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WITTEN LAPLACIAN ON A LATTICE SPIN SYSTEM

by

Ichiro Shigekawa

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — We consider an unbounded lattice spin system with a Gibbs measure. We introduce the Hodge-Kodaira operator acting on differential forms and give a sufficient condition for the positivity of the lowest eigenvalue.

***Résumé* (Laplacien de Witten sur un système de spin sur réseau).** — Nous considérons un réseau de spin muni d'une mesure de Gibbs. Nous introduisons l'opérateur de Hodge-Kodaira agissant sur les formes différentielles, et nous donnons une condition suffisante pour la positivité de la plus petite valeur propre.

1. Introduction

In this paper, we consider the spectral gap problem for a lattice spin system. Here, in our case, the single spin space is $\mathbb{R}$ and so it is non-compact. This is sometimes called an unbounded spin system.

We consider a model that each spin sits on the lattice $\mathbb{Z}^d$, and so the configuration space is $\mathbb{R}^{\mathbb{Z}^d}$. We suppose that a Gibbs measure is given in $\mathbb{R}^{\mathbb{Z}^d}$, which has the following formal expression:

$$\nu = Z^{-1} \exp\Big\{-2\mathscr{J} \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} (x^i - x^j)^2 - 2\sum_{i\in\mathbb{Z}^d} U(x^i)\Big\} \prod_{i\in\mathbb{Z}^d} dx^i. \tag{1.1}$$

Here U is a function of $\mathbb{R}$, called a self potential and $i \sim j$ means that $\|i-j\|_1 = \sum_k |i_k - j_k| = 1$. Under this measure we define the Hodge-Kodaira operator and discuss the positivity of the lowest eigenvalue of the operator. For unbounded spin

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systems, the Poincaré inequality, the logarithmic Sobolev inequality and other properties are well discussed, e.g., Zegarlinski [**11**], Yoshida [**10**], etc. In particular, Helffer [**5, 6, 7, 8**] dealt with this problem in connection to the Witten Laplacian. In fact, he proved the positivity of the lowest eigenvalue of the Hodge-Kodaira operator acting on 1-forms. From this point of view, we generalize his result to any p-forms ($p \geq 1$), i.e., we will prove that the lowest eigenvalue of the Hodge-Kodaira operator acting on p-forms is positive.

The organization of the paper is as follows. In Section 2, we discuss the Witten Laplacian on a finite dimensional space and in Section 3, we summarize differential forms, the Hodge-Kodaira operator and the Weitzenböck formula, which is crucial in the later argument. In Section 4, we give an estimate of spectral gap for 1-dimensional case. Last in Section 5, we prove the positivity of the lowest eigenvalue of the Hodge-Kodaira operator. We only consider the finite region case but we give a uniform estimate. In fact, it is independent of the choice of region and the boundary condition. So the result is valid for the infinite volume case as well.

2. Witten Laplacian in finite dimension

We give a quick review of the Witten Laplacian, which we need later. Details and related topics can be found in Hellfer [**8**], Albeverio-Daletskii-Kondratiev [**1**], Elworthy-Rosenberg [**4**], etc. Simon et al [**3**] is also a good reference for the supersymmetry.

Our interest is in the infinite dimensional case, but we start with the finite dimensional case. Suppose we are given a C^2 function Φ on $\mathbb{R}^N$ and define a measure ν by

$$\nu(dx) = Z^{-1} e^{-2\Phi} dx. \tag{2.1}$$

Here $Z = \int_{\mathbb{R}^N} e^{-2\Phi} dx$ so that ν is a probability measure. Define a Dirichlet form $\mathscr{E}$ by

$$\mathscr{E}(f, g) = \int_{\mathbb{R}^N} (\nabla f, \nabla g) e^{-2\Phi} dx, \tag{2.2}$$

where $\nabla = (\partial_1, \dots, \partial_N)$, $\partial_k = \frac{d}{dx_k}$. $(\nabla f, \nabla g)$ stands for the Euclidean inner product. We must specify the domain of $\mathscr{E}$. (2.2) is well-defined for f, $g \in C_0^\infty(\mathbb{R}^N)$. So at first, $\mathscr{E}$ is defined on $C_0^\infty(\mathbb{R}^N)$. Let us give an explicit form of the dual operator ∂_j^* of ∂_j in $L^2(\nu)$. To do this, note that

$$\int_{\mathbb{R}^N} \partial_j f g e^{-2\Phi} dx = -\int_{\mathbb{R}^N} f \partial_j (g e^{-2\Phi}) dx = -\int_{\mathbb{R}^N} f(\partial_j g - 2\partial_j \Phi g) e^{-2\Phi} dx,$$

which means

$$\partial_j^* = -\partial_j + 2\partial_j\Phi. \tag{2.3}$$

Here ∂_j^* is the dual operator of ∂_j in $L^2(\nu)$.

From this, we can see that the dual operator of ∇ has dense domain and so ∇ is closable. Moreover the generator $\mathfrak{A}$ is given by

$$\mathfrak{A}f = -\sum_j \partial_j^*\partial_j = \sum_j(\partial_j^2 f - 2\partial_j\Phi\partial_j f) = \triangle f - 2(\nabla\Phi, \nabla f). \tag{2.4}$$

This is valid for $f \in C_0^\infty(\mathbb{R}^N)$. We can show that $\mathfrak{A}$ is essentially self-adjoint and so, by taking closure, we may regard $\mathfrak{A}$ as self-adjoint operator. The domain of $\mathscr{E}$ is a set of all functions $f \in L^2(\nu)$ with $\nabla f \in L^2(\nu; \mathbb{R}^N)$.

We now define a Witten Laplacian. Let $I\colon L^2(dx) \longrightarrow L^2(\nu)$ be a unitary operator defined by

$$If(x) = e^\Phi f. \tag{2.5}$$

Let us obtain a operator X_j which satisfies the following commutative diagram:

$$\begin{tikzcd} L^2(dx) \arrow[r, "I"] \arrow[d, "X_j"'] & L^2(\nu) \arrow[d, "\partial_j"] \\ L^2(dx) \arrow[r, "I"] & L^2(\nu) \end{tikzcd} \tag{2.6}$$

It is not hard to see that

$$X_j = e^{-\Phi}\partial_j e^\Phi = \partial_j + \partial_j\Phi.$$

We denote the dual operator of X_j in $L^2(dx)$ by $\tilde{X}_j$. Here we use the following convention. $*$ stands for the dual operator in $L^2(\nu)$ and $\tilde{}$ stands for the dual operator in $L^2(dx)$, dx being the Lebesgue measure in $\mathbb{R}^N$. $\tilde{X}_j$ has the following form:

$$\tilde{X}_j = -\partial_j + \partial_j\Phi.$$

This is also equal to $e^{-\Phi}\partial_j^* e^\Phi$. The operator A associated with the generator $\mathfrak{A} = -\sum_j \partial_j^*\partial_j$ is computed by

$$\begin{aligned} A = e^{-\Phi}\mathfrak{A}e^\Phi &= -e^{-\Phi}(\sum_j \partial_j^*\partial_j)e^\Phi = -\sum_j \tilde{X}_j X_j \\ &= -\sum_j(-\partial_j + \partial_j\Phi)(\partial_j + \partial_j\Phi) = \sum_j(\partial_j^2 + \partial_j^2\Phi - (\partial_j\Phi)^2) \\ &= \triangle + \triangle\Phi - |\nabla\Phi|^2. \end{aligned}$$

Definition 2.1. — $A = \triangle + \triangle\Phi - |\nabla\Phi|^2$ in $L^2(dx)$ is called a Witten Laplacian.

$\mathfrak{A}$ and A are unitarily equivalent to each other but we distinguish them and call A as the Witten Laplacian, which is an operator in $L^2(dx)$.

The following commutation relation is easily checked.

Proposition 2.1. — In $L^2(\nu)$, we have

$$[\partial_i, \partial_j] = 0, \tag{2.7}$$

$$[\partial_i, \partial_j^*] = 2\partial_i\partial_j\Phi, \tag{2.8}$$

$$[\partial_j^*, \partial_k^*] = 0. \tag{2.9}$$

Further, in $L^2(dx)$, we have

$$[X_i, X_j] = 0, \tag{2.10}$$

$$[X_i, \tilde{X}_j] = 2\partial_i\partial_j\Phi, \tag{2.11}$$

$$[\tilde{X}_j, \tilde{X}_j] = 0. \tag{2.12}$$

3. Witten Laplacian acting on differential forms

In Section 2, we have introduced the Witten Laplacian. We now proceed to the Witten Laplacian acting on differential forms.

Let us quickly review the exterior algebra. In the sequel, we will deal with multilinear functionals on $\mathbb{R}^N$. Let t be a p-linear functional and s be a q-linear functional, e.g., t is a functional from $\underbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}_{p}$ into $\mathbb{R}$ which is linear in each coordinate. We define $p+q$-linear functional $t \otimes s$ by

$$t \otimes s(v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q}) = t(v_1, \dots, v_p)s(v_{p+1}, \dots, v_{p+q}). \tag{3.1}$$

$t \otimes s$ is called a tensor product. We also define the alternation mapping A_p by

$$A_p t(v_1, \dots, v_p) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (\operatorname{sgn} \sigma)\, t(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \tag{3.2}$$

for p-linear functional t. Here $\mathfrak{S}_p$ is the symmetric group of degree p and $\operatorname{sgn}\sigma$ stands for the signature. If p-linear functional θ satisfies $A_p\theta = \theta$, θ is called alternating. We denote the set of all alternating functionals of degree p by $\bigwedge^p(\mathbb{R}^N)^*$. For $\theta \in \bigwedge^p(\mathbb{R}^N)^*$ and $\eta \in \bigwedge^q(\mathbb{R}^N)^*$, we define their exterior product $\theta \wedge \eta$ by

$$\theta \wedge \eta = \frac{(p+q)!}{p!q!} A_{p+q}(\theta \otimes \eta). \tag{3.3}$$

Taking an orthonormal basis $\theta_1, \dots, \theta_N$ in $(\mathbb{R}^N)^*$, any element of $\bigwedge^p(\mathbb{R}^N)^*$ is represented as a unique linear combination of the following elements

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_p}. \tag{3.4}$$

We define an inner product in $\bigwedge^p(\mathbb{R}^N)^*$ so that all elements of the form (3.4) become an orthonormal basis in $\bigwedge^p(\mathbb{R}^N)^*$.

$A^p(\mathbb{R}^N) = \mathbb{R}^N \times \bigwedge^p(\mathbb{R}^N)^*$ has a structure of vector bundle and a section of $A^p(\mathbb{R}^N)$ is called a differential form of degree p. The set of all sections is denoted by $\Gamma(A^p(\mathbb{R}^N))$. Since the vector bundle $A^p(\mathbb{R}^N)$ is trivial, any section can be identified with a mapping from $\mathbb{R}^N$ into $\bigwedge^p(\mathbb{R}^N)^*$. In the sequel, we use this convention. $\Gamma^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms and $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ denotes the set of all smooth differential forms with compact support.

We introduce some operators in $\bigwedge^p(\mathbb{R}^N)^*$ as follows. For $\theta \in (\mathbb{R}^N)^*$, we define $\mathrm{ext}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p+1}(\mathbb{R}^N)^*$ by

$$\mathrm{ext}(\theta)\omega = \theta \wedge \omega \tag{3.5}$$

and for $v \in \mathbb{R}^N$, we define $\mathrm{int}(\theta)\colon \bigwedge^p(\mathbb{R}^N)^* \longrightarrow \bigwedge^{p-1}(\mathbb{R}^N)^*$ by

$$(\mathrm{int}(v)\omega)(v_1, \dots, v_{p-1}) = \omega(v, v_1, \dots, v_{p-1}). \tag{3.6}$$

Taking a standard basis $\{e_1, \dots, e_N\}$ of $\mathbb{R}^N$ and its dual basis $\{\theta^1, \dots, \theta^N\}$, we define operators a^i, $(a^i)^*$ by

$$a^i = \mathrm{int}(e_i), \tag{3.7}$$

$$(a^i)^* = \mathrm{ext}(\theta^i). \tag{3.8}$$

Here we regard a^i, $(a^i)^*$ as operators on an exterior algebra $\mathbb{R} \oplus (\mathbb{R}^N)^* \oplus \bigwedge^2(\mathbb{R}^N)^* \oplus \cdots \oplus \bigwedge^N(\mathbb{R}^N)^*$. They satisfy the following commutation relation:

$$[a^i, a^j]_+ = 0, \tag{3.9}$$

$$[a^i, (a^j)^*]_+ = \delta_{ij}, \tag{3.10}$$

$$[(a^i)^*, (a^j)^*]_+ = 0. \tag{3.11}$$

Here $[\ ,\ ]_+$ stands for an anti-commutator, i.e., $[a^i, a^j]_+ = a^i a^j + a^j a^i$.

For differential forms, the covariant differentiation ∇ can be defined. More generally, the covariant differentiation ∇ is defined for tensor fields as follows:

$$\nabla t = \sum_i \theta^i \otimes \partial_i t.$$

Here we remark that the operator is considered in $L^2(\nu)$, i.e., the reference measure is ν. The dual operator of ∇ in $L^2(\nu)$ is given by

$$\nabla^*(\sum_i \theta^i \otimes t_i) = \sum_i \partial_i^* t_i$$

and so we have

$$\nabla^*\nabla t = \sum_i \partial_i^* \partial_i t = -\sum_i (\partial_i^2 - 2\partial_i \Phi \partial_i) t.$$

For differential forms, we can define the exterior differentiation as follows. Let ω be a differential form of degree p. Then its exterior derivative is defined by $d\omega = (p+1)A_{p+1}\nabla\omega$ and it is written as

$$d = \sum_i \operatorname{ext}(\theta^i)\partial_i = \sum_i (a^i)^*\partial_i. \tag{3.12}$$

Hence, its dual operator is expressed as

$$d^* = \sum_i a^i \partial_i^*. \tag{3.13}$$

Using these operators, the Hodge-Kodaira Laplacian is defined as $-(dd^* + d^*d)$. The following formula is called the Weitzenböck formula:

Theorem 3.1. — We have the following identity.

$$dd^* + d^*d = \nabla^*\nabla + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^*a^j. \tag{3.14}$$

Proof. — By (3.12) and (3.13), we have

$$\begin{aligned}
dd^* + d^*d &= \sum_{i,j}\{(a^i)^*\partial_i\, a^j\partial_j^* + a^j\partial_j^*\,(a^i)^*\partial_i\} \\
&= \sum_{i,j}\{(a^i)^*a^j\,\partial_i\partial_j^* - (a^i)^*a^j\,\partial_j^*\partial_i + (a^i)^*a^j\,\partial_j^*\partial_i + a^j(a^i)^*\,\partial_j^*\partial_i\} \\
&= \sum_{i,j}\{(a^i)^*a^j[\partial_i,\partial_j^*] + [(a^i)^*, a^j]_+\partial_j^*\partial_j\} \\
&= \sum_{i,j}\{2(a^i)^*a^j\partial_i\partial_j\Phi + \delta_{ij}\partial_j^*\partial_i\} \\
&= \sum_{i,j} 2\partial_i\partial_j\Phi\,(a^i)^*a^j + \sum_i \partial_i^*\partial_i \\
&= \nabla^*\nabla + \sum_{i,j} 2\partial_i\partial_j\Phi\,(a^i)^*a^j.
\end{aligned}$$

This is what we wanted. □

So far, the reference measure has been ν. The isomorphism $I\colon L^2(dx) \longrightarrow L^2(\nu)$ can be extended to differential forms. Under the Lebesgue measure, the corresponding exterior differentiation and its dual operator are given by

$$D = e^{-\Phi}de^{\Phi}, \tag{3.15}$$

$$\tilde{D} = e^{-\Phi}d^*e^{\Phi}. \tag{3.16}$$

So the operator $\tilde{D}D + D\tilde{D}$ can be defined similarly and it has the following expression:

Theorem 3.2. — We have the following identities:

$$\tilde{D}D + D\tilde{D} = \sum_i \tilde{X}_i X_i + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^* a^j. \tag{3.17}$$

We call the operator $\tilde{D}D + D\tilde{D}$ in $L^2(dx)$ as the Witten Laplacian and distinguish from $dd^* + d^*d$, which is defined in $L^2(d\nu)$ and is called the Hodge-Kodaira Laplacian.

We remark that the operators a^j, $(a^i)^*$ are independent of the underlying measure and so we used the same notation.

Proof. — We easily have

$$\begin{aligned}\tilde{D}D + D\tilde{D} &= e^{-\Phi}(dd^* + d^*d)e^{\Phi}\\ &= e^{-\Phi}\{\nabla^*\nabla + 2\sum_{i,j}\partial_i\partial_j\Phi\,(a^i)^*a^j\}e^{\Phi}\\ &= \sum_i \tilde{X}_iX_i + 2\sum_{i,j}\partial_i\partial_j\Phi\,(a^i)^*a^j.\end{aligned}$$

This is the desired result. □

Due to this unitary equivalence, the following theorem is well-known (see, e.g., [**2**]).

Theorem 3.3. — The Hodge-Kodaira operator $\tilde{D}D + D\tilde{D}$ with a domain $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ is essentially self-adjoint in $L^2(dx; \bigwedge^p(\mathbb{R}^N)^*)$. Furthermore, $d^*d + d^*d$ with a domain $\Gamma_0^\infty(A^p(\mathbb{R}^N))$ is essentially self-adjoint in $L^2(\nu; \bigwedge^p(\mathbb{R}^N)^*)$.

4. Witten Laplacian in one-dimension

In this section, we give an estimate of the bottom of the spectrum in 1-dimensional case. In the sequel, the state space is $\mathbb{R}$ and the underlying measure is Lebesgue measure. We denote the Hamiltonian by ϕ instead of Φ to distinguish. We define an operator $X_\phi = \partial_t + \phi'$, $\partial_t = \frac{d}{dt}$. Using this operator, the Witten Laplacian can be written as

$$-\Box_0 = \tilde{X}_\phi X_\phi, \tag{4.1}$$

which acts on scalar functions, and

$$-\Box_1 = \tilde{X}_\phi X_\phi + 2\phi''(t) \tag{4.2}$$

which acts on 1-forms. Here we identify 1-forms with scalar functions. This is possible since the dimension of fiber space is 1-dimension. Our aim is to give an estimate of the lowest eigenvalue of $-\Box_1$, which we denote by $\lambda_1(\phi)$. From (4.2), we can see that $\lambda_1(\phi) \geq 2c$ if ϕ is convex and $\phi''(t) \geq c$. Noting that $\tilde{X}_\phi = -\partial_t + \partial_t\phi$, we have

$$\tilde{X}_\phi X_\phi - X_\phi\tilde{X}_\phi = (-\partial_t + \phi')(\partial_t + \phi') - (\partial_t + \phi')(-\partial_t + \phi')$$

$$\begin{aligned}&= -\partial_t^2 - \phi'' - \phi'\partial_t + \phi'(\partial_t + \phi') + \partial_t^2 - \phi'' - \phi'\partial_t - \phi'(-\partial_t + \phi')\\ &= -2\phi'',\end{aligned}$$

which means

$$-\Box_1 = X_\phi \tilde{X}_\phi. \tag{4.3}$$

As in Definition 2.1, we have

$$-\Box_0 = \tilde{X}_\phi X_\phi = -\frac{d^2}{dt^2} + \phi'(t)^2 - \phi''(t) \tag{4.4}$$

and further

$$-\Box_1 = X_\phi \tilde{X}_\phi = -\frac{d^2}{dt^2} + \phi'(t)^2 + \phi''(t). \tag{4.5}$$

We note that the pair of $\Box_0$ and $\Box_1$ has a supersymmetric structure. In fact, in the space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, we define

$$Q = \begin{pmatrix} 0 & \tilde{X}_\phi \\ X_\phi & 0 \end{pmatrix}. \tag{4.6}$$

Then Q is symmetric and satisfies

$$Q^2 = \begin{pmatrix} \tilde{X}_\phi X_\phi & 0 \\ 0 & X_\phi \tilde{X}_\phi \end{pmatrix} = \begin{pmatrix} -\Box_0 & 0 \\ 0 & -\Box_1 \end{pmatrix}.$$

By using the following well-known fact (see, e.g., [**3**, Theorem 6.3]), we can see that $-\Box_0$ and $-\Box_1$ have the same spectrum except for 0.

Proposition 4.1. — Let T be a closed operator in a Hilbert space H. Then T^*T and TT^* has the same spectrum except for 0.

By this special structure, we have that eigenvalues of $-\Box_0$ coincide with those of $-\Box_1$ excluding 0.

The next Lemma shows that a bounded perturbation preserves the positivity of the lowest eigenvalue.

Lemma 4.2. — Let χ be a bounded function. We denote by $\chi_{\sup}$, $\chi_{\inf}$ the infimum and the supremum of χ, respectively. Then we have

$$\lambda_1(\phi) \geq e^{-2(\chi_{\sup} - \chi_{\inf})} \lambda_1(\phi + \chi). \tag{4.7}$$

Proof. — Note that

$$\begin{aligned}e^{-\chi}(-\partial_t + \phi' + \chi')e^{\chi} &= -e^{-\chi}\partial_t e^{\chi} + \phi' + \chi'\\ &= -e^{-\chi}(e^{\chi}\chi' + e^{\chi}\partial_t) + \phi' + \chi' = -\partial_t + \phi' = \tilde{X}_\phi.\end{aligned}$$

Hence we have

$$
\begin{aligned}
(-\Box_1 u, u) &= (\tilde{X}_\phi u, \tilde{X}_\phi u) \\
&= ((-\partial_t + \phi')u, (-\partial_t + \phi')u) \\
&= (e^{-\chi}(-\partial_t + \phi' + \chi')e^{\chi}u, e^{-\chi}(-\partial_t + \phi' + \chi')e^{\chi}u) \\
&\geq e^{-2\chi_{\sup}}((-\partial_t + \phi' + \chi')e^{\chi}u, (-\partial_t + \phi' + \chi')e^{\chi}u) \\
&\geq e^{-2\chi_{\sup}}\lambda_1(\phi+\chi)\|e^{\chi}u\|_2^2 \\
&\geq e^{-2\chi_{\sup}}\lambda_1(\phi+\chi)e^{2\chi_{\inf}}\|u\|_2^2 \\
&= e^{-2(\chi_{\sup}-\chi_{\inf})}\lambda_1(\phi+\chi)\|u\|_2^2.
\end{aligned}
$$

This means (4.7). □

The equation (4.7) can be written as

$$\lambda_1(\phi+\chi) \geq e^{-2(\chi_{\sup}-\chi_{\inf})}\lambda_1(\phi).$$

This implies the following. If the function ϕ is a sum of a convex function and a bounded function, then the lowest eigenvalue of $-\Box_1$ is positive, and further the operator $-\Box_0$ has a spectral gap. To be precise, writing $\phi = V + W$ with $V'' \geq c$ and W being bounded, we have the following estimate:

$$\lambda_1(\phi) \geq 2ce^{-2(W_{\sup}-W_{\inf})}.$$

Lastly we give another type of estimate of the lowest eigenvalue for a double well potential of the form $at^4 - bt^2$. To do this, we recall the harmonic oscillator $-\mathfrak{A} = -\frac{d^2}{dt^2} + at^2$ on $L^2(\mathbb{R}, dx)$. It is well-known that the lowest eigenvalue of this operator is $\sqrt{a}$ with an eigenfunction $e^{-\sqrt{a}t^2/2}$. Using this, we have the following:

Proposition 4.3. — If $\phi(t) = at^4 - bt^2$, then we have

$$\lambda_1(\phi) \geq 2\sqrt{3a} - 2b. \tag{4.8}$$

Proof. — From (4.5),

$$
\begin{aligned}
(-\Box_1 u, u) &= ((-\frac{d^2}{dt^2} + \phi''(t) + \phi'(t)^2)u, u) \\
&\geq ((-\frac{d^2}{dt^2} + \phi''(t))u, u) \\
&= ((-\frac{d^2}{dt^2} + 12at^2 - 2b)u, u).
\end{aligned}
$$

Here the operator $-\frac{d^2}{dt^2} + 12at^2$ is a harmonic oscillator and hence the lowest eigenvalue is $2\sqrt{3a}$. This yields that

$$(-\Box_1 u, u) \geq ((2\sqrt{3a} - 2b)u, u),$$

which is the desired result. □

5. Positivity of the lowest eigenvalue for the Witten Laplacian

Lattice spin systems are characterized by Gibbs measures on $X = \mathbb{R}^{\mathbb{Z}^d}$. To define a Gibbs measure, we have to introduce a Hamiltonian. Suppose we are given an potential $U\colon \mathbb{R} \to \mathbb{R}$. Then the Hamiltonian is defined by

$$\Phi(x) = \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} \mathscr{J}(x^i - x^j)^2 + \sum_{i\in\mathbb{Z}^d} U(x^i). \tag{5.1}$$

Here $x = (x^i)_{i\in\mathbb{Z}^d}$ and $i \sim j$ means that $\|i-j\|_1 = |i_1 - j_1| + \cdots + |i_d - j_d| = 1$. $(x^i - x^j)^2$ stands for an interaction between particles. We only deal with this type of nearest neighbor interaction. We can generalize it to finite range interaction but we restrict ourselves to nearest neighbor interaction for the sake of simplicity. The expression of (5.1) involves an infinite sum and it is no more than a formal expression. The Gibbs measure is sometimes expressed as

$$\nu = Z^{-1} e^{-2\Phi(x)} dx. \tag{5.2}$$

But it does not make sense since $\Phi(x)$ diverges and the Lebesgue measure dx is nothing but a fictitious measure.

Precise characterization of Gibbs measures is given by the Dobrushin-Lanford-Ruelle equation. For a given finite region $\Lambda \subseteq \mathbb{Z}^d$ (we denote this fact by $\Lambda \Subset \mathbb{Z}^d$) and a boundary condition $\eta \in X$, we define a Hamiltonian on $\mathbb{R}^\Lambda$ by

$$\Phi_{\Lambda,\eta}(x) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathscr{J}(x^i - x^j)^2 + \sum_{i\in\Lambda} U(x^i) + 2 \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} \mathscr{J}(x^i - \eta^j)^2 \tag{5.3}$$

and introduce a measure on $\mathbb{R}^\Lambda$ by

$$\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda. \tag{5.4}$$

Here dx_Λ denotes the Lebesgue measure on $\mathbb{R}^\Lambda$. Let $\mathscr{F}_{\Lambda^c} = \sigma\{x^i;\, i \in \Lambda^c\}$. We also denote $x_\Lambda = (x^i;\, i \in \Lambda)$ and $x_{\Lambda^c} = (x^i;\, i \in \Lambda^c)$. Then ν is called a Gibbs measure if the conditional probability with respect to $\mathscr{F}_{\Lambda^c}$ is given as

$$E^\nu[\,\cdot\,|x_{\Lambda^c} = \eta_{\Lambda^c}] = \nu_{\Lambda,\eta}(dx_\Lambda) \otimes \delta_{\eta_{\Lambda^c}}(dx_{\Lambda^c}) \tag{5.5}$$

for any $\Lambda \Subset \mathbb{Z}^d$. Here $\delta_{\eta_{\Lambda^c}}$ is the Dirac measure at a point $\eta_{\Lambda^c} \in \mathbb{R}^{\Lambda^c}$, η_{Λ^c} being the restriction of η to Λ^c. The existence and the uniqueness of such measures is a subtle problem. In this paper, we only consider finite region measure and will give uniform estimates. Then our result holds for the infinite system if it exists. In fact, suppose that estimates are uniform. Take any differential form θ which depends on finite variables. We can find a finite region Λ which contains these variables. Then, by the identity (5.5), we have

$$E^\nu[(d\theta, d\theta) + (d^*\theta, d^*\theta)\ |x_{\Lambda^c} = \eta_{\Lambda^c}] \geq kE^\nu[(\theta,\theta)\ |x_{\Lambda^c} = \eta_{\Lambda^c}].$$

Then, by integrating with respect to η_{Λ^c}, we have

$$E^\nu[(d\theta, d\theta) + (d^*\theta, d^*\theta)] \geq kE^\nu[(\theta, \theta)].$$

Now we fix $\Lambda \Subset \mathbb{Z}^d$ and $\eta \in X$ and the Hamiltonian is given by (5.3). As was discussed in the previous section, the Hodge-Kodaira operator $dd^* + d^*d$ is well-defined on $\mathbb{R}^\Lambda$. Our aim is to show that the bottom of the spectrum $\sigma(dd^* + d^*d)$ is positive for p-forms ($p \geq 1$). Under the unitary operator $I\colon L^2(dx_\Lambda) \to L^2(\nu_{\Lambda,\eta})$, we consider the Witten Laplacian $D\tilde{D} + \tilde{D}D$ in $L^2(dx_\Lambda)$.

From now on, we fix $p \geq 1$. Indices $I, J, \ldots$ denote p distinct elements $i_1, i_2, \ldots, i_p$ of Λ. We denote $|I| = p$. When $I = \{i_1, \ldots, i_p\}$, we set $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. So any p-form θ can be written uniquely as $\theta = \sum_I \theta_I dx^I$. From the Weitzenböck formula, we have

$$\begin{aligned}
(\tilde{D}D + D\tilde{D})\theta &= \sum_i \tilde{X}_i X_i \sum_I \theta_I dx^I + 2\sum_{i,j} \partial_i\partial_j\Phi\,(a^i)^* a^j \sum_I \theta_I dx^I \\
&= \sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_i \partial_i^2\Phi\,(a^i)^* a^i dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i \neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I \\
&= \sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_{i\in I} \partial_i^2\Phi dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i \neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I.
\end{aligned}$$

Therefore

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta) \\
&= (\sum_I \sum_i \tilde{X}_i X_i \theta_I dx^I + 2\sum_I \theta_I \sum_{i\in I} \partial_i^2\Phi dx^I \\
&\quad + 2\sum_I \theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&= \sum_I (\sum_i \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I) \\
&\quad + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\,(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&= \sum_I (\sum_{i\in I} \tilde{X}_i X_i \theta_I, \theta_I) + \sum_I (\sum_{i\notin I} \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I) \\
&\quad + 2\sum_{I,J} (\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi(a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&\geq \sum_I (\sum_{i\in I} \tilde{X}_i X_i \theta_I, \theta_I) + 2\sum_I (\theta_I \sum_{i\in I} \partial_i^2\Phi, \theta_I)
\end{aligned}$$

$$+2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi\,(a^i)^*a^j dx^I,\sum_J\theta_J dx^J)$$
$$=\sum_I\sum_{i\in I}((\tilde{X}_iX_i+2\partial_i^2\Phi)\theta_I,\theta_I)+2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi\,(a^i)^*a^j dx^I,\sum_J\theta_J dx^J).$$

To get positivity of the left hand side, we estimate the right hand side term by term. We state our result as a theorem.

Theorem 5.1. — Suppose U is decomposed as $U=V+W$ so that $V''\geq c>0$ and W is bounded. $W_{\sup}$ and $W_{\inf}$ denote the supremum and the infimum of W, respectively. If $2(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}>16d\mathscr{J}$, then the lowest eigenvalue of $\tilde{D}D+D\tilde{D}$ for p-forms is greater than $\{2(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}-16d\mathscr{J}\}p$. Therefore there is no harmonic p-forms for $p\geq 1$.

Proof. — We first estimate the second term. To do this, we first compute $\partial_j\partial_i\Phi$. For $i\neq j$, we have

$$\partial_i\partial_j\Phi(x)=\partial_i\partial_j\Big\{\sum_{\substack{k,l\in\Lambda\\ k\sim l}}\mathscr{J}(x^k-x^l)^2+\sum_{k\in\Lambda}U(x^k)+2\sum_{\substack{k\in\Lambda,l\in\Lambda^c\\ k\sim l}}\mathscr{J}(x^k-\eta^l)^2\Big\}$$
$$=\begin{cases}-4\mathscr{J}, & i\sim j\\ 0, & \text{otherwise.}\end{cases}$$

For $i=j$, we have

$$\partial_i^2\Phi(x)=U''(x^i)+8\mathscr{J}d.$$

Hence

$$2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi(a^i)^*a^j dx^I,\sum_J\theta_J dx^J)$$
$$=-8\mathscr{J}\sum_{I,J}(\theta_I\sum_{i\sim j}(a^i)^*a^j dx^I,\theta_J dx^J)$$
$$=-8\mathscr{J}\sum_{I,J}(\sum_{i\sim j}(a^i)^*a^j dx^I,dx^J)(\theta_I,\theta_J)$$
$$=-8\mathscr{J}\sum_{I,J}c(I,J)(\theta_I,\theta_J).$$

Here we set $c(I,J)=(\sum_{i\sim j}(a^i)^*a^j dx^I,dx^J)$. $c(I,J)=1$ or -1 only when I and J differs by only one element and they are adjacent to each other. Otherwise $c(I,J)=0$. For each fixed I, there are utmost $2dp$ J's with $c(I,J)\neq 0$. Therefore we have

$$-8\mathscr{J}\sum_{I,J}c(I,J)(\theta_I,\theta_J)\geq -4\mathscr{J}\sum_{I,J}|c(I,J)|\{\|\theta_I\|_2^2+\|\theta_J\|_2^2\}$$
$$\geq -4\mathscr{J}\sum_I 2dp\|\theta_I\|_2^2-4\mathscr{J}\sum_J 2dp\|\theta_J\|_2^2$$

$$= -16dp\mathscr{J}\sum_I \|\theta_I\|_2^2.$$

Eventually the second term is estimated as follows:

$$2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi(a^i)^*a^j dx^I, \sum_J \theta_J dx^J) \geq -16dp\mathscr{J}\sum_I |\theta_I|^2.$$

We will show that the first term is greater than the second term if $\mathscr{J}$ is sufficiently small. To estimate the first term, we need to compute $((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I)$ and we regards it as a function of x^i for a moment. So other variables are fixed. We denote other variables by y^i, i.e.,

$$y^i = \{x^j\}_{j\in\Lambda\setminus\{i\}} \in \mathbb{R}^{\Lambda\setminus\{i\}}.$$

The variables $\{x^j\}_{j\in\Lambda}$ are decomposed into x^i and y^i. Then

$$\Phi(x) = \Phi(x^i, y^i) = U(x^i) + 4d\mathscr{J}(x^i)^2 - x^i\Big\{\sum_{\substack{j\in\Lambda\\ j\sim i}} 4x^j + \sum_{\substack{j\in\Lambda^c\\ j\sim i}} 4\eta^j\Big\} + \hat{\Phi}_i(y^i).$$

It is enough to consider the 1-dimensional Hamiltonian of the form

$$\phi(t) = U(t) + 4d\mathscr{J}t^2 - \alpha t.$$

In this case, let us estimate the lowest eigenvalue of a operator $\tilde{X_\phi}X_\phi + 2\phi''(t)$. Here $X_\phi = \partial_t + \phi'$, $\partial_t = \frac{d}{dt}$. But we have already considered the 1-dimensional case in the previous section and so we are ready to estimate $((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I)$. In fact, by Lemma 4.2, the lowest eigenvalue of $\tilde{X}_i X_i + 2\partial_i^2\Phi$ is greater than $(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}$.

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta)\\
&\geq \sum_I\sum_{i\in I}((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I) + 2\sum_{I,J}(\theta_I\sum_{i\neq j}\partial_i\partial_j\Phi\,(a^i)^*a^j dx^I, \sum_J\theta_J dx^J)\\
&\geq \sum_I\sum_{i\in I} 2(c + 8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}\|\theta_I\|_2^2 - 16dp\mathscr{J}\sum_I\|\theta_I\|_2^2\\
&= \sum_I 2p(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})}\|\theta_I\|_2^2 - 16dp\mathscr{J}\sum_I\|\theta_I\|_2^2\\
&= p\{2(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})} - 16d\mathscr{J}\}\|\theta\|_2^2.
\end{aligned}$$

This is what we wanted. □

When U is a double well potential, we can give another kind of estimate. This time, we use Proposition 4.3.

Theorem 5.2. — Assume that U is of the form $U(t) = at^4 - bt^2$. If $\sqrt{3a} - b - 4d\mathscr{J} > 0$, then the lowest eigenvalue of $\tilde{D}D + D\tilde{D}$ for p-forms is not smaller than $2(\sqrt{3a} - b - 4d\mathscr{J})p$. Therefore there is no harmonic p-forms ($p \geq 1$).

Proof. — A proof is almost same as the previous one. This time, one dimensional Hamiltonian is of the form

$$\phi(t) = at^4 - bt^2 + 4d\mathscr{J}t^2 - \alpha t.$$

By Proposition 4.3, the lowest eigenvalue of $\tilde{X}_i X_i + 2\phi''(t) + \phi'(t)^2$ is not smaller than $2\sqrt{3a} - 2b + 8d\mathscr{J}$. Hence we have

$$\begin{aligned}
&((\tilde{D}D + D\tilde{D})\theta, \theta) \\
&\quad \geq \sum_I \sum_{i\in I} ((\tilde{X}_i X_i + 2\partial_i^2\Phi)\theta_I, \theta_I) + 2\sum_{I,J}(\theta_I \sum_{i\neq j} \partial_i\partial_j\Phi\, (a^i)^* a^j dx^I, \sum_J \theta_J dx^J) \\
&\quad \geq \sum_I \sum_{i\in I} (2\sqrt{3a} - 2b - 8d\mathscr{J})\|\theta_I\|_2^2 - 16dp\mathscr{J}\sum_I \|\theta_I\|_2^2 \\
&\quad = 2(\sqrt{3a} - b - 4d\mathscr{J})p\|\theta\|_2^2.
\end{aligned}$$

This completes the proof. □

Lastly we will show that any differential form can be decomposed into three parts; exact, coexact and harmonic, which is usually called the Hodge-Kodaira decomposition. We have seen the positivity of the lowest eigenvalue, the decomposition follows easily. We state it as a theorem.

Theorem 5.3. — Under the assumption of Theorem 5.1 or Theorem 5.2, the following Hodge-Kodaira decomposition holds: For $p = 0$,

$$L^2(\nu) = \{\text{ constant functions }\} \oplus \mathrm{Ran}(d^*) \tag{5.6}$$

and for $p \geq 1$,

$$L^2(\nu; \textstyle\bigwedge^p(\mathbb{R}^\Lambda)^*) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(d^*). \tag{5.7}$$

Proof. — We only give a proof for $p \geq 1$. Set

$$T = (d, d^*)\colon L^2(\nu; \textstyle\bigwedge^p(\mathbb{R}^\Lambda)^*) \to L^2(\nu; \bigwedge^{p+1}(\mathbb{R}^\Lambda)^*) \oplus L^2(\nu; \bigwedge^{p-1}(\mathbb{R}^\Lambda)^*).$$

Here, by taking a closure, T is defined as a closed operator. Then we can have $-\Box_p = T^*T$. In fact, both operator coincides for smooth p-forms with compact support. So the identity follows from the essential self-adjointness of $\Box_p$. Thus we have

$$-\Box_p = dd^* + d^*d$$

on the domain of $\Box_p$. Since the lowest eigenvalue of $-\Box_p$ is positive, it has an inverse operator, which we denote by $-G$. Then for $\omega \in L^2(\nu; \bigwedge^p(\mathbb{R}^\Lambda)^*)$ we have

$$\omega = (dd^* + d^*d)G\omega = d(d^*G\omega) + d^*(dG\omega).$$

The orthogonality between $d(d^*G\omega)$ and $d^*(dG\omega)$ follows easily from the property $d^2 = 0$. This completes the proof. □

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