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CM STABILITY AND THE GENERALIZED FUTAKI INVARIANT II

by

Sean Timothy Paul & Gang Tian

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — The Mabuchi K-energy map is exhibited as a singular metric on the refined CM polarization of any equivariant family $\mathbf{X} \xrightarrow{p} S$. Consequently we show that the generalized Futaki invariant is the leading term in the asymptotics of the reduced K-energy of the generic fiber of the map p. Properness of the K-energy implies that the generalized Futaki invariant is strictly negative.

***Résumé* (CM-stabilité et invariant de Futaki généralisé II).** — On interpréte la K-énergie de Mabuchi comme une métrique singulière sur la CM-polarisation raffinée d'une famille équivariante $\mathbf{X} \xrightarrow{p} S$. Nous montrons que l'invariant de Futaki généralisé est le terme principal de l'asymptotique de la K-énergie réduite de la fibre générique de l'application p. Si la K-énergie est propre, alors l'invariant de Futaki généralisé est strictement négatif.

1. Introduction

1.1. Statement of results. — Throughout this paper $\mathbf{X}$ and S denote smooth, proper complex projective varieties satisfying the following conditions.

1. $\mathbf{X} \subset S \times \mathbb{P}^N$; $\mathbb{P}^N$ denotes the complex projective space of *lines* in $\mathbb{C}^{N+1}$.
2. $p := p_1 : \mathbf{X} \to S$ is flat of relative dimension n, degree d with Hilbert polynomial P.
3. $L|_{\mathbf{X}_z}$ is very ample and the embedding $\mathbf{X}_z := p_1^{-1}(z) \overset{L}{\hookrightarrow} \mathbb{P}^N$ is given by a complete linear system for $z \in S$.

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4. There is an action of $G := SL(N+1,\mathbb{C})$ on the data compatible with the projection and the standard action on $\mathbb{P}^N$.

It is well known that (1) and (3) imply that

$$\mathbb{P}(p_{1*}L) \cong S \times \mathbb{P}^N. \tag{1.1}$$

Which in turn is equivalent to the existence of a line bundle $\mathscr{C}$ on S such that

$$p_{1*}L \cong \underbrace{\bigoplus}_{N+1} \mathscr{C}. \tag{1.2}$$

Below Chow($\mathbf{X}/S$) denotes the Chow form of the family $\mathbf{X}/S$, μ is the coefficient of k^{n-1} in $P(k)$, and $\mathscr{M}_n$ is the coefficient of $\binom{m}{n}$ in the CGKM expansion of $\det(p_{1*}L^{\otimes m})$ for $m >> 0$. A complete discussion of these notions is given in "*CM Stability and the Generalized Futaki Invariant I*". We refer the reader to that paper for the basic definitions and constructions that are used in the present article.

We define an invertible sheaf on S as follows.

Definition 1 **(The Refined CM polarization [1])**. — *We have*

$$\mathbb{L}_1(\mathbf{X}/S) := \{\mathrm{Chow}(\mathbf{X}/S) \otimes \mathscr{C}^{d(n+1)}\}^{n(n+1)+\mu} \otimes \mathscr{M}_n^{-2(n+1)} \tag{1.3}$$

With the family $p_1 : \mathbf{X} \to S$ fixed throughout, we will denote $\mathbb{L}_1(\mathbf{X}/S)$ by $\mathbb{L}_1$ in the remainder of the paper.

Our first result exhibits the Mabuchi energy as a *singular* Hermitian metric on $\mathbb{L}_1$.

Theorem 1. — *Let $\|\ \|$ be any smooth Hermitian metric on $\mathbb{L}_1^{-1}$.*[2] *Then there is a continuous function $\Psi_S : S \setminus \Delta \to (-\infty, c)$ such that for all $z \in S/\Delta$*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{\|\ \|^2(\sigma z)}{\|\ \|^2(z)}\right). \tag{1.4}$$

Here c denotes a constant which depends only on the choice of background Kähler metrics on S and $\mathbf{X}$, Δ denotes the discriminant locus of the map p_1, and $\omega|_{\mathbf{X}_z}$ denotes the restriction of the Fubini Study form of $\mathbb{P}^N$ to the fiber $\mathbf{X}_z$.

Remark 1. — *This should be compared with the main result in Section 8 of* [**17**]. *The principal contribution of our present work is the observation that the whole theory in Section 8 of* [**17**] *should be recast from the beginning with the sheaf $\mathbb{L}_1$.*

Let $X \hookrightarrow \mathbb{P}^N$ be an n dimensional projective variety with Hilbert polynomial P. Let $Hilb_m(X)$ denote the mth Hilbert point of X (see [**12**] for further information). If λ is a one parameter subgroup of G then it is known (see [**12**]) that the weight,

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(1) We use this terminology in order to distinguish this sheaf from one introduced by the second author in ([**17**]).
(2) $\mathbb{L}_1^{-1}$ denotes the dual of $\mathbb{L}_1$.

$w_{\lambda}(m)$, of $Hilb_m(X)$ with respect to λ is a *polynomial* in m of degree at most $n+1$. That is,

$$w_{\lambda}(m) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + \cdots$$

Then the ratio may be expanded as follows.

$$\frac{w_{\lambda}(m)}{mP(m)} = F_0(\lambda) + F_1(\lambda)\frac{1}{m} + \cdots + F_l(\lambda)\frac{1}{m^l} + \ldots$$

***Definition 2* (Donaldson ([5])).** — *$F_1(\lambda)$ is the generalized Futaki invariant of X with respect to λ.*

In our previous paper we have shown the following.

***Theorem* (The weight of the Refined CM polarization).** — i) *There is a natural G linearization on the line bundle $\mathbb{L}_1$.*

ii) *Let λ be a one parameter subgroup of G. Let $z \in \mathfrak{Hilb}^P_{\mathbb{P}^N}(\mathbb{C})$. Let $w_{\lambda}(z)$ denote the weight of the restricted $\mathbb{C}^*$ action (whose existence is asserted in i)) on $\mathbb{L}_1^{-1}|_{z_0}$ where $z_0 = \lambda(0)z$. Then*

$$w_{\lambda}(z) = F_1(\lambda). \tag{1.5}$$

The main result of the paper is the following corollary of (1.4) and (1.5).

***Corollary 1* (Algebraic asymptotics of the Mabuchi energy).** — *Let $\varphi_{\lambda(t)}$ be the Bergman potential associated to an algebraic 1psg λ of G, and let $z \in S \setminus \Delta$. Then there is an asymptotic expansion*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_{\lambda(t)}) - \Psi_S(\lambda(t)) = F_1(\lambda)\log(|t|^2) + O(1) \text{ as } |t| \to 0. \tag{1.6}$$

Moreover $\Psi_S(\lambda(t)) = \psi(\lambda)\log(|t|^2) + O(1)$ where $\psi(\lambda) \in \mathbb{Q}_{\geq 0}$. Moreover, $\psi(\lambda) \in \mathbb{Q}_+$ if and only if $\lambda(0)\mathbf{X}_z = \mathbf{X}_{\lambda(0)z}$ (the limit cycle[(3)] of $\mathbf{X}_z$ under λ) has a component of multiplicity greater than one. Here $O(1)$ denotes any quantity which is bounded as $|t| \to 0$.

Moser iteration and a refined Sobolev inequality (see [**11**] and [**7**]) yield the following.

Corollary 2. — *If $\nu_{\omega|_{\mathbf{X}_z}}$ is proper (bounded from below) then the generalized Futaki invariant of $\mathbf{X}_z$ is strictly negative (nonnegative) for all $\lambda \in G$.*

Remark 2. — *We call the left hand side of (1.6) the* reduced *K-Energy along λ. We also point out that while it is certainly the case that $F_1(\lambda)$ may be defined for any subscheme of $\mathbb{P}^N$ it evidently only controls the behavior of the K-Energy when $\lambda(0)\mathbf{X}_z$ is reduced.*

(3) See [**12**] pg. 61.

Remark 3. — *The precise constant* $d(n+1)$ *in front of* ν_ω *is not really crucial, since what really matters is the sign* of $F_1(\lambda)+\psi(\lambda)$. *That* $\Psi_S(\lambda(t))$ *has logarithmic singularities can be deduced from* [**13**].

Remark 4. — *We emphasize that we do not assume the limit cycle is smooth.*

2. Background and Motivation

Let (X,ω) be a compact Kähler manifold (ω not necessarily a Hodge class) and $P(X,\omega) := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\}$ the space of Kähler potentials. This is the usual description of all Kähler metrics in the same class as ω (up to translations by constants). It is not an overstatement to say that the most basic problem in Kähler geometry is the following

Does there exist $\varphi \in P(X,\omega)$ *such that* $\mathrm{Scal}(\omega_\varphi) \equiv \mu$? $(*)$

This is a fully nonlinear *fourth order* elliptic partial differential equation for φ. μ is a constant, the average of the scalar curvature, it depends only on $c_1(X)$ and $[\omega]$. When $c_1(X)>0$ and ω represents the *anticanonical* class a simple application of the Hodge Theory shows that $(*)$ is equivalent to the *Monge-Ampere equation.*

$$\frac{\det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{F-\kappa\varphi} \quad (\kappa = 1) \qquad (**)$$

where F denotes the Ricci potential. When $\kappa = 0$ this is the celebrated Calabi problem solved by S.T.Yau and when $\kappa < 0$ this was solved by Aubin and Yau independently in the 70's. It is well known that $(*)$ is actually a *variational* problem. There is a natural energy on the space $P(X,\omega)$ whose critical points are those φ such that ω_φ has constant scalar curvature (csc). This energy was introduced by T. Mabuchi ([**10**]) in the 1980's. It is called the *K-Energy map* (denoted by ν_ω) and is given by the following formula

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1\int_X \dot{\varphi}_t(\mathrm{Scal}(\varphi_t)-\mu)\omega_t^n dt.$$

Above, φ_t is a smooth path in $P(X,\omega)$ joining 0 with φ. The K-Energy does not depend on the path chosen. In fact there is the following well known formula for ν_ω where $O(1)$ denotes a quantity which is bounded on $P(X,\omega)$.

$$\nu_\omega(\varphi) = \int_X \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\frac{{\omega_\varphi}^n}{V} - \mu(I_\omega(\varphi)-J_\omega(\varphi)) + O(1)$$

$$J_\omega(\varphi) := \frac{1}{V}\int_X \sum_{i=0}^{n-1}\frac{\sqrt{-1}}{2\pi}\frac{i+1}{n+1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i\wedge{\omega_\varphi}^{n-i-1}$$

$$I_\omega(\varphi) := \frac{1}{V}\int_X \varphi(\omega^n - {\omega_\varphi}^n).$$

We have written down the K-energy in the case when $\omega = c_1(X)$. Observe that ν_ω is essentially the *difference* of two positive terms. What is of interest for us is that the problem $(*)$ is not only a variational problem but a *minimization* problem. With this said we have the following fundamental result.

***Theorem* (S. Bando and T. Mabuchi [1]).** — *If $\omega = c_1(X)$ admits a Kähler Einstein metric then $\nu_\omega \geq 0$. The absolute minimum is taken on the solution to $(**)$ (which is unique up to automorphisms of X).*

Therefore a *necessary* condition for the existence of a Kähler Einstien metric is a bound from below on ν_ω. In order to get a *sufficient* condition one requires that the K-energy *grow* at a certain rate. Precisely, it is required that the K-Energy be *proper.* This concept was introduced by the second author in [**17**].

Definition 3. — *ν_ω is **proper** if there exists a strictly increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (where $\lim_{T\longrightarrow\infty} f(T) = \infty$) such that $\nu_\omega(\varphi) \geq f(J_\omega(\varphi))$ for all $\varphi \in P(M, \omega)$.*

***Theorem* ([17]).** — *Assume that $Aut(X)$ is discrete. Then $\omega = c_1(X)$ admits a Kähler Einstein metric if and only if ν_ω is proper.*

The next result was established by the second author and Xiuxiong Chen. It holds in an *arbitrary* Kähler class ω. An alternative proof of this was given by Donaldson for polarized projective manifolds.

***Theorem* ([3]).** — *If ω admits a metric of csc then $\nu_\omega \geq 0$.*

In this paper our interest is to test for a lower bound of ν_ω along the large but finite dimensional group G of *matrices* in the polarized case. When we restrict our attention to G we make the connection with Mumfords' Geometric Invariant Theory. The past couple of years have witnessed quite a bit of activity on this problem due to this connection.

To put things in historical perspective consider the various formulations of the Futaki invariant.

i) 1983 Futaki ([**6**]) introduces his invariant as a lie algebra character on a Fano manifold X

$$F_\omega : \eta(X) \longrightarrow \mathbb{C}.$$

ii) 1986 Mabuchi (see [**10**]) integrates the Futaki invariant with the introduction of the K-energy map. The linearization of the K-energy along orbits of holomorphic vector fields is the real part of the Futaki invariant.

iii) 1992 Ding and Tian ([**4**]) introduced the *generalized* Futaki invariant. Here the *jumping of complex structures* is introduced. The limit of the derivative of the K-Energy map is identified with the generalized Futaki invariant of $X^{\lambda(0)}$ provided this limit has at most *normal* singularities.

iv) 1997 The CM polarization is defined (see [**17**]) for *smooth* families, as the relative canonical bundle is explicitly involved in the definition. K-Stability is defined in terms of special degenerations and the generalized Futaki invariant.

v) 1999 Yotov formulated the generalized Futaki Invariant in terms of equivariant Chow groups of a *normal* variety.

vi) 2002 For an *arbitrary scheme* Donaldson ([**5**]) defined the weight $F_1(\lambda)$. This is identified with the limit of the derivative of K-energy (by [**4**]) when the limit cycle is a *smooth* (or normal) scheme.

***Remark* 5.** — *We hope that we have clarified the role of the CM polarization. The main point is that once the CM polarization is extended to the Hilbert scheme ([**14**]) the polarization computes the precise asymptotics of the K-energy of any generic fiber of the map* $\mathbf{X} \to S$. *This extension was made possible by an application of the Knudsen Mumford expansion of the determinant of direct images of perfect complexes of sheaves (see [**8**]). In fact,* $\psi(\lambda)$ *already appeared in work of the second author (see [**17**]). Despite this, the role of* $\psi(\lambda)$ *becomes more precise in the present work.*

3. Algebraic potentials

In order to connect these notions to the K-Energy map we now give an account of how to associate an admissible potential $\varphi_{\lambda(t)}$ to a one parameter subgroup of G. In order to detect properness (conjecturally) one restricts attention to the subspace of *Bergman metrics* inside $P(M,\omega)$ since these metrics are *dense* in $P(M,\omega)$ (see [**16**], [**15**], [**19**], [**2**]). By definition these metrics are induced by the Kodaira embeddings furnished by the polarization L. The construction is as follows. We have an embedding

$$X \xrightarrow[L]{} \mathbb{P}(H^0(X,L)^*) = \mathbb{P}^N$$

furnished by some basis $\{S_0,\ldots,S_N\}$ of $H^0(X,L)$. Observe that with the natural Hermitian metric on $H^0(X,L)$, the induced Fubini-Study metric on $\mathbb{P}^N$ is related to the curvature of the initial metric on L by the formula

$$\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^{N}||S_i||^2\right).$$

We conclude that

$$\log\left(\sum_{i=0}^{N}||S_i||^2\right) \in P(X,\omega).$$

Let $\sigma \in SL(N+1,\mathbb{C})$, then

$$\sigma^*(\omega_{FS}) = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\sigma.$$

Where φ_σ is given by the formula

$$\varphi_\sigma = \log\left(\frac{||\sigma z||^2}{||z||^2}\right).$$

We let $\{T_0, \dots T_N\}$ denote the corresponding change of basis

$$\begin{pmatrix} \sigma_{00} & \dots & \sigma_{0N} \\ \sigma_{10} & \dots & \sigma_{1N} \\ \dots & \dots & \dots \\ \sigma_{N0} & \dots & \sigma_{NN} \end{pmatrix} \begin{pmatrix} S_0 \\ \dots \\ \dots \\ S_N \end{pmatrix} = \begin{pmatrix} T_0 \\ \dots \\ \dots \\ T_N \end{pmatrix}.$$

Then we have

$$\varphi_\sigma|_X = \log\left(\frac{\sum_{i=0}^N ||T_i||^2}{\sum_{i=0}^N ||S_i||^2}\right).$$

Putting these ingredients together gives

$$\sigma^*\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^N ||T_i||^2\right). \tag{3.1}$$

Therefore, if we *fix* a basis of $H^0(X, L)$ we get a natural map

$$SL(N+1, \mathbb{C}) \to P(X, \omega).$$

A *one parameter subgroup* of $SL(N+1, \mathbb{C})$ is an algebraic [4] homomorphism

$$\lambda : \mathbb{C}^* \to SL(N+1, \mathbb{C}).$$

Any such $\lambda(t)$ can be diagonalised. That is, we may assume that $\lambda(t)$ takes values in the standard maximal torus $H \cong (\mathbb{C}^*)^N$ of $SL(N+1, \mathbb{C})$.

$$\lambda(t) = \begin{pmatrix} t^{m_0} & \dots & \dots & 0 \\ 0 & t^{m_1} & \dots & 0 \\ 0 & \dots & \dots & t^{m_N} \end{pmatrix}.$$

The exponents m_i satisfy

$$\sum_{0 \le i \le N} m_i = 0.$$

We arrive at the following formula.

$$\varphi_{\lambda(t)}(z) := \log\left(\sum_{0 \le j \le N} |t|^{2m_j} ||S_j||^2(z)\right).$$

Now we may consider the K-energy map as a function on $SL(N+1, \mathbb{C})$.

[4] "Algebraic" means that the matrix coefficients $\lambda(t)_{i,j} \in \mathbb{C}[t, t^{-1}]$.

4. Singular Hermitian metrics

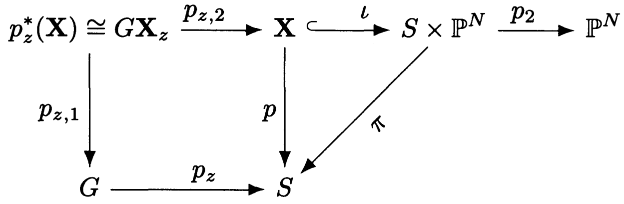
Proof of Theorem 1. — In part I of this work the authors provided the following formula for the first Chern class of $\mathbb{L}_1$.

(4.1)
$$c_1(\mathbb{L}_1) = p_{1*}\left((n+1)c_1(K_{\mathbf{X}/S})c_1(L)^n + \mu c_1(L)^{n+1}\right) \quad K_{\mathbf{X}/S} := K_{\mathbf{X}} \otimes p_1^*(K_S^{\vee}).$$

(4.1) allows us to exhibit the K-energy map as a *singular* metric on the CM polarization (see [**17**]). Recall that $p^{-1}(z) = \mathbf{X}_z \subset \mathbb{P}^N$, where $z \in S_\infty := S \setminus \Delta$. We define

$$G\mathbf{X}_z := \{(\sigma, y) \in G \times \mathbb{P}^N : y \in \sigma\mathbf{X}_z\}.$$

Observe that $G\mathbf{X}_z$ is biholomorphic to $G \times \mathbf{X}_z$. Then we have the following diagram, where p_z denotes the evaluation map, i.e. $p_z(\sigma) := \sigma z$.



Given $z \in B \setminus \Delta$ we can consider $K_{\mathbf{X}_z}$, the canonical bundle of the fiber $\mathbf{X}_z$. These fit together holomorphically into a line bundle K_∞ on $\mathbf{X} \setminus p^{-1}(\Delta)$. On the other hand, the relative canonical bundle K_p of the map p exists and lives on *all* of $\mathbf{X}$.

$$K_p := K_{\mathbf{X}} \otimes p^* K_S^{-1}$$

When we restrict this sheaf to $\mathbf{X} \setminus p^{-1}(\Delta)$ we have an isomorphism

$$K_p \cong K_\infty.$$

$\iota^* p_2^* \omega_{FS}$ restricts to a Kähler metric on $p^{-1}(z)$ ($z \in S_\infty$) and hence induces a Hermitian metric on the bundle K_∞. We denote its curvature by $R(\iota^* p_2^*(\omega_{FS}))$. Let $g_{\mathbf{X}}$ and g_S denote two Kähler metrics on $\mathbf{X}$ and S respectively. In this way we obtain a metric on the relative canonical bundle K_p. We let R_f denote its curvature

$$R_p := R(g_{\mathbf{X}}) - p^* R(g_S).$$

In this way we obtain *two* metrics on the relative canonical bundle over the smooth locus. The crucial point is the following fact.

The curvatures of these metrics are not the same.

The relation between them is given in the following proposition (see [**17**] Lemma 8.5 pg. 31).

***Proposition 1* ("$\partial\bar{\partial}$ lemma along the fibers").** — *There is a smooth function $\Psi : \mathbf{X} \setminus p^{-1}(\Delta) \to \mathbb{R}$ such that*
1) $R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi = R(\iota^* p_2^*(\omega_{FS}))$;
2) $\Psi \leq C$, for some constant C.

Example 1. — *(The universal family of hypersurfaces of degree d in $\mathbb{C}P^{n+1}$)*

$$S := \mathbb{P}(H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d)))$$
$$\mathbf{X} := \{([f],[z]) \in S \times \mathbb{C}P^{n+1} \; | f(z) = 0\}$$
$$p := p_1 \quad (\textit{projection onto the first factor}).$$

Let $|||.|||$ denote any norm on $H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d))$, with associated Fubini-Study metric ω_S. Then a computation shows that

$$\Psi(([f],[z])) = \log\left(\frac{\sum_{i=0}^{n+1} |\frac{\partial f}{\partial z_i}(z)|^2}{|||f|||^2||z||^{2(d-1)}}\right).$$

The next result is a *pointwise* version of (4.1).

Proposition 2. — *There is a continuous Hermitian metric $|| \; ||$ on $\mathbb{L}_1^{-1}$ such that, in the sense of currents we have*

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(|| \; ||^2) = (n+1)p_*(R(g_{\mathbf{X}}) - p^*R(g_S))p_2^*(\omega_{FS})^n + \mu p_* p_2^*(\omega_{FS})^{n+1}.$$

Proof. — See Proposition 4.3 pg. 2576 of [**13**]. □

Now we pull back the curvature form of K_∞ to $G\mathbf{X}_z$

$$R_{G|\mathbf{X}_z} := p_{z,2}^*(R(\pi_2^*(\omega_{FS}))).$$

Recall that for $\sigma \in G$ we define φ_σ by the relation

$$\sigma^*\omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\sigma.$$

Let $\nu_{\omega,z}(\sigma)$ denote the K energy of $(\mathbf{X}_z, \omega_{FS})$ applied to the potential φ_σ. With these notations in place we have the following result.

***Proposition 3* (The complex Hessian of the K-Energy map on G)**
For every smooth compactly supported (N^2+2N-1, N^2+2N-1) form η on G we have

$$d(n+1)\int_G \nu_{\omega,z}(\varphi_\sigma)\partial\bar{\partial}\eta = \int_{G\mathbf{X}_z} ((n+1)R_{G|\mathbf{X}_z} + \mu p_2^*(\omega_{FS})) \wedge p_2^*(\omega_{FS})^n \wedge p_{z,1}^*\eta.$$

The proof of Proposition 3 appears in the next section after some standard preliminaries on Bott Chern classes.

4.1. Bott Chern secondary classes. — Let ϕ be a $GL_N(\mathbb{C})$ invariant polynomial on $M_{N\times N}(\mathbb{C})$ homogeneous of degree d. ϕ_1 denotes the *complete polarization* of ϕ. Let E be a holomorphic vector bundle of rank N over a base X. Let h_1 and h_0 be two Hermitian metrics on E and $\frac{\sqrt{-1}}{2\pi}R(h_i)$ the curvatures. Then we define the *Bott-Chern class* $BC(\phi, E; h_0, h_1)$ by the expression

$$BC(\phi, E; h_0, h_1) := \int_0^1 \phi_1(h_t^{-1}\dot{h}_t, \overbrace{\frac{\sqrt{-1}}{2\pi}R_t, \dots, \frac{\sqrt{-1}}{2\pi}R_t}^{d-1})dt \in \Omega_X^{(d-1,d-1)} \tag{4.2}$$

where h_t is any piecewise C^1 path of Hermitian metrics joining h_0 and h_1. The point of the construction is the following identity:

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}BC(\phi, E; h_0, h_1) = \left(\frac{\sqrt{-1}}{2\pi}\right)^d (\phi(R_{h_0}) - \phi(R_{h_1})).$$

Let $d = n + 1$ where, $n = dim(X)$ in this case $BC(\phi, E; h_0, h_1)$ has top dimension and we may introduce the *Donaldson Functional associated to* ϕ.

$$D_E(h_0, h_1) := \int_X BC(\phi, E; h_0, h_1). \tag{4.3}$$

When h_0 is fixed, we consider it to be a functional on $\mathcal{M}_E$ (the space of hermitian metrics on E). In what follows we take $\phi = Ch_{n+1}$, the $n+1^{st}$ component of the chern character. We can extend the Donaldson functional to "virtual bundles" $\mathcal{E} = E - F$ by observing that a Hermitian metric h on $\mathcal{E}$ is just a pair of metrics, one on E and one on F:

$$h = (h^E, h^F).$$

We set

$$BC(\phi, \mathcal{E}; h_0, h_1) := BC(\phi, E; h_0^E, h_1^E) - BC(\phi, F; h_0^F, h_1^F). \tag{4.4}$$

Let $h : Y \to \mathcal{M}_{\mathcal{E}}$ be a smooth map, where Y is a complex manifold of dimension m.

Lemma 4.1. — *Let ϕ be homogeneous of degree $n + 1$ and h_0 a fixed metric on $\mathcal{E}$. Then for all smooth compactly supported forms ψ of type $(m - 1, m - 1)$ we have the identity*

$$\frac{\sqrt{-1}}{2\pi}\int_Y D_{\mathcal{E}}(\phi; h_0, h(y))\partial_Y\overline{\partial}_Y\psi = \int_{Y\times X} \phi(R(\frac{\sqrt{-1}}{2\pi}h(y))) \wedge \pi_1^*(\psi). \tag{4.5}$$

Next we want to realize the Mabuchi K-energy as the Donaldson functional, with respect to the polynomial $\phi = Ch_{n+1}$, of a certain virtual bundle to be defined below. Then proposition (3) follows at once from the preceding lemma.

Let X be a complex projective manifold (in our present application X is a smooth fiber of $\mathbf{X} \xrightarrow{p} S$), and let L be the restriction of $\underline{O}(1)$ to X. Let φ be a kahler potential.

The two metrics h_{FS} and $e^{-\varphi}h_{FS}$ induce metrics on the canonical bundle $\mathcal{K}$. We consider the virtual bundle

$$2^{n+1}\mathcal{E} := (n+1)(\mathcal{K}^{-1} - \mathcal{K})(L - L^{-1})^n - \mu(L - L^{-1})^{n+1}. \tag{4.6}$$

Here μ is the average of the scalar curvature. We need to calculate the following terms.

$$\begin{aligned} &BC(\phi; \mathcal{K}^{-1} \otimes L^{n-2j}, h_0, h_1) \\ &BC(\phi; \mathcal{K} \otimes L^{n-2j}, h_0, h_1) \\ &BC(\phi; L^{n+1-2j}, h_0, h_1). \end{aligned} \tag{4.7}$$

The path of metrics for the first two expression are given as follows.

$$\begin{aligned} h_{\mathcal{K}^{-1}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha \partial \overline{z_\beta}}\varphi)e^{-t(n-2j)\varphi}h_{FS}^{n-2j} \\ h_{\mathcal{K}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha \partial \overline{z_\beta}}\varphi)^{-1}e^{-t(n-2j)\varphi}h_{FS}^{n-2j}. \end{aligned} \tag{4.8}$$

The complete polarization of ϕ is given by

$$\phi_1(B, A \dots A) = tr(BA^n) \quad A, B \in M_k(\mathbb{C}). \tag{4.9}$$

Therefore,

(4.10)

$$\begin{aligned} BC(\mathcal{K}^{-1} \otimes L^{n-2j}, h_0, h_1) &= \int_0^1 (\Delta_{t\varphi}\varphi - (n-2j)\varphi)((n-2j)\omega_{t\varphi} + Ric_{\omega_t})^n dt \\ BC(\mathcal{K} \otimes L^{n-2j}, h_0, h_1) &= -\int_0^1 (\Delta_{t\varphi}\varphi + (n-2j)\varphi)((n-2j)\omega_{t\varphi} - Ric_{\omega_t})^n dt. \end{aligned}$$

Similarly we have

$$BC(L^{n+1-2j}, h_0, h_1) = -(n+1-2j)^{n+1}\int_0^1 \varphi\omega_t^n dt \quad \omega_t := \omega + t\partial\overline{\partial}\varphi. \tag{4.11}$$

We see that

(4.12)

$$BC((L - L^{-1})^{n+1}, h_{FS}, e^{-\varphi}h_{FS}) = -\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^{n+1}\int_0^1 \varphi\omega_t^n dt.$$

Now we need the following numerical identity.

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^i = \begin{cases} 0 & i < n+1 \text{ or } i = n+2 \\ (n+1)!2^{n+1} & i = n+1. \end{cases} \tag{4.13}$$

It follows at once that

$$\int_X BC((L - L^{-1})^{n+1}, h_{FS}, e^{-\varphi}h_{FS}) = -(n+1)!2^{n+1}\int_0^1\int_X \varphi\omega_t^n dt. \tag{4.14}$$

It follows from (3.9) that

$$BC(\mathcal{K}^{-1}\otimes L^{n-2j}) =$$
$$\int_0^1 \Delta_{t\varphi}\varphi\sum_{i=0}^{n}\binom{n}{i}(n-2j)^i Ric_t^{n-i}\omega_t^i - \int_0^1\sum_{i=0}^{n}\binom{n}{i}(n-2j)^{i+1}\varphi Ric_t^{n-i}\omega_t^i.$$

We use the identity (4.13) to see that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}BC(\mathcal{K}^{-1}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi nRic_t\omega_t^{n-1}\right)dt. \tag{4.15}$$

Similarly we have the second term

$$\sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}BC(\mathcal{K}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi nRic_t\omega_t^{n-1}\right)dt. \tag{4.16}$$

The next lemma follows at once from summing up (4.15), (4.16), and (4.14).

Lemma 4.2. — *Let $D(\mathcal{E},h_{FS},e^{-\varphi}h_{FS})$ denote the Donaldson functional of Ch_{n+1} with respect to $\mathcal{E}$. Then the following identity holds.*

$$D(\mathcal{E},h_{FS},e^{-\varphi}h_{FS}) = \nu_\omega(\varphi) \tag{4.17}$$

Let $\varphi=\varphi_\sigma$ and apply 4.5 to Lemma 4.2 to conclude the proof of Proposition 3. □

Next we observe that the identity

$$R_{G|\mathbf{X}_z} = p_{2,z}^*\left(R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\Psi\right) \tag{4.18}$$

together with the previous lemmas yields the following corollary.

Corollary 3. — *The function*

$$\sigma\in G\to D(\sigma) := d(n+1)\nu_{\omega,z}(\sigma) - \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{\|\ \|^2(\sigma z)}{\|\ \|^2(z)}\right)$$

is pluriharmonic. Where we have defined $\Psi_S(z) := \int_{\{y\in f^{-1}(z)\}}\Psi(y)p_2^*(\omega_{FS})^n$.

Moreover $\Psi_S(z)\le C$ on $S\setminus\Delta$, extends continuously to the locus of reduced fibers, and $\lim_{z\to z_\infty}\Psi_S(z) = -\infty$ whenever $\mathbf{X}_{z_\infty}$ is non-reduced.

Remark 6. — *The construction of Ψ and Ψ_S as well as their behavior on the locus of singular fibers can be seen directly in Example 1. The general case is treated in Lemma 8.5 pg. 31 in* [**17**].

Since $\pi_1(G)=1$ there is a (nonvanishing) entire function ξ on G such that

$$D(\sigma) = \log(|\xi(\sigma)|^2).$$

An analysis of the growth of this function on the standard compactification $\overline{G}$

$$\overline{G} := \{[(w_{ij},z)]\in\mathbb{P}^{(N+1)^2} : \det(w_{ij}) = z^{N+1}\}$$

reveals that it must reduce to a *constant.*

Tying everything together establishes our main result.

***Theorem 1* (The K-Energy as a singular metric on $\mathbb{L}_1^{-1}$).** — *We have*

$$d(n+1)\nu_{\omega,z}(\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right). \tag{4.19}$$

We proceed to the proof of Corollary 1. First substitute $\sigma = \lambda(t)$ in (4.19). Then we have the string of identities.

$$\begin{aligned}
d(n+1)\nu_{\omega,z}(\lambda(t)) &= \log\left(e^{(n+1)\Psi_S(\lambda(t)z)}\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(t^{w_\lambda(z)-w_\lambda(z)}\lambda(t)z)}{||\ ||^2(z)}\right)\\
&= (n+1)\Psi_S(\lambda(t)z) + w_\lambda(z)\log(|t|^2) + O(1)\\
&= F_1(\lambda)\log(|t|^2) + (n+1)\Psi_S(\lambda(t)z) + O(1).
\end{aligned}$$

The passage from line 3 to 4 follows from the defining property of the weight (see the introduction to [**14**]). The passage from line 4 to 5 is the statement of (1.5).

Rationality of the contribution from $\Psi_S(\lambda(t)z)$ follows easily from [**13**] Theorem 3.5 pg. 2564 and Zhiqin Lu's explicit computation of the asymptotics of the K-Energy on hypersurfaces (see [**9**]). This completes the proof of Corollary 1. □

4.2. Properness Implies that $F_1(\lambda) < 0$. — Let $X := \mathbf{X}_z$ a smooth fiber of p. Recall that the algebraic potential associated to a one parameter subgroup λ is given by

$$\varphi_t := \varphi_{\lambda(t)} = \log\Big(\sum_{i=0}^{N} t^{2q_i}||S_i||^2\Big).$$

Then, as we have seen, $\varphi_t \in P(X,\omega)$. Following Yau [**18**], our plan is to use the standard Moser iteration to control $Osc(\varphi_t)$ by $I_\omega(\varphi_t)$. Define

$$\varphi_- := \mathrm{Max}\{-\varphi_t, 1\} \geq 1.$$

Let $p \in \mathbb{Z}_+$. Then we have the (obvious) inequality

$$\varphi_-^p\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\wedge\omega_\varphi^{n-1} \leq \varphi_-^p\omega_\varphi^n.$$

Trivially this implies

$$\int_X \varphi_-^p\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi\wedge\omega_\varphi^{n-1} \leq \int_X \varphi_-^p\omega_\varphi^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

Next integrate by parts on the leftmost side of this inequality

$$\int_X \varphi_-^p \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge \omega_\varphi^{n-1} = -\int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p \wedge \bar{\partial}\varphi\, \omega_\varphi^{n-1}$$
$$= \int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p \wedge \bar{\partial}\varphi_- \omega_\varphi^{n-1}$$
$$= \frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}} \wedge \bar{\partial}\varphi_-^{\frac{p+1}{2}} \wedge \omega_\varphi^{n-1}.$$

Since $\varphi_- \geq 1$ we deduce the gradient estimate

$$\frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}} \wedge \bar{\partial}\varphi_-^{\frac{p+1}{2}} \wedge \omega_\varphi^{n-1} \leq \int_X \varphi_-^p\omega_\varphi^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

We concentrate on the outermost inequality

$$\frac{4p}{n(p+1)^2}\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||^2_{\varphi_t}\omega^n_{\varphi_t} \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

Now we invoke the Sobolev inequality

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}} \leq \mathscr{C}(\varphi_t)\left(\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||^2_{\varphi_t}\frac{\omega^n_{\varphi_t}}{V} + \int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right).$$

$\mathscr{C}(\varphi_t)$ is the Sobolev constant of the metric $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$. Concerning this constant we have the crucial

Proposition 4 **([11], [7]).** — *There is a positive constant $\delta = \delta(n)$ such that for all $\sigma \in SL(N+1,\mathbb{C})$ we have*

$$\mathscr{C}(\varphi_\sigma) < \delta.$$

This follows from the fact the complex projective subvarieties are *minimal* as Riemannian submanifolds of $\mathbb{P}^N$ and hence have vanishing mean curvature.

Therefore,

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}} \leq n(p+1)\delta\int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}.$$

Now extract the $p+1$st root of both sides to get

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n(p+1)}} \leq (n(p+1)\delta)^{\frac{1}{p+1}}\left(\int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right)^{\frac{1}{p+1}}.$$

Now we start the standard iteration: Let $p_0 := 1$ and $p_{j+1}+1 := \frac{n}{n-1}(p_j+1)$. Then we have that

$$||\varphi_-||_{p_{j+1}+1} \leq C^{\frac{1}{p_j+1}}(p_j+1)^{\frac{1}{p_j+1}}||\varphi_-||_{p_j+1} \leq \cdots$$
$$\leq C^{\sum_{i=0}^{j}\frac{1}{p_i+1}}\prod_{i=0}^{j}(p_i+1)^{\frac{1}{p_j+1}}||\varphi_-||_2.$$

That is to say

$$||\varphi_-||_{p_{j+1}+1} \leq C^{\sum_{i=0}^{j} \frac{1}{p_i+1}} \prod_{i=0}^{j} (p_i+1)^{\frac{1}{p_j+1}} ||\varphi_-||_2.$$

It is not hard to check that the infinite product converges. Taking limits as $j \longrightarrow \infty$ gives

$$||\varphi_-||_\infty \leq C\left(\int_X \varphi_-^2 \frac{\omega_\varphi^n}{V}\right)^{\frac{1}{2}} \leq ||\varphi_-||_\infty^{\frac{1}{2}}\, C\left(\int_X \varphi_- \frac{\omega_\varphi^n}{V}\right)^{\frac{1}{2}}.$$

Which implies

$$||\varphi_-||_\infty \leq C^2\left(\int_X \varphi_- \frac{\omega_\varphi^n}{V}\right).$$

Since $\varphi_t \leq C$ as $t \to 0$ we have

$$-\mathrm{inf}_X \varphi_t \leq C_1 \int_X (-\varphi_t)\frac{\omega_\varphi^n}{V} + C_2.$$

Now, by the Green identity we deduce

$$\mathrm{Osc}_X(\varphi_t) := \mathrm{Sup}_X(\varphi_t) - \mathrm{Inf}_X(\varphi_t) \leq C_1\left(\int_X \varphi_t\omega^n - \int_X \varphi_t\omega_\varphi^n\right) + C_2.$$

Using the properness assumption gives:

$$f(\mathrm{Osc}_X(\varphi_t)) \leq \nu_\omega(\varphi_t).$$

Now we are prepared to complete the proof of the corollary.

Case 1: Assume that $X^{\lambda(0)} \neq X$ and moreover that $X^{\lambda(0)}$ is reduced, then by the same argument as in [**17**] we have

$$\mathrm{lim}_{t\to 0}\mathrm{Osc}_X(\varphi_t) \to \infty.$$

Consequently we deduce that

$$\mathrm{lim}_{t\to 0}\nu_\omega(\varphi_t) \to \infty.$$

Corollary 1 yields the precise asymptotics [(5)]

$$\nu_\omega(\varphi_{\lambda(t)}) = F_1(\lambda)\log(t^2) + O(1).$$

This forces the desired sign $F_1(\lambda) < 0$.

Case 2: If $X^{\lambda(0)}$ is nonreduced, then $\Psi(\lambda(t)) \to -\infty$, however, under the properness assumption the K-Energy is bounded from below, and we again have that $F_1(\lambda) < 0$. This completes the proof of Corollary 2. □

(5) Recall that when $X^{\lambda(0)}$ is multiplicity free $\Psi(\lambda(t)) = O(1)$.

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