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THE SPACE OF GENERALIZED FORMAL POWER SERIES SOLUTION OF AN ORDINARY DIFFERENTIAL EQUATION

by

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A José Manuel Aroca, maestro y amigo

Abstract. — We prove that the set of truncations of generalized power series solutions of an ordinary differential equations is contained in a semi-algebraic set of dimension bounded by twice the order of the differential equation.

***Résumé* (L'espace des séries formelles généralisées qui sont solution d'une équation différentielle ordinaire)**
Nous montrons que l'ensemble des troncations de séries généralisées qui sont solutions d'une équation différentielle ordinaire est contenu dans un ensemble semi-algébrique dont la dimension est bornée par le double de l'ordre de l'équation différentielle.

1. Introduction

Consider a polynomial differential equation $F(\partial_0(y),\dots,\partial_n(y)) = 0$, where $F(y_0,\dots,y_n)$ is a polynomial in the variables $y_0,\dots,y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$ (polynomials with real exponents). We are interested in series solutions of $(F=0)$ of the form $\sum_{i=1}^{\infty} c_i\, x^{\mu_i}$, where $c_i \in \mathbb{C}$ and $\mu_i \in \mathbb{R}$ with $\mu_1 < \mu_2 \cdots$ (so called *generalized power series*). D.Y. Grigor'ev and M. Singer describe in [**5**] a parametric version of the Newton polygon process applied to F, which for each integer k, gives rise to a semi-algebraic subset $\mathrm{NIC}_k^{\star}(F) \subseteq \mathbb{R}^{3k}$ so that the space of truncations of length k of generalized power series solution of $(F=0)$ is included in $\mathrm{NIC}_k^{\star}(F)$. The main contribution of this paper is to prove that the dimension of this semi-algebraic set is bounded by $2n$. More precisely, its *adapted dimension* (see subsection 3.2) is bounded by n. The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced

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along the Newton polygon process in a parametric family of power series solution of a differential equation.

Briot and Bouquet [1] in 1856 use the Newton polygon for studying first order and first degree ordinary differential equations and Fine [4] in 1889 gives a description of the method for ordinary differential equation of arbitrary order. In section 2 we present a brief introduction to its classical version. In section 4 we introduce the notion of parametric Newton polygon: specifically, we define it and give some technical results about *parametric polynomials* which will be used in the proof of the main theorem.

In section 3 we state the main theorem and give a straightforward proof for the case $k = 1$. The general case is dealt with in section 5.

2. Newton polygon of an ODE

A *well-ordered* series with complex coefficients and real exponents is a series $\phi(x) = \sum_{\alpha\in S} c_\alpha\, x^\alpha$, where $c_\alpha \in \mathbb{C}$, and S is a well ordered subset of $\mathbb{R}$. If there exist a finitely generated semi-group Γ of $\mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}$, such that, $S \subseteq \gamma+\Gamma$, then we say that $\phi(x)$ is a *grid-based* series (this terminology comes from [6].) Let $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ be the sets of well-ordered series and of grid-based series, respectively. We denote $\mathbb{C}[x^{\mathbb{R}}]$ the subring of series in $\mathbb{C}((x))^g$ with finite support (*polynomials*, so to speak). It is well-know (see [7], for example), that both $\mathbb{C}((x))^w$ and $\mathbb{C}((x))^g$ are actually fields. Both are differential rings with the usual inner operations and the differential operator $\partial = x\frac{d}{dx}$:

$$\partial\left(\sum c_\alpha\, x^\alpha\right) = \sum \alpha\, c_\alpha\, x^\alpha.$$

Denote by ∂_0 the identity operator and for positive integer i, $\partial_i = \partial \circ \partial_{i-1}$.

Let $F(y_0, \dots, y_n)$ be a polynomial in the variables $y_0, \dots, y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$. The differential equation

$$F\big(\partial_0(y), \partial_1(y), \dots, \partial_n(y)\big) = 0$$

will be denoted by $F(y) = 0$. Notice that any polynomial ordinary differential equation can be rewritten in this form.

We are interested in solutions of $F(y) = 0$ in the field $\mathbb{C}((x))^w$. By virtue of [5, 2, 6], all of them are actually in $\mathbb{C}((x))^g$.

Write F in a uniquely, using the standard multiindex notation $y^\rho = y_0^{\rho_0} \cdots y_n^{\rho_n}$ (where $\rho = (\rho_0, \dots, \rho_n)$) as

$$F = \sum_{\alpha,\rho} A_{\alpha,\rho}\, x^\alpha\, y^\rho, \text{ with } A_{\alpha,\rho} \in \mathbb{C},$$

where α and ρ run over finite subsets of $\mathbb{R}$ and $\mathbb{N}^{n+1}$ respectively. The *cloud of points* of F is the set

$$\mathcal{P}(F) = \{(\alpha, |\rho|) : A_{\alpha,\rho} \neq 0\},$$

where $|\rho| = \rho_0 + \cdots + \rho_n$. The Newton polygon $\mathcal{N}(F)$ of F is the convex hull of

$$\bigcup_{P\in\mathcal{P}(F)} (P + \{(a,0) \mid a \geq 0\}) \ .$$

Notice that $\mathcal{N}(F)$ has a finite number of vertices, all of whose ordinates are non-negative integers.

Given a line $L \subseteq \mathbb{R}^2$ with slope $-1/\mu$, we say that μ is the *inclination* of L. Let $\mu \in \mathbb{R}$, we denote $L(F;\mu)$ the supporting line of $\mathcal{N}(F)$ with inclination μ (i.e. the only line L with inclination μ such that $\mathcal{N}(F)$ is contained in the right closed half-plane defined by L and $L \cap \mathcal{N}(F) \neq \varnothing$). More precisely, $L(F;\mu)$ is the set of points (a,b) in $\mathbb{R}^2$ such that $a + \mu b = \nu(F;\mu)$, where $\nu(F;\mu) = \min\{\alpha + \mu\,|\rho|\,; A_{\alpha,\rho} \neq 0\}$.

For any $\mu \in \mathbb{R}$, define the polynomial

$$\Phi_{(F;\mu)}(\mathfrak{c}) = \sum_{(\alpha,|\rho|)\in L(F;\mu)} A_{\alpha,\rho}\, \mu^{w(\rho)}\, \mathfrak{c}^{|\rho|} \in \mathbb{C}[\mathfrak{c}], \tag{1}$$

where $w(\rho) = \rho_1 + 2\rho_2 + \cdots + n\rho_n$. The *Newton polygon data* of F will be the set of vertices $v_0, \ldots, v_t$ (ordered with decreasing ordinate), the sides $[v_i, v_{i+1}]$, $0 \leq i < t$, the *indicial polynomials* associated to each vertex v:

$$\Psi_{(F;v)}(\mathfrak{m}) = \sum_{(\alpha,|\rho|)=v} A_{\alpha,\rho}\, \mathfrak{m}^{w(\rho)} \in \mathbb{C}[\mathfrak{m}]. \tag{2}$$

and the *characteristic polynomials* associated to each side $[v_i, v_{i+1}]$:

$$\Phi_{(F;[v_i,v_{i+1}])}(\mathfrak{c}) = \Phi_{(F;\mu_{[v_i,v_{i+1}]})}(\mathfrak{c}),$$

where $\mu_{[v_i,v_{i+1}]}$ is the inclination of side $[v_i, v_{i+1}]$.

2.1. Necessary Initial Conditions. — Given a well-ordered formal power series $y(x) = \sum_{\alpha\in S} c_\alpha\, x^\alpha$, its *order*, $\operatorname{ord}(y(x))$, is infinity if $y(x) = 0$ and $\min\{\alpha \in S \mid c_\alpha \neq 0\}$ otherwise.

Lemma 1. — *Let $y(x) = c\,x^\mu + \sum_{\alpha>\mu} c_\alpha\, x^\alpha \in \mathbb{C}((x))^w$ be a solution of the differential equation $F(y) = 0$. Then*

$$\Phi_{(F;\mu)}(c) = 0.$$

where c may be zero. In particular, if $y(x) = 0$ is a solution of $F(y) = 0$ then $\Phi_{(F;\mu)}(0) = 0$ for all μ.

Proof. — Developing F

$$F(c\,x^{\mu}+\cdots)=$$
$$\sum_{\alpha,\rho} A_{\alpha,\rho}\,x^{\alpha}\,(c\,x^{\mu}+\cdots)^{\rho_0}(\mu c\,x^{\mu}+\cdots)^{\rho_1}\cdots(\mu^n c\,x^{\mu}+\cdots)^{\rho_n}=$$
$$\sum_{\alpha,\rho}\left\{A_{\alpha,\rho}\,c^{|\rho|}\,\mu^{w(\rho)}x^{\alpha+\mu|\rho|}+\cdots\right\}=$$
$$\left\{\sum_{\alpha+\mu\,|\rho|=\nu(F;\mu)} A_{\alpha,\rho}\,c^{|\rho|}\,\mu^{w(\rho)}\right\}x^{\nu(F;\mu)}+\cdots,$$

where dots $\cdots$ stand for monomials of order greater than the exponent of x in the preceding term. The lemma follows from the fact that $\alpha+\mu\,|\rho|=\nu(F;\mu)$ if and only if $(\alpha,|\rho|)\in L(F;\mu)$. □

Notation 1. — *Let $\varphi\in\mathbb{C}((x))^g$ and $F(y_0,\dots,y_n)\in\mathbb{C}((x))^g[y_0,\dots,y_n]$, denote*

$$F(\varphi+y)=F(\varphi+y_0,\partial(\varphi)+y_1,\dots,\partial_n(\varphi)+y_n)\in\mathbb{C}((x))^g[y_0,\dots,y_n].$$

Definition 1. — *Given $F(y_0,\dots,y_n)$ and a positive integer k, define the set of* necessary k-initials conditions, $\mathrm{NIC}_k(F)$, *to be the subset of $(\mathbb{R}\times\mathbb{C})^k$ of the points $(\mu_1,c_1,\dots,\mu_k,c_k)\in(\mathbb{R}\times\mathbb{C})^k$ such that*

$$\mu_1<\cdots<\mu_k,\ \textit{and}$$
$$\Phi_{(F_1;\mu_1)}(c_1)=0,\dots,\Phi_{(F_k;\mu_k)}(c_k)=0,$$

where $F_1(y)=F(y)$ and $F_{i+1}(y)=F_i(c_i\,x^{\mu_i}+y)$, for $1\le i<k$.
Define the $\mathrm{NIC}^*_k(F)=\mathrm{NIC}_k(F)\cap(\mathbb{R}\times\mathbb{C}^*)^k$, where $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.

Corollary 1. — *If $y(x)=\sum_{i=1}^k c_ix^{\mu_i}+\sum_{\mu_k<\alpha}c_\alpha x^\alpha$ is a solution of $F(y)=0$ with $\mu_1<\cdots<\mu_k$, then*

$$(\mu_1,c_1,\dots,\mu_k,c_k)\in\mathrm{NIC}_k(F).$$

Corollary 2. — *Let $v_0,\dots,v_t$ be the vertices of $\mathcal{N}(F)$, ordered by decreasing ordinate. Let μ_i, $1\le i\le t$ be the inclination of the side $[v_{i-1},v_i]$. Set $\mu_0=-\infty$ and $\mu_{t+1}=+\infty$. The subset $\mathrm{NIC}_1(F)\subseteq(\mathbb{R}\times\mathbb{C})$ is semi-algebraic. Moreover, $\mathrm{NIC}^*_1(F)$ is the finite union of the semi-algebraic sets corresponding to the sides of the Newton polygon of F:*

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^*\,;\mu=\mu_i,\ \textit{and}\ \Phi_{(F;\mu_i)}(c)=0\},\quad 1\le i\le t,$$

and the semi-algebraic sets corresponding to the vertices:

$$\{(\mu,c)\in\mathbb{R}\times\mathbb{C}^*\,;\mu_i<\mu<\mu_{i+1},\ \textit{and}\ \Psi_{(F;v_i)}(\mu)=0\},\quad 0\le i\le t.$$

Proof. — Let $\mu\in\mathbb{R}$, $\mu_i<\mu<\mu_{i+1}$, for some $0\le i\le t$. As $L(F;\mu)\cap\mathcal{N}(F)=v_i$ and $\Phi_{(F;\mu)}(\mathfrak{c})=\mathfrak{c}^h\,\Psi_{(F;v_i)}(\mu)$, (where h is the ordinate of v_i) then, for $c\neq0$ and $\mu_i<\mu<\mu_{i+1}$, one has $\Phi_{(F;\mu)}(c)=0$ if and only if $\Psi_{(F;v_i)}(\mu)=0$, which and we are done. □

Let $\mu \in \mathbb{R}$ be a real number and fix a point $(a, h) \in \mathbb{R} \times \mathbb{N}$.

Definition 2. — *We say that (a, h) belongs to the* red part with respect to μ *of the Newton polygon of $F(y)$ if $h \geq 1$ and either (a, h) is the vertex of $\mathcal{N}(F)$ with minimum ordinate or it belongs to a side of $\mathcal{N}(F)$ with inclination greater than μ.*

Notice that if the red part with respect to μ of $\mathcal{N}(F)$ is empty, then there are no generalized power series solution of $(F = 0)$ of order greater than μ: the vertex (a, h) with minimum ordinate has $h = 0$ and all the sides of $\mathcal{N}(F)$ have inclination less than or equal to μ, hence for $\gamma > \mu$, the polynomial $\Phi_{(F;\mu)}(\mathfrak{c})$ is a non-zero constant and by Corollary 2 the set $\mathrm{NIC}_1^*(F)$ is empty. The reciprocal is not true as Example 1 (page 65) shows.

Lemma 2. — *Let $(\mu_1, c_1, \ldots, \mu_k, c_k) \in \mathrm{NIC}_k^*(F)$, $\varphi = \sum_{j=1}^{k} c_j\, x^{\mu_j}$ and $F_{k+1}(y) = F(\varphi + y)$. The red part of $\mathcal{N}(F_{k+1}(y))$ with respect to μ_k nonempty.*

Proof. — Let $(\mu, c) \in \mathrm{NIC}_1^*(F)$ and consider $G = F(c\, x^{\mu} + y)$. The red part of the Newton polygon of G with respect to μ is not empty. To see this, let $v_0, \ldots, v_t$ be the vertices of $\mathcal{N}(F)$ ordered by decreasing ordinate and let v_k be the vertex with highest ordinate in $L(F;\mu) \cap \mathcal{N}(F)$. The ordinate of this v_k is greater than zero because otherwise $\Phi_{(F;\mu)}(\mathfrak{c})$ would be a nonzero constant, in contradiction with the fact that $\Phi_{(F;\mu)}(c) = 0$.

Returning to the main argument, given a monomial $M = x^{\alpha} y_0^{\rho_0} \cdots y_n^{\rho_n}$, one may write

$$M(c\, x^{\mu} + y) = x^{\alpha} \prod_{i=0}^{n} (c\, \mu^{i}\, x^{\mu} + y_i)^{\rho_i} = M + R, \tag{3}$$

where the points corresponding to the monomials of R have ordinate less than $|\rho|$ and belong to the line with inclination μ passing through $(\alpha, |\rho|)$. If w is the intersection of $L(F;\mu)$ with the axis of abscissas, then the cloud of points $\mathcal{P}(G)$ of G is contained in the positive convex hull of $\{v_0, \ldots, v_k, w\}$. The coefficient of G corresponding to w is precisely $\Phi_{(F;\mu)}(c) = 0$, hence $w \notin \mathcal{P}(G)$. Moreover, $\{v_0, \ldots, v_k\} \subseteq \mathcal{P}(G)$, because of (3). Therefore $v_0, \ldots, v_k$ are vertices of $\mathcal{N}(G)$. Hence either v_k is the vertex of $\mathcal{N}(G)$ with minimum ordinate or there exists a side of $\mathcal{N}(G)$ with inclination greater than μ and we are done. □

***Example 1* (See Figure 1).** — *Let $F = x^{-1}\, y_0^6\, y_1 + y_0^2\, y_1 + x\, y_0^2 - 3\, x\, y_0\, y_1 - x^2\, y_0 + 2\, x^2\, y_1 + x^5$. The point $(1, 1) \in \mathrm{NIC}^*(F)$. Let $G = F(x + y)$. The red part of $\mathcal{N}(G)$ with respect to $\mu = 1$ is vertex v_2' and point p. In this example, there are no solutions of $(G = 0)$ of order greater than 1.*

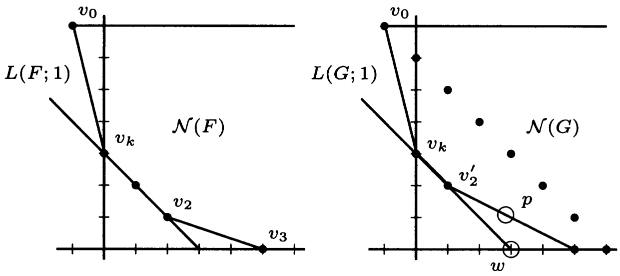

FIGURE 1. Newton polygons of F and G from Lemma 2 and Example 1

3. Main result

In this section we introduce the notions of *truncation* of well-ordered power series and of *adapted dimension*, and proceed to state the main result: the truncation of length k of the solutions of the differential equation $(F=0)$ is contained in a semi-algebraic subset of $(\mathbb{R}\times\mathbb{C})^k$ of adapted dimension less than or equal to the order of F.

The adapted dimension is a proper measure of the *number of free parameters* (real or complex, coefficient or exponent) which have been introduced along the Newton polygon process in a parametric family of power series solution of a differential equation. Heuristically, when one introduces an exponent as a free parameter in the solution space then one must also introduce a coefficient as a free parameter. The simplest non-trivial case is the equation $F(y)=y_1^2-y_0\,y_2=0$, whose solutions are $c\,x^{\mu}$ for any $\mu\in\mathbb{R}$ and $c\in\mathbb{C}$, (*adapted dimension* 2).

3.1. Truncations. — For any positive integer k and real β, the truncation of length k to the right of β is a map $\mathrm{Tr}_{k;\beta}:\mathbb{C}((x))^w\to(\mathbb{R}\times\mathbb{C})^k$ defined as follows. If $y(x)=0$, then

$$\mathrm{Tr}_{k;\beta}(y(x))=((\beta+1,0),(\beta+2,0),\dots,(\beta+k,0)),$$

otherwise, $y(x)=c\,x^{\mu}+\sum_{\alpha>\mu}c_\alpha\,x^\alpha=c\,x^\mu+\bar{y}(x)$, with $c\neq0$ and in this case

$$\mathrm{Tr}_{k;\beta}(y(x))=((\mu,c),\mathrm{Tr}_{k-1;\mu}(\bar{y}(x))).$$

Finally, the *truncation of length* k is $\mathrm{Tr}_k=\mathrm{Tr}_{k;0}$. For instance,

$$\mathrm{Tr}_4(x^{-0.5}+x^{\pi})=((-0.5,1),(\pi,1),(\pi+1,0),(\pi+2,0))\in(\mathbb{R}\times\mathbb{C})^4$$

Remark 1. — *Let $\mathcal{M}_s$ be the subset of $\mathbb{C}[x^{\mathbb{R}}]$ of the elements with exactly s monomials. Then $\mathrm{Tr}_k(\mathcal{M}_s)$ is a semi-algebraic subset of $(\mathbb{R}\times\mathbb{C})^k$.*

Remark 2. — *By corollary 1, if $y(x)$ is a solution of $(F=0)$, then $\mathrm{Tr}_k(y)\subseteq\mathrm{NIC}_k(F)$.*

3.2. Adapted dimension of cells of $(\mathbb{R}\times\mathbb{C})^k$. — In this section we use some known notions results of real algebraic geometry for which we refer to the reader to [**3**] (or any other standard text on the subject).

Given a finite family $P_1,\dots,P_r\in\mathbb{R}[X_1,\dots,X_t]$, we say that a subset C of $\mathbb{R}^t$ is $(P_1,\dots,P_r)$-*invariant* if every polynomial P_i has constant sign (>0, <0, or $=0$) on C. A $\mathcal{C}^\infty$-*cylindrical algebraic decomposition* of $\mathbb{R}^t$ adapted to $P_1,\dots,P_r$ is a cylindrical algebraic decomposition $\mathcal{C}$ all of whose cells are $(P_1,\dots,P_r)$-invariant $\mathcal{C}^\infty$-manifolds and such that the defining functions of the cells of $\mathcal{C}$ are $\mathcal{C}^\infty$.

Algorithms for constructing $\mathcal{C}^\infty$-cylindrical algebraic decomposition for a given family of polynomials are well-known (see for instance [**3**]).

Let $\mathcal{C}=\{\mathcal{C}_1,\dots,\mathcal{C}_t\}$ be a $\mathcal{C}^\infty$-cylindrical algebraic decomposition of $\mathbb{R}^t$. Let C a cell of $\mathcal{C}$. Let $i_1<\dots<i_d\le k$ such that the restriction $\tilde\pi_C : C\to\mathbb{R}^d$ to C of the projection $\tilde\pi(r_1,\dots,r_k)=(r_{i_1},\dots,r_{i_d})$ is a local diffeomorphism of C onto an open subset of $\mathbb{R}^d$. We choose $\tilde\pi_C$ such that $(i_1,i_2,\dots,i_d)$ is minimal with respect the lexicographical order. In particular, $\tilde\pi_C$ is a local system of coordinates of C at any point $\alpha\in C$ and d is the dimension of C. We call $\tilde\pi_C$ the *standard system of coordinates* of the cell C with respect the cylindrical algebraic decomposition $\mathcal{C}$. We denote I_C the d-uple $I_C=(i_1,\dots,i_d)$. The derivations $\left.\frac{\partial}{\partial r_{i_1}}\right|_\alpha,\dots,\left.\frac{\partial}{\partial r_{i_d}}\right|_\alpha$ span the tangent space of C at α. One proves easily that

$$\left.\frac{\partial}{\partial r_{i_j}}\right|_\alpha (r_s|_C)=0,\quad s<i_j,\quad 1\le j\le d. \tag{4}$$

Remark 3. — *The sequence $I_C=(i_1,\dots,i_d)$ is characterized as follows: let i_C be the inclusion of C in $\mathbb{R}^k$ and $\alpha\in C$ any point. Then $j\notin I_C$ if and only if $i_C^*(d\,r_j)_\alpha$ depends linearly on $\{i_C^*(d\,r_s)_\alpha \mid s<j\}$.*

We identify $(\mathbb{R}\times\mathbb{C})^k$ with $\mathbb{R}^{3k}$ as follows: let $(r_1,\dots,r_{3k})$ be the coordinate functions of $\mathbb{R}^{3k}$ and $(\mu_1,c_1,\dots,\mu_k,c_k)$ the coordinate functions of $(\mathbb{R}\times\mathbb{C})^k$. For $1\le t\le k$, let

$$\begin{aligned}\mu_t &= r_{3(t-1)+1}, \quad\text{and}\\ c_t &= r_{3(t-1)+2}+\sqrt{-1}\, r_{3(t-1)+3}.\end{aligned} \tag{5}$$

***Definition 3* (Adapted dimension).** — *Let C be a cell of a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $(\mathbb{R}\times\mathbb{C})^k=\mathbb{R}^{3k}$ and $\tilde\pi_C(r_1,\dots,r_{3k})=(r_{i_1},\dots,r_{i_d})$ be the standard system of coordinates of C with respect to $\mathcal{C}$. For each t, $1\le t\le k$, define d_t as follows:*

1. $d_t=2$ *if* $3(t-1)+1\in\{i_1,\dots,i_d\}$.
2. $d_t=1$ *if either* $3(t-1)+2$ *or* $3(t-1)+3$ *belongs to* $\{i_1,\dots,i_d\}$ *and we are not in case* (1).
3. $d_t=0$ *otherwise.*

The adapted dimension *of C is* $\dim_{\mathrm{a}}(C)=d_1+\dots+d_k$.

Remark 4. — *Certainly,* $\dim(C)\le 2\dim_{\mathrm{a}}(C)$.

Lemma 3. — *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two $\mathcal{C}^\infty$-cylindrical algebraic decompositions of* $(\mathbb{R}\times\mathbb{C})^k$. *Let C be a cell of $\mathcal{C}^1$, and assume that $C = C'_1\cup\cdots\cup C'_s$, where C'_i is a cell of $\mathcal{C}^2$ for all i. Then* $\dim_{\mathrm{a}}(C'_i)\le\dim_{\mathrm{a}}(C)$ *for $1\le i\le s$ and there exists $j\in\{1,\dots,s\}$ such that* $\dim_{\mathrm{a}}(C'_j)=\dim_{\mathrm{a}}(C)$, $\dim C'_j=\dim C$ *and* $I_{C'_j}=I_C$.

Proof. — From the characterization given in Remark 3 and the fact that linear dependency is preserved by the pull-back of the inclusion of C'_i into C, one infers that $I_{C'_i}\subseteq I_C$, which implies that $\dim_{\mathrm{a}} C'_i\le\dim_{\mathrm{a}} C$. Then there must exist an index j with $\dim C'_j=\dim C$, whence $I_{C'_j}=I_C$ and $\dim_{\mathrm{a}} C'_j=\dim_{\mathrm{a}} C$. □

3.3. Main result. — Let $F(y_0,\dots,y_n)$ be a polynomial in the variables $y_0,\dots,y_n$ with coefficients in $\mathbb{C}[x^{\mathbb{R}}]$.

Theorem 1. — *Let $Sol(F)$ the set of solutions of the differential equation $F(y)=0$ in $\mathbb{C}((x))^g$. For any positive integer k, there exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $(\mathbb{R}\times\mathbb{C})^k$ and a finite number number of cells $C_1,\dots,C_s$ of $\mathcal{C}$ such that:*

- $\mathrm{Tr}_k(\mathrm{Sol}(F))\subseteq C_1\cup\cdots\cup C_s$, *and*
- $\dim_{\mathrm{a}}(C_i)\le n$, *for $1\le i\le s$.*

As a consequence, $\dim(C_i)\le 2n$, *for $1\le i\le s$.*

We end this section doing a technical reduction for the proof of Theorem 1 and, for the sake of clarity, giving a simple proof of case $k=1$.

Claim: it is enough to prove the theorem substituting $\mathrm{NIC}^*_k(F)$ for $\mathrm{Tr}_k(\mathrm{Sol}(F))$ in the statement.

Proof of the claim: let $\mathcal{M}_s$ be the subset of $\mathbb{C}[x^{\mathbb{R}}]$ of "polynomials" with exactly s monomials, and $\mathcal{M}_{\ge s}$ the subset of $\mathbb{C}((x))^w$ of series with at least s monomials. Certainly,

$$\mathrm{Tr}_k(\mathrm{Sol}(F))=\mathrm{Tr}_k\big(\mathrm{Sol}(F)\cap\mathcal{M}_{\ge k}\big)\cup\bigcup_{s=0}^{k}\mathrm{Tr}_k\big(\mathrm{Sol}(F)\cap\mathcal{M}_s\big).\tag{6}$$

By definition $\mathrm{Tr}_k\big(\mathrm{Sol}(F)\cap\mathcal{M}_{\ge k}\big)\subseteq(\mathbb{R}\times\mathbb{C}^*)^k$, so that by corollary 1,

$$\mathrm{Tr}_k\big(\mathrm{Sol}(F)\cap\mathcal{M}_{\ge k}\big)\subseteq\mathrm{NIC}^*_k(F).\tag{7}$$

Let $0\le s<k$, and consider the differentiable semi-algebraic function $F_s:(\mathbb{R}\times\mathbb{C}^*)^s\to(\mathbb{R}\times\mathbb{C})^{k-s}$ given by

$$F_s\big((\mu_1,c_1),\dots,(\mu_s,c_s)\big)=\big((\mu_s+1,0),\dots,(\mu_s+k-s,0)\big).$$

One sees easily that $\mathrm{Tr}_k\big(\mathrm{Sol}(F)\cap\mathcal{M}_s\big)$ is the graph of F_s restricted to $\mathrm{Tr}_s\big(\mathrm{Sol}(F)\cap\mathcal{M}_s\big)$. As above, $\mathrm{Tr}_s\big(\mathrm{Sol}(F)\cap\mathcal{M}_s\big)\subseteq\mathrm{NIC}^*_s(F)$ and also, the adapted and the usual dimensions of $\mathrm{NIC}^*_s(F)$ are (respectively) equal to the adapted and usual dimensions of the graph of F_s restricted to $\mathrm{NIC}^*_s(F)$, which finishes. □

Follows a straightforward proof of the theorem for $k=1$.

Proof of the theorem for $k = 1$. — In this case, the first result of the theorem is just Corollary 2.

For the second result, notice that the adapted dimension of any cell in $\mathbb{R} \times \mathbb{C}^*$ is less than or equal to 2. Hence, it is enough to prove that $\dim_a(C) \leq n$ for $n = 0$ and $n = 1$, which we do separately.
If $n = 0$, the only monomial of $F(y)$ corresponding to point $(a, b) \in \mathbb{R} \times \mathbb{N}$ is exactly $A_{a,b} x^a y_0^b$, whence the polynomial $\Phi_{F;\mu_i}(C)$ is nonzero and has only a finite number of roots. The polynomial $\Psi_{(F;v)}(\mathfrak{m}) = A_v$ is clearly a nonzero constant. From these two facts, it follows that the dimension of $NIC_1^*(F)$ is zero
For $n = 1$, in order to prove that $\dim_a(C) \leq 1$, it suffices to show that the projection $(\mu, c) \mapsto \mu$ cannot belong to a local coordinate system of $\mathrm{NIC}_1^*(F)$. Let $v = (a, b)$ be a vertex of the Newton polygon. Since $n = 1$, all the monomials of $F(y)$ corresponding to v are of the form $A_{a,(\rho_0,\rho_1)} x^a y_0^{\rho_0} y_1^{\rho_1}$ with $\rho_0 + \rho_1 = b$. Hence,

$$\Psi_{(F;v)}(\mathfrak{m}) = \sum_{j=0}^{b} A_{a,(b-j,j)}\, \mathfrak{m}^j.$$

and $\Psi_{(F;v)}(\mathfrak{m})$ cannot be zero because for some j, $A_{a,(b-j,j)} \neq 0$. The image of $\mathrm{NIC}_1^*(F)$ by the projection $(\mu, c) \mapsto \mu$ is thus a finite number of points and we are done for $k = 1$. □

4. Parametric polynomials and parametric Newton polygon

In this section we define the parametric Newton polygon data of a parametric differential polynomial. A parametric polynomial is a finite sum of the form $\sum_{i \in I} c_i\, x^{\mu_i}$, where μ_i and c_i are respectively real and complex semi-algebraic $\mathcal{C}^\infty$-functions on a semi-algebraic $\mathcal{C}^\infty$-submanifold C. A parametric differential polynomial H is a polynomial in $y_0, \ldots, y_n$ whose coefficients are parametric polynomials. For any parameter $\phi \in C$, the value H_ϕ of H at ϕ is an ordinary differential polynomial. The parametric polygon data of H will be defined as a family of functional objects on C whose "values" are classical Newton polygon data (vertices, slopes, characteristics polynomials, etc) in such a way that their values at ϕ coincide with the Newton polygon data of H_ϕ.

In order to define the *parametric Newton polygon data* of H, some semi-algebraic properties on the family of exponents of x and on the real and imaginary parts of the coefficients of H are required. They are gathered in the notion of *invariance* on a cell C.

Specifically, for a parametric polynomial $H = \sum_{i \in I} c_i\, x^{\mu_i}$ to be invariant on C we require that the family of exponents $\mathcal{E} = \{\mu_i; i \in I\}$ is totally ordered on C and that none of the coefficients c_i vanishes at any point of C. This way, both the minimum of $\mathcal{E}$ and the (function) coefficient of x^θ in H, provided $\theta \in \mathcal{E}$, are well defined. Moreover, the value of the minimum of $\mathcal{E}$ at every point $\phi \in C$ is the minimum of $\{\mu_i(\phi); i \in I\}$. For technical reasons one needs also to be able to compare the coefficients with functions belonging to some family of functions E (for instance, some constant functions or the coordinate functions of the cell). This gives rise to the notion of invariance with respect a family E.

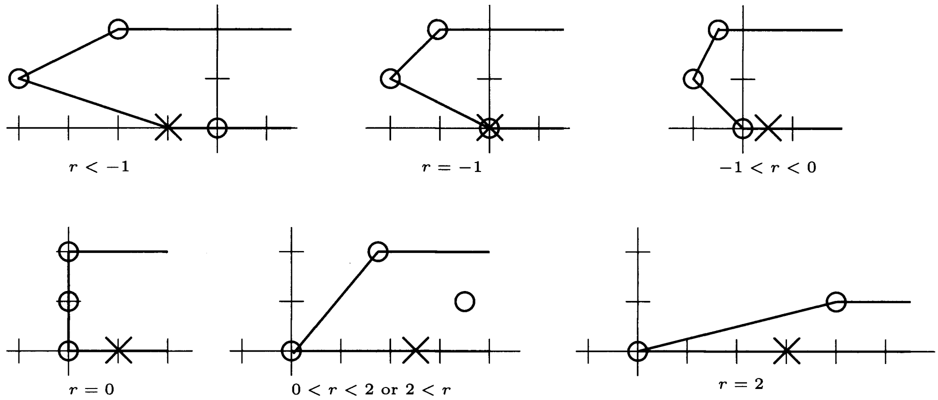

FIGURE 2.

All the above is probably better understood with an example. In Figure 2 are shown the different *shapes* (and points) of the Newton polygon of

$$H = 1 + x^{1+r} + (1+r^2)\, x^{2\,r}\, y_3 + (2-r)\, x^r\, y_0 y_1, \quad r \in \mathbb{R}.$$

One should imagine r "moving" on $\mathbb{R}$ "from left to right" giving rise to the six essentially different shapes of the Newton polygon. One can see how three *exceptional* situations can happen (in the same order as in the figure): (a) two or more points of the polygon collide, (b) a point in the interior of the polygon collides with a side, and (c) a point disappears from the polygon. These are respectively the cases $r = -1$, $r = 0$ and $r = 2$ in Figure 2. All the equations describing those events are of semi-algebraic nature, so that there exists a cylindrical algebraic decomposition of the parameter space such that H is invariant on each cell. In our example, the cells are the sets defined by the equations in r below each diagram.

In the cell $r < -1$, the parametric Newton polygon data is composed of the sequence of functions $V_0 = (r,2), V_1 = (2r,1), V_2 = (1+r,0)$, the indicial polynomials corresponding to each vertex ($\Psi_{(H;V_0)}(\mathfrak{m}) = (2-r)\mathfrak{m}$, $\Psi_{(H;V_1)}(\mathfrak{m}) = (1+r^2)\mathfrak{m}^3$, and $\Psi_{(H;V_2)}(\mathfrak{m}) = 1$), and the characteristic polynomials corresponding to the sides: $\Phi_{(H;[V_0,V_1])}(\mathfrak{c}) = \Psi_{(H;V_0)}(r)\mathfrak{c}^2 + \Psi_{(H;V_1)}(r)\mathfrak{c}$, and $\Phi_{(H;[V_1,V_2])}(\mathfrak{c}) = \Psi_{(H;V_1)}(1-r)\mathfrak{c}^2 + \Psi_{(H;V_0)}(1-r)\mathfrak{c}$. Similarly for the other cells.

In the proof of Theorem 1 we shall differentiate the parametric polynomials with respect to the "parameters". As the class of parametric polynomials is not closed under such derivations (due to $d(x^r)/dr$), we need to consider a larger family including polynomials in $\log x$, which can be related to the space of mappings from $C \times \tilde{\mathbb{C}}$ to $\mathbb{C}$, where $\tilde{\mathbb{C}}$ is the Riemann surface of the logarithm and C is a semi-algebraic smooth manifold.

Throughout this section, $C \subseteq \mathbb{R}^n$ denotes a nonempty semi-algebraic $\mathcal{C}^\infty$ submanifold.

4.1. Parametric polynomials

Definition 4. — *An $\mathcal{N}$-function on C is a semi-algebraic smooth function $f : C \to \mathbb{R}$. An $\mathcal{N}_\mathbb{C}$-function is a function $c : C \to \mathbb{C}$ of the form $c = a + \sqrt{-1}\, b$, where a, b are $\mathcal{N}$-functions on C.*

Let $\tilde{\mathbb{C}}$ be the Riemann surface of the logarithm. A function

$$H : C \times \tilde{\mathbb{C}} \to \mathbb{C}$$

is called an $\mathcal{N}_\mathbb{X}$-function over C if there exist a finite number of functions $c_i \in \mathcal{N}_\mathbb{C}(C)$ and $\mu_i \in \mathcal{N}(C)$, $1 \le i \le k$, such that

$$H(\phi, x) = \sum_{j=1}^{k} c_i(\phi)\, x^{\mu_i(\phi)}, \quad \textit{for all } (\phi, x) \in C \times \tilde{\mathbb{C}}. \tag{8}$$

Denote $\mathcal{N}(C)$, $\mathcal{N}_\mathbb{C}(C)$ and $\mathcal{N}_\mathbb{X}(C)$ the rings of $\mathcal{N}$, $\mathcal{N}_\mathbb{C}$ or $\mathcal{N}_\mathbb{X}$-functions over C, respectively.

$\mathcal{N}_\mathbb{X}(C)[\log x]$ is the set of finite sums

$$H(\phi, x) = \sum_{i,j} c_{i,j}(\phi)\, x^{\mu_{i,j}(\phi)}\, (\log x)^j, \quad \textit{for all } (\phi, x) \in C \times \tilde{\mathbb{C}}.$$

where $c_{i,j} \in \mathcal{N}_\mathbb{C}(C)$ and $\mu_{i,j} \in \mathcal{N}(C)$, $1 \le i \le k$, $0 \le j \le s$.

The following result shows that $\mathcal{N}_\mathbb{X}(c)[\log x]$ is actually the set of "polynomials" in $\log x$ with coefficients in $\mathcal{N}_\mathbb{X}(C)$:

Lemma 4. — *Any $H \in \mathcal{N}_\mathbb{X}(C)[\log x]$, can be written uniquely as $H = \sum_{j=0}^{s} H_j\, (\log x)^j$, where $H_j \in \mathcal{N}_\mathbb{X}(C)$.*

Proof. — Let $\mathcal{O}(\tilde{\mathbb{C}})$ be the differential ring of holomorphic functions on $\tilde{\mathbb{C}}$ with the derivation $\delta = x\frac{\partial}{\partial x}$, which is an integral domain. The map sending $\sum c_\alpha\, x^\alpha \in \mathbb{C}[x^\mathbb{R}]$ to the holomorphic function $\tilde{\mathbb{C}} \ni x \mapsto \sum c_\alpha\, x^\alpha$ is an injective differential ring homomorphism. The result follows from $\log x \in \mathcal{O}(\tilde{\mathbb{C}})$ being algebraically independent over the quotient ring of $\mathbb{C}[x^\mathbb{R}]$. □

4.1.1. *Derivations with respect the parameters.* — Assume that C is a cell of a $\mathcal{C}^\infty$-cylindrical algebraic decomposition and $\pi_C(r_1, \ldots, r_s) = (r_{i_1}, \ldots, r_{i_d})$ its standard system of coordinates. It is known that if f is an $\mathcal{N}$-function then its partial derivatives $\frac{\partial f}{\partial r_{i_j}}$ are also $\mathcal{N}$-functions. The operator $\frac{\partial}{\partial r_{i_j}}$ acts as a derivation on $\mathcal{N}_\mathbb{X}(C)[\log x]$ as follows: if $\mu \in \mathcal{N}(C)$ and $c \in \mathcal{N}_\mathbb{C}(C)$, then

$$\frac{\partial}{\partial r_{i_j}}(c\, x^\mu \log^s x) = \left(\frac{\partial c}{\partial r_{i_j}}\, x^\mu + c\, \frac{\partial \mu}{\partial r_{i_j}}\, x^\mu \log x \right) \log^s x. \tag{9}$$

4.1.2. *The notion of invariance*

***Notation* 2.** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$. For $\phi \in C$, denote by $H(\phi)$ the function $\tilde{\mathbb{C}} \ni x \mapsto H(\phi, x)$.*

If $H \in \mathcal{N}_{\mathbb{X}}(C)$, then $H(\phi) \in \mathbb{C}[x^{\mathbb{R}}]$; for $\tau \in \mathcal{N}(C)$, denote by $[H]_\tau$ the function $[H]_\tau : C \to \mathbb{C}$ such that $[H]_\tau(\phi)$ is the value of the coefficient of $x^{\tau(\phi)}$ in $H(\tau)$. This $[H]_\tau$ is a semi-algebraic function but it is not smooth in general.

If $H = \sum_j H_j \log^j x \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$, we write $[H]_\tau = \sum_j [H_j]_\tau \log^j x$.

We shall denote $<_C$ the partial order over $\mathcal{N}(C)$ given by $\mu <_C \mu'$ if and only if $\mu(\phi) < \mu'(\phi)$ for all $\phi \in C$.

***Definition* 5.** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)$ and E be a finite subset of $\mathcal{N}(C)$. We say that H is* invariant on C with respect to E, *if either $H = 0$ or there exist a finite subset $\mathcal{E} \subseteq \mathcal{N}(C)$ and functions $c_\theta \in \mathcal{N}_{\mathbb{C}}(C)$ for each $\theta \in \mathcal{E}$ with the following properties:*

1. *For all $(\phi, x) \in C \times \tilde{\mathbb{C}}$, $H(\phi, x) = \sum_{\theta \in \mathcal{E}} c_\theta(\phi)\, x^{\theta(\phi)}$.*
2. *The set $\mathcal{E} \cup E$ is totally ordered with respect to $<_C$.*
3. *For every $\theta \in \mathcal{E}$ and every $\phi \in C$, $c_\theta(\phi) \neq 0$.*

We remark that a set $\mathcal{E}$ satisfying (1), (2) and (3) is uniquely determined and independent of E: by (2), its elements are ordered $\theta_1 <_C \theta_2 <_C \cdots <_C \theta_s$ and for each $\phi \in C$, $\theta_1(\phi) < \theta_2(\phi) < \cdots < \theta_s(\phi)$ are the exponents of x in $H(\phi)$.

The set $\mathcal{E}$ will be denoted $\mathcal{E}(H)$. By definition $\mathcal{E}(0) = \varnothing$.

***Definition* 6.** — *An element $H = \sum_{j=0}^{s} H_j \log^j x \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$ is* invariant with respect to E *if each $H_j \in \mathcal{N}_{\mathbb{X}}(C)$, $0 \leq j \leq t$ is invariant on C with respect to E and the set $\mathcal{E}(H) = \cup_j \mathcal{E}(H_j)$ is totally ordered on C.*

***Lemma* 5.** — *Let $H \in \mathcal{N}_{\mathbb{X}}(C)$ be invariant on C with respect to E. If $\tau \in E \cup \mathcal{E}(H)$, then $[H]_\tau \in \mathcal{N}_{\mathbb{C}}(C)$. If $\tau \notin \mathcal{E}(H)$, then $[H]_\tau = 0$. In particular, $H = \sum_{\tau \in \mathcal{E}(H) \cup E} [H]_\tau\, x^\theta$.*

Proof. — If $\tau \notin \mathcal{E}(H)$, then for each $\phi \in C$, $\tau(\phi) \notin \{\theta(\phi) \mid \theta \in \mathcal{E}(H)\}$ because $E \cup \mathcal{E}(H)$ is totally ordered. If $\tau \in \mathcal{E}$ then $[H]_\tau = c_\tau \in \mathcal{N}_{\mathbb{C}}(C)$, using the notation of (1) in Definition 5. □

***Corollary* 3.** — *Let $H, G \in \mathcal{N}_{\mathbb{C}}(C)[\log x]$ be such that H, G and HG are invariant on C with respect to E. Let $\tau \in E$ be such that $\tau - \theta \in E$ for all $\theta \in \mathcal{E}(G)$. Then*

$$[H\, G]_\tau = \sum_{\theta \in \mathcal{E}(G)} [H]_{\tau - \theta}\, [G]_\theta\, .$$

***Corollary* 4.** — *Let $H^1, \ldots, H^t \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$ be invariant on C with respect to E. Assume that $\sum_{j=1}^{s} H^j$, is also invariant on C with respect to E. Let $\tau \in E$, then $\left[\sum_{j=1}^{s} H^j\right]_\tau = \sum_{j=1}^{s} \left[H^j\right]_\tau$. If $\frac{\partial}{\partial r_{i_j}}$ be a vector field on C, then $\frac{\partial}{\partial r_{i_j}} \left[\sum_{j=1}^{s} H^j\right]_\tau = \sum_{j=1}^{s} \left[\frac{\partial}{\partial r_{i_j}} H^j\right]_\tau$.*

The following result is a consequence of the fact that *invariance* is a *semi-algebraic* property.

Lemma 6. — *Let $H^1, \dots, H^t \in \mathcal{N}_{\mathbb{X}}(C)[\log x]$, E a finite subset of $\mathcal{N}(C)$ and $\mathcal{P}$ a finite family of polynomials in $\mathbb{R}[r_1, \dots, r_n]$. There exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $\mathbb{R}^n$ adapted to $\mathcal{P}$ such that C is a finite union of cells of $\mathcal{C}$ and for any cell $C' \in \mathcal{C}$ and any s, $1 \leq s \leq t$, H^s is invariant on C' with respect to $E|_{C'} = \{f|_{C'} \mid f \in E\}$.*

4.2. The Parametric Newton polygon. — A *parametric differential polynomial* is an element $H(y) \in \mathcal{N}_{\mathbb{X}}(C)[y_0 \dots, y_n]$. Let d be the total degree in the indeterminates $y_0, \dots, y_n$ of $H(y)$. Write uniquely

$$H(y) = \sum_{|\rho| \leq d} H_\rho \, y_0^{\rho_0} \cdots y_n^{\rho_n}, \quad H_\rho \in \mathcal{N}_{\mathbb{X}}(C).$$

We proceed to the definition of the parametric Newton polygon (and its data) of $H(y)$ on C. This notion requires several properties on the coefficients of $H(y)$, expressed technically in Definition 8. The first one (condition (a)) is *invariance* on a cell, which lets us speak of *monomials* of $H(y)$ and their coefficients (no monomial disappears or appears inside a cell). In (b) we require that for each height (each ordinate) one can define the leftmost point at that height of the cloud of $H(y)$. We follow the usual algorithm to compute the positive convex hull: starting from the top-leftmost point, which will be the first vertex, we determine inductively the following ones. This is possible, for example (and this is what we impose in (c)) if the "slopes" appearing in the polygon are totally ordered in C:

***Definition** 7.* — *A parametric differential polynomial $H(y)$ is* invariant on C with respect to a finite subset $E \subseteq \mathcal{N}(C)$ *if the following conditions hold:*

(a) *For all $\rho \in \mathbb{R}$, H_ρ is invariant on C with respect to E.*
(b) *Let h be an integer, $0 \leq h \leq d$, and set*

$$\mathcal{E}_h(H) = \bigcup_{|\rho| = h} \mathcal{E}(H_\rho).$$

Then $E \cup \mathcal{E}_h(H)$ is totally ordered for all h, $0 \leq h \leq d$.

(c) *For $\mathcal{E}_h(H) \neq \varnothing$, let $\theta_h = \min \mathcal{E}_h(H)$. Then the union of E with the set of functions on C given by*

$$\frac{\theta_{h_2} - \theta_{h_1}}{h_1 - h_2}, \quad \textit{for } 0 \leq h_2 < h_1 \leq d, \quad \mathcal{E}_{h_1}(H) \neq \varnothing \neq \mathcal{E}_{h_2}(H),$$

is totally ordered.

We say that $H(y) \in \mathcal{N}_{\mathbb{X}}(C)[\log x][y_0 \dots, y_n]$ is invariant on C with respect to E *if, writing $H(y) = \sum_{j=0}^{s} H_j(y) \log^j x$, then each $H_j(y)$ is invariant on C with respect to E. $H(y)$ is just* invariant *if it is invariant with respect to the empty set.*

We proceed to "build up" the Newton polygon. Assume $H(y) \in \mathcal{N}_{\mathbb{X}}(C)[y_0 \dots, y_n]$ is invariant on C. The *parametric Newton polygon of $H(y)$ with respect to C* is just a sequence $V_0, V_1, \dots, V_t$ of vertices, each being a pair (θ, h) where h is an integer, $0 \leq h \leq d$ and θ belongs to $\mathcal{E}_h(C)$. These vertices are defined inductively:

Definition 8. — *Let $V_0 = (\theta_d, d)$, where d is the total degree of $H(y)$ on $y_0, \dots, y_n$. Assume that vertex $V_i = (\theta_{h_i}, h_i)$ has been defined. If $\bigcup_{h<h_i} \mathcal{E}_h(H)$ is empty, then we have finished, Otherwise, set $V_{i+1} = (\theta_{h_{i+1}}, h_{i+1})$, where h_{i+1} is the minimum of those $h < h_i$ such that*

$$\frac{\theta_h - \theta_{h_i}}{h_i - h} = \min\{\frac{\theta_{h'} - \theta_{h_i}}{h_i - h'} \mid h' < h_i,\ \mathcal{E}_{h'}(C) \neq \varnothing\}.$$

The parametric Newton polygon *of $H(y)$ with respect to C as the sequence $V_0, \dots, V_i$. The* sides *are the sets $[V_i, V_{i+1}]$ for $i = 0, \dots, t-1$:*

$$[V_i, V_{i+1}] = \{V_i\} \cup \{(\theta_h, h) \mid h_i > h \geq h_{i+1},\ \frac{\theta_h - \theta_{h_i}}{h_i - h} = \frac{\theta_{h_{i+1}} - \theta_{h_i}}{h_i - h_{i+1}}\}.$$

The inclination *of side $[V_i, V_{i+1}]$ is $\mu_{[V_i, V_{i+1}]} = \frac{\theta_{h_{i+1}} - \theta_{h_i}}{h_i - h_{i+1}} \in \mathcal{N}(C)$.*

One can write uniquely

$$H(y) = \sum_{|\rho| \leq d} \sum_{\theta \in \mathcal{E}_{|\rho|}(H)} H_{\rho,\theta}\, x^{\theta} y_0^{\rho_0} \dots y_n^{\rho_n}, \quad H_{\rho,\theta} \in \mathcal{N}_{\mathbb{C}}(C).$$

Given a vertex $V = (\theta_h, h)$ and a side $[V_i, V_{i+1}]$, define the *indicial* and *characteristic* polynomials as follows (respectively):

$$\begin{aligned} \Psi_{(H;V)}(\mathfrak{m}) &= \sum_{|\rho|=h} H_{\rho,\theta_h}\ \mathfrak{m}^{w(\rho)} \in \mathcal{N}_{\mathbb{C}}[\mathfrak{m}], \\ \Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c}) &= \sum_{(\theta,|\rho|) \in [V_i, V_{i+1}]} H_{\rho,\theta}\ \mu_{[V_i,V_{i+1}]}^{w(\rho)}\ \mathfrak{c}^{|\rho|} \in \mathcal{N}_{\mathbb{C}}[\mathfrak{c}]. \end{aligned}$$

Definition 9. — *The* parametric Newton polygon data of $H(y)$ with respect to C *is the family of vertices $V_0, \dots, V_t$, sides $[V_i, V_{i+1}]$, $0 \leq i < t$, and polynomials $\Psi_{(H;V_i)}(\mathfrak{m})$, $0 \leq i \leq t$ and $\Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c})$, $0 \leq i \leq t-1$.*

Lemma 7. — *Let $H(y)$ be invariant on C with respect to E and $C' \subseteq$ be a semi-algebraic $\mathcal{C}^\infty$-submanifold. Then $H(y)$ is invariant on C' with respect to E and the parametric Newton polygon data of $H(y)$ with respect to C' is the natural restriction of the parametric Newton polygon data of $H(y)$ with respect to C. In particular, if $V = (\theta, h)$ is a vertex with respect to C, then $V|_{C'} = (\theta|_{C'}, h)$ is a vertex with respect to C'.*

Proof. — If $E' \subseteq \mathcal{N}(C)$ is totally ordered with respect to $<_C$, then $E'|_{C'} = \{\tau|_{C'} \mid \tau \in E'\}$ is totally ordered with respect to $<_{C'}$. The minimum of $E'|_{C'}$ is the restriction to C' of the minimum of E. This implies that if $G \in \mathcal{N}_{\mathbb{X}}(C)$ is invariant with respect to E, then $G|_{C'} \in \mathcal{N}_{\mathbb{X}}(C')$ is invariant with respect to $E|_{C'}$ and $\mathcal{E}(G) = \mathcal{E}(G|_{C'})$.

Condition (a) of Definition 7 holds for $H|_{C'}$. Moreover, $\mathcal{E}_h(H) = \mathcal{E}_h(H|_{C'})$ so that also conditions (b) and (c) are satisfied. As one can write

$$H(y) = \sum_{|\rho|\leq d} \sum_{\theta\in\mathcal{E}_{|\rho|}(H)} H_{\rho,\theta}|_{C'}\, x^{\theta|_{C'}} y_0^{\rho_0}\dots y_n^{\rho_n}, \quad H_{\rho,\theta} \in \mathcal{N}_{\mathbb{C}}(C'),$$

then $\Psi_{(H|_{C'};V_i)}(\mathfrak{m})$ and $\Phi_{(H|_{C'};[V_i|_{C'},V_{i+1}|_{C'}])}(\mathfrak{c})$ are (respectively) the restrictions to C' of the polynomials $\Psi_{(H;V_i)}(\mathfrak{m})$ and $\Phi_{(H;[V_i,V_{i+1}])}(\mathfrak{c})$. □

Remark 5. — *The above lemma holds for C' a single point. Namely, for any $\phi \in C$, denote*

$$H_\phi(y) = \sum_{\rho,\theta} H_{\rho,\theta}(\phi)\, x^{\theta(\phi)}\, y^\rho \in \mathbb{C}[x^{\mathbb{R}}][y_0,\dots,y_n].$$

The vertices of the Newton polygon of $H_\phi(y)$ are precisely the points $V_0(\phi),\dots,V_t(\phi)$, where $V_i(\phi) = (\theta_{h_i}(\phi), h_i)$. Moreover, the (differential) monomials of $H_\phi(y)$ whose corresponding points belong to the side $[V_i(\phi), V_{i+1}(\phi)]$ are precisely the monomials $H_{\rho,\theta}(\phi)\, x^{\theta(\phi)}\, y^\rho$, where $(\theta, |\rho|) \in [V_i, V_{i+1}]$. Hence,

$$\Psi_{(H;V)}(\phi,\mathfrak{m}) = \Psi_{(H_\phi(y);V(\phi))}(\mathfrak{m}), \tag{10}$$

$$\Phi_{(H;[V_i,V_{i+1}])}(\phi,\mathfrak{c}) = \Phi_{(H_\phi(y);\mu_{[V_i,V_{i+1}]}(\phi))}(\mathfrak{c}). \tag{11}$$

From the semialgebraic nature of the properties required in Definition 7, one infers

Lemma 8. — *Let $H^1(y),\dots,H^t(y) \in \mathcal{N}_{\mathbb{X}}(C)[\log x][y_0\dots,y_n]$, let $E \subseteq \mathcal{N}(C)$ be a finite subset, and $\mathcal{P}$ a finite set of polynomials. There exists a C^∞-cylindrical decomposition $\mathcal{C}$ adapted to $\mathcal{P}$ such that C is a finite union of cells of $\mathcal{C}$ and for each cell $C' \in \mathcal{C}$ with $C' \subseteq C$, and for each j, $H^j(Y)$ is invariant on C' with respect to E.*

5. Proof of the Main Theorem

We start with the differential equation $F(y) = 0$, where

$$F(y) = \sum_{a\in S}\sum_{|\rho|\leq d} A_{a,\rho}\, x^a\, y_0^{\rho_0}\cdots y_n^{\rho_n} \in \mathbb{C}[x^{\mathbb{R}}][y_0,\dots,y_n],$$

where S is finite subset of $\mathbb{R}$. Rewrite it in the following way

$$F(y) = \sum_{a\in S} f_a(y)\, x^a, \quad \text{where } f_a(y) \in \mathbb{C}[y_0,\dots,y_n].$$

Before proceeding, we need to provide some notation. Then we shall prove in Lemma 9 that $NIC_k^\star(F)$ is semi-algebraic and state and prove Proposition 1, which is the cornerstone of the present paper, from which Theorem 1 will follow.

Let $(\mu_1, c_1,\dots,\mu_k, c_k)$ denote the coordinate functions on $(\mathbb{R}\times\mathbb{C})^k$ and $(r_1,\dots,r_{3k})$ those on $\mathbb{R}^{3k}$ with the identification given in (5). Define

$$\begin{aligned}\varphi = \varphi_0 &= c_1\, x^{\mu_1} + \cdots + c_k\, x^{\mu_k} \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}),\\ \varphi_s &= \mu_1^s\, c_1\, x^{\mu_1} + \cdots + \mu_k^s\, c_k\, x^{\mu_k} \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}),\end{aligned}$$

for any non-negative integer s, and let

$$\mathcal{F}(y) = F(y_0 + \varphi_0, \dots, y_n + \varphi_n) \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k})[y_0, \dots, y_n].$$

For any $\phi \in (\mathbb{R} \times \mathbb{C})^k$, $\mathcal{F}_\phi(y) \in \mathbb{C}[x^{\mathbb{R}}][y_0, \dots, y_n]$ denotes the value of $\mathcal{F}(y)$ at ϕ. One has

$$\mathcal{F}_\phi(y) = F(y_0 + \varphi_0(\phi), \dots, y_n + \varphi_n(\phi)) = F(y + \varphi(\phi))$$

because $\varphi_s(\phi) = \partial_s(\varphi(\phi))$ for $s \in \mathbb{N}$.

Lemma 9. — *The set* $\mathrm{NIC}_k^*(F) \subseteq (\mathbb{R} \times \mathbb{C}^*)^k$ *is semi-algebraic for all* $k \geq 1$.

Proof. — We proceed by induction on k, the case $k = 1$ having already been proved in Corollary 2.

Assume that $\mathrm{NIC}_k^*(F)$ is a semi-algebraic subset of $(\mathbb{R} \times \mathbb{C})^k$. From Lemma 8, there exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}$ of $(\mathbb{R} \times \mathbb{C})^k$ such that $\mathcal{F}(y)$ in invariant on each cell of $\mathcal{C}$ and $\mathrm{NIC}_k^*(F)$ is the union of some of these cells.

Let $\phi \in (\mathbb{R} \times \mathbb{C})^k$ and $\phi' = (\phi, m_{k+1}, b_{k+1}) \in (\mathbb{R} \times \mathbb{C})^{k+1}$. One sees easily that $\phi' \in \mathrm{NIC}_{k+1}^*(F)$ if and only if both $\phi \in \mathrm{NIC}_k^*(F)$ and $(m_{k+1}, b_{k+1}) \in \mathrm{NIC}_1^*(\mathcal{F}_\phi(y))$. Hence it is enough to prove that for any cell $C \in \mathcal{C}$ the set

$$A_C = \{(\phi, \mu, c) \mid \phi \in C,\ (\mu, c) \in \mathrm{NIC}_1^*(\mathcal{F}_\phi(y))\}$$

is semi-algebraic. If C is contained in the complement of $\mathrm{NIC}_k^*(F)$, then $A_C = \varnothing$. Assume that $C \subseteq \mathrm{NIC}_k^*(F)$ and let $V_0, V_1, \dots, V_t$ be the vertices of the parametric Newton Polygon of $\mathcal{F}(y)$ with respect to C. By Remark 5 the vertices of the Newton polygon of $\mathcal{F}_\phi(y)$ are $V_0(\phi), \dots, V_t(\phi)$. From the proof of case $k = 1$ (Corollary 2) and equations (10) and (11), one infers that A_C is the union of the semi-algebraic sets given by the following conditions:

$$\phi \in C,\ \mu = \mu_{[V_{i-1}, V_i]}(\phi), \text{ and } \Phi_{(\mathcal{F}; [V_{i-1}, V_i])}(\phi, c) = 0,$$

for $1 \leq i \leq t$, and

$$\phi \in C,\ \mu_{[V_{i-1}, V_i]}(\phi) < \mu < \mu_{[V_i, V_{i+1}]}(\phi), \text{ and } \Psi_{(\mathcal{F}; V_i)}(\phi, \mu) = 0,$$

for $0 \leq i \leq t$, (where by definition $\mu_{[V_{-1}, V_0]}(\phi) = -\infty$ and $\mu_{[V_t, V_{t+1}]}(\phi) = \infty$). These conditions are semi-algebraic, so A_C is semi-algebraic and so is $\mathrm{NIC}_{k+1}^*(F)$. □

Given a nonempty $\mathcal{C}^\infty$-differentiable semi-algebraic manifold $C \subseteq \mathrm{NIC}_k^*(F)$, let $E_0 = \{\mu_k\} \subseteq \mathcal{N}(C)$. Assume that $\mathcal{F}(y)$ is invariant on C with respect to E_0, let $V_0, \dots, V_t$ be the vertices of the parametric Newton Polygon of $\mathcal{F}(y)$ on C and let $(\theta_h, h) \in [V_i, V_{i+1}]$ for some $0 \leq i < t$.

Definition 10. — *With the above notation, we say that* (θ_h, h) *is in the* red part with respect to μ_k of the parametric Newton Polygon of $\mathcal{F}(y)$ on C *if* $h \geq 1$ *and either the inclination* $\mu_{[V_i, V_{i+1}]} > \mu_k$ *or* $(\theta_h, h) = V_t$.

Notice that the definition makes sense because since $\mathcal{F}(y)$ is invariant on C with respect to E_0, any inclination $\mu_{[V_i, V_{i+1}]}$ can be compared with the function μ_k.

Lemma 10. — *In the conditions of the above definition, the red part with respect to μ_k of the parametric Newton Polygon of $\mathcal{F}(y)$ on C is nonempty.*

Proof. — Let $\phi \in C$. Since $\phi \in \mathrm{NIC}_k^*(\mathcal{F}_\phi(y))$, by Lemma 2, the red part with respect to $\mu_k(\phi)$ of the Newton polygon of $\mathcal{F}_\phi(y)$ is nonempty. The vertices of the Newton polygon of $\mathcal{F}_\phi(y)$ are $V_0(\phi), \dots, V_t(\phi)$. Hence, either there exists a side $[V_i(\phi), V_{i+1}(\phi)]$ with inclination $\mu_{[V_i,V_{i+1}]}(\phi)$ greater than $\mu_k(\phi)$, or $V_t(\phi)$ has ordinate greater than zero. If $\mu_k(\phi) < \mu_{[V_i,V_{i+1}]}(\phi)$, then $\mu_k <_C \mu_{[V_i,V_{i+1}]}$, because $\mu_k \in E_0$ and V_i is in the red part. Otherwise, if $V_t(\phi) = (\theta_t(\phi), h_t)$ with $h_t \geq 1$, then V_t is in the red part. $\square$

Let $\mathrm{NIC}_k^{*,>}(F)$ denote the following semi-algebraic set:

$$\mathrm{NIC}_k^{*,>}(F) = \mathrm{NIC}_k^*(F) \cap \{\mu_1 > 0\}$$

and let $\mathcal{C}$ be a $\mathcal{C}^\infty$-cylindrical algebraic decomposition of $(\mathbb{R}\times\mathbb{C})^k$ such that $\mathrm{NIC}_k^{*,>}(F)$ is the union of some cells of $\mathcal{C}$. $\mathrm{NIC}_k^{*,>}(F)$ is defined as semi-algebraic set by a finite family of polynomials $\mathcal{Q}$ and each cell $C_i \in \mathcal{C}$ by a finite family $\mathcal{P}_i$. Set $\mathcal{P} = \mathcal{Q} \bigcup \cup_i \mathcal{P}_i$. Fix a cell $C \in \mathcal{C}$ and let $I_C = (i_1, \dots, i_d)$ (so that $d = \dim(C)$). Denote $d_a = \dim_{\mathrm{a}}(C)$.

Proposition 1. — *With the notation above, the adapted dimension of C with respect to $\mathcal{C}$ is less than or equal to the order of F: $d_a \leq \mathrm{ord}(F)$.*

Before starting the proof, let us introduce some useful notation.
Given $\lambda \in \mathbb{N}^{n+1}$ and $f(y) \in \mathbb{C}[y_0, \dots, y_n]$, let

$$f^{(\lambda)}(\varphi) = \frac{\partial^{|\lambda|} f}{\partial^{\lambda_0} y_0 \dots \partial^{\lambda_n} y_n}(\varphi_0, \dots, \varphi_n) \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}).$$

By the Taylor expansion formula,

$$f_a(y_0 + \varphi_0, \dots, y_n + \varphi_n) = \sum_{|\lambda| \leq d} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, y^\lambda \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k})[y_0, \dots, y_n],$$

where $\lambda! = \lambda_0! \cdots \lambda_n!$ and $y^\lambda = y_0^{\lambda_0} \cdots y_n^{\lambda_n}$. Hence

$$\mathcal{F}(y) = \sum_{a \in S} \sum_{|\lambda| \leq d} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, x^a\, y^\lambda.$$

F_λ will denote the coefficient of y^λ in $\mathcal{F}(y)$:

$$F_\lambda = \sum_{a \in S} \frac{1}{\lambda!} f_a^{(\lambda)}(\varphi)\, x^a \in \mathcal{N}_{\mathbb{X}}(\mathbb{R}^{3k}),$$

so that $\mathcal{F}(y) = \sum_{|\lambda| \leq d} F_\lambda\, y^\lambda$.

Proof of Proposition 1. — Let $E_0 = \{\mu_k\}$. By Lemma 8 there exists a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}^1$ of $(\mathbb{R} \times \mathbb{C})^k$ adapted to $\mathcal{P}$ such that $\mathcal{F}(y)$ and $\frac{\partial}{\partial r_{i_j}} \mathcal{F}(y)$, for $1 \leq j \leq d$, are all invariant with respect to E_0 on any cell of $\mathcal{C}^1$ contained in C. In particular, C is a finite union of cells of $\mathcal{C}^1$. By Lemma 3, there exists a cell C_1 of

$\mathcal{C}^1$ such that $\dim_{\mathrm{a}} C_1 = d_a$ and $I_{C_1} = I_{C_0}$. Let $\mathcal{P}_1$ be a family of polynomials defining C_1 as semi-algebraic set.

Let $V_0, \dots, V_l$ be the vertices of the parametric Newton polygon of $\mathcal{F}(y)$ on C_1 (in decreasing order or height: $h_0 > h_1 > \cdots > h_l$). Let θ_h be as in Definition 8, so that $V_s = (\theta_{h_s}, h_s)$, $0 \le s \le l$. Given a side $[V_s, V_{s+1}]$ of the Polygon and a height $h \in \mathbb{N}$ with $h_s \ge h \ge h_{s+1}$, we denote by τ_h the following value (see Figure 2): $\tau_h = \theta_{h_s} + (h_s - h)\mu_{[V_s,V_{s+1}]} \in \mathcal{N}(C_1)$. Notice that $\tau_{h_j} = \theta_{h_j}$ for $0 \le j \le l$. Given h with $\mathcal{E}_h(\mathcal{F}) \neq \varnothing$, if $(\theta_h, h) \in [V_s, V_{s+1}]$, then $\tau_h = \theta_h$, otherwise $\tau_h < \theta_h$.

We shall later need to take coefficients with respect to the functions μ_s and $\tau_h - \mu_s$, and compare $\tau_h + \mu_s$ with τ_{h-1}. For simplicity, let E_1 denote the subset of $\mathcal{N}(C_1)$ composed of τ_h, μ_s, $\tau_h - \mu_s$, and $\tau_h + \mu_s$ for $h_0 \ge h \ge h_l$ and $1 \le s \le k$.

Let $\mathcal{H} \subset \mathcal{N}_{\mathbb{X}}(C_1)$ be the set composed of the following functions:

$$F_\lambda,\ \frac{\partial F_\lambda}{\partial r_{i_j}},\ f_a^{(\lambda')}(\varphi)x^a,\ \frac{\partial}{\partial r_{i_j}} f_a^{(\lambda)}(\varphi)\,x^a,\ f_a^{(\lambda)}(\varphi)\,x^a\,\frac{\partial \varphi_s}{\partial r_{i_j}},\ \frac{\partial \varphi_j}{\partial r_i},$$

for all $|\lambda| \le d$, $1 \le j \le d$ and $a \in S$. Fix a $\mathcal{C}^\infty$-cylindrical algebraic decomposition $\mathcal{C}^2$ adapted to $\mathcal{P}_1$ such that any element of $\mathcal{H}$ is invariant with respect to E_1 on any cell of $\mathcal{C}^2$. As above, C_1 is a finite union of cells of $\mathcal{C}^2$ and we may choose a cell C_2 of $\mathcal{C}^2$ such that $C_2 \subseteq C_1$, $\dim_{\mathrm{a}} C_2 = d_a$ and $I_{C_2} = I_{C_0}$.

By Lemma 7, $\mathcal{F}(y)$ and $\frac{\partial \mathcal{F}(y)}{\partial r_{i_j}}$ are invariant on C_2 with respect to E_0 and the parametric Newton Polygon of $\mathcal{F}(y)$ on C_2 has vertices $V_0|_{C_2}, \dots, V_t|_{C_2}$. Therefore, we may write uniquely

$$\mathcal{F}(y) = \sum_{|\lambda|\le d}\ \sum_{\theta \in E_1 \cup \mathcal{E}_{|\lambda|}(\mathcal{F})} F_{\theta,\lambda}\, x^\theta\, y^\lambda, \quad F_{\theta,\lambda} \in \mathcal{N}_{\mathbb{C}}(C).$$

Let $(\theta_h, h) \in [V_s, V_{s+1}]$ be in the red part with respect to μ_k of the parametric Newton polygon of $\mathcal{F}(y)$ on C_2 (recall that $h \ge 1$).

Take $i \in \{i_1, \dots, i_d\}$ and let t the minimum integer greater than or equal to $i/3$, so that the corresponding r_i in (5) is μ_i, $\Re(c_i)$ or $\Im(c_i)$ (real and imaginary parts). Let $\lambda' \in \mathbb{N}^{n+1}$ such that $|\lambda'| = h - 1$. Fix $t \in \{1 \dots, k\}$ and let $\tau = \theta_h + \mu_t \in E_1|_{C_2}$.

We claim that $F_{\tau,\lambda'} = 0$: if $(\theta_h, h) = V_l$ then $\mathcal{E}_{h-1} = \varnothing$ and by Lemma 5 $[F_{\lambda'}]_\tau = 0$; if $(\theta_h, h) \in [V_s, V_{s+1}]$ then $\mu_t < \mu_{[V_s,V_{s+1}]}$, so $\tau < \theta_{h-1}$ and $\tau \notin \mathcal{E}_{h-1}(\mathcal{F})$ and again by Lemma 5 $[F_{\lambda'}]_\tau = 0$. Therefore $0 = \frac{\partial F_{\tau,\lambda'}}{\partial r_i}$.

On the other hand, by direct computation

$$\begin{aligned}
\frac{\partial F_{\tau,\lambda'}}{\partial r_i} &= \frac{\partial}{\partial r_i}\left[\sum_a \frac{1}{\lambda'!} f_a^{(\lambda')}(\varphi)x^a\right]_\tau \overset{(a)}{=} \sum_a \frac{1}{\lambda'!}\left[\frac{\partial}{\partial r_i} f_a^{(\lambda')}(\varphi)x^a\right]_\tau \\
&= \sum_a \left[\frac{1}{\lambda'!}\sum_{j=0}^{n} f_a^{(\lambda'+e_j)}(\varphi)\frac{\partial \varphi_j}{\partial r_i}x^a\right]_\tau
\end{aligned}$$

$$\overset{(b)}{=} \sum_{j=0}^{n}(\lambda'_j + 1)\left[\sum_a \frac{1}{(\lambda'+e_j)!} f_a^{(\lambda'+e_j)}(\varphi)\ x^a\frac{\partial \varphi_j}{\partial r_i}\right]_\tau \tag{12}$$

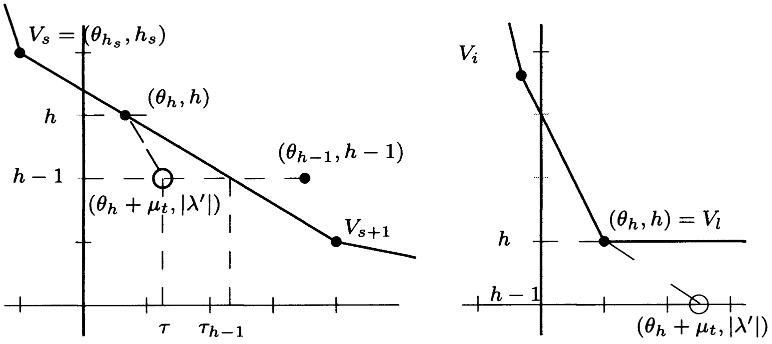

FIGURE 3. Two possibilities for a point in the red part.

$$\overset{(c)}{=} \sum_{j=0}^{n} (\lambda'_j + 1) \sum_{\theta \in \mathcal{E}(\frac{\partial \varphi_j}{\partial r_i})} \left[\sum_{a} \frac{1}{(\lambda' + e_j)!} f_a^{(\lambda' + e_j)}(\varphi)\, x^a \right]_{\tau - \theta} \left[\frac{\partial \varphi_j}{\partial r_i} \right]_\theta$$

$$\overset{(d)}{=} \sum_{j=0}^{n} (\lambda'_j + 1) \sum_{\theta \in \{\mu_1, \ldots, \mu_k\}} F_{\tau - \theta, \lambda' + e_j} \left[\frac{\partial \varphi_j}{\partial r_i} \right]_\theta$$

Where e_j is the element of $\mathbb{N}^{j+1}$ $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 appears in the $j-$th place *counting from 0.*

Equality (a) is a consequence of Corollary 4 and the fact that all members of $\mathcal{H}$ are invariant on C_2 with respect to E_1 (cf. Lemma 7). For (b), we just rewrite $\frac{1}{\lambda!} = \frac{\lambda_j + 1}{(\lambda + e_j)!}$. Equality (d) follows from the definition of $F_{\lambda' + e_j}$ and the fact that it is invariant in C_2 with respect to E_1. Finally, in order to get (c), we use Corollary 3 together with the inclusion $\mathcal{E}(\frac{\partial \varphi_j}{\partial r_j}) \subseteq \{\mu_1, \ldots, \mu_k\}$ which is proved as follows:

From relations (5) and equation (9) one gets

$$\frac{\partial}{\partial r_i}(\mu_j^s\, c_j\, x^{\mu_j}) = (s \mu_j^{s-1} \frac{\partial \mu_j}{\partial r_i}\, c_j + \mu_j^s \frac{\partial c_j}{\partial r_i} + \mu_j^s\, c_j\, \frac{\partial \mu_j}{\partial r_i} \log x)\, x^{\mu_j}, \quad 1 \leq j \leq k.$$

and since $\varphi_s = \sum_{j=1}^{k} \mu_j^s\, c_j\, x^{\mu_j} \in \mathcal{N}_{\mathbb{X}}(C_2)$, then $\mathcal{E}(\frac{\partial \varphi_s}{\partial r_i}) \subseteq \{\mu_1, \ldots, \mu_s\}$.

From equation (4), $\frac{\partial}{\partial r_i}(r_j) = 0$ if $j < i$ and $\frac{\partial}{\partial r_i}(r_i) = 1$, so that

$$\begin{aligned}
&\frac{\partial}{\partial r_i}(\mu_j^s\, c_j\, x^{\mu_j}) = 0, \quad \text{for } j < t, \\
(13) \qquad &\frac{\partial}{\partial r_i}(\mu_t^s\, c_t\, x^{\mu_t}) = (s \mu_t^{s-1}\, c_t + \mu_t^s \frac{\partial c_t}{\partial r_i} + \mu_t^s\, c_t\, \log x)\, x^{\mu_t}, (i = 1 \bmod 3), \\
&\frac{\partial}{\partial r_i}(\mu_t^s\, c_t\, x^{\mu_t}) = \mu_t^s \frac{\partial c_t}{\partial r_i}\, x^{\mu_t}, \quad (i = 2, 3 \mod 3).
\end{aligned}$$

Therefore one may write

$$\frac{\partial \varphi_j}{\partial r_i} = \frac{\partial}{\partial r_i}(\mu_t^j\, c_t\, x^{\mu_t}) + \sum_{s=t+1}^{k} \left[\frac{\partial \varphi_j}{\partial r_i}\right]_{\mu_s}.$$

For $s > t$, $\tau - \mu_s < \theta_h$, so that $F_{\tau-\mu_s,\lambda'+e_j} = 0$ and the last member of equation (12) is

$$\begin{aligned} 0 &= \sum_{j=0}^{n} (\lambda'_j + 1) \sum_{\theta\in\{\mu_1,\ldots,\mu_k\}} F_{\tau-\theta,\lambda'+e_j} \left[\frac{\partial \varphi_j}{\partial r_i}\right]_{\theta} = \\ &= \sum_{j=0}^{n} (\lambda'_j + 1)\, F_{\theta_h,\lambda'+e_j}\, \frac{\partial}{\partial r_i}(\mu_t^j\, c_t\, x^{\mu_t}). \end{aligned}$$

If $i = 1 \mod 3$, from (13) and the fact that c_t does not vanish in $C_2 \subseteq \mathrm{NIC}_k^*(F)$, one infers the following two linear equations

$$0 = \sum_{j=0}^{n} (\lambda'_j + 1)\, \mu_t^j\, F_{\theta_h,\lambda'+e_j}, \tag{14}$$

$$0 = \sum_{j=0}^{n} (\lambda'_j + 1)\, j\, \mu_t^{j-1}\, F_{\theta_h,\lambda'+e_j}. \tag{15}$$

(If $i = 2$ or $0 \mod 3$ then only (14) appears). Letting i run over $\{i_1,\ldots,i_d\}$, one obtains a linear system of equations in the variables $F_{\theta_h,\lambda'+e_0},\ldots,F_{\theta_h,\lambda'+e_n}$. An elementary argument of linear algebra shows that it has rank $d_a = \dim_{\mathrm{a}}(C_2)$. Hence, if $d_a > n$ its only solution is the trivial one, which implies that the red part of the parametric Newton polygon of $\mathcal{F}(y)$ on C_2 is empty, contradicting Lemma 10. □

Proof of Theorem 1. Proceed by contradiction. Assume that there exists a cell C in $\mathrm{NIC}_k^*(F)$ with adapted dimension greater than n and let $\phi \in C$, $m = \mu_1(\phi) - 1$ and consider the differential polynomial $G(y) = F(x^m y)$. The set $\mathrm{NIC}_k^*(G)$ is the image of $\mathrm{NIC}_k^*(F)$ under the translation $T(x,y) = (x-m, y)$ so that $T(C) \cap \mathrm{NIC}_k^{*>}(G)$ is nonempty. Since translations preserve the adapted dimension of cells, there must exist a cell in $\mathrm{NIC}_k^{*>}(G)$ with adapted dimension greater than the order of G, which is equal to the order of F, against Proposition 1. □

References

[1] C. Briot & J. Bouquet – "Propriétés des fonctions définies par des équations différentielles", *Journal de l'École polytechnique* **36** (1856), 133–198.

[2] J. Cano – "On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations", *Analysis* **13** (1993), p. 103–119.

[3] M. Coste – *An introduction to semialgebraic geometry*, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[4] H. B. Fine – "On the Functions Defined by Differential Equations, with an Extension of the Puiseux Polygon Construction to these Equations", *Amer. J. Math.* **11** (1889), p. 317–328.

[5] D. Y. GRIGOR′EV & M. F. SINGER – "Solving ordinary differential equations in terms of series with real exponents", *Trans. Amer. Math. Soc.* **327** (1991), p. 329–351.

[6] J. VAN DER HOEVEN – "Asymptotique automatique", Ph.D. Thesis, Université Paris 7, 1997.

[7] ———, *Transseries and real differential algebra*, Lecture Notes in Mathematics, vol. 1888, Springer, 2006.

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