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INDEX, ETA AND RHO INVARIANTS ON FOLIATED BUNDLES

by

Moulay-Tahar Benameur & Paolo Piazza

Dedicated to Jean-Michel Bismut on the occasion of his sixtieth birthday

Abstract. — We study primary and secondary invariants of leafwise Dirac operators on foliated bundles. Given such an operator, we begin by considering the associated regular self-adjoint operator $\mathcal{D}_m$ on the maximal Connes-Skandalis Hilbert module and explain how the functional calculus of $\mathcal{D}_m$ encodes both the leafwise calculus and the monodromy calculus in the corresponding von Neumann algebras. When the foliation is endowed with a holonomy invariant transverse measure, we explain the compatibility of various traces and determinants. We extend Atiyah's index theorem on Galois coverings to foliations. We define a foliated rho-invariant and investigate its stability properties for the signature operator. Finally, we establish the foliated homotopy invariance of such a signature rho-invariant under a Baum-Connes assumption, thus extending to the foliated context results proved by Neumann, Mathai, Weinberger and Keswani on Galois coverings.

Résumé **(Indices, invariants êta et rho de fibrés feuilletés).** — Nous étudions certains invariants primaires et secondaires associés aux opérateurs de Dirac le long des feuilles de fibrés feuilletés. Etant donné un tel opérateur, nous considérons d'abord l'opérateur auto-adjoint régulier $\mathcal{D}_m$ qui lui est associé sur le module de Hilbert maximal de Connes-Skandalis, puis nous expliquons comment le calcul fonctionnel de $\mathcal{D}_m$ permet de coder le calcul longitudinal ainsi que le calcul sur les fibres de monodromie dans les algèbres de von Neumann correspondantes. Lorsque le feuilletage admet une mesure transverse invariante par holonomie, nous expliquons la compatibilité de diverses traces et déterminants. Nous étendons le théorème de l'indice pour les revêtements d'Atiyah aux feuilletages. Nous définissons l'invariant rho feuilleté et étudions ses propriétés de stabilité lorsque l'opérateur en question est l'opérateur de signature. Finalement, nous établissons l'invariance par homotopie feuilletée de l'invariant rho de l'opérateur de signature le long des feuilles sous une hypothèse de Baum-Connes, prolongeant ainsi au contexte feuilleté des résultats prouvés par Neumann, Mathai, Weinberger et Keswani dans le cadre des revêtements galoisiens.

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Introduction and main results

The Atiyah-Singer index theorem on closed compact manifolds is regarded nowadays as a classic result in mathematics. The original result has branched into several directions, producing new ideas and new results. One of these directions consists in considering elliptic differential operators on the following hierarchy of geometric structures:

- fibrations and operators that are elliptic in the fiber directions; for example, a product fibration $M \times T \to T$ and a family $(D_\theta)_{\theta \in T}$ of elliptic operators on M continuously parametrized by T;
- Galois Γ-coverings and Γ-equivariant elliptic operators;
- measured foliations and operators that are elliptic along the leaves;
- general foliations and, again, operators that are elliptic along the leaves.

One pivotal example, going through all these situations, is the one of foliated bundles. Let $\Gamma \to \tilde{M} \to M$ be a Galois Γ-cover of a smooth compact manifold M, let T be a compact manifold on which Γ acts by diffeomorphism. We can consider the diagonal action of Γ on $\tilde{M} \times T$ and the quotient space $V := \tilde{M} \times_\Gamma T$, which is a compact manifold, a bundle over M and carries a foliation $\mathcal{F}$. This foliation is obtained by considering the images of the fibers of the trivial fibration $\tilde{M} \times T \to T$ under the quotient map $\tilde{M} \times T \to \tilde{M} \times_\Gamma T$ and is known as a *foliated bundle*. More generally, we could allow T to be a compact topological space with an action of Γ by homeomorphisms, obtaining what is usually called a *foliated space* or a *lamination*. We then consider a family of elliptic differential operators $(\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and we assume that it is Γ-equivariant; it therefore yields a leafwise differential operator $D = (D_L)_{L \in V/\mathcal{F}}$ on V, which is elliptic along the leaves of $\mathcal{F}$. Notice that, if $\dim T > 0$ and $\Gamma = \{1\}$ then we are in the family situation; if $\dim T = 0$ and $\Gamma \neq \{1\}$, then we are in the covering situation; if $\dim T > 0$, $\Gamma \neq \{1\}$ and T admits a Γ-invariant Borel measure ν, then we are in the measured foliation situation and if $\dim T > 0$, $\Gamma \neq \{1\}$ then we are dealing with a more general foliation.

In the first three cases, there is first of all a *numeric* index: for families this is quite trivially the integral over T of the locally constant function that associates to θ the index of D_θ; for Γ-coverings we have the Γ-index of Atiyah and for measured foliations we have the measured index introduced by Connes. These last two examples involve the definition of a von Neumann algebra endowed with a suitable trace. More generally, and this applies also to general foliations, one can define *higher indices*, obtained by pairing the index class defined by an elliptic operator with suitable (higher) cyclic cocycles. In the case of foliated bundles there is a formula for these higher indices, due to Connes [**18**], and recently revisited by Gorokhovsky and Lott [**23**] using a generalization of the Bismut superconnection [**13**]. See also [**39**]. Since

our main focus here are numeric (versus higher) invariants, we go back to the case of measured foliated bundles, thus assuming that T admits a Γ-invariant measure ν.

The index is of course a global object, defined in terms of the kernel and cokernel of operators. However, one of its essential features is the possibility of localizing it near the diagonal using the remainders produced by a parametrix for D. On a closed manifold this crucial property is encoded in the so-called Atiyah-Bott formula:

$$\text{ind}(D) = \text{Tr}(R_0^N) - \text{Tr}(R_1^N), \quad \forall N \geq 1 \tag{1}$$

if $R_1 = \text{Id} - DQ$ and $R_0 = \text{Id} - QD$ are the remainders of a parametrix Q. Similar results hold in the other two contexts: Γ-coverings and measured foliations. One important consequence of formula (1) and of the analogous one on Γ-coverings is Atiyah's index theorem on a Γ-covering $\tilde{M} \to M$, stating the equality of the index on M and the von Neumann Γ-index on $\tilde{M}$. Informally, the index upstairs is equal to the index downstairs. On a measured foliation, for example on a foliated bundle $(\tilde{M} \times_\Gamma T, \mathcal{F})$ associated to a Γ-space T endowed with a Γ-invariant measure ν, we also have an index upstairs and an index downstairs, depending on whether we consider the Γ-equivariant family $(\tilde{D}_\theta)_{\theta \in T}$ or the longitudinal operator $D = (D_L)_{L \in V/\mathcal{F}}$; the analogue of formula (1) allows to prove the equality of these two indices. (This phenomenon is well known to experts; we explain it in detail in Section 4.)

Now, despite its many geometric applications, the index remains a very coarse spectral invariant of the elliptic differential operator D, depending only on the spectrum near zero. Especially when considering geometric operators, such as Dirac-type operators, and related geometric questions involving, for example, the diffeomorphism type of manifolds or the moduli space of metrics of positive scalar curvature, one is led to consider more involved spectral invariants. The eta invariant, introduced by Atiyah, Patodi and Singer on odd dimensional manifolds, is such an invariant. This invariant is highly non-local (in contrast to the index) and involves the whole spectrum of the operator. It is, however, too sophisticated: indeed, a small perturbation of the operator produces a variation of the corresponding eta invariant. In geometric questions one considers rather a more stable invariant, the rho invariant, typically a difference of eta invariants having the same local variation. The Cheeger-Gromov rho invariant on a Galois covering $\tilde{M} \to M$ of an odd dimensional manifold M is the most famous example; it is precisely defined as the difference of the Γ-eta invariant on $\tilde{M}$, defined using the Γ-trace of Atiyah, and of the Atiyah-Patodi-Singer eta invariant of the base M. Notice that the analogous difference for the indices (in the even dimensional case) would be equal to zero because of Atiyah's index theorem on coverings; the Cheeger-Gromov rho invariant is thus a genuine *secondary invariant.* The Cheeger-Gromov rho invariant is usually defined for a Dirac-type operator $\tilde{D}$ and

we bound ourselves to this case from now on; we denote it by $\rho_{(2)}(\tilde{D})$. Here are some of the stability properties of rho:

- let (M, g) be an oriented riemannian manifold and let $\tilde{D}^{\text{sign}}$ be the signature operator on $\tilde{M}$ associated to the Γ-invariant lift of g to $\tilde{M}$: then $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is metric independent and a diffeomorphism invariant of M;
- let M be a spin manifold and assume that the space $\mathscr{R}^+(M)$ of metrics with positive scalar curvature is non-empty. Let $g \in \mathscr{R}^+(M)$ and let $\tilde{D}_g^{\text{spin}}$ be the spin Dirac operator associated to the Γ-invariant lift of g. Then the function $\mathscr{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\text{spin}})$ is constant on the connected components of $\mathscr{R}^+(M)$

There are easy examples, involving lens spaces, showing that $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is *not* a homotopy invariant and that $\mathscr{R}^+(M) \ni g \to \rho_{(2)}(\tilde{D}_g^{\text{spin}})$ is *not* the constant function equal to zero. For purely geometric applications of these two results see, for example, [**15**] and [**46**]. These two properties can be proved in general, regardless of the nature of the group Γ. However, when Γ is *torsion-free*, then the Cheeger-Gromov rho invariant enjoys particularly strong stability properties. Let $\Gamma = \pi_1(M)$ and let $\tilde{M} \to M$ be the universal cover. Then in a series of papers [**29**], [**30**], [**31**], Keswani, extending work of Neumann [**41**], Mathai [**36**] and Weinberger [**57**], establishes the following fascinating theorem:

- if M is orientable, Γ is torsion free and the Baum-Connes map $K_*(B\Gamma) \to K_*(C^*_{\max}\Gamma)$ is an isomorphism, then $\rho_{(2)}(\tilde{D}^{\text{sign}})$ is a *homotopy invariant* of M;
- if M is in addition spin and $\mathscr{R}^+(M) \neq \varnothing$ then $\rho_{(2)}(\tilde{D}_g^{\text{spin}}) = 0$ for any $g \in \mathscr{R}^+(M)$.

(The second statement is not explicitly given in the work of Keswani but it follows from what he proves; for a different proof of Keswani's result see the recent paper [**45**].) Informally: *when Γ is torsion free and the maximal Baum-Connes map is an isomorphism, the Cheeger-Gromov rho invariant behaves like an index, i.e. like a primary invariant: more precisely, it is a homotopy invariant for the signature operator and it is equal to zero for the spin Dirac operator associated to a metric of positive scalar curvature.*

Let us now move on in the hierarchy of geometric structures and consider a foliated bundle $(V := \tilde{M} \times_\Gamma T, \mathscr{F})$, with $\tilde{M} \to M$ the universal cover of an odd dimensional compact manifold and T a compact Γ-space endowed with a Γ-invariant Borel (probability) measure ν. We are also given a Γ-equivariant family of Dirac-type operators $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ on the product fibration $\tilde{M} \times T \to T$ and let $D = (D_L)_{L \in V/\mathscr{F}}$ be the induced longitudinally elliptic operator on V. One is then led to the following natural questions:

1. Can one define a foliated rho invariant $\rho_\nu(D; V, \mathscr{F})$?
2. What are its stability properties if $\tilde{D} = \tilde{D}^{\text{sign}}$ and $\tilde{D} = \tilde{D}^{\text{spin}}$?

3. If the isotropy groups of the action of Γ on T are torsion free and the maximal Baum-Connes map with coefficients

$$K_*^\Gamma(E\Gamma; C(T)) \to K_*(C(T) \rtimes_{\max} \Gamma)$$

is an isomorphism, is $\rho_\nu(V, \mathcal{F}) := \rho_\nu(D^{\text{sign}}; V, \mathcal{F})$ a foliated homotopy invariant?

The goal of this paper is to give an answer to these three questions. Along the way we shall present in a largely self-contained manner the main results in index theory and in the theory of eta invariants on foliated bundles.

This work is organized as follows. In Section 1 we introduce the maximal C^*-algebra $\mathcal{A}_m$ associated to the Γ-space T or, more precisely, to the groupoid $\mathcal{G} := T \rtimes \Gamma$. We endow this C^*-algebra with two traces τ^ν_{reg} and τ^ν_{av}, ν denoting as before the Γ-invariant Borel measure on T. We then define two von Neumann algebras $W^*_{\text{reg}}(\mathcal{G})$, $W^*_{\text{av}}(\mathcal{G})$ with their respective traces; we define representations $\mathcal{A}_m \to W^*_{\text{reg}}(\mathcal{G})$, $\mathcal{A}_m \to W^*_{\text{av}}(\mathcal{G})$ and show compatibility of the traces involved.

In Section 2 we move to foliated bundles, giving the definition, studying the structure of the leaves, introducing the monodromy groupoid G and the associated maximal C^*-algebra $\mathcal{B}_m$. We then introduce two von Neumann algebras, $W^*_\nu(G)$ and $W^*_\nu(V, \mathcal{F})$, to be thought of as the one upstairs and the one downstairs respectively, with their respective traces τ^ν, $\tau^\nu_{\mathcal{F}}$. We introduce representations $\mathcal{B}_m \to W^*_\nu(G)$, $\mathcal{B}_m \to W^*_\nu(V, \mathcal{F})$ and define two compatible traces, also denoted τ^ν_{reg} and τ^ν_{av}, on the C^*-algebra $\mathcal{B}_m$. We then prove an explicit formula for these two traces on $\mathcal{B}_m$. We end Section 2 with a proof of the Morita isomorphism $K_0(\mathcal{A}_m) \simeq K_0(\mathcal{B}_m)$ and its compatibility with the morphisms

$$\tau^\nu_{\text{reg},*}, \tau^\nu_{\text{av},*} : K_0(\mathcal{A}_m) \to \mathbb{C}, \quad \tau^\nu_{\text{reg},*}, \tau^\nu_{\text{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$$

induced by the two pairs of traces on $\mathcal{A}_m$ and $\mathcal{B}_m$ respectively.

In Section 3 we move to more analytic questions. We define a natural $\mathcal{A}_m$-Hilbert module $\mathcal{E}_m$ with associated C^*-algebra of compact operators $\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ isomorphic to $\mathcal{B}_m$; we show how $\mathcal{E}_m$ encodes both the L^2-spaces of the fibers of the product fibration $\tilde{M} \times T \to T$ and the L^2-spaces of the leaves of $\mathcal{F}$. We then introduce a Γ-equivariant pseudodifferential calculus, showing in particular how 0-th order operators extend to bounded $\mathcal{A}_m$-linear operators on $\mathcal{E}_m$ and how negative order operators extend to compact operators. We then move to *unbounded regular* operators, for example operators defined by a Γ-equivariant Dirac family $\tilde{D} := (\tilde{D}_\theta)_{\theta \in T}$ and study quite carefully the functional calculus associated to such an operator. We then treat Hilbert-Schmidt operators and trace class operators in our two von Neumann contexts

and give sufficient conditions for an operator to be trace class. We study once again various compatibility issues (this material will be crucial later on).

In Section 4 we introduce, in the even dimensional case, the two indices $\mathrm{ind}_\nu^{\mathrm{up}}(\tilde{D})$, $\mathrm{ind}_\nu^{\mathrm{down}}(D)$ with $\tilde{D} = (\tilde{D}_\theta)_{\theta\in T}$ and $D := (D_L)_{L\in V/\mathcal{F}}$, and show the equality

$$\mathrm{ind}_\nu^{\mathrm{up}}(\tilde{D}) = \mathrm{ind}_\nu^{\mathrm{down}}(D).$$

This is the analogue of Atiyah's index theorem on Galois coverings. We also introduce the relevant index class, in $K_0(\mathcal{B}_m)$, and show how the von Neumann indices can be recovered from it and the two morphisms,

$$\tau^\nu_{\mathrm{reg},*} : K_0(\mathcal{B}_m) \to \mathbb{C}\,, \quad \tau^\nu_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$$

defined by the traces $\tau^\nu_{\mathrm{reg}} : \mathcal{B}_m \to \mathbb{C}$, $\tau^\nu_{\mathrm{av}} : \mathcal{B}_m \to \mathbb{C}$.

In Section 5 we introduce the two eta invariants $\eta^\nu_{\mathrm{up}}(\tilde{D})$, $\eta^\nu_{\mathrm{down}}(D)$ and, finally, the foliated rho-invariant $\rho_\nu(D; V, \mathcal{F})$ as the difference of the two. This answers the first question raised above. We end this section establishing an important link between the rho invariant and the determinant of certain paths.

In Section 6 we study the stability properties of the foliated rho invariant, showing in particular that for the signature operator it is metric independent and a foliated diffeomorphism invariant. This answers the second question raised above.

Finally, in Sections 7, 8 and 9 we prove the foliated homotopy invariance of the signature rho-invariant under a Baum-Connes assumption, following ideas of Keswani. In order to keep this paper in a reasonable size, we establish this result under the additional assumption that the foliated homotopy equivalence is induced by an equivariant fiber homotopy equivalence of the fibration defining the foliated bundle (we call this foliated homotopy equivalences *special*). Thus, Section 7 contains preparatory material on determinants and Bott-periodicity; Section 8 gives a sketch of the proof of the homotopy invariance and Section 9 contains the details. With these three sections we give an answer, at least partially, to the third question raised above. Most of the material explained in the previous part of the paper goes into the rather complicated proof. Some of our results are also meant to clarify statements in the work of Keswani.

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Notations

M	closed manifold	
Γ	fundamental group of M	
$\tilde{M}$	universal cover of M	
F	fundamental domain for the deck transformations	
T	a compact Γ-space	
ν	a Γ-invariant Borel measure on T	
$\mathcal{G}$	the groupoid $T \rtimes \Gamma$	1.1
$\Gamma(\theta)$	isotropy group of Γ at $\theta \in T$	1.1
$\mathcal{A}_c$	$= C_c(T \times \Gamma)$: algebraic crossed product algebra	1.1
$\mathcal{A}_r$	$= C(T) \rtimes_r \Gamma$: reduced crossed product algebra	1.2
$\mathcal{A}_m$	$= C(T) \rtimes_{\max} \Gamma$: maximal crossed product algebra	1.2
V	$= \tilde{M} \times_\Gamma T$: the foliated space	2.1
G	the groupoid $(\tilde{M} \times \tilde{M} \times T)/\Gamma$	2.2
$\mathcal{B}_c$	the compactly supported convolution algebra of G	2.2
$\mathcal{B}_r$	the regular completion of $\mathcal{B}_c$	2.2
$\mathcal{B}_m$	the maximal completion of $\mathcal{B}_c$	2.2
$\mathcal{B}_c^E$	the modified algebra when $E \to V$ is a vector bundle	2.2
τ^ν_{reg}	regular trace of $\mathcal{A}_r$ or $\mathcal{A}_m$	1.4
τ^ν_{av}	trivial or averaged trace on $\mathcal{A}_m$	1.4
also τ^ν_{reg}	regular trace of $\mathcal{B}_r$ or $\mathcal{B}_m$	2.4
also τ^ν_{av}	trivial or averaged trace on $\mathcal{B}_m$	2.4
$\hat{E} \to \tilde{M} \times T$	the Γ-equivariant lift of E	
$W^*_{\mathrm{av}}(\mathcal{G})$	average von Neumann algebra of $\mathcal{G}$	1.3
$W^*_{\mathrm{reg}}(\mathcal{G})$	regular von Neumann algebra of $\mathcal{G}$	1.3
π^{reg}	regular representation of $\mathcal{A}_r$ in $W^*_{\mathrm{reg}}(\mathcal{G})$	1.3
π^{av}	average representation of $\mathcal{A}_m$ in $W^*_{\mathrm{av}}(\mathcal{G})$	1.3
τ^ν	trace on $W^*_{\mathrm{reg}}(\mathcal{G})$	1.4
also τ^ν	trace on $W^*_{\mathrm{av}}(\mathcal{G})$	1.4
$W^*_\nu(G; E)$	regular von Neumann algebra of G with coefficients in E	2.3
$W^*_\nu(V, \mathcal{F}; E)$	leafwise von Neumann algebra with coefficients in E	2.3
τ^ν	trace on $W^*_\nu(G; E)$	2.4

$\tau^{\nu}_{\mathcal{F}}$	trace on $W^*_{\nu}(V,\mathcal{F};E)$	2.4
$\mathcal{E}_c$	the prehilbertian $\mathcal{A}_c$-module $C^{\infty,0}_c(\tilde{M}\times T,\hat{E})$	3.1
$\mathcal{E}_r$	the completion of $\mathcal{E}_c$ into a Hilbert $\mathcal{A}_r$-module	3.1
$\mathcal{E}_m$	the completion of $\mathcal{E}_c$ into a Hilbert $\mathcal{A}_m$-module	3.1
$D=(D_L)_{L\in V/\mathcal{F}}$	leafwise geometric operator	3.3
$\tilde{D}=(\tilde{D}_\theta)_{\theta\in T}$	Γ-invariant fiberwise geometric operator	3.3
$\mathcal{D}_m$	the induced regular operator on $\mathcal{E}_m$	3.3
$\mathcal{D}_r$	the induced regular operator on $\mathcal{E}_r$	3.3
$\mathrm{Ind}(\mathcal{D}_m)$	the index class of $\mathcal{D}_m$ in $K_*(\mathcal{A}_m)$	4.2
$\mathrm{IND}(\mathcal{D}_m)$	the index class of $\mathcal{D}_m$ in $K_*(\mathcal{B}_m)$	4.2
$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D})$	measured index upstairs	4.2
$\mathrm{ind}^{\nu}_{\mathrm{down}}(\tilde{D})$	measured index downstairs	4.2
$\eta^{\nu}_{\mathrm{up}}(\tilde{D})$	eta invariant upstairs	5.1
$\eta^{\nu}_{\mathrm{down}}(D)$	eta invariant downstairs	5.1
$\rho^{\nu}(D;V,\mathcal{F})$	foliated rho invariant associated to D	5.2
$\rho^{\nu}(V,\mathcal{F})$	foliated rho invariant for the signature operator	6.2

1. Group actions

1.1. The discrete groupoid $\mathcal{G}$. — Let Γ be a discrete group. Let T be a compact topological space on which the group Γ acts by homeomorphisms on the left. We shall assume that T is endowed with a Γ-invariant Borel measure ν; this is a non-trivial hypothesis. Thus (T,ν) is a compact Borel measured space on which Γ acts by measure preserving homeomorphisms. We shall assume that ν is a probability measure. We consider the crossed product groupoid $\mathcal{G}:=T\rtimes\Gamma$; thus the set of arrows is $T\times\Gamma$, the set of units is T,

$$s(\theta,\gamma)=\gamma^{-1}\theta \quad \text{and} \quad r(\theta,\gamma)=\theta.$$

The composition law is given by

$$(\gamma'\theta,\gamma')\circ(\theta,\gamma)=(\gamma'\theta,\gamma'\gamma)\,.$$

We denote by $\mathcal{A}_c$ the convolution $\star$-algebra of compactly supported continuous functions on $\mathcal{G}$ and by $L^1(\mathcal{G})$ the Banach $\star$-algebra which is the completion of $\mathcal{A}_c$ with respect to the Banach norm $\|\cdot\|_1$ defined by

$$\|f\|_1:=\max\{\sup_{\theta\in T}\sum_{\gamma\in\Gamma}|f(\theta,\gamma)|;\sup_{\theta\in T}\sum_{\gamma\in\Gamma}|f(\gamma^{-1}\theta,\gamma^{-1})|\}.$$

The convolution operation and the adjunction are fixed by the following formulae

$$(f*g)(\theta,\gamma)=\sum_{\gamma_1\in\Gamma}f(\theta,\gamma_1)g(\gamma_1^{-1}\theta,\gamma_1^{-1}\gamma) \text{ and } f^*(\theta,\gamma)=\overline{f(\gamma^{-1}\theta,\gamma^{-1})}.$$

For $\theta \in T$ we shall denote by $\Gamma(\theta)$ the isotropy group of the point θ: $\Gamma(\theta) := \{\gamma \in \Gamma \,|\, \gamma\theta = \theta\}$. So, $\Gamma(\theta)$ is a subgroup of Γ and the orbit of θ under the action of Γ, denoted $\Gamma\theta$, can be identified with $\Gamma/\Gamma(\theta)$. Finally, we recall that $\mathcal{G}^\theta := r^{-1}(\theta)$ and that $\mathcal{G}_\theta := s^{-1}(\theta)$.

1.2. C^*-algebras associated to the discrete groupoid $\mathcal{G}$. — For any $\theta \in T$, we define the regular $*$-representation $\pi_\theta^{\mathrm{reg}}$ of $\mathcal{A}_c$ in the Hilbert space $\ell^2(\Gamma)$, viewed as $\ell^2(\mathcal{G}_\theta)$, by the following formula

$$\pi_\theta^{\mathrm{reg}}(f)(\xi)(\gamma) := \sum_{\gamma' \in \Gamma} f(\gamma\theta, \gamma\gamma'^{-1})\xi(\gamma').$$

It is easy to check that this formula defines a $*$-representation $\pi_\theta^{\mathrm{reg}}$ which is L^1 continuous. Moreover, we complete $L^1(\mathcal{G})$ with respect to the norm $\sup_{\theta \in T} \|\pi_\theta^{\mathrm{reg}}(\cdot)\|$ and obtain a C^*-algebra $\mathcal{A}_r$. The C^*-algebra $\mathcal{A}_r$ is usually called the regular C^*-algebra of the groupoid $\mathcal{G}$, it will also be denoted with the symbols $C_r^*(\mathcal{G})$ or $C(T) \rtimes_r \Gamma$.

If we complete the Banach $*$-algebra $L^1(\mathcal{G})$ with respect to all continuous $*$-representations, then we get the C^*-algebra $\mathcal{A}_m$, usually called the maximal C^*-algebra of the groupoid $\mathcal{G}$. See [**49**] for more details on these constructions. Other notations for $\mathcal{A}_m$ are $C_m^*(\mathcal{G})$ and $C(T) \rtimes_m \Gamma$.

By construction, any continuous $*$-homomorphism from $L^1(\mathcal{G})$ to a C^*-algebra B yields a C^*-algebra morphism from $\mathcal{A}_m$ to B. In particular, the homomorphism π^{reg} yields a C^*-algebra morphism

$$\pi^{\mathrm{reg}} : \mathcal{A}_m \longrightarrow \mathcal{A}_r.$$

1.3. von Neumann algebras associated to the discrete groupoid $\mathcal{G}$. — At the level of measure theory, recall that we have fixed once for all a Γ-invariant borelian probability measure ν on T. We associate with $\mathcal{G}$ two von Neumann algebras that will be important for our purpose.

The first one is the regular von Neumann algebra $W_{\mathrm{reg}}^*(\mathcal{G})$. It is the algebra $L^\infty(T, B(\ell^2\Gamma); \nu)^\Gamma$ of Γ-equivariant essentially bounded families of bounded operators on $\ell^2\Gamma$, so it acts on the Hilbert space $L^2(T \times \Gamma, \nu)$. An element T of $W_{\mathrm{reg}}^*(\mathcal{G})$ is thus (a class of) a familly $(T_\theta)_{\theta \in T}$ of operators in $\ell^2(\Gamma)$, which satisfies the following properties:

- For any $\xi \in L^2(T \times \Gamma)$ the map $\theta \mapsto <T_\theta \xi_\theta, \xi_\theta>$ is Borel measurable where $\xi_\theta(\gamma) := \xi(\theta, \gamma)$;
- $\theta \mapsto \|T_\theta\|$ is ν-essentially bounded on T;
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$.

Notice that if we denote by $R_\gamma^* : \ell^2\Gamma \to \ell^2\Gamma$ the operator

$$(R_\gamma^*\xi)(\alpha) := \xi(\alpha\gamma),$$

then $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$ for any $T \in B(\ell^2\Gamma)$. That $W^*_{\rm reg}(\mathcal{G})$ is a von Neumann algebra is clear since it is the commutant of a unitary group associated with the action of Γ. The $*$-representation $\pi^{\rm reg}$ is then valued in $W^*_{\rm reg}(\mathcal{G})$ as can be checked easily, and we have the $*$-representation

$$\pi^{\rm reg} : \mathcal{A}_{\rm r} \longrightarrow W^*_{\rm reg}(\mathcal{G}).$$

This $*$-representation then extends to the maximal C^*-algebra $\mathcal{A}_m$.

The second von Neumann algebra that will be important for us will be called the average von Neumann algebra $W^*_{\rm av}(\mathcal{G})$ and we proceed now to define it. We set $\mathcal{G}_0 := (T \times \Gamma)/\sim$ where we identify (θ, γ) with $(\theta, \gamma\alpha)$ whenever $\alpha\theta = \theta$. Then $\mathcal{G}_0$ is Borel and an element T of $W^*_{\rm av}(\mathcal{G})$ is (a class of) a family $(T_\theta)_{\theta \in T}$ of operators in $\ell^2(\Gamma\theta)$, which satisfies the properties:

- For any measurable (as a function on $\mathcal{G}_0$) ν-square integrable section ξ of the Borel field $\ell^2(\Gamma/\Gamma(\theta))$ over T, the map $\theta \mapsto <T_\theta \xi_\theta, \xi_\theta>$ is Borel measurable where $\xi_\theta[\gamma] := \xi[\theta, \gamma]$
- $\theta \mapsto \|T_\theta\|$ is ν-essentially bounded on T;
- For any $\gamma \in \Gamma$, we have $T_{\gamma\theta} = \gamma T_\theta$;

Here we denote by $R^*_\gamma : \ell^2(\Gamma/\Gamma(\theta)) \to \ell^2(\Gamma/\Gamma(\gamma\theta))$ the isomorphism given by $(R^*_\gamma \xi)[\alpha] := \xi[\alpha\gamma]$, and $\gamma T := R^*_\gamma \circ T \circ R^*_{\gamma^{-1}}$. Again $W^*_{\rm av}(\mathcal{G})$ is a von Neumann algebra; for more details on this constructions see for instance [**21**], [**20**]

There is an interesting representation π^{av} of $L^1(\mathcal{G})$ in $W^*_{\rm av}(\mathcal{G})$ defined as follows. Let $f \in C_c(\mathcal{G})$; for any $\theta \in T$, we set

$$\pi^{av}_\theta(f)(\xi)(x = [\alpha]) := \sum_{y \in \Gamma/\Gamma(\theta)} \sum_{[\beta]=y} f(\alpha\theta, \alpha\beta^{-1})\xi(y), \quad \xi \in \ell^2(\Gamma/\Gamma(\theta)).$$

Remark 1.1. — *If we identify $\Gamma/\Gamma(\theta)$ with the orbit $\Gamma\theta$ then π^{av} becomes*

$$\pi^{av}_\theta(f)(\xi)(\theta') = \sum_{\theta'' \in \Gamma\theta} \sum_{\alpha\theta''=\theta'} f(\theta', \alpha)\xi(\theta'') = \sum_{\alpha \in \Gamma} f(\theta', \alpha)\xi(\alpha^{-1}\theta').$$

Proposition 1.2. — *For any $f \in L^1(\mathcal{G})$ and any $\theta \in T$, the operator $\pi^{av}_\theta(f)$ is bounded and the family $\pi^{av}(f) = (\pi^{av}_\theta(f))_{\theta \in T}$ defines a continuous $*$-representation of $L^1(\mathcal{G})$ with values in $W^*_{\rm av}(\mathcal{G})$. Hence, π^{av} yields a $*$-representations of the maximal C^*-algebra $\mathcal{A}_m$ in $W^*_{\rm av}(\mathcal{G})$.*

Proof. — If we set for any $f \in C_c(\mathcal{G})$, $f_0(\theta,\theta') := \sum_{\gamma\theta=\theta'} f(\theta',\gamma)$, then for $g \in C_c(\mathcal{G})$ we have:

$$\begin{aligned}
(f*g)_0(\theta,\theta') &= \sum_{\gamma.\theta=\theta'} (f*g)(\theta',\gamma) \\
&= \sum_{\gamma\theta=\theta'} \sum_{\gamma_1\in\Gamma} f(\theta',\gamma_1) g(\gamma_1^{-1}\theta', \gamma_1^{-1}\gamma) \\
&= \sum_{\theta''\in\Gamma.\theta} \sum_{\gamma_1^{-1}.\theta'=\theta'',\ \gamma_2^{-1}.\theta''=\theta} f(\theta',\gamma_1) g(\theta'',\gamma_2) \\
&= \sum_{\theta''\in\Gamma.\theta} f_0(\theta'',\theta') g_0(\theta,\theta'') \\
&= (f_0 * g_0)(\theta,\theta').
\end{aligned}$$

Since $\pi^{av}(f)$ is simply convolution by the kernel f_0, we deduce that π is a representation of the convolution algebra $\mathcal{A}_c$. Now, the kernel $(f^*)_0$ is given by

$$(f^*)_0(\theta,\theta') = \sum_{\gamma\theta=\theta'} \overline{f}(\gamma^{-1}\theta',\gamma^{-1}) = \sum_{\alpha\theta'=\theta} \overline{f}(\theta,\alpha) = \overline{f_0(\theta',\theta)}.$$

It remains to prove that π^{av} is L^1-continuous. But, we have:

$$\begin{aligned}
\|\pi^{\mathrm{av}}_\theta(f)\xi\|_2^2 &= \sum_{\theta'\in\Gamma\theta} |\sum_{\gamma\in\Gamma} f(\theta',\gamma)\xi(\gamma^{-1}\theta')|^2 \\
&\le \sum_{\theta'\in\Gamma\theta} (\sum_{\gamma\in\Gamma} |f(\theta',\gamma)|) \times (\sum_{\gamma\in\Gamma} |f(\theta',\gamma)|.|\xi(\gamma^{-1}\theta')|^2) \\
&\le \|f\|_1 \sum_{\theta'\in\Gamma\theta} \sum_{\gamma\in\Gamma} |f(\theta',\gamma)|.|\xi(\gamma^{-1}\theta')|^2 \\
&\le \|f\|_1 \sum_{\gamma\in\Gamma} \sum_{\theta''\in\Gamma\theta} |\xi(\theta'')|^2 |f(\gamma\theta'',\gamma)| \\
&\le \|f\|_1^2 \|\xi\|_2^2.
\end{aligned}$$

So, $\|\pi^{\mathrm{av}}(f)\| = \sup_{\theta\in T} \|\pi^{\mathrm{av}}_\theta(f)\| \le \|f\|_1$. □

We therefore deduce the existence of a $*$-homomorphism of C^*-algebras:

$$\pi^{\mathrm{av}} : \mathcal{A}_m \longrightarrow W^*_{\mathrm{av}}(\mathcal{G}).$$

1.4. Traces. — For any non negative element $T = (T_\theta)_{\theta\in T}$ of the von Neumann algebra $W^*_{\mathrm{reg}}(\mathcal{G})$ (resp. $W^*_{\mathrm{av}}(\mathcal{G})$), we set

$$\tau^\nu(T) := \int_T <T_\theta(\delta_e), \delta_e> d\nu(\theta),$$

where in the regular case, δ_e stands for the δ function at the unit e of Γ, while in the second case it is the δ function of the class $[e]$ in $\Gamma/\Gamma(\theta)$.

Proposition 1.3. — *The functional τ^ν induces a faithful normal positive finite trace.*

Proof. — Positivity is clear since T is non negative in the von Neumann algebra if and only if for ν-almost every θ the operator T_θ is non negative. If the non negative element $T = (T_\theta)_{\theta \in T}$ satisfies $\tau^\nu(T) = 0$ then $<T_\theta(\delta_e), \delta_e> = 0$ for ν-almost every θ. But, the Γ-equivariance of T implies that

$$<T_\theta(\delta_\gamma), \delta_\gamma> = 0, \quad \forall \gamma \in \Gamma \text{ and } \nu \ a.e.$$

Therefore, $T_\theta = 0$ for ν-almost every θ and hence $T = 0$ in $W^*_{\mathrm{reg}}(\mathcal{G})$. In the second case, the proof is similar again by Γ-equivariance and by replacing δ_γ by $\delta_{[\gamma]}$.

If $T(n) \uparrow T$ is an increasing sequence of non negative operators which converges in the von Neumann algebra to T, then for ν-almost every θ, the sequence $T(n)_\theta$ increases to T_θ. But then since the state $<\cdot(\delta_e), \delta_e>$ is normal, the conclusion follows by Beppo-Levi's property for ν.

If now T is in the von Neumann algebra $W^*_{\mathrm{reg}}(\mathcal{G})$ then writing T_θ as an infinite matrix in $\ell^2\Gamma$ and using the Γ equivariance we deduce that

$$T^{\alpha,\beta}_{\gamma\theta} = T^{\alpha\gamma,\beta\gamma}_{\theta}.$$

If we now consider a second operator S in $W^*_{\mathrm{reg}}(\mathcal{G})$, then we have

$$(T_\theta S_\theta)^{e,e} = \sum_{\gamma\in\Gamma} T^{e,\gamma}_\theta \, S^{\gamma,e}_\theta = \sum_{\gamma\in\Gamma} S^{e,\gamma^{-1}}_{\gamma\theta} T^{\gamma^{-1},e}_{\gamma\theta},$$

by the Γ-equivariance property. The Γ-invariance of measure ν can now be applied to yield that $\tau^\nu(TS) = \tau^\nu(ST)$. A similar proof works for the von Neumann algebra $W^*_{\mathrm{av}}(\mathcal{G})$. □

We define the functionals τ^ν_{reg} and τ^ν_{av} on $\mathcal{A}_c$ by setting for $f \in \mathcal{A}_c$

$$\tau^\nu_{\mathrm{reg}}(f) := \int_T f(\theta, e) d\nu(\theta), \tag{2}$$

$$\tau^\nu_{\mathrm{av}}(f) := \int_T \left[\sum_{g\in\Gamma(\theta)} f(\theta, g) \right] d\nu(\theta). \tag{3}$$

Lemma 1.4. — *1. We have $\tau^\nu \circ \pi^{\mathrm{reg}} = \tau^\nu_{\mathrm{reg}}$ and $\tau^\nu \circ \pi^{\mathrm{av}} = \tau^\nu_{\mathrm{av}}$.*

2. Hence, τ^ν_{reg} and τ^ν_{av} extend to finite traces on $\mathcal{A}_r$ and $\mathcal{A}_m$.

Proof. — The statement for the regular trace is classical and we thus omit the (easy) proof. We consider for any $f \in L^1(\mathcal{G})$ the Borel family of operators $(\pi^{\mathrm{av}}_\theta(f))_{\theta\in T}$ defined in the previous paragraph. For any $f \in \mathcal{A}_c$, denote as before by f_0 the function

$$f_0(\theta, \theta') := \sum_{\gamma\theta=\theta'} f(\theta', \gamma).$$

Then we know that $\pi^{\mathrm{av}}(f)$ is given as convolution with f_0. If $f \in \mathcal{A}_c$, then we have, using the identification $\Gamma/\Gamma(\theta) \equiv \Gamma\theta$:

$$\begin{aligned}\int_T <\pi^{\mathrm{av}}_\theta(f)\delta_\theta, \delta_\theta> d\nu(\theta) &= \int_T f_0(\theta,\theta) d\nu(\theta) \\ &= \int_T \sum_{\gamma\in\Gamma(\theta)} f(\theta,\gamma) d\nu(\theta) \\ &= \tau^\nu_{\mathrm{av}}(f).\end{aligned}$$

□

As a Corollary of the above Lemma notice that the traces $\tau^\nu_{\mathrm{reg}} : \mathcal{A}_r \to \mathbb{C}$ and $\tau^\nu_{\mathrm{av}} : \mathcal{A}_m \to \mathbb{C}$ induce group homomorphisms

$$\tau^\nu_{\mathrm{reg},*} : K_0(\mathcal{A}_r) \to \mathbb{R}\,, \quad \tau^\nu_{\mathrm{av},*} : K_0(\mathcal{A}_m) \to \mathbb{R}. \tag{4}$$

2. Foliated spaces

2.1. Foliated spaces. — Let M be a compact manifold without boundary and let Γ denote its fundamental group and $\tilde{M}$ its universal cover. The group Γ acts by homemorphisms on the compact topological space T and hence acts on the right, freely and properly, on the space $\tilde{M} \times T$ by the formula

$$(\tilde{m},\theta)\gamma := (\tilde{m}\gamma, \gamma^{-1}\theta), \quad (\tilde{m},\theta) \in \tilde{M} \times T \text{ and } \gamma \in \Gamma.$$

The quotient space of $\tilde{M} \times T$ under this action is denoted by V. We assume as before the existence of a Γ-invariant probability measure ν. If we want to be specific about the action of Γ on T we shall consider it as a homomorphism $\Psi : \Gamma \to \mathrm{Homeo}(T)$. We do not assume the action to be locally free [1] .

If $p : \tilde{M} \times T \to V$ is the natural projection then the leaves of a lamination on V are given by the projections $L_\theta = p(\tilde{M}_\theta)$, where θ runs through the compact space T, and

$$\tilde{M}_\theta := \tilde{M} \times \{\theta\}\,. \tag{5}$$

It is easy to check that this is a lamination of V with smooth leaves and possibly complicated transverse structure according to the topology of T, see for instance [**12**]. By definition, it is easy to check that the leaf L_θ coincides with the leaf $L_{\theta'}$ if and only if θ' belongs to the orbit $\Gamma\theta$ of θ under the action of Γ in T. We shall refer to this lamination by $(V, \mathcal{F})$ and sometimes shall call it a foliated space or, more briefly, a foliation. If $\Gamma(\theta)$ is the isotropy group of $\theta \in T$ then we see from the definition of L_θ that L_θ is diffeomorphic to the quotient manifold $\tilde{M}/\Gamma(\theta)$ through the map $L_\theta \to \tilde{M}/\Gamma(\theta)$ given by $[\tilde{m}', \theta'] \to [\tilde{m}'\gamma]$, if $\theta' = \gamma\theta$. Note however that L_θ is also

[1] Recall that an action is locally free if given $\gamma \in \Gamma$ and open set U in T such that $\gamma(\theta) = \theta$ for any $\theta \in U$ then $\gamma = 1$.

diffeomorphic to $\tilde{M}/\Gamma(\theta')$ for any $\theta' \in \Gamma\theta$. Moreover the monodromy cover of a leaf L is obtained by choosing $\theta \in T$ such that $L = L_\theta$ and by using the composite map

$$\tilde{M} \to \tilde{M}_\theta \to (\tilde{M}_\theta)/\Gamma(\theta) \simeq L_\theta = L.$$

which is a monodromy cover of L corresponding to θ.

Notice that the set of $\theta \in T$ for which $\Gamma(\theta)$ is non-trivial has in general positive measure. This is the case, for instance, when there exists a subgroup Γ_1 of Γ whose action on T has the property that $\nu(T^{\Gamma_1}) > 0$, where T^{Γ_1} is the fixed-point subspace defined by Γ_1. In fact, one can construct simple examples where the measure of the set of $\theta \in T$ for which $\Gamma(\theta)$ is non-trivial is any value in $(0, 1)$. See Example 2.2 for a specific situation.

Example 2.1. — *As an easy example where this situation occurs naturally, consider any Galois covering $\tilde{M}'$ of M with structure group Γ' such that $\pi_1(\tilde{M}') \neq 1$. Assume the existence of a locally free Γ'-action $\Psi' : \Gamma' \to \mathrm{Homeo}(T)$ on T and let V be the resulting foliated space. Assume the existence of an invariant measure ν on T. Since Γ' is a quotient of $\Gamma := \pi_1(M)$ we have a natural group homomorphism $\pi : \Gamma \to \Gamma'$ and thus an action $\Psi := \Psi' \circ \pi$ of Γ on T. By definition ν is also Γ-invariant. The isotropy group of this action at $\theta \in T$ is at least as big as the fundamental group of $\tilde{M}'$. Notice that one can show that*

$$(\tilde{M} \times T)/\Gamma = (\tilde{M}' \times T)/\Gamma' \equiv V.$$

Summarizing: V is a lamination where the set of leaves with non-trivial monodromy has measure equal to $\nu(T) = 1$.

Example 2.2. — *Take M to be any manifold whose fundamental group is a free product of copies of $\mathbb{Z}$, for example a connected sum of $\mathbb{S}^1 \times \mathbb{S}^2$'s, so that now Γ is the free group of rank k. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ be the generators. Let T be $\mathbb{S}^2$. Let $C \subset \mathbb{S}^2$ be a parallel and let $U \subset \mathbb{S}^2$ one of the two hemispheres bounded by C. Let $\Psi(\gamma_1)$ be any measure-preserving diffeomorphism of $\mathbb{S}^2$ that fixes U. We then define Ψ on the other generators in an arbitrary measure-preserving way. Then any point θ in U would have nontrivial isotropy group $\Gamma(\theta)$. Clearly, one can jazz up this example by selecting any T and finding a single homeomorphism whose fixed point set is a set of nonzero measure.*

Example 2.3. — *Following* [**38**] *we now give an example of a lamination with the set of leaves with non-trivial monodromy of positive measure and, in addition, of a rather complicated sort. Take a (generalized) Cantor set K of positive Lebesgue measure in the unit circle. Choose now a homeomorphism ϕ of the circle admitting K as the fixed point set. Let M be any closed odd dimensional manifold with $\pi_1(M) = \mathbb{Z}$. Consider the foliated space V obtained by suspension of ϕ: thus $V = \tilde{M} \times_{\mathbb{Z}} \mathbb{S}^1$ with $\mathbb{Z} = \pi_1(\mathbb{S}^1)$*

acting on $\mathbb{S}^1$ via ϕ and acting by deck transformations on $\tilde{M}$. The set of $\theta \in S^1$ such that $\{\gamma \in \mathbb{Z} | \gamma\theta = \theta\}$ is non-trivial is equal to K, hence it has positive measure. Using [**38**] *page 105/106, we can find a Radon ϕ-invariant measure ν on $\mathbb{S}^1$ and $\nu(K) > 0$. Notice that in this class of examples, although the measure is diffuse, one can even ensure that the set of leaves with non-trivial holonomy has positive transverse measure. These laminations show up in the study of aperiodic tillings and especially of quasi-crystals. In* [**12**] *for instance, the measured foliated index for such laminations, a primary invariant, is used to solve the gap-labelling conjecture. The authors expect potential applications of the foliated rho invariant to aperiodic solid physics.*

2.2. The monodromy groupoid and the C^*-algebra of the foliation. — Let $\tilde{M}$, Γ and T be as before. We define the monodromy groupoid G as the quotient space $(\tilde{M} \times \tilde{M} \times T)/\Gamma$ of $\tilde{M} \times \tilde{M} \times T$ by the right diagonal action

$$(\tilde{m}, \tilde{m}', \theta)\gamma := (\tilde{m}\gamma, \tilde{m}'\gamma, \gamma^{-1}\theta).$$

The groupoid structure is clear: the space of units $G^{(0)}$ is the space $V = \tilde{M} \times_\Gamma T$, the source and range maps are given by

$$s[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}', \theta] \text{ and } r[\tilde{m}, \tilde{m}', \theta] = [\tilde{m}, \theta],$$

where the brackets denote equivalence classes modulo the action of the group Γ

It is not difficult to show that G can be identified in a natural way with the usual monodromy groupoid associated to the foliated space $(V, \mathcal{F})$, as defined, for example, in [**44**]. More precisely given a smooth path $\alpha : [0,1] \to L$, with L a leaf, choose any lift $\tilde{\beta} : [0,1] \to \tilde{M}$ of the projection of the path α in M through the natural projection $V \to M$. Then there exists a unique $\theta \in T$ with $\alpha(0) = [\tilde{\beta}(0), \theta]$ and we obtain in this way a well defined element $[\tilde{\beta}(0), \tilde{\beta}(1), \theta]$ of G which only depends on the leafwise homotopy class of α with fixed end-points. This furnishes the desired isomorphism.

We fix now a Lebesgue class measure dm on M and the corresponding Γ-invariant measure $d\tilde{m}$ on $\tilde{M}$. We denote by $\mathcal{B}_c$ the convolution $*$-algebra of continuous compactly supported functions on G. For $f, g \in \mathcal{B}_c$ we have:

$$(f * g)[\tilde{m}, \tilde{m}', \theta] = \int_{\tilde{M}} f[\tilde{m}, \tilde{m}'', \theta] g[\tilde{m}'', \tilde{m}', \theta] d\tilde{m}'' \quad \text{and} \quad f^*[\tilde{m}, \tilde{m}', \theta] = \overline{f[\tilde{m}', \tilde{m}, \theta]}.$$

More generally, let E be a hermitian continuous longitudinally smooth vector bundle over V; thus E is a continuous bundle over V such that its restriction to each leaf is smooth [**38**]. Consider $\mathrm{END}(E) := (s^*E)^* \otimes (r^*E) = \mathrm{Hom}(s^*E, r^*E)$, a bundle of endomorphisms over G. We consider $\mathcal{B}_c^E := C_c^{\infty,0}(G, \mathrm{END}(E))$ the space of continuous longitudinally smooth sections of $\mathrm{END}(E)$; this is also a $*$-algebra with product

and adjoint given by

$$(f_1 * f_2)[\tilde{m}, \tilde{m}', \theta] = \int_{\tilde{M}} f_1[\tilde{m}, \tilde{m}'', \theta] \circ f_2[\tilde{m}'', \tilde{m}', \theta]\, d\tilde{m}'',$$

$$f^*[\tilde{m}, \tilde{m}', \theta] = (f[\tilde{m}', \tilde{m}, \theta])^*.$$

Let $\widehat{E}$ be its lift to $\widetilde{M} \times T$; denote by H_θ the Hilbert space $H_\theta = L^2(\widetilde{M} \times \{\theta\}; \widehat{E}_{|\widetilde{M} \times \{\theta\}})$. Any $f \in \mathcal{B}_c^E$ can be viewed as a smooth kernel acting on H_θ by the formula

$$\pi_\theta^{\mathrm{reg}}(f)(\xi)(\tilde{m}) := \int_{\tilde{M}} f[\tilde{m}, \tilde{m}', \theta](\xi(\tilde{m}'))d\tilde{m}', \quad \text{for any } \xi \in H_\theta\,,$$

and this defines a $*$-representation π_θ^{reg} in H_θ . We point out that the representation π_θ^{reg} is continuous for the L^1 norm defined by:

$$\|f\|_1 := \max\{ \sup_{(\tilde{m},\theta) \in \tilde{M} \times T} \int_{\tilde{M}} \|f[\tilde{m}, \tilde{m}', \theta]\|_E d\tilde{m}' ; \sup_{(\tilde{m},\theta) \in \tilde{M} \times T} \int_{\tilde{M}} \|f[\tilde{m}', \tilde{m}, \theta]\|_E d\tilde{m}' \}.$$

If we complete $\mathcal{B}_c$ with respect to the C^* norm

$$\|f\|_{\mathrm{reg}} := \sup_{\theta \in T} \|\pi_\theta^{reg}(f)\|,$$

then we get $\mathcal{B}_r^E$, the regular C^*-algebra of the groupoid G with coefficients in E. When $E = V \times \mathbb{C}$ then we denote this C^*-algebra simply by $\mathcal{B}_r$ In the same way, if we complete $\mathcal{B}_c$ with respect to all L^1 continuous $*$-representations, then we get the maximal C^*-algebra of the groupoid G, that will be denoted by $\mathcal{B}_m^E$ and simply by $\mathcal{B}_m$ when $E = V \times \mathbb{C}$.

2.3. von Neumann Algebras of foliations. — The material in this paragraph is classical; for more details see for instance [**18**], [**25**] [**9**], [**17**], [**35**].

The representation π^{reg} defined above takes value in the regular von Neumann algebra of the groupoid G. More precisely, the regular von Neumann algebra $W_\nu^*(G; E)$ of G with coefficients in E, acts on the Hilbert space $H = L^2(T \times \tilde{M}, \widehat{E}; \nu \otimes d\tilde{m})$, and is by definition the space of families $(S_\theta)_{\theta \in T}$ of bounded operators on $L^2(\tilde{M}, \widehat{E})$ such that

- For any $\gamma \in \Gamma$, $S_{\gamma\theta} = \gamma S_\theta$ where γS_θ is defined using the action of Γ on the equivariant vector bundle $\widehat{E}$;
- The map $\theta \mapsto \|S_\theta\|$ is ν-essentially bounded on T;
- For any $(\xi, \eta) \in H^2$, the map $\theta \mapsto <S_\theta(\xi_\theta), \eta_\theta>$ is Borel measurable.

The von Neumann algebra $W_\nu^*(G; E)$ is a type II_∞ von Neumann algebra as we shall see later. It is easy to see that for any $S \in \mathcal{B}_r^E$, the operator $\pi^{\mathrm{reg}}(S)$ belongs to $W_\nu^*(G; E)$.

In the same way we define a leafwise von Neumann algebra that we shall denote by $W_\nu^*(V, \mathcal{F}; E)$; this algebra acts on the Hilbert space [**21**] $H = \int^\oplus L^2(L_\theta, E|_{L_\theta}) d\nu(\theta)$

where L_θ is, as before, the leaf in V corresponding to θ. Equivalently, and using the identification of the leaves with quotient of $\tilde{M}$ under isotropy, $W^*_\nu(V,\mathcal{F};E)$ can be described as the set of families $(S_\theta)_{\theta\in T}$ of bounded operators on $(L^2(\tilde{M}_\theta/\Gamma(\theta),E|_\theta)_{\theta\in T}$ such that

- The map $\theta\mapsto\|S_\theta\|$ is ν-essentially bounded on T.
- For any square integrable sections ξ,η of the Borel field $(L^2(\tilde{M}_\theta/\Gamma(\theta),E_\theta))_{\theta\in T}$, the map $\theta\mapsto <S_\theta(\xi_\theta),\eta_\theta>$ is Borel measurable.
- $S_{\gamma\theta}=\gamma S_\theta$, for any $(\theta,\gamma)\in T\times\Gamma$.

Notice that $\Gamma(\gamma\theta)=\gamma\Gamma(\theta)\gamma^{-1}$ and hence the definition of γS_θ is clear.

Proposition 2.4. — *There is a well defined representation $\pi_{\rm av}$ from the maximal C^* algebra $\mathcal{B}^E_m$ to the leafwise von Neumann algebra $W^*_\nu(V,\mathcal{F};E)$ such that for $f\in C_c(G,\mathrm{END}(E))$ the operator $(\pi_{\rm av}(f))_\theta$ is given by the kernel*

$$f_0(x,y)=0 \text{ if } L_x\neq L_y \text{ and } f_0([\tilde{m},\theta],[\tilde{m}',\theta]):=\sum_{\gamma\in\Gamma(\theta)} f[\tilde{m},\tilde{m}'\gamma,\theta].$$

Proof. — For simplicity we take E the product line bundle. For $f\in C_c(G)$ the formula

$$((\pi_{\rm av}(f))_\theta\,\xi)(x):=\int_{L_\theta} f_0(x,y)\xi(y)dy,\quad \xi\in L^2(L_\theta), x\in L_\theta\subset V.$$

defines a bounded operator on $L^2(L_\theta)$. Indeed the sum on the RHS in the definition of f_0 is finite since f is compactly supported. Moreover, when restricted to the leaf L_x the kernel f_0 is supported within a uniform neighborhood of the diagonal of L_x. We have:

$$\begin{aligned}
\|\pi_{\rm av}(f))_\theta(\xi)\|_2^2 &= \int_{L_\theta}|\int_{L_\theta} f_0(x,x')\xi(x')dx'|^2dx\\
&\leq \int_{L_\theta}\left(\int_{L_\theta}|f_0(x,x')|dx'\right)\left(\int_{L_\theta}|f_0(x,x')||\xi(x')|^2dx'\right)dx\\
&\leq \|f_0\|_1\int_{L_\theta}|\xi(x')|^2\int_{L_\theta}|f_0(x,x')|dxdx'\\
&\leq \|f_0\|_1^2\|\xi\|_2^2.
\end{aligned}$$

Here $\|f_0\|_1$ stands for the L^1-norm

$$\mathrm{Max}(\sup_{x'\in V}\int_{L_x}|f_0(x,x')|dx,\sup_{x\in V}\int_{L_x}|f_0(x,x')|dx').$$

Therefore, we have

$$\sup_{\theta\in T}\|\pi_{\rm av}(f)\|\leq\|f_0\|_1.$$

But now it is easy to check that $\|f_0\|_1 \leq \|f\|_1$. On the other hand π_{av} is a $*$-representation; since for $f, g \in C_c(G)$ one has, with proof similar to the one given for $\mathcal{G}$,

$$(f * g)_0 = f_0 * g_0 \text{ and } (f^*)_0 = (f_0)^*.$$

To sum up, these arguments prove that π_{av} on $\mathcal{B}_c$ extends to a continuous $*$-representation of the $\mathcal{B}_m$ in the von Neumann algebra $W^*_\nu(V, \mathcal{F})$. This completes the proof. □

2.4. Traces. — We fix once and for all a fundamental domain F for the free and proper action of Γ on $\tilde{M}$. Let χ be the characteristic function of F. Then we set for any non-negative element $S \in W^*_\nu(G; E)$,

$$\tau^\nu(S) := \int_T \mathrm{tr}(M_\chi \circ S_\theta \circ M_\chi) d\nu(\theta),$$

where tr is the usual trace of a non-negative operator on a Hilbert space.

We shall also denote by χ the induced function $\chi \otimes 1_T$, i.e. the characteristic function of $F \times T$ in $\tilde{M} \times T$. Since $F \times T$ is a fundamental domain for the free and proper action of Γ on $\tilde{M} \times T$, we shall also denote by χ_θ the same function χ but viewed as the characteristic function of F inside a given leaf L_θ, which is the image under the projection $\tilde{M} \times T \to V$ of $\tilde{M} \times \{\theta\}$. We define a functional τ^ν on the leafwise von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$, by setting for any non-negative element $S \in W^*_\nu(V, \mathcal{F}; E)$

$$\tau^\nu_{\mathcal{F}}(S) := \int_T \mathrm{tr}(M_{\chi_\theta} \circ S_\theta \circ M_{\chi_\theta}) d\nu(\theta),$$

where the M_{χ_θ} appearing in the integrand is the multiplication operator in the L^2 space of sections over $\tilde{M}_\theta/\Gamma(\theta)$, by the characteristic function χ_θ of F viewed in $\tilde{M}_\theta/\Gamma(\theta)$.

Proposition 2.5. — *With the above notations we have:*

- *the functional τ^ν yields a positive semifinite normal faithful trace on $W^*_\nu(G, E)$;*
- *the functional $\tau^\nu_{\mathcal{F}}$ yields a positive semifinite normal faithful trace on $W^*_\nu(V, \mathcal{F}; E)$.*

Proof. — If $R = S^*S \in W^*_\nu(G; E)$, then for any $\theta \in T$,

$$M_\chi \circ R_\theta \circ M_\chi = (S_\theta M_\chi)^*(S_\theta M_\chi) \geq 0.$$

Therefore, $\mathrm{tr}(M_\chi \circ R_\theta \circ M_\chi) \geq 0$ and hence $\tau^\nu(R) \geq 0$. Moreover, $\tau^\nu(R) = 0$ if and only if $M_\chi R_\theta M_\chi = 0$ for ν-almost every θ. The Γ equivariance of R implies the relations

$$M_{\gamma_1 \chi} R_{\gamma\theta} M_{\gamma_2 \chi} = U_\gamma \left[M_{\gamma^{-1}\gamma_1 \chi} R_\theta M_{\gamma^{-1}\gamma_2 \chi} \right] U_{\gamma^{-1}}, \quad \gamma, \gamma_1, \gamma_2 \in \Gamma.$$

The same relations hold for S. In particular,

$$M_{\gamma\chi}R_\theta M_{\gamma\chi} = U_\gamma \left[M_\chi R_{\gamma^{-1}\theta} M_\chi\right] U_{\gamma^{-1}} = 0.$$

Since ν is Γ-invariant, we deduce that $M_{\gamma\chi}R_\theta M_{\gamma\chi} = 0$ ν almost everywhere. Thus

$$\sum_{\gamma'\in\Gamma} (M_{\gamma'\chi}S_\theta M_{\gamma\chi})^*(M_{\gamma'\chi}S_\theta M_{\gamma\chi}) = M_{\gamma\chi}R_\theta M_{\gamma\chi} = 0, \quad \nu - \text{a.e. } \theta \in T.$$

As a consequence, we get that for ν almost every $\theta \in T$ and for any $\gamma, \gamma' \in \Gamma$,

$$M_{\gamma'\chi}S_\theta M_{\gamma\chi} = 0,$$

which proves that $S = 0$ in $W^*_\nu(G,E)$ and whence $R = 0$ in $W^*_\nu(G,E)$. On the other hand for any non negative $A, B \in W^*_\nu(G;E)$, we have

$$\begin{aligned} M_\chi A_\theta B_\theta M_\chi &= \sum_{\gamma\in\Gamma} M_\chi A_\theta M_{\gamma\chi} B_\theta M_\chi \\ &= \sum_{\gamma\in\Gamma} M_\chi A_\theta (U_\gamma M_\chi U_{\gamma^{-1}}) B_\theta M_\chi \\ &= \sum_{\gamma\in\Gamma} M_\chi U_\gamma A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} U_{\gamma^{-1}} M_\chi \\ &= \sum_{\gamma\in\Gamma} U_\gamma \left[M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi}\right] U_{\gamma^{-1}} \end{aligned}$$

and so,

$$\begin{aligned} \mathrm{tr}(M_\chi A_\theta B_\theta M_\chi) &= \sum_{\gamma\in\Gamma} \mathrm{tr}\left[M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi}\right] \\ &= \sum_{\gamma\in\Gamma} \mathrm{tr}\left[M_\chi B_{\gamma^{-1}\theta} M_{\gamma^{-1}\chi} A_{\gamma^{-1}\theta} M_\chi\right]. \end{aligned}$$

Now the Γ-invariance of ν yields again

$$\begin{aligned} \tau^\nu(AB) &= \int_T \mathrm{tr}(M_\chi A_\theta B_\theta M_\chi) d\nu(\theta) = \int_T \sum_{\gamma\in\Gamma} \mathrm{tr}\left[M_\chi B_\theta M_{\gamma^{-1}\chi} A_\theta M_\chi\right] d\nu(\theta) \\ &= \int_T \mathrm{tr}(M_\chi B_\theta A_\theta M_\chi) d\nu(\theta) = \tau^\nu(BA). \end{aligned}$$

The normality is a consequence of normality of tr and of the Beppo-Levi property. That τ^ν is semi-finite is straightforward.

Finally, according to our description of the leafwise von Neumann algebra $W^*_\nu(V,\mathcal{F};E)$, its elements are also equivariant Borel families. So, the proof of the first item is readily adapted to take care of the quotients by the isotropy groups. □

Recall the two $*$-representations

$$\pi_{\mathrm{reg}} : \mathcal{B}^E_r \to W^*_\nu(G,E), \quad \pi_{\mathrm{av}} : \mathcal{B}^E_m \to W^*_\nu(V,\mathcal{F};E).$$

Corollary 2.6. — *The two functionals* $\tau^\nu_{\mathrm{reg}} := \tau^\nu \circ \pi_{\mathrm{reg}}$ *and* $\tau^\nu_{\mathrm{av}} := \tau^\nu_{\mathcal{G}} \circ \pi_{\mathrm{av}}$ *are traces on the* C^**-algebras* $\mathcal{B}^E_r$ *and* $\mathcal{B}^E_m$ *respectively*[(2)]. *Moreover they are explicitly given, for* $f \in \mathcal{B}^E_c$ *longitudinally smooth by the formulas*

$$\tau^\nu_{\mathrm{reg}}(f) := \int_{F\times T} \mathrm{tr}_{E_{[\tilde m,\theta]}}(f[\tilde m,\tilde m,\theta])d\tilde m d\nu(\theta) \tag{6}$$

$$\tau^\nu_{\mathrm{av}}(f) := \int_{F\times T} \sum_{\gamma\in\Gamma(\theta)} \mathrm{tr}_{E_{[\tilde m,\theta]}}(f[\tilde m,\tilde m\gamma,\theta])d\tilde m d\nu(\theta). \tag{7}$$

Proof. — We only need to show the two formulas (6) and (7). The first one is tautological, so we only sketch the proof of the second one. Let then $f \in \mathcal{B}_c$ longitudinally smooth be fixed. The operator $[\pi_{\mathrm{av}}(f)]_\theta$ acts on $L^2(L_\theta, E)$ with Schwartz kernel f_0 given by

$$f_0([\tilde m,\theta],[\tilde m',\theta]) = \sum_{\gamma\in\Gamma(\theta)} f[\tilde m,\tilde m'\gamma,\theta].$$

Therefore, the operator $M_\chi[\pi_{\mathrm{av}}(f)]_\theta M_\chi$ has Schwartz kernel supported in $F\times F$ viewed in $L_\theta \times L_\theta$. Recall that L_θ is identified with $\tilde M/\Gamma(\theta)$. We deduce

$$\tau^\nu[\pi_{\mathrm{av}}(f)] = \int_{F\times T} f_0([\tilde m,\theta],[\tilde m,\theta])d\mu_\theta(\tilde m)d\nu(\theta),$$

with $d\mu_\theta(\tilde m)$ being the measure induced by $d\tilde m$ on the leaf through θ. Whence, the formula is readily deduced. □

In the sequel we shall also denote by τ^ν_{reg} the resulting trace on the maximal C^*-algebra $\mathcal{B}^E_m$, obtained via the natural epimorphism $\mathcal{B}^E_m \to \mathcal{B}^E_r$.

Remark 2.7. — *The proof of the tracial property of* τ^ν_{reg} *and* τ^ν_{av} *can also be carried out directly. Here are the details (we only treat the averaged trace* τ^ν_{av} *and for simplicity we take* E *equal to the product line bundle). Let* f, f' *be two elements of* $C_c(G)$. *We have:*

$$(f * f')[\tilde m,\tilde m',\theta] = \int_F \sum_{\alpha\in\Gamma} f[\tilde m,\tilde m''\alpha,\theta]f'[\tilde m''\alpha,\tilde m',\theta]d\tilde m''.$$

(2) These traces will not be finite in general.

Hence we deduce

$$\begin{aligned}
\tau^{\nu}_{\mathrm{av}}(f*f') &= \int_{F\times F\times T}\sum_{\gamma\in\Gamma(\theta)}\sum_{\alpha\in\Gamma} f[\tilde m,\tilde m'\alpha,\theta]f'[\tilde m'\alpha,\tilde m\gamma,\theta]d\tilde m'd\tilde m d\nu(\theta)\\
&= \int_{F\times F\times T}\sum_{\gamma\in\Gamma(\theta)}\sum_{\alpha\in\Gamma} f'[\tilde m',\tilde m\gamma\alpha^{-1},\alpha\theta]f[\tilde m\alpha^{-1},\tilde m',\alpha\theta]d\tilde m'd\tilde m d\nu(\theta)\\
&= \int_{F\times F}\sum_{\alpha\in\Gamma}\int_T\sum_{\gamma'\in\Gamma(\theta')} f'[\tilde m'\gamma'^{-1},\tilde m\alpha^{-1},\theta']f[\tilde m\alpha^{-1},\tilde m',\theta']d\tilde m'd\tilde m d\nu(\theta')\\
&= \int_{F\times T}\sum_{\gamma'\in\Gamma(\theta')}(f'*f)[\tilde m\gamma'^{-1},\tilde m',\theta']d\tilde m'd\tilde m d\nu(\theta').
\end{aligned}$$

Now note that since $\gamma'\in\Gamma(\theta')$, *we have*

$$(f'*f)[\tilde m\gamma'^{-1},\tilde m',\theta'] = (f'*f)[\tilde m,\tilde m'\gamma',\theta'].$$

Therefore, we get

$$\tau^{\nu}_{\mathrm{av}}(f*f') = \tau^{\nu}_{\mathrm{av}}(f'*f).$$

Proposition 2.8. — *1. The trace* $\tau^{\nu}_{\mathrm{reg}}$ *induces a group homomorphism* $\tau^{\nu}_{\mathrm{reg},*}: K_0(\mathcal{B}^E_r)\longrightarrow\mathbb{R}$.

2. The trace τ^{ν}_{av} *induces a group homomorphism* $\tau^{\nu}_{\mathrm{av},*}: K_0(\mathcal{B}^E_m)\longrightarrow\mathbb{R}$.

Proof. — We only sketch the proof of this classical result: one shows, for instance, that $L^1(W^*_\nu(G;E))\cap\mathcal{B}^E_r$, with $L^1(W^*_\nu(G;E))$ the Schatten-ideal of τ^ν-trace class operators, is dense holomorphically closed in $\mathcal{B}^E_r$. Similarly $L^1(W^*_\nu(V,\mathcal{F};E))\cap\pi_{\mathrm{av}}(\mathcal{B}^E_m)$ is dense and holomorphically closed in $\pi_{\mathrm{av}}(\mathcal{B}^E_m)$; this finishes the proof by using the definition of τ^ν_{av}. □

2.5. Compatibility with Morita isomorphisms. — The goal of this subsection is to prove the compatibility between the different traces defined so far and the isomorphisms induced in K-theory by Morita equivalence.

Recall the C^*-algebras $\mathcal{A}_r$ and $\mathcal{A}_m$ associated to the groupoid $\mathcal{G}:=T\rtimes\Gamma$. Let $\mathcal{K}$ denote as usual the C^*-algebra of compact operators on a Hilbert space.

Proposition 2.9. — *There are isomorphisms of* C^**-algebras:*

$$\mathcal{B}_r\simeq\mathcal{A}_r\otimes\mathcal{K}\,,\qquad \mathcal{B}_m\simeq\mathcal{A}_m\otimes\mathcal{K}. \tag{8}$$

Proof. — We fix $\tilde m_0\in\tilde M$ and consider the subgroupoid $G(\tilde m_0)$ consisting of the elements which start and end in the image of $\{\tilde m_0\}\times T$ in V:

$$G(\tilde m_0)=\{[\tilde m_0,\tilde m_0\alpha,\theta],\theta\in T\text{ and }\alpha\in\Gamma\}.$$

Notice that the composition in $G(\tilde m_0)$ can be expressed in the following way:

$$[\tilde m_0,\tilde m_o\alpha',\theta']\circ[\tilde m_0,\tilde m_o\alpha,\alpha'\theta'] = [\tilde m_0,\tilde m_0\alpha\alpha',\theta']\,.$$

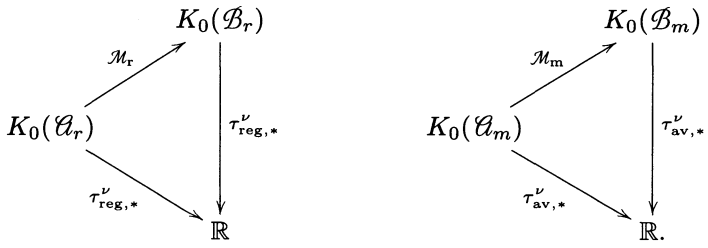
Then there is a *groupoid isomorphism* between $G(\tilde{m}_0)$ and the groupoid $\mathcal{G}$ given by

$$[\tilde{m}_0, \tilde{m}_0\alpha, \theta] \longmapsto (\theta, \alpha^{-1}).$$

In particular the reduced (respectively maximal) C^*-algebras associated to $G(\tilde{m}_0)$ and $\mathcal{G}$ are isomorphic: $C^*_r(G(\tilde{m}_0)) \simeq \mathcal{A}_r$ (respectively $C^*_m(G(\tilde{m}_0)) \simeq \mathcal{A}_m$). Now the main result in [**27**], see also [**8**], together with the fact that the image of $\{\tilde{m}_0\} \times T$ in V intersects every leaf of the foliation, we deduce that the stable C^*-algebra $\mathcal{B}_r$ is isomorphic to the tensor product C^*-algebra $\mathcal{A}_r \otimes \mathcal{K}$. In the same way, the C^*-algebra $\mathcal{B}_m$ is isomorphic to the tensor product C^*-algebra $\mathcal{A}_m \otimes \mathcal{K}$, using the maximal version of the stability theorem which is valid as pointed out in [**27**]. □

Denote by $\mathcal{M}_{\mathrm{r}} : K_0(\mathcal{A}_r) \to K_0(\mathcal{B}_r)$ and $\mathcal{M}_{\mathrm{m}} : K_0(\mathcal{A}_m) \to K_0(\mathcal{B}_m)$ the isomorphisms induced in K-theory by the isomorphisms (8).

Proposition 2.10. — *The following diagrams are commutative*



Proof. — Let us identify T with a fiber of the flat bundle $V = \tilde{M} \times_\Gamma T \to M$. Let Ω be an open connected submanifold of $\tilde{M}$ contained in a fundamental domain F of the action of Γ. Let U be the projection in V of $\Omega \times T$. Then $U \to T$ is an open neighborhood of T in V such that the induced foliation on U is given by the fibres of $U \to T$. The subgroupoid G^U_U of G consisting of homotopy classes of paths drawn in leaves, starting and ending in U, can be describe as

$$G^U_U = \{[\tilde{m}, \tilde{m}'\gamma, \theta] \in \frac{\Omega \times \tilde{M} \times T}{\Gamma}, [\tilde{m}, \theta] \in U \text{ and } [\tilde{m}'\gamma, \theta] \in U\}.$$

An easy inspection of the groupoid laws in G^U_U shows that the bijection

$$[\tilde{m}, \tilde{m}'\gamma, \theta] \longmapsto (\tilde{m}, \tilde{m}', \theta, \gamma^{-1}) \in \Omega \times \Omega \times (T \rtimes \Gamma),$$

is an isomorphism of groupoids, so that the reduced (resp. maximal) C^*-algebra of G^U_U is isomorphic to $\mathcal{K}(L^2\Omega) \otimes [C(T) \rtimes_r \Gamma]$ (resp. $\mathcal{K}(L^2\Omega) \otimes [C(T) \rtimes_m \Gamma]$). Recall that $\mathcal{K}(L^2\Omega)$ denotes the nuclear C^*-algebra of compact operators in the Hilbert space $L^2\Omega$.

If we now fix a continuous compactly supported function φ on Ω with L^2 norm equal to 1 then for any continuous compactly supported function $\xi \in \mathcal{C}_c$, we set:

$$T(\xi)[\tilde{m},\tilde{m}',\theta] := \sum_{\gamma,\gamma'\in\Gamma} \varphi(\tilde{m}\gamma)\overline{\varphi(\tilde{m}'\gamma')}\xi(\gamma^{-1}\theta,\gamma^{-1}\gamma').$$

Since φ is supported in a fundamental domain, it is clear that only one couple (γ,γ') gives a non trivial contribution. Moreover, the function $T(\xi)$ is well defined on G and is supported inside G_U^U. The map T is a $*$-homomorphism from the algebra $\mathcal{C}_c$ to the algebra $\mathcal{B}_c$ which implements the Morita isomorphisms $\mathcal{M}_{\mathrm{r}}$ and $\mathcal{M}_{\mathrm{m}}$ in K-theory. Indeed, we have:

$$\begin{aligned}
T(\xi)*T(\xi')[\tilde{m},\tilde{m}',\theta] &= \int_{\tilde{M}} T(\xi)[\tilde{m},\tilde{m}'',\theta]T(\xi')[\tilde{m}'',\tilde{m}',\theta]d\tilde{m}'' \\
&= \sum_{\alpha,\alpha',\beta,\beta'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\beta')}\int_{\tilde{M}} \overline{\varphi(\tilde{m}''\beta)}\varphi(\tilde{m}''\alpha')d\tilde{m}'' \times \\
&\quad \xi(\alpha^{-1}\theta,\alpha^{-1}\beta)\xi'(\alpha'^{-1}\theta,\alpha'^{-1}\beta') \\
&= \sum_{\alpha,\alpha'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\alpha')}\sum_{\beta\in\Gamma}\xi(\alpha^{-1}\theta,\alpha^{-1}\beta)\xi'(\beta^{-1}\theta,\beta^{-1}\alpha') \\
&= \sum_{\alpha,\alpha'\in\Gamma} \varphi(\tilde{m}\alpha)\overline{\varphi(\tilde{m}'\alpha')}(\xi*\xi')(\alpha^{-1}\theta,\alpha^{-1}\alpha') \\
&= T(\xi*\xi')[\tilde{m},\tilde{m}',\theta].
\end{aligned}$$

Hence, we conclude that

$$T(\xi)*T(\xi') = T(\xi*\xi').$$

In a similar way one checks that $(T(\xi))^* = T(\xi^*)$.
T extends to a morphism between the corresponding reduced C^*-algebras. More precisely, let $f \in L^2(\tilde{M})$, then the regular representation π^{reg} is given for any $\tilde{m} \in \tilde{M}$ by:

$$(\pi^{\mathrm{reg}}T(\xi))_\theta(f)(\tilde{m}) = \int_{\tilde{M}} \sum_{\gamma',\gamma\in\Gamma} \varphi(\tilde{m}\gamma)\overline{\varphi(\tilde{m}'\gamma')}\xi(\gamma^{-1}\theta,\gamma^{-1}\gamma')f(\tilde{m}')d\tilde{m}'.$$

Denote by $g : \Gamma \to \mathbb{C}$ the function given by

$$g(\gamma') := \int_{\tilde{M}} \overline{\varphi(\tilde{m}'\gamma'^{-1})}f(\tilde{m}')d\tilde{m}',$$

then, one easily shows that the function g belongs to $\ell^2(\Gamma)$ and that its ℓ^2-norm can be estimated as follows:

$$\begin{aligned}\|g\|_2^2 &= \sum_{\gamma'} |g(\gamma')|^2 = \sum_{\gamma'} \left| \int_{\tilde{M}} \overline{\phi(\tilde{m}'\gamma'^{-1})} f(\tilde{m}')\, d\tilde{m} \right|^2 \\ &= \sum_{\gamma'} \left| \int_{F\gamma'} \overline{\phi(\tilde{m}'\gamma'^{-1})} f(\tilde{m}')\, d\tilde{m}' \right|^2 \leq \sum_{\gamma'} \int_{F\gamma'} |f(\tilde{m}')|^2 d\tilde{m}' = \|f\|_2^2.\end{aligned}$$

If we recall the regular representation of the algebra $\mathcal{A}_c$, denoted also by π^{reg}, then, using g we can write:

$$(\pi_{\mathrm{reg}} T(\xi))_\theta(f)(\tilde{m}) = \sum_{\gamma\in\Gamma} \phi(\tilde{m}\gamma) \sum_{\gamma'\in\Gamma} \xi(\gamma^{-1}\theta, \gamma^{-1}\gamma') g(\gamma'^{-1}) = \sum_{\gamma\in\Gamma} \phi(\tilde{m}\gamma)(\pi_\theta^{\mathrm{reg}}(\xi))(g)(\gamma^{-1})$$

Therefore, if we compute the L^2-norm of the function $(\pi^{\mathrm{reg}} T(\xi))_\theta(f)$ we get:

$$\begin{aligned}\|(\pi_{\mathrm{reg}} T(\xi))_\theta(f)\|_2^2 &= \int_{\tilde{M}} \left| \sum_{\gamma\in\Gamma} \phi(\tilde{m}\gamma) \pi_\theta^{\mathrm{reg}}(\xi)(g)(\gamma^{-1}) \right|^2 d\tilde{m} \\ &= \sum_{\alpha\in\Gamma} \int_{F\alpha^{-1}} \left| \phi(\tilde{m}\alpha) \pi_\theta^{\mathrm{reg}}(\xi)(g)(\alpha^{-1}) \right|^2 d\tilde{m} \\ &= \sum_{\alpha\in\Gamma} \left| \pi_\theta^{\mathrm{reg}}(\xi)(g)(\alpha^{-1}) \right|^2 \int_{F\alpha^{-1}} |\phi(\tilde{m}\alpha)|^2 d\tilde{m} \\ &= \|\pi_\theta^{\mathrm{reg}}(\xi)(g)\|_2^2 \\ &\leq \|\xi\|_{\mathcal{A}_r}^2 \|g\|_2^2 \leq \|\xi\|_{\mathcal{A}_r}^2 \|f\|_2^2.\end{aligned}$$

Summarizing: $\sup_{\theta\in T} \|(\pi^{\mathrm{reg}}(T\xi))_\theta\| \leq \|\xi\|_{\mathcal{A}_r}$ so that $\|T(\xi)\|_{\mathcal{B}_r} \leq \|\xi\|_{\mathcal{A}_r}$ as required. It thus remains to show compatibility of the traces with respect to the homomorphism T, and only on the compactly supported functions. Let us start with the regular trace. We have:

$$\begin{aligned}\tau_{\mathrm{reg}}^\nu(T(\xi)) &= \int_{F\times T} T(\xi)[\tilde{m}, \tilde{m}, \theta] d\tilde{m} d\nu(\theta) \\ &= \int_T \xi(\theta, 1) \int_{\tilde{M}} |\varphi(\tilde{m})|^2 d\tilde{m} d\nu(\theta) \\ &= \int_T \xi(\theta, 1) d\nu(\theta) \\ &= \tau_{\mathrm{reg}}^\nu(\xi).\end{aligned}$$

Note that when $\tilde{m} \in \Omega$, only $\gamma = 1$ contributes to the sum defining $T(\xi)$.

Let us now check, briefly, that T induces a morphism between the maximal C^*-algebras. It suffices to show that T is continuous with respect to the L^1-norms on the

groupoids $\mathcal{G}$ and G. But for $\xi \in \mathcal{A}_c$ and for any $\tilde{m} \in \Omega$ we have

$$\begin{aligned}\int_{\tilde{M}} |(T\xi)[\tilde{m}, \tilde{m}', \theta]|\, d\tilde{m}' &\leq |\phi(\tilde{m})| \int_{\tilde{M}} |\phi(\tilde{m}')|\, d\tilde{m}' \left(\sum_{\gamma' \in \Gamma} |\xi(\theta, \gamma')| \right) \\ &\leq \|\phi\|_1 \|\phi\|_\infty \|\xi\|_1 .\end{aligned}$$

Hence,

$$\|T(\xi)\|_1 \leq \|\phi\|_1 \|\phi\|_\infty \|\xi\|_1 .$$

Now let us check the compatibility with the average trace τ^ν_{av}. We have, for $\xi \in \mathcal{A}_c$:

$$\begin{aligned}\tau^\nu_{\mathrm{av}}(T(\xi)) &= \int_{F \times T} \sum_{\gamma \in \Gamma(\theta)} T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta] d\tilde{m}\, d\nu(\theta) \\ &= \int_{\Omega \times T} \sum_{\gamma \in \Gamma(\theta)} T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta] d\tilde{m}\, d\nu(\theta)\end{aligned}$$

$$= \int_T \sum_{\gamma \in \Gamma(\theta)} \xi(\theta, \gamma) \int_\Omega |\varphi(\tilde{m})|^2 d\tilde{m}\, d\nu(\theta)$$
$$= \int_T \sum_{\gamma \in \Gamma(\theta)} \xi(\theta, \gamma) d\nu(\theta)$$
$$= \tau^\nu_{\mathrm{av}}(\xi).$$

Note that in the expression $T(\xi)[\tilde{m}, \tilde{m}\gamma, \theta]$ for $\tilde{m} \in \Omega$, only the couple $(1, \gamma)$ contributes non trivially to the sum. □

3. Hilbert modules and Dirac operators

3.1. Connes-Skandalis Hilbert module. — Recall that $V = \tilde{M} \times_\Gamma T$ where $\tilde{M} \to M$ is the universal Γ-covering of the closed manifold M and where Γ acts by homeomorphisms on the compact space T. We fix a hermitian vector bundle E over V and we denote by $\widehat{E}$ its pull-back to $\tilde{M} \times T$. We define a right action of the convolution algebra $\mathcal{A}_c = C_c(T \rtimes \Gamma) \equiv C_c(\mathcal{G})$ on the space $\mathcal{E}_c = C_c^{\infty,0}(\widetilde{M} \times T; \widehat{E})$, of compactly supported sections of the vector bundle $\widehat{E}$ which are smooth with respect to the $\tilde{M}$ variable and continuous with respect to the T variable, as follows.

$$(\xi f)(\tilde{m}, \theta) = \sum_{\gamma \in \Gamma} \xi(\tilde{m}\gamma^{-1}, \gamma\theta) f(\gamma\theta, \gamma), \quad \xi \in \mathcal{E}_c, \quad f \in \mathcal{A}_c.$$

A $\mathcal{A}_c$-valued inner product $<.;.>$ on $\mathcal{E}_c$ is also defined by [**27**]

$$<\xi_1; \xi_2>(\theta, \gamma) := \int_{\widetilde{M}} <\xi_1(\tilde{m}, \gamma^{-1}\theta); \xi_2(\tilde{m}\gamma^{-1}, \theta)>_{E_{[\tilde{m}, \theta]}} d\tilde{m},$$

where $<.;.>_E$ is the hermitian scalar product that we have fixed of the vector bundle E. A classical computation shows that these operations endow the space $\mathcal{E}_c$ with the structure of a pre-Hilbert module over the algebra $\mathcal{A}_c$.

As in the previous sections, we denote by $\mathcal{A}_r$ and $\mathcal{A}_m$ the reduced and maximal C^*-algebras of the groupoid $\mathcal{G}$. Recall that there is a natural C^*-algebra morphism

$$\lambda : \mathcal{A}_m \longrightarrow \mathcal{A}_r.$$

The pre-Hilbert $\mathcal{A}_c$-module $\mathcal{E}_c$ can be completed with respect to the reduced C^*-norm to yield a right Hilbert C^*-module over $\mathcal{A}_r$ that we shall denote by $\mathcal{E}_r$. In the same way, we can complete $\mathcal{E}_c$ with respect to the maximal C^*-norm and define the Hilbert C^*-module $\mathcal{E}_m$ over the C^*-algebra $\mathcal{A}_m$. It is then clear that the natural map $\mathcal{E}_c \longrightarrow \mathcal{E}_r$, extends to a morphism of Hilbert modules $\mathcal{E}_m \to \mathcal{E}_r$. More precisely, we have a well defined linear map

$$\varrho : \mathcal{E}_m \longrightarrow \mathcal{E}_r \text{ such that } \varrho(\xi f) = \varrho(\xi)\lambda(f) \quad f \in \mathcal{A}_m \text{ and } \xi \in \mathcal{E}_m.$$

We denote as in the previous sections by G the monodromy groupoid

$$G := \frac{\tilde{M} \times \tilde{M} \times T}{\Gamma}.$$

The algebra $\mathcal{B}_c^E$ of smooth compactly supported sections of the bundle $\mathrm{END}(E)$ over G is faithfully represented in $\mathcal{E}_c$ by the formula [**19**]

$$\chi(\varphi)(\xi)(\tilde{m},\theta) := \int_{\tilde{M}} \varphi[\tilde{m},\tilde{m}',\theta]\xi(\tilde{m}',\theta)d\tilde{m}', \quad \varphi \in C_c^\infty(G,\mathrm{END}(E)), \xi \in \mathcal{E}_c.$$

Recall that $\mathcal{B}_r^E$ and $\mathcal{B}_m^E$ are respectively the reduced and maximal C^*-algebras associated with G and with coefficients in E. Given a C^*-algebra A and a Hilbert A-module $\mathcal{E}$, the algebra $B_A(\mathcal{E})$ consists of bounded adjointable A-linear morphisms of $\mathcal{E}$. Recall also that the C^*-algebra $\mathcal{K}_A(\mathcal{E})$ of A-compact operators is the completion in $B_A(\mathcal{E})$ of the subalgebra of A-finite rank operators. The following proposition is proved in [**27**], see also [**39**] and [**8**].

Proposition 3.1. — *For any $\varphi \in \mathcal{B}_c^E$, the map $\chi(\varphi) : \mathcal{E}_c \to \mathcal{E}_c$ is $\mathcal{A}_c$-linear and the morphism χ extends to continuous $*$-representations*

$$\chi_r : \mathcal{B}_r^E \longrightarrow \mathcal{K}_{\mathcal{A}_r}(\mathcal{E}_r) \text{ and } \chi_m : \mathcal{B}_m^E \longrightarrow \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m),$$

which are C^*-algebra isomorphisms.

Notice that the proof of this proposition is usually given for the holonomy groupoid of the foliation; however the same argument applies to the monodromy groupoid. Note also that the proof is usually given for the reduced C^*-algebra but it remains valid for the maximal C^*algebra too [**27**] [Remarque 5].

For any $\theta \in T$, we have defined in Subsection 1.3 a representation π_θ^{av} of the maximal C^*-algebra $\mathcal{A}_m$ in the Hilbert space $\ell^2(\Gamma/\Gamma(\theta))$. By using Remark 1.1 we can write

$$\pi_\theta^{av}(f)(\xi)(\theta') := \sum_{\theta'' \in \Gamma.\theta} \sum_{\gamma\theta''=\theta'} f(\theta',\gamma)\xi(\theta''), \quad f \in \mathcal{A}_c, \xi \in \ell^2(\Gamma\theta) \text{ and } \theta' \in \Gamma\theta.$$

Using the $\mathcal{A}_m$-Hilbert module $\mathcal{E}_m$ together with the representation π_θ^{av}, one defines the Hilbert space

$$\mathcal{H}_\theta^{av} := \mathcal{E}_m \otimes_{\pi_\theta^{av}} \ell^2(\Gamma\theta).$$

Similarly

$$\mathcal{H}_\theta^{reg} := \mathcal{E}_m \otimes_{\pi_\theta^{reg}} \ell^2(\Gamma).$$

Lemma 3.2. — *There exists an isomorphism of Hilbert spaces, Φ_θ, between $\mathcal{H}_\theta^{av}$ and the Hilbert space $L^2(L_\theta, E)$ of square integrable sections of the vector bundle E over*

the leaf L_θ through θ, induced by the formula

$$\Phi_\theta(\xi \otimes f)(\tilde{m}, \theta) := \sum_{\gamma \in \Gamma} f(\gamma\theta)\ \left[\xi(\tilde{m}\gamma^{-1}, \gamma\theta)\right], \quad \xi \in \mathcal{E}_c \text{ and } f \in C_c(\Gamma\theta).$$

Similarly there exists an isomorphism Ψ_θ of Hilbert spaces between $\mathcal{H}_\theta^{reg}$ and $L^2(\tilde{M}_\theta, \widehat{E})$ induced by the formula

$$\Psi_\theta(\xi \otimes \delta_\gamma)(\tilde{m}) := \xi(\tilde{m}\gamma^{-1}, \gamma\theta).$$

where δ_γ denotes the delta function at Γ.

Proof. — If $\alpha \in \Gamma(\theta)$ then we can write for $\xi \in \mathcal{E}_c$:

$$\begin{aligned}
\Phi_\theta(\xi \otimes f)(\tilde{m}\alpha^{-1}, \theta) &= \sum_{\gamma \in \Gamma} f(\gamma\theta)\ \left[\xi(\tilde{m}\alpha^{-1}\gamma^{-1}, \gamma\theta)\right] \\
&= \sum_{\beta \in \Gamma} f(\beta\theta)\ \xi(\tilde{m}\beta^{-1}, \beta\theta) \\
&= \Phi_\theta(\xi \otimes f)(\tilde{m}, \theta).
\end{aligned}$$

Hence, $\Phi_\theta(\xi \otimes f)$ is a smooth section of $\hat{E}$ over $\tilde{M}_\theta$ which is $\Gamma(\theta)$-invariant. Moreover, if $f = \delta_{\gamma\theta}$ and if we denote by $K_{\gamma\theta}$ the (compact) support of ξ in $\tilde{M} \times \{\gamma\theta\}$ then the support of $\Phi_\theta(\xi \otimes \delta_{\gamma\theta})$ is contained in

$$[K_{\gamma\theta} \cdot \gamma] \cdot \Gamma(\theta),$$

and hence is $\Gamma(\theta)$-compact.

Let now $g \in \mathcal{C}_c$ be given. Then we have

$$\begin{aligned}
\Phi_\theta(\xi g \otimes f)(\tilde{m}, \theta) &= \sum_{\gamma \in \Gamma} f(\gamma\theta)(\xi g)(\tilde{m}\gamma^{-1}, \gamma\theta) \\
&= \sum_{\gamma \in \Gamma} f(\gamma\theta) \sum_{\alpha \in \Gamma} g(\alpha\gamma\theta, \alpha)\xi(\tilde{m}\gamma^{-1}\alpha^{-1}, \alpha\gamma\theta) \\
&= \sum_{\theta', \theta'' \in \Gamma\theta} \sum_{\gamma\theta = \theta', \alpha\theta' = \theta''} f(\theta')g(\theta'', \alpha)\xi(\tilde{m}(\alpha\gamma)^{-1}, \theta'') \\
&= \sum_{\theta', \theta'' \in \Gamma\theta} \sum_{\beta\theta = \theta'', \alpha\theta' = \theta''} f(\theta')g(\theta'', \alpha)\xi(\tilde{m}\beta^{-1}, \theta'').
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
\Phi_\theta(\xi \otimes \pi_\theta^{\mathrm{av}}(g)(f))(\tilde{m}, \theta) &= \sum_{\theta'' \in \Gamma\theta} \pi_\theta^{\mathrm{av}}(g)(f)(\theta'') \sum_{\gamma_1\theta = \theta''} \xi(\tilde{m}\gamma_1^{-1}, \theta'') \\
&= \sum_{\theta'', \theta' \in \Gamma\theta} \sum_{\delta\theta' = \theta'', \delta_1\theta = \theta''} f(\theta')g(\theta'', \delta)\xi(\tilde{m}\delta_1^{-1}, \theta'').
\end{aligned}$$

Hence, we obtain the equality $\Phi_\theta(\xi g \otimes f) = \Phi_\theta(\xi \otimes \pi_\theta^{\mathrm{av}}(g)(f))$.

In order to finish the proof, we need to identify the scalar product on the Hilbert space $\mathcal{H}_\theta^{av}$. We have

$$\begin{aligned}
<\xi\otimes f,\xi\otimes f> &= <\pi_\theta^{av}(<\xi,\xi>)(f),f> \\
&= \sum_{\theta'\in\Gamma\theta}\pi_\theta^{av}(<\xi,\xi>)(f)(\theta')\overline{f(\theta')} \\
&= \sum_{\theta'\in\Gamma\theta}\overline{f(\theta')}\sum_{\beta\in\Gamma}<\xi,\xi>(\theta',\beta)f(\beta^{-1}\theta') \\
&= \sum_{\theta',\theta''\in\Gamma\theta}\overline{f(\theta')}f(\theta'')\sum_{\beta\theta''=\theta'}\int_{\tilde M}<\xi(\tilde m,\beta^{-1}\theta'),\xi(\tilde m\beta^{-1},\theta')>d\tilde m \\
&= \sum_{\theta',\theta''\in\Gamma\theta}\overline{f(\theta')}f(\theta'')\sum_{\alpha\theta'=\theta''}\int_{\tilde M}<\xi(\tilde m,\alpha\theta'),\xi(\tilde m\alpha,\theta')>d\tilde m
\end{aligned}$$

On the other hand, if we view $\Phi_\theta(\xi\otimes f)$ as a section over the leaf L_θ through θ, then we can use a fundamental domain F_θ for the free and proper action of the isotropy group $\Gamma(\theta)$ on $\tilde M$ and write

$$\begin{aligned}
&<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> = \int_{F_\theta}<\Phi_\theta(\xi\otimes f)(\tilde m,\theta),\Phi_\theta(\xi\otimes f)(\tilde m,\theta)>d\tilde m \\
= &\sum_{\theta_1,\theta_2\in\Gamma\theta}f(\theta_1)\overline{f(\theta_2)}\sum_{\gamma_1\theta=\theta_1,\gamma_2\theta=\theta_2}\int_{F_\theta}<\xi(\tilde m\gamma_1^{-1},\theta_1),\xi(\tilde m\gamma_2^{-1},\theta_2)>d\tilde m
\end{aligned}$$

We fix a section $\varphi:\Gamma\theta\to\Gamma$ of the map $\gamma\mapsto\gamma\theta$. Then $\beta=\varphi(\theta_1)^{-1}\gamma_1$ is an element of the isotropy group $\Gamma(\theta)$ and we have

$$\begin{aligned}
&<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> \\
= &\sum_{\theta'',\theta'\in\Gamma\theta}f(\theta'')\overline{f(\theta')}\sum_{\gamma_2\theta=\theta'}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta}<\xi(\tilde m\beta^{-1}\varphi(\theta'')^{-1},\theta''),\xi(\tilde m\gamma_2^{-1},\theta')>d\tilde m \\
= &\sum_{\theta'',\theta'\in\Gamma\theta}f(\theta'')\overline{f(\theta')}\sum_{\gamma_2\theta=\theta'}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta\beta^{-1}\varphi(\theta'')^{-1}}\times \\
&<\xi(\tilde m_1,\theta''),\xi(\tilde m_1\varphi(\theta'')\beta\gamma_2^{-1},\theta')>d\tilde m_1 \\
= &\sum_{\theta'',\theta'\in\Gamma\theta}f(\theta'')\overline{f(\theta')}\sum_{\alpha\theta'=\theta''}\sum_{\beta\in\Gamma(\theta)}\int_{F_\theta\beta^{-1}\varphi(\theta'')^{-1}}<\xi(\tilde m_1,\theta''),\xi(\tilde m_1\alpha,\theta')>d\tilde m
\end{aligned}$$

Setting $\delta=\varphi(\theta_1)\beta^{-1}\varphi(\theta_1)^{-1}$ and noticing that a fundamental domain $F_{\theta''}$ is equal to $F_\theta\varphi(\theta'')^{-1}$ we get

$$\begin{aligned}
&<\Phi_\theta(\xi\otimes f),\Phi_\theta(\xi\otimes f)> \\
&= \sum_{\theta',\theta''\in\Gamma\theta}f(\theta'')\overline{f(\theta')}\sum_{\alpha\theta'=\theta''}\sum_{\delta\in\Gamma(\theta'')}\int_{(F_\theta\varphi(\theta'')^{-1})\delta}<\xi(\tilde m_1,\theta''),\xi(\tilde m_1\alpha,\alpha^{-1}\theta'')>d\tilde m
\end{aligned}$$

$$= \sum_{\theta',\theta''\in\Gamma\theta} f(\theta'')\overline{f(\theta')} \sum_{\alpha\theta'=\theta''} \sum_{\delta\in\Gamma(\theta'')} \int_{F_{\theta''\delta}} <\xi(\tilde{m}_1,\theta''), \xi(\tilde{m}_1\alpha, \alpha^{-1}\theta'')> d\tilde{m}$$

$$= \sum_{\theta',\theta''\in\Gamma\theta} f(\theta'')\overline{f(\theta')} \sum_{\alpha\theta'=\theta''} \int_{\tilde{M}} <\xi(\tilde{m}_1,\alpha\theta'), \xi(\tilde{m}_1\alpha,\theta')> d\tilde{m}.$$

Hence $<\xi\otimes f, \xi\otimes f> = <\Phi_\theta(\xi\otimes f), \Phi_\theta(\xi\otimes f)>$. It now remains to show that Φ_θ is surjective. Let η be a smooth compactly supported section over the leaf L_θ and denote by $\tilde{\eta}$ its lift into a $\Gamma(\theta)$-invariant section over $\tilde{M}\times\theta$ and by ξ_0 any extension of $\tilde{\eta}$ into a leafwise smooth continuous section over $\tilde{M}\times T$. Let φ be a smooth function on $\tilde{M}$ such that $\sum_{\alpha\in\Gamma(\theta)} \alpha\varphi = 1$ and such that for any compact set K in $L_\theta \simeq \tilde{M}/\Gamma(\theta)$, the intersection of the support of φ with the inverse image of K, under the projection $\tilde{M}\to L_\theta$, is compact in $\tilde{M}$. We view φ as a function on $\tilde{M}\times T$ independent of the T variable and set

$$\xi := \varphi\xi_0.$$

Then $\xi\in C_c^{\infty,0}(\tilde{M}\times T, \tilde{E})$ and one checks immediately that $\Phi_\theta(\xi\otimes\delta_\theta) = \eta$. The proof of the second isomorphism is simpler and is left as an exercise. □

Recall that we have defined two representations, that we have both denoted π^{av}, respectively of the C^*-algebras $\mathcal{A}_m$ and $\mathcal{B}_m^E$ in the corresponding von Neumann algebras of the discrete groupoid $\mathcal{G}$ and of the monodromy groupoid G with coefficients in the vector bundle E:

$$\pi^{\mathrm{av}} : \mathcal{A}_m \to W^*_{\mathrm{av}}(\mathcal{G}), \quad \pi^{\mathrm{av}} : \mathcal{B}_m^E \to W^*_\nu(V,\mathcal{F};E).$$

Recall also that we have defined a $*$-representation χ_m of $\mathcal{B}_m^E$ in the compact operators of the Hilbert module $\mathcal{E}_m$:

$$\chi_m : \mathcal{B}_m^E \to \mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m).$$

Proposition 3.3. — *Let S be a given element of $\mathcal{B}_m^E$. Then we have*

$$\pi_\theta^{av}(S) = \Phi_\theta \circ \left[\chi_m(S)\otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma\theta)}\right] \circ \Phi_\theta^{-1}.$$

with $\Phi_\theta : \mathcal{E}_m\otimes_{\pi_\theta^{\mathrm{av}}} \ell^2(\Gamma\theta) \to L^2(L_\theta, E)$ the isomorphism given in Lemma 3.2. In the same way, we have

$$\pi_\theta^{reg}(S) = \Psi_\theta \circ \left[\chi_r(S)\otimes_{\pi_\theta^{reg}} I_{\ell^2(\Gamma)}\right] \circ \Psi_\theta^{-1}.$$

with $\Psi_\theta : \mathcal{E}_m\otimes_{\pi_\theta^{reg}} \ell^2(\Gamma) \to L^2(\tilde{M}_\theta, \widehat{E})$ the second isomorphism given in Lemma 3.2.

Proof. — Let us fix an element $k \in C_c^{\infty,0}(G; \mathrm{END}(E))$ and give the proof for $S = k$. We compute for $\xi \in \mathcal{E}_c$ and $f \in C_c[\Gamma\theta]$:

$$\begin{aligned}
\Phi_\theta(\chi(k)(\xi) \otimes f)(\tilde{m}, \theta) &= \sum_{\gamma \in \Gamma} f(\gamma\theta)\chi(k)(\xi)(\tilde{m}\gamma^{-1}, \gamma\theta) \\
&= \sum_{\gamma \in \Gamma} f(\gamma\theta) \int_{\tilde{M}} k[\tilde{m}\gamma^{-1}, \tilde{m}', \gamma\theta]\xi(\tilde{m}', \gamma\theta) d\tilde{m}' \\
&= \sum_{\gamma \in \Gamma} f(\gamma\theta) \int_{\tilde{M}} k[\tilde{m}, \tilde{m}'\gamma, \theta]\xi(\tilde{m}', \gamma\theta) d\tilde{m}' \\
&= \sum_{\gamma \in \Gamma} f(\gamma\theta) \int_{\tilde{M}} k[\tilde{m}, \tilde{m}_1, \theta]\xi(\tilde{m}_1\gamma^{-1}, \gamma\theta) d\tilde{m}_1.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\pi_\theta^{av}(k)(\Phi_\theta(\xi \otimes f))(\tilde{m}, \theta) &= \int_{F_\theta} \sum_{\alpha \in \Gamma(\theta)} k[\tilde{m}, \tilde{m}'\alpha, \theta] \sum_{\gamma \in \Gamma} f(\gamma\theta)\xi(\tilde{m}'\gamma^{-1}, \gamma\theta) d\tilde{m}' \\
&= \sum_{\gamma \in \Gamma} f(\gamma\theta) \sum_{\alpha \in \Gamma(\theta)} \int_{F_\theta \alpha} k[\tilde{m}, \tilde{m}'', \theta]\xi(\tilde{m}''\alpha^{-1}\gamma^{-1}, \gamma\theta) d\tilde{m}'' \\
&= \sum_{\gamma' \in \Gamma} f(\gamma'\theta) \sum_{\alpha \in \Gamma(\theta)} \int_{F_\theta \alpha} k[\tilde{m}, \tilde{m}'', \theta]\xi(\tilde{m}''\gamma'^{-1}, \gamma'\theta) d\tilde{m}'' \\
&= \sum_{\gamma' \in \Gamma} f(\gamma'\theta) \int_{\tilde{M}} k[\tilde{m}, \tilde{m}'', \theta]\xi(\tilde{m}''\gamma'^{-1}, \gamma'\theta) d\tilde{m}''.
\end{aligned}$$

So we get

$$\Phi_\theta(\chi(k)(\xi) \otimes f) = \pi_\theta^{av}(k)(\Phi_\theta(\xi \otimes f))$$

which proves the first statement by continuity. We omit the proof of the second statement as it is similar and in fact easier. □

3.2. Γ-equivariant pseudodifferential operators. — This subsection is devoted to a brief overview of the pseudodifferential calculus relevant to our study. All stated results are known and we therefore only sketch the proofs.

Let $\mathcal{E}_c$ be as before $C_c^{\infty,0}(\tilde{M} \times T, \widehat{E})$ endowed with its structure of pre-Hilbert $\mathcal{A}_c$-module. Recall that if we complete the prehilbertian module $\mathcal{E}_c$ with respect to the regular norm on $\mathcal{A}_c$ then we get a Hilbert C^*-module $\mathcal{E}_r$ over the regular C^*-algebra $\mathcal{A}_r$. In the same way, completing $\mathcal{A}_c$ with respect to the maximal C^*-norm yields a Hilbert C^*-module $\mathcal{E}_m$ over the maximal C^*-algebra $\mathcal{A}_m$. We fix two vector bundles E and F over V and we denote by $\widehat{E}$ and $\widehat{F}$ their pullbacks to $\tilde{M} \times T$ into Γ-equivariant vector bundles; we let $\widehat{E}_\theta$ be the restriction of $\widehat{E}$ to $\tilde{M}_\theta$. We set, as before, $\tilde{M}_\theta := \tilde{M}_\theta$.

Definition 3.4. — *Let $P : C_c^{\infty,0}(\tilde{M} \times T, \widehat{E}) \to C^{\infty,0}(\tilde{M} \times T, \widehat{F})$ be a linear map. We shall say that P defines a pseudodifferential operator of order m on the monodromy groupoid G if there is a continuous family of order m pseudodifferential operators $(P_\theta)_{\theta \in T}$,*

$$P_\theta : C_c^\infty(\tilde{M}_\theta, \widehat{E}_\theta) \to C^\infty(\tilde{M}_\theta, \widehat{F}_\theta)$$

satisfying:
(1) $(P\xi)(\tilde{m}, \theta) = (P_\theta \xi(\cdot, \theta))(\tilde{m} \times \{\theta\})$
(2) *P is Γ-equivariant: $R_\gamma^* P R_{\gamma^{-1}}^* = P$;*
(3) *the Schwartz kernel of P, K_P, which can be thought of as a Γ-invariant distributional section on $\tilde{M} \times \tilde{M} \times T$, is of Γ-compact support, i.e. the image of the support in $(\tilde{M} \times \tilde{M} \times T)/\Gamma =: G$ is a compact set.*

Notice that (2) can be then restated as: $P_{\gamma\theta} = \gamma P_\theta \ \forall \theta \in T$, $\forall \gamma \in \Gamma$, exactly as in the definition of the regular von Neumann algebra. The notion of continuity for families of pseudodifferential operators is classical and will not be recalled here, see, for example, [**11**], [**42**], [**32**], [**55**], [**56**]. Finally, because of the third condition P maps $C_c^{\infty,0}(\tilde{M} \times T, \widehat{E})$ into $C_c^{\infty,0}(\tilde{M} \times T, \widehat{F})$.

Notice that a Γ-equivariant continuous family of differential operators acting between the sections of two equivariant vector bundles is an example of a pseudodifferential operator on G.

If $m \in \mathbb{Z}$, we shall denote by $\Psi_c^m(G; \widehat{E}, \widehat{F})$ the space of pseudodifferential operators of order $\leq m$ from $\widehat{E}$ to $\widehat{F}$ [(3)] . We set

$$\Psi_c^\infty(G; \widehat{E}, \widehat{F}) := \bigcup_{m \in \mathbb{Z}} \Psi_c^m(G; \widehat{E}, \widehat{F}) \text{ and } \Psi_c^{-\infty}(G; \widehat{E}, \widehat{F}) := \bigcap_{m \in \mathbb{Z}} \Psi_c^m(G; \widehat{E}, \widehat{F}).$$

Using condition (3) , it is not difficult to check that the space $\Psi_c^\infty(G; \widehat{E}, \widehat{E})$ is a filtered algebra. Moreover, assigning to P its formal adjoint $P^* = (P_\theta^*)_{\theta \in T}$ gives $\Psi_c^\infty(G; \widehat{E}, \widehat{E})$ the structure of an involutive algebra; the formal adjoint is defined also for $P \in \Psi_c^m(G; \widehat{E}, \widehat{F})$ and it is then an alement in $\Psi_c^m(G; \widehat{F}, \widehat{E})$.

Remark 3.5. — *Notice that Definition 3.4 fits into the general framework of pseudodifferential calculus on groupoids, as developed by Connes and many others. More precisely, let $P = (P_\theta)_{\theta \in T}$ be a pseudodifferential operator on G as in Definition 3.4. For any $\theta \in T$ and any $x = [\tilde{m}, \theta] \in L_\theta$ the diffeomorphism*

$$\rho_{x,\theta} : \tilde{M} \to G^x = r^{-1}(x) \quad \textit{given by} \quad \rho_{x,\theta}(\tilde{m}') = [\tilde{m}, \tilde{m}', \theta],$$

(3) The notation for this space of operators is not unique: in [**34**] it is denoted $\Psi^*_{\rtimes,c}(\tilde{M} \times T; \widehat{E}, \widehat{F})$ with $\rtimes$ denoting equivariance and c denoting again *of Γ-compact support*; in [**39**] it is simply denoted as $\Psi^*_\Gamma(\widehat{E}, \widehat{F})$.

allows to define a pseudodifferential operator P_x on G^x with coefficients in s^*E, viz. $P_x := (\rho_{x,\theta}^{-1})^* \circ P_\theta \circ (\rho_{x,\theta})^*$. It is easy to check that P_x only depends on x and that the family $(P_x)_{x\in V}$ is a pseudodifferential operator on G in the sense of Connes. Conversely if we are given now a pseudodifferential operator $(P_x)_{x\in V}$ in the sense of Connes, then a choice of a base point m_0 in M allows to construct $P = (P_\theta)_{\theta\in T}$ satisfying the assumptions of Definition 3.4, viz. $P_\theta := \rho^*_{x(\theta),\theta} \circ P_{x(\theta)} \circ (\rho^{-1}_{x(\theta),\theta})^*$ with $x(\theta) = [\tilde{m}_0, \theta]$ and $[\tilde{m}_0] = m_0$.

Remark 3.6. — *According to* [**18**] *a pseudodifferential operator as in Connes, admits a well defined distributional Schwartz kernel over G. It is easy to check that this Schwartz kernel coincides with our K_P when the two families correspond as in the previous remark.*

Remark 3.7. — *The construction explained in remark 3.5. also allows to establish an identification between Connes' von Neumann algebra* [**18**] *for the groupoid G and our von Neumann algebra $W^*_\nu(G,E)$. It is easy to check that Connes' trace* [**18**] *corresponds to our trace τ^ν through this identification.*

Lemma 3.8. — *A pseudodifferential operator P of order m yields an $\mathcal{C}_c$-linear map $\mathcal{P} : \mathcal{E}_c \to \mathcal{F}_c$. Moreover, the following identity holds in $\mathcal{C}_c$: $<\mathcal{P}u, v> = <u, \mathcal{P}^* v>$ $\forall u \in \mathcal{E}_c$, $\forall v \in \mathcal{F}_c$.*

Proof. — Let $\xi \in \mathcal{E}_c$ and let $f \in \mathcal{C}_c$. By definition $(\xi f)(\cdot) = \sum_\gamma (R^*_{\gamma^{-1}}\xi)(\cdot) f(\gamma\pi(\cdot), \gamma)$ with $\pi : \tilde{M} \times T \to T$ the projection. Hence:

$$\begin{aligned}
\mathcal{P}(\xi f) &= \mathcal{P}\left(\sum_\gamma (R^*_{\gamma^{-1}}\xi)(\cdot) f(\gamma\pi(\cdot), \gamma)\right) \\
&= \sum_\gamma \left(\mathcal{P}\left(R^*_{\gamma^{-1}}\xi\right)(\cdot)\right) f(\gamma\pi(\cdot), \gamma) \\
&= \sum_\gamma \left(R^*_{\gamma^{-1}}\mathcal{P}\xi\right)(\cdot) f(\gamma\pi(\cdot), \gamma) = (\mathcal{P}\xi) f
\end{aligned}$$

where in the second equality we have used the fact that $\mathcal{P}$ commutes with multiplication by functions in $C(T)$ (indeed, $\mathcal{P}$ is given by a continuous family) and in the third equality we have used the Γ-equivariance. The equality $<\mathcal{P}u, v> = <u, \mathcal{P}^* v>$ is established in a straightforward way. □

Proposition 3.9. — *Let $\mathcal{P}$ be a pseudodifferential operator of order m between $\mathcal{E}_c$ and $\mathcal{F}_c$. Then we have:*

1. *If $m \leq 0$ then $\mathcal{P}$ extends to a bounded adjointable $\mathcal{C}_m$-linear operator $\overline{\mathcal{P}}_m$ from $\mathcal{E}_m$ to $\mathcal{F}_m$ and to a bounded adjointable $\mathcal{C}_r$-linear operator $\overline{\mathcal{P}}_r$ from $\mathcal{E}_r$ to $\mathcal{F}_r$.*

2. *If $m < 0$, then $\overline{\mathcal{P}}_m$ is an $\mathcal{A}_m$-linear compact operator from $\mathcal{E}_m$ to $\mathcal{F}_m$ and $\overline{\mathcal{P}}_r$ is an $\mathcal{A}_r$-linear compact operator from $\mathcal{E}_r$ to $\mathcal{F}_r$.*

Proof. — We only sketch the arguments, following [**55**]. For simplicity we take E and F to be the trivial line bundles, so that $\mathcal{E}_c = \mathcal{F}_c$. We give the proof for the maximal completion, the proof for the regular completion being the same.

For the first item, one applies the classical argument of Hörmander, see for example [**53**], reducing the continuity of order zero pseudodifferential operators to that of the smoothing operators. We omit the details.

For the second item, one starts with $\mathcal{P}$ of order $< -n$, with n equal to the dimension of $\tilde{M}$. Then $\mathcal{P}$ is given by integration against a continuous compactly supported element in G; in other words $\mathcal{P} = \chi(K)$, with $K \in C_c(G)$. We already know that such an element extends to a compact operator $\overline{\mathcal{P}}$ on $\mathcal{E}_m$, see 3.1. If $\mathcal{P}$ is of order less than $-n/2$ then we consider $Q := \mathcal{P}^*\mathcal{P}$ which is of order less then $-n$ and symmetric. We know that Q extends to a (compact) bounded operator on $\mathcal{E}_m$; thus if $f \in \mathcal{E}_c$ then, in particular, $\|\mathcal{P}f\|^2 \leq C\|f\|^2$ which means that $\mathcal{P}$ extends to a bounded operator $\overline{\mathcal{P}}$ on $\mathcal{E}_m$. Similarly $\mathcal{P}^*$ extends to a bounded operator $\overline{\mathcal{P}^*}$ and by density we obtain that $\overline{\mathcal{P}}$ is adjointable with adjoint equal to $\overline{\mathcal{P}^*}$. Now, again by density, we have $\overline{Q} = \overline{\mathcal{P}}^*\overline{\mathcal{P}}$; thus we can take the square root of $\overline{Q}$ which will be again compact since $\overline{Q}$ is. Using the polar decomposition for $\overline{\mathcal{P}}$ we can finally conclude that $\overline{\mathcal{P}}$ is compact which is what we need to prove.

If the order of $\mathcal{P}$ is $m < 0$ then we fix $\ell \in \mathbb{N}$ such that $m2^\ell < -n$; then we proceed inductively, considering $(\mathcal{P}^*\mathcal{P})^{2^\ell}$ and applying the above argument. □

Let $P = (P_\theta)_{\theta \in T}$ be an element in $\Psi_c^\ell(G)$; its principal symbol $\sigma_\ell(P)$ defines a Γ-equivariant function on the vertical cotangent bundle $T_V^*(\tilde{M} \times T)$ to the trivial fibration $\tilde{M} \times T \to T$; equivalently, $\sigma_\ell(P)$ is a function on the longitudinal cotangent bundle $T^*\mathcal{F}$ to the foliation $(V, \mathcal{F})$. If, more generally, $P \in \Psi_c^\ell(G; \widehat{E}, \widehat{F})$, then its principal symbol will be a Γ-equivariant section of the bundle $\mathrm{Hom}(\pi_V^*(\widehat{E}), \pi_V^*(\widehat{F})) := \pi_V^*(\widehat{E^*}) \otimes \pi_V^*(\widehat{F})$ with $\pi_V : T_V^*(\tilde{M} \times T) \to (\tilde{M} \times T)$ the natural projection; equivalently, $\sigma_\ell(P)$ is a section of the bundle $\mathrm{Hom}(\pi_{\mathcal{F}}^*E, \pi_{\mathcal{F}}^*F)$ over the longitudinal cotangent bundle $\pi_{\mathcal{F}} : T^*\mathcal{F} \to V$. We shall say that P is elliptic if its principal symbol $\sigma_\ell(P)$ is invertible on non-zero cotangent vectors. We end this subsection by stating the following fundamental and classical result whose proof can be found, for example in the work of Connes [**17**], see also [**38**]. (Notice that in this particular case the proof can be easily done directly, mimicking the classic one on a closed compact manifold.)

Theorem 3.10. — *Let $P \in \Psi_c^\ell(G; \widehat{E}, \widehat{F})$ be elliptic; then there exists $Q \in \Psi_c^{-\ell}(G; \widehat{F}, \widehat{E})$ such that*

$$\mathrm{Id} - PQ \in \Psi_c^{-\infty}(G; \widehat{F}, \widehat{F}), \qquad \mathrm{Id} - QP \in \Psi_c^{-\infty}(G; \widehat{E}, \widehat{E}). \tag{9}$$

Notice that in our definition elements in $\Psi_c^{-\infty}$ are of Γ-compact support: this applies in particular to both $S := \mathrm{Id} - PQ$ and $R := \mathrm{Id} - QP$.

We end this subsection by observing that it is also possible to introduce Sobolev modules $\mathcal{E}^{(k)}$ and prove the usual properties of pseudodifferential operators, see [**56**]. For simplicity we consider the case $k \in \mathbb{N}$. In order to give the definition, we fix an elliptic differential operator of order k, P; for example $P = D^k$, with D a Dirac type operator. This is a regular unbounded operator (see the next subsection). We consider the domain of its extension $\mathrm{Dom}\mathcal{P}_m$ and we endow it with the $\mathcal{C}_m$-valued scalar product

$$<s,t>_k := <s,t> + <\mathcal{P}_m s, \mathcal{P}_m t>.$$

This defines the Sobolev module of order k, $\mathcal{E}^{(k)}$. One can prove for these modules the usual properties:

- different choices of P yield compatible Hilbert module structures;
- if $k > \ell$ we have $\mathcal{E}^{(k)} \hookrightarrow \mathcal{E}^{(\ell)}$ and the inclusion in $\mathcal{C}_m$-compact
- if $R \in \Psi_c^m(G,E)$ then R extends to a bounded operator $\mathcal{E}^{(k)} \to \mathcal{E}^{(k-m)}$.

Since we shall make little use of these properties, we omit the proofs.

3.3. Functional calculus for Dirac operators. — Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be a Γ-equivariant family of Dirac-type operators acting on the sections of a Γ-equivariant vertical hermitian Clifford module $\widehat{E}$ endowed with a Γ-equivariant connection. We shall make the usual assumptions on the connection and on the Clifford action ensuring that each $\tilde{D}_\theta$ is formally self-adjoint. Recall that $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T} \in \Psi_c^1(G; \widehat{E})$ and that $\tilde{D}$ induces a $\mathcal{C}_c$-linear operator on $\mathcal{E}_c$ that we have denoted by $\mathcal{D}$.

Proposition 3.11. — *The operator $\mathcal{D}$ is closable in $\mathcal{E}_r$ and in $\mathcal{E}_m$. Moreover, the closures $\mathcal{D}_r$ and $\mathcal{D}_m$ on the Hilbert modules $\mathcal{E}_r$ and $\mathcal{E}_m$ respectively, are regular and self-adjoint operators.*

Proof. — We give a classical proof based on general results described for instance in [**55**]. Since the densely defined operator $\mathcal{D}$ is formally self-adjoint, it is closable with symmetric closures in $\mathcal{E}_r$ and $\mathcal{E}_m$ respectively. Let $\tilde{Q} \in \Psi_c^{-1}(G, \widehat{E})$ be a formally self-adjoint parametrix for $\tilde{D}$:

$$\mathrm{Id} - \tilde{D}\tilde{Q} = \tilde{S}\,, \qquad \mathrm{Id} - \tilde{Q}\tilde{D} = \tilde{R}\,.$$

For simplicity, we denote by π the regular or the maximal representation, by $\mathcal{E}_\pi$ the corresponding Hilbert module and by $\mathcal{D}_\pi$ the closure of $\mathcal{D}$. Since $\tilde{Q}$ has negative order, it extends into a bounded operator on $\mathcal{E}_\pi$, denoted by $\overline{Q}_\pi$, or simply by Q_π. On the other hand, we know that the zero-th order pseudodifferential operator $\tilde{D}\tilde{Q}$ extends to a bounded $\mathcal{C}_\pi$-linear operator on $\mathcal{E}_\pi$. If ξ belongs to the domain of this closure (which is $\mathcal{E}_\pi$) then there exists ξ_n in $C_c^{\infty,0}(\tilde{M} \times T, \widehat{E})$ converging in the π-norm

to ξ and such that $(\tilde{D}\tilde{Q})\xi_n$ is convergent in the π norm. We deduce that $\mathcal{Q}_\pi(\xi)$ is well defined and is the limit of $\tilde{Q}\xi_n$. Hence we deduce that $\mathcal{Q}_\pi\xi$ belongs to the domain of $\mathcal{D}_\pi$ and that $\mathrm{Im}\mathcal{Q}_\pi \subset \mathrm{Dom}\,\mathcal{D}_\pi$. Hence, $\mathcal{D}_\pi\mathcal{Q}_\pi$ is a bounded operator which coincides with the extension of $\tilde{D}\tilde{Q}$ and we have with obvious notation,

$$\mathcal{D}_\pi\mathcal{Q}_\pi = I - \mathcal{S}_\pi\,,$$

so $\mathcal{Q}_\pi^*\mathcal{D}_\pi^* \subset (\mathcal{D}_\pi\mathcal{Q}_\pi)^* = I - \mathcal{S}_\pi^*$ and hence $\mathrm{Dom}(\mathcal{D}_\pi^*) \subset \mathrm{Im}(\mathcal{Q}_\pi^*) + \mathrm{Im}(\mathcal{S}_\pi^*)$. Since $\mathcal{Q}_\pi$ is self-adjoint we deduce that

$$\mathrm{Dom}(\mathcal{D}_\pi^*) \subset \mathrm{Im}(\mathcal{Q}_\pi) + \mathrm{Im}(\mathcal{S}_\pi^*) \subset \mathrm{Dom}(\mathcal{D}_\pi).$$

The last inclusion is a consequence of the fact that $\mathcal{S}_\pi^*$ is induced by a smoothing Γ-compactly supported operator. So $\mathcal{D}_\pi$ is self-adjoint. Now, the graph of $\mathcal{D}_\pi$, $G(\mathcal{D}_\pi)$, is given by

$$G(\mathcal{D}_\pi) = \{(\mathcal{Q}_\pi(\eta) + \mathcal{S}_\pi(\eta'), \mathcal{D}_\pi(\mathcal{Q}_\pi(\eta)) + \mathcal{D}_\pi(\mathcal{S}_\pi(\eta')), \eta, \eta' \in \mathcal{E}_\pi\}.$$

Hence $G(\mathcal{D}_\pi)$, which is closed in $\mathcal{E}_\pi \times \mathcal{E}_\pi$, coincides with the image of a bounded morphism U of $\mathcal{C}_\pi$-modules given by

$$U = \begin{pmatrix} \mathcal{Q}_\pi & \mathcal{S}_\pi \\ \mathcal{D}_\pi\mathcal{Q}_\pi & \mathcal{D}_\pi\mathcal{S}_\pi \end{pmatrix}.$$

Now, as a general fact, the image of such morphism, when closed, is always orthocomplemented. Thus $\mathcal{D}_\pi$ is regular. □

Recall that we established in Lemma 3.2 isomorphisms of Hilbert spaces

$$\Phi_\theta : \mathcal{E}_m \otimes_{\pi_\theta^{\mathrm{av}}} \ell^2(\Gamma\theta) \to L^2(L_\theta, E)\,, \quad \Psi_\theta : \mathcal{E}_m \otimes_{\pi_\theta^{reg}} \ell^2(\Gamma) \to L^2(\tilde{M}_\theta, \widehat{E}).$$

Proposition 3.12. — *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a continuous bounded function. Then for any $\theta \in T$, the bounded operator, acting on $L^2(L_\theta, E)$, given by*

$$\Phi_\theta \circ \left[\psi(\mathcal{D}_m) \otimes_{\pi_\theta^{av}} I_{\ell^2(\Gamma\theta)}\right] \circ \Phi_\theta^{-1},$$

coincides with the operator $\psi(D_{L_\theta})$ where D_{L_θ} is our Dirac type operator acting on the leaf L_θ.
In the same way the operator, acting on $L^2(\tilde{M}_\theta, \widehat{E}_\theta)$, given by

$$\Psi_\theta \circ \left[\psi(\mathcal{D}_m) \otimes_{\pi_\theta^{reg}} I_{\ell^2(\Gamma)}\right] \circ \Psi_\theta^{-1},$$

coincides with the operator $\psi(\tilde{D}_\theta)$.

Proof. — We prove only the first result, the proof of the second is similar. Since the operator $\mathcal{D}_m$ is a regular self-adjoint operator, its continuous functional calculus is

well defined. See [**55**]. Let $\xi \in \mathcal{E}_c$ and let $f \in C_c[\Gamma\theta]$, then we have

$$\begin{aligned}\Phi_\theta(\mathcal{D}_m(\xi)\otimes f)(\tilde{m},\theta) &= \sum_{\gamma\in\Gamma} f(\gamma\theta)(\mathcal{D}\xi)(\tilde{m}\gamma^{-1},\gamma\theta)\\ &= \sum_{\gamma\in\Gamma} f(\gamma\theta)[R^*_{\gamma^{-1}}(\mathcal{D}\xi)](\tilde{m},\theta)\\ &= \sum_{\gamma\in\Gamma} f(\gamma\theta)\mathcal{D}(R^*_{\gamma^{-1}}\xi)(\tilde{m},\theta).\end{aligned}$$

On the other hand, the action of the operator D_{L_θ} on the image of Φ_θ is given by

$$(D_{L_\theta}\circ\Phi_\theta)(\xi\circ f)(\tilde{m},\theta) = \sum_{\gamma\in\Gamma} f(\gamma\theta)\tilde{D}_\theta([\gamma^{-1}\xi]_\theta)(\tilde{m}).$$

Since by definition of $\mathcal{D}$ we have $\mathcal{D}(\gamma^{-1}\xi)(\tilde{m},\theta) = \tilde{D}_\theta([\gamma^{-1}\xi]_\theta)(\tilde{m})$ we obtain that

$$\Phi_\theta\circ(\mathcal{D}_m\otimes I)\circ\Phi_\theta^{-1} = D_{L_\theta}.$$

If ψ is as above then we get as a consequence of the definition of functional calculus,

$$\begin{aligned}\psi(D_{L_\theta}) &= \psi\left(\Phi_\theta\circ(\mathcal{D}_m\otimes I)\circ\Phi_\theta^{-1}\right)\\ &= \Phi_\theta\circ\psi(\mathcal{D}_m\otimes I)\circ\Phi_\theta^{-1}.\end{aligned}$$

By uniqueness of the functional calculus we also deduce that $\psi(\mathcal{D}_m\otimes I) = \psi(\mathcal{D}_m)\otimes I$, and hence the proof is complete. □

Before proving the main result of this Subection, we recall two technical results about trace class operators. First we establish two useful Lemmas. The first one is classical and generalizes [**53**] Proposition A.3.2 while the second one is an easy extension of similar results for coverings established in [**1**].

Lemma 3.13. — *Let $S \in W^*_\nu(G,E)$; then the following statements are equivalent:*

- *S is τ^ν Hilbert-Schmidt (i.e. $\tau^\nu(S^*S) < +\infty$);*
- *there exists a measurable section K_S of $\mathrm{END}(E)$ over G such that for ν-almost every θ the operator S_θ is given on $L^2(\tilde{M}_\theta,\widehat{E}_\theta)$ by*

$$(S_\theta\xi)(\tilde{m}) = \int_{\tilde{M}} K_S(\tilde{m},\tilde{m}',\theta)\xi(\tilde{m}')d\tilde{m}',$$

with

$$\int_{\tilde{M}\times F\times T} \mathrm{tr}\left(K_S(\tilde{m},\tilde{m}',\theta)^*K_S(\tilde{m},\tilde{m}',\theta)\right)\, d\tilde{m}\, d\tilde{m}'\, d\nu(\theta) < +\infty$$

where we interpret K_S as a Γ-equivariant section on $\tilde{M}\times\tilde{M}\times T$.

Moreover in this case the τ^ν Hilbert-Schmidt norm of S, $\|S\|^2_{\nu-\mathrm{HS}} := \tau^\nu(S^*S)$, is given by

$$\|S\|^2_{\nu-\mathrm{HS}} = \int_{\tilde{M}\times F\times T} \mathrm{tr}\left(K_S(\tilde{m},\tilde{m}',\theta)^*K_S(\tilde{m},\tilde{m}',\theta)\right)\, d\tilde{m}\, d\tilde{m}'\, d\nu(\theta).$$

Proof. — We have, by definition,

$$\|S\|^2_{\nu-\mathrm{HS}} = \tau^\nu(S^*S) = \int_T \|S_\theta M_\chi\|^2_{\mathrm{HS}}\, d\nu(\theta)$$

where the integrand involves the usual Hilbert-Schmidt norm in $L^2(\tilde{M}_\theta, \widehat{E}_\theta)$. Therefore the proof is easily deduced using [**53**][page 251] □

Lemma 3.14. — *Let S be a positive selfadjoint operator in $W^*_\nu(G,E)$; then the following statements are equivalent:*

- *$\tau^\nu(S) < +\infty$;*
- *for any smooth compactly supported function ϕ on $\tilde{M}$, the measurable function*

$$T \ni \theta \longrightarrow \mathrm{Tr}(M_{\overline{\phi}} \circ S_\theta \circ M_\phi)$$

  is ν-integrable on T, where the trace is the usual trace for bounded operators on the Hilbert space $L^2(\tilde{M}, \widehat{E})$;
- *for any smooth compactly supported function ϕ on $\tilde{M}$, the function*

$$T \ni \theta \longrightarrow \|S_\theta^{1/2} \circ M_\phi\|^2_{\mathrm{HS}}$$

  is ν-integrable on T.

Proof. — We follow the techniques in [**1**] and use Lemma 3.13. The second and third items are clearly equivalent. Assume that $\tau^\nu(S) < +\infty$ and let ϕ be a smooth compactly supported function on $\tilde{M}$ with uniform norm $\|\phi\|_\infty$. We let Γ_ϕ be a finite subset of Γ such that the support of ϕ lies in the union $\cup_{\gamma\in\Gamma_\phi} F\gamma$. Here F is a fundamenal domain as before. Then $S^{1/2}$ is τ^ν Hilbert-Schmidt and if $K_{S^{1/2}}$ is its Schwartz kernel, then we easily deduce

$$\begin{aligned}
&\int_{\tilde{M}\times\tilde{M}\times T} |\phi(\tilde{m}')|^2 \ \mathrm{tr}\,(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta))\ d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ &\sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\gamma\times T} |\phi(\tilde{m}')|^2 \ \mathrm{tr}\,(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta))\ d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
\leq\ &\|\phi\|^2_\infty \times \sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\gamma\times T} \mathrm{tr}\,(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta))\ d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ &\|\phi\|^2_\infty \times \sum_{\gamma\in\Gamma_\phi} \int_{\tilde{M}\times F\times T} \mathrm{tr}\,\big(K_{S^{1/2}}(\tilde{m}\gamma^{-1},\tilde{m}',\gamma\theta)^* K_{S^{1/2}}(\tilde{m}\gamma^{-1},\tilde{m}',\gamma\theta)\big)\ d\tilde{m}\, d\tilde{m}'\, d\nu(\theta)\\
=\ &\|\phi\|^2_\infty \times \mathrm{Card}(\Gamma_\phi) \times \tau^\nu(S) \quad < +\infty.
\end{aligned}$$

Conversely, let ϕ be any nonnegative smooth compactly supported function on $\tilde{M}$ such that $\phi\chi = \chi$, where χ is the characteristic function of F. Then we have

$$\begin{aligned}\tau^\nu(S) &= \int_T \mathrm{Tr}(M_\chi \circ S_\theta \circ M_\chi)\, d\nu(\theta)\\ &= \int_T \mathrm{Tr}(M_\chi \circ M_\phi \circ S_\theta \circ M_\phi \circ M_\chi)\, d\nu(\theta)\\ &\leq \int_T \mathrm{Tr}(M_\phi \circ S_\theta \circ M_\phi)\, d\nu(\theta) < +\infty. \qquad \square\end{aligned}$$

Proposition 3.15. — *Let $S = (S_\theta)_{\theta\in T}$ be an element of the von Neumann algebra $W^*_\nu(G;E)$. We assume that S_θ is an integral operator with smooth kernel for any θ in T and that the resulting Schwartz kernel K_S is a Borel bounded section over G.*

- *If S is positive and self-adjoint, then S is τ^ν trace class and we have*

$$\tau^\nu(S) = \int_{F\times T} \mathrm{tr}(K_S(\tilde{m},\tilde{m},\theta))d\tilde{m}d\nu(\theta), \tag{10}$$

  where F is a fundamental domain in $\tilde{M}$ and where in the right hand side we interpret $K(S)$ as a Γ-equivariant section on $\tilde{M}\times\tilde{M}\times T$.
- *If S is assumed to be τ^ν trace class, then formula* (10) *holds.*

Proof. — Let us prove the first item. Let ϕ be a smooth compactly supported function on $\tilde{M}$. The operator $M_\phi \circ S_\theta \circ M_{\overline{\phi}}$ acting on $L^2(\tilde{M}_\theta, \widehat{E})$ has a smooth compactly supported Schwartz kernel and is therefore trace class with

$$\mathrm{Tr}(M_\phi \circ S_\theta \circ M_{\overline{\phi}}) = \int_{\tilde{M}_\theta} |\phi(\tilde{m})|^2 K_S(\tilde{m},\tilde{m},\theta)\, d\tilde{m}\,.$$

Since K_S is bounded as a section over G and since ν is a borelian measure, we have

$$\int_T \mathrm{Tr}(M_\phi \circ S_\theta \circ M_{\overline{\phi}})d\nu(\theta) < +\infty\,.$$

This shows, using Lemma 3.14, that S is τ^ν trace class and also that $S^{1/2}$ is τ^ν Hilbert-Schmidt. By Lemma 3.13 we deduce that the $S^{1/2}$ is an integral operator with measurable Schwartz kernel $K_{S^{1/2}}$ satisfying

$$\|S^{1/2}\|^2_{\mathrm{HS}} := \int_{\tilde{M}\times F\times T} \mathrm{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)\right) d\tilde{m}\, d\tilde{m}' d\nu(\theta) < +\infty\,.$$

On the other hand we also have

$$\begin{aligned}K_S(\tilde{m},\tilde{m},\theta) &= \int_{\tilde{M}} K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta) K_{S^{1/2}}(\tilde{m}',\tilde{m},\theta)\, d\tilde{m}\\ &= \int_{\tilde{M}} K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta) K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^*\, d\tilde{m}\end{aligned}$$

The last equality employs the fact that $S^{1/2}$ is selfadjoint. Taking pointwise traces we get:

$$\begin{aligned} \operatorname{tr} K_S(\tilde{m},\tilde{m},\theta) &= \int_{\tilde{M}} \operatorname{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^*\right) d\tilde{m} \\ &= \int_{\tilde{M}} \operatorname{tr}\left(K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)^* K_{S^{1/2}}(\tilde{m},\tilde{m}',\theta)\right) d\tilde{m}\,. \end{aligned}$$

Therefore

$$\tau^\nu(S) = \|S^{1/2}\|^2_{\mathrm{HS}} = \int_{F\times T} \operatorname{tr} K_S(\tilde{m},\tilde{m},\theta) d\tilde{m} d\nu(\theta)\,.$$

This finishes the proof of the first item.

Regarding the second item, assume now that S is τ^ν trace class i.e. $\tau^\nu(|S|)$ is finite. Write $S = U|S|$ for the polar decomposition of S in $W^*_\nu(G,E)$. Then the operators $|S|^{1/2}$ and $U|S|^{1/2}$ are τ^ν Hilbert-Schmidt and thus have L^2 Schwartz kernels $K_{|S|^{1/2}}$ and $K_{U|S|^{1/2}}$. Using Lemma 3.13 and the polarization identity we deduce:

$$\begin{aligned} & <U|S|^{1/2}, |S|^{1/2}>_{\mathrm{HS}} \\ = & \int_{\tilde{M}\times F\times T} \operatorname{tr} K_{U|S|^{1/2}}(\tilde{m},\tilde{m}',\theta) K_{|S|^{1/2}}(\tilde{m},\tilde{m}',\theta)^*\, d\tilde{m}\, d\tilde{m}'\, d\nu(\theta) \\ = & \int_{\tilde{M}\times F\times T} \operatorname{tr} K_{U|S|^{1/2}}(\tilde{m},\tilde{m}',\theta) K_{|S|^{1/2}}(\tilde{m}',\tilde{m},\theta)\, d\tilde{m}\, d\tilde{m}'\, d\nu(\theta) \\ = & \int_{F\times T} \operatorname{tr} K_S(\tilde{m},\tilde{m},\theta)\, d\tilde{m}\, d\nu(\theta)\,. \end{aligned}$$

Hence

$$\tau^\nu(S) = <U|S|^{1/2}, |S|^{1/2}>_{\mathrm{HS}} = \int_{F\times T} \operatorname{tr} K_S(\tilde{m},\tilde{m},\theta)\, d\tilde{m}\, d\nu(\theta)\,.$$

The proof is complete. □

Remark 3.16. — *A proof similar to the one given above shows, as in* [1] *(Proposition 4.16), that if $R = (R_\theta)_{\theta\in T}$ has a continuous (or even Borel bounded) leafwise smooth Schwartz kernel with Γ-compact support, then R is τ^ν trace class with $\tau^\nu(R) = \int_{F\times T} \operatorname{tr} K_R(\tilde{m},\tilde{m},\theta) d\tilde{m}\, d\nu(\theta)$.*

A similar statement holds for a leafwise operator in $W^*_\nu(V,\mathcal{F};E)$ with a Borel bounded leafwise smooth Schwartz kernel which is supported within a uniform C-neighbourhood, $C\in\mathbb{R}, C>0$, of the diagonal of every leaf.

Proposition 3.17. — *Let $\psi:\mathbb{R}\to\mathbb{C}$ be a measurable rapidly decreasing function. The the operator $\psi(\tilde{D}) := (\psi(\tilde{D}_\theta))_{\theta\in T}$ satisfies the assumption of Proposition 3.15 (second item). In particular $\psi(\tilde{D})$ has a bounded fiberwise-smooth Schwartz kernel K_ψ and we have*

$$\tau^\nu(\psi(\tilde{D})) = \int_{F\times T} \operatorname{tr}(K_\psi[\tilde{m},\tilde{m},\theta]) d\tilde{m}\; d\nu(\theta).$$

Proof. — Using [**38**], Theorem 7.36 (which is in fact valid for any measurable rapidly decreasing function) we know that K_ψ is bounded and fiberwise smooth and that $\psi(\tilde{D}) \in W^*_\nu(G,E)$. Therefore it remains to show that $\psi(\tilde{D})$ is τ^ν trace class since then we can simply apply Proposition 3.15 (second item). But $|\psi(\tilde{D})| = |\psi|(\tilde{D})|$ and $|\psi|$ is a measurable rapidly decreasing function; therefore $|\psi(\tilde{D})|$ has a bounded fiberwise smooth Schwartz and thus satisfies the assumptions of Proposition 3.15 (first item). We conclude that $\psi(\tilde{D})$ is τ^ν trace class. □

A statement similar to the one just proved holds for the leafwise Dirac-type operator $D := (D_L)_{L\in V/\mathcal{F}}$. In order to keep this paper to a reasonable size we state the corresponding proposition without proof. See [**52**].

Proposition 3.18. — *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a measurable rapidly decreasing function. Then the operator $\psi(D) := (\psi(D_L))_{L\in V/\mathcal{F}}$ is $\tau^\nu_{\mathcal{G}}$ trace class, has a leafwise smooth Schwartz kernel K_ψ which is bounded as a measurable section over the equivalence relation $\cup_{L\in V/\mathcal{F}} L \times L$, and we have*

$$\tau^\nu_{\mathcal{G}}(\psi(D)) = \int_{F\times T} \mathrm{tr}(K_\psi([\tilde{m},\theta],[\tilde{m},\theta]))d\tilde{m}\, d\nu(\theta)$$

where now $F \times T$ is viewed as a subset in V.

We are now in position to prove the main results of this section.

Theorem 3.19. — *Let for simplicity $\psi : \mathbb{R} \to \mathbb{C}$ be a Schwartz class function. Then $\psi(\mathcal{D}_m) \in \mathcal{K}_{\mathcal{G}_m}(\mathcal{E}_m)$ and the element $\chi_m^{-1}(\psi(\mathcal{D}_m)) \in \mathcal{B}^E_m$ admits a finite τ^ν_{av} trace and also a finite τ^ν_{reg} trace. Moreover*

- *$\tau^\nu_{av}(\chi_m^{-1}(\psi(\mathcal{D}_m))) = \tau^\nu_{\mathcal{G}}\left[(\psi(D_L))_{L\in V/F}\right]$ where $(\psi(D_L))_{L\in V/F}$ is the corresponding element in the leafwise von Neumann algebra $W^*_\nu(V,\mathcal{F};E)$ and $\tau^\nu_{\mathcal{G}}$ is the trace on this von Neumann algebra as defined in Subsection 2.4.*
- *$\tau^\nu_{reg}(\chi_m^{-1}(\psi(\mathcal{D}_m))) = \tau^\nu\left[(\psi(\tilde{D}_\theta))_{\theta\in T}\right]$ where $(\psi(\tilde{D}_\theta))_{\theta\in T}$ is the corresponding element in the regular von Neumann algebra $W^*_\nu(G,E)$ and τ^ν is the trace on this von Neumann algebra as defined in Subsection 2.4.*

Proof. — We know from Corollary 2.6 that $\tau^\nu_{av} = \tau^\nu_{\mathcal{G}} \circ \pi^{av}$. Therefore

$$\begin{aligned}\tau^\nu_{av}(\chi_m^{-1}(|\psi(\mathcal{D}_m)|)) &= \tau^\nu_{\mathcal{G}}\left[(\pi^{av}\circ\chi_m^{-1})(|\psi(\mathcal{D}_m)|)\right]\\ &= \tau^\nu_{\mathcal{G}}\left((\Phi_\theta \circ \left[|\psi|(\mathcal{D}_m)\otimes_{\pi^{av}_\theta} I\right]\circ\Phi_\theta^{-1})_{\theta\in T}\right).\end{aligned}$$

The last equality is a consequence of Proposition 3.3. Now, using Proposition 3.12, we finally deduce

$$\tau^\nu_{av}(\chi_m^{-1}(|\psi(\mathcal{D}_m)|)) = \tau^\nu_{\mathcal{G}}\left((|\psi|(D_L))_{L\in V/F}\right) < +\infty.$$

Hence we see from Proposition 3.1 that $\chi_m^{-1}(|\psi(\mathcal{D}_m)|)$ is trace class and the same computation with ψ instead of $|\psi|$ finishes the proof of the first item. The second item is proved repeating the same argument. □

4. Index theory

Let $\tilde{M}$, Γ, and T be as in the previous sections and let $(V, \mathcal{F})$, with $V = \tilde{M} \times_\Gamma T$, the associated foliated bundle. We assume in this section only that the manifold M is even dimensional and hence that the leaves of our foliation are even dimensional. Let E be a continuous longitudinally smooth hermitian vector bundle on V and let $\widehat{E}$ be its lift to $\tilde{M} \times T$. Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be as in the previous section a Γ-equivariant continuous family of Dirac-type operators. The bundle E is $\mathbb{Z}_2$-graded, $E = E^+ \oplus E^-$, and the operator $\tilde{D}$ is odd and essentially self-adjoint, i.e.

$$\tilde{D}_\theta = \begin{pmatrix} 0 & \tilde{D}_\theta^- \\ \tilde{D}_\theta^+ & 0 \end{pmatrix} \quad \forall \theta \in T$$

and $(\tilde{D}_\theta^-)^* = \tilde{D}_\theta^+$. Let $D := (D_L)_{L \in V/F}$ be the longitudinal operator induced by $\tilde{D}$ on the leaves of the foliation $(V, \mathcal{F})$.

4.1. The numeric index. — We consider for each θ the orthogonal projection $\tilde{\Pi}_\theta^\pm$ onto the L^2-null space of the operator $\tilde{D}_\theta^\pm$. Similarly, on each leaf L, we consider the orthogonal projections $\Pi_L^\pm$ onto the L^2-null space of the operator D_L. It is well known that these orthogonal projections are smoothing operators, but of course are not localized in a compact neighborhood of the unit space V, viewed as a subspace of the graph of the foliation equivalence relation.

Proposition 4.1

- *The family* $\tilde{\Pi}^\pm := (\tilde{\Pi}_\theta^\pm)_{\theta \in T}$ *belongs to the regular von Neumann algebra* $W_\nu^*(G, E^\pm)$. *Moreover it is a* τ^ν *trace class operator.*
- *The family* $\Pi^\pm := (\Pi_L^\pm)_{L \in V/F}$ *belongs to the leafwise von Neumann algebra* $W_\nu^*(V, \mathcal{F}; E^\pm)$. *Moreover it is a* $\tau_{\mathcal{G}}^\nu$ *trace class operator.*

Proof. — As we have already mentioned, for any Borel bounded function $f : \mathbb{R} \to \mathbb{C}$, the operator $f(\tilde{D})$ (respectively $f(D)$) belongs to the von Neumann algebra $W_\nu^*(G, E)$ (to the von Neumann algebra $W_\nu^*(V, \mathcal{F}; E)$). Hence, $\tilde{\Pi}^\pm$ belongs to $W_\nu^*(G, E^\pm)$ and $\Pi^\pm$ belongs to $W_\nu^*(V, \mathcal{F}; E^\pm)$.

Recall on the other hand from Propositions 3.17, 3.18 that $e^{-\tilde{D}^2}$ is τ^ν trace class and that e^{-D^2} is $\tau_{\mathcal{G}}^\nu$ trace class. Hence the proof is complete since

$$\tilde{\Pi} = \tilde{\Pi} e^{-\tilde{D}^2} \quad \text{and} \quad \Pi = \Pi e^{-D^2}. \qquad \square$$

Definition 4.2. — *We define the monodromy index of $\tilde{D}$ as*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) := \tau^{\nu}(\tilde{\Pi}^{+}) - \tau^{\nu}(\tilde{\Pi}^{-}) \tag{11}$$

We define the leafwise index of D as

$$\mathrm{ind}^{\nu}_{\mathrm{down}}(D) := \tau^{\nu}_{\mathcal{G}}(\Pi^{+}) - \tau^{\nu}_{\mathcal{G}}(\Pi^{-}). \tag{12}$$

As $\tilde{D}^{+}$ is elliptic, we can find a Γ-equivariant family of parametrices $\tilde{Q} := (\tilde{Q}_{\theta})_{\theta\in T}$ of Γ-compact support with remainders $\tilde{R}_{+}$ and $\tilde{R}_{-}$; the remainder families are Γ-equivariant, smoothing and of Γ-compact support, i.e.

$$\tilde{R}_{+} = I - \tilde{Q}\tilde{D}^{+} \quad \text{and} \quad \tilde{R}_{-} = I - \tilde{D}^{+}\tilde{Q}; \quad \tilde{R}_{\pm} \in \Psi_{c}^{-\infty}(G, E^{\pm}).$$

We know that $\tilde{R}_{\pm}$ are both τ^{ν} trace class. Let Q, R_{+}, R_{-} be the longitudinal operators induced on $(V, \mathcal{F})$; thus $Q, R_{+} \in W^{*}_{\nu}(V, \mathcal{F}; E^{+})$ and $R_{-} \in W^{*}_{\nu}(V, \mathcal{F}; E^{-})$ with $R_{\pm}$ $\tau^{\nu}_{\mathcal{G}}$ trace class, see Remark 3.16.

Proposition 4.3. — *For any $N \in \mathbb{N}, N \geq 1$, the following formulas hold:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \tau^{\nu}(\tilde{R}_{+})^{N} - \tau^{\nu}(\tilde{R}_{-})^{N}, \qquad \mathrm{ind}^{\nu}_{\mathrm{down}}(D) = \tau^{\nu}_{\mathcal{G}}(R_{+})^{N} - \tau^{\nu}_{\mathcal{G}}(R_{-})^{N}. \tag{13}$$

Proof. — Let $N = 1$; then the proof given by Atiyah in [**1**] extends easily to the present context. Replacing the parametrix $\tilde{Q}$ by $\tilde{Q}_{N} := \tilde{Q}(1 + \tilde{R}_{-} + \tilde{R}_{-}^{2} + \cdots + \tilde{R}_{-}^{N-1}$, which is again a parametrix, reduces the general case to the one treated by Atiyah. □

Using these formulas we shall now sketch the proof of the precise analogue of Atiyah's index theorem on coverings.

Proposition 4.4. — *The monodromy index and the leafwise index coincide:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \mathrm{ind}^{\nu}_{\mathrm{down}}(D). \tag{14}$$

Proof. — Given $\epsilon > 0$ we can choose a parametrix $\tilde{Q} \in \Psi_{c}^{-1}(G; \widehat{E}^{-}, \widehat{E}^{+})$ with the property that the two remainders $\tilde{R}_{\pm} = (\tilde{R}_{\pm})_{\theta}$, $\theta \in T$ are such that each $(\tilde{R}_{\pm})_{\theta}$ is supported within an ϵ-neighbourhood of the diagonal in $\tilde{M}_{\theta} \times \tilde{M}_{\theta}$. Let $\mathcal{R}_{\pm,m} : \mathcal{E}^{\pm}_{m} \to \mathcal{E}^{\mp}_{m}$ be the induced operators on the $\mathcal{C}_{m}$-Hilbert modules $\mathcal{E}^{\pm}_{m}$; since $\tilde{R}_{\pm}$ are smoothing and of Γ-compact support we certainly know that $\mathcal{R}_{\pm,m}$ are $\mathcal{C}_{m}$-compact operators. Let $K^{\pm} := \chi_{m}^{-1}(\mathcal{R}_{\pm,m}) \in \mathcal{B}^{E^{\pm}}_{m}$; $K^{\pm}$ is simply given by the Schwartz kernel of $\tilde{R}_{\pm}$ and is in fact an element in $\mathcal{B}^{E^{\pm}}_{c}$. In particular $K^{\pm}$ has finite $\tau^{\nu}_{\mathrm{reg}}$ trace and τ^{ν}_{av} trace. By arguments very similar (in fact easier) to those establishing Theorem 3.19 we know that

$$\tau^{\nu}(\tilde{R}_{\pm}) = \tau^{\nu}_{\mathrm{reg}}(\chi_{m}^{-1}\mathcal{R}_{\pm,m}) \equiv \tau^{\nu}_{\mathrm{reg}}(K^{\pm}), \quad \tau^{\nu}_{\mathcal{G}}(R_{\pm}) = \tau^{\nu}_{\mathrm{av}}(\chi_{m}^{-1}\mathcal{R}_{\pm,m}) \equiv \tau^{\nu}_{\mathrm{av}}(K^{\pm}). \tag{15}$$

Thus, from (13), it suffices to show that

$$\tau^{\nu}_{\mathrm{reg}}(K^{\pm}) = \tau^{\nu}_{\mathrm{av}}(K^{\pm}).$$

We can write

$$\begin{aligned}\tau^{\nu}_{\mathrm{av}}(K^{\pm}) &= \int_{F\times T}\sum_{\gamma\in\Gamma(\theta)} K^{\pm}[\tilde{m},\tilde{m}\gamma,\theta]\,d\tilde{m}\,d\nu(\theta)\\ &= \int_{F\times T} K^{\pm}[\tilde{m},\tilde{m},\theta]\,d\tilde{m}\,d\nu(\theta)+\int_{F\times T}\sum_{\gamma\in\Gamma(\theta);\gamma\neq e} K^{\pm}[\tilde{m},\tilde{m}\gamma,\theta]\,d\tilde{m}\,d\nu(\theta)\\ &= \tau^{\nu}_{\mathrm{reg}}(K^{\pm})+\int_{F\times T}\sum_{\gamma\in\Gamma(\theta);\gamma\neq e} K^{\pm}[\tilde{m},\tilde{m}\gamma,\theta]\,d\tilde{m}\,d\nu(\theta).\end{aligned}$$

Choosing ϵ small enough we can ensure that $K^{\pm}[\tilde{m},\tilde{m}\gamma,\theta]=0\ \forall\gamma\in\Gamma(\theta),\ \gamma\neq e$. The proof is complete. □

Remark 4.5. — *The possibility of localizing a parametrix in an arbitrary small neighbourhood of the diagonal plays a crucial role in the proof of the above proposition. There are more general situations, for example foliated flat bundles $\tilde{M}\times_\Gamma T$ with $\tilde{M}$ a manifold with boundary, where it is not possible to localize the parametrix. In these cases the analogue of Atiyah's index theorem does* **not** *hold.*

4.2. The index class in the maximal C^*-algebra. — Let $\tilde{D}^+$ be as in the previous subsection. As before we consider a parametrix $\tilde{Q}:=(\tilde{Q}_\theta)_{\theta\in T}\in\Psi_c^{-1}(G;\widehat{E}^-,\widehat{E}^+)$ with remainders $\tilde{R}_+$ and $\tilde{R}_-$. The family $\tilde{Q}$ defines a bounded $\mathcal{C}_m$-linear operator $\mathcal{Q}_m$ from $\mathcal{E}_m^-$ to $\mathcal{E}_m^+$. The families $\tilde{R}_+$ and $\tilde{R}_-$ define $\mathcal{C}_m$-linear compact operators $\mathcal{R}_{\pm,m}$ on the Hilbert modules $\mathcal{E}_m^{\pm}$ respectively.

We now define idempotents p,p_0 in $M_{2\times 2}(\mathcal{K}_{\mathcal{C}_m}(\mathcal{E}_m)\oplus\mathbb{C})$ by setting

$$p:=\begin{pmatrix}\mathcal{R}_{+,m}^2 & \mathcal{R}_{+,m}(I+\mathcal{R}_{+,m})\mathcal{Q}_m\\ \mathcal{R}_{-,m}\mathcal{D}_m^+ & I-\mathcal{R}_{-,m}^2\end{pmatrix},\quad p_0:=\begin{pmatrix}0&0\\0&I\end{pmatrix}. \tag{16}$$

We thus get a K_0-class $[p-p_0]\in K_0(\mathcal{K}_{\mathcal{C}_m}(\mathcal{E}_m))$.

Definition 4.6. — *The (maximal) index class* $\mathrm{IND}(\mathcal{D}_m)\in K_0(\mathcal{B}_m)$ *associated to the family $\tilde{D}$ is, by definition, the image under the composite isomorphism*

$$K_0(\mathcal{K}_{\mathcal{C}_m}(\mathcal{E}_m))\to K_0(\mathcal{B}_m^E)\to K_0(\mathcal{B}_m)$$

of the class $[p-p_0]$.

One also considers the index class in $K_0(\mathcal{C}_m)$:

$$\mathrm{Ind}(\mathcal{D}_m):=\mathcal{M}_{\max}^{-1}(\mathrm{IND}(\mathcal{D}_m))\in K_0(\mathcal{C}_m) \tag{17}$$

with $\mathcal{M}_{\max}: K_0(\mathcal{C}_m)\to K_0(\mathcal{B}_m)$ the Morita isomorphism considered in Proposition 2.10.

Recall now the morphisms $\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}$ and $\tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{B}_r) \to \mathbb{C}$. Using the natural morphism $K_0(\mathcal{B}_m) \to K_0(\mathcal{B}_r)$ we view both morphisms with domain $K_0(\mathcal{B}_m)$:

$$\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{B}_m) \to \mathbb{C}\,, \qquad \tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{B}_m) \to \mathbb{C}. \tag{18}$$

Recall also that using the natural morphism $K_0(\mathcal{A}_m) \to K_0(\mathcal{A}_r)$ we have induced morphisms

$$\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{A}_m) \to \mathbb{C}\,, \qquad \tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{A}_m) \to \mathbb{C}. \tag{19}$$

Proposition 4.7. — *Let* $\mathrm{IND}(\mathcal{D}_m) \in K_0(\mathcal{B}_m)$ *and* $\mathrm{Ind}(\mathcal{D}_m) \in K_0(\mathcal{A}_m)$ *be the two index classes introduced above. Then the following formulas hold:*

$$\mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) = \tau^{\nu}_{\mathrm{reg},*}(\mathrm{IND}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{reg},*}(\mathrm{Ind}(\mathcal{D}_m)), \tag{20}$$

$$\mathrm{ind}^{\nu}_{\mathrm{down}}(D) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{IND}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{Ind}(\mathcal{D}_m)). \tag{21}$$

Consequently, from (14)*, we have the following fundamental equality:*

$$\tau^{\nu}_{\mathrm{reg},*}(\mathrm{Ind}(\mathcal{D}_m)) = \tau^{\nu}_{\mathrm{av},*}(\mathrm{Ind}(\mathcal{D}_m)). \tag{22}$$

Proof. — We only need to prove the first equality in each equation, for the second one is a consequence of the definition of $\mathrm{Ind}(\mathcal{D}_m) \in K_0(\mathcal{A}_m)$ and the compatibilty result explained in Proposition 2.10. For the first equality we apply (13) with $N = 2$ and the parametrix $\tilde{Q}$. Using now (15), (16) we get

$$\begin{aligned} \mathrm{ind}^{\nu}_{\mathrm{up}}(\tilde{D}) &= \tau^{\nu}((\tilde{R}_+)^2) - \tau^{\nu}((\tilde{R}_-)^2) \\ &= \tau^{\nu}_{\mathrm{reg}}((\mathcal{R}_{+,m})^2) - \tau^{\nu}_{\mathrm{reg}}((\mathcal{R}_{-,m})^2) \\ &= \tau^{\nu}_{\mathrm{reg},*}(\mathrm{IND}(\mathcal{D}_m)). \end{aligned}$$

The proof of the other one is similar. □

Remark 4.8. — *The equalities in Proposition 4.7 can be rephrased as the equality between the numeric C^*-algebraic index and the von Neumann index. Notice once again that there are more general situations where this proposition does not hold, in the sense that there exists a well defined von Neumann index but there does not exist a well-defined C^*-algebraic index we can equate it to. The simplest example is given by a fibration of compact manifolds $L \to V \to T$ with V and L manifolds with boundary. The von Neumann index defined by the family of Atiyah-Patodi-Singer boundary conditions is certainly well defined (this is the integral over T of the function that assigns to $\theta \in T$ the APS index of D^+_θ). On the other hand, unless the boundary family associated to $(D^+_\theta)_{\theta \in T}$ is invertible, there is not a well defined Atiyah-Patodi-Singer index class in $K_0(C(T)) = K^0(T)$. For more on higher Atiyah-Patodi-Singer index theory on foliated bundles see* [**34**], [**33**].

4.3. The signature operator for odd foliations. — We briefly review the definition of the leafwise signature operator in the odd case. Recall that when $\dim(M) = 2m-1$, the leafwise signature operator is defined as the operator D^{sign} acting on leafwise differential forms on V, defined on even forms of degree $2k$ by

$$D^{\text{sign}}_{ev} = i^m(-1)^{k+1}(* \circ d - d \circ *),$$

and on odd forms of degree $2k-1$ by

$$D^{\text{sign}}_{od} = i^m(-1)^{m-k}(d \circ * + * \circ d),$$

where d is the leafwise de Rham differential and $*$ is the usual Hodge operator along the leaves associated with the Riemannian metric on the foliation [**38**]. An easy computation shows that the two operators D^{sign}_{odd} and D^{sign}_{ev} are conjugate so that their invariants coincide and it is sufficient to work with one of them. In contrast with [**3**], D^{sign} will be in the sequel the operator D^{sign}_{od}. Using the lifted structure to the fibers of the monodromy covers $\tilde{M} \times \{\theta\}$ of the leaves, we consider in the same way the Γ-equivariant family of signature operators $\tilde{D}^{\text{sign}} = (\tilde{D}^{\text{sign}}_{\theta})_{\theta \in T}$ which actually coincides with the lift of D^{sign} as can be easily checked. The following is well known, see [**2**], [**3**] for the first part and [**28**] for the second:

Recall that the K_1 index of D^{sign} is the class of the Cayley transform of D^{sign}, see for instance [**28**].

Proposition 4.9. — *The operator D^{sign} is a leafwise elliptic essentially self-adjoint operator whose K_1 index class is a leafwise homotopy invariant of the foliation.*

The square of D^{sign} is proportional to the Laplace operator along the leaves and hence it is leafwise elliptic. The proof that D^{sign}_{ev} is formally self-adjoint is straightforward, see [**3**], and classical elliptic theory on foliations of compact spaces allows to deduce that it is essentially self-adjoint. Now D^{sign} is unitarily equivalent to D^{sign}_{ev} and hence is also formally self-adjoint. We shall get back to the index class later on. The homotopy invariance means that if $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a leafwise oriented leafwise homotopy equivalence between odd dimensional foliations, then with obvious notations we have

$$f_* \operatorname{Ind}(D^{\text{sign}}) = \operatorname{Ind}(D^{\text{sign}'})$$

where f_* is the isomorphism induced by the Morita equivalence implemented by f [**28**].

5. Foliated rho invariants

Recall that T is a compact Hausdorff space on which the discrete countable group Γ acts by homeomorphisms, M is a compact closed manifold with fundamental group Γ and universal cover $\tilde{M}$ and that $V = \tilde{M} \times_\Gamma T$ is the induced foliated space. We are

also given a Borel measure ν on T which is Γ-invariant. We assume in the present section that M is odd dimensional and whence that the leaves of the induced foliation $\mathcal{F}$ of V are odd dimensional. We fix as in the previous section a Dirac-type operator along the leaves of the foliation $(V, \mathcal{F})$ acting on the vector bundle E. We denote by D this operator acting leafwise, so $D = (D_L)_{L \in V/F}$ where each D_L is an elliptic Dirac-type operator on the leaf L acting on the restriction of E to L. We also consider the lifted operator $\tilde{D}$ to the monodromy groupoid G of the foliation (V, F) as defined in Section 3.2. More precisely, $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ is a Γ-equivariant continuous family of Dirac type operators on $\tilde{M}$.

5.1. Foliated eta and rho invariants. — The construction of foliated eta invariants was first given independently in the two references [**47**] [**43**] and the two definitions work in fact for general measured foliations. Notice that [**47**] works with the measurable groupoid defined by foliation, whereas [**43**] works with the holonomy groupoid. As we shall clarify in a moment, the choice of the groupoid does make a difference for these non-local invariants. We give in this paragraph a self-contained treatment of these two definitions following our set-up, but using the monodromy groupoid instead of the holonomy groupoid considered in [**43**].

We denote by k_t and $\tilde{k}_t$ the longitudinally smooth uniformly bounded Schwartz kernels of the operators $\varphi_t(D)$ and $\varphi_t(\tilde{D})$ obtained using the function $\varphi_t(x) := xe^{-t^2x^2}$ for $t > 0$. See Lemma 3.17.

Lemma 5.1. — *(Bismut-Freed estimate) There exists a constant $C \geq 0$ such that for any $(\tilde{m}, \theta) \in \tilde{M} \times T$, we have:*

$$|\operatorname{tr}(k_t([\tilde{m}, \theta], [\tilde{m}, \theta]))| \leq C \quad \text{and} \quad |\operatorname{tr}(\tilde{k}_t([\tilde{m}, \tilde{m}, \theta]))| \leq C, \text{ for } t \leq 1.$$

Proof. — A proof of these estimates appear already in [**47**]. We give nevertheless a sketch of the argument.
The Bismut-Freed estimate on a closed odd dimensional compact manifold M is a pointwise estimate on the vector-bundle trace of the Schwartz kernel of $D\exp(-t^2D^2)$ restricted to the diagonal. See the original article [**14**] but also [**37**]. As explained for example in the latter reference the Bismut-Freed estimate is ultimately a consequence of Getzler rescaling for the heat kernel of a Dirac laplacian on the even dimensional manifold obtained by crossing M with S^1. Since these arguments are purely local, they easily extend to our foliated case, using the compactness of $V := \tilde{M} \times_\Gamma T$ in order to control uniformly the constants appearing in the poinwise estimate. □

The operators D^2 and $\tilde{D}^2$ (as well as the operators $|D|$ and $|\tilde{D}|$) are non negative operators which are affiliated respectively with the von Neumann algebra $W^*_\nu(V, \mathcal{F}; E)$ and the von Neumann algebra $W^*_\nu(G; E)$. (This means that their sign operators as

well as their spectral projections belong to the von Neumann algebra.) Moreover, according to the usual pseudodifferential estimates along the leaves (see for instance [**55**], [**9**]), the resolvents of these operators belong respectively to the C^*-algebras $\mathcal{K}(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{F}})$ of $\tau^\nu_{\mathcal{F}}$-compact elements in $W^*_\nu(V,\mathcal{F};E)$ and $\mathcal{K}(W^*_\nu(G;E),\tau^\nu)$ of τ^ν-compact elements of $W^*_\nu(G;E)$. We recall that these compact operators are roughly defined using for instance the vanishing at infinity of the singular numbers, and we refer, for example, to [**9**] for the precise definition of these ideals. Set

$$D^2=\int_0^{+\infty}\lambda dE_\lambda \quad \text{and} \quad \tilde{D}^2=\int_0^{+\infty}\lambda d\tilde{E}_\lambda,$$

for the spectral decompositions in their respective von Neumann algebras. So E_λ and $\tilde{E}_\lambda$ are the spectral projections corresponding to $(-\infty,\lambda)$. Since the traces are normal on both von Neumann algebras,

$$N(\lambda)=\tau^\nu_{\mathrm{av}}(E_\lambda) \quad \text{and} \quad \tilde{N}(\lambda)=\tau^\nu_{\mathrm{reg}}(\tilde{E}_\lambda),$$

are well defined finite (Proposition 5.6 in the next subsection) non-decreasing and non-negative functions, which are right continuous. Hence there are Borel-Stieljes measures ϑ and $\tilde{\vartheta}$ on $\mathbb{R}$, such that:

$$\tau^\nu_{\mathcal{F}}(f(D))=\int_{\mathbb{R}} f(x)d\vartheta(x) \text{ and } \tau^\nu(f(\tilde{D}))=\int_{\mathbb{R}} f(x)d\tilde{\vartheta}(x),$$

for any Borel function $f:\mathbb{R}\to[0,+\infty]$. Since N and $\tilde{N}$ are finite, the measures ϑ and $\tilde{\vartheta}$ are easily proved to be σ-finite.

Proposition 5.2. — *The functions* $t\mapsto\tau^\nu_{\mathcal{F}}(De^{-t^2D^2})$ *and* $t\mapsto\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})$ *are Lebesgue integrable on* $(0,+\infty)$.

Proof. — We have

$$|\tau^\nu_{\mathcal{F}}(De^{-t^2D^2})|\leq\tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2}) \quad \text{and} \quad |\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})|\leq\tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2}).$$

Therefore and since the function $x\mapsto|x|e^{-t^2x^2}$ is rapidly decreasing, we know from Propositions 3.17 and 3.18 that for any $t>0$

$$\tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2})<+\infty \text{ and } \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2})<+\infty.$$

We also have the formulae

$$\tau^\nu_{\mathcal{F}}(|D|e^{-t^2D^2})=\int_{\mathbb{R}_+}\sqrt{x}e^{-t^2x}d\vartheta(x) \quad \text{and} \quad \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2})=\int_{\mathbb{R}_+}\sqrt{x}e^{-t^2x}d\tilde{\vartheta}(x).$$

Therefore, by Tonelli's theorem

$$\begin{aligned}\int_1^{+\infty} \tau^\nu_{\mathcal{G}}(|D|e^{-t^2D^2})dt &= \int_0^\infty \sqrt{x}\int_1^\infty e^{-t^2x}\,dt\,d\vartheta(x)\\ &= \int_0^\infty \sqrt{x}e^{-x}\int_1^\infty e^{-(t^2-1)x}\,dt\,d\vartheta(x)\\ &= \frac{1}{2}\int_0^\infty \sqrt{x}e^{-x}\int_0^\infty x^{-1/2}(u+x)^{-1/2}e^{-u}\,du\,d\vartheta(x)\\ &\leq \frac{1}{2}\left(\int_0^\infty e^{-x}d\vartheta(x)\right)\left(\int_0^\infty u^{-1/2}e^{-u}du\right)\\ &= \frac{\sqrt{\pi}}{2}\tau^\nu_{\mathcal{G}}(e^{-D^2}).\end{aligned}$$

The same proof show that

$$\int_1^{+\infty} \tau^\nu(|\tilde{D}|e^{-t^2\tilde{D}^2})dt < +\infty.$$

On the other hand, we have

$$\begin{aligned}\int_0^1 |\tau^\nu_{\mathcal{G}}(De^{-t^2D^2})|dt &\leq \int_0^1\int_{F\times T} |\operatorname{tr}(k_t([\tilde{m},\theta],[\tilde{m},\theta])|\,d\tilde{m}\,d\nu(\theta)\,dt\\ &\leq \int_0^1\int_{F\times T} C\,d\tilde{m}\,d\nu(\theta)\,dt\\ &= C\times \operatorname{vol}(V,d\tilde{m}\otimes\nu) < +\infty.\end{aligned}$$

Again, the same proof works as well for the regular trace and the regular von Neumann algebra. □

We are now in position to define the foliated eta invariants.

Definition 5.3. — *We define the up and down eta invariants of our longitudinal Dirac type operator by the formulae*

$$\eta^\nu_{\mathrm{up}}(\tilde{D}) := \frac{2}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu(\tilde{D}e^{-t^2\tilde{D}^2})dt \quad \textit{and} \quad \eta^\nu_{\mathrm{down}}(D) := \frac{2}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu_{\mathcal{G}}(De^{-t^2D^2})dt.$$

Since the traces on both von Neumann algebras are positive, the two eta invariants are real numbers.

Definition 5.4. — *The foliated rho invariant associated to the longitudinal Dirac type operator D on the foliated flat bundle $(V,\mathcal{F})$ is defined as*

$$\rho^\nu(D;V,\mathcal{F}) := \eta^\nu_{\mathrm{up}}(\tilde{D}) - \eta^\nu_{\mathrm{down}}(D)$$

with $\tilde{D}$ the lift of D to the monodromy cover.

We are mainly interested in the present paper in the leafwise signature operator D^{sign} and its leafwise lift to the monodromy groupoid $\tilde{D}^{\mathrm{sign}}$. In this case, we can state the following convenient result.

Lemma 5.5. — *Denote by Δ_j the Laplace operator on leafwise j-forms. Then the foliated eta invariant of the operator D^{sign} on $(V,\mathcal{F})$ is given by*

$$\eta^{\nu}(D^{\mathrm{sign}};V,\mathcal{F})=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^{\nu}_{\mathcal{G}}(*de^{-t^2\Delta_{m-1}})dt=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^{\nu}_{\mathcal{G}}(d*e^{-t^2\Delta_m})dt.$$

Similar statements hold for the lifted family $\tilde{D}^{\mathrm{sign}}$.

Proof. — This is an immediate consequence of a straightforward leafwise version of the computation made in [**2**, p. 67-68]. □

5.2. Eta invariants and determinants of paths. — We review the notion of (log-)determinants of paths, adapting the work of de La Harpe-Skandalis [**24**] to our context. Recall that M is odd dimensional. For any von Neumann algebra $\mathcal{M}$ endowed with a positive semifinite faithful normal trace τ, we denote by $L^1(\mathcal{M},\tau)$ the Schatten space of summable τ-measurable operators in the sense of [**22**]. Recall that $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ is a two sided $*$-ideal in $\mathcal{M}$. By Propositions 3.17, 3.18 we have for any rapidly decreasing Borel function ψ

$$\psi(\tilde{D}):=(\psi(\tilde{D}_\theta))_{\theta\in T}\in L^1(W^*_\nu(G;E),\tau^\nu)\cap W^*_\nu(G;E)$$

$$\psi(D):=(\psi(D_{L_\theta}))_{\theta\in T}\in L^1(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{G}})\cap W^*_\nu(V,\mathcal{F};E).$$

We set $\tilde{D}=\tilde{U}|\tilde{D}|$ and $D=U|D|$ for the polar decompositions in the corresponding von Neumann algebras. Then, this decomposition obviously coincides with the leafwise decompositions

$$\tilde{D}_\theta=\tilde{U}_\theta|\tilde{D}_\theta| \text{ and } D_L=U_L|D_L|.$$

For any $\theta\in T$ with $L=L_\theta$, we write the spectral decompositions:

$$|\tilde{D}_\theta|=\int_0^{+\infty}\lambda d\tilde{E}^\theta_\lambda \text{ and } |D_L|=\int_0^{+\infty}\lambda dE^L_\lambda.$$

As we have already remarked, the collection of partial isometries $\tilde{U}=(\tilde{U}_\theta)_{\theta\in T}$ (resp. $U=(U_{L_\theta})_{\theta\in T}$) as well as that of spectral projections $\tilde{E}_\lambda=(\tilde{E}^\theta_\lambda)_{\theta\in T}$ (resp. $E_\lambda=(E^{L_\theta}_\lambda)_{\theta\in T}$), all belong to $W^*_\nu(G;E)$ (resp. $W^*_\nu(V,\mathcal{F};E)$).

Proposition 5.6. — *We have $\tau^\nu(\tilde{E}_\lambda)<+\infty$ and $\tau^\nu_{\mathcal{G}}(E_\lambda)<+\infty$ for any $\lambda\in\mathbb{R}_+$.*

Proof. — We know that for any $\lambda<0$ the operator $(|\tilde{D}|-\lambda)^{-1}$ is τ^ν-compact in $W^*_\nu(G;E)$. In the same way, the operator $(|D|-\lambda)^{-1}$ is $\tau^\nu_{\mathcal{G}}$-compact in $W^*_\nu(V,\mathcal{F};E)$ [**17**]. Hence the spectral projections of $(|\tilde{D}|-\lambda)^{-1}$ are τ^ν-finite and the spectral projections of $(|D|-\lambda)^{-1}$ are $\tau^\nu_{\mathcal{G}}$-finite. This completes the proof. □

For any $t > 0$, the t-th singular number of the operator $\tilde{D}$ with respect to the probability measure ν is defined by [**22**]

$$\mu_t(\tilde{D}) = \inf\{\|\tilde{D}\tilde{p}\|,\ \tilde{p} = \tilde{p}^2 = \tilde{p}^* \in W^*_\nu(G;E) \text{ and } \int_T \mathrm{tr}(M_\chi(I-\tilde{p}_\theta)M_\chi)d\nu(\theta) \leq t\}.$$

In the same way, we define

$$\mu_t(D) = \inf\{\|Dp\|,\ p = p^2 = p^* \in W^*_\nu(V,\mathcal{F};E) \text{ and } \int_T \mathrm{tr}(M_\chi(I-p_{L_\theta})M_\chi)d\nu(\theta) \leq t\}.$$

From Proposition 5.6, we deduce that $0 \leq \mu_t(\tilde{D}) = \mu_t(|\tilde{D}|) < +\infty$ and $0 \leq \mu_t(D) = \mu_t(|D|) < +\infty$. The spectral measure of $|\tilde{D}|$ with respect to the probability measure ν is denoted $\tilde{\mu}$, while the spectral measure of $|D|$ is denoted μ. So for $\tilde{D}$ for instance we have

$$\mu(B) = \int_T \mathrm{tr}(M_\chi 1_B(|D_{L_\theta}|)M_\chi)d\nu(\theta)$$

for any Borel subset B of the spectrum of $|D|$ and

$$\tilde{\mu}(\tilde{B}) = \int_T \mathrm{tr}(M_\chi 1_{\tilde{B}}(|\tilde{D}_\theta|)M_\chi)d\nu(\theta)$$

for any Borel subset $\tilde{B}$ of the spectrum of $|\tilde{D}|$.

We denote by $\mathcal{IK}_{E,\mathrm{reg}}$ (resp. $\mathcal{IK}_{E,\mathrm{triv}}$) the subgroup of invertible operators in $W^*_\nu(G;E)$ (resp. in $W^*_\nu(V,\mathcal{F};E)$) which differ from the identity by an element of the ideal $\mathcal{K}(W^*_\nu(G;E),\tau^\nu)$ (resp. $\mathcal{K}(W^*_\nu(V,\mathcal{F};E),\tau^\nu_{\mathcal{F}})$). The subgroup of bounded operators which differ from the identity by a τ^ν-summable (resp. $\tau^\nu_{\mathcal{F}}$-summable) operator will be denoted $\mathcal{I}L^1_{E,\mathrm{reg}}$ (resp. $\mathcal{I}L^1_{E,\mathrm{triv}}$).

Whenever possible we shall refer to both von Neumann algebras $W^*_\nu(G;E)$ and $W^*_\nu(V,\mathcal{F};E)$ as $\mathcal{M}$. We shall then use the notation $\mathcal{IK}$ (resp. $\mathcal{I}L^1$) and denote by τ the corresponding trace.

Lemma 5.7. — *The space $\mathcal{I}L^1$ (resp. $\mathcal{IK}$) is a subgroup of the group of invertibles $\mathrm{GL}(\mathcal{M})$ of the von Neumann algebra $\mathcal{M}$.*

Proof. — We only need to check the stability for taking inverses. Let then $I+T$ be an invertible element in $\mathcal{M}$ such that $T \in L^1(\mathcal{M},\tau)$ (resp. $\mathcal{K}(\mathcal{M},\tau)$). Then we can write

$$(I+T)^{-1} - I = (I+T)^{-1}(I-(I+T)) = -(I+T)^{-1}T \in L^1(\mathcal{M},\tau) \text{ (resp. } \mathcal{K}(\mathcal{M},\tau)).$$

□

Proposition 5.8. — *Let $\gamma : [0,1] \to \mathcal{IK}$ be a continuous path for the uniform norm. For any $\epsilon > 0$, there exists a continuous piecewise affine path $\gamma_\epsilon : [0,1] \to \mathcal{I}L^1$ such*

that for any $t \in [0,1]$ we have $\|\gamma(t) - \gamma_\epsilon(t)\| \leq \epsilon$. Moreover, if $\gamma(0)$ and $\gamma(1)$ belong to $\mathscr{I}L^1$, then we can insure that $\gamma_\epsilon(i) = \gamma(i)$ for $i = 0, 1$.

Proof. — Since γ is continous for the operator norm, we can find $\delta > 0$ such that

$$|t - s| \leq \delta \Rightarrow \|\gamma(t) - \gamma(s)\| \leq \epsilon/3.$$

We subdivide $[0,1]$ into $0 = x_0 < x_1 < \cdots < x_n = 1$ so that $|x_{j+1} - x_j| \leq \delta$ for any j. On the other hand, the ideal $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is dense in $\mathcal{K}(\mathcal{M}, \tau)$ for the uniform norm. Therefore, for any $j = 0, \cdots, n$, we can find $\gamma_\epsilon(x_j) \in B(\gamma(x_j), \epsilon/9)$, the ball centered at $\gamma(x_j)$ with radius $\epsilon/9$, such that $\gamma_\epsilon(x_j) \in \mathscr{I}L^1$. We then define a path $\gamma_\epsilon : [0,1] \to \mathscr{I}L^1$ which is affine on every interval $[x_j, x_{j+1}]$ and prescribed by the values $\gamma_\epsilon(x_j)$ for $j = 0, \cdots, n$. The path γ_ϵ is then continuous and differentiable outside the finite set $\{x_j, j = 0, \cdots, n\}$. Moreover, for $t \in [x_j, x_{j+1}]$ we have

$$\begin{aligned} \|\gamma_\epsilon(t) - \gamma_\epsilon(x_j)\| &= t \times \|\gamma_\epsilon(x_{j+1}) - \gamma_\epsilon(x_j)\| \leq \|\gamma_\epsilon(x_{j+1}) - \gamma_\epsilon(x_j)\| \\ &\leq \|\gamma(x_{j+1}) - \gamma(x_j)\| + 2\epsilon/9 \leq 5\epsilon/9. \end{aligned}$$

Therefore,

$$\|\gamma(t) - \gamma_\epsilon(t)\| \leq \|\gamma(t) - \gamma(x_j)\| + \|\gamma(x_j) - \gamma_\epsilon(x_j)\| + \|\gamma_\epsilon(x_j) - \gamma_\epsilon(t)\| \leq \epsilon/3 + \epsilon/9 + 5\epsilon/9 = \epsilon.$$

□

Definition 5.9. — *Given a continuous piecewise C^1 path $\gamma : [0,1] \to \mathscr{I}L^1$ for the L^1-norm in $\mathcal{M}$, we define the determinant $w^\tau(\gamma)$ by the formula*

$$w^\tau(\gamma) := \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\gamma(t)^{-1}\gamma'(t))dt.$$

When $\mathcal{M}$ is $W^*_\nu(G, E)$ this determinant will be denoted by $w^\nu(\gamma)$ while when $\mathcal{M}$ is equal to $W^*_\nu(V, \mathcal{F}; E)$ this determinant will be denoted $w^\nu_{\mathcal{F}}(\gamma)$.

We summarize the properties of the determinant in the following

Proposition 5.10. — *Let $\gamma : [0,1] \to \mathscr{I}L^1$ be a continuous piecewise C^1 path for the L^1-norm.*

1. *Assume that*

$$\|\gamma(t) - I\|_1 < 1, \quad \text{for any } t \in [0,1].$$

Then, for any $t \in [0,1]$ the operator $\mathrm{Log}(\gamma(t))$ is well defined in the von Neumann algebra and we have

$$w^\tau(\gamma) = \frac{1}{2\pi\sqrt{-1}} \left[\tau(\mathrm{Log}(\gamma(1))) - \tau(\mathrm{Log}(\gamma(0)))\right].$$

2. *There exists $\delta_\gamma > 0$ such that for any continuous piecewise C^1 path $\alpha : [0,1] \to \mathcal{J}L^1$ for the L^1 norm, with*

$$\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma \text{ and } \alpha(i) = \gamma(i), i = 0, 1,$$

we have $w^\tau(\alpha) = w^\tau(\gamma)$.

3. *If γ is a continuous piecewise C^1 path for the uniform norm, then the determinant $w^\tau(\gamma)$ is well defined. Moreover, $w^\tau(\gamma)$ only depends on the homotopy class of γ with fixed endpoints and with respect to the uniform norm.*

Proof. — This proposition is a straightforward extension of the corresponding results in [**24**]. We give a brief outline of the proof here for the benefit of the reader. It is clear in the first item, since τ is a positive trace, that the function $t \mapsto \mathrm{Log}(\gamma(t))$ is well defined (using for instance the series) and is a piecewise smooth path. Moreover, we have

$$\frac{d}{dt}\tau(\mathrm{Log}(\gamma(t)) = \tau(\gamma^{-1}(t)\frac{d\gamma}{dt}(t).$$

This completes the proof of the first item.

Let α be a continuous piecewise C^1 path satisfying the assumptions of the second item. We consider the continuous piecewise C^1 loop $\beta : [0,1] \to \mathcal{J}L^1$ given by $\beta(t) = \gamma(t)^{-1}\alpha(t)$ which satisfies $\beta(0) = \beta(1) = I$. We have

$$\|\beta(t) - I\|_1 \leq \|\gamma(t)^{-1}\| \times \|\gamma(t) - \beta(t)\|_1.$$

Therefore, with $\delta_\gamma = \frac{1}{\inf_{t\in[0,1]} \|\gamma(t)^{-1}\|}$, we are done using the first item.

The rest of the proof is similar and is omitted. □

Definition 5.11. — *Let $\gamma : [0,1] \to \mathcal{J}\mathcal{K}$ be a continous path for the uniform norm such that $\gamma(0)$ and $\gamma(1)$ are in $\mathcal{J}L^1$. We define the determinant $w^\tau(\gamma)$ by $w^\tau(\gamma) := w^\tau(\alpha)$, for any continuous piecewise C^1 path $\alpha : [0,1] \to \mathcal{J}L^1$ such that*

$$\|\alpha(t) - \gamma(t)\|_1 \leq \delta_\gamma \text{ and } \alpha(i) = \gamma(i), i = 0, 1.$$

Remark 5.12. — *It is clear from the previous proposition that the above definition is well posed.*

We now set

$$\varphi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds, \quad \psi_t(x) := -e^{i\pi\varphi(tx)} \quad \text{and}$$
$$f_t(x) := xe^{-t^2x^2} \quad \text{for } x \in \mathbb{R}, \text{ and any } t \geq 0.$$

Then the function $1 - \psi_t$, the derivative ψ_t' and the function f_t are Schwartz class functions for any $t > 0$. Using the results of the previous sections, we deduce that the operators $I - \psi_t(\mathcal{D}_m)$, $\psi_t{}'(\mathcal{D}_m)$ and $f_t(\mathcal{D}_m)$ are $\mathcal{C}_m$-compact operators on the Hilbert module $\mathcal{E}_m$. Moreover, their images under the representations in the von Neumann

algebras $W^*_\nu(G;E)$ and $W^*_\nu(V,\mathcal{F};E)$ are trace class operators. Note also that the operator $\psi_t(\mathcal{D}_m)$ is invertible with inverse given by $-e^{-i\pi\varphi(t\mathcal{D}_m)}$, so $\psi_t(\mathcal{D}_m)$ is a smooth path of invertibles in $\mathcal{J}\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$ whose image under $\pi^{reg}\circ\chi_m^{-1}$ in $W^*_{\rm reg}(G;E)$ is also a smooth path of invertibles in $\mathcal{J}L^1_{E,{\rm reg}}$. The same result holds for the image under $\pi^{av}\circ\chi_m^{-1}$ in $W^*_\nu(V,\mathcal{F};E)$. We denote by

$$\gamma^{\rm reg}(\mathcal{D}_m)\equiv(\gamma^{\rm reg}_t(\mathcal{D}_m))_{t\geq 0}:=\left((\pi^{reg}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\right)_{t\geq 0}$$

and

$$\gamma^{\rm av}(\mathcal{D}_m)\equiv(\gamma^{\rm av}_t(\mathcal{D}_m))_{t\geq 0}:=\left((\pi^{av}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\right)_{t\geq 0}$$

the resulting smooth paths in the two von Neumann algebras. Using the traces τ^ν and $\tau^\nu_{\mathcal{G}}$, we define

$$w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m):=w^\nu(\gamma^{{\rm reg},\epsilon}(\mathcal{D}_m))\text{ and }w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m):=w^\nu_{\mathcal{G}}(\gamma^{{\rm av},\epsilon}(\mathcal{D}_m)).$$

with $\gamma^{{\rm reg},\epsilon}(\mathcal{D}_m)$ the path $\left((\pi^{reg}\circ\chi_m^{-1})(\psi_t(\mathcal{D}_m))\right)_{t\geq\epsilon}^{t\leq 1/\epsilon}$ and similarly for $\gamma^{{\rm av},\epsilon}(\mathcal{D}_m)$

Theorem 5.13. — *The following relations hold:*

$$\lim_{\epsilon\to 0}w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)=\frac{1}{2}\eta^\nu_{\rm up}(\tilde{D})\quad\textit{and}\quad\lim_{\epsilon\to 0}w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m)=\frac{1}{2}\eta^\nu_{\rm down}(D)$$

and hence

$$2\rho^\nu(D;V,\mathcal{F})=\lim_{\epsilon\to 0}[w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)-w^\nu_{{\rm av},\epsilon}(\mathcal{D}_m)].$$

Proof. — We have by definition and by straightforward computation

$$\begin{aligned}\gamma^{\rm reg}_t(\mathcal{D}_m)^{-1}\frac{d}{dt}\gamma^{\rm reg}_t(\mathcal{D}_m)&=(\pi^{\rm reg}\circ\chi_m^{-1})\left(i\pi\mathcal{D}_m\frac{2}{\sqrt{\pi}}e^{-t^2\mathcal{D}_m^2}\right)\\&=2i\sqrt{\pi}(\pi^{\rm reg}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right).\end{aligned}$$

But we know by Proposition 3.12 that

$$(\pi^{\rm reg}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right)=(f_t(\tilde{D}_\theta))_{\theta\in T},$$

where $(\tilde{D}_\theta)_{\theta\in T}$ is the Γ-invariant Dirac type family. Hence we get

$$\gamma^{\rm reg}_t(\mathcal{D}_m)^{-1}\frac{d}{dt}\gamma^{\rm reg}_t(\mathcal{D}_m)=2i\sqrt{\pi}(f_t(\tilde{D}_\theta))_{\theta\in T},$$

where this equality holds in the von Neumann algebra $W^*_\nu(G;E)$. Applying the trace τ^ν, integrating over $(0,+\infty)$ and dividing by $2i\pi$, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}w^\nu_{{\rm reg},\epsilon}(\mathcal{D}_m)&=\lim_{\epsilon\to 0}\frac{1}{\sqrt{\pi}}\int_\epsilon^{1/\epsilon}\tau^\nu((f_t(\tilde{D}_\theta))_{\theta\in T})dt\\&=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\tau^\nu((f_t(\tilde{D}_\theta))_{\theta\in T})dt=\frac{1}{2}\eta^\nu_{\rm up}(\tilde{D}).\end{aligned}$$

The proof of the second equality is similar and one uses the equality

$$(\pi^{\rm av}\circ\chi_m^{-1})\left(f_t(\mathcal{D}_m)\right)=(f_t(D_L))_{L\in V/F},$$

which is proved in Proposition 3.12 . □

6. Stability properties of ρ^ν for the signature operator

6.1. Leafwise homotopies. — Let Γ, T and $\tilde{M}$ be as in the previous sections. Let $V := \tilde{M} \times_\Gamma T$ be the associated foliated flat bundle. Assume that $\tilde{M}'$ is another Γ-coverings and let T' be a compact space endowed with a continuous action of Γ by homeomorphisms. We consider $\tilde{M}' \times T$ and the foliated flat bundle $V' := \tilde{M}' \times_\Gamma T$.

Definition 6.1. — *Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be two foliated spaces. A leafwise map $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a continuous map such that*

- *The image under f of any leaf of $(V, \mathcal{F})$ is contained in a leaf of $(V', \mathcal{F}')$.*
- *The restriction of f to any leaf of $(V, \mathcal{F})$ is a smooth map between smooth leaves.*

Remark 6.2. — 1. *We do not assume, that the leafwise derivatives to all orders of f are also continuous.*
2. *If V and V' are smooth manifolds and $f : V \to V'$ is a differentiable map, then f is a leafwise map if and only if $f_* : T(V) \to T(V')$ sends $T\mathcal{F}$ to $T\mathcal{F}'$.*

Roughly speaking, a leafwise map induces a "continuous map" between the quotient spaces of leaves. When the foliations are trivial, a leafwise map $f : M \times T \to M' \times T'$ is given by

$$f(m, \theta) = (h(m, \theta), k(\theta)), \quad (m, \theta) \in M \times T,$$

where k and h are continous and h is smooth with respect to the first variable.

An easy example of a leafwise map occurs when f is the quotient of a leafwise map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ between the two trivial foliations, which is (Γ, Γ')-equivariant with respect to a group homomorphism $\alpha : \Gamma \to \Gamma'$. We shall get back to this example more explicitely later on. It is easy to construct a leafwise map between V and V' which is not the quotient of a (Γ, Γ') equivariant leafwise map $\tilde{f}$. Moreover, if $\tilde{f}$ exists then it is not unique: indeed, for example, if $\delta \in Z(\Gamma) \subset \Gamma$ is an element in the center of Γ, then the leafwise map $\tilde{f}_\delta := \tilde{f} \circ \delta^*$ (where $\delta^* : \tilde{M} \times T \to \tilde{M} \times T$ is the diffeomorphism induced by the action of δ on the right), is equivariant with respect to the same homomorphism $\alpha : \Gamma \to \Gamma'$ (because $\delta \in Z(\Gamma)$) and also induces f.

Given a foliated space $(V, \mathcal{F})$ in the sense of [**38**], a subspace W of $(V, \mathcal{F})$ will be called a transversal to the foliation if for any $w \in W$ there exists a distinguished neighborhood U_w of w in V which is homeomorphic to $\mathbb{R}^p \times (U_w \cap W)$. Then one can show that the intersection of W with any leaf L of $(V, \mathcal{F})$ is a discrete subspace of L, that is a zero dimensional submanifold of L. Such a transversal is complete if it intersects all the leaves. In our example of foliated bundle $V = \tilde{M} \times_\Gamma T$, any fiber of

$V \to M$ is a complete transversal which is in addition compact, and any open subset of such fiber is a transversal.

Definition 6.3. — *1. Let $(V, \mathcal{F})$ be a foliated space. Two leafwise maps $f, g : (V, \mathcal{F}) \to (V', \mathcal{F}')$ are leafwise homotopic if there exists a leafwise map $H : (V \times [0,1], \mathcal{F} \times [0,1]) \to (V', \mathcal{F}')$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.*

2. Let $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ be two foliated spaces. A leafwise map $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ is a leafwise homotopy equivalence, if there exists a leafwise map $g : (V', \mathcal{F}') \to (V, \mathcal{F})$ such that

- *$g \circ f$ is leafwise homotopic to the identity of $(V, \mathcal{F})$.*
- *$f \circ g$ is leafwise homotopic to the identity of $(V', \mathcal{F}')$.*

3. We shall say that the foliations $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ are (strongly) leafwise homotopy equivalent if there exists a leafwise homotopy equivalence from $(V, \mathcal{F})$ to $(V', \mathcal{F}')$.

Note that according to the above definition, the homotopies in (2) are supposed to preserve the leaves.

It is a classical fact that two leafwise homotopy equivalent compact foliated spaces $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ have necessarily the same leaves dimension [**10**]. Note also that each leafwise homotopy equivalence sends a transversal to a transversal.

Lemma 6.4. — *A leafwise homotopy equivalence induces a local homeomorphism between transversals to the foliations.*

Proof. — See also [**10**]. Let f be the leafwise homotopy equivalence with homotopy inverse g, and denote by $h : [0,1] \times V \to V$ the $C^{\infty,0}$ homotopy between gf and the identity. Let $w \in V$. Let W be an open transversal of $(V, \mathcal{F})$ through $w \in W$. Take a distinguished chart U' in $(V', \mathcal{F}')$ which is an open neighborhood of $f(w)$ and which is homeomorphic to $D' \times W'$ for some transversal W' at $f(w)$. Then one finds an open distinguished chart U in $(V, \mathcal{F})$ such that $f(U) \subset U'$. Reducing W if necessary we can assume that U is homeomorphic to $D \times W$ for some disc D. Now, it is clear that since f is leafwise, it induces a map $\hat{f} : W \to W'$. By the same reasonning, we can assume furthermore that $g(U')$ is contained in a distinguished chart U_1 in $(V, \mathcal{F})$, homeomorphic to $D_1 \times W_1$.

The homotopy h induces a continous map $\hat{h} : W \to W_1$ and this map (or its reduction to a smaller domain) is simply the holonomy of the path $t \mapsto h(t, w)$. Hence $\hat{h}$ is locally invertible and it s clear that $\hat{h}^{-1}\hat{g}$ is a continuous inverse for $\hat{f}$. □

When $V = \tilde{M} \times_\Gamma T$ and $V' = \tilde{M}' \times_{\Gamma'} T'$, a particular case of leafwise homotopy equivalence is given by the quotient of an equivariant leafwise homotopy equivalence between $\tilde{M} \times T$ and $\tilde{M}' \times T'$. Recall that a fiberwise smooth map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$

is a continous map which can be written in the form

$$f(\tilde{m},\theta) = (h(\tilde{m},\theta), k(\theta)), \quad (\tilde{m},\theta) \in \tilde{M} \times T,$$

with h smooth with respect to the first variable. If $\alpha : \Gamma \to \Gamma'$ is a group homomorphism, then the fiberwise map $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ is α-equivariant if $\tilde{f}((\tilde{m},\theta)\gamma) = (\tilde{f}(\tilde{m},\theta))\alpha(\gamma)$.

In the following definition we extend the action of Γ and Γ' on $\tilde{M} \times T$ and $\tilde{M}' \times T'$ to $\tilde{M} \times [0,1] \times T$ and $\tilde{M}' \times [0,1] \times T'$ respectively, by declaring the action trivial on the $[0,1]$ factor.

Definition 6.5. — *We shall say that $f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is a special homotopy equivalence if there exist continuous maps $\tilde{f} : \tilde{M}\times T \to \tilde{M}'\times T'$, $\tilde{g} : \tilde{M}'\times T' \to \tilde{M}\times T$, $H : \tilde{M}\times[0,1]\times T \to \tilde{M}\times T$, $H' : \tilde{M}'\times[0,1]\times T' \to \tilde{M}'\times T'$, and group homomorphisms $\alpha : \Gamma \to \Gamma'$, $\beta : \Gamma' \to \Gamma$ such that:*

- *$\tilde{f}$, $\tilde{g}$, H and H' are fiberwise smooth;*
- *$\tilde{f}$ is α-equivariant; $\tilde{g}$ is β-equivariant; H is Γ-equivariant, H' is Γ'-equivariant;*
- *the restriction of H to $\tilde{M}\times\{0\}\times T$ (resp. of H' to $\tilde{M}'\times\{0\}\times T'$) is the identity map and the restriction of H to $\tilde{M}\times\{1\}\times T$ (resp. of H' to $\tilde{M}'\times\{1\}\times T'$) is $\tilde{g}\circ\tilde{f}$ (resp. $\tilde{f}\circ\tilde{g}$);*
- *$f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is induced by $\tilde{f} : \tilde{M}\times T \to \tilde{M}'\times T'$.*

If there exists such a special homotopy equivalence, we say that $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ are special homotopy equivalent.

Lemma 6.6. — *If the pairs $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ are special homotopy equivalent, then they are leafwise homotopic equivalent.*

Proof. — The equivariance of $\tilde{H}$ and $\tilde{H}'$ with respect to α and β, and the trivial action on the $[0,1]$ factor, allows to induce leafwise maps $H : V \times [0,1] \to V$ and $H' : V' \times [0,1] \to V'$ by setting,

$$H([\tilde{m},\theta];t) := [H(\tilde{m},t,\theta)] \text{ and } H'([\tilde{m}',\theta'];t) := [H'(\tilde{m}',t,\theta')].$$

In the same way the maps $\tilde{f}$ and $\tilde{g}$ induce leafwise maps f and g which are leafwise homotopy equivalences through the homotopies H and H'. □

Lemma 6.7. — *If $f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ is a special homotopy equivalence induced by $\tilde{f}(\tilde{m},\theta) = (h(\tilde{m},\theta),k(\theta))$ as in the previous definition, then $\alpha : \Gamma \to \Gamma'$ is an isomorphism and $k : T \to T'$ is an equivariant homeomorphism.*

Proof. — Let $\tilde{f}$ and $\tilde{g}$ be equivariant leafwise smooth maps which give a special homotopy equivalence as in the above definition. We denote by k and k' the continuous

equivariant maps induced by $\tilde{f}$ and $\tilde{g}$ on T and T' respectively. So,

$$k : T \to T' \text{ and } k' : T' \to T.$$

Since our homotopies send leaves to leaves, the composite maps $k' \circ k$ and $k \circ k'$ are identity maps. Moreover, if α and β are the group homomorphisms corresponding to the equivariance property of $\tilde{f}$ and $\tilde{g}$ respectively, then the homotopy $\tilde{H}$ satisfies

$$\tilde{H}((\tilde{m}, t, \theta)\gamma) = \tilde{H}(\tilde{m}, t, \theta)(\beta \circ \alpha)(\gamma), \quad \forall t \in [0, 1].$$

Therefore, applying this relation to $t = 0$, we get $\beta \circ \alpha = id_\Gamma$. The same argument gives the relation $\alpha \circ \beta = id_{\Gamma'}$. □

Remark 6.8. — *As already remaked, easy examples show that the foliations $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ can be leafwise homotopy equivalent with non isomorphic groups Γ and Γ' and non homeomorphic spaces T and T'.*

6.2. $\rho^\nu(V, \mathcal{F})$ is metric independent. — We fix a continuous leafwise smooth Riemannian metric g on (V, F). g is lifted to a Γ-equivariant leafwise metric $\tilde{g}$ on $\tilde{M} \times T$, see [**38**]. So $\tilde{g} = (\tilde{g}(\theta))_{\theta \in T}$, where $\tilde{g}(\theta)$ is a metric on $\tilde{M} \times \{\theta\}$ and we assume that this structure is transversely continuous and equivariant with respect to the action of Γ. In what follows we shall refer to the bundle of exterior powers of the cotangent bundle as the Grassmann bundle. Consider the Γ-equivariant vector bundle $\widehat{E}$ over $\tilde{M} \times T$, obtained by pulling back from V the longitudinal Grassmann bundle E of the foliation $(V, \mathcal{F})$. Assume for the sake of simplicity of signs that the dimension of M is $4\ell - 1$ that is in the notations of Section 4, $m = 2\ell$. Consider the associated Γ-equivariant family of signature operators $(\tilde{D}_\theta^{\text{sign}})_{\theta \in T}$ associated with $\tilde{g}$, as defined in Section 4. We denote by D^{sign} the longitudinal signature operator on $(V, \mathcal{F})$ associated with the leafwise metric g acting on leafwise $2\ell - 1$ forms.

Recall that ν is a Γ-invariant Borel measure on T. We have defined in Subsection 5.1 a foliated rho-invariant $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$. We want to investigate the behavior of $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$ under a change of metric and under a leafwise diffeomorphism. First, we deal with the invariance of ρ^ν with respect to a change of metric. Up to constant, we can replace $\rho^\nu(D^{\text{sign}}; V, \mathcal{F})$, as it is usual, see [**3**] [**16**], by the ρ invariant of the foliation $(V, \mathcal{F})$ defined as:

$$\rho^\nu(V, \mathcal{F}; g) := \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\tau^\nu(\tilde{*}\tilde{d}e^{-t\tilde{\Delta}}) - \tau^\nu_{\mathcal{F}}(*de^{-t\Delta})\right] \frac{dt}{\sqrt{t}},$$

where $\tilde{\Delta}$ and Δ are the Laplace operators on leafwise $2\ell - 1$ forms, associated with the metrics $\tilde{g}$ and g respectively.

Proposition 6.9. — *Let Γ, $\tilde{M}$, T, ν and $(V, \mathcal{F})$ be as above. Let $(g_u)_{u \in [0,1]}$ be a continuous leafwise smooth one-parameter family of continuous leafwise smooth metrics*

on $(V, \mathcal{F})$. *Then*

$$\rho^{\nu}(V, \mathcal{F}; g_0) = \rho^{\nu}(V, \mathcal{F}; g_1). \tag{23}$$

Proof. — The proof of this proposition in the case where T is reduced to a point was first given by J. Cheeger and M. Gromov in [**16**]. The Cheeger-Gromov proof extends to the general case of measured foliations and in particular to the case of foliated bundles and we proceed to explain the easy modifications needed for foliated bundles. Let D_u, for $0 \leq u \leq 1$, be the leafwise operator on $2\ell - 1$ leafwise differential forms of $(V, \mathcal{F})$, given by $D_u = *_u \circ d$, where d is the leafwise de Rham operator and $*_u$ is the leafwise Hodge operator associated with the metric g_u. It is easy to see that $u \mapsto \tau^{\nu}_{\mathcal{G}}(D_u e^{-t\Delta_u})$ is smooth. Since V is compact, the elliptic estimates along the leaves are uniform and we have for instance

$$\mathcal{R}(e^{-r\Delta_0}) \subset \mathrm{Dom}(\Delta_u), \quad \forall r > 0 \text{ and } u \in [0, 1].$$

Here $\mathcal{R}$ denotes the range of an operator and Dom the domain. Therefore, we can follow the steps of the proof in [**16**] and deduce the fundamental relation

$$\frac{d}{du}|_{u=0}\tau^{\nu}_{\mathcal{G}}(D_u e^{-t\Delta_u}) = \tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-t\Delta_0}) + 2t\frac{d}{dt}\tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-t\Delta_0})).$$

Using integration by parts, we then deduce

$$\sqrt{\pi}\frac{d}{du}|_{u=0}\int_{\epsilon}^{A}\tau^{\nu}_{\mathcal{G}}(D_u e^{-t\Delta_u})\frac{dt}{\sqrt{t}} = 2\sqrt{A}\tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-A\Delta_0}) - 2\sqrt{\epsilon}\tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-\epsilon\Delta_0}).$$

Using the normality of the trace $\tau^{\nu}_{\mathcal{G}}$ and the spectral decomposition in the type II_{∞} von Neumann algebra $W^*_{\nu}(V, \mathcal{F}; E)$, we deduce that

$$\lim_{A \to +\infty} 2\sqrt{A}\tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-A\Delta_0}) = 0.$$

Now, the same estimates are as well valid in the type II_{∞} von Neumann algebra $W^*_{\nu}(G; E)$ with the normal trace τ^{ν}. Hence, we are reduced to comparing the limits as $\epsilon \to 0$ of the difference

$$2\sqrt{\epsilon}\tau^{\nu}_{\mathcal{G}}(\frac{d*}{du}(0)de^{-\epsilon\Delta_0}) - 2\sqrt{\epsilon}\tau^{\nu}(\frac{d\tilde{*}}{du}(0)de^{-\epsilon\tilde{\Delta}_0}).$$

Replacing the heat operators by corresponding parametrices which are localized near the units V, in the two groupoids involved, see for instance [**17**], the limit of the two terms in the above difference is proved to be the same by classical arguments, which finishes the proof. □

According to the previous proposition we can now denote by $\rho^{\nu}(V, \mathcal{F})$ *the signature rho invariant associated to any metric as before. All the leafwise maps considered in the rest of the paper are assumed to respect the orientations.*

If we are now given a leafwise smooth homeomorphism $f : V \longrightarrow V'$, then we can transport the leafwise metric g from V to f_*g on V' and form the corresponding signature operator $D^{\mathrm{sign}\,\prime}$ along the leaves of $(V', \mathcal{F}')$ and also the Γ-equivariant signature operator $\tilde{D}^{\mathrm{sign}\,\prime} = (\tilde{D}^{\mathrm{sign}\,\prime}_{\theta'})_{\theta' \in T'}$ corresponding to the lifted Γ-invariant metric. Finally, the Γ-invariant measure ν on T, yields a holonomy invariant transverse measure $\Lambda(\nu)$ on the foliation $(V, \mathcal{F})$. The leafwise smooth homeomorphism f sends transversals to transversals and allows to transport the measure $\Lambda(\nu)$ into a holonomy invariant transverse measure $f_*\Lambda(\nu)$ on $(V', \mathcal{F}')$. Such a measure yields by restriction to a fiber a Γ'-invariant measure ν' on T' so that $f_*\Lambda(\nu) = \Lambda(\nu')$. More precisely, a fiber $V'_{m'_0}$ of the fibration $V' \to M'$ is a transversal to the foliation $\mathcal{F}'$ and hence the holonomy invariant transverse measure $f_*\Lambda(\nu)$ restricts to a measure on $V'_{m'_0}$. On the other hand, by fixing $\tilde{m}'_0$ with $[\tilde{m}'_0] = m'_0$ we get an identification of $V'_{m'_0}$ with the space T'. It is an easy exercise to check that the corresponding mesure on T' through this identification is Γ'-invariant and that the associated holonomy invariant transverse measure on the foliation $(V', \mathcal{F}')$ is precisely $f_*\Lambda(\nu)$.

Proposition 6.10. — *With the above notations, we have the following equalities for the eta invariants associated with the two signature operators D^{sign} and $D^{\mathrm{sign}\,\prime}$:*

$$\eta^{\nu}_{\mathrm{down}}(D^{\mathrm{sign}}) = \eta^{\nu'}_{\mathrm{down}}(D^{\mathrm{sign}\,\prime}) \quad \textit{and} \quad \eta^{\nu}_{\mathrm{up}}(\tilde{D}^{\mathrm{sign}}) = \eta^{\nu'}_{\mathrm{up}}(\tilde{D}^{\mathrm{sign}\,\prime}).$$

Proof. — Let us prove, for example, the second equality (the first one will be obtained in a similar way). Let W be the regular von Neumann algebra associated to $(V, \mathcal{F})$, the vertical Grassmann bundle $\widehat{E}$ and g. Let τ be the trace defined by g and ν and let W' and τ' be the corresponding objects, associated to $(V', \mathcal{F}')$, f_*g and the transported measure ν' under the leafwise smooth homeomorphism f. The leafwise smooth homeomorphism f lifts to a leafwise smooth homeomorphism $\tilde{f}$ between the monodromy groupoids G and G'. More precisely, for any $x \in V$ f lifts to a diffeomorphism $\tilde{f}_x : G_x \to G'_{f(x)}$ which induces, by the pull-back of forms, a unitary U_x between the spaces of L^2-forms. Recall that the metric on $(V', \mathcal{F}')$ is f_*g. The signature operator on $G'_{f(x)}$ associated with the metric f_*g is easily identified with the push-forward operator under $\tilde{f}$, that is the conjugation of the signature operator on G_x by the unitary U_x. Hence the functional calculus of $\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)}$ is also the conjugation of the functional calculus of $\tilde{D}^{\mathrm{sign}}_x$ by U_x. So, in particular, for any $x \in V$ we have

$$\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)} \exp(-t(\tilde{D}^{\mathrm{sign}\,\prime}_{f(x)})^2) = U_x \tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2) U_x^{-1}.$$

Now, by definition of the trace τ' associated with the image measure ν', one easily shows that

$$\tau'(U_x \tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2) U_x^{-1}) = \tau(\tilde{D}^{\mathrm{sign}}_x \exp(-t(\tilde{D}^{\mathrm{sign}}_x)^2)).$$

Therefore, the $f_*\Lambda(\nu)$ measured eta invariant of the G'-invariant family $(\tilde{D}^{\mathrm{sign}\,\prime}_{x'})_{x'\in V'}$ as defined by Peric in [**43**] coincides with the $\Lambda(\nu)$ measured eta invariant of the G-invariant family $(\tilde{D}^{\mathrm{sign}}_x)_{x\in V}$. On the other hand and as we already observed, these Peric measured eta invariants coincide with ours for the Γ'-invariant and Γ-invariant families of signature operators on $\tilde{M}'\times T'$ and $\tilde{M}\times T$ respectively. Hence the proof is complete. □

Corollary 6.11. — *Let $(V,\mathcal{F},\nu)$ and $(V',\mathcal{F}',\nu')$ be two foliated bundles as above and assume that there exists a leafwise smooth homeomorphism between $(V,\mathcal{F})$ and $(V',\mathcal{F}')$ such that $f_*\nu=\nu'$. Then*

$$\rho^{\nu}(V,\mathcal{F})=\rho^{\nu'}(V',\mathcal{F}')\,.$$

Proof. — We use the two previous propositions. The first one allows to compute $\rho^{\nu}(V,\mathcal{F})$ using any metric g. Then we apply the same proposition to $\rho^{f_*\nu}(V',\mathcal{F}')$ and compute it using the image metric f_*g. Finally, the second proposition allows to finish the proof. □

7. Loops, determinants and Bott periodicity

As before, let $\mathcal{A}_m$ be the maximal C^*-algebra of the groupoid $T\rtimes\Gamma$; let $\mathcal{E}_m$ be the $\mathcal{A}_m$-Hilbert module considered in the previous sections. Thus $\mathcal{E}_m$ is obtained by completion of the $\mathcal{A}_c$-Module $C^\infty_c(\widetilde{M}\times T,\widehat{E})$. Let $\mathcal{D}_m$ be the regular unbounded $\mathcal{A}_m$-linear operator induced by a Γ-equivariant family of Dirac operators. Let

$$\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m):=\{A\in\mathcal{B}_{\mathcal{A}_m}(\mathcal{E}_m)\text{ such that }A-\mathrm{Id}\in\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)\text{ and }A\text{ is invertible}\}.$$

Let $\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))$ be the space of homotopy classes of loops in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m)$ which contain the identity operator. Then, using the inverse of the Bott isomorphism $\beta^{-1}:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))$, the isomorphism $(\chi_m^{-1})_*:K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{B}^E_m)$ induced by $\chi_m:\mathcal{B}^E_m\to\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m)$, and the inverse of the Morita isomorphism $\mathcal{M}_m:K_0(\mathcal{A}_m)\to K_0(\mathcal{B}^E_m)$ of Proposition 2.10, we obtain an isomorphism

$$\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\xrightarrow{\beta^{-1}}K_0(\mathcal{K}_{\mathcal{A}_m}(\mathcal{E}_m))\xrightarrow{(\chi_m^{-1})_*}K_0(\mathcal{B}^E_m)\xrightarrow{\mathcal{M}_m^{-1}}K_0(\mathcal{A}_m).$$

We denote by $\Theta:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to K_0(\mathcal{A}_m)$ the composition of these isomorphisms. Recall the representations

$$\pi^{\mathrm{reg}}:\mathcal{B}^E_m\to W^*_\nu(G;E)\,;\quad \pi^{\mathrm{av}}:\mathcal{B}^E_m\to W^*_\nu(V,\mathcal{F};E).$$

Given a morphism α between two C^*-algebras, we denote, with obvious abuse of notation, by $\Omega\alpha$ the induced map on homotopy classes of loops. We thus obtain maps $\Omega\pi^{\mathrm{reg}}$, $\Omega\chi_m^{-1}$, $\Omega\pi^{\mathrm{av}}$; we define

$$\sigma^{\mathrm{reg}}:\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))\to\Omega(\mathcal{IK}(W^*_\nu(G;E)))\,;$$

$$\sigma^{\mathrm{av}} : \Omega(\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)) \to \Omega(\mathcal{IK}(W^*_\nu(V,\mathcal{F};E)))$$

with

$$\sigma^{\mathrm{reg}} := \Omega\pi^{\mathrm{reg}} \circ \Omega\chi_m^{-1}\,, \quad \sigma^{\mathrm{av}} := \Omega\pi^{\mathrm{av}} \circ \Omega\chi_m^{-1}\,.$$

Recall, finally, that if ℓ is a loop in $\mathcal{I}L^1(W^*_\nu(V,\mathcal{F};E))$, or more generally in $\mathcal{IK}(W^*_\nu(V,\mathcal{F};E))$, then ℓ has a well defined (log-)determinant $w^\nu_{\mathcal{F}}(\ell) \in \mathbb{C}$. Similarly, if ℓ is a loop in $\mathcal{I}L^1(W^*_\nu(G;E))$, or more generally in $\mathcal{IK}(W^*_\nu(G;E))$, then ℓ has a well defined (log-)determinant $w^\nu(\ell) \in \mathbb{C}$.

Proposition 7.1. — *The following diagram commutes:*

$$\begin{array}{ccc} \Omega(\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)) & \xrightarrow{\ \Theta\ } & K_0(\mathcal{G}_m) \\ \Big\downarrow{\scriptstyle \sigma^{\mathrm{av}}} & & \Big\downarrow{\scriptstyle \tau^\nu_{\mathrm{av},*}} \\ \Omega(\mathcal{IK}(W^*_\nu(V,\mathcal{F};E))) & \xrightarrow{\ w^\nu_{\mathcal{F}}\ } & \mathbb{C}. \end{array}$$

Similarly, the following diagram commutes:

$$\begin{array}{ccc} \Omega(\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)) & \xrightarrow{\ \Theta\ } & K_0(\mathcal{G}_m) \\ \Big\downarrow{\scriptstyle \sigma^{\mathrm{reg}}} & & \Big\downarrow{\scriptstyle \tau^\nu_{\mathrm{av},*}} \\ \Omega(\mathcal{IK}(W^*_\nu(G;E))) & \xrightarrow{\ w^\nu\ } & \mathbb{C}. \end{array}$$

Proof. — Recall that for a C^*-algebra A the (inverse of the) Bott isomorphism $\beta : K_0(A) \to K_1(SA)$ is given by the map $[p] \to [(\exp(2\pi itp)]$; as there will be several C^*-algebras involved, we denote this map by β_A. We observe that

$$\beta_{\mathcal{B}^E_m} \circ (\chi_m^{-1})_* = \Omega(\chi_m^{-1})_* \circ \beta_{\mathcal{K}_{\mathcal{G}_m}(\mathcal{E}_m)}\,.$$

Therefore,

$$\begin{aligned} \Omega\pi^{\mathrm{av}} \circ \beta_{\mathcal{B}^E_m} \circ (\chi_m^{-1})_* \circ \beta^{-1}_{\mathcal{K}_{\mathcal{G}_m}(\mathcal{E}_m)} &= \Omega\pi^{\mathrm{av}} \circ \Omega(\chi_m^{-1})_* \circ (\beta_{\mathcal{K}_{\mathcal{G}_m}(\mathcal{E}_m)} \circ \beta^{-1}_{\mathcal{K}_{\mathcal{G}_m}(\mathcal{E}_m)}) \\ &= \Omega\pi^{\mathrm{av}} \circ \Omega(\chi_m^{-1})_* \\ &= \sigma^{\mathrm{av}}. \end{aligned}$$

On the other hand, by definition of $\Omega\pi^{\mathrm{av}}$,

$$\Omega\pi^{\mathrm{av}} \circ \beta_{\mathcal{B}^E_m} = \beta_{\mathcal{K}(W^*_\nu(V,\mathcal{F};E))} \circ \pi^{\mathrm{av}}_*;$$

therefore

$$\begin{aligned} w^\nu_{\mathcal{F}} \circ \Omega\pi^{\mathrm{av}} \circ \beta_{\mathcal{B}^E_m} &= w^\nu_{\mathcal{F}} \circ \beta_{\mathcal{K}(W^*_\nu(V,\mathcal{F};E))} \circ \pi^{\mathrm{av}}_* \\ &= \tau^\nu_{\mathcal{F}} \circ \pi^{\mathrm{av}}_* \\ &= \tau^\nu_{\mathrm{av},*} \end{aligned}$$

where τ^{ν}_{av} is the trace on $\mathcal{B}^{E}_{m}$ as defined in Subsection 2.4 and with the equality $w^{\nu}_{\mathcal{G}} \circ \beta_{\mathcal{K}(W^{*}_{\nu}(V,\mathcal{F};E))} = \tau^{\nu}_{\mathcal{G}}$ proved by direct computation. To finish the proof we simply apply Proposition 2.10. □

Definition 7.2. — *We shall denote by*

$$w^{\nu}_{\mathrm{av}} : \Omega(\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)) \to \mathbb{C} \quad \textit{and} \quad w^{\nu}_{\mathrm{reg}} : \Omega(\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)) \to \mathbb{C}$$

the compositions $w^{\nu}_{\mathcal{G}} \circ \sigma^{\mathrm{av}}$ *and* $w^{\nu} \circ \sigma^{\mathrm{reg}}$ *respectively.*

We can summarize the previous proposition by the following two equations

$$w^{\nu}_{\mathrm{av}} = \tau^{\nu}_{\mathrm{av},*} \circ \Theta\,, \qquad w^{\nu}_{\mathrm{reg}} = \tau^{\nu}_{\mathrm{reg},*} \circ \Theta. \tag{24}$$

Remark 7.3. — *Definition 7.2 can be extended to a path in* $\mathcal{IK}_{\mathcal{G}_m}(\mathcal{E}_m)$ *provided the two extreme points are mapped by* $\pi^{\mathrm{reg}} \circ \chi_m^{-1}$ *and* $\pi^{\mathrm{av}} \circ \chi_m^{-1}$ *into* τ^{ν} *trace class and* $\tau^{\nu}_{\mathcal{G}}$ *trace class perturbations of the identity respectively.*

8. On the homotopy invariance of rho on foliated bundles

Before plunging into foliated bundles and the foliated homotopy invariance of the signature rho invariant defined in Section 5, we digress briefly and treat a general orientable measured foliation $(V, \mathcal{F})$. We denote by Λ the holonomy invariant transverse measure. We fix a longitudinal riemannian metric on $(V, \mathcal{F})$ and we denote by D^{sign} the associated longitudinal signature operator. Let G be the monodromy groupoid associated to $(V, \mathcal{F})$. Then, as already remarked, Peric has defined in [**43**] a foliated eta invariant $\eta^{\Lambda}(\tilde{D}^{\mathrm{sign}})$, with $\tilde{D}^{\mathrm{sign}}$ the lift of D^{sign} to the monodromy covers, a G-equivariant operator on G. The work of Peric employs the holonomy groupoid, but is is not difficult to see that his arguments apply to the monodromy groupoid as well. Ramachandran, on the other hand, has defined in [**47**] an eta invariant $\eta^{\Lambda}(D^{\mathrm{sign}})$ using the measurable groupoid defined by the foliation, as we have already observed. We infer that the definition of foliated rho invariant is basically present in the literature. It suffices to define $\rho^{\Lambda}(D^{\mathrm{sign}}) := \eta^{\Lambda}(\tilde{D}^{\mathrm{sign}}) - \eta^{\Lambda}(D^{\mathrm{sign}})$. Assume now that G^{x}_{x} is torsion-free for any $x \in V$, then Connes has defined in [**18**] a Baum-Connes map $K_*(BG) \to K_*(C^{*}_{\mathrm{reg}}(V, \mathcal{F}))$ which factors through a maximal Baum-Connes map with values in the K-theory of the maximal C^*-algebra $C^{*}_{\mathrm{max}}(V, \mathcal{F})$. Here BG is the classifying space of the monodromy groupoid, see [**18**], page 126. If $(V, \mathcal{F})$ is equal to the foliated bundle $V = \tilde{M} \times_{\Gamma} T$, then BG is given by the homotopy quotient $E\Gamma \times_{\Gamma} T$, with $E\Gamma$ equal to the universal space for Γ principal bundles. The Baum-Connes conjecture states that the Baum-Connes map is an isomorphism. We shall make a stronger assumption here, namely that the *maximal* Baum-Connes map is an isomorphism. This is a non trivial assumption and even if it is known to be satisfied

for instance for amenable actions, there are examples where it fails to be true. The general conjecture one would then like to make goes as follows.

Let $(V', \mathcal{F}')$ be another foliation, endowed with a holonomy invariant transverse measure Λ' and let $f : (V, \mathcal{F}) \to (V', \mathcal{F}')$ be a leafwise measure preserving homotopy equivalence.

Conjecture. — *If G_x^x is torsion-free for any $x \in V$ and $K_*(BG) \to K_*(C^*_{\max}(V, \mathcal{F}))$ is an isomorphism, then $\rho^{\Lambda}(D^{\mathrm{sign}}) = \rho^{\Lambda'}(D^{\mathrm{sign}\,\prime})$.*

We shall now specialize to foliated bundles. Let Γ, T and $\tilde{M}$ be as in the previous sections. Let $V := \tilde{M} \times_\Gamma T$ and let $(V, \mathcal{F})$ be the associated foliated bundle. We assume the existence of a Γ-invariant measure on T; let $\Lambda(\nu)$ be the associated holonomy invariant transverse measure on $(V, \mathcal{F})$. Let $D = (D_L)_{L \in V/\mathcal{F}}$ be a longitudinal Dirac-type operator. Let $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$ be the associated Γ-equivariant family of Dirac operators. As already remarked the rho invariant $\rho^{\Lambda(\nu)}(D)$ defined above, is indeed equal to our rho invariant $\rho^\nu(D; V, \mathcal{F})$. Assume now that $\tilde{M}'$ is the Γ' universal covering of a compact manifold M' and let T' be a compact space endowed with a continuous action of Γ' by homeomorphisms. We consider $\tilde{M}' \times T'$ and the foliated bundle $V' := \tilde{M}' \times_{\Gamma'} T'$. Let $(V', \mathcal{F}')$ be the associated foliated space. We assume the existence of a Γ'-invariant measure ν' on T' and we let $\Lambda(\nu')$ be the associated transverse measure on $(V', \mathcal{F}')$. Given a measure preserving foliated homotopy equivalence $f : V \to V'$, we can apply the general conjecture stated above to the invariants $\rho^{\Lambda(\nu)}(D)$, $\rho^{\Lambda(\nu')}(D')$ with D and D' denoting now the signature operators. We obtain in this way a conjecture about the homotopy invariance of the signature rho invariant $\rho^\nu(V, \mathcal{F})$ defined and studied in this paper; we shall deal with the general conjecture on foliated spaces in a different paper. In the rest of this section, we shall tackle the homotopy invariance of rho for the special homotopy equivalences descending from equivariant homotopies $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T'$ as described in the previous section.

8.1. The Baum-Connes map for the discrete groupoid $T \rtimes \Gamma$. — In order to tackle the homotopy invariance of our $\rho^\nu(V, \mathcal{F})$ we first need to describe in the most geometric way the Baum-Connes map relevant to foliated bundles. This subsection is thus devoted to recall the definition of the Baum-Connes map with coefficients in the $\Gamma - C^*$-algebra $C(T)$ and, more importantly, to give a very geometric description of it. There are indeed several definitions available in the literature, with proofs of their compatibility sometime missing. The differences are all concentrated in the domain and, consequently, in the definition of the application; the target is always the same, namely $K_*(C(T) \rtimes_r \Gamma)$ (which is nothing but $K_*(\mathcal{G}_r)$ in our notation). Notice that if T is a point, we also have two different possibilities for the classical Baum-Connes map, depending on whether we consider, on the left hand side, the

Baum-Douglas definition of K-homology or, instead, Kasparov's definition; although the compatibility of the two pictures has been assumed for many years, a complete proof only appeared recently, see the paper [**6**]. Going back to our more general situation, we begin with the Baum-Connes-Higson definition [**5**], which is given is terms of Kasparov KK-theory and the intersection product:

$$\mu_{BCH} : K_j^\Gamma(\underline{E}\Gamma; C(T)) \to K_j(C(T) \rtimes_r \Gamma). \tag{25}$$

The group on the left is, by definition,

$$\lim_{X \subset \underline{E}\Gamma} KK_\Gamma^j(C_0(X), C(T))$$

with the direct limit taken over the directed system of all Γ-compact subset of $\underline{E}\Gamma$. Similarly, there is a maximal Baum-Connes-Higson map:

$$\mu_{BCH} : K_j^\Gamma(\underline{E}\Gamma; C(T)) \to K_j(C(T) \rtimes_m \Gamma). \tag{26}$$

Next, we have the original definition of Baum and Connes [**4**], with the left hand side defined in terms of Gysin maps:

$$\mu_{BC} : K^j(T, \Gamma) \to K_j(C(T) \rtimes_r \Gamma). \tag{27}$$

We are not aware of a published proof of the compatibility of these two maps.

There is a third description of the Baum-Connes map with coefficients in $C(T)$: consider as set of cycles the (isomorphism classes of) pairs $(X, E \to X \times T)$ where X is a spin$_c$ proper Γ-manifold and E is a Γ-equivariant vector bundle on $X \times T$; define the usual Baum-Douglas equivalence relation on these cycles, bordism, direct sum and bundle modification; we obtain a group that we denote by $K_j^{\text{geo}}(T \rtimes \Gamma)$ with $j = \dim M \bmod 2$. The Baum-Connes map in this case is denoted

$$\mu_\rtimes : K_j^{\text{geo}}(T \rtimes \Gamma) \to K_j(C(T) \rtimes_r \Gamma) \tag{28}$$

and is very simply described as the map that associates to $[X, E \to X \times T]$ the index class of the Γ-equivariant family $(D_\theta)_{\theta \in T}$, with D_θ the spin$_c$ Dirac operator on X twisted by $E\big|_{X \times \{\theta\}}$. Also in this case we have a maximal version of the map:

$$\mu_\rtimes : K_j^{\text{geo}}(T \rtimes \Gamma) \to K_j(C(T) \rtimes_m \Gamma). \tag{29}$$

Thanks to the Ph.D. thesis of Jeff Raven [**48**] it is now established that the two groups $K_j^\Gamma(\underline{E}\Gamma; C(T))$ and $K_j^{\text{geo}}(T \rtimes \Gamma)$ are isomorphic and the two pairs of maps (25), (28) and (26), (29) are compatible; the proof of Raven's isomorphism is far from being trivial. Notice that, as in [**29**], we can consider orientable manifolds instead of spin$_c$ manifolds; thus the set of cycles for this version of Raven's group is given by pairs $(X, E \to X \times T)$ with X an orientable proper riemannian Γ-manifold and E a Γ-equivariant vector bundle on $X \times T$ endowed with an equivariant Clifford-module structure with respect to the Clifford algebra bundle of T^*X. Introduce on these cycles the equivalence relation given by bordism, direct sum and bundle modification

as in [**29**] (Subsection 2.2, pages 59 and 60). The resulting group will be isomorphic to $K_j^{\mathrm{geo}}(T \times \Gamma)$ and the resulting Baum-Connes map will be compatible. In the rest of this work we look at the stability properties of our foliated rho-invariant for the signature operator under a bijectivity hypothesis on the map (29). However, in order to exhibit examples we do need to use the compatibility between (26) and (29); indeed almost all examples where the **B**aum-Connes assumption is satisfied are proved using the Baum-Connes-Higson description.

8.2. Homotopy invariance of $\rho^\nu(V,\mathcal{F})$ for special homotopy equivalences. — We can state the main result of this section as follows:

Theorem 8.1. — *Let $V := \tilde{M} \times_\Gamma T$ and $V' := \tilde{M}' \times_{\Gamma'} T'$ be two foliated flat bundles, with Γ and Γ' discrete* **torsion-free** *groups*[(4]. *Assume that there exists a* **special** *leafwise homotopy equivalence $f : (V,\mathcal{F}) \to (V',\mathcal{F}')$ and let $k : T \to T'$ be the induced equivariant homeomorphism . Let ν' be a Γ'-invariant measure on T'; let $\nu := k^*\nu'$ be the corresponding Γ-invariant measure on T. Assume that the Baum-Connes map* (28) *for the maximal C^*-algebra*

$$\mu_\rtimes : K_j^{\mathrm{geo}}(T \rtimes \Gamma) \longrightarrow K_*(C(T) \rtimes_{\max} \Gamma)$$

is **bijective**. *Then*

$$\rho^\nu(V,\mathcal{F}) = \rho^{\nu'}(V',\mathcal{F}'). \tag{30}$$

Remark 8.2. — *This theorem has been extended by Benameur and Roy to non special leafwise homotopy equivalences in* [**7**]. *The main ingredient is the use of an appropriate Hilbert bimodule associated with the equivalence.*

Sketch of the proof. — We follow the method of Keswani, see [**30**], [**31**] and [**29**]. We simply denote the relevant signature operators by $D' = (D'_{L'})_{L' \in V'/\mathcal{F}'}$, $\tilde{D}' = (\tilde{D}'_\theta)_{\theta \in T}$, $\mathcal{D}'_m$ and $D = (D_L)_{L \in V/\mathcal{F}}$, $\tilde{D} = (\tilde{D}_\theta)_{\theta \in T}$, $\mathcal{D}_m$. We shall first assume that $T = T'$ and $\Gamma = \Gamma'$. Consider, with obvious notation, the trivial Γ-equivariant fibration $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ as well as the foliated space $(X, \mathcal{F}^\sqcup)$, with $X := V' \sqcup (-V)$ and $\mathcal{F}^\sqcup$ induced by $\mathcal{F}$ and $\mathcal{F}'$. The longitudinal Grassmann bundles on V' and $-V$ define a longitudinally smooth bundle H over the foliated space X. Let $\widehat{H}$ be the equivariant vector bundle on $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ obtained by pulling back the bundle H. All the constructions explained in the previous sections extend to $(\tilde{M}' \sqcup -\tilde{M}) \times T \to T$ and $\widehat{H}$ as well as to $(X, \mathcal{F}^\sqcup)$ and H. More precisely, we treat $(\tilde{M}' \sqcup -\tilde{M}) \times \{\theta\}$ as the leaf of the product foliation even if it is not connected and we consider the induced lamination $\mathcal{F}^\sqcup$. So the leaves are not connected for us. Clearly, we can define the

(4) The assumption on Γ and Γ' can be replaced by the weaker assumption that the isotropy groups are torsion-free, as can be checked in the proof.

C^*-algebra $\mathcal{B}_m^H$ as the completion of the convolution algebra of compactly supported continuous sections over the corresponding monodromy groupoid $G^{\sqcup}$, with respect to the direct sum of the regular representations in $L^2(\tilde{M}', \tilde{E}') \oplus L^2(\tilde{M}, \tilde{E})$. Note that $G^{\sqcup}$ can be identified with the space

$$G^{\sqcup} = [(\tilde{M}' \sqcup -\tilde{M}) \times (\tilde{M}' \sqcup -\tilde{M}) \times T]/\Gamma.$$

The reader should note that $\mathcal{B}_m^H$ is different from the C^*-algebra of the monodromy groupoid of the disjoint union of the two foliations $(V', \mathcal{F}')$ and $(V, \mathcal{F})$, and that $\mathcal{B}_m^H$ is Morita equivalent to the C^*-algebra $\mathcal{A}_m$. Indeed, we then have a well defined $\mathcal{A}_m$-Hilbert module $\mathcal{H}_m$ (this is nothing but $\mathcal{E}'_m \oplus \mathcal{E}_m$) as well as an isomorphism $\chi_m : \mathcal{B}_m^H \to \mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$ constructed in the same way as in the previous sections. Now, there are again representations

$$\pi^{\mathrm{reg}} = (\pi_\theta^{\mathrm{reg}})_{\theta \in T} : \mathcal{B}_m^H \to W_\nu^*(G^{\sqcup}; H), \quad \pi^{\mathrm{av}} = (\pi_\theta^{\mathrm{av}})_{\theta \in T} : \mathcal{B}_m^H \to W_\nu^*(X, \mathcal{F}^{\sqcup}; H).$$

Here, the von Neumann algebras $W_\nu^*(G^{\sqcup}; H)$ and $W_\nu^*(X, \mathcal{F}^{\sqcup}; H)$ are defined using ν-essentially bounded families over T as in the previous sections, except that the operators act on the direct sums of the Hilbert spaces. Said differently, we are again simply allowing disconnected leaves. Finally, the previous constructions of traces and determinants on foliations, work as well for these two von Neumann algebras. So, extending obviously the constructions of Section 7, using the composition operation of Hilbert modules, we can consider determinants

$$w_{\mathrm{reg}}^{\sqcup} : \Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)) \to \mathbb{C}, \quad w_{\mathrm{av}}^{\sqcup} : \Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)) \to \mathbb{C}.$$

Following the notation of Subsection 5.2, consider the path in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$

$$\mathcal{W}_\epsilon := \left(\psi_t(\mathcal{D}'_m) \oplus (\psi_t(\mathcal{D}_m))^{-1}\right)_{t=\epsilon}^{t=1/\epsilon}.$$

Consider $w_{\mathrm{reg}}^{\sqcup}(\mathcal{W}_\epsilon)$ and $w_{\mathrm{av}}^{\sqcup}(\mathcal{W}_\epsilon)$ (one can easily show that the determinants of these paths are indeed well defined, see Remark 7.3). The proof proceeds along the following steps:

- we connect $\psi_\epsilon(\mathcal{D}'_m) \oplus (\psi_\epsilon(\mathcal{D}_m))^{-1}$ to the identity using the small time path ST_ϵ. This step is based on the injectivity of the Baum-Connes map and on the homotopy invariance of the signature index class;
- we connect $\psi_{1/\epsilon}(\mathcal{D}'_m) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m))^{-1}$ to the identity via the large time path $LT_{1/\epsilon}$. This step is based on the surjectivity of the Baum-Connes map, on the foliated homotopy invariance of the space of leafwise harmonic forms and on the homotopy invariance of the signature index class;
- we obtain in this way a loop ℓ in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$, i.e. an element of $\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m))$;
- we prove that $w_{\mathrm{reg}}^{\sqcup}(LT_{1/\epsilon})$ and $w_{\mathrm{av}}^{\sqcup}(LT_{1/\epsilon})$ are well defined and that

$$w_{\mathrm{reg}}^{\sqcup}(LT_{1/\epsilon}) \to 0 \quad \text{and} \quad w_{\mathrm{av}}^{\sqcup}(LT_{1/\epsilon}) \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \tag{31}$$

– we prove that $w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon)$ and $w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)$ are well defined and

$$(w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon) - w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)) \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \tag{32}$$

Now consider the map $\Theta : \Omega(\mathcal{IK}_{\mathcal{C}_m}(\mathcal{H}_m)) \to K_0(\mathcal{C}_m)$. By the surjectivity of the Baum-Connes map one proves, using Θ, the following fundamental equality:

$$w^{\sqcup}_{\mathrm{reg}}(\ell) - w^{\sqcup}_{\mathrm{av}}(\ell) = 0 \tag{33}$$

which means that

$$(w^{\sqcup}_{\mathrm{reg}}(\mathcal{W}_\epsilon) - w^{\sqcup}_{\mathrm{av}}(\mathcal{W}_\epsilon)) + (w^{\sqcup}_{\mathrm{reg}}(LT_{1/\epsilon}) - w^{\sqcup}_{\mathrm{av}}(LT_{1/\epsilon}) + (w^{\sqcup}_{\mathrm{reg}}(ST_\epsilon) - w^{\sqcup}_{\mathrm{av}}(ST_\epsilon)) = 0.$$

Taking the limit as $\epsilon \downarrow 0$, using (31), (32) and recalling that

$$\lim_{\epsilon \downarrow 0}(w^{\sqcup}_{\mathrm{reg}}(\mathcal{W}_\epsilon) - w^{\sqcup}_{\mathrm{av}}(\mathcal{W}_\epsilon)) = \rho^{\nu'}(V', \mathcal{F}') - \rho^{\nu}(V, \mathcal{F})$$

we end the proof in the particular case $T = T'$ and $\Gamma = \Gamma'$. In the general case we know that, since we have assumed the special homotopy equivalence, T and T' are homeomorphic and that the two groups are isomorphic. Therefore, the above proof can be adapted easily. □

9. Proof of the homotopy invariance for special homotopy equivalences: details

We shall now provide more details for the proof of Theorem 8.1; most of our work in the previous sections will go into the proof. We shall work under the additional assumption that $T = T'$ and $\Gamma = \Gamma'$.

9.1. Consequences of surjectivity I: equality of determinants. — The following proposition will play a crucial role in our analysis. Recall that we have defined traces $\tau^{\nu}_{\mathrm{reg},*} : K_0(\mathcal{C}_m) \to \mathbb{C}$ and $\tau^{\nu}_{\mathrm{av},*} : K_0(\mathcal{C}_m) \to \mathbb{C}$; where in our notation $\mathcal{C}_m := C(T) \rtimes_m \Gamma$.

Proposition 9.1. — *Assume the Baum-Connes map*

$$\mu_{\rtimes} : K_0^{\mathrm{geo}}(T \rtimes \Gamma) \to K_0(C(T) \rtimes_m \Gamma)$$

surjective; then

$$\tau^{\nu}_{\mathrm{reg},*} = \tau^{\nu}_{\mathrm{av},*}.$$

Proof. — According to the definition of $K_0^{\mathrm{geo}}(T \rtimes \Gamma)$, we know that each K-theory class $\alpha \in K_0(C(T) \rtimes_m \Gamma)$ is, by the surjectivity of $\mu_{\rtimes}$, the index class associated to a Γ-equivariant family of Dirac-type operators on manifolds without boundary. Using formula (22) (which is a consequence of the analogue of Atiyah's index theorem on coverings and the Atiyah-Bott formula), we end the proof. □

Proposition 9.2. — *If the Baum-Connes map $\mu_{\rtimes} : K_0^{\mathrm{geo}}(T \rtimes \Gamma) \to K_0(C(T) \rtimes_m \Gamma)$ is surjective, then w^{ν}_{av} and w^{ν}_{reg} coincide on $\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m))$.*

Proof. — Recall that $w^{\nu}_{\mathrm{av}} : \Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m)) \to \mathbb{C}$ and $w^{\nu}_{\mathrm{reg}} : \Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{E}_m)) \to \mathbb{C}$ are defined by passing to the respective von Neumann algebras and then taking the de La Harpe–Skandalis (log-)determinant there (see Definition 7.2): in formulae

$$w^{\nu}_{\mathrm{av}} := w^{\nu} \circ \sigma^{\mathrm{av}}, \quad w^{\nu}_{\mathrm{reg}} := w^{\nu} \circ \sigma^{\mathrm{reg}}.$$

Using the commutative diagram of Proposition 7.1, as summarized in formula (24), and the equality of traces on K_0 given by Proposition 9.1, we immediately conclude the proof. □

Corollary 9.3. — *Let $V = \tilde{M} \times_{\Gamma} T$ and $V' = \tilde{M}' \times_{\Gamma} T$ be two homotopy equivalent foliated bundles as in the previous subsection, i.e. through a special homotopy equivalence. Let $\mathcal{H}_m = \mathcal{E}'_m \oplus \mathcal{E}_m$ be the $\mathcal{A}_m$-Hilbert module associated to the disjoint union of $\tilde{M}' \times T$ and $-(\tilde{M} \times T)$. Let ℓ be a loop in $\Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m))$.*
If the Baum-Connes map $\mu_{\rtimes}$ is surjective, then

$$w^{\sqcup}_{\mathrm{av}}(\ell) = w^{\sqcup}_{\mathrm{reg}}(\ell). \tag{34}$$

If we consider, in particular, the loop $\ell \in \Omega(\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m))$ defined in the sketch of the proof of Theorem 8.1, then we have justified formula (33).

9.2. Consequences of surjectivity II: the large time path. — Let $V = \tilde{M} \times_{\Gamma} T$ and $V' = \tilde{M}' \times_{\Gamma} T$ be two homotopy equivalent foliated bundles as in the previous subsection, i.e. through a special homotopy equivalence with $\Gamma = \Gamma'$ and $T = T'$. We consider the Cayley transforms of the regular operators $\mathcal{D}_m : \mathcal{E}_m \to \mathcal{E}_m$ and $\mathcal{D}'_m : \mathcal{E}'_m \to \mathcal{E}'_m$:

$$\mathcal{U} := (\mathcal{D}_m - i\mathrm{Id})(\mathcal{D}_m + i\mathrm{Id})^{-1}, \quad \mathcal{U}' := (\mathcal{D}'_m - i\mathrm{Id})(\mathcal{D}'_m + i\mathrm{Id})^{-1}.$$

Let $\tilde{f} : \tilde{M} \times T \to \tilde{M}' \times T$ be a fiberwise smooth equivariant map inducing the special homotopy equivalence between $(V, \mathcal{F})$ and $(V', \mathcal{F}')$; let g and $\tilde{g}$ be choices for the homotopy inverses of f and $\tilde{f}$, with $\tilde{g} : \tilde{M}' \times T \to \tilde{M} \times T$ inducing g. This notation should not cause any trouble even if the metrics are denoted by the same letters. Following [**30**] (Section 3) one can construct a path of unitaries in $\mathcal{H}_m = \mathcal{E}'_m \oplus \mathcal{E}_m$, $\mathcal{V}(t), t \in [0, 2]$, connecting $\mathcal{U}' \oplus \mathcal{U}^{-1} = \mathcal{V}(0)$ to the identity $\mathrm{Id}_{\mathcal{H}_m} = \mathcal{V}(2)$. The path $\mathcal{V}(t), t \in [0, 2]$ (which is denoted $\mathbf{W}(t)$ in [**30**]) is obtained by defining a perturbation $\sigma(t)$ of the grading operator defining the signature operator; the definition of $\sigma(t)$, which is due to Higson and Roe, employs the pull back operator defined by the homotopy equivalence $\tilde{g}$ (precomposed and composed respectively with an extension to $\mathcal{E}_m$ and $\mathcal{E}'_m$ of the smoothing operators $(\phi(\tilde{D}_\theta))_{\theta \in T}$, $(\phi(\tilde{D}'_\theta))_{\theta \in T}$, ϕ being a rapidly decreasing smooth function with compactly supported Fourier transform). We omit

the actual definition of $\mathcal{V}(t)$ since it is somewhat lengthy and refer instead to [**30**], pages 968-969.

Recall that our goal is to construct a path connecting $\psi_{1/\epsilon}(\mathcal{D}'_m) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m))^{-1}$, (where $\psi_\alpha(x) = -\exp(i\pi \frac{2}{\sqrt{\pi}} \int_0^{\alpha x} e^{-u^2} du)$, to the identity on $\mathcal{H}_m$.

To this end, notice that the Cayley transform of the operator $\mathcal{D}_m$ can be expressed as $-\exp(i\pi\chi(\mathcal{D}_m))$, with $\pi\chi(x) = 2\arctan(x)$.

Definition 9.4. — [**30**] *A chopping function is an odd continuous function $\mu : \mathbb{R} \to \mathbb{C}$ such that $|\mu(x)| \leq 1$ and $\lim_{x\to\pm\infty} \mu(x) = \pm 1$.*

Both $\chi(x) := \frac{2}{\pi}\arctan(x)$ and $\phi(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ are chopping functions. Two chopping functions μ_1 and μ_2 can be homotoped one to the other via the straight line homotopy $k_s = (1-s)\mu_1 + s\mu_2$. Thus $\mathcal{U}' \oplus \mathcal{U}^{-1}$, which is equal to $-\exp(i\pi\chi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\mathcal{D}_m))$, can be joined to

$$-\exp(i\pi\phi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\phi(\mathcal{D}_m)), \quad \phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

via the path $\mathcal{K}(s) := -\exp(i\pi k_s(\mathcal{D}'_m)) \oplus -\exp(-i\pi k_s(\mathcal{D}_m))$. We denote by LT the concatenation of $\mathcal{K}(s)$ and $\mathcal{V}(t)$. So LT is a path joining $-\exp(i\pi\phi(\mathcal{D}'_m)) \oplus -\exp(-i\pi\phi(\mathcal{D}_m))$, with $\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$, to the identity.

Definition 9.5. — *Let $\epsilon > 0$ be fixed. The large time path $LT_{1/\epsilon}$ is the path obtained from the above construction but with the operators $\mathcal{D}_m$ and $\mathcal{D}'_m$ replaced by $\frac{1}{\epsilon}\mathcal{D}_m$ and $\frac{1}{\epsilon}\mathcal{D}'_m$ respectively. The large time path connects*

$$\psi_{1/\epsilon}(\mathcal{D}'_m) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m))^{-1}, \quad \text{with} \quad \psi_{1/\epsilon}(x) = -\exp\left(i\pi(\frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-u^2} du)\right),$$

to the identity.

For later use, we notice that

$$\psi_{1/\epsilon} = -\exp(i\pi\phi_{1/\epsilon}), \quad \text{with} \quad \phi_{1/\epsilon}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-u^2} du. \tag{35}$$

For each fixed $\epsilon > 0$ $LT_{1/\epsilon}$ is a path in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$ (we recall that this is the group consisting of the operators $A \in \mathcal{B}_{\mathcal{A}_m}(\mathcal{H}_m)$ such that $A - \mathrm{Id} \in \mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$ and A is invertible). In order to show this property we first recall that at the end of Subsection 3.2, Sobolev modules $\mathcal{E}_m^{(\ell)}$ were introduced and the compactness of the inclusion $\mathcal{E}_m^{(\ell)} \hookrightarrow \mathcal{E}_m^{(k)}$, $\ell > k$ was stated. Observe then that if χ is any chopping function with the property that $\chi' \sim 1/x^2$ as $|x| \to \infty$, then, using the compactness of the inclusion of the Sobolev module $\mathcal{E}_m^{(1)}$ into $\mathcal{E}_m$, one proves easily that $-\exp(i\pi\chi(\mathcal{D}_m)) \in \mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$. Notice now that both $\frac{2}{\pi}\arctan(x)$ and $\frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ satisfy this condition; thus $LT_{1/\epsilon} \in \mathcal{IK}_{\mathcal{A}_m}$.

9.3. The determinants of the large time path. — Recall the isomorphism $\chi_m : \mathcal{B}_m^H \to K_{\mathcal{A}_m}(\mathcal{H}_m)$, and the representations

$$\pi^{\mathrm{reg}} : \mathcal{B}_m^H \to W_\nu^*(G^\sqcup; H)\,; \quad \pi^{\mathrm{av}} : \mathcal{B}_m^E \to W_\nu^*(X, \mathcal{F}^\sqcup; H)\,, \text{ with } X = [\tilde{M} \sqcup (-\tilde{M}')] \times_\Gamma T.$$

Proceeding as in Section 7, we can use χ_m^{-1} and π^{reg} in order to define a path $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ in $\mathcal{IK}(W_\nu^*(G^\sqcup; H))$. The end-points of this path are τ^ν trace class perturbations of the identity; thus, see Remark 7.3, the determinant $w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon}))$ is well defined and we can set

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) := w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon}))\,.$$

Similarly,

$$w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) := w^\nu_{\mathcal{F}^\sqcup}(\sigma^{\mathrm{av}}(LT_{1/\epsilon}))$$

is well defined (and we recall that $\mathcal{F}^\sqcup$ is the foliation induced on X by the foliations $\mathcal{F}$ and $\mathcal{F}'$ on V and V' respectively).

Proposition 9.6. — *As $\epsilon \downarrow 0$ we have*

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) \longrightarrow 0\,, \quad w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) \longrightarrow 0. \tag{36}$$

Proof. — Fix $\epsilon > 0$ and recall that $LT_{1/\epsilon}$ is the composition of two paths: the path $\mathcal{V}_{1/\epsilon}$, connecting

$$-\exp(i\pi\chi(\frac{1}{\epsilon}\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\frac{1}{\epsilon}\mathcal{D}_m)) \text{ (with } \chi(x) = \frac{2}{\pi}\arctan(x)) \quad \text{to} \quad \mathrm{Id}_{\mathcal{H}_m}\,,$$

and the straight line path $\mathcal{K}_{1/\epsilon}$, connecting $\psi_{1/\epsilon}(\mathcal{D}'_m)) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m)))^{-1}$ to

$$-\exp(i\pi\chi(\frac{1}{\epsilon}\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi(\frac{1}{\epsilon}\mathcal{D}_m)).$$

Consider $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ in $\mathcal{IK}(W_\nu^*(G^\sqcup; H))$; for the signature family $\tilde{P}$ associated to $\tilde{M}' \sqcup (-\tilde{M}) \times T \to T$ denote by $\tilde{\Pi} := (\tilde{\Pi}_\theta)_{\theta \in T}$ the element in $W_\nu^*(G^\sqcup; H)$ defined by the family of orthogonal projections onto the null space. Then, proceeding as in Keswani [**30**], one can show that $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$ converges strongly to the path

$$\tilde{\mathcal{V}}_\infty(t) = \begin{cases} -\tilde{\Pi} + \tilde{\Pi}^\perp\,, & t \in [-1, 3/2] \\ -e(t)\tilde{\Pi} + \tilde{\Pi}^\perp\,, & t \in [3/2, 2] \end{cases} \tag{37}$$

with

$$e(t) = -\begin{pmatrix} \exp(2\pi i t) & 0 \\ 0 & \exp(-2\pi i t) \end{pmatrix}.$$

More precisely: $\sigma^{\mathrm{reg}}(\mathcal{K}_{1/\epsilon})$ converges strongly to the constant path $\tilde{\Pi} + \tilde{\Pi}^\perp$, whereas $\sigma^{\mathrm{reg}}(\mathcal{V}_{1/\epsilon})$ (is homotopic, with fixed end-points, to a path that) converges strongly to $\tilde{\mathcal{V}}_\infty(t)$. Similarly, if we denote by $\Pi \in W_\nu^*(X, \mathcal{F}^\sqcup; H)$ the projection onto the null

space of the longitudinal signature operator on X, then $\sigma^{\mathrm{av}}(LT_{1/\epsilon})$ converges strongly to the path

$$\mathcal{V}_\infty(t) = \begin{cases} -\Pi + \Pi^\perp\,, & t \in [-1, 3/2] \\ -e(t)\Pi + \Pi^\perp\,, & t \in [3/2, 2]. \end{cases} \tag{38}$$

We can now end the proof[(5)]. Recall the function $\phi_{1/\epsilon}(x) := \frac{2}{\sqrt{\pi}} \int_0^{x/\epsilon} e^{-u^2} du$, see formula (35); consider the function $\alpha(x)$ equal to zero for $x = 0$, equal to 1 for $x > 0$ and equal to -1 for $x < 0$; let $\varpi_{1/\epsilon}(t) = (1-t)\phi_{1/\epsilon} + t\alpha$ be the straight-line path joining $\phi_{1/\epsilon}$ to α; consider the path

$$X_{1/\epsilon}(t) := -\exp(i\pi\varpi_{1/\epsilon}(t)(\tilde{D}')) \oplus -\exp(-i\pi\varpi_{1/\epsilon}(t)(\tilde{D}))\,.$$

We notice that as $\epsilon \to 0$, $\phi_{1/\epsilon}$ converges pointwise to α. Using once again the spectral theorem for unbounded operators this means that, in the strong topology,

$$\phi_{1/\epsilon}(\tilde{P}) \longrightarrow \alpha(\tilde{P}) \quad \text{as} \quad \epsilon \downarrow 0 \tag{39}$$

where we recall that $\tilde{P}$ denotes the signature family on $(\tilde{M}' \sqcup (-\tilde{M})) \times T \to T$. We go back to the path $X_{1/\epsilon}(t)$, which is a path in $W^*_\nu(G^\sqcup; H)$ joining $\sigma^{\mathrm{reg}}(\psi_{1/\epsilon}(\mathcal{D}'_m)) \oplus (\psi_{1/\epsilon}(\mathcal{D}_m)))^{-1})$, i.e. $\psi_{1/\epsilon}(\tilde{D}')) \oplus (\psi_{1/\epsilon}(\tilde{D})))^{-1}$, to the constant path $-\tilde{\Pi} + \tilde{\Pi}^\perp$. Consider the loop $\gamma_{1/\epsilon}$ in $W^*_\nu(G^\sqcup; H)$ obtained by the concatenation of $X_{1/\epsilon}(t)$, $\tilde{\mathcal{V}}_\infty(t)$ and the reverse of $\sigma^{\mathrm{reg}}(LT_{1/\epsilon})$. By the above results the loop $\gamma_{1/\epsilon}$ is strongly null homotopic, thus its determinant is equal to zero. Summarizing:

$$w^\nu(\sigma^{\mathrm{reg}}(LT_{1/\epsilon})) = w^\nu(\tilde{\mathcal{V}}_\infty) + w^\nu(X_{1/\epsilon})$$

which can be rewritten as

$$w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) = w^\nu(\tilde{\mathcal{V}}_\infty) + w^\nu(X_{1/\epsilon}).$$

Computing

$$\tilde{\mathcal{V}}_\infty(t)^{-1} \frac{d\tilde{\mathcal{V}}_\infty(t)}{dt} = \begin{cases} 0\,, & t \in [-1, 3/2] \\ (2\pi i) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \tilde{\Pi}\,, & t \in [3/2, 2] \end{cases}$$

and recalling that the von Neumann dimension of the null space of the signature operator is a foliated homotopy invariant, see [**26**], we deduce that $w^\nu(\tilde{\mathcal{V}}_\infty(t)) = 0$. Thus the first part of the proposition will follow from the following result:

$$w^\nu(X_{1/\epsilon}) \longrightarrow 0 \quad \text{as} \quad \epsilon \downarrow 0\,.$$

(5) Notice that the proof given by Keswani for coverings contains a few imprecisions; the argument given here corrects them.

However, this is clear from (39) and the normality of the trace, given that, by direct computation,

$$w^\nu(X_{1/\epsilon}) = \frac{1}{2\pi i}\tau^\nu\left(\phi_{1/\epsilon}(\tilde{P}) - \alpha(\tilde{P})\right).$$

Essentially the same argument, using the strong convergence of $\sigma^{\mathrm{av}}(LT_{1/\epsilon})$ to $\mathcal{V}_\infty$ (see (38)), shows that $w^\nu_{\mathrm{av}}(LT_{1/\epsilon}) \longrightarrow 0$. □

9.4. Consequences of injectivity: the small time path. — So far, we have connected the $t = 1/\epsilon$ endpoint of the path

$$\mathcal{W}_\epsilon := \left(\psi_t(\mathcal{D}'_m) \oplus (\psi_t(\mathcal{D}_m))^{-1}\right)_{t=\epsilon}^{t=1/\epsilon} \quad \text{in} \quad \mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$$

to the identity using the large time path $LT_{1/\epsilon}$. We also showed that

$$\lim_{\epsilon\to 0}(w^\nu_{\mathrm{reg}}(LT_{1/\epsilon}) - w^\nu_{\mathrm{av}}(LT_{1/\epsilon})) = 0\,.$$

We now wish to close up the concatenation of $\mathcal{W}_\epsilon$ and $LT_{1/\epsilon}$ to a loop based at the identity. This step will be achieved through the small time path ST_ϵ, a path in $\mathcal{IK}_{\mathcal{A}_m}(\mathcal{H}_m)$ connecting the $t = \epsilon$ end point of $\mathcal{W}_\epsilon$ to the identity. We shall want to ensure that

$$\lim_{\epsilon\to 0}(w^\nu_{\mathrm{reg}}(ST_\epsilon) - w^\nu_{\mathrm{av}}(ST_\epsilon)) = 0\,. \tag{40}$$

The existence of *a path* connecting $\left(\psi_\epsilon(\mathcal{D}'_m) \oplus (\psi_\epsilon(\mathcal{D}_m))^{-1}\right)$ to the identity is in fact not difficult and follows from the proof of the Hilsum-Skandalis theorem; what is more delicate is the construction of a path satisfying the crucial property (40). It is here that the injectivity of the Baum-Connes map is used, as we proceed now to explain in more details.

Let $V = \tilde{M} \times_\Gamma T$ and $V' = \tilde{M}' \times_\Gamma T$ be two homotopy equivalent foliated bundles as in the previous subsections, with $\tilde{M}$ and $\tilde{M}'$ orientable. We fix leafwise Γ-equivariant metrics on $\tilde{M} \times T$ and $\tilde{M}' \times T$. We denote by $\tilde{D} = (\tilde{D}_\theta)$, $D = (D_L)_{L\in V/\mathcal{F}}$ and $\mathcal{D}_m$ respectively the Γ-equivariant signature family, the longitudinal signature operator on $(V, \mathcal{F})$ and the $\mathcal{A}_m$-linear signature operator on the $\mathcal{A}_m$-Hilbert module $\mathcal{E}_m$. We fix similar notations for $V' = \tilde{M}' \times_\Gamma T$ and we let as usual $\mathcal{H}_m = \mathcal{E}'_m \oplus \mathcal{E}_m$. We denote only in the rest of this paragraph by Λ and Λ' the vertical Grassmann bundles on $\tilde{M} \times T$ and $\tilde{M}' \times T$ respectively. Consider the index classes $\mathrm{Ind}(\mathcal{D}_m)$, $\mathrm{Ind}(\mathcal{D}'_m)$, two elements in $K_1(\mathcal{A}_m)$. By the foliated homotopy invariance of the signature index class we know that $\mathrm{Ind}(\mathcal{D}_m) = \mathrm{Ind}(\mathcal{D}'_m)$. On the other hand, using the very definition of the Baum-Connes map $\mu_\rtimes$, we have $\mathrm{Ind}(\mathcal{D}_m) = \mu_\rtimes[\tilde{M}, \Lambda \to \tilde{M} \times T]$ and $\mathrm{Ind}(\mathcal{D}'_m) = \mu_\rtimes[\tilde{M}', \Lambda' \to \tilde{M}' \times T]$, so that, by the assumed injectivity of $\mu_\rtimes$ we infer that

$$[\tilde{M}, \Lambda \to \tilde{M} \times T] = [\tilde{M}', \Lambda' \to \tilde{M}' \times T] \quad \text{in} \quad K_1^{\mathrm{geo}}(T \rtimes \Gamma)\,. \tag{41}$$

This is the information we want to use. Before stating the main result of this subsection we give a convenient definition.

Definition 9.7. — *We shall say that a chopping function χ is controlled if*

- *the derivative of χ is a Schwartz function;*
- *the Fourier transform of χ is supported in $[-1,1]$* [(6)]*;*
- *the functions χ^2-1 and $\chi(\chi^2-1)$ are Schwartz and their Fourier transforms are supported in $[-1,1]$.*

For the existence of such a function, see [**40**].

Theorem 9.8. — *If $[\tilde{M},\Lambda\to\tilde{M}\times T]=[\tilde{M}',\Lambda'\to\tilde{M}'\times T]$ in $K_1^{\mathrm{geo}}(T\rtimes\Gamma)$ then there exist a Γ-proper manifold Y, a longitudinally smooth Γ-equivariant vector bundle $\widehat{L}\to Y\times T$ and a continuous s-path of Γ-equivariant families on Y*

$$\tilde{B}_s:=(\tilde{B}_{s,\theta})_{\theta\in T}\quad s\in(0,1)$$

such that

1. *for each $s\in(0,1)$ and $\theta\in T$, $(\tilde{B}_{s,\theta})$ is a first order elliptic differential operator on Y acting on the sections of $\widehat{L}|_{\tilde{Y}\times\{\theta\}}$*
2. *the $\mathcal{A}_m$-Hilbert bundle $\mathcal{L}_m$ obtained by completing $C_c^\infty(Y\times T,\widehat{L})$ contains $\mathcal{E}'_m\oplus\mathcal{E}_m$ as an orthocomplemented submodule; thus there is an orthogonal decomposition $\mathcal{L}_m=(\mathcal{E}'_m\oplus\mathcal{E}_m)\oplus(\mathcal{E}'_m\oplus\mathcal{E}_m)^\perp$*
3. *for any controlled chopping function χ the path $-\exp(i\pi\chi(\mathcal{B}_s))$ is norm continuous in the space of bounded operators in $\mathcal{L}_m$ (here, for $s\in(0,1)$, $\mathcal{B}_s$ denotes the regular $\mathcal{A}_m$-linear operator defined by the family $(\tilde{B}_{s,\theta})_{\theta\in T}$);*
4. *we have, in norm topology,*

$$\begin{aligned}\lim_{s\to1}(-\exp(i\pi\chi(\mathcal{B}_s)))&=\mathrm{Id}_{\mathcal{L}_m}\\ \lim_{s\to0}(-\exp(i\pi\chi(\mathcal{B}_s)))&=\big(-\exp(i\pi\chi(\mathcal{D}'_m))\oplus-\exp(-i\pi\chi(\mathcal{D}_m)\big)\oplus\mathrm{Id}_\perp\end{aligned}$$

   with $\mathrm{Id}_\perp$ denoting the identity on $(\mathcal{E}'_m\oplus\mathcal{E}_m)^\perp$.
5. *$-\exp(i\pi\chi(\mathcal{B}_s))\in\mathcal{JK}_{\mathcal{A}_m}$.*

Proof. — If $[\tilde{M},\Lambda\to\tilde{M}\times T]=[\tilde{M}',\Lambda'\to\tilde{M}'\times T]$ in $K_1^{\mathrm{geo}}(T\rtimes\Gamma)$, then we know that we can pass from $(\tilde{M},\Lambda\to\tilde{M}\times T)$ to $(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a finite number of equivalences. The most delicate one is bordism, so we assume directly that we have a manifold X endowed with a proper action of Γ, a Γ-equivariant bundle $\widehat{H}$ on $X\times T$, a proper Γ-manifold with boundary Z' and an equivariant vector bundle $\widehat{F}'$ on $Z'\times T$

(6) Notice that it is impossible to have, as required in [**30**], that $\hat{\chi}$ is smooth and compactly supported (since, otherwise, χ itself , which is the Fourier transform of $\hat{\chi}$, would be rapidly decreasing and thus **not** a chopping function).

such that the boundary of Z' is equal to X and $\widehat{F}'$ restricted to $\partial Z' \times T$ is equal to $\widehat{H}$. Consider the manifold with cylindrical ends, Z, obtained by attaching to Z' a cylinder $[0,\infty) \times X$; consider the cylinder $W = X \times \mathbb{R}$; these are proper Γ manifolds if we extend the action to be trivial in the cylindrical direction; extend bundles to the cylindrical parts in the obvious way. The Γ manifold Y appearing in the statement of the theorem is the disjoint union of Z, $-Z$ and W, as in [**29**]. The bundle $\widehat{L}$ is given in terms of $\widehat{H}$ and its extension to the cylindrical parts. The equivariant families $\tilde{B}_s$, $s \in (0,1)$, appearing in the statement of the theorem are *explicitly defined* (in [**29**] see: the last displayed formula page 70; the last displayed formula page 72; the second displayed formula page 76 and the first displayed formula page 77). We shall see an example in a moment. The common feature of these operators is that they are Dirac-type on all of Y but look like an harmonic oscillator along the cylindrical ends. Since we have extended the action in a trivial way to the $\mathbb{R}$-direction of the cylindrical ends we can decompose the Hilbert module defined on the cylinder $(X \times \mathbb{R}) \times T$ as $\mathcal{E}_m(X) \otimes_{\mathbb{C}} L^2(\mathbb{R})$. Using the spectral decomposition of the harmonic oscillator we see, as in [**29**], that there is an orthogonal decomposition of Hilbert modules $\mathcal{E}_m(X \times \mathbb{R}) = (\mathcal{E}_m(X \times \mathbb{R}))' \oplus^{\perp} (\mathcal{E}_m(X \times \mathbb{R}))''$ with $(\mathcal{E}_m(X \times \mathbb{R}))'$ equal to the tensor product of $\mathcal{E}_m(X)$ with the 1-dimensional space generated by the kernel of the harmonic oscillator and $(\mathcal{E}_m(X \times \mathbb{R}))''$ equal to the tensor product of $\mathcal{E}_m(X)$ with the orthogonal space to this kernel in $L^2(\mathbb{R})$. In particular, $(\mathcal{E}_m(X \times \mathbb{R}))' \simeq \mathcal{E}_m(X)$, so that the Hilbert module $\mathcal{L}_m$ obtained by completing $C_c^{\infty,0}(Y \times T, \widehat{H})$ does contain $\mathcal{E}_m(X)$ as an orthocomplemented submodule. Regarding the statements involving the continuity and limiting properties of $-\exp(i\pi\mathcal{B}_s)$, we shall treat only the first of the four steps proving Theorem 5.1.10 in [**29**]. Thus Y is the cylinder $X \times \mathbb{R}$ and

$$\mathcal{B}_t = \begin{pmatrix} 0 & \mathcal{D}_X \\ \mathcal{D}_X & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix} \quad \text{with} \quad t \in (0,1].$$

The operator $\mathcal{B}_t$ restricted to $(\mathcal{E}_m(X \times \mathbb{R}))'$ is precisely $\begin{pmatrix} 0 & \mathcal{D}_X \\ \mathcal{D}_X & 0 \end{pmatrix}$. Let us consider $\mathcal{B}_t$ restricted to the orthocomplement $(\mathcal{E}_m(X \times \mathbb{R}))''$ and denote it $\mathcal{C}_t$, so that

$$\mathcal{B}_t = \begin{pmatrix} 0 & \mathcal{D}_X \\ \mathcal{D}_X & 0 \end{pmatrix} \oplus \mathcal{C}_t\,.$$

We can prove the norm-resolvent continuity of $\mathcal{C}_t$ (this notion extends to the C^*-algebraic framework) exactly as in [**29**]; we also obtain that $f(\mathcal{C}_t)$ goes to 0 in norm as $t \to 0$ for any rapidly decreasing function f. Using the fact that $\chi^2 - \mathrm{Id}$ is indeed rapidly decreasing we see that $\chi^2(\mathcal{C}_t) - \mathrm{Id}$ goes to zero in norm as $t \to 0$. A similar

statement holds for $\chi(\mathscr{C}_t)(\chi^2(\mathscr{C}_t) - \mathrm{Id})$. Then, writing as in [**30**]

$$-\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$$

with h and g entire, we prove that $-\exp(i\pi\chi(\mathscr{C}_t))$ converges in norm to the identity on $(\mathscr{E}_m(X\times\mathbb{R}))''$, so that $-\exp(i\pi\chi(\mathscr{B}_t))$ converges to (two copies of) $-\exp(i\pi\mathscr{D}_X)\oplus\mathrm{Id}_\perp$ as $t \to 0$. Of course, it is not true in this case that $-\exp(i\pi\chi(\mathscr{B}_t))$ converges to the identity as $t \to 1$ but the idea is that there will be further paths of operators in $\mathscr{IK}_{\mathscr{A}_m}$ with the property that their concatenation will produce the desired path, joining $-\exp(i\pi\mathscr{D}_X)\oplus\mathrm{Id}_\perp$ to the identity up to stabilization. For the bordism relation these paths are obtained by adapting to our context, as we have done above, the remaining three paths appearing in the treatment of the bordism relation in [**29**]; see in particular the Subsections 5.1.2, 5.1.3, 5.1.4 there.

Finally, let us comment about cycles that are equivalent through a bundle modification. We are thus considering, in general,

$$(X, E \to X\times T) \sim (X', E' \to X'\times T) \equiv (\widehat{X}, \widehat{E} \to \widehat{X}\times T)$$

where, as explained for example in [**29**], $\widehat{X}$ is a sphere bundle $S^{2n} \to \widehat{X} \xrightarrow{\pi} X$ and $\widehat{E}$ is the tensor product of $(\pi\times \mathrm{Id}_T)^*(E)$ and a certain bundle V built out of the Grassmann bundle of $\widehat{X}$; V is defined originally on $\widehat{X}$ and then extended trivially on all of $\widehat{X}\times T$. Consider the two T-families of Dirac-type operators defined by the equivariant Clifford modules $E \to X\times T$ and $E' \to X'\times T$ respectively and denote them briefly by $P = (P_\theta)_{\theta\in T}$ and $P' = (P'_\theta)_{\theta\in T}$ (for this argument we thus forget about the tilde). Let $\mathscr{E}_m$ and $\mathscr{E}'_m$ be the two Hilbert modules associated to these data and let $\mathscr{P}$ and $\mathscr{P}'$ be the regular operators defined by the two families above. Then we want to show that there exist

(i) an orthogonal decomposition of Hilbert modules $\mathscr{E}'_m = \mathscr{E}_m \oplus \mathscr{E}_m^\perp$;

(ii) a continuous s-path of Γ-equivariant first order differential operators $R_s := (R_{s,\theta})_{\theta\in T}$, $s\in[0,2)$, on

$\widehat{X}$ with $R_0 = P'$ and with regular extensions $\mathscr{R}_s$, $s\in[0,2)$;

(iii) for any controlled chopping function χ the path $-\exp(i\pi\chi(\mathscr{R}_s))$ is norm continuous in the space of

bounded operators in $\mathscr{E}'_m$;

(iv) $(-\exp(i\pi\chi(\mathscr{R}_s))) \longrightarrow (-\exp(i\pi\chi(\mathscr{P})))\oplus\mathrm{Id}_\perp$ as $s\to 2$.

The existence of the s-path $R_s := (R_{s,\theta})_{\theta\in T}$, $s\in[0,2)$, is proved following the arguments in [**29**], Subsection 5.2: thus we write $P' = P^0 + P^1 + Z^0$ where for each $\theta\in T$, P^1_θ is a vertical operator on the fiber bundle $S^{2n} \to \widehat{X} \xrightarrow{\pi} X$, P^0_θ is a horizontal operator defined in terms of P_θ and Z^0_θ is a 0-th order operator. Define R_s, for $s\in[0,1]$ as $R_s := P^0 + P^1 + (1-s)Z^0$ so that $R_0 = P'$ as required. Next observe, as in [**29**], that for each $\theta\in T$ the vertical operator P^1_θ has a one-dimensional

kernel, when restricted to each sphere of the sphere bundle $S^{2n} \to \widehat{X} \xrightarrow{\pi} X$; using the orthogonal projection onto the null space of these operators on spheres we obtain an orthogonal decomposition $\mathcal{E}'_m = \mathcal{U} \oplus \mathcal{U}^\perp$ with $\mathcal{U}$ isomorphic to $\mathcal{E}_m$. We can now define R_s for $s \in [1,2)$; consider R_1 and its extension to $\mathcal{E}'_m$ which is diagonal with respect to the orthogonal decomposition. The restriction of $\mathcal{R}_1$ to $\mathcal{U}$ is, by definition, $\mathcal{P}^0$, given that $\mathcal{P}^1$ is zero on $\mathcal{U}$; using the isomorphism between $\mathcal{U}$ and $\mathcal{E}_m$, $\mathcal{P}^0$ can be connected to $\mathcal{P}$, since they differ by the extension of a 0-th order operator Z_1 (it will suffice to consider $P^0 + (s-1)Z_1$, $s \in [1,2]$). For the restriction of $\mathcal{R}_1 = \mathcal{P}^1 + \mathcal{P}^0$ to $\mathcal{U}^\perp$ we consider instead the open path $\mathcal{P}^0 + \frac{1}{2-s}\mathcal{P}^1$, $s \in [1,2)$. Summarizing, we have defined a continuous s-path of regular operators $\mathcal{R}_s$, $s \in [0,2)$. Using the fact that $(\mathcal{P}^1)^2$ is strictly positive on $\mathcal{U}^\perp$ one can prove the stated continuity properties, as well as the crucial fact that $(-\exp(i\pi\chi(\mathcal{R}_s))) \longrightarrow (-\exp(i\pi\chi(\mathcal{P}))) \oplus \mathrm{Id}_\perp$ as $s \to 2$. Putting together the above two constructions, the one for the bordism relation and the one for the bundle modification relation, one can end the proof of the first four items in the statement of the theorem. We finally tackle the property that $-\exp(i\pi\chi(\mathcal{B}_s)) \in \mathcal{IK}_{\mathcal{G}_m}$. From the fact that χ is controlled, it suffices to show that $f(\mathcal{B}_s)$ is in $\mathcal{K}_{\mathcal{G}_m}$ if f is rapidly decreasing; let us see this property for the case of the cylinder considered above. With respect to the above decomposition,

$$f(\mathcal{B}_s) = f\left(\begin{pmatrix} 0 & \mathcal{D}_X \\ \mathcal{D}_X & 0 \end{pmatrix}\right) \oplus f(\mathcal{C}_t)\,.$$

and it suffices to see that $f(\mathcal{C}_t)$ is compact. Write $f(\mathcal{C}_t) = (f(\mathcal{C}_t)(\mathcal{C}_t^2)^N) \circ (\mathcal{C}_t^2)^{-N}$, where we recall that $\mathcal{C}_t^2$ is positive. Since f is rapidly decreasing the first operator is bounded; thus we are left with the task of proving that $(\mathcal{C}_t^2)^{-N}$ is compact. Recall that $\mathcal{C}_t^2$ is the restriction to $(\mathcal{E}_m(X \times \mathbb{R}))''$ of $(\mathcal{D}^2 \otimes \mathrm{Id}_{2\times 2} + t^{-2}X^2)$, with $X = \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix}$. Write $(\mathcal{C}_t^2)^{-N}$ in terms of the heat kernel, using the inverse Mellin transform:

$$(\mathcal{C}_t^2)^{-N} = \frac{1}{(N-1)!}\int_0^\infty \exp(-t\mathcal{C}_t^2)t^{N-1}dt\,.$$

Observe that the heat kernel of $(\mathcal{D}^2 \otimes \mathrm{Id}_{2\times 2} + t^{-2}X^2)$ decouples. Using again the invertibility of $\mathcal{C}_t^2$, the properties of the heat kernel of $\mathcal{D}^2$ and, more importantly, of the heat kernel of the harmonic oscillator, it is not difficult to end the proof. □

Let $\chi_\epsilon(x) := \chi(\epsilon x)$. Then, up to a harmless stabilization, the above theorem allows us to connect $\big(-\exp(i\pi\chi_\epsilon(\mathcal{D}'_m)) \oplus -\exp(-i\pi\chi_\epsilon(\mathcal{D}_m)\big)$ to the identity; we denote by $\gamma_1^\epsilon \in \mathcal{IK}_{\mathcal{G}_m}$, $\gamma_1^\epsilon \equiv (\gamma_1^\epsilon(s))_{s\in[0,1]}$ the resulting path. Recall, however, that our goal is rather to connect $\big(-\exp(i\pi\phi_\epsilon(\mathcal{D}'_m)) \oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_m)\big)$ to the identity, with $\phi_\epsilon(x) = \frac{2}{\sqrt{\pi}}\int_0^{\epsilon x} e^{-u^2}du$. Take the linear homotopy between the two chopping

functions χ and ϕ and set

$$\mathcal{M}(t) := t\left(\chi(\epsilon\mathcal{D}'_m) \oplus -\chi(\epsilon\mathcal{D}_m)\right) + (1-t)\left(\phi(\epsilon\mathcal{D}'_m) \oplus -\phi(\epsilon\mathcal{D}_m)\right).$$

Consider then the path

$$\gamma_2^\epsilon(t) = -\exp(i\pi\mathcal{M}(t)).$$

Definition 9.9. — *The small time path ST_ϵ is the path obtained by the concatenation of γ_1^ϵ and γ_2^ϵ.*

So ST_ϵ is a path in $\mathcal{IK}_{\mathcal{A}_m}$ and connects $\psi_\epsilon(\mathcal{D}'_m)\oplus(\psi_\epsilon(\mathcal{D}_m))^{-1} \equiv \big(-\exp(i\pi\phi_\epsilon(\mathcal{D}'_m))$ $\oplus -\exp(-i\pi\phi_\epsilon(\mathcal{D}_m))$ to the identity.

9.5. The determinants of the small time path. — Let $(X,\mathcal{F})$, $X = Z\times_\Gamma T$, be a foliated bundle as in the proof of Theorem 9.8. Let L be a continuous longitudinally smooth vector bundle on X as in Theorem 9.8 and let $\mathcal{L}_m$ be the associated Hilbert $\mathcal{A}_m$-module. Let $\mathcal{B}_m^L$ be the maximal C^*-algebra associated to the groupoid $G^Z := (Z\times Z\times T)/\Gamma$. Recall the isomorphism $\chi_m : \mathcal{B}_m^L \to K_{\mathcal{A}_m}(\mathcal{L}_m)$, and the representations

$$\pi^{\mathrm{reg}} : \mathcal{B}_m^L \to W_\nu^*(G^Z;L)\,; \quad \pi^{\mathrm{av}} : \mathcal{B}_m^L \to W_\nu^*(X,\mathcal{F};L).$$

Proceeding as in Section 7, we can use χ_m^{-1} and π^{reg} in order to define a path $\sigma^{\mathrm{reg}}(ST_\epsilon)$ in $\mathcal{IK}(W_\nu^*(G^Z;L))$. The end-points of this path are τ^ν trace class perturbations of the identity; thus, see Remark 7.3, the determinant $w^\nu(\sigma^{\mathrm{reg}}(ST_\epsilon))$ is well defined and we can set

$$w^\nu_{\mathrm{reg}}(ST_\epsilon) := w^\nu(\sigma^{\mathrm{reg}}(ST_\epsilon)).$$

Similarly,

$$w^\nu_{\mathrm{av}}(ST_\epsilon) := w^\nu_{\mathcal{F}}(\sigma^{\mathrm{av}}(ST_\epsilon))$$

is well defined. The goal of this subsection is to indicate a proof of the following

Theorem 9.10. — *As $\epsilon\downarrow 0$ we have*

$$w^\nu_{\mathrm{reg}}(ST_\epsilon) - w^\nu_{\mathrm{av}}(ST_\epsilon) \longrightarrow 0\,. \tag{42}$$

Proof. — To simplify the notation we shall assume that the injectivity radius of $(\tilde{M}_\theta,\tilde{g}_\theta)$ is greater or equal to 1 for each $\theta\in T$; we also assume that for each $\theta\in T$ the distance between $\tilde{m}$ and $\tilde{m}\,\gamma$ is greater than 1 for each $\tilde{m}\in\tilde{M}_\theta$ and for each $\gamma\in\Gamma(\theta)$, $\gamma\neq e$. We begin by a few preliminary remarks. Recall that ST_ϵ is the concatenation of two paths: γ_1^ϵ and γ_2^ϵ. Using the fundamental Proposition 3.12 we observe that

$$\sigma^{\mathrm{reg}}(\gamma_1^\epsilon(t)) \equiv \sigma^{\mathrm{reg}}(-\exp(i\pi\chi(\mathcal{B}_t)) = -\exp(i\pi\chi(\tilde{B}_t))$$

and

$$\sigma^{\mathrm{av}}(\gamma_1^\epsilon(t)) \equiv \sigma^{\mathrm{av}}(-\exp(i\pi\chi(\mathcal{B}_t)) = -\exp(i\pi\chi(B_t))$$

with $B_t = ((B_t)_L)_{L\in X/\mathcal{G}}$ the longitudinal differential operator induced by the Γ-equivariant family $\tilde{B}_t$. (Once again, here and before the statement of Theorem 9.10 we are using, is a slight extension of the results proved in Section 3, allowing for manifolds with cylindrical ends and operators that are modeled like harmonic oscillators along the ends). Similarly, up to a harmless stabilization by $\mathrm{Id}_\perp$ (that will in any case disappear after taking determinants), we can write

$$\sigma^{\mathrm{reg}}(\gamma_2^\epsilon(t)) = -\exp i\pi\left(t\chi(\epsilon\tilde{P}) + (1-t)\phi(\epsilon\tilde{P})\right),$$

$$\sigma^{\mathrm{av}}(\gamma_2^\epsilon(t)) = -\exp i\pi\left(t\chi(\epsilon P) + (1-t)\phi(\epsilon P)\right)$$

where $\tilde{P}$ and P are the signature operators on $(\tilde{M}'\sqcup(-\tilde{M}))\times T \to T$ and on $(X,\mathcal{G}^\sqcup)$ respectively (this is the notation we had introduced in the subsection on the large time path). One can prove that for $j=1,2$ the paths $\sigma^{\mathrm{reg}}(\gamma_j^\epsilon)$ and $\sigma^{\mathrm{av}}(\gamma_j^\epsilon)$ are all made of *trace class* perturbations of the identity. Moreover, the determinants of these two paths are well defined *individually* and without the regularizing procedure explained in Proposition 5.8. We shall justify this claim in a moment. This property granted, we can break the proof of (42) into two distinct statements:

$$w^\nu_{\mathrm{reg}}(\gamma_1^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_1^\epsilon) \longrightarrow 0\,. \tag{43}$$

$$w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_2^\epsilon) \longrightarrow 0\,. \tag{44}$$

We now tackle (44) which is slightly easier since it involves exclusively operators on manifolds without boundary.

First we observe that to each operator $\tilde{P}_\theta$ and P_L we can apply the results of [**54**], [**51**]. In particular, using the properties of χ, which is of controlled type, and ϕ we have:

1. $\chi(\tilde{P}_\theta)$ and $\chi(P_L)$, are given by 0-th order pseudodifferential operators with Schwartz kernel localized in an uniform R-neighbourhood of the diagonal (remember that the Fourier transform of χ is compactly supported); we shall assume without loss of generality that $R=1$;
2. $\phi(\tilde{P}_\theta)$ and $\phi(P_L)$ are each one the sum of a 0-th order pseudodifferential operators with Schwartz kernel localized in an uniform $R=1$-neighbourhood of the diagonal and of an integral operator with smooth kernel;
3. if $\tilde{\chi}$ denotes the linear chopping function equal to $sign(x)$ for $|x|>1$ and equal to x for $|x|\leq 1$ then $(\chi(\tilde{P}_\theta)-\tilde{\chi}(\tilde{P}_\theta))_{\theta\in T}$ and $(\phi(\tilde{P}_\theta)-\tilde{\chi}(\tilde{P}_\theta))_{\theta\in T}$ are τ^ν trace class elements given by longitudinally smooth kernels (indeed, the differences $\chi-\tilde{\chi}$ and $\phi-\tilde{\chi}$ are rapidly decreasing);
4. similarly, $(\chi(P_L)-\tilde{\chi}(P_L))_{L\in X/\mathcal{G}^\sqcup}$ and $(\phi(P_L)-\tilde{\chi}(P_L))_{L\in X/\mathcal{G}^\sqcup}$ are $\tau^\nu_{\mathcal{G}^\sqcup}$ trace class elements given by uniformly bounded longitudinally smooth kernels;

5. consequently, $(\chi(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T}$ and $(\chi(P_L) - \phi(P_L))_{L\in X/\mathcal{G}^\sqcup}$ are both trace class elements given by longitudinally smooth kernels; indeed it suffices to write $(\chi(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T} = ((\chi(\tilde{P}_\theta) - \tilde{\chi}(\tilde{P}_\theta))_{\theta\in T} + (\tilde{\chi}(\tilde{P}_\theta) - \phi(\tilde{P}_\theta))_{\theta\in T}$.

Notice that these properties imply easily the claim we have made about the determinants of $\sigma^{\mathrm{reg}}(\gamma_2^\epsilon)$ and $\sigma^{\mathrm{av}}(\gamma_2^\epsilon)$. We go back to our goal, i.e. proving (44). We observe that since γ_2^ϵ is defined in terms of a linear homotopy, we have, by direct computation,

$$w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) = -\frac{1}{2}\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) \quad w^\nu_{\mathrm{av}}(\gamma_2^\epsilon) = -\frac{1}{2}\tau^\nu_{\mathcal{G}^\sqcup}\left(\chi(\epsilon P) - \phi(\epsilon P)\right).$$

Write

$$\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) = \tau^\nu\left((\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - (\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)\right)$$

with $\phi(\epsilon\tilde{P})_\epsilon$ a compression of $\phi(\epsilon\tilde{P})$ to a Γ-equivariant ϵ-neighbourhood of $\{(\tilde{m},\tilde{m},\theta), \tilde{m}\in\tilde{M}, \theta\in T\}$ in $\tilde{M}\times\tilde{M}\times T$. Both $\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ and $\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ are individually τ^ν trace class: indeed the first term is the ϵ-compression of a longitudinally smooth kernel (since $\chi(\epsilon\tilde{P})$ is already ϵ-local) and it is therefore τ^ν trace class; the second term can be written as the sum $(\phi(\epsilon\tilde{P}) - \chi(\epsilon\tilde{P})) + (\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon)$ and both terms are trace class; thus

$$\tau^\nu\left(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})\right) = \tau^\nu\left((\chi(\epsilon\tilde{P}) - (\phi(\epsilon\tilde{P})_\epsilon\right) - \tau^\nu\left(\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon\right).$$

A similar expression can be written for $\tau^\nu_{\mathcal{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P))$. Consider now the difference $w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_2^\epsilon)$ which is the sum

$$\left(\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathcal{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P)_\epsilon)\right) + \tag{45}$$

$$\left(\tau^\nu(\phi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathcal{G}^\sqcup}(\phi(\epsilon P) - \phi(\epsilon P)_\epsilon)\right). \tag{46}$$

As already remarked the two differences $\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon$ and $\chi(\epsilon P) - (\phi(\epsilon P))_\epsilon$ are given by longitudinally smooth kernel which are supported in an ϵ-neighbourhood of the diagonal. Proceeding as in the proof of Proposition 4.4 we shall now prove that $\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P})_\epsilon) - \tau^\nu_{\mathcal{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P)_\epsilon)$ is in fact equal to zero for ϵ small enough. Indeed, consider the Γ-equivariant family $\chi(\epsilon\tilde{P})$; we know that $\chi(\epsilon\tilde{P}) \in \Psi^0_c(G,E)$. Similarly, consider $\phi(\epsilon\tilde{P})_\epsilon \in \Psi^0_c(G,E)$. We know that $\chi(\epsilon\tilde{P}) - (\phi(\epsilon\tilde{P}))_\epsilon \in \Psi_c^{-\infty}(G,E)$ and that this operator extends to an element $\mathscr{P}^\epsilon_{\chi,\phi} \in \mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m)$. Observe now that

$$(\mathscr{P}^\epsilon_{\chi,\phi})\otimes_{\pi^{\mathrm{reg}}_\theta}\mathrm{Id} = \chi(\epsilon\tilde{P}_\theta) - \phi(\epsilon\tilde{P}_\theta)_\epsilon, \quad (\mathscr{P}^\epsilon_{\chi,\phi})\otimes_{\pi^{\mathrm{av}}_\theta}\mathrm{Id} = \chi(\epsilon P_L) - \phi(\epsilon P_L)_\epsilon \text{ with } L = L_\theta.$$

Using Theorem 3.19 we thus can write

$$\tau^\nu(\chi(\epsilon\tilde{P}) - \phi(\epsilon\tilde{P}))_\epsilon) - \tau^\nu_{\mathcal{G}^\sqcup}(\chi(\epsilon P) - \phi(\epsilon P))_\epsilon) = \tau^\nu_{\mathrm{reg}}(\mathscr{P}^\epsilon_{\chi,\phi}) - \tau^\nu_{\mathrm{av}}(\mathscr{P}^\epsilon_{\chi,\phi})$$

where we have omitted the isomorphism $\chi_m^{-1} : \mathcal{K}_{\mathcal{A}_m}(\mathcal{H}_m) \to \mathcal{B}^H_m$. Taking ϵ small enough and proceeding precisely as in the proof of Proposition 4.4 we see that the right hand side is equal to zero for ϵ small enough (it is in this last step that we use the fact that $\mathscr{P}^\epsilon_{\chi,\phi}$ is given by an ϵ-localized smoothing kernel). Finally, the terms in the

second summand of (45) are individually zero since they are trace class elements given by longitudinally smooth kernels which restrict to zero on the diagonal. Summarizing: $w^\nu_{\mathrm{reg}}(\gamma_2^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_2^\epsilon) = 0$ for ϵ small enough.

We are left with the task of proving that γ_1^ϵ has well defined determinants and that

$$\lim_{\epsilon\to 0} w^\nu_{\mathrm{reg}}(\gamma_1^\epsilon) - w^\nu_{\mathrm{av}}(\gamma_1^\epsilon) = 0\,. \tag{47}$$

To this end we begin by writing explicitly the left hand side:

$$w^\nu_{\mathrm{reg}}(\gamma_1^\epsilon) = \frac{1}{2\pi i}\int_0^1 \tau^\nu\left((-\exp(-i\pi\chi(\epsilon\tilde{B}_t)))\frac{d}{dt}(-\exp(i\pi\chi(\epsilon\tilde{B}_t)))\right)dt \tag{48}$$

$$w^\nu_{\mathrm{av}}(\gamma_1^\epsilon) = \frac{1}{2\pi i}\int_0^1 \tau^\nu_{\mathcal{G}}\left((-\exp(-i\pi\chi(\epsilon B_t)))\frac{d}{dt}(-\exp(i\pi\chi(\epsilon B_t)))\right)dt \tag{49}$$

provided the right hand sides make sense. To see why the last statement is true, we begin by making a general comment on the traces we are using. Remember that the two paths of operators $\tilde{B}_s$ and B_s, $s\in(0,1)$, are defined on foliated bundles that might have as leaves manifolds with cylindrical ends. We define the two relevant von Neumann algebras in the obvious way and we define the two traces τ^ν and $\tau^\nu_{\mathcal{G}}$ as we did in Subsection 2.4. Needless to say, an arbitrary smoothing operator will not be trace class on such a foliation, since its Schwartz kernel might not be integrable in the cylindrical direction. (This is the typical situation for the heat kernel associated to a Dirac operator which restrict to a $\mathbb{R}^+$-invariant operator $\frac{d}{dt}+D_\partial$ along the cylindrical ends.) We now write

$$\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$$

with h and g entire. Recall that χ is of controlled type; we shall now see that this implies that $1-\chi^2(\tilde{B}_t)$ is τ^ν trace class and $1-\chi^2(B_t)$ is $\tau^\nu_{\mathcal{G}}$ trace class. Moreover these operators are given by longitudinally smooth kernels that are supported within a uniform ($R=1$)-neighbourhood of the diagonal. These statements are clear when $(\tilde{M},\Lambda\to\tilde{M}\times T)\sim(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a bundle modification or a direct sum of vector bundles (indeed, from our discussion of the bundle modification relation in the proof of Theorem 9.8, it is clear that in this case we remain within the category of foliations of compact manifolds without boundary and it suffices to apply [**50**] for the latter property and [**22**] for the first). If $(\tilde{M},\Lambda\to\tilde{M}\times T)\sim(\tilde{M}',\Lambda'\to\tilde{M}'\times T)$ through a bordism, then we use the fact that $\tilde{B}_{\theta,t}$ and $(B_t)_L$ are again of bounded propagation speed and restrict to harmonic oscillators along the cylinders of the relevant manifolds with cylindrical ends (this is needed in order to make claims about the trace class property). For the trace class property we also make use of the results in [**22**], proceeding as in [**29**] but using singular numbers instead of eigenvalues.

Using $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$ we can then conclude, as in [**30**] Lemma 4.1.7, that

$$\sigma^{\mathrm{reg}}(\gamma_1^\epsilon(t)) \equiv -\exp(i\pi\chi(\epsilon\tilde{B}_t)) \quad \text{and} \quad \sigma^{\mathrm{av}}(\gamma_1^\epsilon(t)) \equiv -\exp(i\pi\chi(\epsilon B_t))\,, \quad t \in [0,1]$$

are piecewise continuosly differentiable in the L^1 norm and that they both have a well defined (log-)determinant, as we had claimed (notice that in the proof of Lemma 4.1.7 in [**30**] only the controlled property of χ is used).

Having justified (48) and (49), we next make the following:

Claim. — *There exists polynomials p_1, p_2 such that, uniformly in $s \in [0,1]$,*

$$||\chi(\tilde{B}_s) - \chi(\epsilon\tilde{B}_s)||_1 < p_1(\frac{1}{\epsilon})\,, \quad ||\chi(B_s) - \chi(\epsilon B_s)||_1 < p_2(\frac{1}{\epsilon}). \tag{50}$$

Assume the claim; then using the inequality

$$||AB||_1 \leq ||A||_1 ||B||_\infty\,, \quad A \in L^1(\mathcal{M},\tau) \cap \mathcal{M}\,, \quad B \in \mathcal{M}$$

which is valid in any Von Neumann algebra $\mathcal{M}$ endowed with a faithful normal trace τ, one can show, proceeding exactly as in Lemma 4.2.8 of [**30**], that there exist polynomials q_1 and q_2 such that, uniformly in $s \in [0,1]$,

$$||\chi^2(\epsilon\tilde{B}_s) - \mathrm{Id}||_1 < q_1(\frac{1}{\epsilon})\,, \quad ||\chi^2(\epsilon B_s) - \mathrm{Id}||_1 < q_2(\frac{1}{\epsilon}). \tag{51}$$

We first end the proof of (47) using (51).
For any entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ we define $[f(z)]_N := \sum_{n=0}^N a_n z^n$. Consider the entire function h in the decomposition $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$. Proceeding as in Lemma 4.2.6 in [**30**] we show using the first inequality in (51) that for each $\alpha > 0$ there exists an $\epsilon > 0$ and an integer N_ϵ such that

- $||h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s))) - [h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s)))]_{N_\epsilon}||_1 < \alpha$
- $[h(\pi^2(\mathrm{Id} - \chi^2(\epsilon\tilde{B}_s)))]_{N_\epsilon}$ is of propagation less than 1.

Remark here that N_ϵ is in fact fixed by ϵ and, with our conventions, can be set to be equal to the integral part of $1/\epsilon$. Thus the left hand side of the above inequality can be thought of as a positive function of ϵ, converging to 0 when $\epsilon \downarrow 0$. A similar statement can be made for the derivative of $h(\epsilon\tilde{B}_s)$ with respect to s. Applying the same reasoning to the second summand in the decomposition $\exp(i\pi z) = h(\pi^2(1-z^2)) + (i\pi z)g(\pi^2(1-z^2))$ we conclude as in [**30**] Lemma 4.2.10, that for each $\alpha > 0$ there exists an $\epsilon > 0$ and an integer N_ϵ such that

$$\begin{aligned} &\Big| \int_0^1 \tau^\nu \Big((-\exp(-i\pi\chi(\epsilon\tilde{B}_t))) \frac{d}{dt} (-\exp(i\pi\chi(\epsilon\tilde{B}_t))) \Big)\, dt \\ &\quad - \int_0^1 \tau^\nu \Big(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \Big)\, dt \Big| < \alpha. \end{aligned} \tag{52}$$

Similarly, using the second inequality in the claim and the second inequality in (51), we can prove that for each $\alpha > 0$ there exists a $\delta > 0$ and an integer N_δ such that

$$\begin{aligned}(53)\qquad &\Big| \int_0^1 \tau_{\mathcal{G}}^{\nu} \Big((-\exp(-i\pi\chi(\delta B_t))) \frac{d}{dt} (-\exp(i\pi\chi(\delta B_t))) \Big)\, dt \\ &\quad - \int_0^1 \tau_{\mathcal{G}}^{\nu} \Big(([-\exp(-i\pi\chi(\delta B_t))]_{N_\delta}) \frac{d}{dt} ([-\exp(i\pi\chi(\delta B_t))]_{N_\delta}) \Big)\, dt \Big| < \alpha.\end{aligned}$$

Since the left hand sides of the inequalities (52), (53) can be thought of as positive functions of ϵ and δ converging to 0 as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, it is clear that we can ensure the existence of a common value, say η and N_η, for which both inequalities are satisfied. Consider again the difference $|w_{\mathrm{reg}}^{\nu}(\gamma_1^{\epsilon}) - w_{\mathrm{av}}^{\nu}(\gamma_1^{\epsilon})|$ that we rewrite as $|A_\epsilon + B_\epsilon + C_\epsilon|$ with

$$\begin{aligned} A_\epsilon &:= w_{\mathrm{reg}}^{\nu}\gamma_1^{\epsilon} - \int_0^1 \tau^{\nu} \Big(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \Big)\, dt \\ B_\epsilon &:= \int_0^1 \tau^{\nu} \Big(([-\exp(-i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon\tilde{B}_t))]_{N_\epsilon}) \Big)\, dt \\ &\quad - \int_0^1 \tau_{\mathcal{G}}^{\nu} \Big(([-\exp(-i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \Big)\, dt \\ C_\epsilon &:= \int_0^1 \tau_{\mathcal{G}}^{\nu} \Big(([-\exp(-i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \frac{d}{dt} ([-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}) \Big)\, dt - w_{\mathrm{av}}^{\nu}\gamma_1^{\epsilon}. \end{aligned}$$

We know that for each $\alpha > 0$ there exists a common ϵ such that $|A_\epsilon| < \alpha$ and $|C_\epsilon| < \alpha$. On the other hand, using the fact that $[-\exp(i\pi\chi(\epsilon B_t))]_{N_\epsilon}$ is of propagation equal to 1, we can prove, proceeding as in Proposition 4.4, that there exists ϵ such that $B_\epsilon = 0$. Thus we have proved (47) modulo the claim.

We shall prove the claim for the particular case of the cylinder; let us prove, for example, the first inequality. Consider

$$\tilde{B}_t = \begin{pmatrix} 0 & \tilde{D} \\ \tilde{D} & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix} \quad \text{with} \quad t \in (0, 1].$$

Observe that the left hand side of the first inequality in the claim is nothing but the last term in inequality (4.3) in [**29**]. Proceed now exactly as in the part of the proof of Lemma 4.7 in [**29**] that begins with the inequality (4.3). It is not difficult to realize that the proof given there, i.e. the proof of the first inequality in the claim, can be easily adapted to our von Neumann context using singular numbers and the results of Fack and Kosaki. More precisely, the operator $\tilde{B}_t^2$ can be diagonalized with respect to the eigenfunctions of the operator X^2, with

$$X = \begin{pmatrix} x & \partial_x \\ -\partial_x & -x \end{pmatrix}.$$

The functional calculus of $\tilde{B}_t^2$ is then reduced to the functional calculus of the operator

$$\tilde{D}' + \frac{1}{t}\lambda_k, \quad \text{with} \quad \tilde{D}' = \begin{pmatrix} \tilde{D}^2 & 0 \\ 0 & \tilde{D}^2 \end{pmatrix}$$

and where λ_k is an eigenvalue of X^2 as in [**29**] . Now the L^1-norm $\|\chi(\tilde{B}_s) - \chi(\epsilon\tilde{B}_s)\|_1$ is given by the sum over k of L^1-norms in corresponding von Neumann algebras of the operator $(\chi - \chi_\epsilon)(\tilde{D}' + \lambda_k)$. By [**22**], this L^1-norm is expressed in terms of the singular numbers $\mu_s^\nu(\tilde{D}' + \lambda_k) = \mu_s^\nu(\tilde{D}') + \lambda_k$. This reduces the estimate to the similar estimate of the singular numbers of $\tilde{D}'$ exactly as in [**29**]. This latter being a leafwise elliptic second order differential operator, we can use the estimate $\mu_s(\tilde{D}') \sim s^{2/p}$ where p is the dimension of the leaves, see for instance [**9**]. Hence the proof of the first inequality of the claim is completed following the steps of [**29**]. The proof of the second inequality in the claim is similar. Thus we have proved the claim and thus (47) in the case of cylinders. For manifolds with cylindrical ends we split the relevant statements into purely cylindrical ones and statements on compact foliated bundles, as in [**29**]. We end here our explanation of the proof of (47). The proof of Theorem 8.1 is now complete. □

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