightarrow & L(\mathscr{P}M, t\times s)_+ \odot ((M,\triangle)_+ \bar{\wedge} S^n) \odot R(\mathscr{P}M, t\times s)_+ \\ & & \downarrow \\ & & S^n \bar{\wedge} (\mathscr{P}M, t\times s)_+ \end{array}$$

is given by $(\alpha, v, \beta) \mapsto (\epsilon(v, \beta(1)), \alpha H(\beta(1), \alpha(0))\beta)$ where H is as in Lemma 3.1.4 and ϵ is the evaluation map for (S^0_M, tS^ν). □

This lemma completes the proof of Lemma 3.3.3.

Lemma 5.4.3 is very similar to Lemma 5.4.1. In both lemmas a dual pair defined using a monoid is composed with the dual pair $(S^0_M, S\nu)$. Let $c : \mathscr{P}M \to \Pi M$ be the map that takes a path to its homotopy class with end points fixed. This map induces a map

$$N(\mathscr{P}M, t\times s)_+ \to N(\Pi M, t\times s)_+$$

and similarly for the corresponding left and right modules. Functoriality implies that the following diagrams commute

$$\begin{array}{ccccc} U_M & \longrightarrow & N(\mathscr{P}M, t\times s)_+ & \xrightarrow{\sim} & R(\mathscr{P}M, t\times s)_+ \odot L(\mathscr{P}M, t\times s)_+ \\ & \searrow & \downarrow & & \downarrow \\ & & N(\Pi M, t\times s)_+ & \xrightarrow{\sim} & R(\Pi M, t\times s)_+ \odot L(\Pi M, t\times s)_+ \end{array}$$

$$\begin{array}{ccc} L(\mathscr{P}M, t\times s)_+ \odot R(\mathscr{P}M, t\times s)_+ & \longrightarrow & (\mathscr{P}M, t\times s)_+ \\ \downarrow & & \downarrow \\ L(\Pi M, t\times s)_+ \odot R(\Pi M, t\times s)_+ & \longrightarrow & (\Pi M, t\times s)_+ \end{array}$$

Composing with the dual pair (S^0_M, S^ν) gives similar diagrams showing compatibility of the dual pairs $((\mathscr{P}M, s)_+, T_M s^* S^\nu)$ and $((\Pi M, s)_+, T_M s^* S^\nu)$.

5.5. Shadows and traces for Ranicki dualizable bimodules

The shadows in $\mathscr{M}_{\mathsf{Ex}}$ are very similar to $\odot$ in $\mathscr{M}_{\mathsf{Ex}}$. In special cases they are a 'derived form' of the semiconjugacy classes of the fundamental group. They also relate to the target of the Hattori-Stallings trace.

Definition 5.5.1. — Let $\mathscr{Z}$ be an $\mathscr{A}$-$\mathscr{A}$-bimodule. Then $\langle\!\langle \mathscr{Z} \rangle\!\rangle$ is the cyclic bar resolution $C(\mathscr{Z},\mathscr{A})$.

The maps θ are induced by

$$\begin{array}{ccccc} r_!\Delta^*(\mathscr{X}\boxtimes\mathscr{Y}) & \longrightarrow & r_!\Delta^*(\mathscr{X}\odot\mathscr{Y}) & \longrightarrow & \langle\!\langle \mathscr{X}\odot\mathscr{Y}\rangle\!\rangle \\ \downarrow & & & & \vdots\ \theta \\ r_!\Delta^*(\mathscr{Y}\boxtimes\mathscr{X}) & \longrightarrow & r_!\Delta^*(\mathscr{Y}\odot\mathscr{X}) & \longrightarrow & \langle\!\langle \mathscr{Y}\odot\mathscr{X}\rangle\!\rangle \end{array}$$

The target of the shadow functor is the category of based spaces.

The cyclic bar resolution $C(\mathscr{Z},\mathscr{A})$ is the geometric realization of the simplicial based space with n simplices

$$r_!\Delta^*((\mathscr{A})^n\boxtimes\mathscr{Z}),$$

face maps

$$\begin{aligned} \partial_0 &= \mathrm{id}_{\mathscr{A}}^{n-1}\odot\kappa \\ \partial_i &= \mathrm{id}_{\mathscr{A}}^{n-i-1}\odot\mu\odot\mathrm{id}_{\mathscr{A}}^{i-1}\odot\mathrm{id}_{\mathscr{Z}} \text{ for } 0<i<n \\ \partial_n &= (\mathrm{id}_{\mathscr{A}}^{n-1}\odot\kappa')\gamma \end{aligned}$$

where $\gamma\colon(\mathscr{A})^n\boxtimes\mathscr{Z}\to(\mathscr{A})^{n-1}\boxtimes\mathscr{Z}\boxtimes\mathscr{A}$ is the twist map, and degeneracy maps

$$s_i = \mathrm{id}_{\mathscr{A}}^{i}\odot\iota\odot\mathrm{id}_{\mathscr{A}}^{n-i}\odot\mathrm{id}_{\mathscr{Z}}.$$

As before, we use n-duality to define the trace of a map.

Definition 5.5.2. — Let $\mathscr{X}$ be an n-dualizable $\mathscr{A}$-$\mathscr{B}$-bimodule with dual $\mathscr{Y}$, coevaluation and evaluation

$$\eta\colon S^n\bar{\wedge}U_{\mathscr{A}}\longrightarrow\mathscr{X}\odot\mathscr{Y} \qquad\text{and}\qquad \epsilon\colon\mathscr{Y}\odot\mathscr{X}\longrightarrow S^n\bar{\wedge}U_{\mathscr{B}}.$$

Suppose $\mathscr{Q}$ is a $\mathscr{A}$-$\mathscr{A}$-bimodule, $\mathscr{P}$ is an $\mathscr{B}$-$\mathscr{B}$-bimodule and $f\colon\mathscr{Q}\odot\mathscr{X}\to\mathscr{X}\odot\mathscr{P}$ is a map of bimodules. Then the *trace* of f is the stable homotopy class of the composite

$$\begin{array}{l} \langle\!\langle\mathscr{Q}\bar{\wedge}S^n\rangle\!\rangle \xrightarrow{\langle\!\langle\mathrm{id}\odot\eta\rangle\!\rangle} \langle\!\langle\mathscr{Q}\odot\mathscr{X}\odot\mathscr{Y}\rangle\!\rangle \\ \qquad\qquad\qquad \downarrow \langle\!\langle f\odot\mathrm{id}\rangle\!\rangle \\ \qquad\qquad \langle\!\langle\mathscr{X}\odot\mathscr{P}\odot\mathscr{Y}\rangle\!\rangle \xrightarrow{\cong} \langle\!\langle\mathscr{Y}\odot\mathscr{X}\odot\mathscr{P}\rangle\!\rangle \xrightarrow{\langle\!\langle\epsilon\odot\mathrm{id}\rangle\!\rangle} \langle\!\langle S^n\bar{\wedge}\mathscr{P}\rangle\!\rangle \end{array}$$

We give examples of this trace in the next section.

CHAPTER 6

CLASSICAL FIXED POINT THEORY

This chapter implements the plan described in the introduction for proving Theorem D, the converse of the Lefschetz fixed point theorem that uses the Reidemeister trace. We use the dual pairs from Chapter 5 to interpret the fixed point invariants we defined in Chapter 2 as examples of the trace in bicategories with shadows. Then we use functoriality to identify the algebraic, geometric, and homotopy Reidemeister traces.

In Chapter 5 we used the results on composites of dual pairs to produce new dual pairs. In this chapter we will use the corresponding results for the compatibility of composites of dual pairs and traces to compare the based and unbased versions of different forms of the Reidemeister trace. We also show how to use properties of the trace in bicategories with shadows to recover some standard fixed point theory results.

6.1. The geometric Reidemeister trace

In this section we define the unbased geometric Reidemeister trace using trace in a bicategory. For this invariant we use the topologized fundamental groupoid.

Let $f\colon M \to M$ be an endomorphism of a closed smooth manifold and

$$\Pi^f M = \{(\gamma, x) \in \Pi M \times M | \gamma(0) = f(x)\}.$$

There is a Hurewicz fibration $t \times s\colon \Pi^f M \to M \times M$ given by $s(\gamma, x) = x$, $t(\gamma, x) = \gamma(1)$. This defines a $(\Pi M, t \times s)_+$-$(\Pi M, t \times s)_+$-bimodule $(\Pi^f M, t \times s)_+$ with the usual left action of ΠM on itself and the right action given by first composing with f and then composing paths. Then $(\Pi M, s)_+ \odot (\Pi^f M, t \times s)_+$ is equivalent to the right $(\Pi M, t \times s)_+$-module $(\Pi^f M, s)_+$. The map f induces a map of right $(\Pi M, t \times s)_+$-modules

$$f_*\colon (\Pi M, s)_+ \to (\Pi M, s)_+ \odot (\Pi^f M, t \times s)_+.$$

If M is n-dualizable, the trace of f_* is the stable homotopy class of the map

$$S^n \to \langle\!\langle S^n \bar{\wedge} (\Pi^f M, t \times s)_+ \rangle\!\rangle \cong S^n \wedge \langle\!\langle (\Pi^f M, t \times s)_+ \rangle\!\rangle$$

given by $v \mapsto (\epsilon[\eta(v), f(\rho\eta(v))], H[f\rho(\eta(v)), \rho(\eta(v))])$ where $H(f\rho(\eta(v)), \rho(\eta(v)))$ is a path from $f\rho\eta(v)$ to $\rho\eta(v)$ as in Lemma 3.1.4.

Definition 6.1.1. — The *unbased geometric Reidemeister trace*, $R^{U,geo}(f)$, is the element $\mathrm{tr}(f_*) \in \pi_0^s(\langle\!\langle \Pi^f M, t \times s)_+ \rangle\!\rangle)$.

Proposition 6.1.2. — *A choice of base point $* \in M$ determines an isomorphism*

$$\langle\!\langle S^n \bar{\wedge} (\Pi^f M, t \times s)_+ \rangle\!\rangle \to \vee_{\langle\!\langle \pi_1 M^\phi \rangle\!\rangle} S^n.$$

Under this identification $R^{U,geo}(f) = R^{geo}(f)$.

Proof. — In Lemma 5.4.2 we compared the dual pairs

$$((\Pi M, s)_+, T_M s^* S^\nu)$$

and

$$(\tilde{M}_+, T\pi^* S^\nu)$$

using a third dual pair, $((\Pi M_x, t)_+, ({}_x\Pi M, s)_+)$. The dual pair

$$((\Pi M_x, t)_+, ({}_x\Pi M, s)_+)$$

has the property that the evaluation is an isomorphism. Then the result follows from Lemma 4.5.6. □

Remark 6.1.3. — The classical definition of the geometric Reidemeister trace described in Chapter 2 suggests that composites of dual pairs might be relevant since the geometric Reidemeister trace is defined using the index and information about the fundamental group. The index can be defined using the classical dual pair of a closed smooth manifold and the fundamental group information can be described using a standard dual pair related to the fundamental groupoid regarded as a monoid.

6.2. The homotopy Reidemeister trace

The Moore paths monoid can also be used to define a trace. Given a map $f\colon M \to M$, let

$$\mathcal{P}^f M = \{(\gamma, u, x) \in \mathcal{P}M \times M | \gamma(0) = f(x)\}.$$

There are maps $s, t\colon \mathcal{P}^f M \to M$ given by $s(\gamma, u, x) = x$ and $t(\gamma, u, x) = \gamma(u)$ and these define a $(\mathcal{P}M, t \times s)_+$-$(\mathcal{P}M, t \times s)_+$-bimodule $(\mathcal{P}^f M, t \times s)_+$ with the usual left action of $\mathcal{P}M$ and the right action of $\mathcal{P}M$ given by first composing with f and then composing paths. The right $(\mathcal{P}M, t \times s)_+$-module $(\mathcal{P}M, s)_+ \odot (\mathcal{P}^f M, t \times s)_+$ is equivalent to the right $(\mathcal{P}M, t \times s)_+$-module $(\mathcal{P}^f M, s)_+$. The map f induces a map of right $(\mathcal{P}M, t \times s)_+$-modules

$$\tilde{f}\colon (\mathcal{P}M, s)_+ \to (\mathcal{P}M, s)_+ \odot (\mathcal{P}^f M, t \times s)_+.$$

There is a map from the shadow of $(\mathcal{P}^f M, t \times s)$ to

$$\Lambda^f M := \{(\gamma, u) \in \mathcal{P}M | \gamma(0) = f(\gamma(u))\} \simeq \{\alpha \in M^I | \alpha(0) = f(\alpha(1))\}.$$

For n-simplices this map is defined by

$$\begin{aligned}\mathcal{P}M \boxtimes \cdots \boxtimes \mathcal{P}M \boxtimes \mathcal{P}^f M &\to \Lambda^f M\\ (\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma) &\mapsto (\alpha_1\alpha_2\ldots\alpha_n\gamma).\end{aligned}$$

This map is surjective. It is also an isomorphism on components: two paths γ_1 and γ_2 in $\mathcal{P}^f M$ have the same image in $\pi_0(\Lambda^f M)$ if and only if there is a path α from $\gamma_1(0)$ to $\gamma_2(0)$ such that $\gamma_1\alpha$ is homotopic to $f(\alpha)\gamma_2$. This is similar to the comparison of $C(\pi^\phi, \pi)$ with $\langle\!\langle \pi^\phi \rangle\!\rangle$ in Section 3.2.

The trace of $\tilde{f}$ is the stable homotopy class of the map

$$S^n \to S^n \wedge \Lambda^f M_+$$

given by $v \mapsto (\epsilon(\eta(v), f\rho\eta(v)), H(f(\rho\eta(v)), \rho\eta(v)))$, where η is the Pontryagin-Thom map for an embedding $M \to \mathbb{R}^n$, ϵ is the evaluation for the dual pair (S^0_M, S^ν), and $H(f(\rho\eta(v)), \rho\eta(v))$ is a path from $f\rho\eta(v)$ to $\rho\eta(v)$ as in Lemma 3.1.4.

Definition 6.2.1. — The *homotopy Reidemeister trace*, $R^{\mathrm{htpy}}(f)$, is the trace of $\tilde{f}$.

Proposition 6.2.2. — *The map*

$$\mathcal{P}^f M \to \Pi^f M$$

that takes a path to its homotopy class with end points fixed induces an isomorphism

$$\pi_0^s(\langle\!\langle(\mathcal{P}^f M, t \times s)_+\rangle\!\rangle) \to \pi_0^s(\langle\!\langle(\Pi^f M, t \times s)_+\rangle\!\rangle).$$

The image of $R^{\mathrm{htpy}}(f)$ under this isomorphism is $R^{U,geo}(f)$.

Proof. — In Proposition 3.2.3 we defined an isomorphism $\pi_0^s(X) \cong H_0(X)$. The maps that define this isomorphism are all natural and so the following diagram commutes.

$$\begin{array}{ccccccc}
\pi_0^s(\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+) & \longleftarrow & \pi_q(\Sigma^q\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+) & \longrightarrow & H_q(\Sigma^q\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+) & \longleftarrow & H_0(\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\pi_0^s(\langle\!\langle\Pi^f M\rangle\!\rangle_+) & \longleftarrow & \pi_q(\Sigma^q\langle\!\langle\Pi^f M\rangle\!\rangle_+) & \longrightarrow & H_q(\Sigma^q\langle\!\langle\Pi^f M\rangle\!\rangle_+) & \longleftarrow & H_0(\langle\!\langle\Pi^f M\rangle\!\rangle_+)
\end{array}$$

Here q is chosen so that the Freudenthal suspension theorem implies that the first horizontal maps are isomorphisms. The second horizontal maps are the Hurewicz maps. The third maps are the homology isomorphism. The map $H_0(\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+) \to H_0(\langle\!\langle\Pi^f M\rangle\!\rangle_+)$ is an isomorphism since the map $\langle\!\langle\mathcal{P}^f M\rangle\!\rangle \to \langle\!\langle\Pi^f M\rangle\!\rangle$ is an isomorphisms on components. Then

$$\pi_0^s(\langle\!\langle\mathcal{P}^f M\rangle\!\rangle_+) \to \pi_0^s(\langle\!\langle\Pi^f M\rangle\!\rangle_+)$$

is also an isomorphism.

At the end of Section 5.4 we showed that the diagrams

$$\begin{array}{ccccc} S^0 & \longrightarrow & S^0_M \odot U_M \odot S^\nu & \longrightarrow & (\mathcal{P}M,s)_+ \odot T_M s^* S^\nu \\ & & & \searrow & \downarrow \\ & & & & (\Pi M,s)_+ \odot T_M s^* S^\nu \end{array}$$

and

$$\begin{array}{ccccc} T_M s^* S^\nu \odot (\mathcal{P}M,s)_+ & \to & S^n \odot L(\mathcal{P}M,t\times s)_+ \odot R(\mathcal{P}M,t\times s)_+ & \to & S^n \odot (\mathcal{P}M,t\times s)_+ \\ \downarrow & & \downarrow & & \downarrow \\ T_M s^* S^\nu \odot (\Pi M,s)_+ & \to & S^n \odot L(\Pi M,t\times s)_+ \odot R(\Pi M,t\times s)_+ & \to & S^n \odot (\Pi M,t\times s)_+ \end{array}$$

commute. The diagram

$$\begin{array}{ccc} (\mathcal{P}M,s)_+ & \longrightarrow & (\mathcal{P}^f M,s)_+ \\ \downarrow & & \downarrow \\ (\Pi M,s)_+ & \longrightarrow & (\Pi^f M,s)_+ \end{array}$$

also commutes. The compatibility of shadows and these diagrams imply that the image of $R^{\mathrm{htpy}}(f)$ under the map to components is $R^{geo}(f)$. □

6.3. The Klein and Williams invariant as a trace

To identify $R^{KW}(f)$ with $R^{\mathrm{htpy}}(f)$ we need to fill in some of the details we omitted in Section 2.4. We will still not give a complete proof here. All of the details can be found in Section 8.2.

Recall that S_B is the unreduced fiberwise suspension.

***Proposition 6.3.1* (Proposition 2.4.4).** — *There is an isomorphism*

$$\{S^0_M, S_M(\Gamma_f^*(N(i)))\}_M \cong \{S^0, \Lambda^f M_+\}.$$

Sketch of proof. — Since M is a closed smooth manifold Corollary 5.1.3 implies there is an isomorphism

$$\{S^0_M, S_M(\Gamma_f^*(N(i)))\}_M \cong \{S^n, tS^\nu \odot S_M(\Gamma_f^*(N(i)))\}$$

where ν is the normal bundle of M.

It remains to identify $S_M(\Gamma_f^*(N(i)))$. Let τ be the normal bundle of the inclusion of the diagonal of M into $M\times M$. The ex-space $S_{M\times M}(N(i))$ is weakly equivalent to

$$\triangle_! S^\tau \boxtimes (\mathcal{P}M, t\times s)_+.$$

Taking pullbacks of both sides gives a weak equivalence

$$S_M(\Gamma_f^*(N(i))) \simeq \triangle_! S^\tau \boxtimes \Lambda^f M_+.$$

Combining this with the isomorphism above, we have an isomorphism

$$\{S^0_M, S_M(\Gamma^*_f(N(i)))\}_M \cong \{S^n, S^\nu \boxtimes \triangle_! S^\tau \boxtimes \Lambda^f M_+\}.$$

By construction of the $\boxtimes$ product $S^\nu \boxtimes \triangle_! S^\tau$ is equivalent to $S^\nu \wedge_M S^\tau$ and this bundle is trivial. So we have an isomorphism

$$\{S^0_M, S_M(\Gamma^*_f(N(i)))\}_M \cong \{S^n, S^n \bar{\wedge} \Lambda^f M_+\}.$$ □

Theorem 6.3.2. — *Let M be a closed smooth manifold and $f\colon M \to M$ a continuous map. Under the identification in Proposition 6.3.1*

$$R^{KW}(f) = R^{\mathrm{htpy}}(f).$$

Proof. — To define the stable homotopy Euler class we used the map

$$\sigma_+ \amalg \sigma_- : S^0_M \to f^* S_{M\times M} N(M \times M - \triangle).$$

The corresponding map $\varsigma\colon S^0_M \to \triangle_! S^\tau \odot \Lambda^f M_+$ under the identification in Proposition 6.3.1 takes the section of S^0_M to the section of $\triangle_! S^\tau \odot \Lambda^f M_+$ and on the other copy of M,

$$\varsigma(m) = ((m, f(m)), H(f(m), m))$$

where $H(f(m), m)$ is a path from $f(m)$ to m as in Lemma 3.1.4.

Let ϕ denote the equivalence in Proposition 6.3.1. Then we have the following isomorphisms.

$$\begin{array}{ccc}
[S^0_M, S_M f^* N(M \times M - \triangle)]_M & \xrightarrow{\phi_*} & [S^0_M, \triangle_! S^\tau \odot \Lambda^f M_+]_M \\
\downarrow F & & \downarrow F \\
\{S^0_M, S_M f^* N(M \times M - \triangle)\}_M & \xrightarrow{\phi_*} & \{S^0_M, \triangle_! S^\tau \odot \Lambda^f M_+\}_M \\
\downarrow D & & \downarrow D \\
\{S^n, S^\nu \odot S_M f^* N(M \times M - \triangle)\} & \xrightarrow[(1\odot\phi)_*]{} & \{S^n, S^\nu \odot \triangle_! S^\tau \odot \Lambda^f M_+\}
\end{array}$$

The map F is the stabilization isomorphism. The map D is the isomorphism that defines the dual pair (S^0_M, tS^ν) as in Corollary 5.1.3.

In the top left corner we have the stable cohomotopy Euler class and in the bottom right corner the corresponding map is

$$S^n \xrightarrow{\eta} S^\nu \odot S^0_M \xrightarrow{\mathrm{id}\odot\varsigma} S^\nu \odot \triangle_! S^\tau \odot \Lambda^f M_+ \simeq (S^\nu \wedge_M S^\tau) \odot \Lambda^f M_+ \simeq S^n \times \Lambda^f M$$

This map is the trace of the map induced by $f\colon M \to M$ on the space of free Moore paths in M. □

6.4. Duality for unbased bimodules enriched in chain complexes

There is an unbased version of the algebraic Reidemeister trace which can be defined using the trace in a bicategory with shadows. This invariant was defined by Coufal in [4, 5] using a different approach.

The unbased algebraic Reidemeister trace is an example of the trace in the bicategory of categories, bimodules and natural transformations enriched in chain complexes of modules over a commutative ring R. In this bicategory the 0-cells are categories enriched in chain complexes of modules over R. If $\mathscr{A}$ and $\mathscr{B}$ are categories enriched in Ch_R then $\mathscr{A}\otimes\mathscr{B}$ is the category with objects $\mathrm{ob}\mathscr{A}\times\mathrm{ob}\mathscr{B}$ and morphisms chain complexes given by the tensor product of the morphism chain complexes of $\mathscr{A}$ and $\mathscr{B}$. The 1-cells are enriched functors of the form

$$\mathscr{A}\otimes\mathscr{B}^{\mathrm{op}}\to\mathsf{Ch}_R.$$

We refer to functors of this form as $\mathscr{A}$-$\mathscr{B}$-bimodules. The 2-cells are enriched natural transformations. We denote this bicategory $\mathscr{E}_{\mathsf{Ch}}$.

The bicategory composition $\odot$ is the enriched tensor product of functors. If $\mathscr{X}\colon\mathscr{A}\otimes\mathscr{B}^{\mathrm{op}}\to\mathsf{Ch}_R$ and $\mathscr{Y}\colon\mathscr{B}\otimes\mathscr{C}^{\mathrm{op}}\to\mathsf{Ch}_R$ are enriched functors $\mathscr{X}\odot\mathscr{Y}$ is the coequalizer of the maps

$$\coprod_{b,b'\in\mathrm{ob}(\mathscr{B})}\mathscr{X}(a,b')\otimes\mathscr{B}(b,b')\otimes\mathscr{Y}(b,c)\ \underset{\coprod\mathrm{id}\otimes\kappa_{\mathscr{B}}}{\overset{\coprod\kappa_{\mathscr{B}}\otimes\mathrm{id}}{\rightrightarrows}}\ \coprod_{b\in\mathrm{ob}(\mathscr{B})}\mathscr{X}(a,b)\otimes\mathscr{Y}(b,c).$$

This bicategory is the 'many object' generalization of the bicategory of rings, chain complexes of bimodules and maps of chain complexes. A complete definition of this bicategory can be found in Section 9.2 or [38].

Let M be a connected CW complex and let $\mathbb{Z}\Pi M$ be the category with objects the points of M and $\mathbb{Z}\Pi M(x,y)$ the free abelian group on the homotopy classes of paths in M from x to y. We have forgotten the topology on ΠM.

We think of $\mathbb{Z}\Pi M$ as a category enriched in chain complexes of abelian groups concentrated in degree 0. Define a right $\mathbb{Z}\Pi M$-module

$$\mathscr{C}M\colon\mathbb{Z}\Pi M\to\mathsf{Ch}_{\mathbb{Z}}$$

where $\mathscr{C}M(x)$ is the cellular chains on the universal cover of M based at x. The action of a homotopy class of paths is the chain map that is induced by the action on the universal cover.

Note that

$$\mathbb{Z}\Pi M(x,x)\cong\mathbb{Z}\pi_1(M,x)$$

and

$$\mathscr{C}M(x)\cong C_*(\tilde{M}).$$

Lemma 6.4.1. — *The right $\mathbb{Z}\Pi M$-module $\mathscr{C}M$ is dualizable if and only if the right $\mathbb{Z}\pi_1(M,x)$-module $C_*(\tilde{M})$ is dualizable.*

Let M be a finite, connected CW complex. Then $\mathscr{C}M(x)$ is dualizable as a right $\mathbb{Z}\Pi M(x,x)$-module for any $x\in M$ and $\mathscr{C}M$ is dualizable as a right $\mathbb{Z}\Pi M$-module.

Proof. — For any $x \in M$, the groupoid $\mathbb{Z}\Pi M$ defines a $\mathbb{Z}\Pi M$-$\mathbb{Z}\Pi M(x,x)$-bimodule $\mathbb{Z}\Pi M(x,-)$ and a $\mathbb{Z}\Pi M(x,x)$-$\mathbb{Z}\Pi M$-bimodule $\mathbb{Z}\Pi M(-,x)$. These form a dual pair

$$(\mathbb{Z}\Pi M(-,x), \mathbb{Z}\Pi M(x,-)).$$

The coevaluation map

$$\eta\colon \mathbb{Z}\Pi M \to \mathbb{Z}\Pi M(x,-) \odot \mathbb{Z}\Pi M(-,x)$$

takes a representative α of a homotopy class to $(\alpha|_{[\frac{1}{2},1]} \circ \beta^{-1}, \beta \circ \alpha|_{[0,\frac{1}{2}]})$ where β is any path from $\alpha(\frac{1}{2})$ to x. This is independent of the choice of β since different choices are identified by the $\odot$ product. The evaluation map

$$\epsilon\colon \mathbb{Z}\Pi M(-,x) \odot \mathbb{Z}\Pi M(x,-) \to \mathbb{Z}\Pi M(x,x)$$

is composition.

Note that $\mathscr{C}M \odot \mathbb{Z}\Pi M(x,-)$ is isomorphic to $\mathscr{C}M(x)$ for any $x \in M$ and that the coevaluation and evaluation maps for the dual pair $(\mathbb{Z}\Pi M(x,-), \mathbb{Z}\Pi M(-,x))$ are isomorphisms. The first statement then follows from Theorems 4.3.3 and 4.3.4.

For the second part of the lemma it is enough to show that one of the modules is dualizable. For any $x \in M$, $\mathscr{C}M(x)$ is a finitely generated chain complex of free $\mathbb{Z}\pi_1(M,x)$-modules and so is dualizable with dual $\mathrm{Hom}_{\mathbb{Z}\pi_1(M,x)}(\mathscr{C}M(x), \mathbb{Z}\pi_1(M,x))$. Composing this dual pair with the dual pair $(\mathbb{Z}\Pi M(x,-), \mathbb{Z}\Pi M(-,x))$ gives a dual pair for $\mathscr{C}M$. □

This lemma is a special case of Lemma 9.2.5.

Remark 6.4.2. — In the previous section we assumed that M was a closed smooth manifold. In this section we assumed that M is a connected finite CW complex. The different assumptions only reflect that these categories have different 'practical generalities' that we want to work in.

6.5. The unbased algebraic Reidemeister trace

Before we can define the unbased algebraic Reidemeister trace we need to define the shadow in the bicategory of enriched categories, bimodules and homomorphisms. Let $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{A}$-bimodule. Then $\langle\!\langle \mathscr{X} \rangle\!\rangle$ is the coequalizer of the maps

$$\coprod_{a,a' \in \mathscr{A}} \mathscr{A}(a,a') \otimes \mathscr{X}(a,a') \underset{\coprod(\kappa_{\mathscr{A}} \circ \gamma)}{\overset{\coprod \kappa_{\mathscr{A}}}{\rightrightarrows}} \coprod_{a \in \mathscr{A}} \mathscr{X}(a,a).$$

Let $f\colon M \to M$ be a continuous map and f_* the induced map on $\mathbb{Z}\Pi M$. Let $U^{f_*}_{\mathbb{Z}\Pi M}$ be the $\mathbb{Z}\Pi M$-$\mathbb{Z}\Pi M$-bimodule defined by $U^{f_*}_{\mathbb{Z}\Pi M}(x,y) = \mathbb{Z}\Pi M(f(y),x)$ with action of $\mathbb{Z}\Pi M \otimes \mathbb{Z}\Pi M^{\mathrm{op}}$ given by $(\gamma,\alpha,\beta) \mapsto (\gamma\alpha f(\beta))$. The map $f\colon M \to M$ defines a natural transformation

$$\tilde{f}_*\colon \mathscr{C}M \to \mathscr{C}^{f_*}M := \mathscr{C}M \odot U^{f_*}_{\mathbb{Z}\Pi M}.$$

The trace of $\tilde{f}_*$ in the bicategory of enriched categories, bimodules, and natural transformations is a map

$$\mathbb{Z} \to \langle\!\langle U^{f_*}_{\mathbb{Z}\Pi M} \rangle\!\rangle.$$

Definition 6.5.1. — The *unbased algebraic Reidemeister trace* of f, $R^{U,alg}(f)$, is the image of 1 under the trace of $\tilde{f}_*$.

Remark 6.5.2. — Using a particular description of the coevaluation map for the dual pair $(\mathscr{C}M, \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M))$ we can compare the unbased algebraic Reidemeister with the unbased algebraic generalized Lefschetz number in **[4]**. Just as in Equation 2.3.2, there is a map

$$\nu\colon \mathscr{C}M \odot \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathscr{C}M)\ .$$

Since $\mathscr{C}M$ is dualizable, this map is an isomorphism. We can chose the coevaluation for the dual pair $(\mathscr{C}M, \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M)$ to be

$$\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathscr{C}M) \xrightarrow{\nu^{-1}} \mathscr{C}M \odot \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M).$$

Then unbased algebraic Reidemeister trace is the image of 1 under the map

$$\begin{array}{ccc}
\mathbb{Z} & & \langle\!\langle U^{f_*}_{\mathbb{Z}\Pi M} \rangle\!\rangle \\
\downarrow & & \uparrow{\scriptstyle \epsilon} \\
\mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathscr{C}M) & & \langle\!\langle \mathscr{C}^{f_*}M \odot \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M) \rangle\!\rangle \\
{\scriptstyle \tilde{f}_*}\downarrow & & \uparrow{\scriptstyle \cong} \\
\mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathscr{C}^{f_*}M) & \xrightarrow{\nu^{-1}} & \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M) \odot \mathscr{C}^{f_*}M
\end{array}\ .$$

The invariant defined in **[4]** is the image of $\tilde{f}_*$ under the map

$$\mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathscr{C}^{f_*}M) \xrightarrow{\nu^{-1}} \mathscr{C}^{f_*}M \odot \mathrm{Hom}_{\mathbb{Z}\Pi M}(\mathscr{C}M, \mathbb{Z}\Pi M) \xrightarrow{\mathrm{ev}} \langle\!\langle U^{f_*}_{\mathbb{Z}\Pi M} \rangle\!\rangle.$$

The map $\tilde{f}_*$ is of the form described in Lemma 4.5.6 and the coevaluation for the dual pair $(\mathbb{Z}\Pi M(x,-), \mathbb{Z}\Pi M(-,x))$ is an isomorphism so we have the following corollary.

Corollary 6.5.3. — *A choice of base point $* \in M$ determines an isomorphism*

$$\langle\!\langle U^{f_*}_{\mathbb{Z}\Pi M} \rangle\!\rangle \to \langle\!\langle \pi_1(M,*)^{\phi} \rangle\!\rangle$$

which takes $R^{U,alg}(f)$ to $R^{alg}(f)$.

6.6. The proof of Theorem D and some properties of the trace

We can now give a conceptual proof of Theorem D. We will start by connecting the geometric Reidemeister trace with the algebraic Reidemeister trace. Here we use the functoriality of the trace in bicategories with shadows.

The rational cellular chains functor is a symmetric monoidal functor which commutes with trace. We can use this functor to define a lax functor of bicategories $C_*(-;\mathbb{Q})\colon \mathscr{M}_{\mathsf{Ex}} \to \mathscr{E}_{\mathsf{Ch}}$. A monoid $\mathscr{A}$ in Ex with projection map $t\times s\colon \mathscr{A}\to A\times A$ is taken to the category with objects the set A (forget the topology). For $a,a'\in A$, the morphism chain complex is

$$C_*((t\times s)^{-1}(a,a');\mathbb{Q}),$$

the rational cellular chains on the inverse image of (a,a'). An $\mathscr{A}$-$\mathscr{A}'$-bimodule $\mathscr{X}$ with projection $p\times p'$ is taken to the bifunctor which on a pair of objects $(a,a')\in A\times A'$ is the chain complex $C_*((p\times p')^{-1}(a,a');\mathbb{Q})$. A map of modules is taken to the induced map on cellular chains.

For any monoid $\mathscr{A}$ the map $\phi_{\mathscr{A}}$ is the identity. For bimodules $\mathscr{X}$ and $\mathscr{Y}$ the map $\phi_{\mathscr{X},\mathscr{Y}}$ is induced by the inclusion maps

$$(p_{\mathscr{X}}\times p'_{\mathscr{X}})^{-1}(a,a')\wedge(p_{\mathscr{Y}}\times p'_{\mathscr{Y}})^{-1}(a',a'')\to(p_{\mathscr{X}})^{-1}(a)\odot(p'_{\mathscr{Y}})^{-1}(a'').$$

Lemma 6.6.1. — *The functor $C_*(-;\mathbb{Q})$ defines a map*

$$\mathbb{Z}\langle\!\langle \Pi^{f_*}M\rangle\!\rangle\to\langle\!\langle \mathbb{Z}\Pi^{f_*}M\rangle\!\rangle$$

and under this map $R^{U,geo}(f)=R^{U,alg}(f)$. The same functor defines a map

$$\mathbb{Z}\langle\!\langle \pi_1M^{\phi}\rangle\!\rangle\to\langle\!\langle \mathbb{Z}\pi_1M^{\phi}\rangle\!\rangle.$$

Under this map $R^{geo}(f)=R^{alg}(f)$.

We can now complete the proof of Theorem D from the introduction.

Proof of Theorem D. — By Lemma 6.6.1 there is a map

$$\mathbb{Z}\langle\!\langle \pi_1M^{\phi}\rangle\!\rangle\to\langle\!\langle \mathbb{Z}\pi_1M^{\phi}\rangle\!\rangle$$

and under this map

$$R^{alg}(f)=R^{geo}(f).$$

Proposition 6.2.2 implies $R^{geo}(f)$ is zero if and only if $R^{\mathrm{htpy}}(f)$ is zero. By Theorem 6.3.2 there is an equivalence

$$S_Mf^*N(M\times M-\triangle)\simeq\triangle_!S^{\tau}\odot\Lambda^fM_+$$

and under this equivalence

$$R^{KW}(f)=R^{\mathrm{htpy}}(f).$$

By Corollary 2.4.5 $R^{KW}(f)$ is zero if and only if f is homotopic to a map with no fixed points. □

In addition to demonstrating the compatibility of various forms of the Reidemeister trace, we can use the trace in bicategories with shadows to give new proofs of various basic results in fixed point theory. These results are applications of Lemmas 4.5.5 and 4.5.4.

Since these results follow from properties of trace in bicategories they hold for any of the forms of the Reidemeister trace.

Corollary 6.6.2. — *For a product of continuous maps*

$$f_M \times f_N \colon M \times N \to M \times N$$

of closed smooth manifolds, one of which is simply connected,

$$R(f_M \times f_N) = R(f_M) \times R(f_N).$$

This follows from Lemma 4.5.5 and is a very special case of results in [**9, 17, 18, 36**].

According to [**21**, I.5.2] the Nielsen number satisfies a commutativity property. Let X and Y be compact connected ENR's and $f\colon X \to Y$, $g\colon Y \to X$ be continuous maps. Then $N(g \circ f) = N(f \circ g)$. We can recover this result from Lemma 4.5.4.

Corollary 6.6.3. — *If M and N are closed smooth manifolds and $f\colon M \to N$ and $g\colon N \to M$ are continuous maps, there is a bijection between the fixed point classes of $f \circ g$ and $g \circ f$ and under this identification*

$$R(f \circ g) = R(g \circ f).$$

In particular, $N(f \circ g) = N(g \circ f)$.

6.7. The Reidemeister trace for regular covering spaces

In addition to the Reidemeister trace defined using the universal cover, there is a Reidemeister trace for all regular covering spaces. The theory for regular covers is very similar to the theory for universal covers, except maps do not always lift to regular covers. To resolve this problem, we will restrict attention to those maps that do have lifts. This means that for a normal subgroup K of $\pi_1 M$ we will only consider maps $f\colon M \to M$ such that $\phi(K) \subset K$. Here ϕ is the same as in Chapter 2; it is the map induced on $\pi_1 M$ by f after choosing a base point and a path ζ from that base point to its image under f.

Definition 6.7.1. — [**21**, III.2.1] Two fixed points x and y of $f\colon M \to M$ are in the same *mod K fixed point class* if there exists a lift of f to $\tilde{f}/K\colon \tilde{M}/K \to \tilde{M}/K$ and lifts of x, y to $\tilde{x}, \tilde{y} \in \tilde{M}/K$ such that $\tilde{f}/K(\tilde{x}) = \tilde{x}$ and $\tilde{f}/K(\tilde{y}) = \tilde{y}$.

If K is the trivial subgroup of $\pi_1 M$, this is the usual definition of fixed point classes.

Lemma 6.7.2. — [**21**, III.2.2] *Two fixed points x and y are in the same mod K fixed point class if and only if there is path γ in M from x to y such that $\gamma f(\gamma^{-1})$ is in K.*

Since $\phi(K) \subset K$, ϕ induces a map $\phi\colon \pi_1M/K \to \pi_1M/K$. Let $\langle\!\langle(\pi_1M/K)^\phi\rangle\!\rangle$ be the semiconjugacy classes of π_1M/K with respect to the induced homomorphism ϕ. For each fixed point x pick a path γ_x in M from the base point $*$ to x. Then there is a well defined injection from the mod K fixed point classes of f to $\langle\!\langle(\pi_1M/K)^\phi\rangle\!\rangle$ that takes a fixed point x to the homotopy class of the path $\gamma_x^{-1}f(\gamma_x)\zeta$.

Definition 6.7.3. — The *mod K geometric Reidemeister trace* of f, $R_K^{geo}(f)$, is

$$\sum_{\text{mod K fixed point classes } F_j} i(F_j)\cdot F_j \in \mathbb{Z}\langle\!\langle(\pi_1M/K)^\phi\rangle\!\rangle$$

For spaces with a universal cover there is a bijection between regular covers and normal subgroups of the fundamental group. These regular covers provide more examples of dual pairs.

Lemma 6.7.4. — *Suppose M is a space with a universal cover $\tilde{M}$ and $\tilde{M}_+$ is dualizable as a π_1M space. If $K \triangleleft \pi_1M$, then $(\tilde{M}/K)_+$ is dualizable as a $(\pi_1M)/K$ space.*

Proof. — This proof uses a composite of dual pairs. The group π_1M/K has actions by π_1M and π_1M/K on both the left and the right. We think of π_1M/K as a π_1M-π_1M/K-bimodule and $t\pi_1M/K$ as a π_1M/K-π_1M-bimodule. Then

$$(\pi_1M/K, t(\pi_1M/K))$$

is a dual pair. The coevaluation

$$\pi_1M \to \pi_1M/K \odot t(\pi_1M/K)$$

is the quotient map. The evaluation

$$t(\pi_1M/K) \odot \pi_1M/K \to \pi_1M/K$$

is composition.

Note that $\tilde{M}_+ \odot (\pi_1M/K)$ is a cover of M corresponding to the subgroup $K \subset \pi_1M$. Since both $\tilde{M}_+$ and π_1M/K are dualizable Theorem 4.3.3 implies the composite $\tilde{M}/K$ is dualizable with dual $(t\pi_1M/K) \odot T\pi^*\nu$. □

The proof of Proposition 3.2.3 also implies the following result.

Lemma 6.7.5. — *The map induced on homology by the trace of $\tilde{f}/K\colon \tilde{M}/K \to \tilde{M}/K$ is the Mod K geometric Reidemeister trace of f.*

Lemma 6.7.6. — *The map $\langle\!\langle\pi_1M^\phi\rangle\!\rangle \to \langle\!\langle(\pi_1M/K)^\phi\rangle\!\rangle$ that takes a semiconjugacy class to the corresponding mod K semiconjugacy class takes the Reidemeister trace of f to the mod K Reidemeister trace of f. In particular, the sum of the coefficients of the Reidemeister trace is the Lefschetz number.*

Proof. — The map used to define the Reidemeister trace is of the form described in Lemma 4.5.6 so this result follows from Lemmas 6.7.4 and 4.5.6. If K is π_1M the mod K Reidemeister trace is the Lefschetz number, so the sum of the coefficients of $R^{geo}(f)$ is the Lefschetz number of f. □

CHAPTER 7

DUALITY FOR FIBERWISE PARAMETRIZED MODULES

In this chapter we describe fiberwise generalizations of some of the results from the previous two chapters. Unfortunately, since we are now interested in fiberwise maps not all invariants defined in Chapter 5 make sense. For example, it is no longer possible to choose a base point and so we will now only use unbased invariants. Another challenge, and benefit, of the fiberwise generalization is that the invariant that gives a converse to the fiberwise Lefschetz fixed point theorem, a generalization of the homotopy Reidemeister trace, is much richer than the classical invariant. One consequence of this is that it isn't clear what invariants, if any, deserve to be called the fiberwise geometric Reidemeister trace or the fiberwise algebraic Reidemeister trace. We describe one candidate invariant for the fiberwise geometric Reidemeister trace. This invariant was defined by Scofield. It does not give a converse to the fiberwise Lefschetz fixed point theorem.

We define the fiberwise homotopy Reidemeister trace using the approach of the previous chapters. This invariant can be identified with the fiberwise invariant defined by Klein and Williams. The definitions of these invariants, their comparison, and even the proof that these invariants give a converse to the fiberwise Lefschetz fixed point theorem are almost identical to the approach in the classical case.

7.1. Fiberwise Costenoble-Waner duality

The bicategory we use to study fiberwise spaces is closely related to the bicategory Ex we used to study classical fixed point theory. The 0-cells are spaces over B. The 1-cells are spaces over and under the 0-cells. The 2-cells are maps of total spaces that commute with the section and projection. This bicategory was introduced in **[34]**, where a more sophisticated stable version was also studied.

More formally, the 0-cells of Ex_B are spaces over B. That is, a space C with a map $C \to B$. A 1-cell from $C \to B$ to $D \to B$ is a space X and maps

$$D \times_B C \xrightarrow{\sigma} X \xrightarrow{p} D \times_B C$$

such that the composite $p \circ \sigma$ is the identity map of $D \times_B C$. For two 1-cells X and Y from C to D, a 2-cell from X to Y is a map $f\colon X \to Y$ such that

$$\begin{array}{ccccc} C \times_B D & \longrightarrow & X & \longrightarrow & C \times_B D \\ \| & & \downarrow{\scriptstyle f} & & \| \\ C \times_B D & \longrightarrow & Y & \longrightarrow & C \times_B D \end{array}$$

commutes.

As in Remark 1.4.1, for a 1-cell X over C and D we require that X and $C \times_B D$ are of the homotopy types of CW-complexes, the projection $X \to C \times_B D$ is a Hurewicz fibration, and the section $C \times_B D \to X$ is a fiberwise cofibration. When these conditions are not satisfied we implicitly use the model structures and approximation techniques from **[34]** to maintain homotopical control.

The bicategory composition in Ex_B is very similar to the bicategory composition in Ex. The external smash product of two 1-cells in Ex_B, written $\bar{\wedge}$, is defined by taking the fiberwise smash product over the 0-cells. This is not the fiberwise smash product over B. If X is a 1-cell from C to D and Y is a 1-cell from D to E then we define $X \boxtimes Y$, a 1-cell from C to E, as the pullback along $\triangle\colon D \to D \times_B D$ and then pushout along $r\colon D \to B$ of $X \bar{\wedge} Y$.

$$\begin{array}{ccccc} C \times_B E & \xleftarrow{\mathrm{id} \times r \times \mathrm{id}} & C \times_B D \times_B E & \xrightarrow{\mathrm{id} \times \triangle \times \mathrm{id}} & C \times_B D \times_B D \times_B E \\ \downarrow & & \downarrow & & \downarrow \\ X \boxtimes Y & \longleftarrow & (\mathrm{id} \times \triangle \times \mathrm{id})^*(X \bar{\wedge} Y) & \longrightarrow & X \bar{\wedge} Y \\ \downarrow & & \downarrow & & \downarrow \\ C \times_B E & \xleftarrow[\mathrm{id} \times r \times \mathrm{id}]{} & C \times_B D \times_B E & \xrightarrow[\mathrm{id} \times \triangle \times \mathrm{id}]{} & C \times_B D \times_B D \times_B E \end{array}$$

The unit 1-cell associated to a 0-cell $C \to B$ is $(C, \triangle)_+$ and we will denote this U_C.

Definition 7.1.1. — A 1-cell X in Ex_B over C is *fiberwise Costenoble-Waner n-dualizable* if there is a 1-cell Y over C and maps

$$S^n_B \xrightarrow{\eta} X \boxtimes tY \qquad \text{and} \qquad tY \boxtimes X \xrightarrow{\epsilon} \triangle_! S^n_C$$

such that

$$\begin{array}{ccc} S^n_B \boxtimes X & \xrightarrow{\eta \boxtimes \mathrm{id}} & (X \boxtimes tY) \boxtimes X \\ & & \downarrow{\scriptstyle \cong} \\ {\scriptstyle \gamma}\downarrow & & X \boxtimes (tY \boxtimes X) \\ & & \downarrow{\scriptstyle \mathrm{id} \boxtimes \epsilon} \\ X \boxtimes S^n & \xrightarrow[\cong]{} & X \boxtimes \triangle_! S^n_C \end{array} \qquad \begin{array}{ccc} tY \boxtimes S^n_B & \xrightarrow{\mathrm{id} \boxtimes \eta} & tY \boxtimes (X \boxtimes tY) \\ & & \downarrow{\scriptstyle \cong} \\ {\scriptstyle (\sigma \boxtimes \mathrm{id})\gamma}\downarrow & & (tY \boxtimes X) \boxtimes tY \\ & & \downarrow{\scriptstyle \epsilon \boxtimes \mathrm{id}} \\ S^n \boxtimes tY & \xrightarrow[\cong]{} & \triangle_! S^n_C \boxtimes tY \end{array}$$

stably commute up to fiberwise homotopy over C.

To give examples of this kind of duality we will need to consider equivariant Costenoble-Waner duality. A bundle construction gives a connection between dual pairs in the bicategory $G\mathsf{Ex}$ and dual pairs in the bicategory Ex_B.

Let G be a compact Lie group. There is a bicategory $G\mathsf{Ex}$ with 0-cells G-spaces. The 1-cells in $G\mathsf{Ex}$ are ex-spaces X with an action by G such that the section and projection maps are equivariant. The 2-cells are equivariant maps of total spaces that commute with the section and projection maps. The bicategory composition is induced from that in Ex. The group G acts by the diagonal action.

This bicategory also has a stable version and duality in that bicategory has an interpretation as V-duality in $G\mathsf{Ex}$.

Let S^V denote the one point compactification of a representation V of G.

Definition 7.1.2. — [**34**, 18.3.1] A 1-cell X in $G\mathsf{Ex}$ is *V-dualizable* for a representation V of G if there is a 1-cell Y in $G\mathsf{Ex}$ and maps

$$S^V \xrightarrow{\eta} X \boxtimes tY \qquad \text{and} \qquad tY \boxtimes X \xrightarrow{\epsilon} \triangle_! S^V_B$$

such that

$$\begin{array}{ccc} S^V \boxtimes X & \xrightarrow{\eta\boxtimes \mathrm{id}} & (X \boxtimes tY) \boxtimes X \\ & & \downarrow{\cong} \\ \gamma\downarrow & & X \boxtimes (tY \boxtimes X) \\ & & \downarrow{\mathrm{id}\boxtimes\epsilon} \\ X \boxtimes S^V & \xrightarrow[\cong]{} & X \boxtimes \triangle_! S^V_B \end{array} \qquad \begin{array}{ccc} tY \boxtimes S^V & \xrightarrow{\mathrm{id}\boxtimes \eta} & tY \boxtimes (X \boxtimes tY) \\ & & \downarrow{\cong} \\ (\sigma\boxtimes\mathrm{id})\gamma\downarrow & & (tY \boxtimes X) \boxtimes tY \\ & & \downarrow{\epsilon\boxtimes\mathrm{id}} \\ S^V \boxtimes tY & \xrightarrow[\cong]{} & \triangle_! S^V_B \boxtimes tY \end{array}$$

commute stably up to equivariant fiberwise homotopy.

Theorem 5.1.2 has a generalization to the bicategory $G\mathsf{Ex}$.

Theorem 7.1.3. — [**34**, 18.6.1] *Let M be a closed smooth G-manifold embedded in a representation V. Then (S^0_M, S^ν) is a Costenoble-Waner V-dual pair.*

Let P be a principal G-bundle and B be P/G. Then there is a lax functor

$$\mathbb{P}\colon G\mathsf{Ex} \to \mathsf{Ex}_B.$$

This functor takes a G-space F to $P \times_G F$, where $P \times_G F$ is $P \times F$ quotiented by the diagonal action of G. On 1-cells and 2-cells $\mathbb{P}$ is also given by the functor $P \times_G (-)$, which converts an ex-G-space E over a G-space F into a 1-cell in Ex_B. The section and projection maps of E over F induce section and projection maps for $\mathbb{P}(E)$ over B.

Theorem 7.1.4. — [**34**, 19.4.4] *If (X, Y) is a dual pair in $G\mathsf{Ex}$, $(\mathbb{P}(X), \mathbb{P}(Y))$ is a dual pair in Ex_B.*

Combining this theorem with Theorem 7.1.3 we have the following result.

Corollary 7.1.5. — *If $p\colon M \to B$ is a fiber bundle with closed smooth manifold fibers, then the dual of $(M, \mathrm{id})_+ \in \mathsf{Ex}_B(M,*)$ is the fiberwise one point compactification of the fiberwise normal bundle.*

The coevaluation and evaluation maps for this dual pair are similar to those for the dual pair in Theorem 5.1.2. The evaluation is defined using the following generalization of Lemma 3.1.4.

Lemma 7.1.6. — [**6**, II.5.17] *Let $L \to B$ be a fiberwise ENR. Then there is an open neighborhood W of the diagonal in $L \times_B L$ and a fiberwise homotopy $H_t\colon W \to L$ such that $H_0(x,y) = x$, $H_1(x,y) = y$ and $H_t(x,x) = x$.*

Since fiberwise Costenoble-Waner duality is duality in a bicategory there are other descriptions of dual pairs. The only other characterization we will need is given in the following corollary.

Corollary 7.1.7. — *If (X,Y) is a Costenoble-Waner dual pair in Ex_B, then the coevaluation map of the dual pair induces an isomorphism*

$$\{W \odot X, Z\}_M \to \{W, Z \odot tY\}_B$$

for $W \in \mathsf{Ex}_B(B,B)$ and $Z \in \mathsf{Ex}_B(M,B)$.

7.2. Ranicki duality for fiberwise spaces

The bicategory $\mathscr{M}_{\mathsf{Ex}_B}$ of monoids, bimodules, and maps in Ex_B is defined exactly as the bicategory $\mathscr{M}_{\mathsf{Ex}}$ is defined. The bicategory composition and shadow are defined in analogy with the composition and shadow in Chapter 5. We also have comparisons of the bar resolutions and cyclic bar resolutions with colimits. Like dual pairs of spaces and dual pairs in Ex, the dual pairs in Ex_B and dual pairs of modules in Ex_B are defined using n-duality.

If M is a space over B, $p\colon M \to B$, instead of considering the topologized fundamental groupoid or the free Moore path space we will use the fiberwise free Moore paths,

$$\mathscr{P}_B M = \{(\gamma, u) \in \mathrm{Map}_B(B \times [0,\infty), M) \times [0,\infty) | \gamma(t) = \gamma(u) \,\text{for all}\, t \geq u\}.$$

This is a space over B with the map to B given by $(\gamma, u) \mapsto p\gamma(0)$. The space $\mathscr{P}_B M$ is the free Moore paths in M that are each contained in a single fiber over B. For more details on Map_B see [**34**, 1.3.7].

Lemma 7.2.1. — *If $p\colon M \to B$ is a fibration, the map*

$$t \times s\colon \mathscr{P}_B M \to M \times_B M$$

given by $(t \times s)(\gamma, u) = (\gamma(u), \gamma(0))$ is a fibration.

To minimize notation we will use paths rather than Moore paths in this proof.

Proof. — We must show that all diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathscr{P}_B M \\ {\scriptstyle i_0}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle (t,s)} \\ X \times I & \xrightarrow[H]{} & M \times_B M \end{array}$$

have a lift k.

The map f has an adjoint $\bar{f}\colon X \times I \to M$ which satisfies $p(f(x,t)) = p(f(x,0))$ for all $x \in X$, $t \in I$. Let $H = H_1 \times_B H_0$. Since the diagram commutes $\bar{f}$ must also satisfy $\bar{f}(x,0) = H_0(x,0)$ and $\bar{f}(x,1) = H_1(x,0)$. Let J be the subset $(0,I) \cup (I,0) \cup (1,I)$ of $I \times I$. Then a lift k in the diagram above corresponds to a lift $\bar{k}$ in the diagram

$$\begin{array}{ccc} X \times J & \xrightarrow{g} & M \\ {\scriptstyle \iota}\downarrow & \overset{\bar{k}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I \times I & \xrightarrow[\tilde{H}]{} & B \end{array}$$

where $g\colon J \times X \to M$ is defined by

$$\begin{aligned} g(x,0,s) &= H_0(x,s) \\ g(x,t,0) &= \bar{f}(x,t) \\ g(x,1,s) &= H_1(x,s) \end{aligned}$$

and $\tilde{H}(x,t,s) = pH_1(x,s)$.

Let $\phi\colon X \times J \times I \to X \times I \times I$ be a homeomorphism such that

$$\begin{array}{ccccc} X \times J & & \xrightarrow{\iota} & & X \times I \times I \\ & {\scriptstyle i_0}\searrow & & \nearrow{\scriptstyle \phi} & \\ & & X \times J \times I & & \end{array}$$

commutes. Then there is a lift $\tilde{k}$ in the diagram

$$\begin{array}{ccccc} X \times J & & \xrightarrow{g} & & M \\ \Big\downarrow{\scriptstyle \iota} & {\scriptstyle i_0}\searrow & & \overset{\tilde{k}}{\nearrow} & \Big\downarrow{\scriptstyle p} \\ & & X \times J \times I & & \\ & {\scriptstyle \phi}\swarrow & & \searrow{\scriptstyle \tilde{H}\circ\phi} & \\ X \times I \times I & & \xrightarrow[\tilde{H}]{} & & B \end{array}$$

since p is a fibration, and $\tilde{k} \circ \phi^{-1}$ defines the lift $\bar{k}$. □

Composition of paths gives a strictly associative product $\mathscr{P}_B M \times_M \mathscr{P}_B M \to \mathscr{P}_B M$. The inclusion of M into $\mathscr{P}_B M$ is the unit. Adding a disjoint section to $\mathscr{P}_B M$ gives a monoid in Ex_B.

Recall that for a monoid $\mathscr{A}$, $R(\mathscr{A})$ is $\mathscr{A}$ regarded as a right $\mathscr{A}$-module and $L(\mathscr{A})$ is $\mathscr{A}$ regarded as a left $\mathscr{A}$-module.

***Corollary** 7.2.2.* — *Let $M \to B$ be a fiber bundle with closed smooth manifold fibers. Then we have the following dual pairs.*

(*i*) $(R(\mathscr{P}_B M, t \times s)_+, L(\mathscr{P}_B M, t \times s)_+)$
(*ii*) $((\mathscr{P}_B M, s)_+, T_M s^* S_M^{\nu_B})$.

Proof. — Here $T_M s^* S_M^{\nu_B}$ is defined to be $L(\mathscr{P}_B M, t \times s)_+ \odot S_M^{\nu_B}$ where ν_B is the fiberwise normal bundle of M over B and $S_M^{\nu_B}$ is the fiberwise one point compactification of ν_B over M.

The first of these dual pairs comes from the monoid $(\mathscr{P}_B M, t \times s)_+$ as in Lemma 5.3.2. The second is the composite of the dual pairs

$$(S_M^0, tS_M^{\nu_B})$$

and

$$(R(\mathscr{P}_B M, t \times s)_+, L(\mathscr{P}_B M, t \times s)_+). \qquad \square$$

CHAPTER 8

FIBERWISE FIXED POINT THEORY

In Chapter 6, the corresponding chapter for classical invariants, we described many invariants and gave several applications of trace in bicategories. This section is much shorter. One of the reasons is that based invariants cannot be defined for fiberwise space. Another is that is not clear what invariants should be the generalization of the algebraic and geometric Reidemeister traces.

In contrast to the algebraic and geometric Reidemeister traces, the homotopy Reidemeister trace has a straightforward fiberwise generalization. The invariant defined by Klein and Williams also has a fiberwise generalization and their proof of the converse to the Lefschetz fixed point theorem easily generalizes.

8.1. Fiberwise fixed point theory invariants

The fiberwise homotopy Reidemeister trace is based on the fiberwise free Moore paths monoid. This is the Nielsen-Reidemeister invariant defined by Crabb and James in [**6**, II.6] and in Section 8.3 it is identified with the invariant defined by Klein and Williams in [**25**].

Let $M \to B$ be a fiber bundle with dualizable fibers and $f\colon M \to M$ a fiberwise map. Then f can be used to define a module $(\mathcal{P}^f_B M, t\times s)_+$ in Ex_B. This is analogous to the definition of $(\mathcal{P}^f M, t\times s)_+$. The map f also defines a map of right $(\mathcal{P}_B M, t\times s)_+$ modules

$$\tilde{f}\colon (\mathcal{P}_B M, s)_+ \to (\mathcal{P}_B M, s)_+ \odot (\mathcal{P}^f_B M, t\times s)_+.$$

Since $(\mathcal{P}_B M, s)_+$ is dualizable, the trace of $\tilde{f}$ is defined. It is a fiberwise stable map over B,

$$S^n \times B \to \langle\!\langle S^n \bar{\wedge} (\mathcal{P}^f_B M, t\times s)_+ \rangle\!\rangle.$$

As in Section 6.2, there is a map from the shadow of $(\mathcal{P}^f_B M, t\times s)$ to the twisted path space,

$$\Lambda^f_B M = \{\gamma \in M^I | f(\gamma(1)) = \gamma(0), p(\gamma(t)) = p(\gamma(0)) \text{ for all } t \in I\}.$$

Definition 8.1.1. — The *fiberwise homotopy Reidemeister trace*, $R_B^{\mathrm{htpy}}(f)$, is the image of the trace of $\tilde{f}$ in $\{S_B^0, \Lambda_B^f M\}_B$.

The fiberwise homotopy Reidemeister trace is a very rich invariant and it should be possible to use this invariant to define other, simpler, invariants. One invariant we can extract from the fiberwise homotopy Reidemeister trace is the fiberwise Nielsen number defined by Scofield.

The fiberwise homotopy Reidemeister trace is a map

$$S^n \times B \to S^n \wedge \Lambda_B^f M_+.$$

Each connected component of $\Lambda_B^f M$ has a map to B and so we get a map

$$H_0(B_+) \cong H_n(S^n \wedge (B_+)) \longrightarrow H_n(S^n \wedge \Lambda_B^f M_+) \xrightarrow{\phi} \oplus H_n(S^n \wedge (B_+)) \cong \oplus H_0(B_+)$$

where ϕ is induced by the decomposition of $\Lambda_B^f M_+$ into path components followed by the map to B. The image of this map on one component $H_0(B)$ in $\oplus H_0(B)$ is the fiberwise index of the corresponding 'fiberwise fixed point class'. The fiberwise Nielsen number is the number of fiberwise fixed point classes with nonzero index.

Scofield showed this invariant does not give a converse to the fiberwise Lefschetz fixed point theorem.

Example 8.1.2. — [**41**, V.3.16] Let $f\colon S^3 \times S^3 \to S^3 \times S^3$ be the map $f(b, z) = (b, b^{24} z)$. If $S^3 \times S^3$ is a space over S^3 via the first coordinate projection, f is a fiberwise map. All maps fiberwise homotopic to f have a fixed point, but the fiberwise Nielsen number of f is zero.

8.2. The converse to the fiberwise Lefschetz fixed point theorem

In this section we describe the fiberwise generalization of Klein and Williams' proof of the converse to the Lefschetz Fixed Point Theorem. The intuition and general structure here are identical to that in Section 2.4.

Proposition 8.2.1. — [**12, 25**] *Let $M \to B$ be a continuous map. Then fiberwise homotopies of a fiberwise map $f\colon M \to M$ to a fixed point free map correspond to liftings which make the diagram below commute up to fiberwise homotopy.*

$$\begin{array}{ccc} & & M \times_B M - \triangle \\ & \nearrow & \downarrow \\ M & \xrightarrow{\Gamma_f} & M \times_B M. \end{array}$$

Here Γ_f is the graph of f.

Proof. — Let $(\mathsf{Top}^*/B)(M,M)$ be the fiberwise maps $M \to M$ that are fixed point free. Let $\mathrm{proj}_1 \colon M \times_B M \to M$ be the first coordinate projection. We have the following map of fibration sequences.

$$\begin{array}{ccc}
(\mathsf{Top}^*/B)(M,M) & \hookrightarrow & (\mathsf{Top}/B)(M,M) \\
\downarrow{\scriptstyle \text{graph}} & & \downarrow{\scriptstyle \text{graph}} \\
(\mathsf{Top}/B)(M, M \times_B M - \triangle) & \hookrightarrow & (\mathsf{Top}/B)(M, M \times_B M) \\
\downarrow{\scriptstyle \mathrm{proj}_{1*}} & & \downarrow{\scriptstyle \mathrm{proj}_{1*}} \\
(\mathsf{Top}/B)(M,M) & = & (\mathsf{Top}/B)(M,M)
\end{array}$$

The fibers are taken over the identity map.

The top square is homotopy cartesian and so the homotopy fibers of the inclusions

$$(\mathsf{Top}^*/B)(M,M) \to (\mathsf{Top}/B)(M,M)$$

and

$$(\mathsf{Top}/B)(M, M \times_B M - \triangle) \to (\mathsf{Top}/B)(M, M \times_B M)$$

coincide up to homotopy. □

We can convert this lifting question into a question about the existence of sections. For a fiberwise map $f \colon X \to Y$, let $r_B(f) \colon N_B(f) \to Y$ be a Hurewicz fiberwise fibration such that

$$\begin{array}{ccc}
X & \longrightarrow & N_B(f) \\
& {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle r_B(f)} & \\
& Y &
\end{array}$$

commutes and $X \to N_B(f)$ is an equivalence. Liftings up to fiberwise homotopy in a diagram

$$\begin{array}{ccc}
 & & X \\
 & \nearrow & \downarrow{\scriptstyle f} \\
Z & \xrightarrow[g]{} & Y
\end{array}$$

correspond to sections of the fiberwise fibration $g^* N_B(f) \to Z$.

Suppose $p \colon E \to M$ is a fiberwise Hurewicz fibration over a space B. The unreduced fiberwise suspension of E over M is the double mapping cylinder

$$S_M E := M \times \{0\} \cup_p E \times [0,1] \cup_p M \times \{1\}.$$

This has a map to M. Let

$$\sigma_-, \sigma_+ \colon M \to S_M E$$

be the sections of $S_M E \to M$ given by the inclusions of $M \times \{0\}$ and $M \times \{1\}$ into $S_M E$.

Proposition 8.2.2. — *If $E \to M$ admits a section then σ_- and σ_+ are homotopic over M.*

Conversely, assume $M \to B$ is a fibration, $p\colon E \to M$ is $(r+1)$-connected, and M is a cell complex over B of dimension less than or equal to $2r+1$. If σ_- and σ_+ are homotopic over M, then p has a section.

For the definition of a cell complex over B, see [**34**, 24.1].

Proof. — If $E \to M$ admits a section ς this section defines a map

$$S_M\varsigma\colon S_M M = M \times I \to S_M E$$

which is a homotopy over M between σ_- and σ_+.

Let $X_1 = M \cup_p (E \times [0,1/2])$ and $X_2 = M \cup_p (E \times [1/2,1])$. Then

$$S_M E = X_1 \cup_{E\times\{1/2\}} X_2.$$

Since $p\colon E \to M$ is $(r+1)$-connected the pairs (X_1, E) and (X_2, E) are also $(r+1)$-connected. By the Blakers-Massey Theorem, for any choice of base point,

$$\pi_i(X_1, E) \to \pi_i(S_M E, X_2)$$

is an isomorphism for $i < 2r+2$ and a surjection for $i = 2r+2$.

From the pairs (X_1, E) and $(S_M E, X_2)$ we get two long exact sequences of homotopy groups for any choice of base point in E

$$\cdots \to \pi_i(E) \to \pi_i(X_1) \to \pi_i(X_1, E) \to \pi_{i-1}(E) \to \cdots$$

$$\cdots \to \pi_i(X_2) \to \pi_i(S_M E) \to \pi_i(S_M E, X_2) \to \pi_{i-1}(X_2) \to \cdots$$

Diagram chasing and the isomorphism from the Blakers-Massey Theorem give an exact sequence

$$\pi_{2r+1}(E) \to \pi_{2r+1}(X_1) \oplus \pi_{2r+1}(X_2) \to \pi_{2r+1}(S_M E) \to \pi_{2r}(E) \to \cdots.$$

The exact sequence continues to the right but does not continue further to the left. Using the retractions of X_1 and X_2 to M we get an exact sequence

$$\pi_{2r+1}(E) \to \pi_{2r+1}(M) \oplus \pi_{2r+1}(M) \to \pi_{2r+1}(S_M E) \to \pi_{2r}(E) \to \cdots.$$

We would like to compare E to the homotopy pullback of the maps σ_- and σ_+. The homotopy pullback P is the pullback in the diagram

$$\begin{array}{ccc} P & \longrightarrow & NM \\ {\scriptstyle \sigma_-^* r(\sigma_+)}\downarrow & & \downarrow{\scriptstyle r(\sigma_+)} \\ M & \xrightarrow[\sigma_-]{} & S_M E \end{array}$$

where $r(\sigma_+)\colon NM \to S_M E$ is the map $\sigma_+\colon M \to S_M E$ converted into a fibration (not necessarily a fiberwise fibration). The map $\sigma_-^* r(\sigma_+)\colon P \to M$ is also a fibration with the same fiber.

For any choice of base point in P we get two long exact sequences associated to these fibrations

$$\cdots \to \pi_i(F) \to \pi_i(NM) \to \pi_i(S_M E) \to \pi_{i-1}(F) \to \cdots$$

$$\cdots \to \pi_i(F) \to \pi_i(M) \to \pi_i(P) \to \pi_{i-1}(F) \to \cdots$$

where F is the fiber of $r(\sigma_+)$. The same diagram chase as above gives a long exact sequence

$$\cdots \to \pi_i(P) \to \pi_i(M) \oplus \pi_i(M) \to \pi_i(S_M E) \to \pi_{i-1}(P) \to \cdots$$

The diagram

$$\begin{array}{ccc} E & \xrightarrow{p} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \sigma_-} \\ M & \xrightarrow[\sigma_+]{} & S_M E \end{array}$$

is commutative up to preferred fiberwise homotopy given by the homotopy from σ_- to σ_+ and so there is a map $q\colon E \to P$ such that

$$\begin{array}{ccc} E & \xrightarrow{q} & P \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \sigma_-^* r(\sigma_+)} & \\ & M & \end{array}$$

commutes. In particular, q is a fiberwise map over B. By comparing our exact sequences we see that for any choice of base point in E, $q_*\colon \pi_i(E) \to \pi_i(P)$ is a bijection for $i < 2r + 1$ and a surjection for $i = 2r + 1$.

If $[-, -]_B$ denotes (unsectioned) fiberwise homotopy classes of fiberwise maps, the fiberwise Whitehead theorem [**34**, 24.1.2] and the fact that the map $E \to P$ is $(2r + 1)$-connected imply that

$$q_*\colon [M, E]_B \to [M, P]_B$$

is a surjection. The fiberwise homotopy between the sections σ_- and σ_+ defines a fiberwise map $h\colon M \to P$ and so there is a fiberwise map $\varsigma\colon M \to E$ such that $q\varsigma\colon M \to P$ is fiberwise homotopic to h. Then

$$\mathrm{id}_M \simeq_B \sigma_-^* r(\sigma_+) \circ h \simeq_B \sigma_-^* r(\sigma_+) \circ q \circ \varsigma \simeq_B p \circ \varsigma$$

We can use the fiberwise homotopy lifting property of p to deform ς into an actual section. $\square$

We now assume that all spaces over M are sectioned and all maps over M preserve this section. In particular, $S_M E$ is an ex-space over M with section σ_- and the notation $[-, -]_B$ is now used for the sectioned fiberwise homotopy classes of maps.

There is a fiberwise fibration replacing the inclusion

$$i\colon M \times_B M - \triangle \to M \times_B M$$

that is also a Hurewicz fibration of spaces. Since i can be replaced with a map that is both a fibration and a fiberwise fibration

$$[S^0_M, S_M\Gamma_{f*}(N_M(M\times_B M-\triangle))]_M$$

is both the fiberwise homotopy classes of maps and the maps in the homotopy category for a model structure, see [**34**, 9.1]. This connects the geometric description above with duality in Ex.

The candidate for the fiberwise fibration replacing i is the map

$$(M\times_B M-\triangle)\times_{M\times_B M} P_B(M\times_B M)\longrightarrow M\times_B M$$

where $(M\times_B M-\triangle)\times_{M\times_B M} P_B(M\times_B M)$ is

$$\{((m_1,m_2),\gamma)\in(M\times_B M-\triangle)\times P_B(M\times_B M)|\gamma(1)=(m_1,m_2)\}$$

and the map to $M\times_B M$ is given by evaluation at 0. For this result we use the path space rather than the space of Moore paths.

We need a preliminary lemma.

Lemma 8.2.3. — *Suppose $p_1\colon E_1\to B$ and $p_2\colon E_2\to B$ are fibrations and $f\colon E_1\to E_2$ is a map over B. Then the map*

$$s\colon E_1\times_{E_2} P_BE_2=\{(e,\gamma)|\gamma(1)=f(e)\}\to E_2$$

given by $(e,\gamma)\mapsto\gamma(0)$ is a fibration.

Proof. — Recall that P_BE_2 is the subspace of paths in E_2 consisting of paths γ such that $p_2(\gamma(t))=p_2(\gamma(0))$ for all $t\in I$.

The space $E_1\times_{E_2} P_BE_2$ is the pullback of

$$\begin{array}{ccc} & & P_BE_2 \\ & & \downarrow{\scriptstyle t\times s} \\ E_1\times_B E_2 & \xrightarrow{f\times\mathrm{id}} & E_2\times_B E_2 \end{array}$$

and the map $E_1\times_{E_2} P_BE_2\to E_2$ is the composite of

$$\mathrm{id}\times s\colon E_1\times_{E_2} P_BE_2\to E_1\times_B E_2$$

and the second coordinate projection

$$\mathrm{proj}_2\colon E_1\times_B E_2\to E_2.$$

So it is enough to show that both of these maps are fibrations.

The projection map is a fibration since it is the pullback

$$\begin{array}{ccc} E_1\times_B E_2 & \longrightarrow & E_1 \\ {\scriptstyle\mathrm{proj}_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ E_2 & \xrightarrow[p_2]{} & B \end{array}$$

along a fibration. Lemma 7.2.1 shows that $t \times s \colon P_B E_2 \to E_2 \times_B E_2$ is a fibration. This implies $E_1 \times_{E_2} P_B E_2 \to E_1 \times_B E_2$ is a fibration. □

Lemma 8.2.4. — *Let $p \colon M \to B$ be a fiber bundle. Then there is a fiberwise fibration replacing the inclusion*

$$M \times_B M - \triangle \to M \times_B M$$

that is also a Hurewicz fibration.

Proof. — Since $p \colon M \to B$ is a bundle there is a cover $\{U_i\}$ of B and homeomorphisms $f_i \colon p^{-1}(U_i) \to F \times U_i$. The maps f_i are maps to a product so projection to F gives maps

$$f_{i,F} \colon p^{-1}(U_i) \to F.$$

The fiber product $q \colon M \times_B M \to B$ is locally trivial with respect to this open cover and the trivialization homeomorphisms are

$$f_{i,F} \times f_{i,F} \times p \colon q^{-1}(U_i) \to F \times F \times U_i.$$

These homeomorphisms restrict to give a local trivialization of

$$M \times_B M - \triangle \to B$$

since the maps $f_{i,F}$ are injective. So $M \times_B M - \triangle \to B$ is a fiber bundle with fiber $F \times F - \triangle$. Then Lemma 8.2.3 completes the proof. □

Under the assumptions in Proposition 8.2.2, the fiberwise Freudenthal suspension theorem gives the following isomorphism:

$$[S^0_M, S_M E]_M \cong \{S^0_M, S_M E\}_M.$$

If we further assume that $M \to B$ is a space over B such that S^0_M is Costenoble-Waner dualizable in Ex_B with dual $T_{M,B}$, then we have an isomorphism

$$\{S^0_M, S_M E\}_M \cong \{S^0_B, T_{M,B} \odot S_M E\}_B.$$

Definition 8.2.5. — **[25**, 4.3] Let E and M be as in Proposition 8.2.2 and assume S^0_M is Costenoble-Waner dualizable in Ex_B. The *fiberwise stable homotopy Euler characteristic* of $p \colon E \to M$ is the class

$$\chi_B(p) \in \{S^0_B, T_{M,B} \odot S_M E\}_B$$

which corresponds to the map

$$\sigma_+ \amalg \sigma_- \colon S^0_M \to S_M E$$

via the isomorphisms above.

If M is a space over B, $f \colon M \to M$ is a fiberwise map, and

$$i \colon M \times_B M - \triangle \to M \times_B M$$

is the inclusion we denote the fiberwise stable homotopy Euler characteristic of $\Gamma^*_f r_B(i)$ by $R^{KW}_B(f)$.

Corollary 8.2.6. — *Let $f\colon M \to M$ be a fiberwise map of a smooth fiber bundle with closed smooth manifold fibers F. If $\dim(B) \leq \dim(F) - 3$, the map f is fiberwise homotopic to a fixed point free map if and only if $R_B^{KW}(f)$ is trivial.*

Remark 8.2.7. — In **[12]** Fadell and Husseini defined a different fiberwise invariant using obstruction theory. For this invariant they require that the dimension of the fiber is at least three.

8.3. Identification of R^{KW} with R^{htpy}

Let τ_B be the fiberwise normal bundle of the inclusion of the diagonal into $M \times_B M$. Regard the sphere bundle $S(\tau_B)$ and the disk bundle $D(\tau_B)$ as spaces over $M \times_B M$ by inclusion. Let $N_B S(\tau_B)$ and $N_B D(\tau_B)$ be the total spaces of the fiberwise fibrations corresponding to the inclusions. Also using this notation, let $N_B(M \times_B M - \triangle)$ be the total space of the fiberwise fibration corresponding to $M \times_B M - \triangle \to M \times_B M$.

Lemma 8.3.1. — *As an ex-space over $M \times_B M$, $S_{M \times_B M} N_B(M \times_B M - \triangle)$ is weakly equivalent to $\triangle_! S^{\tau_B} \odot (\mathcal{P}_B M, t \times s)_+$.*

Proof. — There is a fiberwise homotopy cocartesian square

$$\begin{array}{ccc} S(\tau_B) & \longrightarrow & D(\tau_B) \\ \downarrow & & \downarrow \\ M \times_B M - \triangle & \longrightarrow & M \times_B M \end{array}$$

This is a diagram of inclusions over $M \times_B M$. Replacing all of the maps to $M \times_B M$ by fibrations, we get the following fiberwise homotopy cocartesian square.

$$\begin{array}{ccc} N_B S(\tau_B) & \longrightarrow & N_B D(\tau_B) \\ \downarrow & & \downarrow \\ N_B(M \times_B M - \triangle) & \longrightarrow & M \times_B M \end{array}$$

The fiberwise homotopy cofiber **[6**, II.2.1**]** of the bottom arrow is

$$S_{M \times_B M} N_B(M \times_B M - \triangle)$$

This is weakly equivalent to the fiberwise homotopy cofiber of the top arrow.

The top arrow is a fiberwise cofibration. To see this, observe that the inclusion $S(\tau_B) \to D(\tau_B)$ is a cofibration. Pulling back along

$$s\colon \mathcal{P}_B(M \times_B M) \to M \times_B M$$

preserves cofibrations and in this case converts a cofibration into a fiberwise cofibration. The fiberwise homotopy cofiber of $N_B S(\tau_B) \to N_B D(\tau_B)$ is weakly equivalent to its fiberwise cofiber. The fiberwise cofiber is

$$N_B D(\tau_B)/\sim = \{(\gamma, u) \in \mathcal{P}_B(M \times_B M) | \gamma(u) \in D(\tau_B)\}/\sim .$$

Here $(\gamma_1, u_1) \sim (\gamma_2, u_2)$ if $\gamma_1(u_1), \gamma_2(u_2) \in S(\tau_B)$ and $\gamma_1(0) = \gamma_2(0)$.

There is a map

$$N_B D(\tau_B) \to (\mathscr{P}_B M, t \times s) \times_M (D(\tau_B), \triangle \circ p) \times_M (\mathscr{P}_B M, t \times s)$$

given by

$$((\gamma_1, \gamma_2), u) \mapsto (\gamma_2^{-1}, (\gamma_1(u), \gamma_2(u)), H(\gamma_1(u), \gamma_2(u))\gamma_1).$$

The path $H(\gamma_1(u_1), \gamma_2(u_2))$ is as in Lemma 7.1.6. This map descends to an equivalence

$$N_B D(\tau_B)/\sim \to (\mathscr{P}_B M, t \times s)_+ \odot \triangle_! S^{\tau_B} \odot (\mathscr{P}_B M, t \times s)_+.$$

The inclusion of M into $\mathscr{P}_B M$ as constant paths defines a map

$$(M, \triangle)_+ \to (\mathscr{P}_B M, t \times s)_+.$$

This map is an equivalence and so there is an equivalence between $N_B D(\tau_B)/\sim$ and $\triangle_! S^{\tau_B} \odot (\mathscr{P}_B M, t \times s)_+$. □

When we defined the stable homotopy Euler characteristic we pulled the fibration back before taking the fiberwise suspension. These operations commute, so we have a weak equivalence between $S_M f^* N(M \times_B M - \triangle)$ and $\triangle_! S^{\tau_B} \odot \Lambda_B^f M_+$ where

$$\Lambda_B^f M = \{(\gamma, u) \in \mathscr{P}_B M | f(\gamma(u)) = \gamma(0)\}.$$

This is a space over M by $\gamma \mapsto \gamma(u)$.

Theorem 8.3.2. — *Let $M \to B$ be a smooth fiber bundle with closed smooth manifold fibers and $f\colon M \to M$ a fiberwise map. Then there is an isomorphism*

$$\{S_B^0, T_{M,B} \odot S_M E\}_B \cong \{S_B^0, S^n \wedge_B \Lambda_B^f M_+\}_B$$

and under this isomorphism

$$R_B^{KW}(f) = R_B^{\mathrm{htpy}}(f).$$

Proof. — To define the stable homotopy Euler characteristic we used the map

$$\sigma_+ \amalg \sigma_- \colon S_M^0 \to f^* S_{M \times_B M} N_B(M \times_B M - \triangle).$$

The corresponding map $\varsigma\colon S_M^0 \to \triangle_! S^{\tau_B} \odot \Lambda_B^f M_+$ takes the section of S_M^0 to the section of $\triangle_! S^{\tau_B} \odot \Lambda_B^f M_+$. On the other copy of M, ς is defined by

$$\varsigma(m) = ((m, f(m)), H(f(m), m))$$

where H is as in Lemma 7.1.6.

Let ϕ_B be the weak equivalence of Lemma 8.3.1. Then the following diagram of isomorphisms commutes.

$$\begin{array}{ccc}
[S^0_M, S_M f^* N_B(M\times_B M - \triangle)]_M & \xrightarrow{\phi_{B*}} & [S^0_M, \triangle_! S^{\tau_B} \odot \Lambda^f_B M_+]_M \\
\downarrow F & & \downarrow F \\
\{S^0_M, S_M f^* N_B(M\times_B M - \triangle)\}_M & \xrightarrow{\phi_{B*}} & \{S^0_M, \triangle_! S^{\tau_B} \odot \Lambda^f_B M_+\}_M \\
\downarrow D & & \downarrow D \\
\{S^0_B, T_{M,B}\odot S_M f^* N(M\times M - \triangle)\}_B & \xrightarrow[(\mathrm{id}\,\odot\phi_B)_*]{} & \{S^0_B, T_{M,B}\odot \triangle_! S^{\tau_B} \odot \Lambda^f_B M_+\}_B
\end{array}$$

The stabilization map F is an isomorphism because of dimension assumptions. The map D is the isomorphism from Corollary 7.1.7.

In the top left corner we have the stable cohomotopy Euler class and in the bottom right corner the corresponding map is

$$S^0_B \xrightarrow{\eta} T_{M,B}\odot S_M \xrightarrow{\mathrm{id}\,\odot\varsigma} T_{M,B}\odot \triangle_! S^{\tau_B} \odot \Lambda^f_B M_+ \cong S^n \wedge_B \Lambda^f_B M_+ \ .$$

This map is the trace of a lift of $f\colon M\to M$ to the space of fiberwise Moore paths. □

Proof of Theorem E. — By Theorem 8.3.2

$$R^{\mathrm{htpy}}_B(f) = R^{KW}_B(f).$$

By Corollary 8.2.6, $R^{KW}_B(f)$ is zero if and only if f is homotopic to a map with no fixed points. □

CHAPTER 9

A REVIEW OF BICATEGORY THEORY

In this chapter we will give several examples of bicategories with shadows. We used the bicategories in Sections 9.2 and 9.3 earlier. The bicategory in Section 9.1 is not necessary for what came earlier, but it may be helpful as an alternative source of motivation. Additional information about these types of constructions can be found in **[38]**.

The first example, in Section 9.1, can be interpreted as a generalization of the bicategory of rings, bimodules, and homomorphisms to a symmetric monoidal category that is not the category of abelian groups.

The example in Section 9.2 is also a generalization of the bicategory of rings, bimodules, and homomorphisms and of the example in Section 9.1. It is an enriched version of the bicategory of categories, bimodules, and natural transformations. In this context a bimodule is an enriched functor

$$\mathscr{A} \otimes \mathscr{B}^{\mathrm{op}} \to \mathscr{V}$$

where $\mathscr{V}$ is a symmetric monoidal category and $\mathscr{A}$ and $\mathscr{B}$ are categories enriched in $\mathscr{V}$. This is the 'many object' version of the bicategory of rings, bimodules, and homomorphisms and was used to define the unbased algebraic Reidemeister trace.

The last example comes back to the definitions of Section 9.1 and is a generalization of these definitions from symmetric monoidal categories to bicategories.

9.1. Bicategory of enriched monoids, bimodules, and maps

In this section and the following section we will describe two bicategories that can be defined from a symmetric monoidal category. These bicategories all have shadows and any symmetric monoidal functor induces a lax shadow functor of the associated bicategories.

In this section and Section 9.2 let $\mathscr{V}$ be a symmetric monoidal category with unit object I, tensor product $\otimes$, and symmetry isomorphism γ. The category $\mathscr{V}$ must also have all colimits.

A *monoid* in $\mathscr{V}$ is an object A in $\mathscr{V}$ with maps $\mu\colon A \otimes A \to A$ and $\iota\colon I \to A$ which are unital and associative. An A-B-*bimodule* is an object X in $\mathscr{V}$ with a pair of maps

$\kappa_A\colon A\otimes X\to X$ and $\kappa_B\colon X\otimes B\to X$ that are unital and associative with respect to the maps μ and ι for A and B. We also require that

$$\kappa_B(\kappa_A\otimes \mathrm{id}_B)=\kappa_A(\mathrm{id}_A\otimes\kappa_B).$$

Let X and Y be A-B-bimodules. A map $f\colon X\to Y$ in $\mathscr{V}$ is a *map of bimodules* if the following diagram

$$\begin{array}{ccc} A\otimes X & \xrightarrow{\kappa_A} & X \\ {\scriptstyle \mathrm{id}\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A\otimes Y & \xrightarrow{\kappa_A} & Y \end{array}$$

and the corresponding diagram for the map κ_B commute.

Definition 9.1.1. — Let X be an A-B-bimodule and Y a B-C-bimodule. Then $X\odot Y$ is the coequalizer

$$X\otimes B\otimes Y \underset{\mathrm{id}\otimes\kappa_B}{\overset{\kappa_B\otimes\mathrm{id}}{\rightrightarrows}} X\otimes Y\xrightarrow{\pi} X\odot Y.$$

If $A\otimes -$ preserves coequalizers, $X\odot Y$ is a left A module and the map

$$A\otimes X\odot Y\to X\odot Y$$

is induced by the map $\kappa_A\otimes\mathrm{id}$. The right C module structure is similar.

Define a bicategory $\mathscr{N}_{\mathscr{V}}$ with 0-cells the monoids in $\mathscr{V}$ and $\mathscr{N}_{\mathscr{V}}(A,B)$ the category of B-A-bimodules and bimodule maps. The $\odot$ product is described in Definition 9.1.1, and the unit functor associated to a monoid A is that monoid considered as an A-A-bimodule using the monoid multiplication μ. [(1)]

Duality in this bicategory was considered in [**37**, 2.1].

Definition 9.1.2. — For $Z\in\mathscr{N}_{\mathscr{V}}(A,A)$ the *shadow* of Z, $\langle\langle Z\rangle\rangle$, is the coequalizer

$$A\otimes Z\underset{\kappa_A\gamma}{\overset{\kappa_A}{\rightrightarrows}} Z\longrightarrow \langle\langle Z\rangle\rangle$$

The target of the shadow is the category $\mathscr{V}$.

Let X be an A-B-bimodule and Y a B-A-bimodule. Noting that

$$A\otimes(X\odot Y)\cong(A\otimes X)\odot Y,$$

the natural transformations $\theta_{B,A}$ are induced by:

$$\begin{array}{ccccc} A\otimes X\odot Y & \rightrightarrows & X\odot Y & \longrightarrow & \langle\langle X\odot Y\rangle\rangle \\ & & \downarrow & \swarrow{\scriptstyle \theta_{B,A}} & \\ & & \langle\langle Y\odot X\rangle\rangle & & \end{array}$$

(1) An alternative bicategory composition and shadows for these 0-cells, 1-cells, and 2-cells is described in [**35**]. This is related to the homotopy colimits we used earlier.

where the map $X \odot Y \to \langle\!\langle Y \odot X \rangle\!\rangle$ is induced by the composite

$$X \otimes Y \to Y \otimes X \to Y \odot X \to \langle\!\langle Y \odot X \rangle\!\rangle.$$

Let Ab be the category of abelian groups. The monoids in Ab are the associative rings with unit and $\odot$ is the usual tensor product over a ring. The bicategory $\mathscr{N}_{\mathsf{Ab}}$ is Mod, the bicategory of rings, bimodules, and homomorphisms. If R is a commutative ring and Mod_R is the symmetric monoidal category of R modules, the bicategory $\mathscr{N}_{\mathsf{Mod}_R}$ is the bicategory with 0-cells algebras over R, 1-cells bimodules and 2-cells homomorphisms.

Let Ch be the category of chain complexes of abelian groups and chain maps. A ring R, thought of as a chain complex concentrated in degree 0, is a monoid in Ch. Then $\mathscr{N}_{\mathsf{Ch}}(\mathbb{Z}, R)$ is the category of chain complexes of left R-modules and chain maps. The functor $\odot$ is the usual tensor product of chain complexes over R.

Definition 9.1.3. — Let $F\colon \mathscr{V} \to \mathscr{U}$ be a lax monoidal functor. Define a lax functor of bicategories $\mathscr{N}_{\mathscr{V}} \to \mathscr{N}_{\mathscr{U}}$ as follows.

(i) If A is a monoid with composition $\mu\colon A \otimes A \to A$ and unit $\iota\colon I \to A$ then FA is a monoid with composition

$$FA \otimes FA \xrightarrow{\phi} F(A \otimes A) \xrightarrow{F(\mu)} F$$

and unit

$$I' \longrightarrow F(I) \xrightarrow{F(\iota)} F(A)\ .$$

(ii) If X is an A-B-bimodule with action $\kappa_A\colon A \otimes X \to X$ by A and action $\kappa_B\colon X \otimes B \to X$ by B then FX is an FA-FB-bimodule with action $F(\kappa_A)\phi$ by $F(A)$ and action $F(\kappa_B)\phi$ by $F(B)$.

(iii) The natural transformations $\phi_{X,Y}$ are induced by ϕ.

(iv) The natural transformations ϕ_A are the identity.

Lemma 9.1.4. — *The lax functor induced by a lax symmetric monoidal functor is a shadow functor.*

Proof. — The natural map $\langle\!\langle FX \rangle\!\rangle \to F\langle\!\langle X \rangle\!\rangle$ is defined using the following diagram.

$$\begin{array}{ccccc}
FX \otimes FA & \overset{F(\kappa_A)\phi}{\underset{F(\kappa_A)\phi\gamma'}{\rightrightarrows}} & FX & \longrightarrow & \langle\!\langle FX \rangle\!\rangle \\
\Big\downarrow{\scriptstyle \phi} & & \Big\| & & \vdots \\
F(X \otimes A) & \overset{F(\kappa_A)}{\underset{F(\kappa_A)\gamma}{\rightrightarrows}} & FX & \longrightarrow & F\langle\!\langle X \rangle\!\rangle
\end{array}$$

□

9.2. Bicategory of enriched categories, bimodules, and maps

In some ways the example in this section is modeled on the 2-category of categories, functors, and natural transformations, but this is not the most helpful motivation. It is much more useful to think about the bicategory of rings, bimodules, and homomorphisms and view these bicategories as "many object" generalizations.

A category $\mathscr{A}$ is *enriched* in $\mathscr{V}$ if for each $a, b \in \mathrm{ob}(\mathscr{A})$, $\mathscr{A}(a,b) \in \mathrm{ob}(\mathscr{V})$ and for $a, b, c \in \mathrm{ob}(\mathscr{A})$ composition

$$\mathscr{A}(b,c) \otimes \mathscr{A}(a,b) \to \mathscr{A}(a,c)$$

is a map in $\mathscr{V}$. We also require that for each $a \in \mathrm{ob}(\mathscr{A})$, there is a map $I \to \mathscr{A}(a,a)$ in $\mathscr{V}$ and these maps and the composition maps satisfy unit and associativity conditions. For more details see [**24**, p. 23].

For two enriched categories $\mathscr{A}$ and $\mathscr{B}$, an *enriched $\mathscr{A}$-$\mathscr{B}$-bimodule* is an enriched functor $\mathscr{X}\colon \mathscr{A} \otimes \mathscr{B}^{\mathrm{op}} \to \mathscr{V}$. It consists of an object $\mathscr{X}(a,b)$ in $\mathscr{V}$ for each $a \in \mathrm{ob}(\mathscr{A})$, $b \in \mathrm{ob}(\mathscr{B})$ and maps

$$\kappa\colon \mathscr{A}(a,a') \otimes \mathscr{B}(b,b') \to \mathscr{V}(\mathscr{X}(a,b'), \mathscr{X}(a',b))$$

in $\mathscr{V}$. These maps must be compatible with composition and the unit objects in $\mathscr{A}$ and $\mathscr{B}$. Functors of this form are sometimes called *distributors* or *profunctors*.

As discussed in the beginning of this section, these objects should be thought of as many object generalizations of monoids and bimodules. This is reflected in the use of the term bimodule here and is compatible with the previous use in the sense that a category with one object enriched in $\mathscr{V}$ is a monoid in $\mathscr{V}$ and a bimodule (enriched functor) from a pair of categories each with one object is a bimodule (1-cell in $\mathscr{N}_{\mathscr{V}}$).

Let $\mathscr{X}, \mathscr{Y}$ be $\mathscr{A}$-$\mathscr{B}$-bimodules. An *enriched natural transformation* is a collection of morphisms in $\mathscr{V}$

$$\{\delta_{a,b}\colon \mathscr{X}(a,b) \to \mathscr{Y}(a,b) \in \mathrm{Mor}\,\mathscr{V}\}_{a\in \mathrm{ob}(\mathscr{A}), b \in \mathrm{ob}(\mathscr{B})}$$

such that the diagram below commutes for all $a, a' \in \mathrm{ob}(\mathscr{A})$ and $b, b' \in \mathrm{ob}(\mathscr{B})$.

$$\begin{array}{ccc}
\mathscr{A}(a,a') \otimes \mathscr{B}(b,b') & \xrightarrow{\kappa} & \mathscr{V}(\mathscr{X}(a,b'), \mathscr{X}(a',b)) \\
{\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle \mathscr{V}(\mathrm{id}, \delta_{a',b})} \\
\mathscr{V}(\mathscr{Y}(a,b'), \mathscr{Y}(a',b)) & \xrightarrow[\mathscr{V}(\delta_{a,b'}, \mathrm{id})]{} & \mathscr{V}(\mathscr{X}(a,b'), \mathscr{Y}(a',b))
\end{array}$$

Definition 9.2.1. — Let $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{B}$ bimodule and $\mathscr{Y}$ a $\mathscr{B}$-$\mathscr{C}$ bimodule. Then $\mathscr{X} \odot \mathscr{Y}$ is the coequalizer of the maps.

$$\coprod_{b,b' \in \mathrm{ob}(\mathscr{B})} \mathscr{X}(a,b') \otimes \mathscr{B}(b,b') \otimes \mathscr{Y}(b,c) \underset{\coprod \mathrm{id} \otimes \kappa_{\mathscr{B}}}{\overset{\coprod \kappa_{\mathscr{B}} \otimes \mathrm{id}}{\rightrightarrows}} \coprod_{b \in \mathrm{ob}(\mathscr{B})} \mathscr{X}(a,b) \otimes \mathscr{Y}(b,c).$$

If coequalizers are preserved by the tensor product, $\mathscr{X} \odot \mathscr{Y}$ is an $\mathscr{A}$-$\mathscr{C}$ bimodule. The $\mathscr{A}$-$\mathscr{C}$-bimodule structure is induced by the map

$$\mathscr{A}(a,a')\otimes\mathscr{C}(c,c') \xrightarrow{\kappa} \mathscr{V}(\mathscr{X}(a,b)\otimes\mathscr{Y}(b,c'),\mathscr{X}(a',b)\otimes\mathscr{Y}(b,c))$$
$$\longrightarrow \mathscr{V}(\mathscr{X}(a,b)\otimes\mathscr{Y}(b,c'),\mathscr{X}\odot\mathscr{Y}(a,c'))$$

This is the usual tensor product of functors except the first coproduct is indexed over pairs of objects rather than morphisms since $\mathscr{B}(b,b')$ is an object in $\mathscr{V}$ rather than a set.

We define a bicategory $\mathscr{E}_{\mathscr{V}}$ with 0-cells categories enriched in $\mathscr{V}$, 1-cells enriched bimodules, and 2-cells enriched natural transformations. The $\odot$ product for $\mathscr{E}_{\mathscr{V}}$ is given in Definition 9.2.1. For any category $\mathscr{A}$ enriched over $\mathscr{V}$ define an $\mathscr{A}$-$\mathscr{A}$-bimodule by $U_{\mathscr{A}}(a,a')=\mathscr{A}(a',a)\in\mathscr{V}$. The associativity of composition gives a single map

$$\mathscr{A}(a,a')\otimes\mathscr{A}(b',a)\otimes\mathscr{A}(b,b')\to\mathscr{A}(b,a').$$

Using symmetry and the $\mathscr{V}(-,-)-\otimes$ adjunction, we get a map

$$\mathscr{A}(a,a')\otimes\mathscr{A}(b,b')\to\mathscr{V}(\mathscr{A}(b',a),\mathscr{A}(b,a')).$$

The target of this map is $\mathscr{V}(U_{\mathscr{A}}(a,b'),U_{\mathscr{A}}(a',b))$ and this gives the required action. This is the unit functor for $\mathscr{E}_{\mathscr{V}}$.

This bicategory also has shadows.

Definition 9.2.2. — [4, 2.2.5] Let $\mathscr{Z}\colon\mathscr{A}\otimes\mathscr{A}^{\mathrm{op}}\to\mathscr{V}$ be an $\mathscr{A}$-$\mathscr{A}$-bimodule. The *shadow* of $\mathscr{Z}$, $\langle\!\langle\mathscr{Z}\rangle\!\rangle$, is the coequalizer of the pair of maps

$$\coprod_{a,a'\in\mathscr{A}}\mathscr{A}(a,a')\otimes\mathscr{Z}(a,a') \underset{\coprod(\kappa_{\mathscr{A}}\circ\gamma)}{\overset{\coprod\kappa_{\mathscr{A}}}{\rightrightarrows}} \coprod_{a\in\mathscr{A}}\mathscr{Z}(a,a).$$

The target of the shadow is the category $\mathscr{V}$. A composition of maps similar to those used in $\mathscr{N}_{\mathscr{V}}$ induces the maps $\theta_{\mathscr{A},\mathscr{B}}$ in $\mathscr{E}_{\mathscr{V}}$.

The bicategory Mod of rings, bimodules, and homomorphisms is the bicategory $\mathscr{N}_{\mathsf{Ab}}$. The corresponding bicategory $\mathscr{E}_{\mathsf{Ab}}$ is less familiar. We can think of the 0-cells in $\mathscr{E}_{\mathsf{Ab}}$ as rings with many objects and the 1-cells as modules with many objects. For an enriched category the abelian group structure on the hom sets gives the "addition" and the category composition gives the "multiplication". Similarly, we think of an enriched functor with target Ab as a module with many objects. The addition comes from the hom sets enrichment and the action of the category corresponds to the action of the ring.

Let $\mathscr{V},\mathscr{U}$ be symmetric monoidal categories and $F\colon\mathscr{V}\to\mathscr{U}$ be a lax symmetric monoidal functor. For a category $\mathscr{A}$ enriched in $\mathscr{V}$ define a category $F(\mathscr{A})$ enriched in $\mathscr{U}$ with the same objects as $\mathscr{A}$ and with morphisms defined by

$$F(\mathscr{A})(a,a')=F(\mathscr{A}(a,a')).$$

Composition in $F(\mathscr{A})$ is defined by

$$\begin{array}{ccc} F(\mathscr{A})(a',a'')\otimes F(\mathscr{A})(a,a') & & F(\mathscr{A}(a,a'')) \\ \| & & \uparrow \\ F(\mathscr{A}(a',a''))\otimes F(\mathscr{A}(a,a')) & \xrightarrow{\ \phi\ } & F(\mathscr{A}(a',a'')\otimes \mathscr{A}(a,a')) \end{array}$$

Let $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule. Then $F(\mathscr{X})$ is the $F(\mathscr{A})$-$F(\mathscr{B})$-bimodule defined by $F(\mathscr{X})(a,b) = F(\mathscr{X}(a,b))$. The action of $F(\mathscr{A})$ and $F(\mathscr{B})$ is given by

$$\begin{array}{ccc} F(\mathscr{A})(a,a')\otimes F(\mathscr{B})(b,b') & & \mathscr{V}(F(\mathscr{X}(a,b')),F(\mathscr{X}(a',b))) \\ \downarrow{\scriptstyle \phi} & & \uparrow \\ F(\mathscr{A}(a,a')\otimes \mathscr{B}(b,b')) & \xrightarrow{\ F(\kappa)\ } & F(\mathscr{V}(\mathscr{X}(a,b'),\mathscr{X}(a',b))) \end{array}$$

Definition 9.2.3. — Given a symmetric monoidal functor $F\colon \mathscr{V} \to \mathscr{U}$ define a lax functor of bicategories $\mathscr{E}_{\mathscr{V}} \to \mathscr{E}_{\mathscr{U}}$ as follows.

(*i*) The function $\mathrm{ob}\mathscr{E}_{\mathscr{V}} \to \mathrm{ob}\mathscr{E}_{\mathscr{U}}$ is given by $\mathscr{A} \mapsto F(\mathscr{A})$.
(*ii*) The functor $\mathscr{E}_{\mathscr{V}}(\mathscr{A},\mathscr{B}) \to \mathscr{E}_{\mathscr{U}}(F(\mathscr{A}),F(\mathscr{B}))$ is given by composition with F.
(*iii*) The natural transformation $\phi_{X,Y}$ is induced by ϕ via the composites

$$F(\mathscr{X}(a,b))\otimes F(\mathscr{Y}(b,c)) \xrightarrow{\phi} F(\mathscr{X}(a,b)\otimes \mathscr{Y}(b,c)) \to F(\mathscr{X}\odot\mathscr{Y})(a,c).$$

(*iv*) The natural transformation ϕ_A is the identity.

Lemma 9.2.4. — *A lax functor $\mathscr{E}_{\mathscr{V}} \to \mathscr{E}_{\mathscr{U}}$ induced by a symmetric monoidal functor $F\colon \mathscr{V} \to \mathscr{U}$ is a shadow functor.*

Proof. — The natural transformations $\psi_{\mathscr{A}}\colon \langle\!\langle F(\mathscr{X})\rangle\!\rangle \to F\langle\!\langle \mathscr{X}\rangle\!\rangle$ are induced by the composites $F\mathscr{X}(a,a) \to F(\coprod \mathscr{X}(a',a')) \to F\langle\!\langle \mathscr{X}\rangle\!\rangle$. □

Lax functors of bicategories give one way of comparing dual pairs, but as in Theorem 4.3.3, composites of dual pairs give another way. In this bicategory there is one example of a composite of dual pairs that is particularly relevant.

Let $\Phi\colon \mathscr{C} \to \mathscr{A}$ be an enriched functor. If $\mathscr{X}$ is a $\mathscr{B}$-$\mathscr{A}$-bimodule let $\mathscr{X}_\Phi$ be the $\mathscr{B}$-$\mathscr{C}$-bimodule defined by $\mathscr{X}_\Phi(b,c) = \mathscr{X}(b,\Phi(c))$. If $\mathscr{Y}$ is an $\mathscr{A}$-$\mathscr{B}$-bimodule let ${}_\Phi\mathscr{Y}$ be the $\mathscr{C}$-$\mathscr{B}$-bimodule defined by ${}_\Phi\mathscr{Y}(c,b) = \mathscr{Y}(\Phi(c),b)$. In particular, Φ can be used to define an $\mathscr{A}$-$\mathscr{C}$-bimodule $U_{\mathscr{A}\Phi}$ and a $\mathscr{C}$-$\mathscr{A}$-bimodule ${}_\Phi U_{\mathscr{A}}$. These bimodules form a dual pair with coevaluation

$$\eta\colon U_{\mathscr{C}} \to {}_\Phi U_{\mathscr{A}} \odot U_{\mathscr{A}\Phi}$$

induced by the functor Φ and evaluation

$$\epsilon\colon U_{\mathscr{A}\Phi} \odot {}_\Phi U_{\mathscr{A}} \to U_{\mathscr{A}}$$

which is composition in $\mathscr{A}$.

Lemma 9.2.5. — *Let $\mathscr{V}$ be a symmetric monoidal category and $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be categories enriched in $\mathscr{V}$. Suppose $\Phi\colon \mathscr{C} \to \mathscr{A}$ is an enriched functor and $\mathscr{X}$ is a $\mathscr{B}$-$\mathscr{C}$-bimodule with dual $\mathscr{Y}$ and coevaluation and evaluation maps*

$$U_{\mathscr{B}} \xrightarrow{\chi} \mathscr{X} \odot \mathscr{Y} \qquad \text{and} \qquad \mathscr{Y} \odot \mathscr{X} \xrightarrow{\theta} U_{\mathscr{C}}.$$

(i) *If Φ has right adjoint $\Psi\colon \mathscr{A} \to \mathscr{C}$ then $\mathscr{X}_\Phi$ is dualizable with dual $U_{\mathscr{A}\Phi} \odot \mathscr{Y}$.*
(ii) *If Ψ is also a left adjoint for Φ, then $\mathscr{X}_\Phi$ is dualizable with dual ${}_\Psi\mathscr{Y}$.*
(iii) *Suppose Ψ is both left and right adjoint to Φ. Given a 2-cell*

$$f\colon \mathscr{Y} \odot \mathscr{Q} \to \mathscr{P} \odot \mathscr{Y}$$

let f' be the 2-cell

$$\begin{array}{ccc} {}_\Psi\mathscr{Y} \odot \mathscr{Q} \xrightarrow{f(\Psi-,-)} {}_\Psi\mathscr{P} \odot \mathscr{Y} & & {}_\Psi\mathscr{P}_\Psi \odot {}_\Psi\mathscr{Y} \\ \downarrow \cong & & \uparrow \cong \\ {}_\Psi\mathscr{P} \odot U_{\mathscr{C}} \odot \mathscr{Y} & \xrightarrow{\mathrm{id}\odot\eta\odot\mathrm{id}} & {}_\Psi\mathscr{P} \odot {}_\Phi U_{\mathscr{A}} \odot U_{\mathscr{A}\Phi} \odot \mathscr{Y}. \end{array}$$

Then

$$\begin{array}{ccccccc} \langle\!\langle \mathscr{Q} \rangle\!\rangle & & & & & & \langle\!\langle {}_\Psi\mathscr{P}_\Psi \rangle\!\rangle \\ \downarrow tr(f) & & & & & & \uparrow \cong \\ \langle\!\langle \mathscr{P} \rangle\!\rangle & \xrightarrow{\cong} & \langle\!\langle \mathscr{P} \odot U_{\mathscr{C}} \rangle\!\rangle & \xrightarrow{\langle\!\langle \mathrm{id}\odot\eta \rangle\!\rangle} & \langle\!\langle \mathscr{P} \odot {}_\Phi U_{\mathscr{A}} \odot U_{\mathscr{A}\Phi} \rangle\!\rangle & \xrightarrow{\cong} & \langle\!\langle U_{\mathscr{A}\Phi} \odot \mathscr{P} \odot {}_\Phi U_{\mathscr{A}} \rangle\!\rangle \end{array}$$

is the trace of f'.

In Lemma 6.4.1 we proved a special case of this result. The following corollary of Lemma 9.2.5 generalizes Lemma 6.4.1.

Corollary 9.2.6. — *Let Π be a connected groupoid enriched in $\mathscr{V}$ and $\mathscr{X}\colon \Pi \to \mathscr{V}$ a right Π-module. If $\mathscr{X}(x)$ is dualizable as a right $\Pi(x,x)$-module in $\mathscr{N}_{\mathscr{V}}$ for some $x \in \Pi$ then $\mathscr{X}$ is dualizable in $\mathscr{E}_{\mathscr{V}}$.*

If $\mathscr{P}$ is a Π-Π-bimodule, $f\colon \mathscr{X} \to \mathscr{X} \odot \mathscr{P}$ is a map of right modules, and $f_x\colon \mathscr{X}(x) \to (\mathscr{X} \odot \mathscr{P})(x)$ the map f restricted to $x \in \mathrm{ob}\Pi$, then $\mathrm{tr}(f) = \mathrm{tr}(f_x)$.

9.3. Bicategory of bicategorical monoids

In this section we generalize both the bicategory from Section 9.1 and the bicategory of monoids, bimodules, and maps in Ex and Ex_B from Chapters 5 and 7 to any bicategory.

Remark 9.3.1. — There is one important difference between this section and Chapters 5, 6, 7, and 8. In Chapters 5, 6, 7, and 8 we used homotopy colimits. In this section we will use colimits. Despite this difference, many of the results in this section have analogues in Chapters 5, 6, 7, and 8.

This bicategory includes many of the best features of the two previous examples and additional elements that are necessary for topological applications. As we saw in Chapters 5 and 7 this bicategory eliminates the need to choose base points but retains the structure necessary to define topological duality.

In this section $\mathscr{W}$ is a bicategory with bicategory composition $\boxtimes$. The hom categories of $\mathscr{W}$ must have all coequalizers.

Definition 9.3.2. — Let A be a 0-cell in $\mathscr{W}$. A *monoid* in $\mathscr{W}$ is a 1-cell $\mathscr{A} \in \mathscr{W}(A,A)$ with 2-cells

$$U_A \xrightarrow{\iota} \mathscr{A} \qquad \text{and} \qquad \mathscr{A} \boxtimes \mathscr{A} \xrightarrow[\mu]{} \mathscr{A}$$

which are unital and associative. That is,

$$\mathscr{A} \cong U_A \boxtimes \mathscr{A} \xrightarrow{\iota \boxtimes \mathrm{id}} \mathscr{A} \boxtimes \mathscr{A} \xrightarrow{\mu} \mathscr{A}$$

and

$$\mathscr{A} \cong \mathscr{A} \boxtimes U_A \xrightarrow{\mathrm{id} \boxtimes \iota} \mathscr{A} \boxtimes \mathscr{A} \xrightarrow{\mu} \mathscr{A}$$

are the identity map of $\mathscr{A}$ and

$$\begin{array}{ccc} \mathscr{A} \boxtimes \mathscr{A} \boxtimes \mathscr{A} & \xrightarrow{\mu \boxtimes \mathrm{id}} & \mathscr{A} \boxtimes \mathscr{A} \\ \Big\downarrow{\scriptstyle \mathrm{id} \boxtimes \mu} & & \Big\downarrow{\scriptstyle \mu} \\ \mathscr{A} \boxtimes \mathscr{A} & \xrightarrow{\mu} & \mathscr{A} \end{array}$$

commutes. We call ι the *unit* and μ the *composition.*

The simplest example of a monoid is the 1-cell U_A for a 0-cell A in $\mathscr{W}$. The unit map is the identity and the composition is the unit isomorphism.

Definition 9.3.3. — Let $\mathscr{A}$ and $\mathscr{B}$ be monoids in $\mathscr{W}$. An *$\mathscr{A}$-$\mathscr{B}$-bimodule* in $\mathscr{W}$ is a 1-cell $\mathscr{X} \in \mathscr{W}(B,A)$ and two 2-cells

$$\kappa_A \colon \mathscr{A} \boxtimes \mathscr{X} \to \mathscr{X}$$

and

$$\kappa_B \colon \mathscr{X} \boxtimes \mathscr{B} \to \mathscr{X}$$

that are unital and associative with respect to the monoid structure of $\mathscr{A}$ and $\mathscr{B}$. We also require that the actions of κ_A and κ_B commute.

Any 1-cell X in $\mathscr{W}(B,A)$ is a U_A-U_B-bimodule. The monoids U_A and U_B act by the unit isomorphisms. Any monoid $\mathscr{A}$ is an $\mathscr{A}$-$\mathscr{A}$-bimodule with the left and right actions given by μ.

By neglect of structure an $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{X}$ is also a left $\mathscr{A}$-module or a right $\mathscr{B}$-module. Let $L\mathscr{X}$ be the monoid $\mathscr{X}$ regarded as an $\mathscr{A}$-U_B-bimodule with left action given by κ and right action the unit isomorphism. Let $R\mathscr{X}$ be the monoid $\mathscr{X}$ regarded as a U_A-$\mathscr{B}$-bimodule with left action the unit isomorphism and right action κ.

Definition 9.3.4. — Let $\mathscr{A}$ and $\mathscr{B}$ be monoids and $\mathscr{X}$ and $\mathscr{Y}$ be $\mathscr{A}$-$\mathscr{B}$-bimodules. A *map of bimodules* is a 2-cell $f\colon \mathscr{X} \to \mathscr{Y}$ such that

$$\begin{array}{ccc} \mathscr{A} \boxtimes \mathscr{X} & \xrightarrow{\kappa} & \mathscr{X} \\ \downarrow{\scriptstyle \mathrm{id} \boxtimes f} & & \downarrow{\scriptstyle f} \\ \mathscr{A} \boxtimes \mathscr{Y} & \xrightarrow[\kappa']{} & \mathscr{Y} \end{array}$$

and the corresponding diagram for $\mathscr{B}$ commute in $\mathscr{W}(B, A)$.

Definition 9.3.5. — Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be monoids, $\mathscr{X}$ be an $\mathscr{A}$-$\mathscr{B}$-bimodule, and $\mathscr{Y}$ be a $\mathscr{B}$-$\mathscr{C}$-bimodule. Then $\mathscr{X} \odot \mathscr{Y}$ is defined by the coequalizer

$$\mathscr{X} \boxtimes \mathscr{B} \boxtimes \mathscr{Y} \rightrightarrows \mathscr{X} \boxtimes \mathscr{Y} \longrightarrow \mathscr{X} \odot \mathscr{Y}$$

in $\mathscr{W}(C, A)$.

If the $\boxtimes$ product preserves coequalizers, $\mathscr{X} \odot \mathscr{Y}$ is an $\mathscr{A}$-$\mathscr{C}$-bimodule. The left $\mathscr{A}$ action is induced by the left action of $\mathscr{A}$ on $\mathscr{X}$. The right $\mathscr{C}$ action is induced by the right action of $\mathscr{C}$ on $\mathscr{Y}$.

If $\mathscr{B}$ is the monoid U_B then $\mathscr{X} \odot \mathscr{Y} = \mathscr{X} \boxtimes \mathscr{Y}$.

This defines a bicategory $\mathscr{M}_{\mathscr{W}}$ with 0-cells monoids in $\mathscr{W}$, 1-cells bimodules in $\mathscr{W}$, and 2-cells maps of bimodules. The unit for a monoid $\mathscr{A}$ is given by regarding that monoid as a bimodule over itself. The bicategory composition is $\odot$.

In the bicategory $\mathscr{M}_{\mathscr{W}}$ we have some very simple examples of dualizable objects given by the monoids of $\mathscr{W}$.

Proposition 9.3.6. — *For a monoid $\mathscr{A}$, $(R(U_{\mathscr{A}}), L(U_{\mathscr{A}}))$ is a dual pair.*

Proof. — Let $N(U_{\mathscr{A}})$ be $\mathscr{A}$ regarded as a U_A-U_A-bimodule. Then the unit map

$$\iota\colon U_A \to N(U_{\mathscr{A}})$$

is a map of U_A-U_A-bimodules. The associativity diagram for the composition μ implies that

$$\mu\colon L(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) = L(U_{\mathscr{A}}) \boxtimes R(U_{\mathscr{A}}) \to U_{\mathscr{A}}$$

is a map of $\mathscr{A}$-$\mathscr{A}$-bimodules. The unit conditions imply the composites

$$R(U_{\mathscr{A}}) \cong U_A \odot R(U_{\mathscr{A}}) \xrightarrow{\iota \odot \mathrm{id}} N(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) \xrightarrow{\mu} R(U_{\mathscr{A}})$$

$$L(U_{\mathscr{A}}) \cong L(U_{\mathscr{A}}) \odot U_A \xrightarrow{\mathrm{id} \odot \iota} L(U_{\mathscr{A}}) \odot N(U_{\mathscr{A}}) \xrightarrow{\mu} L(U_{\mathscr{A}})$$

are identity maps.

The coevaluation map

$$\eta\colon U_A \to N(U_{\mathscr{A}}) \cong R(U_{\mathscr{A}}) \odot L(U_{\mathscr{A}})$$

is the unit. The evaluation map

$$\epsilon\colon L(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) \cong L(U_{\mathscr{A}}) \boxtimes R(U_{\mathscr{A}}) \to U_{\mathscr{A}}$$

is the composition. The diagrams demonstrating that this is a dual pair commute by the unit and associativity conditions of the monoid.

$$\begin{array}{ccccc} R(U_{\mathscr{A}}) & \xleftarrow{\cong} & U_A \odot R(U_{\mathscr{A}}) & \xrightarrow{\iota \odot \mathrm{id}} & N(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \cong \odot\, \mathrm{id}} \\ R(U_{\mathscr{A}}) & \xleftarrow[\mu]{} & R(U_{\mathscr{A}}) \odot U_{\mathscr{A}} & \xleftarrow[\mathrm{id} \odot \mu]{} & R(U_{\mathscr{A}}) \odot L(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) \end{array}$$

$$\begin{array}{ccccc} L(U_{\mathscr{A}}) & \xleftarrow{\cong} & L(U_{\mathscr{A}}) \odot U_A & \xrightarrow{\mathrm{id} \odot \iota} & L(U_{\mathscr{A}}) \odot N(U_{\mathscr{A}}) \\ \downarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id} \odot \cong} \\ L(U_{\mathscr{A}}) & \xleftarrow[\mu]{} & U_{\mathscr{A}} \odot U_L(\mathscr{A}) & \xleftarrow[\mu \odot \mathrm{id}]{} & L(U_{\mathscr{A}}) \odot R(U_{\mathscr{A}}) \odot L(U_{\mathscr{A}}) \end{array}$$

□

Lemma 9.3.7. — *Suppose $\mathscr{W}$ is a bicategory with shadows $[[-]]$. If the target of the shadow $[[-]]$ has all coequalizers then $\mathscr{M}_{\mathscr{W}}$ is a bicategory with shadows.*

Proof. — Let $\mathbf{T}$ be the target of the shadow $[[-]]$. The shadows of $\mathscr{M}_{\mathscr{W}}$, $\langle\!\langle - \rangle\!\rangle$, are defined by the coequalizers

$$[[\mathscr{A} \boxtimes \mathscr{X}]] \rightrightarrows [[\mathscr{X}]] \longrightarrow \langle\!\langle \mathscr{X} \rangle\!\rangle$$

in $\mathbf{T}$. The composite

$$[[\mathscr{X} \boxtimes \mathscr{Y}]] \to [[\mathscr{Y} \boxtimes \mathscr{X}]] \to [[\mathscr{Y} \odot \mathscr{X}]] \to \langle\!\langle \mathscr{Y} \odot \mathscr{X} \rangle\!\rangle$$

induces a map

$$[[\mathscr{X} \odot \mathscr{Y}]] \to \langle\!\langle \mathscr{Y} \odot \mathscr{X} \rangle\!\rangle.$$

This map induces the isomorphisms $\theta_{\mathscr{B},\mathscr{A}}$. □

Note that the shadows constructed in Section 9.1 are of this form. A symmetric monoidal category $\mathscr{V}$ is a bicategory with a single 0-cell and the identity functor defines shadows for this bicategory. The shadows constructed in Chapter 5 for the bicategory $\mathscr{M}_{\mathsf{Ex}}$ are also of this form. The functors $[[-]]$ are $r_! \triangle^*$.

Lemma 9.3.8. — *Let $\mathscr{W}$ and $\mathscr{U}$ be bicategories. A lax functor $F \colon \mathscr{W} \to \mathscr{U}$ of bicategories defines a lax functor $\mathscr{M}_F \colon \mathscr{M}_{\mathscr{W}} \to \mathscr{M}_{\mathscr{U}}$.*

If the shadows of $\mathscr{M}_{\mathscr{W}}$ are defined as in Lemma 9.3.7 and F is a shadow functor then $\mathscr{M}_F$ is a shadow functor.

Proof. — The functor $\mathscr{M}_F$ takes a monoid $\mathscr{A}$ to the monoid $F\mathscr{A}$ with unit

$$U_{FA} \to F(U_A) \to F\mathscr{A}$$

and composition

$$F\mathscr{A} \boxtimes F\mathscr{A} \to F(\mathscr{A} \boxtimes \mathscr{A}) \to F\mathscr{A}.$$

An $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{X}$ is taken to the $F\mathscr{A}$-$F\mathscr{B}$-bimodule $F\mathscr{X}$ with actions

$$F\mathscr{A} \boxtimes F\mathscr{X} \to F(\mathscr{A} \boxtimes \mathscr{X}) \to F\mathscr{X}$$

$$F\mathscr{X} \boxtimes F\mathscr{B} \to F(\mathscr{X} \boxtimes \mathscr{B}) \to F\mathscr{X}$$

The map $F\mathscr{X} \boxtimes F\mathscr{Y} \to F(\mathscr{X} \boxtimes \mathscr{Y})$ induces a map

$$F\mathscr{X} \odot F\mathscr{Y} \to F(\mathscr{X} \odot \mathscr{Y}).$$

If shadows in $\mathscr{M}_{\mathscr{W}}$ are defined from the shadows of $\mathscr{W}$ and F is a shadow functor, $\mathscr{M}_F$ is a shadow functor. The maps $\langle\!\langle F\mathscr{X}\rangle\!\rangle \to F\langle\!\langle \mathscr{X}\rangle\!\rangle$ are induced by the corresponding maps

$$[[F\mathscr{X}]] \to F[[\mathscr{X}]]. \qquad \square$$

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