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ORDINARY PARTS OF ADMISSIBLE REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS I. DEFINITION AND FIRST PROPERTIES

by

Matthew Emerton

Abstract. — If G is a connected reductive p-adic group, P is a parabolic subgroup of G, and M is a Levi factor of P, and if A is an Artinian local ring having a finite residue field of characteristic p, then we define a functor Ord_P from the category of admissible smooth P-representations over A to the category of admissible smooth M-representations of A, which we call the functor of ordinary parts. We show that this functor is right adjoint to the functor of parabolic induction $\mathrm{Ind}_{\overline{P}}^{G}$, where $\overline{P}$ is an opposite parabolic to P.

***Résumé* (Parties ordinaires de représentations admissibles de groupes réductifs p-adiques I. Définitions et premières propriétés)**

Soit G un groupe p-adique connexe réductif, P un sous-groupe parabolique de G, et M un facteur de Levi de P. Si A est un anneau local artinien ayant un corps résiduel fini de caractéristique p, alors nous définissons un foncteur Ord_P de la catégorie des P-représentations sur A lisses et admissibles vers la catégorie des M-représentations de A lisses et admissibles, que nous appelons foncteur des parties ordinaires. Nous montrons que ce foncteur est adjoint à droite du foncteur d'induction parabolique $\mathrm{Ind}_{\overline{P}}^{G}$, où $\overline{P}$ est un opposé parabolique de P.

1. Introduction

This paper is the first of two in which we define and study the functors of ordinary parts on categories of admissible smooth representations of p-adic reductive groups over fields of characteristic p, as well as their derived functors. These functors of ordinary parts, which are characterized as being right adjoint to parabolic induction, play an important role in the study of smooth representation theory in characteristic p. They also have important global applications: when applied in the context of the

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p-adically completed cohomology spaces introduced in [**6**], they provide a representation theoretic approach to Hida's theory of (nearly) ordinary parts of cohomology [**11, 12, 13**]. The immediate applications that we have in mind are for the group GL_2: the results of our two papers have applications to the construction of the mod p and p-adic local Langlands correspondences for the group $\mathrm{GL}_2(\mathbb{Q}_p)$, and to the investigation of local-global compatibility for p-adic Langlands over the group GL_2 over $\mathbb{Q}$.

To describe our results more specifically, let G be (the $\mathbb{Q}_p$-valued points of) a connected reductive p-adic group, P a parabolic subgroup of G, $\overline{P}$ an opposite parabolic to P, and $M = P \bigcap \overline{P}$ the corresponding Levi factor of P and $\overline{P}$. If k is a finite field of characteristic p, we let $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ (resp. $\mathrm{Mod}_M^{\mathrm{adm}}(k)$) denote the category of admissible smooth G-representations (resp. M-representations) over k. Parabolic induction yields a functor $\mathrm{Ind}_{\overline{P}}^G : \mathrm{Mod}_M^{\mathrm{adm}}(k) \to \mathrm{Mod}_G^{\mathrm{adm}}(k)$. The functor of ordinary parts associated to P is then a functor $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(k) \to \mathrm{Mod}_M^{\mathrm{adm}}(k)$, which is right adjoint to $\mathrm{Ind}_{\overline{P}}^G$.

In this paper, we define Ord_P and study its basic properties. In the sequel [**10**] we investigate the derived functors of Ord_P, and present some applications to the computation of Ext spaces in the category $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ in the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, computations which in turn play a role in the construction of the mod p and p-adic local Langlands correspondences (see [**4**] and [**8**]). The applications to local-global compatibility are part of the arguments of [**8**]. We hope to discuss the applications in the context of p-adically completed cohomology in a future paper. Let us only mention here that Theorem 3.4.8 below is an abstract formulation of Hida's general principle that the ordinary part of cohomology should be finite over weight space.

1.1. Arrangement of the paper. — In order to allow for maximum flexibility in applications, we actually work throughout the paper with representations defined not just over the field k, but over general Artinian local rings, or even complete local rings, having residue field k. This necessitates a development of the foundations of the theory of admissible representations over such coefficient rings. Such a development is the subject of Section 2. We also take the opportunity in that section to present some related representation theoretic notions that do not seem to be in the literature, and which will be useful in this paper, its sequel, and future applications. The functors Ord_P are defined, and some of their basic properties established, in Section 3. Their characterization as adjoint functors is proved in Section 4. In the appendix we establish some simple functional analytic results about modules over p-adic integer rings.

1.2. Notation and terminology. — Throughout the paper, we fix a prime p, as well as a finite extension E of $\mathbb{Q}_p$, with ring of integers $\mathscr{O}$. We let $\mathbb{F}$ denote the residue field of $\mathscr{O}$, and ϖ a choice of uniformizer of $\mathscr{O}$. Let $\mathrm{Comp}(\mathscr{O})$ denote the category of complete Noetherian local $\mathscr{O}$-algebras having finite residue fields, and let $\mathrm{Art}(\mathscr{O})$

denote the full subcategory of $\mathrm{Comp}(\mathscr{O})$ consisting of those objects that are Artinian (or equivalently, of finite length as $\mathscr{O}$-modules).

If A is an object of $\mathrm{Comp}(\mathscr{O})$, and V is an A-module which is torsion as an $\mathscr{O}$-module, then we write $V^* := \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})$ to denote the Pontrjagin dual of V (where V is endowed with its discrete topology), equipped with its natural profinite topology. If V is a G-representation over A, for some group G, then the contragredient action makes V^* a G-representation over A (with each element of G acting via a continuous automorphism).

If V is any $\mathscr{O}$-module, then we let V_{fl} denote the maximal $\mathscr{O}$-torsion free quotient of V. We write $V[\varpi^i]$ to denote the kernel of multiplication by ϖ^i on V, and $V[\varpi^\infty] := \bigcup_{i\geq 0} V[\varpi^i]$. If $\mathscr{C}$ is any $\mathscr{O}$-linear category, then for any integer $i \geq 0$ (resp. $i = \infty$), we let $\mathscr{C}[\varpi^i]$ denote the full subcategory of $\mathscr{C}$ consisting of objects that are annihilated by ϖ^i (resp. by some power of ϖ).

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2. Representations of p-adic analytic groups

Let G be a p-adic analytic group. Throughout this section, we will let A denote an object of $\mathrm{Comp}(\mathscr{O})$, with maximal ideal $\mathfrak{m}$. We denote by $\mathrm{Mod}_G(A)$ the abelian category of representations of G over A (with morphisms being A-linear G-equivariant maps); equivalently, $\mathrm{Mod}_G(A)$ is the abelian category of $A[G]$-modules, where $A[G]$ denotes the group ring of G over A. In this section we introduce, and study some basic properties of, various categories of G-representations, as well as of certain related categories of what we will call augmented G-representations.

2.1. Augmented G-representations. — If H is a compact open subgroup of G, then as usual we let $A[[H]]$ denote the completed group ring of H over A, i.e.

$$A[[H]] := \varprojlim_{H'} A[H/H'], \tag{2.1.1}$$

where H' runs over all normal open subgroups of H. We equip $A[[H]]$ with the projective limit topology obtained by endowing each of the rings $A[H/H']$ appearing on the right hand side of (2.1.1) with its $\mathfrak{m}$-adic topology. The rings $A[H/H']$ are then profinite (since each H/H' is a finite group), and hence this makes $A[[H]]$ a profinite, and so in particular compact, topological ring.

2.1.2. *Theorem.* — *The completed group ring $A[[H]]$ is Noetherian.*

Proof. — In the case when $A = \mathscr{O} = \mathbb{Z}_p$, this is proved by Lazard [**14**]: after replacing H by an open subgroup, if necessary, Lazard equips $\mathbb{Z}_p[[H]]$ with an exhaustive decreasing filtration $F^\bullet$ whose associated graded ring $\mathrm{Gr}^\bullet_F\mathbb{Z}_p[[H]]$ is isomorphic to a polynomial algebra (with generators in degree 1) over the graded ring $\mathbb{F}_p[t]$ (which is the graded ring associated to $\mathbb{Z}_p$ with its p-adic filtration).

For general A, we may write $A[[H]] \xrightarrow{\sim} A\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[[H]]$, and so equip $A[[H]]$ with an exhaustive decreasing filtration obtained as the completed tensor product of the $\mathfrak{m}$-adic filtration on A and Lazard's filtration $F^\bullet$ on $\mathbb{Z}_p[[H]]$. The graded ring associated to this filtration on the completed tensor product is naturally isomorphic to a quotient of the tensor product of the individual associated graded rings, i.e. is isomorphic to a quotient of $(\mathrm{Gr}^\bullet_{\mathfrak{m}} A) \otimes_{\mathbb{F}_p[t]} \mathrm{Gr}^\bullet_F \mathbb{Z}_p[[H]]$, where $\mathrm{Gr}^\bullet_{\mathfrak{m}} A$ denotes the graded ring associated to the $\mathfrak{m}$-adic filtration on A (which is also naturally an algebra over the graded ring $\mathbb{F}_p[t]$ associated to the p-adic filtration of $\mathbb{Z}_p$, since $p \in \mathfrak{m}$).

Thus the associated graded ring to $A[[H]]$ is isomorphic to a quotient of a polynomial algebra over the Noetherian ring $\mathrm{Gr}^\bullet_{\mathfrak{m}} A$, and thus is again Noetherian. Consequently $A[[H]]$ itself is Noetherian, as claimed. □

2.1.3. *Proposition.* — 1. *Any finitely generated $A[[H]]$-module admits a unique profinite topology with respect to which the $A[[H]]$-action on it becomes jointly continuous.*
2. *Any $A[[H]]$-linear morphism of finitely generated $A[[H]]$-modules is continuous with respect to the profinite topologies on its source and target given by part* (1).

Proof. — Since $A[[H]]$ is Noetherian, we may find a presentation

$$A[[H]]^s \to A[[H]]^r \to M \to 0$$

of M, for some $r, s \geq 0$. Since $A[[H]]$ is profinite, it follows that the first arrow has closed image, and hence that M is the quotient of the profinite module $A[[H]]^r$ by a closed $A[[H]]$-submodule. If we equip M with the induced quotient topology, then it becomes a profinite module. The resulting profinite topology on M is clearly independent of the chosen presentation, and satisfies the requirements of the proposition. □

2.1.4. *Definition.* — If M is a finitely generated $A[[H]]$-module, we refer to the topology given by the preceding lemma as the canonical topology on M.

2.1.5. *Definition.* — By an augmented representation of G over A we mean an $A[G]$-module M equipped with an $A[[H]]$-module structure for some (equivalently, any) compact open subgroup H of G, such that the two induced $A[H]$-actions (the first induced by the inclusion $A[H] \subset A[[H]]$ and the second by the inclusion $A[H] \subset A[G]$) coincide.

2.1.6. *Definition.* — By a profinite augmented G-representation over A we mean an augmented G-representation M over A that is also equipped with a profinite topology, so that the $A[[H]]$-action on M is jointly continuous[(1)] for some (equivalently any) compact open subgroup H over A.

(1) It is equivalent to ask that the profinite topology on M admits a neighborhood basis at the origin consisting of $A[[H]]$-submodules.

The equivalence of the conditions "some" and "any" in the preceding definitions follows from the fact that if H_1 and H_2 are two compact open subgroups of G, then $H := H_1 \bigcap H_2$ has finite index in each of H_1 and H_2.

We denote by $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ the abelian category of augmented G-representations over A, with morphisms being maps that are simultaneously G-equivariant and $A[[H]]$-linear for some (equivalently, any) compact open subgroup H of G, and by $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ the abelian category of profinite augmented G-representations, with morphisms being continuous A-linear G-equivariant maps (note that since $A[H]$ is dense in $A[[H]]$, any such map is automatically $A[[H]]$-linear for any compact open subgroup H of G). Forgetting the topology induces a forgetful functor

$$\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \longrightarrow \mathrm{Mod}_G^{\mathrm{aug}}(A).$$

We let $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ consisting of augmented G-modules that are finitely generated over $A[[H]]$ for some (equivalently any) compact open subgroup H of G. Proposition 2.1.3 shows that by equipping each object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ with its canonical topology, we may lift the inclusion $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \to \mathrm{Mod}_G^{\mathrm{aug}}(A)$ to an inclusion $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \to \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$.

2.1.7. *Proposition.* — *The category $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)$ forms a Serre subcategory of each of the abelian categories $\mathrm{Mod}_G^{\mathrm{aug}}(A)$ and $\mathrm{Mod}_G^{\mathrm{pro\,aug}}(A)$ (i.e. it is closed under passing to subobjects, quotients, and extensions). In particular, it itself is an abelian category.*

Proof. — Closure under the formation of quotients and extensions is evident, and closure under the formation of subobjects follows from Theorem 2.1.2. □

2.2. Smooth G-representations.

— In this subsection we give the basic definitions, and state the basic results, related to smooth, and admissible smooth, representations of G over A. In the case when A is Artinian, our definitions will agree with the standard ones. Otherwise, they may be slightly unorthodox, but will be the most useful ones for our later purposes.

2.2.1. *Definition.* — Let V be a representation of G over A. We say that a vector $v \in V$ is smooth if:

1. v is fixed by some open subgroup of G.
2. v is annihilated by some power $\mathfrak{m}^i$ of the maximal ideal of A.

We let V_{sm} denote the subset of V consisting of smooth vectors.

2.2.2. *Remark.* — The following equivalent way of defining smoothness can be useful: a vector $v \in V$ is smooth if and only if v is annihilated by the intersection $J \bigcap A[H]$ for some open ideal $J \subset A[[H]]$, where H is some (equivalently, any) compact open subgroup of G.

2.2.3. *Remark.* — If A is Artinian, then $\mathfrak{m}^i = 0$ for sufficiently large i, and so automatically $\mathfrak{m}^i v = 0$ for any element v of any A-module V. Thus condition (2) can be omitted from the definition of smoothness for such A, and our definition coincides with the usual one.

2.2.4. *Lemma*. — *The subset V_{sm} is an $A[G]$-submodule of V.*

Proof. — If $v_i \in V_{\mathrm{sm}}$ $(i = 1, 2)$ are fixed by open subgroups $H_i \subset G$, and annihilated by $\mathfrak{m}^{i_1}$ and $\mathfrak{m}^{i_2}$, then any A-linear combination of v_1 and v_2 is fixed by $H_1 \bigcap H_2$, and annihilated by $\mathfrak{m}^{\max i_1, i_2}$, and hence also lies in V_{sm}. Thus V_{sm} is an A-submodule of V. Also, if $v \in V_{\mathrm{sm}}$ is fixed by the open subgroup $H \subset G$, and annihilated by $\mathfrak{m}^i$, then gv is fixed by $gHg^{-1} \subset G$ for any $g \in G$, and again annihilated by $\mathfrak{m}^i$. Thus V_{sm} is closed under the action of G. □

2.2.5. *Definition*. — We say that a G-representation V of G over A is smooth if $V = V_{\mathrm{sm}}$; that is, if every vector of V is smooth.

We let $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G(A)$ consisting of smooth G-representations. The association $V \mapsto V_{\mathrm{sm}}$ is a left-exact functor from $\mathrm{Mod}_G(A)$ to $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, which is right adjoint to the inclusion of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ into $\mathrm{Mod}_G(A)$.

2.2.6. *Lemma*. — *The full subcategory $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ of $\mathrm{Mod}_G(A)$ is closed under the formation of subobjects, quotients, and inductive limits (and so in particular is an abelian category).*

Proof. — This is clear. □

The following lemma provides some useful ways of thinking about smooth representations of G.

2.2.7. *Lemma*. — *Let V be a G-representation over A, and suppose that V is torsion as an $\mathscr{O}$-module. (Note that this holds automatically if A is torsion as an $\mathscr{O}$-module, e.g. if A is an object of $\mathrm{Art}(\mathscr{O})$.) The following conditions on V are equivalent.*

1. *V is smooth.*
2. *The A-action on V and the G-action on V are both jointly continuous, when V is given its discrete topology.*
3. *The A-action and the G-action on V^* are both jointly continuous when V^* is given its natural profinite topology.*
4. *For some (equivalently, every) compact open subgroup H of G, the $A[H]$-action on V extends (in a necessarily unique manner) to an $A[[H]]$-action which is continuous when V is equipped with the discrete topology.*
5. *For some (equivalently, every) compact open subgroup H of G, the H-action on V^* extends (in a necessarily unique manner) to a continuous action of the completed group ring $A[[H]]$ on V^*.*

(In both conditions (2) and (3), the ring A is understood to be equipped with its $\mathfrak{m}$-adic topology.)

Proof. — The straightforward (and well-known) proofs are left to the reader. □

Lemma 2.2.7 implies that passing to Pontrjagin duals induces an anti-equivalence

$$\mathrm{Mod}_G^{\mathrm{sm}}(A) \xrightarrow{\text{anti}\sim} \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A). \tag{2.2.8}$$

2.2.9. *Definition.* — We say that a smooth G-representation V over A is admissible if $V^H[\mathfrak{m}^i]$ (the $\mathfrak{m}^i$-torsion part of the subspace of H-fixed vectors in V) is finitely generated over A for every open subgroup H of G and every $i \geq 0$.

2.2.10. *Remark.* — If A is Artinian, so that $\mathfrak{m}^i = 0$ for i sufficiently large, then $V^H[\mathfrak{m}^i] = V^H$ for i sufficiently large, and so the preceding definition agrees with the usual definition of an admissible smooth A-module.

We let $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of admissible representations.

2.2.11. *Lemma.* — *A smooth G-representation V over A is admissible if and only if V^* is finitely generated as an $A[[H]]$-module for some (equivalently, every) compact open subgroup H of G.*

Proof. — This is well-known. □

By Lemma 2.2.11, the anti-equivalence (2.2.8) restricts to an anti-equivalence

$$\mathrm{Mod}_G^{\mathrm{adm}}(A) \xrightarrow{\quad\text{anti}\sim\quad} \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A). \tag{2.2.12}$$

2.2.13. *Proposition.* — *The category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ forms a Serre subcategory of the abelian category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, it itself is an abelian category.*

Proof. — This follows directly from the anti-equivalences (2.2.8) and (2.2.12), together with Proposition 2.1.7 and Lemma 2.2.6. □

2.2.14. *Remark.* — It follows from Lemma 2.2.11, together with the topological version of Nakayama's lemma, that if H is an open pro-p subgroup of G (so that $A[[H]]$ is a – typically non-commutative – local ring), then V is admissible if and only if $V^H[\mathfrak{m}]$ is finite dimensional over $A/\mathfrak{m}$. In particular, if the condition of Definition 2.2.9 holds when $i = 1$, then it holds automatically for all values of i, and so V is admissible.

2.2.15. *Definition.* — Let V be a G-representation over A.

1. We say that an element $v \in V$ is locally admissible if v is smooth, and if the smooth G-subrepresentation of V generated by v is admissible.
2. We let $V_{\mathrm{l\,adm}}$ denote the subset of V consisting of locally admissible elements.

2.2.16. *Lemma.* — *The subset $V_{\mathrm{l\,adm}}$ is an $A[G]$-submodule of V.*

Proof. — Let $v_i \in V_{\mathrm{l\,adm}}$ $(i = 1, 2)$, and let W_i denote the G-subrepresentation of V generated by v_i. Since $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ is a Serre subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, we see that the image of the map $W_1 \bigoplus W_2 \to V$ induced by the inclusions $W_i \subset V$ is admissible. This image certainly contains any A-linear combination of the v_i, and thus $V_{\mathrm{l\,adm}}$ is an A-submodule of V. It is also clearly G-invariant, and thus is an $A[G]$-submodule of V. □

2.2.17. *Definition*. — We say that a G-representation V of G over A is locally admissible if $V = V_{\mathrm{l\,adm}}$; that is, if every vector of V is locally admissible. [2]

We let $\mathrm{Mod}_G^{\mathrm{l\,adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ consisting of locally admissible G-representations.

2.2.18. *Proposition*. — *The category* $\mathrm{Mod}_G^{\mathrm{l\,adm}}(A)$ *is closed under passing to subrepresentations, quotients, and inductive limits in* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. *In particular, it itself is an abelian category.*

Proof. — Since $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ is closed under passing to subrepresentations and quotients in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, the same is evidently true for $\mathrm{Mod}_G^{\mathrm{l\,adm}}(A)$. The closure under formation of inductive limits is also clear, since the property of being locally admissible is checked element-wise. □

2.2.19. *Remark*. — Clearly any finitely generated locally admissible representation is in fact admissible smooth. Since any representation is the inductive limit of finitely generated ones, we see that a representation is locally admissible if and only if it can be written as an inductive limit of admissible smooth representations.

The association $V \mapsto V_{\mathrm{l\,adm}}$ gives rise to a left-exact functor from $\mathrm{Mod}_G(A)$ to $\mathrm{Mod}_G^{\mathrm{l\,adm}}(A)$, which is right adjoint to the inclusion of $\mathrm{Mod}_G^{\mathrm{l\,adm}}(A)$ into $\mathrm{Mod}_G(A)$.

The following diagram summarizes the relations between the various categories that we have introduced:

$$
\begin{tikzcd}
\mathrm{Mod}_G(A) \arrow[d, bend right, "(\ )_{\mathrm{sm}}"'] \arrow[dd, bend right=70, "(\ )_{\mathrm{l\,adm}}"'] & & \\
\mathrm{Mod}_G^{\mathrm{sm}}(A) \arrow[u, hook] \arrow[r, leftrightarrow, "\mathrm{anti}\sim"] & \mathrm{Mod}_G^{\mathrm{pro\,aug}}(A) \arrow[r] & \mathrm{Mod}_G^{\mathrm{aug}}(A) \\
\mathrm{Mod}_G^{\mathrm{l\,adm}}(A) \arrow[u, hook] & & \\
\mathrm{Mod}_G^{\mathrm{adm}}(A) \arrow[u, hook] \arrow[r, leftrightarrow, "\mathrm{anti}\sim"] & \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A) \arrow[uu, hook, "\text{canonical topology}"] \arrow[uur, hook, bend right] &
\end{tikzcd}
$$

[2] As was pointed out by a referee, in the context of smooth representations of p-adic groups over the field of complex numbers, such representations have been called *quasi-admissible* by Harish-Chandra.

2.3. Some finiteness conditions.

— Let Z denote the centre of G.

2.3.1. *Definition.* — Let V be an object of $\mathrm{Mod}_G(A)$.

1. We say that V is Z-finite if, writing I to denote the annihilator of V in $A[Z]$, the quotient $A[Z]/I$ is finitely generated as an A-module.
2. We say that $v \in V$ is locally Z-finite if the $A[Z]$-submodule of V generated by v is finitely generated as an A-module. We let $V_{Z-\mathrm{fin}}$ denote the subset of locally Z-finite elements of V.
3. We say that V is locally Z-finite if every $v \in V$ is locally finite over Z, i.e. if $V = V_{Z-\mathrm{fin}}$.

2.3.2. *Lemma.* — 1. *For any object V of $\mathrm{Mod}_G(A)$, the subset $V_{Z-\mathrm{fin}}$ is an $A[G]$-submodule of V.*

2. *If V_1 and V_2 are two (locally) Z-finite representations, then $V_1 \bigoplus V_2$ is again (locally) Z-finite.*
3. *Any $A[G]$-invariant submodule or quotient of a (locally) Z-finite representation is again (locally) Z-finite.*

Proof. — If V_1 and V_2 are $A[G]$-modules annihilated by the A-finite quotients $A[Z]/I_1$ and $A[Z]/I_2$ of $A[Z]$, respectively, then $V_1 \bigoplus V_2$ is annihilated by $A[Z]/(I_1 \bigcap I_2)$, which is again A-finite (since it embeds into $(A[Z]/I_1) \times (A[Z]/I_2)$). This proves (2) in the case of Z-finite representations.

Now suppose that v_1 and v_2 are two elements of $V_{Z-\mathrm{fin}}$, for some $A[G]$-module V. Let I_i denote the annihilator of v_i in $A[Z]$, and let W_i be the G-subrepresentation of V generated by v_i. Since Z is the centre of G, we see that I_i annihilates all of W_i. Thus each W_i is Z-finite, and thus so is $W_1 \bigoplus W_2$. The image of the natural map $W_1 \bigoplus W_2 \to V$ contains all linear combinations of the v_i, and is G-invariant. Thus $V_{Z-\mathrm{fin}}$ is an $A[G]$-submodule of V, proving (1).

Now let V_1 and V_2 be locally Z-finite $A[G]$-modules. Then $(V_1 \bigoplus V_2)_{Z-\mathrm{fin}}$ contains V_1 and V_2, and so, since it is an $A[G]$-submodule of $V_1 \bigoplus V_2$, equals all of $V_1 \bigoplus V_2$. This proves (2) in the case of locally Z-finite representations.

Part (3) is evident (and was already used implicitly in the proof of part (1)). □

2.3.3. *Lemma.* — *If V is an object of $\mathrm{Mod}_G(A)$ which is finitely generated over $A[G]$, then V is Z-finite if and only if V is locally Z-finite.*

Proof. — As we already observed in the proof of the preceding lemma, if an $A[G]$-module is generated by a single locally Z-finite vector, then it is Z-finite. If V is generated by finitely many such vectors, then it is a quotient of a direct sum of finitely many Z-finite representations, and so is again Z-finite (by parts (2) and (3) of the preceding lemma). □

2.3.4. *Lemma.* — *If V is an object of $\mathrm{Mod}_G(A)$, then any locally admissible vector in V is locally Z-finite.*

Proof. — Suppose that $v \in V$ is locally admissible, let $H \subset G$ be a compact open subgroup that fixes v, and let $\mathfrak{m}^i$ annihilate v, for some $i \geq 0$. If $W \subset V$ denotes the $A[G]$-submodule generated by v, then W is admissible smooth, by assumption, and is annihilated by $\mathfrak{m}^i$, and so W^H is finitely generated over A. Since W^H is Z-invariant, we conclude that v is locally Z-finite, as claimed. □

2.3.5. *Lemma.* — *If an object V of* $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ *is finitely generated over $A[G]$, then V is Z-finite.*

Proof. — This follows directly from the preceding lemmas. □

2.3.6. *Lemma.* — *Let V be an object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*, and consider the following conditions on V.*

1. *V is of finite length (as an object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*, or equivalently, as an object of* $\mathrm{Mod}_G(A)$*), and is admissible.*
2. *V is finitely generated as an $A[G]$-module, and is admissible.*
3. *V is of finite length (as an object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$*), and is Z-finite.*

Then (1) *implies* (2) *and* (3).

Proof. — The implication (1) $\Rightarrow$ (2) is immediate. The implication (1) $\Rightarrow$ (3) then follows from this together with the preceding lemma. □

We make the following conjecture:

2.3.7. *Conjecture.* — *If G is a reductive p-adic group, then the three conditions of Lemma* 2.3.6 *are mutually equivalent.*

2.3.8. *Theorem.* — *Conjecture* 2.3.7 *holds in the following two cases:*

1. *G is a torus.*
2. $G = \mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. — Suppose first that G is a torus, so that $G = Z$. Let V be an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. Taking into account Lemma 2.3.5, we see that each of conditions (2) and (3) of Lemma 2.3.6 implies that V is both finitely generated over $A[Z]$ and Z-finite. These conditions, taken together, imply in their turn that V is finitely generated as an A-module. In particular, it is a finite length object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Thus conditions (2) and (3) of Lemma 2.3.6 each imply condition (1).

Suppose now that $G = \mathrm{GL}_2(\mathbb{Q}_p)$. The results of [**1**] and [**2**] imply that any irreducible smooth G-representation that is Z-finite is admissible (see [**2**, Cor. 1.2], and note that each of the representations listed in this corollary is admissible), and thus condition (3) of Lemma 2.3.6 implies condition (1). It remains to show that the same is true of condition (2) of Lemma 2.3.6. To this end, let V be an object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ that is finitely generated over $A[G]$. We must show that it is of finite length.

Write $H = \mathrm{GL}_2(\mathbb{Z}_p)Z \subset G$. Let $S \subset V$ be a finite set that generates V over $A[G]$, and let W be the $A[H]$-submodule of V generated by S. Since V is smooth (by assumption), and Z-finite (by Lemma 2.3.5), we see that W is finitely generated as an

A-module, annihilated by some power of $\mathfrak{m}$. Since W contains the generating set S, the H-equivariant embedding $W \hookrightarrow V$ induces a G-equivariant surjection $c\text{–}\mathrm{Ind}_H^G W \to V$. Thus it suffices to show that any admissible quotient V of $c\text{–}\mathrm{Ind}_H^G W$ is of finite length as an $A[G]$-module. Taking into account Proposition 2.2.13, we see in fact that it suffices to prove the analogous result with W replaced by one of its (finitely many) Jordan-Hölder factors, which will be a finite dimensional irreducible representation of H over $k := A/\mathfrak{m}$. Extending scalars to a finite extension of k, if necessary, we may furthermore assume that W is an absolutely irreducible representation of H over k, and this we do from now on.

Since V is a quotient of $c\text{–}\mathrm{Ind}_H^G W$, the evaluation map

$$\mathrm{Hom}_{k[G]}(c\text{–}\mathrm{Ind}_H^G W, V) \otimes_k c\text{–}\mathrm{Ind}_H^G W \to V \tag{2.3.9}$$

is surjective. Lemma 2.3.10 below shows that $\mathrm{Hom}_{k[G]}(c\text{–}\mathrm{Ind}_H^G W, V)$ is a finite dimensional k-vector space. It is also naturally a module over the endomorphism algebra $\mathscr{H} := \mathrm{End}_{k[G]}(c\text{–}\mathrm{Ind}_H^G W)$. In [**1**] it is proved that $\mathscr{H} \cong k[T]$ (where T denotes a certain specified endomorphism whose precise definition need not concern us here). Thus $\mathrm{Hom}_{k[G]}(c\text{–}\mathrm{Ind}_H^G W, V)$ is a finite dimensional $k[T]$-module, and hence is a direct sum of $k[T]$-modules of the form $k[T]/f(T)k[T]$, for various non-zero polynomials $f(T) \in k[T]$.

Taking into account the surjection (2.3.9), we see that it suffices to show that any G-representation of the form $(c\text{–}\mathrm{Ind}_H^G W)/f(T)(c\text{–}\mathrm{Ind}_H^G W)$ is of finite length. Further extending scalars, if necessary, we may assume that $f(T)$ factors completely in $k[T]$ as a product of linear factors, and hence are reduced to showing that for any scalar $\lambda \in k$, the quotient $(c\text{–}\mathrm{Ind}_H^G W)/(T-\lambda)(c\text{–}\mathrm{Ind}_H^G W)$ is of finite length. This, however, follows from the results of [**1**] and [**2**]. (Indeed, these quotients are either irreducible or of length two – see Theorems 1.1 and 2.7.1 of [**2**].) Thus the theorem is proved in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. □

2.3.10. *Lemma.* — *If U and V are smooth representations of G over A, such that U is finitely generated over $A[G]$ and V is admissible, then* $\mathrm{Hom}_{A[G]}(U, V)$ *is a finitely generated A-module.*

Proof. — Let $W \subset U$ be a finitely generated A-submodule that generates U over $A[G]$, let $H \subset G$ be a compact open subgroup of G that fixes W, and let $i \geq 0$ be such that $\mathfrak{m}^i$ annihilates W. (Since U is a smooth representation of G over A, such H and i exist.) Restricting homomorphisms to W induces an embedding

$$\mathrm{Hom}_{A[G]}(U, V) \hookrightarrow \mathrm{Hom}_A(W, V^H[\mathfrak{m}^i]). \tag{2.3.11}$$

Since V is an admissible smooth G-representation, its submodule $V^H[\mathfrak{m}^i]$ is finitely generated over A, and so the target of (2.3.11) is finitely generated over A. The same is thus true of the source, and the lemma follows. □

2.4. ϖ-adically admissible G-representations. — In this subsection we introduce a class of representations related to the considerations of [**15**].

2.4.1. ***Definition.*** — We say that a G-representation V over A is ϖ-adically continuous if:

1. V is ϖ-adically separated and complete.
2. The $\mathscr{O}$-torsion subspace $V[\varpi^\infty]$ of V is of bounded exponent (i.e. is annihilated by a sufficiently large power of ϖ).
3. The G-action $G \times V \to V$ is continuous, when V is given its ϖ-adic topology.
4. The A-action $A \times V \to V$ is continuous, when A is given its $\mathfrak{m}$-adic topology and V is given its ϖ-adic topology.

2.4.2. ***Remark.*** — Conditions (3) and (4) of the definition can be expressed more succinctly as follows: for any $i \geq 0$, the G-action on $V/\varpi^i V$ is smooth, in the sense of 2.2.1. Also, the reader may easily check, in Definition 2.4.1, that condition (3) is equivalent to the apparently weaker condition that the map $G \times V \to V$ be left-continuous (i.e. that the map $G \to V$ defined by $g \mapsto gv$ is continuous for each $v \in V$) when V is equipped with its ϖ-adic topology, and that condition (4) is equivalent to the apparently weaker condition that the map $A \times V \to V$ be left-continuous when A is equipped with its $\mathfrak{m}$-adic topology and V is equipped with its ϖ-adic topology.

We let $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ denote the full subcategory of $\mathrm{Mod}_G(A)$ consisting of ϖ-adically continuous representations of G over A.

2.4.3. ***Remark.*** — If V is a $\mathscr{O}$-torsion object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, then $V = V[\varpi^i]$ for some $i \geq 0$, hence $V = V/\varpi^i V$, and so V is in fact a smooth representation of G. Thus $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)[\varpi^\infty] = \mathrm{Mod}_G^{\mathrm{sm}}(A)[\varpi^\infty]$. In particular, when A is an object of $\mathrm{Art}(\mathscr{O})$, the categories $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ and $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ coincide.

We now establish some basic results concerning the category $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.

2.4.4. ***Proposition.*** — *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence in $\mathrm{Mod}_G(A)$. If V_2 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, then the following are equivalent:*

1. *V_1 is closed in the ϖ-adic topology of V_2, and $V_3[\varpi^\infty]$ has bounded exponent.*
2. *V_3 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.*

Furthermore, if these equivalent conditions hold, then the G-representation V_1 is also an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, and the ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1.

Proof. — The surjection $V_2 \to V_3$ induces a surjection $V_2/\varpi^i V_2 \to V_3/\varpi^i V_3$ for each $i \geq 0$. Thus $V_3/\varpi^i V_3$ is a smooth $A[G]$-representation for each $i \geq 0$, since this is true of $V_2/\varpi^i V_2$ by assumption. Since V_2 is assumed to be ϖ-adically complete and separated, we see furthermore that V_1 is closed in V_2 if and only if V_3 (which is isomorphic to the quotient V_2/V_1) is ϖ-adically complete and separated. From this we conclude the equivalence of (1) and (2).

If these conditions hold, then it follows from Lemma A.1 that the ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1, and hence that V_1 is ϖ-adically separated and complete, since V_2 is by assumption. This shows that V_1 satisfies condition (1) of Definition 2.4.1. It obviously satisfies condition (2), since it is a submodule of V_2, which satisfies this condition by assumption. Conditions (3) and (4) are also inherited from the corresponding conditions for V_2, again taking into account the fact that the ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1. Thus V_1 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, as claimed. □

2.4.5. *Corollary*. — *The category* $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ *is closed under the formation of kernels and images in the abelian category* $\mathrm{Mod}_G(A)$.

Proof. — Suppose that $0 \to V_1 \to V_2 \to V_3$ is an exact sequence of objects in $\mathrm{Mod}_G(A)$, with V_2 and V_3 lying in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. Let V_3' denote the image of V_2 in V_3. Since $V_3[\varpi^\infty]$ has bounded exponent, by assumption, so does $V_3'[\varpi^\infty]$. The preceding proposition then implies that both V_1 and V_3' lie in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. This proves the closure of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ under the formation of kernels and images. □

2.4.6. *Corollary*. — *If V is any object of* $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, *then each of* $V[\varpi^\infty]$ *and* V_{fl} *is also an object of* $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.

Proof. — Since $V[\varpi^\infty]$ is of bounded exponent, the discussion of Remark A.2 show that $V[\varpi^\infty]$ is ϖ-adically closed in V. Since $(V_{\mathrm{fl}})[\varpi^\infty] = 0$, the result is now seen to follow directly from Proposition 2.4.4, applied to the short exact sequence $0 \to V[\varpi^\infty] \to V \to V_{\mathrm{fl}} \to 0$. □

2.4.7. *Definition*. — A ϖ-adically admissible representation of G over A is an object V of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ for which the induced G-representation on $(V/\varpi V)[\mathfrak{m}]$ (which is then a smooth G-representation over $A/\mathfrak{m}$, by Remark 2.4.2) is in fact an admissible smooth G-representation over $A/\mathfrak{m}$.

2.4.8. *Remark*. — Since admissible smooth representations form a Serre subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, we see (taking into account Remarks 2.2.14 and 2.4.2) that if V is a ϖ-adically admissible G-representation over A, then in fact $V/\varpi^i V$ is an admissible smooth representation for every $i \geq 0$.

We let $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ denote the full subcategory of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ consisting of ϖ-adically admissible representations.

2.4.9. *Remark*. — Combining the observations of Remarks 2.4.3 and 2.4.8, one sees that $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)[\varpi^\infty] = \mathrm{Mod}_G^{\mathrm{adm}}(A)[\varpi^\infty]$. In particular, when A is an object of $\mathrm{Art}(\mathscr{O})$, the categories $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ and $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ coincide.

2.4.10. *Proposition*. — *The functor* $V \mapsto \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O})$ *induces an A-linear anti-equivalence of categories* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)^{\mathrm{fl}} \xrightarrow{\sim} \mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)^{\mathrm{fl}}$.

Proof. — As is explained in the proof of [**15**, Thm. 1.2], the functor $V \mapsto \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O})$, the latter space being equipped with the weak topology (i.e. the topology of point-wise convergence), induces an equivalence of categories between the category of ϖ-adically separated and complete flat $\mathscr{O}$-modules and the category of compact linear-topological flat $\mathscr{O}$-modules. The reader may now easily check that this functor induces the required equivalence. □

2.4.11. *Proposition.* — *The category* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ *is closed under the formation of kernels, images, and cokernels in the abelian category* $\mathrm{Mod}_G(A)$, *and under extensions in the category* $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. *In particular,* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ *is an abelian category.*

Proof. — Suppose first that $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of objects in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, with V_1 and V_3 being objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. For any $i \geq 0$, we have the exact sequence $V_1/\varpi^i V_1 \to V_2/\varpi^i V_2 \to V_3/\varpi^i V_3 \to 0$. Since $V_1/\varpi^i V_1$ and $V_3/\varpi^i V_3$ are assumed to be admissible smooth G-representations over A, the same is true of $V_2/\varpi^i V_2$, by Proposition 2.2.13. Thus V_2 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$.

Suppose now that $0 \to V_1 \to V_2 \to V_3$ is an exact sequence of objects of $\mathrm{Mod}_G(A)$, with V_2 and V_3 being objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. Let V_3' denote the image of V_2 in V_3. It follows from Corollary 2.4.5 that V_1 and V_3' are objects of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. Since $V_3[\varpi^\infty]$ is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)[\varpi^\infty]$, which by Remark 2.4.9 equals $\mathrm{Mod}_G^{\mathrm{adm}}(A)[\varpi^\infty]$, so is $V_3'[\varpi^\infty]$, by Proposition 2.2.13. Choose $i \geq 0$ so that $V_3'[\varpi^i] = V_3'[\varpi^\infty]$. For each $j \geq i$, our given exact sequence yields an exact sequence

$$V_3'[\varpi^i] = V_3'[\varpi^j] \to V_1/\varpi^j V_1 \to V_2/\varpi^j V_2 \to V_3'/\varpi^j V_3' \to 0.$$

Since the first and third terms of the exact sequence lie in $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, we conclude from Proposition 2.2.13 that the other two terms do also. We conclude that V_1 and V_3' are both objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}$.

Suppose finally that $V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of objects of $\mathrm{Mod}_G(A)$, with V_1 and V_2 being objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. We wish to show that V_3 is also an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. By what we have already proved, we know that the image of V_1 in V_2 again lies in $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. Thus, replacing V_1 by this image, we may assume that our sequence is in fact short exact.

We intend to apply Theorem A.11, but before doing so, we must verify that V_3 is complete, with $\mathscr{O}$-torsion of bounded exponent. Since $V_2[\varpi^\infty]$ has bounded exponent by assumption, it is no loss of generality to do this after replacing V_1 and V_2 by $(V_1)_{fl}$ and $(V_2)_{fl}$ respectively, and replacing V_3 by its quotient by the image of $V_2[\varpi^\infty]$. We may thus assume that V_1 and V_2 are $\mathscr{O}$-torsion free.

Applying $\mathrm{Hom}_{\mathscr{O}}(-, \mathscr{O})$ to the embedding $V_1 \hookrightarrow V_2$, and taking into account the anti-equivalence of categories of Proposition 2.4.10, we obtain a map $M_2 \to M_1$ of objects of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)^{\mathrm{fl}}$. Let M denote the image of M_2 in M_1, and let $\widetilde{M}$ denote the saturation of M in M_1 as an $\mathscr{O}$-module, i.e.

$$\widetilde{M} = \{m \in M_1 \mid \varpi^i m \in M \text{ for some } i \geq 0\}.$$

The quotient $M_1/\widetilde{M}$ is again an object of $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)^{\mathrm{fl}}$, and thus the complex $M_2 \to M_1 \to M_1/\widetilde{M}$ in $\mathrm{Mod}_G^{\mathrm{fg\,aug}}(A)^{\mathrm{fl}}$, with the last map being a surjection, arises via the anti-equivalence of Proposition 2.4.10 from a complex $V \to V_1 \to V_2$ in $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)^{\mathrm{fl}}$, with the first arrow being an injection, by Lemma A.13. Since the second arrow is also an injection, by assumption, we conclude that $V = 0$, and hence that $M_1/\widetilde{M} = 0$, i.e. that the saturation in M_1 of the image M of M_2 is equal to M_1. Thus the quotient M_1/M is $\mathscr{O}$-torsion. Since it is furthermore finitely generated over $A[[H]]$, for any compact open subgroup H of G (as this is true of M_1), it is of bounded exponent. It follows from Lemma A.14 that $V_3[\varpi^\infty]$ has bounded exponent. Lemma A.1 then implies that ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1. Since V_1 is ϖ-adically complete by assumption, we find that it is ϖ-adically closed in V_2, and so it follows from Proposition 2.4.4 that V_3 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, and in particular is ϖ-adically complete.

We now apply $\mathrm{Hom}_{\mathscr{O}}(-,\mathscr{O})$ to the exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ to obtain (by Theorem A.11) the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{O}}\big((V_3)_{\mathrm{fl}},\mathscr{O}\big) \to \mathrm{Hom}_{\mathscr{O}}\big((V_2)_{\mathrm{fl}},\mathscr{O}\big) \to \mathrm{Hom}_{\mathscr{O}}\big((V_1)_{\mathrm{fl}},\mathscr{O}\big) \\ \to \mathrm{Hom}_{\mathscr{O}}(V_3[\varpi^\infty],E/\mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_2[\varpi^\infty],E/\mathscr{O}). \end{aligned}$$

(Note that $\mathrm{Hom}_{\mathscr{O}}\big((V_i)_{\mathrm{fl}},\mathscr{O}\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}}(V_i,\mathscr{O})$ for $i = 1,2,3$, since $\mathscr{O}$ is torsion free as a module over itself.) Proposition 2.4.10 and Lemma 2.2.11 (together with Remark 2.4.9) imply that, for any compact open subgroup H of G, each of $\mathrm{Hom}_{\mathscr{O}}\big((V_1)_{\mathrm{fl}},\mathscr{O}\big)$, $\mathrm{Hom}_{\mathscr{O}}\big((V_2)_{\mathrm{fl}},\mathscr{O}\big)$, and $\mathrm{Hom}_{\mathscr{O}}(V_2[\varpi^\infty],E/\mathscr{O})$ is a finitely generated $A[[H]]$-module. Since $A[[H]]$ is Noetherian, a consideration of this exact sequence shows that the $A[[H]]$-modules $\mathrm{Hom}_{\mathscr{O}}\big((V_3)_{\mathrm{fl}},\mathscr{O}\big)$ and $\mathrm{Hom}_{\mathscr{O}}(V_3[\varpi^\infty],E/\mathscr{O})$ are also finitely generated, and thus that $(V_3)_{\mathrm{fl}}$ and $V_3[\varpi^\infty]$ are both objects of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$ (by the same proposition and lemma). We have already observed that since $V_3[\varpi^\infty]$ has bounded exponent, Proposition 2.4.4 implies that V_3 lies in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. Since it is furthermore an extension of the objects $(V_3)_{\mathrm{fl}}$ and $V_3[\varpi^\infty]$ of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, it follows from the first paragraph of the proof that V_3 is itself an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$. □

2.4.12. *Proposition.* — *If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence in $\mathrm{Mod}_G(A)$, and if V_2 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, then the following are equivalent:*

1. *V_1 is closed in the ϖ-adic topology of V_2, and $V_3[\varpi^\infty]$ has bounded exponent.*
2. *V_1 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$.*
3. *V_3 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.*
4. *V_3 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$.*

If these equivalent conditions hold, then the ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1.

Proof. — The equivalence of (1) and (3) follows from Proposition 2.4.4, and clearly (4) implies (3). Since V_1 (resp. V_3) is the kernel of the map $V_2 \to V_3$ (resp. the cokernel of the map $V_1 \to V_2$), the equivalence of (2) and (4) follows from Proposition 2.4.11. Finally, note that if (3) holds, then $V_3/\varpi^i V_3$ is a quotient of $V_2/\varpi^i V_2$ for any $i \geq 0$. The latter module is an admissible $A[G]$-representation by assumption, and thus so is the former. Consequently, V_3 is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, and so (3) implies (4). The remaining claim of the proposition follows from Proposition 2.4.4. □

2.4.13. *Corollary*. — *If V is any object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, then each of $V[\varpi^\infty]$ and V_{fl} is also an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$.*

Proof. — This follows directly from the preceding proposition, applied to the short exact sequence $0 \to V[\varpi^\infty] \to V \to V_{\mathrm{fl}} \to 0$. (Compare the proof of Corollary 2.4.6.) □

3. Ordinary parts

In this section we take G to be a p-adic reductive group (by which we always mean the group of $\mathbb{Q}_p$-points of a connected reductive algebraic group over $\mathbb{Q}_p$). We suppose that P is a parabolic subgroup of G (more precisely, the group of $\mathbb{Q}_p$-valued points of a parabolic subgroup of G defined over $\mathbb{Q}_p$), with unipotent radical N, that M is a Levi factor of P (so that $P = MN$), and that $\overline{P}$ is an opposite parabolic to P, with unipotent radical $\overline{N}$, chosen so that $M = P \bigcap \overline{P}$. (This condition uniquely determines $\overline{P}$.) Our goal in this section is to define the functor of ordinary parts, which (for any object A of $\mathrm{Comp}(\mathscr{O})$) is a functor $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_M^{\mathrm{adm}}(A)$. In the following section we will show that it is right adjoint to parabolic induction $\mathrm{Ind}_{\overline{P}}^G : \mathrm{Mod}_M^{\mathrm{adm}}(A) \to \mathrm{Mod}_G^{\mathrm{adm}}(A)$.

In fact, it is technically convenient to define Ord_P in a more general context, as a functor $\mathrm{Ord}_P : \mathrm{Mod}_P^{\mathrm{sm}}(A) \to \mathrm{Mod}_M^{\mathrm{sm}}(A)$, and this we do in Subsection 3.1. In Subsection 3.2 we establish some elementary properties of Ord_P. In Subsection 3.3, we prove that Ord_P, when applied to admissible smooth G-representations, yields admissible smooth M-representations. Finally, Subsection 3.4 extends the main definitions and results of the preceding subsections to the context of ϖ-adically continuous and ϖ-adically admissible representations over A.

The functor Ord_P is analogous to the locally analytic Jacquet functor J_P studied in [**5**] and [**7**], and many of the constructions and arguments of this section and the next are suitable modifications of the constructions and arguments found in those papers. Typically, the arguments become simpler. (As a general rule, the theory of Ord_P is more elementary than the theory of the Jacquet functors J_P, just as the theory of p-adic modular forms is more elementary in the ordinary case than in the more general finite slope case.)

3.1. The definition of Ord_P. — We fix G, P, $\overline{P}$, etc., as in the preceding discussion. We let A denote an object of $\mathrm{Comp}(\mathscr{O})$.

3.1.1. *Definition*. — If V is a representation of N over A, and if $N_1 \subset N_2$ are compact open subgroups of N, then let $h_{N_2,N_1} : V^{N_1} \to V^{N_2}$ denote the operator defined by $h_{N_2,N_1}(v) := \sum_{n \in N_2/N_1} nv$. (Here we are using n to denote both an element of N_2/N_1, and a choice of coset representative for this element in N_2. Since $v \in V^{N_1}$, the value nv is well-defined independently of the choice of coset representative.)

3.1.2. *Lemma*. — *If $N_1 \subset N_2 \subset N_3$ are compact open subgroups of N, then*

$$h_{N_3,N_2} h_{N_2,N_1} = h_{N_3,N_1}.$$

Proof. — This is immediate from the definition of the operators. $\square$

Fix a compact open subgroup P_0 of P, and set $M_0 := M \bigcap P_0$, $N_0 := N \bigcap P_0$, and $M^+ := \{m \in M \mid mN_0m^{-1} \subset N_0\}$. Let Z_M denote the centre of M, and write $Z_M^+ := M^+ \bigcap Z_M$. Note that each of M_0 and N_0 is a subgroup of G, while M^+ and Z_M^+ are submonoids of G.

3.1.3. *Definition*. — If V is a representation of P over A, then for any $m \in M^+$, we define $h_{N_0,m} : V^{N_0} \to V^{N_0}$ via the formula $h_{N_0,m}(v) := h_{N_0,mN_0m^{-1}}(mv)$.

3.1.4. *Lemma*. — *If $m_1, m_2 \in M^+$, then $h_{N_0,m_1m_2} = h_{N_0,m_1} h_{N_0,m_2}$.*

Proof. — We compute that

$$\begin{aligned} h_{N_0,m_1m_2} &= h_{N_0,m_1m_2N_0m_2^{-1}m_1^{-1}} m_1m_2 \\ &= h_{N_0,m_1N_0m_1^{-1}} h_{m_1N_0m_1^{-1},m_1m_2N_0m_2^{-1}m_1^{-1}} m_1m_2 \\ &= h_{N_0,m_1N_0m_1^{-1}} m_1 h_{N_0,m_2N_0m_2^{-1}} m_2 = h_{N_0,m_1} h_{N_0,m_2}. \quad \square \end{aligned}$$

If $m \in M_0$, then (since in this case $mN_0m^{-1} = N_0$) the operator $h_{N_0,m}$ coincides with the given action of m on V^{N_0}. In general, the preceding lemma shows that the operators $h_{N_0,m}$ induce an action of the monoid M^+ on V^{N_0}, which we refer to as the Hecke action of M^+.

Regarding V^{N_0} as a module over the monoid algebra $A[M^+]$, and so in particular its central subalgebra $A[Z_M^+]$, via this action, we may consider the $A[Z_M]$-module $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$, as well as its submodule $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})_{Z_M-\mathrm{fin}}$.

3.1.5. *Lemma*. — *If U is an $A[Z_M^+]$-module which is finitely generated over A, then evaluation at the element $1 \in Z_M$ induces an embedding $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U) \hookrightarrow U$. In particular, $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U)$ is finitely generated over A.*

Proof. — If B denotes the image of $A[Z_M^+]$ in $\mathrm{End}_A(U)$, then B is a commutative finite A-algebra, and so is isomorphic to the product of its localizations at its finitely many maximal ideals, i.e. $B \xrightarrow{\sim} \prod_{\mathfrak{m} \text{ maximal}} B_{\mathfrak{m}}$. This factorization of B induces a corresponding factorization of the B-module U, namely $U = \prod_{\mathfrak{m} \text{ maximal}} U_{\mathfrak{m}}$, and a consequent factorization

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U) \xrightarrow{\sim} \prod_{\mathfrak{m} \text{ maximal}} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U_{\mathfrak{m}}).$$

We will say that a maximal ideal $\mathfrak{m}$ is ordinary (resp. non-ordinary) if and only if no (resp. some) element $z \in Z_M^+$ has its image in B lying in $\mathfrak{m}$. If $\mathfrak{m}$ is ordinary, then the $A[Z_M^+]$-module structure on $U_{\mathfrak{m}}$ extends in a unique manner to an $A[Z_M]$-module structure (since Z_M^+ generates Z_M as a group [**5**, Prop. 3.3.2 (i)]), and evaluation at $1 \in Z_M$ induces an isomorphism

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U_{\mathfrak{m}}) \xrightarrow{\sim} U_{\mathfrak{m}}.$$

On the other hand, if $\mathfrak{m}$ is non-ordinary, then it is easily seen that

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U_{\mathfrak{m}}) = 0.$$

Thus evaluation at 1 induces an embedding

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U) \xrightarrow{\sim} \prod_{\mathfrak{m} \text{ ordinary}} U_{\mathfrak{m}} \subset U,$$

as claimed. □

3.1.6. *Lemma*. — *Let W be an $A[Z_M^+]$-module, and let $\phi \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)$.*

1. *The image $\mathrm{im}(\phi)$ of ϕ is an $A[Z_M^+]$-submodule of W.*
2. *ϕ is locally Z_M-finite if and only if $\mathrm{im}(\phi)$ is finitely generated as an A-module.*
3. *If $z \in Z_M^+$, then $z\,\mathrm{im}(\phi) = \mathrm{im}(\phi)$.*

Proof. — Claim (1) is clear, since we have $z_1\phi(z_2) = \phi(z_1 z_2)$ for $z_1 \in Z_M^+$ and $z_2 \in Z_M$. As for claim (2), note that we may regard ϕ as an element of the $A[Z_M]$-submodule $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], \mathrm{im}(\phi))$ of $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)$. If $\mathrm{im}(\phi)$ is finitely generated over A, then Lemma 3.1.5 shows that this submodule is finitely generated over A, and thus that ϕ is locally Z_M-finite. On the other hand, let $\mathrm{ev}_1 : \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}} \to W$ denote the map given by evaluation at $1 \in Z_M^+$. For any $\phi \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$, if U denotes the $A[Z_M]$-submodule of $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$ generated by ϕ, then clearly $\mathrm{ev}_1(U)$ coincides with the image $\mathrm{im}(\phi)$ of ϕ. Thus if ϕ is locally Z_M-finite, we see that U is finitely generated over A, and hence the same is true of $\mathrm{im}(\phi)$. This completes the proof of (2). Claim (3) follows from the following calculation:

$$\begin{aligned} \mathrm{im}(\phi) &:= A\text{-linear span of the elements } \phi(z'), z' \in Z_M \\ &= A\text{-linear span of the elements } \phi(zz'), z' \in Z_M \\ &= A\text{-linear span of the elements } z\phi(z'), z' \in Z_M \\ &= z\,\mathrm{im}(\phi). \end{aligned}$$

□

3.1.7. *Lemma*. — *Let W be an $A[M^+]$-module.*

1. *The $A[Z_M]$-module structure on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)$ extends naturally to an $A[M]$-module structure, characterized by the property that for any $z \in Z_M$, the map $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W) \to W$ induced by evaluation at z is M^+-equivariant.*
2. *If the M_0-action on W is furthermore smooth, then the induced M-action on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$ is smooth.*

Proof. — From [**5**, Prop. 3.3.6], we see that the natural map $A[M^+] \otimes_{A[Z_M^+]} A[Z_M] \to A[M]$ is an isomorphism, and thus that the restriction map induces an isomorphism

$$\mathrm{Hom}_{A[M^+]}(A[M], W) \xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W). \tag{3.1.8}$$

The source of this isomorphism has a natural $A[M]$-module structure, which we may then transport to the target. This $A[M]$-module structure clearly satisfies the condition of (1).

Suppose now that W is M_0-smooth. Part (4) of Lemma 2.2.7 shows that the $A[M_0]$-action on W then extends to an $A[[M_0]]$-action, and that any element of W is annihilated by an open ideal in $A[[M_0]]$. Let ϕ be an element of $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$. Since $\mathrm{im}(\phi)$ is a finitely generated A-submodule of W, by Lemma 3.1.6, we may find an open ideal I in $A[[M_0]]$ which annihilates $\mathrm{im}(\phi)$. Hence the intersection $I \bigcap A[M_0]$ annihilates ϕ itself (as follows directly from the definition of the M-action on ϕ, together with the fact that $M_0 \subset M^+$), and so the M_0-action on ϕ is smooth (by Remark 2.2.2). Thus the M-action on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$ is smooth, proving (2). □

We now give the main definition of this section.

3.1.9. *Definition.* — If V is a smooth representation of P over A, then we write

$$\mathrm{Ord}_P(V) := \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})_{Z_M-\mathrm{fin}},$$

and refer to $\mathrm{Ord}_P(V)$ as the P-ordinary part of V (or just the ordinary part of V, if P is understood).

Lemma 3.1.7 shows that $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$, and hence $\mathrm{Ord}_P(V)$, is naturally an M-representation over A. Since the Hecke M_0-action on V^{N_0} coincides with the given M_0-action, it is smooth, and so the same lemma shows that the M-action on $\mathrm{Ord}_P(V)$ is smooth. Thus the formation of $\mathrm{Ord}_P(V)$ yields a functor from $\mathrm{Mod}_P^{\mathrm{sm}}(A)$ to $\mathrm{Mod}_M^{\mathrm{sm}}(A)$.

3.1.10. *Definition.* — We define the canonical lifting

$$\mathrm{Ord}_P(V) \to V^{N_0}$$

to be the composite of the inclusion $\mathrm{Ord}_P(V) \subset \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$ with the map $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}) \to V^{N_0}$ given by evaluation at the element $1 \in A[Z_M]$.

Our choice of terminology is motivated by that of [**3**, §4].

Although the definition of the functor Ord_P depends on the choice of compact open subgroup P_0 of P, we now show that it is independent of this choice, up to natural isomorphism. To this end, suppose that P_0' is an open subgroup of P_0. Write $M_0' := M \bigcap P_0'$, $N_0' := N \bigcap P_0'$, $(M^+)' := \{m \in M \,|\, mN_0'm^{-1} \subset N_0'\}$, $(Z_M^+)' := (M^+)' \bigcap Z_M$. We define the Hecke action of $(M^+)'$ on $V^{N_0'}$ in analogy to the Hecke action of M^+ on V^{N_0}.

Let $z \in Z_M$ be such that $zN_0'z^{-1} \subset N_0$, and for any object V of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, define a map $h_{N_0,N_0',z} : V^{N_0'} \to V^{N_0}$ via the formula $v \mapsto h_{N_0,zN_0'z^{-1}}zv$, for $v \in V^{N_0'}$.

3.1.11. *Lemma*. — 1. *The map $h_{N_0,N_0',z}$ intertwines the Hecke action of the intersection $M^+ \bigcap (M^+)'$ on its source and target.*
2. *If P_0'' is another compact open subgroup of P, giving rise to N_0'', $(M^+)''$, etc., and if $z, z' \in Z_M$ are such that $zN_0'z^{-1} \subset N_0$ and $z'N_0''(z')^{-1} \subset N_0'$, then $h_{N_0,N_0',z}h_{N_0',N_0'',z'} = h_{N_0,N_0'',zz'}$.*

Proof. — If $m \in M^+ \bigcap (M^+)'$, then

$$\begin{aligned}
h_{N_0,m}h_{N_0,N_0',z} &= h_{N_0,mN_0m^{-1}}mh_{N_0,zN_0'z^{-1}}z \\
&= mh_{m^{-1}N_0m,N_0}h_{N_0,zN_0'z^{-1}}z \\
&= mh_{m^{-1}N_0m,zN_0'z^{-1}}z \\
&= zmh_{(zm)^{-1}N_0zm,N_0'} \\
&= zmh_{(zm)^{-1}N_0zm,m^{-1}N_0'm}h_{m^{-1}N_0'm,N_0'} \\
&= zh_{z^{-1}N_0z,N_0'}h_{N_0',mN_0'm^{-1}}m \\
&= h_{N_0,zN_0'z^{-1}}zh_{N_0',mN_0'm^{-1}}m \\
&= h_{N_0,N_0',z}h_{N_0',m}.
\end{aligned}$$

This proves claim (1). Claim (2) is proved by an analogous computation. □

If we write $Y' = M^+ \bigcap (M^+)'$ and $Y = Z_M^+ \bigcap (Z_M^+)'$, then Y generates Z_M as a group [**5**, Prop. 3.3.2 (i)], and Y' and Z_M generate M [**5**, Prop. 3.3.2 (ii)]. In particular, the inclusions

$$\mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}) \subset \mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0'})$$

and

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}) \subset \mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0})$$

are actually equalities. Since $h_{N_0,N_0',z}$ is Y'-equivariant, it induces an $M = Y'Z_M$-equivariant map

$$\mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0'}) \to \mathrm{Hom}_{A[Y]}(A[Z_M], V^{N_0})$$

and hence an M-equivariant map

$$\mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}) \to \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}),$$

which we denote by $\imath_{N_0,N_0',z}$. Now define

$$\imath_{N_0,N_0'} := z^{-1}\imath_{N_0,N_0',z} : \mathrm{Hom}_{A[(Z_M^+)']}(A[Z_M], V^{N_0'}) \to \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0}).$$

3.1.12. *Proposition.* — 1. *The map $\imath_{N_0,N_0'}$ is well-defined independently of the choice of element $z \in Z_M$ such that $zN_0'z^{-1} \subset N_0$.*

2. *$\imath_{N_0,N_0'}$ is an M-equivariant isomorphism.*
3. *If P_0'' is another choice of compact open subgroup of P, giving rise to N_0'', etc., then $\imath_{N_0,N_0'}\imath_{N_0',N_0''} = \imath_{N_0,N_0''}$.*

Proof. — Since z^{-1} is central in M, and since $\imath_{N_0,N_0',z}$ is M-equivariant, the same is true of the map $z^{-1}\imath_{N_0,N_0',z}$. This proves the M-equivariance claim of part (2).

If P_0'' is as in part (3), and $z' \in Z_M$ is such that $z'N_0''(z')^{-1} \subset N_0'$, then part (2) of Lemma 3.1.11 shows that $\imath_{N_0,N_0',z}\imath_{N_0',N_0'',z'} = \imath_{N_0,N_0'',zz'}$. Thus

$$z^{-1}\imath_{N_0,N_0',z}(z')^{-1}\imath_{N_0',N_0'',z'} = (zz')^{-1}\imath_{N_0,N_0'',zz'}.$$

(Here we have also used the fact that $\imath_{N_0,N_0',z}$ is M-equivariant, and so in particular Z_M-equivariant.) This proves part (3).

If we take $P_0'' = P_0$ in part (3), then we find that $\imath_{N_0,N_0,zz'}$ is induced by $h_{N_0,zz'}$, and thus gives the action of $zz' \in Z_M^+$ on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$. Consequently $(zz')^{-1}\imath_{N_0,N_0,zz'}$ acts as the identity on $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})$, and we find that $z^{-1}\imath_{N_0,N_0',z}$ and $(z')^{-1}\imath_{N_0',N_0,z'}$ are mutually inverse isomorphisms. This completes the proof of claim (2). Also, since z and z' are independent of one another, we find that the isomorphism $z^{-1}\imath_{N_0,N_0',z}$ does not depend on the choice of z. This proves claim (1), and completes the proof of the proposition. □

The preceding result shows that Ord_P is well-defined, up to canonical isomorphism, independently of the choice of the open subgroup P_0 of P. It is similarly independent of the choice of the Levi factor M of P. Indeed, if M' is another Levi factor of P, then we may canonically identify M and M' via the canonical isomorphisms $M \xleftarrow{\sim} P/N \xrightarrow{\sim} M'$. Furthermore, there is a uniquely determined element $n \in N$ such that $nMn^{-1} = M'$, and the isomorphism induced by conjugation by n yields the same identification of M and M'. Via this canonical identification, we may and do regard any M'-representation equally well as an M-representation.

Given a choice of compact open subgroup $P_0 \subset P$, write $P_0' := nP_0n^{-1}$, $M_0' := nM_0n^{-1} = nMn^{-1} \bigcap P_0'$, $N_0' := nN_0n^{-1} = N \bigcap P_0'$, $(M')^+ := nM^+n^{-1} = \{m' \in M' \mid m'N_0'(m')^{-1} \subset N_0'\}$, and $Z_{M'}^+ := nZ_M^+n^{-1} = (M')^+ \bigcap Z_{M'}$. We define the Hecke action of $(M')^+$ on $V^{N_0'}$ in analogy to the Hecke action of M^+ on V^{N_0}, and denote by

$$\mathrm{Ord}_P' : \mathrm{Mod}_P^{\mathrm{sm}}(A) \to \mathrm{Mod}_M^{\mathrm{sm}}(A)$$

the functor

$$V \mapsto \mathrm{Hom}_{A[Z_{M'}^+]}(A[Z_{M'}], V^{N_0'})_{Z_{M'}-\mathrm{fin}}.$$

(The analogue of Lemma 3.1.7 for P'_0 and M' shows that $\mathrm{Hom}_{A[Z^+_{M'}]}(A[Z_{M'}], V^{N'_0})$, and so also $\mathrm{Hom}_{A[Z^+_{M'}]}(A[Z_{M'}], V^{N'_0})_{Z_{M'}-\mathrm{fin}}$, is naturally an M'-representation, and hence an M-representation.) Thus Ord'_P is the analogue of the functor Ord_P, but computed using the Levi factor M' of P rather than M. (Note that we have already seen that this functor is well-defined independently of the particular choice of compact open subgroup P'_0 used to compute it.)

Multiplication by n provides an isomorphism $V^{N_0} \xrightarrow{\sim} V^{N'_0}$, which intertwines the operator h_m on V^{N_0} with the operator $h_{nmn^{-1}}$ on $V^{N'_0}$, for each $m \in M^+$. Consequently, multiplication by n induces an M-equivariant isomorphism

$$\mathrm{Hom}_{A[Z^+_M]}(A[Z_M], V^{N_0}) \xrightarrow{\sim} \mathrm{Hom}_{A[Z^+_{M'}]}(A[Z_{M'}], V^{N'_0}),$$

and hence an M-equivariant isomorphism $\mathrm{Ord}_P(V) \xrightarrow{\sim} \mathrm{Ord}'_P(V)$. This shows that the functor Ord_P is well-defined, up to canonical isomorphism, independently of the choice of Levi factor of P.

3.2. Some elementary properties of Ord_P. — In this subsection, we record some simple properties of the functor Ord_P. We maintain the notation of the preceding subsection (A, P, M, N, P_0, M_0, N_0, M^+, Z_M, Z^+_M, etc.), and begin with a lemma that records some useful facts related to the monoid Z^+_M.

3.2.1. ***Lemma***. — 1. *The monoid Z^+_M contains a compact open subgroup of Z_M.*

2. *If Z_0 is a compact open subgroup of Z^+_M, then we may find a submonoid $Z_0 \subset Y \subset Z^+_M$ for which the quotient Y/Z_0 is a finitely generated monoid, and such that Y generates Z_M as a group.*

Proof. — Since M_0 is an open subgroup of M, the intersection $M_0 \bigcap Z_M$ is a compact open subgroup of Z_M, which is clearly contained in Z^+_M. This proves (1).

Suppose that Z_0 is any compact open subgroup of Z_M contained in Z^+_M. We claim that Z_M/Z_0 is a finitely generated abelian group. (The proof will be given in the following paragraph.) This quotient is generated by Z^+_M/Z_0 (since, as we have already recalled, Z^+_M generates Z_M as a group). Thus we may find a finitely generated submonoid of Z^+_M/Z_0 which generates Z_M/Z_0. The preimage of this submonoid in Z^+_M is a submonoid Y of Z^+_M satisfying the requirements of (2).

To see that Z_M/Z_0 is finitely generated, let Z^0_M denote the ($\mathbb{Q}_p$-points of the) connected component of (the algebraic group underlying) Z_M; so Z^0_M is (the group of $\mathbb{Q}_p$-points of) a torus. If we write $Z^0_0 := Z_0 \bigcap Z^0_M$, then Z^0_M/Z^0_0 has finite index in Z_M/Z_0, and so it suffices to show that the former group is finitely generated. Let S denote the maximal split quotient of Z^0_M (in the category of tori), and let S' denote the kernel of the map $Z^0_M \to S$. The torus S' is then anisotropic, and hence (its groups of $\mathbb{Q}_p$-points is) compact. The intersection $S' \bigcap Z^0_0$ is compact and open in the compact group S', and hence of finite index. Thus, if we let S_0 denote the image of Z^0_0 in S, it suffices to show that S/S_0 is finitely generated. Let S_1 denote the maximal compact subgroup of S. Since S_0 is of finite index in S_1, it suffices in turn to show that S/S_1 is finitely generated. Since S is a product of copies of $\mathbb{Q}_p^\times$, this follows from the isomorphism $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \xrightarrow{\sim} \mathbb{Z}$. (We remark that in fact Z_M itself is topologically finitely generated. Indeed, this follows from what we have just proved, together with [**9**, Prop. 6.4.1].) $\square$

3.2.2. *Lemma*. — *If $\{W_i\}_{i\in I}$ is an inductive system of smooth Z_M^+-modules*[(3)]*, then the natural map of $A[Z_M]$-modules*

$$\varinjlim_i \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_i)_{Z_M-\mathrm{fin}} \to \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], \varinjlim_i W_i)_{Z_M-\mathrm{fin}} \tag{3.2.3}$$

is an isomorphism.

Proof. — Let $\phi \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}}$ for some index j in the directed set I, and suppose that the image of ϕ under (3.2.3) vanishes, or equivalently, that the image of $\mathrm{im}(\phi)$ under the map $W_j \to \varinjlim_i W_i$ vanishes. Since Lemma 3.1.6 shows that $\mathrm{im}(\phi)$ is a finitely generated A-submodule of W_j, we may find an index k lying over j such that the image of $\mathrm{im}(\phi)$ under the transition map $W_j \to W_k$ vanishes, or equivalently, such that the image of ϕ under the map $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}} \to \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_k)_{Z_M-\mathrm{fin}}$ vanishes. This proves the injectivity of (3.2.3).

Now let $\phi' \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], \varinjlim_i W_i)_{Z_M-\mathrm{fin}}$. Lemma 3.1.6 shows that $\mathrm{im}(\phi')$ is a finitely generated and Z_M^+-invariant A-submodule of $\varinjlim_i W_i$. Thus we may find a finitely generated A-module $U \subset W_j$ that maps isomorphically onto $\mathrm{im}(\phi')$, for some sufficiently large index j. Let $Z_0 \subset Z_M^+$ be a compact open subgroup that fixes U element-wise, so that the Z_M^+-action on U factors through Z_M^+/Z_0. Fix a submonoid Y of Z_M^+ satisfying the conditions of Lemma 3.2.1 (2).

If we choose k over j large enough, then, as the monoid ring $A[Y/Z_0]$ is Noetherian (since Y/Z_0 is finitely generated), the image of the finitely generated $A[Y/Z_0]$-module $A[Y/Z_0]U$ in W_k will map isomorphically onto its image in $\varinjlim_i W_i$, which is to say, onto $\mathrm{im}(\phi')$. Consequently, if we choose j large enough, then we may choose U to be an $A[Y]$-submodule of W_j that maps isomorphically onto $\mathrm{im}(\phi')$. We may then lift ϕ' to an element

$$\phi \in \mathrm{Hom}_{A[Y]}(A[Z_M], U)_{Z_M-\mathrm{fin}} \subset \mathrm{Hom}_{A[Y]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}}.$$

Since Y generates Z_M as a group, the natural inclusion

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}} \subset \mathrm{Hom}_{A[Y]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}}$$

is an isomorphism. Thus ϕ is in fact an element of $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W_j)_{Z_M-\mathrm{fin}}$, which maps to the given element ϕ'. This shows that (3.2.3) is surjective. $\square$

(3) I.e., smooth under the action of a compact open subgroup of Z_M^+; such a subgroup exists, by Lemma 3.2.1 (1).

3.2.4. *Proposition*. — $\mathrm{Ord}_P : \mathrm{Mod}_P^{\mathrm{sm}}(A) \to \mathrm{Mod}_M^{\mathrm{sm}}(A)$ *is left-exact and additive, and commutes with inductive limits.*

Proof. — The functors $V \mapsto V^{N_0}$ and $W \mapsto \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], W)_{Z_M-\mathrm{fin}}$ (on the categories of smooth P-representations and M^+-modules respectively) are left-exact and additive, and (taking into account Lemma 3.2.2) commute with inductive limits. Thus Ord_P also has these properties. □

3.3. Preservation of admissibility. — If V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, then we may also regard it as an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, and so define $\mathrm{Ord}_P(V)$, an object of $\mathrm{Mod}_M^{\mathrm{sm}}(A)$. The main result of this subsection is Theorem 3.3.3 below, which shows that if V is an admissible smooth G-representation, then $\mathrm{Ord}_P(V)$ is an admissible smooth M-representation.

We begin with a key lemma, which allows us to gain some control over the Hecke Z_M^+-action on V^{N_0}. Recall that if I is a compact open subgroup of G, then we say that I admits an Iwahori decomposition with respect to P and $\overline{P}$ if the product map

$$(I \bigcap \overline{N}) \times (I \bigcap M) \times (I \bigcap N) \to I$$

is a bijection. Note that, by applying the bijection $g \mapsto g^{-1}$ from I to itself, we find that the product map

$$(I \bigcap N) \times (I \bigcap M) \times (I \bigcap \overline{N}) \to I$$

is also a bijection.

3.3.1. *Remark*. — Note that the product map induces an injection

$$\overline{N} \times M \times N \hookrightarrow G.$$

Thus the following pair of conditions is equivalent to I admitting an Iwahori decomposition:

1. $I \subset \overline{N}MN$.
2. If $\overline{n}mn \in I$, with $\overline{n} \in \overline{N}$, $m \in M$, $n \in N$, then each of $\overline{n}, m, n$ lies in I.

From this it follows immediately that if I and I' are two subgroups, both admitting an Iwahori decomposition, then the same is true of $I \cap I'$, i.e. $I \cap I' = (I \cap I' \cap \overline{N}) \times (I \cap I' \cap M) \times (I \cap I' \cap N)$.

3.3.2. *Lemma*. — *Let I_0 and I_1 be two compact open subgroups of G, both of which admit an Iwahori decomposition with respect to P and $\overline{P}$. Suppose furthermore that $I_1 \bigcap \overline{N} \subset I_0 \bigcap \overline{N}$, that $I_1 \bigcap M = I_0 \bigcap M$, and that $I_1 \bigcap N = I_0 \bigcap N = N_0$. If $z_0 \in Z_M^+$ is such that $(I_0 \bigcap \overline{N}) \subset z_0(I_1 \bigcap \overline{N})z_0^{-1}$, then $h_{N_0,z_0}(V^{I_1}) \subset V^{I_0}$.*

Proof. — To ease notation, write $M_1 := I_1 \bigcap M = I_0 \bigcap M$, and $\overline{N}_i := I_i \bigcap \overline{N}$, for $i = 0, 1$. Thus $I_0 = \overline{N}_0 M_1 N_0$, while $I_1 = \overline{N}_1 M_1 N_0$. Also, since z_0 centralizes M, we see that $z_0 I_1 z_0^{-1} = z_0 \overline{N}_1 z_0^{-1} M_1 z_0 N_0 z_0^{-1}$, and so our assumption that $z_0 \overline{N}_1 z_0^{-1} \supset \overline{N}_0$ implies that $z_0 I_1 z_0^{-1} \bigcap I_0 = \overline{N}_0 M_1 z_0 N_0 z_0^{-1}$.

Evidently $z_0I_1z_0^{-1} \supset z_0I_1z_0^{-1}\bigcap I_0$, and so $z_0V^{I_1} = V^{z_0I_1z_0^{-1}} \subset V^{z_0I_1z_0^{-1}\bigcap I_0}$. Thus to prove the lemma, it suffices to show that $h_{N_0,z_0N_0z_0^{-1}}(V^{z_0I_1z_0^{-1}\bigcap I_0}) \subset V^{I_0}$. Since M_1 normalizes both N_0 and $z_0N_0z_0^{-1}$, we see that $h_{N_0,z_0N_0z_0^{-1}}(V^{z_0I_1z_0^{-1}\bigcap I_0})$ is invariant under M_1 and N_0. Thus it suffices to show that it is furthermore invariant under $\overline{N}_0$.

Let $\overline{n} \in \overline{N}_0$, and $v \in V^{z_0I_1z_0^{-1}\bigcap I_0}$. Fix a set $\{n_l\}$ of coset representatives for $N_0/z_0N_0z_0^{-1}$. Then

$$\overline{n}h_{N_0,z_0N_0z_0^{-1}}v = \overline{n}\sum_l n_lv = \sum_l \overline{n}n_lv = \sum_l n_l'\overline{p}_lv = \sum_l n_l'v,$$

where we have used the Iwahori decomposition of I_0 to rewrite each product $\overline{n}n_l$ in the form $n_l'\overline{p}_l$, with $n_l' \in N_0$ and $\overline{p}_l \in \overline{N}_0M_1$. We claim that the n_l' again form a set of coset representatives for $N_0/z_0N_0z_0^{-1}$. Given this, we find that

$$\overline{n}h_{N_0,z_0N_0z_0^{-1}}v = h_{N_0,z_0N_0z_0^{-1}}v,$$

and thus that $V^{z_0I_1z_0^{-1}\bigcap I_0}$ is fixed by $\overline{N}_0$, as required.

Suppose now that $(n_l')^{-1}n_{l'}' \in z_0N_0z_0^{-1}$ for some $l \neq l'$. Then

$$n_l^{-1}n_{l'} = \overline{p}_l^{-1}(n_l')^{-1}n_{l'}'\overline{p}_{l'},$$

and the right hand side of this equation is a product of elements all lying in $z_0I_1z_0^{-1}\bigcap I_0$. Since the left hand side lies in N, we see from the Iwahori decomposition of $z_0I_1z_0^{-1}\bigcap I_0$ that in fact $n_l^{-1}n_{l'} \in z_0N_0z_0^{-1}$, contradicting the fact that they are distinct coset representatives for $N_0/z_0N_0z_0^{-1}$. Thus $\{n_l'\}$ is indeed a set of distinct coset representatives for $N_0/z_0N_0z_0^{-1}$, and the lemma is proved. □

We now choose a cofinal sequence $\{I_i\}_{i\geq 0}$ of compact open subgroups of G, with each I_i normal in I_0, and such that each I_i admits an Iwahori decomposition with respect to P and $\overline{P}$. We write $M_i := I_i\bigcap M$, $N_i := I_i\bigcap N$, and $\overline{N_i} := I_i\bigcap\overline{N}$, for each $i \geq 0$.

For each $i \geq j \geq 0$, we write $I_{i,j} := \overline{N}_iM_jN_0$. Each $I_{i,j}$ is a compact open subgroup of I_0 (since we may rewrite it as $I_{i,j} = I_iM_jN_0$, noting that M_jN_0 is a subgroup of M_0N_0, and thus of I_0, and so normalizes I_i). The bijection

$$\overline{N}_i \times M_j \times N_0 \xrightarrow{\sim} I_{i,j}$$

provides an Iwahori decomposition of $I_{i,j}$.

3.3.3. *Theorem.* — *If V is an admissible smooth representation of G over A, then* $\mathrm{Ord}_P(V)$ *is an admissible smooth representation of M, and the canonical lifting of Definition* 3.1.10 *is an embedding.*

Proof. — Since V is smooth by assumption, it is the inductive limit of its submodules $V[\mathfrak{m}^i]$ $(i \geq 0)$. Since Ord_P is left-exact and preserves inductive limits (Proposition 3.2.4), the embedding $\mathrm{Ord}_P(V[\mathfrak{m}^i]) \hookrightarrow \mathrm{Ord}_P(V)[\mathfrak{m}^i]$ is an isomorphism, and $\mathrm{Ord}_P(V)$ is the inductive limit of its submodules $\mathrm{Ord}_P(V)[\mathfrak{m}^i]$. It thus suffices to prove the theorem with V replaced by $V[\mathfrak{m}^i]$ for some $i \geq 0$. We must show that, for each $j \geq 0$, $\mathrm{Ord}_P(V)^{M_j}$ is finitely generated over A, and that the canonical lifting induces an embedding $\mathrm{Ord}_P(V)^{M_j} \hookrightarrow V^{N_0}$. We will compute $\mathrm{Ord}_P(V)$ using the compact open subgroup $P_0 := M_0 N_0$ of P. Observe that $V^{M_j N_0}$ is invariant under the Hecke Z_M^+-action on V^{N_0} (since Z_M^+ is central in M), and thus

$$\mathrm{Ord}_P(V)^{M_j} = \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})^{M_j}_{Z_M-\mathrm{fin}} = \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{M_j N_0})_{Z_M-\mathrm{fin}}.$$

Let $\phi \in \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{M_j N_0})_{Z_M-\mathrm{fin}}$. Lemma 3.1.6 (2) shows that $\mathrm{im}(\phi)$ is a finitely generated Z_M^+-invariant A-module, and thus is contained in $V^{I_{i,j}}$ for some sufficiently large $i \geq j$. If we choose $z_0 \in Z_M^+$ so that $z_0 \overline{N}_i z_0^{-1} \supset \overline{N_j}$, then Lemma 3.1.6 (3) shows that in fact $\mathrm{im}(\phi) = h_{N_0,z_0}\,\mathrm{im}(\phi) \subset h_{N_0,z_0}(V^{I_{i,j}}) \subset V^{I_{j,j}}$ (the final inclusion holding by Lemma 3.3.2). Thus if we let U denote the maximal A-submodule of $V^{I_{j,j}}$ that is invariant under the Hecke Z_M^+-action, then we see that in fact $\mathrm{im}(\phi) \subset U$. Consequently, $\mathrm{Ord}_P(V)^{M_j} \subset \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], U)$, and so is finitely generated over A, by Lemma 3.1.5. The same lemma shows that the canonical lifting induces an embedding $\mathrm{Ord}_P(V)^{M_j} \hookrightarrow U \subset V^{I_{j,j}} \subset V^{N_0}$. □

3.4. Extension to the case of ϖ-adically complete representations. — We begin by making the following definition.

3.4.1. *Definition*. — If V is an object of $\mathrm{Mod}_P^{\varpi-\mathrm{cont}}(A)$, then we define

$$\mathrm{Ord}_P(V) := \varprojlim_i \mathrm{Ord}_P(V/\varpi^i V).$$

(Note that each of the quotients $V/\varpi^i V$ is an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A/\mathfrak{m}^i)$, by Remark 2.4.2, and hence its P-ordinary part is defined.)

We now establish some basic properties of the functor Ord_P on the category $\mathrm{Mod}_P^{\varpi-\mathrm{cont}}(A)$. It will ease the notation, and lend itself to further applications, if we work in a more general situation. Thus, we let H denote an arbitrary p-adic analytic group, and let F denote any $\mathcal{O}$-linear, $\mathcal{O}$-module valued, left-exact functor on $\mathrm{Mod}_H^{\mathrm{sm}}(A)[\varpi^\infty]$. We extend F to a functor on $\mathrm{Mod}_H^{\varpi-\mathrm{cont}}(A)$ via the formula

$$F(V) := \varprojlim_i F(V/\varpi^i V), \tag{3.4.2}$$

for any object V of $\mathrm{Mod}_H^{\varpi-\mathrm{cont}}(A)$. For any such V, the $\mathcal{O}$-module $F(V)$ has a natural projective limit topology, obtained by endowing each of the terms $F(V/\varpi^i V)$ appearing in the projective limit that defines $F(V)$ with its discrete topology. We also note that $F(V)$ is ϖ-adically complete and separated (since each term $F(V/\varpi^i V)$ appearing in the projective limit is ϖ-adically discrete [4], hence ϖ-adicallly complete and separated, and a projective limit of ϖ-adically complete and separated $\mathcal{O}$-modules is itself ϖ-adically complete and separated).

The following results establish some additional properties of $F(V)$.

[4] I.e. the ϖ-adic topology on this module coincides with the discrete topology.

3.4.3. *Proposition.* — 1. *The natural map $F(V[\varpi^\infty]) \to F(V)[\varpi^\infty]$ is an isomorphism. In particular, $F(V)[\varpi^\infty]$ is of bounded exponent.*

2. *There is a natural embedding $F(V)_{\mathrm{fl}} \hookrightarrow F(V_{\mathrm{fl}})$, whose cokernel is of bounded exponent.*
3. *The projective limit topology and the ϖ-adic topology on $F(V)$ coincide.*

Proof. — Suppose first that V is $\mathscr{O}$-torsion free. Then for any $i, j \geq 0$, the short exact sequence

$$0 \longrightarrow V/\varpi^i V \xrightarrow{\varpi^j} V/\varpi^{i+j} V \longrightarrow V/\varpi^j V \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow F(V/\varpi^i V) \xrightarrow{\varpi^j} F(V/\varpi^{i+j} V) \longrightarrow F(V/\varpi^j V).$$

Passing to the projective limit in i, we obtain an exact sequence

$$0 \longrightarrow F(V) \xrightarrow{\varpi^j} F(V) \longrightarrow F(V/\varpi^j V).$$

Thus $F(V)$ is $\mathscr{O}$-torsion free, which verifies conditions (1) and (2), and for any $j \geq 0$, we obtain an embedding

$$F(V)/\varpi^j F(V) \hookrightarrow F(V/\varpi^j V).$$

This verifies condition (3), and completes the proof of the lemma for $\mathscr{O}$-torsion free V.

Let us turn to the general case. Fix $i \geq 0$ so that $V[\varpi^i] = V[\varpi^\infty]$. For any $j \geq i$, we have the diagram

$$\begin{tikzcd} 0 \arrow[r] & V[\varpi^i] \arrow[r] & V/\varpi^j V \arrow[r] \arrow[dr, "\varpi^i"] & V_{\mathrm{fl}}/\varpi^j V_{\mathrm{fl}} \arrow[r] \arrow[d] & 0 \\ & & & V/\varpi^{i+j} V & \end{tikzcd}$$

whose top row is exact, which gives rise to the diagram

$$\begin{tikzcd} 0 \arrow[r] & F(V[\varpi^i]) \arrow[r] & F(V/\varpi^j V) \arrow[r] \arrow[dr, "\varpi^i"] & F(V_{\mathrm{fl}}/\varpi^j V_{\mathrm{fl}}) \arrow[d] \\ & & & F(V/\varpi^{i+j} V) \end{tikzcd} \tag{3.4.4}$$

whose top row is again exact. Passing to the inductive limit over j, we obtain the diagram

$$\begin{tikzcd} 0 \arrow[r] & F(V[\varpi^i]) \arrow[r] & F(V) \arrow[r] \arrow[dr, "\varpi^i"] & F(V_{\mathrm{fl}}) \arrow[d] \\ & & & F(V) \end{tikzcd}$$

with an exact top row. The result of the first paragraph shows that $F(V_{\mathrm{fl}})$ is $\mathscr{O}$-torsion free, and so we see that $F(V[\varpi^i])$ maps isomorphically onto $F(V)[\varpi^\infty]$, verifying condition (1), and also that there is an induced injection $F(V)_{\mathrm{fl}} \hookrightarrow F(V_{\mathrm{fl}})$, whose cokernel is annihilated by ϖ^i, verifying condition (2).

We now derive (3). For any $j \geq 0$, let $K_j \subset F(V)$ denote the kernel of the natural map $F(V) \to F(V/\varpi^j V)$. The sequence $\{K_j\}_{j\geq 0}$ forms a basis of neighborhoods of 0 in the projective limit topology on $F(V)$. Clearly $\varpi^j F(V) \subset K_j$.

Taking into account condition (1), which we have already proved, and the exactness of the top row of (3.4.4), we find that the map $F(V)[\varpi^\infty] \to F(V/\varpi^j V)$ is injective if $j \geq i$, and thus that $K_j \bigcap F(V)[\varpi^\infty] = 0$. If we let K'_j denote the image of K_j under the surjection $F(V) \to F(V)_{\mathrm{fl}}$, it follows that, when $j \geq i$, the induced surjection $K_j \to K'_j$ is in fact an isomorphism.

It follows from the results of the first paragraph of the proof that the kernel of the map $F(V_{\mathrm{fl}}) \to F(V_{\mathrm{fl}}/\varpi^j V_{\mathrm{fl}})$ is precisely $\varpi^j F(V_{\mathrm{fl}})$, and thus that the image of K'_j under the injection $F(V)_{\mathrm{fl}} \hookrightarrow F(V_{\mathrm{fl}})$ of (2) is contained in $\varpi^j F(V_{\mathrm{fl}})$. Since, as we have shown above, the cokernel of this injection is annihilated by ϖ^i, we find that $\varpi^i K'_j \subset \varpi^j F(V)_{\mathrm{fl}}$, and thus (since $F(V)_{\mathrm{fl}}$ is $\mathscr{O}$-torsion free) that $K'_j \subset \varpi^{j-i} F(V)_{\mathrm{fl}}$, provided that $j \geq i$.

Both K_j and $\varpi^{j-i}F(V)$ are subsets of K_{j-i}, and we have just shown that the image of the first in K'_{j-i} is contained in the image of the second. If we now suppose that $j \geq 2i$, so that the surjection $K_{j-i} \to K'_{j-i}$ is also an isomorphism, we conclude that $K_j \subset \varpi^{j-i}F(V)$. Thus the two decreasing sequences of $\mathscr{O}$-submodules $\{K_j\}_{j\geq 0}$ and $\{\varpi^j F(V)\}_{j\geq 0}$ of $F(V)$ do indeed define the same topology on $F(V)$, proving (3). □

3.4.5. *Corollary.* — *If H_1 and H_2 are two p-adic analytic groups, if $F : \mathrm{Mod}^{\mathrm{sm}}_{H_1}(A) \to \mathrm{Mod}^{\mathrm{sm}}_{H_2}(A)$ is a left-exact functor, and if we extend F to a functor on $\mathrm{Mod}^{\varpi-\mathrm{cont}}_{H_1}(A)$ via the formula (3.4.2), then the extended functor takes values in $\mathrm{Mod}^{\varpi-\mathrm{cont}}_{H_2}(A)$. If in addition the restriction of F to $\mathrm{Mod}^{\mathrm{adm}}_{H_1}(A)$ takes values in $\mathrm{Mod}^{\mathrm{adm}}_{H_2}(A)$, then the restriction to $\mathrm{Mod}^{\varpi-\mathrm{adm}}_{H_1}(A)$ of the extension of F takes values in $\mathrm{Mod}^{\varpi-\mathrm{adm}}_{H_2}(A)$.*

Proof. — The preceding proposition implies, for any object V of $\mathrm{Mod}^{\varpi-\mathrm{cont}}_{H_1}(A)$, that $F(V)$ is ϖ-adically complete and separated and that $F(V)[\varpi^\infty]$ has bounded exponent, verifying conditions (1) and (2) of Definition 2.4.1 for $F(V)$. We also conclude that the projective limit topology on $F(V)$ coincides with the ϖ-adic topology. Consequently, for any $i \geq 0$, we may find $j \geq 0$ such that $F(V)/\varpi^i F(V)$ is a subquotient of $F(V/\varpi^j V)$. Since the latter representation is a smooth H_2-representation, so is the former, by Lemma 2.2.6. Remark 2.4.2 now shows that $F(V)$ is an object of $\mathrm{Mod}^{\varpi-\mathrm{cont}}_{H_2}(A)$.

Suppose now that F restricts to a functor $\mathrm{Mod}^{\mathrm{adm}}_{H_1}(A) \to \mathrm{Mod}^{\mathrm{adm}}_{H_2}(A)$. If V is an object of $\mathrm{Mod}^{\varpi-\mathrm{adm}}_{H_1}(A)$, then $V/\varpi^j V$ is an object of $\mathrm{Mod}^{\mathrm{adm}}_{H_1}(A)$, for each $j \geq 0$, and so $F(V/\varpi^j V)$ is an object of $\mathrm{Mod}^{\mathrm{adm}}_{H_2}(A)$. As shown in the preceding paragraph, for any $i \geq 0$, the quotient $F(V)/\varpi^i F(V)$ is a subquotient of $F(V/\varpi^j V)$ for some $j \geq 0$, and thus is also an object of $\mathrm{Mod}^{\mathrm{adm}}_{H_2}(A)$, by Proposition 2.2.13. We conclude that $F(V)$ is an object of $\mathrm{Mod}^{\varpi-\mathrm{adm}}_{H_2}(A)$, as claimed. □

3.4.6. *Proposition*. — *Taking ordinary parts is a functor* $\mathrm{Ord}_P : \mathrm{Mod}_P^{\varpi-\mathrm{cont}}(A) \to \mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$.

Proof. — Since Ord_P is a left-exact functor on $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, this follows directly from Corollary 3.4.5, together with the fact that the definition of Ord_P on $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$ is made according to the formula (3.4.2). □

For each $i \geq 0$, we have the canonical lifting $\mathrm{Ord}_P(V/\varpi^i V) \to (V/\varpi^i V)^{N_0}$. These are compatible in an evident way as we change i, and so passing to the projective limit, we obtain an M^+-equivariant map

$$\mathrm{Ord}_P(V) \to V^{N_0}, \tag{3.4.7}$$

which we again refer to as the canonical lifting.

3.4.8. *Theorem*. — *The passage to ordinary parts gives rise to a functor* Ord_P : $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A) \to \mathrm{Mod}_M^{\varpi-\mathrm{adm}}(A)$. *Furthermore, for any object V of* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, *the canonical lifting* (3.4.7) *is a closed embedding, when source and target are given their ϖ-adic topologies.*

Proof. — The first claim of the theorem follows from Corollary 3.4.5, together with Theorem 3.3.3. That same theorem shows that if V is an object of $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, then the canonical lifting $\mathrm{Ord}_P(V/\varpi^j V) \to (V/\varpi^j V)^{N_0}$ is an embedding, for any $j \geq 0$. Passing to the projective limit over j, we find that the canonical lifting $\mathrm{Ord}_P(V) \to V^{N_0}$ is a topological embedding, when source and target are given their corresponding projective limit topologies. Proposition 3.4.3 (applied to each of the left-exact functors Ord_P and $(-)^{N_0}$) implies that these topologies coincide with the ϖ-adic topologies on source and target, completing the proof of the theorem. □

4. Adjunction formulas

4.1. Parabolic induction. — Let A denote an object of $\mathrm{Comp}(\mathscr{O})$, with maximal ideal $\mathfrak{m}$. If U is an object of $\mathrm{Mod}_M^{\mathrm{sm}}(A)$, then we regard U as a $\overline{P}$-representation, via the projection $\overline{P} \to \overline{P}/\overline{N} \xrightarrow{\sim} M$, and define:

$$\mathrm{Ind}_{\overline{P}}^G U := \{f : G \to U \mid f \text{ locally constant },$$
$$f(\overline{p}g) = \overline{p}f(g) \text{ for all } \overline{p} \in \overline{P}, g \in G\}.$$

The right regular action of G on functions induces a natural G-action on $\mathrm{Ind}_{\overline{P}}^G U$, making it an $A[G]$-module.

If U is an object of $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$, then again regarding U as a $\overline{P}$-representation, we define:

$$\mathrm{Ind}_{\overline{P}}^G U := \{f : G \to U \mid f \text{ continuous when } U \text{ is given its } \varpi\text{-adic topology },$$
$$f(\overline{p}g) = \overline{p}f(g) \text{ for all } \overline{p} \in \overline{P}, g \in G\}.$$

Again, the right regular action of G on functions induces a natural G-action on $\mathrm{Ind}_{\overline{P}}^{G} U$, making it an $A[G]$-module.

In order to establish some basic properties of the functor $\mathrm{Ind}_{\overline{P}}^{G}$, it will be convenient to choose a continuous section $\sigma : \overline{P}\backslash G \to G$ to the projection $G \to \overline{P}\backslash G$. (Such a section exists, since G is a locally trivial $\overline{P}$-bundle over $\overline{P}\backslash G$, and hence we may find a finite cover of $\overline{P}\backslash G$ by disjoint open and closed sets over which this bundle is trivialized.) If we write $\mathscr{C}^{\mathrm{sm}}(\overline{P}\backslash G, U)$ (resp. $\mathscr{C}(\overline{P}\backslash G, U)$) to denote the space of locally constant (resp. continuous) U-valued functions on $\overline{P}\backslash G$, for an object U of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ (resp. an object U of $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, equipped with its ϖ-adic topology), then pulling back functions along σ induces natural A-linear isomorphisms

$$\mathrm{Ind}_{\overline{P}}^{G} U \xrightarrow{\sim} \mathscr{C}^{\mathrm{sm}}(\overline{P}\backslash G, U), \quad \text{for any object } U \text{ of } \mathrm{Mod}_M^{\mathrm{sm}}(A), \tag{4.1.1}$$

and

$$\mathrm{Ind}_{\overline{P}}^{G} U \xrightarrow{\sim} \mathscr{C}(\overline{P}\backslash G, U), \quad \text{for any object } U \text{ of } \mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A). \tag{4.1.2}$$

4.1.3. *Lemma.* — *If U is an object of $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$, then for each $i \geq 0$, there is a natural isomorphism*

$$(\mathrm{Ind}_{\overline{P}}^{G} U)/\varpi^i(\mathrm{Ind}_{\overline{P}}^{G} U) \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^{G}(U/\varpi^i U).$$

Proof. — Taking into account the isomorphism (4.1.2), it suffices to show that

$$\mathscr{C}(X, U)/\varpi^i \mathscr{C}(X, U) \xrightarrow{\sim} \mathscr{C}(X, U/\varpi^i U)$$

for any profinite space X. For any $j \geq i$, we have the pair of short exact sequences

$$0 \longrightarrow (\varpi^{j-i}U + U[\varpi^i])/\varpi^j U \longrightarrow U/\varpi^j U \xrightarrow{\varpi^i} \varpi^i U/\varpi^j U \longrightarrow 0$$

and

$$0 \to \varpi^i U/\varpi^j U \to U/\varpi^j U \to U/\varpi^i U \to 0.$$

Note that the projective systems $\{(\varpi^{j-i}U + U[\varpi^i])/\varpi^j U\}_{j\geq i}$ and $\{\varpi^i U/\varpi^j U\}_{j\geq i}$ both satisfy the Mittag–Leffler condition. Indeed, for the latter projective system, the transition maps are even surjective. For the former projective system, denoting the jth term in the system by M_j, the image of the map $M_k \to M_j$ (for $k \geq j \geq i$) is $(\varpi^{\min(k-i,j)}U + U[\varpi^i])/\varpi^j U$. If $k \geq j + i$, then $k - i \geq j$, and so this image simplifies to $(\varpi^j U + U[\varpi^i])/\varpi^j U$, which is independent of k.

Upon applying the (manifestly exact) functor $\mathscr{C}^{\mathrm{sm}}(X, -)$, we obtain the short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathscr{C}^{\mathrm{sm}}\big(X, (\varpi^{j-i}U + U[\varpi^i])/\varpi^j U\big) &\longrightarrow \mathscr{C}^{\mathrm{sm}}(X, U/\varpi^j U) \\ &\xrightarrow{\varpi^i} \mathscr{C}^{\mathrm{sm}}(X, \varpi^i U/\varpi^j U) \longrightarrow 0 \end{aligned}$$

and

$$0 \to \mathscr{C}^{\mathrm{sm}}(X, \varpi^i U/\varpi^j U) \to \mathscr{C}^{\mathrm{sm}}(X, U/\varpi^j U) \to \mathscr{C}^{\mathrm{sm}}(X, U/\varpi^i U) \to 0.$$

Furthermore, the two projective systems $\{\mathscr{C}^{\mathrm{sm}}\big(X, (\varpi^{j-i}U + U[\varpi^i])/\varpi^j U\big)\}_{j\geq i}$ and $\{\mathscr{C}^{\mathrm{sm}}(X, \varpi^i U/\varpi^j U)\}_{j\geq i}$ again satisfy the Mittag–Leffler condition, since $\mathscr{C}^{\mathrm{sm}}(X, -)$ is exact. Thus, passing to the projective limit in j, we obtain short exact sequences

$$0 \longrightarrow \mathscr{C}(X, U[\varpi^i]) \longrightarrow \mathscr{C}(X, U) \xrightarrow{\varpi^i} \mathscr{C}(X, \varpi^i U) \longrightarrow 0$$

and

$$0 \to \mathscr{C}(X, \varpi^i U) \to \mathscr{C}(X, U) \to \mathscr{C}(X, U/\varpi^i U) \to 0.$$

(The exactness of the second sequence also follows directly from the discreteness of $U/\varpi^i U$.) Putting these together yields the exact sequence

$$0 \longrightarrow \mathscr{C}(X, U[\varpi^i]) \longrightarrow \mathscr{C}(X, U) \xrightarrow{\varpi^i} \mathscr{C}(X, U) \longrightarrow \mathscr{C}(X, U/\varpi^i U) \longrightarrow 0,$$

proving the lemma. □

4.1.4. *Lemma.* — *For objects U of $\mathrm{Mod}_M^{\mathrm{sm}}(A)$, the formation of $\mathrm{Ind}_{\overline{P}}^G U$ commutes with the formation of inductive limits.*

Proof. — Taking into account the isomorphism (4.1.1), it suffices to show that the formation of $\mathscr{C}^{\mathrm{sm}}(X, U)$ commutes with the formation of inductive limits for any profinite space X. However, this is clear, as any element of $\mathscr{C}^{\mathrm{sm}}(X, U)$ assumes only finitely many different values in U (since X is compact). □

4.1.5. *Proposition.* — *Parabolic induction $U \mapsto \mathrm{Ind}_{\overline{P}}^G U$ gives rise to exact functors $\mathrm{Mod}_M^{\mathrm{sm}}(A) \to \mathrm{Mod}_G^{\mathrm{sm}}(A)$ and $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A) \to \mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.*

Proof. — We first consider the case when U is smooth. As already noted in the proof of Lemma 4.1.3, the formation of locally constant functions is manifestly an exact functor. It follows from the isomorphism (4.1.1) that the formation of $\mathrm{Ind}_{\overline{P}}^G U$ is exact in U. We must show that it furthermore yields an object in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. Fix a compact open subgroup H of G. Since $\overline{P}\backslash G$ is compact, it is the union of finitely many H-orbits, say $\overline{P}g_1 H, \ldots, \overline{P}g_r H$. Restricting functions from G to $\coprod_{i=1}^r g_i H$ thus induces the first of two isomorphisms in the following sequence of maps:

$$\begin{aligned}
&\mathrm{Ind}_{\overline{P}}^G U \\
&\xrightarrow{\sim} \bigoplus_{i=1}^r \{f \in \mathscr{C}^{\mathrm{sm}}(g_i H, U) \mid f(\overline{p}g_i h) = \overline{p}f(g_i h) \text{ for all } \overline{p} \in \overline{P}\bigcap g_i H g_i^{-1}\} \\
&\xrightarrow{\sim} \bigoplus_{i=1}^r \{f \in \mathscr{C}^{\mathrm{sm}}(H, U) \mid f(g_i^{-1}\overline{p}g_i h) = \overline{p}f(h) \text{ for all } \overline{p} \in \overline{P}\bigcap g_i H g_i^{-1}\} \\
&\hookrightarrow \bigoplus_{i=1}^r \mathscr{C}^{\mathrm{sm}}(H, U),
\end{aligned} \tag{4.1.6}$$

in which the second isomorphism is induced by left translation by g_i^{-1} on the ith summand, and the third map is the obvious embedding. If we equip each of the summands with the H-action given by right translation, then these maps become H-equivariant.

The right regular H-action on $\mathscr{C}^{\mathrm{sm}}(H,U)$ makes this space an object of $\mathrm{Mod}_H^{\mathrm{sm}}(A)$, since any element of $\mathscr{C}^{\mathrm{sm}}(H,U)$ is uniformly locally constant (since H is compact) and assumes only finitely many values in U. Thus $\mathrm{Ind}_{\overline{P}}^G U$ embeds into a finite direct sum of smooth H-representations, and so is itself a smooth H-representation.

We now consider the case of ϖ-adically continuous representations. If $0 \to U_1 \to U_2 \to U_3 \to 0$ is a short exact sequence in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$, then for any $i \geq 0$ we obtain the short exact sequence $0 \to U_1/(U_1 \bigcap \varpi^i U_2) \to U_2/\varpi^i U_2 \to U_3/\varpi^i U_3 \to 0$, which in turn yields a short exact sequence

$$0 \to \mathscr{C}^{\mathrm{sm}}\big(\overline{P}\backslash G, U_1/(U_1 \textstyle\bigcap \varpi^i U_2)\big) \to \mathscr{C}^{\mathrm{sm}}\big(\overline{P}\backslash G, U_2/\varpi^i U_2\big) \\ \to \mathscr{C}^{\mathrm{sm}}\big(\overline{P}\backslash G, U_3/\varpi^i U_3\big) \to 0.$$

Since the projective system $\{U_1/(U_1 \bigcap \varpi^i U_2)\}_{i\geq 0}$ has surjective transition maps, so does the projective system $\{\mathscr{C}^{\mathrm{sm}}\big(\overline{P}\backslash G, U_1/(U_1 \bigcap \varpi^i U_2)\big)\}_{i\geq 0}$. Thus, passing to the projective limit in i, we obtain a short exact sequence

$$0 \to \mathscr{C}(\overline{P}\backslash G, U_1) \to \mathscr{C}(\overline{P}\backslash G, U_2) \to \mathscr{C}(\overline{P}\backslash G, U_3) \to 0.$$

(Here we have used Proposition 2.4.4, which shows that the topology on U_1 induced by the decreasing sequence of $\mathscr{O}$-submodules $U_1 \bigcap \varpi^i U_2$ coincides with the ϖ-adic topology on U_1.) Taking into account the isomorphism (4.1.2), we see that $\mathrm{Ind}_{\overline{P}}^G$ induces an exact functor on $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$. It remains to show that it takes values in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$.

If U is any object of $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$, then the space $\mathrm{Ind}_{\overline{P}}^G U$ is ϖ-adically complete and separated, since the isomorphism (4.1.2) identifies it with a space of continuous functions with values in the ϖ-adically complete and separated space U. Lemma 4.1.3 then shows that there is a natural isomorphism

$$\mathrm{Ind}_{\overline{P}}^G U \xrightarrow{\sim} \varprojlim_i \mathrm{Ind}_{\overline{P}}^G (U/\varpi^i U),$$

and thus that the functor $\mathrm{Ind}_{\overline{P}}^G$ on $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$ is naturally isomorphic to the functor on $\mathrm{Mod}_M^{\varpi-\mathrm{cont}}(A)$ obtained from the functor $\mathrm{Ind}_{\overline{P}}^G$ on $\mathrm{Mod}_M^{\mathrm{sm}}(A)$ according to formula (3.4.2). Corollary 3.4.5 now implies that $\mathrm{Ind}_{\overline{P}}^G$ takes values in $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$. □

4.1.7. *Proposition.* — *If U is an admissible smooth (resp. locally admissible smooth, resp. ϖ-adically admissible) $A[M]$-module, then $\mathrm{Ind}_{\overline{P}}^G U$ is an admissible smooth (resp. locally admissible smooth, resp. ϖ-adically admissible) $A[G]$-module.*

Proof. — In light of Proposition 4.1.5 and Lemmas 4.1.3 and 4.1.4, it suffices to consider the case when U is an object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Let H be a compact open subgroup of G, and let $j \geq 0$. As in the proof of Proposition 4.1.5, let $\{\overline{P}g_1, \ldots, \overline{P}g_r\} \subset G$ be a set of representatives for the finitely many orbits of H on $\overline{P}\backslash G$. If we write $M_i := M \cap g_i H g_i^{-1}$, for $i = 1, \ldots, r$, then the composition of the first two maps of (4.1.6) induces an embedding

$$(\mathrm{Ind}_{\overline{P}}^{G} U)^{H}[\mathfrak{m}^{j}] \hookrightarrow \bigoplus_{i=1}^{r} U^{M_i}[\mathfrak{m}^{j}].$$

Since each M_i is an open subgroup of M, and since U is assumed admissible, we see that the target of this embedding is finitely generated over A, and thus so is the source. This proves the proposition. □

Suppose now that U is an object of $\mathrm{Mod}_{M}^{\mathrm{sm}}(G)$, so that elements of $\mathrm{Ind}_{\overline{P}}^{G} U$ are locally constant. If f is such an element, then its support, which is an open and closed subset of G, is invariant under left translation by $\overline{P}$, and so corresponds to a compact open subset of the quotient $\overline{P}\backslash G$. We will thus frequently speak of the support of f as a subset of $\overline{P}\backslash G$, rather than of G.

Composing the closed embedding $N \hookrightarrow G$ with the quotient map $G \to \overline{P}\backslash G$, we obtain an open immersion

$$\jmath : N \hookrightarrow \overline{P}\backslash G,$$

which identifies N with an open subset $\jmath(N)$ of $\overline{P}\backslash G$. Let $\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big)$ denote the A-submodule of $\mathrm{Ind}_{\overline{P}}^{G} U$ consisting of elements whose support (thought of as a compact open subset of $\overline{P}\backslash G$) is contained in $\jmath(N)$. Pulling back such a function to N via $\jmath$, we obtain a locally constant compactly supported function $N \to U$. Thus, if we denote $\mathscr{C}_c^{\mathrm{sm}}(N, U)$ the A-module of all such functions, then we have a map of A-modules

$$\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big) \to \mathscr{C}_c^{\mathrm{sm}}(N, U). \tag{4.1.8}$$

4.1.9. *Lemma*. — *The map* (4.1.8) *is an isomorphism.*

Proof. — Clearly this map is injective. On the other hand, given a function $f \in \mathscr{C}_c^{\mathrm{sm}}(N, U)$, we may extend it to a function on $\overline{P}N$ via the formula $f(\overline{p}n) = \overline{p}f(n)$, and then extend it by zero to a function on G. Since f is compactly supported, this extension is locally constant, and yields an element of $\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big)$ that maps to f under (4.1.8). Thus (4.1.8) is also surjective. □

We denote the inverse isomorphism to (4.1.8) by $\jmath_*$. Thus

$$\jmath_* : \mathscr{C}_c^{\mathrm{sm}}(N, U) \xrightarrow{\sim} \big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big).$$

The translation action of P on $\overline{P}\backslash G$ leaves its open subset $\jmath(N)$ invariant. Concretely, we see that if $mn \in MN = P$, and $n' \in N$, then

$$\jmath(n')mn = \jmath(m^{-1}n'mn)$$

(where the left side denotes the translation action of mn on the element $\jmath(n') \in \overline{P}\backslash G$). Thus $\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big)$ is P-invariant. We may transport this P-action via the isomorphism of Lemma 4.1.9 to a P-action on $\mathscr{C}_c^{\mathrm{sm}}(N, U)$. Of course, this P-action also admits a direct description, as follows:

$$(mnf)(n') = mf(m^{-1}n'mn), \text{ for } m \in M,\ n, n' \in N, \text{ and } f \in \mathscr{C}_c^{\mathrm{sm}}(N, U).$$

There is a natural isomorphism of $A[P]$-modules

$$\mathscr{C}_c^{\mathrm{sm}}(N, A) \otimes_A U \xrightarrow{\sim} \mathscr{C}_c^{\mathrm{sm}}(N, U),$$

if we equip the left hand side with the tensor product P-action, where P acts on $\mathscr{C}_c^{\mathrm{sm}}(N, A)$ via the above description (taking the M-action on the coefficients A to be trivial), and on U through its quotient M.

4.2. A simple adjunction formula. — Fix a compact open subgroup P_0 of P, and define M_0, N_0, M^+, etc., as in Subsection 3.1. We regard $\mathscr{C}_c^{\mathrm{sm}}(N, A)$ as an $A[P]$-module via the formula of the preceding subsection (taking the M-action on A to be trivial). For any compact open subset $\Psi \subset N$, we denote by $1_\Psi \in \mathscr{C}_c^{\mathrm{sm}}(N, A)$ the characteristic function of Ψ (i.e. 1_Ψ is identically equal to 1 on Ψ, and vanishes elsewhere). If V is an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, then (since M normalizes N), the space $\mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N, A), V)$ has a natural M-action.

4.2.1. *Lemma.* — *The map* $\mathrm{ev}_{1_{N_0}} : \mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N, A), V) \to V^{N_0}$, *induced by evaluation at the function* $1_{N_0} \in \mathscr{C}_c^{\mathrm{sm}}(N, A)$, *is* M^+*-equivariant, if we equip the target with its Hecke* M^+*-action.*

Proof. — This is a simple calculation. □

Since the source of $\mathrm{ev}_{1_{N_0}}$ is an $A[M]$-module, it induces an M-equivariant map

$$\mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N, A), V) \to \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0}) \quad \left(\xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})\right), \tag{4.2.2}$$

where the isomorphism, which was observed in the proof of Lemma 3.1.7 (1), follows from [**5**, Prop. 3.3.6].

4.2.3. *Proposition.* — *The map* (4.2.2) *is an isomorphism.*

Proof. — Given $\phi \in \mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N, A), V)$, let $\tilde{\phi} \in \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0})$ denote the image of ϕ under (4.2.2). If $m \in M^+$, then

$$\tilde{\phi}(m) = (m\tilde{\phi})(1) = (m\phi)(1_{N_0}) = \phi(m1_{N_0}) = \phi(1_{mN_0m^{-1}}) \tag{4.2.4}$$

(where $1_{mN_0m^{-1}}$ denotes the characteristic function of mN_0m^{-1}). Since any element of $\mathscr{C}_c^{\mathrm{sm}}(N, A)$ may be written in the form $\sum_i a_i n_i 1_{mN_0m^{-1}}$ for some finite collections $\{a_i\} \subset A$ and $\{n_i\} \subset N$, and some $m \in M^+$, we see that ϕ is completely determined by the function $\tilde{\phi}$, and thus (4.2.2) is injective. Conversely, given any $\tilde{\phi} \in \mathrm{Hom}_{A[M^+]}(A[M], V^{N_0})$, we can define $\phi \in \mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N, A), V)$ via the formula (4.2.4), i.e.

$$\phi(f) := \sum_i a_i n_i \tilde{\phi}(m),$$

for any function $f = \sum_i a_i n_i 1_{mN_0m^{-1}} \in \mathscr{C}_c^{\mathrm{sm}}(N, A)$; the M^+-equivariance of $\tilde{\phi}$ ensures that $\phi(f)$ is well-defined, independently of the choice of representation of f as a sum of this form. Thus (4.2.2) is also surjective. □

4.2.5. ***Remark.*** — Since the source of (4.2.2) is independent of the choice of N_0, this isomorphism yields another proof of Proposition 3.1.12.

4.2.6. ***Proposition.*** — *If U and V are objects of* $\mathrm{Mod}_M^{\mathrm{sm}}(A)$ *and* $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, *respectively, then there is a natural isomorphism*

$$\mathrm{Hom}_{A[P]}\big(\mathscr{C}_c^{\mathrm{sm}}(N,U),V\big) \xrightarrow{\sim} \mathrm{Hom}_{A[M]}\big(U,\mathrm{Hom}_{A[Z_M^+]}(A[Z_M],V^{N_0})\big).$$

Proof. — We compute

$$\begin{aligned}\mathrm{Hom}_{A[P]}\big(\mathscr{C}_c^{\mathrm{sm}}(N,U),V\big) &\xrightarrow{\sim} \mathrm{Hom}_{A[P]}\big(\mathscr{C}_c^{\mathrm{sm}}(N,A)\otimes_A U,V\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{A[P]}\big(U,\mathrm{Hom}_A(\mathscr{C}_c^{\mathrm{sm}}(N,A),V)\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{A[M]}\big(U,\mathrm{Hom}_{A[N]}(\mathscr{C}_c^{\mathrm{sm}}(N,A),V)\big)\\ &\xrightarrow{\sim} \mathrm{Hom}_{A[M]}\big(U,\mathrm{Hom}_{A[Z_M^+]}(A[Z_M],V^{N_0})\big),\end{aligned}$$

where the third isomorphism arises from the fact that the P-action on M factors through $M = P/N$, and the final isomorphism is provided by Proposition 4.2.3. □

4.2.7. ***Proposition.*** — *If U is a locally Z_M-finite smooth representation of M over A then there is a natural M-equivariant isomorphism $U \xrightarrow{\sim} \mathrm{Ord}_P(\mathscr{C}_c^{\mathrm{sm}}(N,U))$, characterized by the fact that its composite with the canonical lifting $\mathrm{Ord}_P(\mathscr{C}_c^{\mathrm{sm}}(N,U)) \to \mathscr{C}_c^{\mathrm{sm}}(N,U)^{N_0}$ is given by mapping an element $u \in U$ to the constant function u supported on N_0.*

Proof. — If we write $U = \bigcup_{i\in I} U_i$ as the directed union of finitely generated, Z_M-invariant, A-modules, then we may write

$$\mathscr{C}_c^{\mathrm{sm}}(N,U)^{N_0} \xrightarrow{\sim} \varinjlim_{i\in I,z\in Z_M^+} \mathscr{C}^{\mathrm{sm}}(z^{-1}N_0z,U_i)^{N_0} \xrightarrow{\sim} \varinjlim_{i\in I,z\in Z_M^+} A[z^{-1}N_0z/N_0]\otimes_A U_i,$$

where, in forming the inductive limits, we direct Z_M^+ via the relation of divisibility. Lemma 3.2.2 yields an isomorphism

$$\varinjlim_{i,z} \mathrm{Hom}_{A[Z_M^+]}(A[Z_M],A[z^{-1}N_0z/N_0]\otimes_A U_i)_{Z_M-\mathrm{fin}} \xrightarrow{\sim} \mathrm{Ord}_p(\mathscr{C}_c^{\mathrm{sm}}(N,U)).$$

It is easily checked that evaluation at 1 induces an isomorphism

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M],A[z^{-1}N_0z/N_0]\otimes_A U_i)_{Z_M-\mathrm{fin}} \xrightarrow{\sim} U_i,$$

where U_i is embedded into $A[z^{-1}N_0z/N_0]\otimes_A U_i$ via $u\mapsto 1\otimes u$. Thus we find that

$$U \xrightarrow{\sim} \varinjlim_i U_i \xrightarrow{\sim} \mathrm{Ord}_p(\mathscr{C}_c^{\mathrm{sm}}(N,U))$$

in the manner claimed. We leave it to the reader to verify that this isomorphism is M-equivariant. □

4.2.8. *Corollary*. — *If U is a locally Z_M-finite smooth representation of M over A, and if V is a smooth representation of P over A, then passing to ordinary parts induces a natural isomorphism*

$$\mathrm{Hom}_{A[P]}\big(\mathscr{C}_c^{\mathrm{sm}}(N,U),V\big) \xrightarrow{\sim} \mathrm{Hom}_{A[M]}\big(U,\mathrm{Ord}_P(V)\big).$$

Proof. — This follows from the preceding propositions, together with the fact that, since U is locally Z_M-finite by assumption, the inclusion

$$\mathrm{Hom}_{A[M]}\big(U,\mathrm{Ord}_P(V)\big) \subset \mathrm{Hom}_{A[M]}\big(U,\mathrm{Hom}_{A[Z_M^+]}(A[Z_M],V^{N_0})\big)$$

is in fact an equality. □

4.3. Ordinary parts of parabolically induced representations. — In this subsection we will strengthen Proposition 4.2.7 by computing the P-ordinary part of a representation parabolically induced from $\overline{P}$.

Fix a minimal parabolic subgroup P_0 of G (defined over $\mathbb{Q}_p$) contained in P, choose a Levi factor M_0 of P_0 contained in M, and let W denote the Weyl group of G with respect to M_0. We may choose a subset $W_P \subset W$ so that G decomposes as the disjoint union of locally closed subsets

$$G = \coprod_{w\in w_P} \overline{P}wP.$$

We may and do choose W_P so that it contains the identity $1 \in W$, and also the longest element $w_\ell \in W$. We define a partial ordering on W_P as follows: $w \succ w'$ if and only if $\overline{P}wP$ is contained in the closure of $\overline{P}w'P$. (See for instance the discussion at the beginning of [**3**, §6.3]; note, though, that the parabolic subgroups P_Θ and P_Ω considered there are both supposed to be attached to subsets Θ and Ω of positive simple roots. Thus to compare our situation with that of [**3**, §6.3], one should write $P = P_\Omega$ for some set of positive simple roots, as well as $\overline{P} = w_\ell^{-1}Pw_\ell$, so that the above decomposition of G gets rewritten as $G = \coprod_{w\in w_P} P_\Omega w_\ell w P_\Omega$. In [**3**, §1.1] a subgroup W_Ω of W associated to Ω is defined, and a particular set of double-coset representatives $[W_\Omega\backslash W/W_\Omega]$ of the double quotient $W_\Omega\backslash W/W_\Omega$ is chosen. If $w \in W$, let $[w] \in [W_\Omega\backslash W/W_\Omega]$ denote the representative of $W_\Omega w W_\Omega$. The map $w \mapsto [w_\ell w]$ then induces a bijection between the subset W_P of W and the set $[W_\Omega\backslash W/W_\Omega]$, which is order-*reversing*, when W_P is equipped with the order introduced above, and $[W_\Omega\backslash W/W_\Omega]$ is equipped with the order defined in [**3**, §6.3].)

For any $w \in W_P$, write

$$S_w := \coprod_{w'\in W_P,\, w\succ w'} \overline{P}\backslash\overline{P}w'P.$$

Then each S_w is an open subset of $\overline{P}\backslash G$, $S_1 := \jmath(N)$, $S_{w_\ell} := \overline{P}\backslash G$, and $S_{w'} \subset S_w$ if and only if $w \succ w'$.

If U is a locally Z_M-finite smooth representation of M over A, then for each $w \in W_P$, we let $(\mathrm{Ind}_{\overline{P}}^G U)(S_w)$ denote the subspace of $\mathrm{Ind}_{\overline{P}}^G U$ consisting of functions whose support (when regarded as a subset of $\overline{P}\backslash G$) lies in S_w. Since each S_w is a union of P-orbits in $\overline{P}\backslash G$, each subspace $(\mathrm{Ind}_{\overline{P}}^G U)(S_w)$ is P-invariant.

4.3.1. *Lemma*. — *If $w \succ w'$ and w is an immediate successor of w' in W_P (i.e. if $w \neq w'$, and if there is no element $w'' \in W_P$ such that $w \succneqq w'' \succneqq w'$), then*

$$\mathrm{Ord}_P\big((\mathrm{Ind}_{\overline{P}}^G U)(S_w)/(\mathrm{Ind}_{\overline{P}}^G U)(S_{w'})\big) = 0.$$

Proof. — Restricting elements of $(\mathrm{Ind}_{\overline{P}}^G U)(S_w)$ to $\overline{P}wP$ induces an exact sequence of P-representations $0 \to (\mathrm{Ind}_{\overline{P}}^G U)(S_{w'}) \to (\mathrm{Ind}_{\overline{P}}^G U)(S_w) \to \mathscr{C}$, where

$$\mathscr{C} := \{f : \overline{P}wP \to U \mid f(\overline{n}mwp) = mf(wp) \text{ for all } \overline{n} \in \overline{N}, m \in M, p \in P, \\ \text{and the support of } f \text{ is compact modulo } \overline{P}\}$$

is equipped with the right regular P-action. Since Ord_P is a left-exact functor, it suffices to show that $\mathrm{Ord}_P(\mathscr{C}) = 0$. Suppose that

$$\phi \in \mathrm{Ord}_P(\mathscr{C}) := \mathrm{Hom}_{A[Z_M]^+}(A[Z_M], \mathscr{C}^{N_0})_{Z_M-\mathrm{fin}}.$$

Lemma 3.1.6 (2) shows that $\mathrm{im}(\phi)$ is a finitely generated A-submodule of $\mathscr{C}$. In particular, we may find a compact open subset Ω of $\overline{P}\backslash\overline{P}wP$ such that the support of every element of $\mathrm{im}(\phi)$, taken modulo $\overline{P}$, lies in Ω. Since Ω is compact, we may find $z \in Z_M^+$ such that $\Omega z^{-1} \subset \overline{P}\backslash\overline{P}wN_0$. Since Lemma 3.1.6 (3) implies that

$$\mathrm{im}(\phi) = h_{N_0,z}\,\mathrm{im}(\phi), \tag{4.3.2}$$

we see that in fact every element of $\mathrm{im}(\phi)$ has its support contained in $\overline{P}\backslash\overline{P}wN_0$. Since ϕ takes values in $\mathscr{C}^{N_0}$, every element f of $\mathrm{im}(\phi)$ is furthermore invariant under the right regular action of elements of N_0. Thus, if we write $f(w) = u$, then, when restricted to its support (i.e. to $\overline{P}\backslash\overline{P}wN_0$), the function f is given by the formula

$$\overline{n}mwn_0 \mapsto mu. \tag{4.3.3}$$

We write $f =: f_u$.

If $z \in Z_M^+$, then $h_{N_0,z}f_u$ is again supported on $\overline{P}\backslash\overline{P}wN_0$, and invariant under N_0, and so we may write $h_{N_0,z}f_u = f_{u'}$, for some $u' \in U$. To compute u', we evaluate $h_{N_0,z}f_u$ at w:

$$u' = (h_{N_0,z}f)(w) = \sum_{n\in N_0/zN_0z^{-1}} (nzf_u)(w) = \sum_{n\in N_0/zN_0z^{-1}} f_u(wnz).$$

Since f_u is supported on $\overline{P}wN_0$, we see (taking into account the formula (4.3.3)) that

$$f_u(wnz) = \begin{cases} 0 \text{ if } n \notin w^{-1}\overline{N}w \bigcap N_0 \\ (wzw^{-1})u \text{ if } n \in w^{-1}\overline{N}w \bigcap N_0. \end{cases}$$

Thus

$$u' = [\big(N_0 \bigcap w^{-1}\overline{N}w\big) : \big(zN_0z^{-1} \bigcap w^{-1}\overline{N}w\big)] \cdot (wzw^{-1})u.$$

Since w is a non-trivial element of W_P (since it is strictly greater than the element $w' \in W_P$), the intersection $N \bigcap w^{-1}\overline{N}w$ is non-trivial, and so the intersection

$N_0 \bigcap w^{-1}\overline{N}w$ is a non-trivial pro-p group, which is topologically finitely generated, but of infinite order. As z ranges over the elements of Z_M^+, the groups zN_0z^{-1} range over a cofinal collection of open subgroups of N_0. Thus the intersections $zN_0z^{-1} \bigcap w^{-1}\overline{N}w$ range over a cofinal collection of open subgroups of $N_0 \bigcap w^{-1}\overline{N}w$, and so the index $[(N_0 \bigcap w^{-1}\overline{N}w) : (zN_0z^{-1} \bigcap w^{-1}\overline{N}w)]$ can be made an arbitrarily large power of p by choosing $z \in Z_M^+$ appropriately. Since u is annihilated by some power of p, we conclude that $u' = 0$. From (4.3.2) we conclude that in fact $\mathrm{im}(\phi) = 0$, and the lemma follows, since ϕ was an arbitrarily chosen element of $\mathrm{Ord}_P(\mathscr{C})$. □

4.3.4. *Proposition.* — *If U is a locally Z_M-finite smooth M-representation over A, then there is a natural isomorphism of M-representations $U \xrightarrow{\sim} \mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G U)$.*

Proof. — It follows from the preceding lemma, together with the left-exactness of Ord_P, that the inclusion

$$(\mathrm{Ind}_{\overline{P}}^G U)(\jmath(N)) \subset \mathrm{Ind}_{\overline{P}}^G U$$

induces an isomorphism on P-ordinary parts. The proposition now follows from Proposition 4.2.7 and Lemma 4.1.9. □

4.3.5. *Corollary.* — *If U is an object of $\mathrm{Mod}_M^{\varpi-\mathrm{adm}}(A)$, then there is a natural isomorphism of M-representations $U \xrightarrow{\sim} \mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G U)$.*

Proof. — For each integer $i \geq 0$, the preceding proposition, together with Lemma 2.3.4, yields an isomorphism

$$U/\varpi^i U \xrightarrow{\sim} \mathrm{Ord}_P\big(\mathrm{Ind}_{\overline{P}}^G(U/\varpi^i U)\big).$$

Passing to the projective limit over i, and taking into account Definition 3.4.1 and Lemma 4.1.3, we obtain the required isomorphism. □

4.4. The main adjunction formula.

— The main result of this subsection is Theorem 4.4.6, showing that $\mathrm{Ind}_{\overline{P}}^G$ and Ord_P are adjoint functors.

Let U be a smooth, admissible representation of M over A, and let V be a smooth representation of G over A. Recall from Subsection 4.1 that $\jmath_*$ provides a P-equivariant isomorphism between $\mathscr{C}_c^{\mathrm{sm}}(N, U)$ and $(\mathrm{Ind}_{\overline{P}}^G U)(\jmath(N))$.

If Ω is a compact open subset of $\overline{P}\backslash G$, and $f \in \mathrm{Ind}_{\overline{P}}^G U$, then we will let $f_{|\Omega}$ denote the restriction of f to the preimage (under the canonical projection $G \to \overline{P}\backslash G$) of Ω in G, extended by zero to a function on G. The function $f_{|\Omega}$ again lies in $\mathrm{Ind}_{\overline{P}}^G U$, and the support of $f_{|\Omega}$, thought of as a subset of $\overline{P}\backslash G$, is equal to the intersection of Ω and the support of f. The reader may verify the following formula, which we will apply repeatedly: for any $g \in G$, we have that

$$(gf)_{|\Omega} = g(f_{|\Omega g}). \tag{4.4.1}$$

We will apply similar notation when considering elements of $\mathscr{C}_c^{\mathrm{sm}}(N, U)$: i.e. if $f \in \mathscr{C}_c^{\mathrm{sm}}(N, U)$, and if Ψ is a compact open subset of N, then we will let $f_{|\Psi}$ denote the restriction of f to Ψ, extended by zero over N. We have the following formula, analogous to (4.4.1): if $mn \in MN = P$, then

$$(mnf)_{|\Psi} = mn(f_{|m^{-1}\Psi mn}). \tag{4.4.2}$$

4.4.3. *Lemma*. — *Let $\phi \in \mathrm{Hom}_{A[P]}\big((\mathrm{Ind}_{\overline{P}}^G U)(\jmath(N)), V\big)$. If $f \in (\mathrm{Ind}_{\overline{P}}^G U)(\jmath(N))$ and $g \in G$ are chosen so that gf also lies in $(\mathrm{Ind}_{\overline{P}}^G U)(\jmath(N))$, then $\phi(gf) = g\phi(f)$.*

Proof. — Let Ω denote the support of f, which by assumption is contained in $\jmath(N)$, and write $\Psi = \jmath^{-1}(\Omega) \subset N$. As in Subsection 3.3, choose a cofinal sequence $\{I_i\}_{i\geq 0}$ of compact open subgroups of G, each admitting an Iwahori decomposition $\overline{N}_i \times M_i \times N_i \xrightarrow{\sim} I_i$. Since Ω, and hence Ψ, is compact, we may choose i sufficiently large so that Ψ is a union of left cosets of N_i, i.e. so that $N_i n \subset \Psi$ for every $n \in \Psi$. Write $f = \jmath_* f'$, with $f' \in \mathscr{C}_c^{\mathrm{sm}}(N, U)$. Since f' is a locally constant function, and in particular assumes only finitely many distinct values in U, enlarging i if necessary, and choosing i' sufficiently large, we may furthermore assume that f' is *left-N_i-invariant* [5], and takes values in $U^{M_i}[\mathfrak{m}^{i'}]$. If $n \in \Psi$, we then see that $(nf')_{|N_i}$ is a constant $U^{M_i}[\mathfrak{m}^{i'}]$-valued function on N_i.

We let X denote the subspace of $\mathscr{C}_c^{\mathrm{sm}}(N, U)$ consisting of constant $U^{M_i}[\mathfrak{m}^{i'}]$-valued functions on N_i, regarded as functions on N via extension by zero. The previous paragraph shows that $(nf')_{|N_i} \in X$.

We claim that $\jmath_*(X)$ is element-wise-fixed by I_i. Note that $\jmath_*(X)$ equals the set of functions $f_u : G \to U$ which are supported on $\overline{P}N_i = \overline{N}MN_i$, and which on that set are defined by the formula $f_u(\overline{n}mn) = mu$, for $\overline{n} \in \overline{N}$, $m \in M$, and $n \in N_i$, where u ranges over the elements of $U^{M_i}[\mathfrak{m}^{i'}]$. Note next (taking into account the Iwahori decomposition of I_i) that $\overline{P}N_i = \overline{P}I_i$, and thus that if F is any function on G supported on $\overline{P}N_i$, the same is true of gF, if $g \in I_i$. Furthermore, if $g \in I_i$ and $n \in N_i$ then the product ng again lies in I_i, and so (again taking into account the Iwahori decomposition of I_i) may be written in the form $g = \overline{n}'m'n'$, for some $\overline{n}' \in \overline{N}_i, m' \in M_i$, and $n' \in N_i$. We thus compute that, for any $\overline{n} \in \overline{N}$ and $m \in M$, that

$$\begin{aligned}(gf_u)(\overline{n}mn) = f_u(\overline{n}mng) = f_u(\overline{n}m\overline{n}'m'n') &= f_u(\overline{n}(m\overline{n}'m^{-1})mm'n') \\ &= mm'u = mu = f_u(\overline{n}mn).\end{aligned}$$

(Here the first equality holds by definition, since G acts by right translation; the second equality is just applying the Iwahori decomposition for ng given above; the third equality is clear; the fourth and sixth equalities follow from the definition of f_u as a function; and the fifth equality follows from the fact that $u \in U^{M_i}$.) Thus $\jmath_*(X)$ is indeed fixed element-wise by I_i.

Since U is assumed to be admissible smooth, the space $U^{M_i}[\mathfrak{m}^{i'}]$ is a finitely generated A-module, and thus the same is true of X and $\jmath_*(X)$. Thus $\phi(\jmath_*(X))$ is

[5] I.e. constant on left N_i-cosets.

a finitely generated A-submodule of V, and we may choose $j \geq i$ so that I_j fixes $\phi(\jmath_*(X))$ element-wise.

The support of gf is equal to the translate Ωg^{-1} of Ω, which by assumption again lies in $\jmath(N)$. Note that this translate is equal to the image under the natural projection $\overline{P}\backslash G \to G$ of the translate $\Psi g^{-1} \subset N g^{-1} \subset G$. Since by assumption it lies $\jmath(N)$, we see that Ψg^{-1} is a compact subset of $\overline{P}N$. Let $\overline{\Psi}$ denote the projection of Ψg^{-1} onto $\overline{N}$ (i.e. the projection onto the first factor in the source of the isomorphism $\overline{N} \times M \times N \xrightarrow{\sim} \overline{P}N$). Choose an element $z \in Z_M$ so that $z\overline{\Psi}z^{-1} \subset \overline{N}_j$, while $z^{-1}N_i z \subset N_i$.

Fix an $n \in \Psi$. Since $\Psi g^{-1} \subset \overline{\Psi}P$, we may write $gn^{-1} = p\overline{n}$ for some $\overline{n} \in \overline{\Psi}^{-1}$ and $p \in P$. By virtue of our choice of z, we have $z\overline{n}z^{-1} \in \overline{N}_j^{-1} = \overline{N}_j$. We now compute:

$$(4.4.4)\quad g(f_{|\jmath(N_i)zn}) = gn^{-1}z^{-1}(znf)_{|\jmath(N_i)} = p\overline{n}z^{-1}(znf)_{|\jmath(N_i)} \\ = pz^{-1}z\overline{n}z^{-1}(znf)_{|\jmath(N_i)} = pz^{-1}(znf)_{|\jmath(N_i)}.$$

All but the final equality are trivial manipulations (taking into account the formula (4.4.1)). The final equality holds because $(znf)_{|\jmath(N_i)}$ lies in $\jmath_*(X)$ (as we will show in a moment), and so is fixed by the element $z\overline{n}z^{-1} \in \overline{N}_j \subset \overline{N}_i$. To see that $(znf)_{|\jmath(N_i)} \in \jmath_*(X)$, note that since $(nf')_{|N_i}$ is a constant $U^{M_i}[\mathfrak{m}^{i'}]$-valued function on N_i, an application of (4.4.2) shows that $(znf')_{|zN_i z^{-1}} = z((nf')_{|N_i})$ is a constant $U^{M_i}[\mathfrak{m}^{i'}]$-valued function on $zN_i z^{-1} \supset N_i$. Thus $(znf')_{|N_i}$ is again a constant $U^{M_i}[\mathfrak{m}^{i'}]$-valued on N_i, and so lies in X. Consequently $(znf)_{|\jmath(N_i)} = \jmath_*((znf')_{|N_i})$ lies in $\jmath_*(X)$, as claimed.

Applying ϕ to (4.4.4), we compute that

$$(4.4.5)\quad \phi(g(f_{|\jmath(N_i)zn})) = \phi(pz^{-1}(znf)_{|\jmath(N_i)}) = pz^{-1}\phi((znf)_{|\jmath(N_i)}) \\ = pz^{-1}(z\overline{n}z^{-1})\phi((znf)_{|\jmath(N_i)}) = gn^{-1}z^{-1}\phi((znf)_{|\jmath(N_i)}) \\ = g\phi(n^{-1}z^{-1}(znf)_{|\jmath(N_i)}) = g\phi(f_{|\jmath(N_i)zn}).$$

Here the first equality is given by (4.4.4), the second and fifth equalities follow from the P-equivariance of ϕ, and the third equality arises from the fact that $\phi((znf)_{|\jmath(N_i)}) \in \phi(\jmath_*(X))$ (by the discussion of the preceding paragraph), and hence is fixed by the element $z\overline{n}z^{-1} \in \overline{N}_j$. The fourth equality is a trivial manipulation, while the final equality is an application of (4.4.1).

Since $z^{-1}N_i z \subset N_i$, we may write Ψ as a finite disjoint union of left $z^{-1}N_i z$ cosets, say $\Psi = \coprod_l z^{-1}N_i z n_l$, and thus decompose Ω as a union of translates of $\jmath(N_i)$, namely $\Omega = \coprod_l \jmath(N_i)zn_l$. Accordingly, we may write $f = \sum_l f_{|\jmath(N_i)zn_l}$, and then compute (taking into account (4.4.5)) that

$$\phi(gf) = \sum_l \phi(gf_{|\jmath(N_i)zn_l}) = \sum_l g\phi(f_{|\jmath(N_i)zn_l}) = g\phi(f).$$

This proves the lemma. □

We now prove our main result.

4.4.6. *Theorem.* — *Let A be an object of* $\mathrm{Comp}(\mathscr{O})$. *If U is an object of* $\mathrm{Mod}_M^{\mathrm{adm}}(A)$ (*resp.* $\mathrm{Mod}_M^{\varpi-\mathrm{adm}}(A)$) *and V is an object of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ (*resp.* $\mathrm{Mod}_G^{\varpi-\mathrm{cont}}(A)$) *then passage to ordinary parts induces an isomorphism*

$$\mathrm{Hom}_{A[G]}\big(\mathrm{Ind}_{\overline{P}}^G U, V\big) \xrightarrow{\sim} \mathrm{Hom}_{A[M]}\big(U, \mathrm{Ord}_P(V)\big).$$

Consequently, the functor Ord_P *on* $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ (*resp.* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$) *is right adjoint to the functor* $U \mapsto \mathrm{Ind}_{\overline{P}}^G U$ (*which takes* $\mathrm{Mod}_M^{\mathrm{adm}}(A)$, *resp.* $\mathrm{Mod}_M^{\varpi-\mathrm{adm}}(A)$, *to* $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, *resp.* $\mathrm{Mod}_G^{\varpi-\mathrm{adm}}(A)$, *by Proposition* 4.1.7).

Proof. — We first consider the case when U and V are smooth. Restricting maps from $\mathrm{Ind}_{\overline{P}}^G U$ to $\big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big)$ induces a map

$$(4.4.7)\quad \mathrm{Hom}_{A[G]}\big(\mathrm{Ind}_{\overline{P}}^G U, V\big) \to \mathrm{Hom}_{A[P]}\big(\big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big), V\big) \xrightarrow{\sim} \mathrm{Hom}_{A[G]}\big(A[G] \otimes_{A[P]} \big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big), V\big).$$

The natural map

$$A[G] \otimes_{A[P]} \big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big) \to \mathrm{Ind}_{\overline{P}}^G U$$

is surjective, as $\big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big)$ generates $\mathrm{Ind}_{\overline{P}}^G U$ over G, since the G-translates of $\jmath(N)$ cover $\overline{P}\backslash G$. Thus (4.4.7) is injective. We claim that it is in fact an isomorphism. To prove this, we have to show that any map of $A[G]$-modules

$$\phi : A[G] \otimes_{A[P]} \big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big) \to V,$$

in which the target is smooth, necessarily factors through the quotient $\mathrm{Ind}_{\overline{P}}^G U$ of the source. In other words, we have to show that if $g_1, \ldots, g_l$ is a finite sequence of elements of G, and $f_1, \ldots, f_l$ is a finite sequence of elements of $\big(\mathrm{Ind}_{\overline{P}}^G U\big)\big(\jmath(N)\big)$, such that

$$(4.4.8)\qquad \sum_{i=1}^{l} g_i f_i = 0 \text{ in } \mathrm{Ind}_{\overline{P}}^G U,$$

then

$$(4.4.9)\qquad \sum_{i=1}^{l} g_i \phi(f_i) \stackrel{?}{=} 0 \text{ in } V.$$

It suffices to show that each point x of $\overline{P}\backslash G$ has a compact open neighborhood Ω_x such that for any neighborhood $\Omega'_x \subset \Omega_x$ of x we have

$$(4.4.10)\qquad \sum_{i=1}^{l} g_i \phi(f_{i|\Omega'_x g_i}) \stackrel{?}{=} 0.$$

Indeed, we can then partition $\overline{P}\backslash G$ into a finite disjoint union of such neighborhoods, say $\overline{P}\backslash G = \coprod_{j=1}^{s} \Omega'_{x_j}$, and writing

$$g_i f_i = \sum_{j=1}^{s} (g_i f_i)_{|\Omega'_{x_j}} = \sum_{j=1}^{s} g_i f_{i|\Omega'_{x_j} g_i}$$

(where we have used (4.4.1)), so that

$$f_i = \sum_{j=1}^{s} f_{i|\Omega'_{x_j} g_i},$$

we conclude that if (4.4.10) holds, then so does (4.4.9).

If $x = \overline{P}g$ for some $g \in G$, then replacing $(g_1, \ldots, g_l)$ by $(gg_1, \ldots, gg_l)$, we see that it suffices to treat the case when x equals the identity coset, which is equal to the image $\jmath(e)$ of the identity e of N under the open immersion $\jmath : N \hookrightarrow \overline{P}\backslash G$. Let Ω_e be a compact open neighborhood of $\jmath(e)$ in $\jmath(N)$, chosen so that $\Omega_e g_i \subset \jmath(N)$ for all i for which $g_i \in \overline{P}N$, and so that $\Omega_e g_i$ is disjoint from the support of f_i for all other i. It follows from (4.4.8) (and an application of (4.4.1)) that

$$\sum_{i=1}^{l} (g_i f_i)_{|\Omega_e} = \sum_{i=1}^{l} g_i f_{i|\Omega_e g_i} = 0 \text{ in } \big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big).$$

Applying the map ϕ to the second of these equalities (which is an equation involving elements of $\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big)$, since $f_{i|\Omega_e g_i} = 0$ if $\Omega_e g_i \not\subset \jmath(N)$, by virtue of our choice of Ω_e), and taking into account Lemma 4.4.3, we deduce that indeed $\sum_{i=1}^{l} g_i \phi(f_{i|\Omega_e g_i}) = 0$, and thus we have shown that (4.4.7) is an isomorphism. Now composing (4.4.7) with the isomorphism

$$\mathrm{Hom}_{A[P]}\big(\big(\mathrm{Ind}_{\overline{P}}^{G} U\big)\big(\jmath(N)\big), V\big) \xrightarrow{\sim} \mathrm{Hom}_{A[P]}\big(\mathscr{C}_c^{\mathrm{sm}}(N, U), V\big)$$

induced by $\jmath_*$, together with the isomorphism of Corollary 4.2.8, and taking into account Proposition 4.3.4 (note that these results apply, since Lemma 2.3.4 implies that U is locally Z_M-finite), completes the proof of the theorem in the case when U and V are smooth.

We now consider the case when U is ϖ-adically admissible and V is ϖ-adically continuous. There is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{A[G]}\big(\mathrm{Ind}_{\overline{P}}^{G} U, V\big) & \xrightarrow{\sim} & \varprojlim_i \mathrm{Hom}_{(A/\varpi^i A)[G]}\big(\mathrm{Ind}_{\overline{P}}^{G}(U/\varpi^i U), V/\varpi^i V\big) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Hom}_{A[M]}\big(U, \mathrm{Ord}_P(V)\big) & \xrightarrow{\sim} & \varprojlim_i \mathrm{Hom}_{(A/\varpi^i A)[M]}\big(U/\varpi^i U, \mathrm{Ord}_P(V/\varpi^i V)\big),
\end{array}$$

in which the vertical arrows are given by passing to ordinary parts (here we are taking into account Proposition 4.3.4 and Corollary 4.3.5), and the horizontal arrows arise from the isomorphism of Lemma 4.1.3 and Definition 3.4.1. By what we have already proved, the right hand vertical arrow is also an isomorphism. Thus so is the left hand vertical arrow. □

Appendix A
Functional analysis

In this appendix we establish some simple functional analytic results regarding (not necessarily finitely generated) modules over the ring $\mathscr{O}$.

A.1. Lemma. — *If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of $\mathscr{O}$-modules such that $V_3[\varpi^\infty]$ has bounded exponent, then the ϖ-adic topology on V_2 induces the ϖ-adic topology on V_1.*

Proof. — If we choose i so large that $V_3[\varpi^i] = V_3[\varpi^\infty]$, then we deduce the second of the inclusions

$$\varpi^j V_1 \subset V_1 \bigcap \varpi^j V_2 \subset \varpi^{j-i} V_1;$$

the first is evident. This proves the lemma. □

In the remainder of the appendix, we will restrict our attention to $\mathscr{O}$-modules whose submodule of torsion elements is of bounded exponent. We begin with a general remark about such modules.

A.2. Remark. — If V is any $\mathscr{O}$-module then there is a canonical short exact sequence

$$0 \to V[\varpi^\infty] \to V \to V_{\mathrm{fl}} \to 0. \tag{A.3}$$

Suppose now that the torsion submodule of V is of bounded exponent, i.e. that $V[\varpi^\infty] = V[\varpi^i]$ for some sufficiently large i. Then, for any $j \geq i$, tensoring the above short exact sequence by $\mathscr{O}/\varpi^j$ over $\mathscr{O}$ induces a short exact sequence

$$0 \to V[\varpi^\infty] = V[\varpi^i] \to V/\varpi^j V \to V_{\mathrm{fl}}/\varpi^j V_{\mathrm{fl}} \to 0. \tag{A.4}$$

We now turn to proving Theorem A.11 below. We begin by first making a definition, and then establishing some preliminary results.

A.5. Definition. — If V is an $\mathscr{O}$-module, then we say that an $\mathscr{O}$-linear map $\phi : V \to E$ is bounded if the image is contained in $\dfrac{1}{\varpi^i}\mathscr{O}$ for some $i \geq 0$. We let $\mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E)$ denote the $\mathscr{O}$-submodule of $\mathrm{Hom}_{\mathscr{O}}(V, E)$ consisting of bounded maps.

A.6. Lemma. — *If A is an Artinian local ring, then an A-module is flat if and only if it is free.*

Proof. — Any free module is flat; we must prove the converse. Let $\mathfrak{m}$ denote the maximal ideal of A, and choose a basis $\{\bar{e}_i\}_{i\in I}$ of $V/\mathfrak{m}V$, as well as elements $e_i \in V$ lifting the basis elements $\bar{e}_i$. The elements e_i give rise to a map $\phi : A^{\oplus I} \to V$, which we will show is an isomorphism.

Let U denote the kernel of ϕ, and W its image. By construction $V = W + \mathfrak{m}V$. Iterating this inequality, and using the fact that $\mathfrak{m}^i = 0$ for sufficiently large i, we find that in fact $V = W$, i.e. that ϕ is surjective. Thus we have a short exact sequence

$$0 \longrightarrow U \longrightarrow A^{\oplus I} \xrightarrow{\phi} V \longrightarrow 0,$$

which when tensored with $A/\mathfrak{m}$ over A, yields a short exact sequence

$$0 \longrightarrow U/\mathfrak{m}U \longrightarrow A^{\oplus I} \xrightarrow{\overline{\phi}} V/\mathfrak{m}V \longrightarrow 0$$

(since V is flat over A, by assumption). Now $\overline{\phi}$ is an isomorphism, since the $\overline{e}_i$ form a basis of $V/\mathfrak{m}V$. Thus we find that $U/\mathfrak{m}U = 0$, and hence that $U = \mathfrak{m}U$. Iterating this inequality (and again using the fact that $\mathfrak{m}^i = 0$ for some i) shows that $U = 0$, and thus that ϕ is injective. Consequently ϕ is an isomorphism, as claimed. □

A.7. Lemma. — *If V is a ϖ-adically complete and separated, torsion free $\mathscr{O}$-module, then there is a short exact sequence*

$$0 \to \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E) \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty] \to 0.$$

Proof. — The short exact sequence $0 \to \mathscr{O} \to E \to E/\mathscr{O} \to 0$ gives rise to an exact sequence $0 \to \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V, E) \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})$, which in turn induces an exact sequence

$$\text{(A.8)} \qquad 0 \to \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E) \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty].$$

We will show that this sequence is also exact on the right.

To this end, let $\phi \in \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty]$, and suppose that ϕ is annihilated by ϖ^i, so that in fact

$$\phi : V/\varpi^i V \to \frac{1}{\varpi^i}\mathscr{O}/\mathscr{O}.$$

Since V is torsion free, and hence flat over $\mathscr{O}$, it follows that $V/\varpi^j V$ is flat over $\mathscr{O}/\varpi^j\mathscr{O}$, for each $j > 0$, and so Lemma A.6 implies that $V/\varpi^j V$ is free over $\mathscr{O}/\varpi^j\mathscr{O}$, for each $j > 0$. Proceeding inductively on j, we see that we may construct a projective system of maps $\phi_j : V/\varpi^j V \to \dfrac{1}{\varpi^i}\mathscr{O}/\varpi^{j-i}\mathscr{O}$, for each $j \geq i$, such that $\phi_i = \phi$. Passing to the projective limit in j, we obtain a map $V \to \dfrac{1}{\varpi^i}\mathscr{O} \subset E$, which is an element of $\mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E)$ lifting ϕ. □

A.9. Lemma. — *If V is an $\mathscr{O}$-module whose torsion submodule is of bounded exponent, then there is an exact sequence*

$$0 \to \mathrm{Hom}_{\mathscr{O}}(V_{\mathrm{fl}}, E/\mathscr{O})[\varpi^\infty] \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty] \to \mathrm{Hom}_{\mathscr{O}}(V[\varpi^\infty], E/\mathscr{O}) \to 0.$$

Proof. — Since $E/\mathscr{O}$ is an injective $\mathscr{O}$-module, the short exact sequence (A.3) gives rise to a short exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{O}}(V_{\mathrm{fl}}, E/\mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V[\varpi^\infty], E/\mathscr{O}) \to 0,$$

which in turn gives rise to an exact sequence

$$0 \to \mathrm{Hom}_{\mathscr{O}}(V_{\mathrm{fl}}, E/\mathscr{O})[\varpi^\infty] \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty] \to \mathrm{Hom}_{\mathscr{O}}(V[\varpi^\infty], E/\mathscr{O}).$$

We will show that this latter exact sequence is also short exact.

To this end, suppose that $\phi : V[\varpi^\infty] \to E/\mathscr{O}$. By assumption $V[\varpi^\infty] = V[\varpi^i]$ for some i, and so in fact $\phi : V[\varpi^\infty] \to \dfrac{1}{\varpi^i}\mathscr{O}/\mathscr{O}$. The short exact sequence (A.4) (taken with $j = i$) induces an embedding $V[\varpi^\infty] = V[\varpi^i] \hookrightarrow V/\varpi^i V$, and thus, using the fact that $\frac{1}{\varpi^i}\mathscr{O}/\mathscr{O}$ is injective over $\mathscr{O}/\varpi^i\mathscr{O}$, we may extend ϕ to a map $\phi' : V/\varpi^i V \to \frac{1}{\varpi^i}\mathscr{O}/\mathscr{O}$. Composing ϕ' with the projection $V \to V/\varpi^i V$ yields an element of $\mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty]$ which maps onto ϕ. $\square$

A.10. Proposition. — *If V is a ϖ-adically complete and separated $\mathscr{O}$-module whose torsion submodule is of bounded exponent, then there is an exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E) & \\ \to \mathrm{Hom}_{\mathscr{O}}(V, E/\mathscr{O})[\varpi^\infty] &\to \mathrm{Hom}_{\mathscr{O}}(V[\varpi^\infty], E/\mathscr{O}) \to 0. \end{aligned}$$

Proof. — Since $\mathscr{O}$ and E are torsion free, the surjection $V \to V_{\mathrm{fl}}$ induces isomorphisms $\mathrm{Hom}_{\mathscr{O}}(V_{\mathrm{fl}}, \mathscr{O}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O})$ and $\mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V_{\mathrm{fl}}, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V, E)$. Thus we obtain the required four term exact sequence by gluing together the short exact sequences obtained by applying Lemma A.7 to V_{fl} and Lemma A.9 to V. $\square$

A.11. Theorem. — *If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of ϖ-adically complete and separated $\mathscr{O}$-modules, the torsion submodule of each of which has bounded exponent, then there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{O}}(V_3, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_2, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_1, \mathscr{O}) & \\ \to \mathrm{Hom}_{\mathscr{O}}(V_3[\varpi^\infty], E/\mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_2[\varpi^\infty], E/\mathscr{O}) & \\ \to \mathrm{Hom}_{\mathscr{O}}(V_1[\varpi^\infty], E/\mathscr{O}) &\to 0. \end{aligned}$$

Proof. — The injective resolution $E \to E/\mathscr{O}$ of $\mathscr{O}$ gives rise to a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_3, E) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}}(V_2, E) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}}(V_1, E) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_3, E/\mathscr{O}) \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_2, E/\mathscr{O}) \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_1, E/\mathscr{O}) \arrow[r] & 0. \end{tikzcd}$$

Passing to bounded maps in the top row, and to torsion submodules in the bottom row, we obtain a morphism of exact sequences

(A.12)

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V_3, E) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V_2, E) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{O}-\mathrm{bd}}(V_1, E) \arrow[d] \\ 0 \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_3, E/\mathscr{O})[\varpi^\infty] \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_2, E/\mathscr{O})[\varpi^\infty] \arrow[r] & \mathrm{Hom}_{\mathscr{O}}(V_1, E/\mathscr{O})[\varpi^\infty]. \end{tikzcd}$$

The top row of this diagram is canonically identified with the exact sequence of continuous dual spaces

$$0 \to (E \otimes_{\mathscr{O}} V_3)' \to (E \otimes_{\mathscr{O}} V_2)' \to (E \otimes_{\mathscr{O}} V_1)'$$

associated to the exact sequence of Banach spaces

$$0 \to E \otimes_{\mathscr{O}} V_1 \to E \otimes_{\mathscr{O}} V_2 \to E \otimes_{\mathscr{O}} V_3 \to 0,$$

and so is exact on the right, by the Hahn–Banach theorem. (One could also prove this exactness directly, using the style of argument employed in the proof of Lemmas A.7 and A.9.)

We claim that the bottom row of (A.12) is also exact on the right. To see this, let $\phi \in \mathrm{Hom}_{\mathscr{O}}(V_1, E/\mathscr{O})[\varpi^\infty]$, and suppose that $\varpi^j \phi = 0$, so that in fact $\phi : V_1 \to \frac{1}{\varpi^j}\mathscr{O}/\mathscr{O}$. Now, if $i \geq 0$ is chosen so that $V_3[\varpi^\infty] = V_3[\varpi^i]$, then (as was noted in the proof of Lemma A.1) we have an inclusion $V_1 \bigcap \varpi^{i+j} V_2 \subset \varpi^j V_1$. Since $\frac{1}{\varpi^{i+j}}\mathscr{O}/\mathscr{O}$ is injective over $\mathscr{O}/\varpi^{i+j}\mathscr{O}$, we find that we may lift the composite map

$$V_1/V_1 \bigcap \varpi^{i+j} V_2 \longrightarrow V_1/\varpi^j V_1 \xrightarrow{\phi} \frac{1}{\varpi^i}\mathscr{O}/\mathscr{O} \subset \frac{1}{\varpi^{i+j}}\mathscr{O}/\mathscr{O}$$

to a map $\phi' : V_2/\varpi^{i+j} V_2 \to \frac{1}{\varpi^{i+j}}\mathscr{O}/\mathscr{O}$. The map ϕ' then gives an element of $\mathrm{Hom}_{\mathscr{O}}(V_2, E/\mathscr{O})[\varpi^\infty]$ which maps onto ϕ.

We have thus seen that both the top and bottom row of (A.12) are short exact. Applying the snake lemma, and taking into account Proposition A.10, we obtain the six term exact sequence in the statement of the theorem. □

We close the appendix by noting two further lemmas. They are both simple consequences of the Hahn–Banach theorem (and could also be deduced directly using arguments of the type used in the proof of Lemmas A.7 and A.9).

A.13. Lemma. — *Let $V_1 \to V_2$ be an $\mathscr{O}$-linear map of ϖ-adically complete and torsion free $\mathscr{O}$-modules. If the induced map $\mathrm{Hom}_{\mathscr{O}}(V_2, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_1, \mathscr{O})$ is surjective, then the given map is injective.*

Proof. — Let V denote the kernel of the given map; it is a ϖ-adically closed and saturated $\mathscr{O}$-submodule of V_1, and hence is again ϖ-adically complete. Thinking of V and V_1 as the unit balls of the corresponding E-Banach spaces $E \otimes_{\mathscr{O}} V$ and $E \otimes_{\mathscr{O}} V_1$, we see by the Hahn-Banach theorem that if $V \neq 0$ then the image of the restriction $\mathrm{Hom}_{\mathscr{O}}(V_1, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V, \mathscr{O})$ is non-zero, contradicting the hypothesis of the lemma. □

A.14. Lemma. — *Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of $\mathscr{O}$-modules, such that V_1 and V_2 are ϖ-adically complete and torsion free. If the cokernel of the induced map $\mathrm{Hom}_{\mathscr{O}}(V_2, \mathscr{O}) \to \mathrm{Hom}_{\mathscr{O}}(V_1, \mathscr{O})$ is torsion, with bounded exponent, then $V_3[\varpi^\infty]$ is also of bounded exponent.*

Proof. — Let $\widetilde{V_1}$ denote the saturation in V_2 of (the image of) V_1, i.e.

$$\widetilde{V_1} = \{v \in V_2 \,|\, \varpi^i \in V_1 \text{ for some } i \geq 0\}.$$

(To simplify the notation, we identify V_1 with its image in V_2.) There is an evident isomorphism $\widetilde{V_1}/V_1 \xrightarrow{\sim} V_3[\varpi^\infty]$, and hence we must show that $\widetilde{V_1}/V_1$ has bounded exponent. By assumption, there exists $j \geq 0$ such that if $\phi : V_1 \to \mathscr{O}$, then $\varpi^j \phi$ is the restriction of some $\psi : V_2 \to \mathscr{O}$. Let $v \in \widetilde{V_1}$, and choose $i \geq 0$ minimally such that $\varpi^i v \in V_1$. Suppose that $i > j$. Since $\varpi^i v \notin \varpi V_1$ (by the minimality of i), the Hahn–Banach theorem allows us to choose $\phi : V_1 \to \mathscr{O}$ such that $\phi(\varpi^i v) = 1$. Suppose that $\varpi^j \phi$ is the restriction of ψ, as above. We compute that $\varpi^i \psi(v) = \psi(\varpi^i v) = \varpi^j \phi(\varpi^i v) = \varpi^j$, and hence that $\psi(v) = \varpi^{j-i}$. In particular we conclude that $\varpi^{j-i} \in \mathscr{O}$, and thus that $i \leq j$, a contradiction. Consequently it must be that $V_3[\varpi^\infty] = V_3[\varpi^j]$, proving the lemma. $\square$

References

[1] L. Barthel & R. Livné – "Irreducible modular representations of GL_2 of a local field", *Duke Math. J.* **75** (1994), p. 261–292.

[2] C. Breuil – "Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I", *Compositio Math.* **138** (2003), p. 165–188.

[3] W. Casselman – "Introduction to the theory of admissible representations of p-adic reductive groups", unpublished notes distributed by P. Sally, draft May 1 1995, `http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`.

[4] P. Colmez – "Représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ,Γ)-modules", *Astérisque* **330** (2010), p. 281–509.

[5] M. Emerton – "Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties", *Ann. Sci. École Norm. Sup.* **39** (2006), p. 775–839.

[6] ______, "On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms", *Invent. Math.* **164** (2006), p. 1–84.

[7] ______, "Jacquet modules of locally analytic representations of p-adic reductive groups II. The relation to parabolic induction", preprint `http://www.math.northwestern.edu/~emerton/pdffiles/jacquet-two.pdf`, 2007.

[8] ______, "Local-global compatibility in the p-adic langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$", in preparation.

[9] ______, "Locally analytic vectors in representations of locally p-adic analytic groups", to appear in *Memoirs of the Amer. Math. Soc.*

[10] ______, "Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors", this volume.

[11] H. Hida – "Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms", *Invent. Math.* **85** (1986), p. 545–613.

[12] ______, "p-ordinary cohomology groups for $\mathrm{SL}(2)$ over number fields", *Duke Math. J.* **69** (1993), p. 259–314.

[13] ______, "Control theorems of p-nearly ordinary cohomology groups for $\mathrm{SL}(n)$", *Bull. Soc. Math. France* **123** (1995), p. 425–475.

[14] M. LAZARD – "Groupes analytiques p-adiques", *Publl. Math. I.H.É.S.* **26** (1965), p. 389–603.

[15] P. SCHNEIDER & J. TEITELBAUM – "Banach space representations and Iwasawa theory", *Israel J. Math.* **127** (2002), p. 359–380.

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