Astérisque

Javier Ribón

Unfoldings of tangent to the identity diffeomorphisms

Astérisque, tome 323 (2009), p. 325-370

<http://www.numdam.org/item?id=AST_2009__323__325_0>

Astérisque
323, 2009, p. 325–370

UNFOLDINGS OF TANGENT TO THE IDENTITY DIFFEOMORPHISMS

by

Javier Ribón

Abstract. — This paper is devoted to classify one dimensional unfoldings of tangent to the identity analytic diffeomorphisms, in other words elements φ of $\mathrm{Diff}(\mathbb{C}^2, 0)$ of the form $(x, f(x,y))$ with $(\partial f/\partial y)(0,0) = 1$. We provide the topological classification in absence of small divisors phenomena and an analytic classification of the finite codimension unfoldings. Such results are based on the study of the stable structures preserved by the diffeomorphisms. The main tool is the use of real flows. In both the topological and the analytic cases a non-wandering property is required, namely the Rolle property in the topological setting and infinitesimal stability in the analytic one.

We also prove that under generic hypotheses the analytic class of an unfolding φ depends only on the analytic classes of the germs of 1-dimensional diffeomorphisms obtained by localizing along an irreducible component of the fixed points set of φ.

***Résumé* (Déploiements des difféomorphismes tangents à l'identité).** — Cet article est consacré à la classification des déploiements à un paramètre des difféomorphismes analytiques tangents à l'identité, en d'autres termes, les éléments φ de $\mathrm{Diff}(\mathbb{C}^2, 0)$ qui sont de la forme $(x, f(x,y))$, avec $(\partial f/\partial y)(0,0) = 1$. Nous fournissons une classification topologique en l'absence de phénomènes de petits diviseurs et la classification analytique des déploiements de codimension finie. Les preuves sont basées sur l'étude des structures stables qui sont invariants par l'action des difféomorphismes. L'outil principal est le recours aux flots réels. Une propriété de non-errance est nécessaire, à savoir la propriété de Rolle dans le cas topologique et la stabilité infinitésimale dans le cas analytique.

On prouve aussi que, sous des hypothèses génériques, la classe analytique d'un déploiement φ ne dépend que des classes analytiques des germes de difféomorphismes en dimension 1 obtenus en localisant le long d'une composante irréductible de l'ensemble des points fixes de φ.

1. Introduction

We classify one dimensional unfoldings of tangent to the identity diffeomorphisms, i.e. elements φ of $\mathrm{Diff}(\mathbb{C}^2, 0)$ of the form $(x, f(x,y))$ with $(\partial f/\partial y)(0,0) = 1$. The

2000 Mathematics Subject Classification. — 37F75, 32H02, 32S65, 32A05.
Key words and phrases. — Analytic classification, topological classification, diffeomorphisms, local dynamics, vector fields.

set of diffeomorphisms of the previous form is denoted by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. We provide a topological classification of the multi-parabolic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ and an analytic classification of the non-degenerate ones (see section 2). Such results are based on the study of the stable structures preserved by the diffeomorphisms whose main tool is the use of real flows (section 3). This part of the paper is a survey of the papers [**10**] (topological classification) and [**12**] (analytic classification).

In section 6 we present a new result. We are interested on the local behavior of global objects. For instance in our setting we are interested on describing the nature of $\varphi = (x, f(x,y)) \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ in a neighborhood of $\gamma \setminus \{(0,0)\}$ for an irreducible component γ of the fixed points set $\mathrm{Fix}(\varphi)$ of φ. Given $(x_0,y_0) \in \gamma \setminus \{(0,0)\}$ we denote by $\varphi_{(x_0,y_0)}$ the germ of $\varphi_{|x=x_0}$ in the neighborhood of $y = y_0$. Suppose γ is parabolic, i.e. $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. Then "part" of the Ecalle-Voronin invariants of $\varphi_{(x,y)}$, where (x,y) belongs to $\gamma \setminus \{(0,0)\}$, can be extended continuously to $x = 0$ in good sectors S in the parameter space. Under the proper hypothesis (see theorem 6.1) we can prove that the analytic class of φ in the neighborhood of $\gamma \setminus \{(0,0)\}$ determines the analytic class of φ in $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. By varying the parameter x we show that the Ecalle Voronin invariants of $\varphi_{(x,y)}$ "turn" with respect to the Ecalle Voronin invariants of φ (see subsection 6.2). Section 6 is a small glimpse of a more detailed work to be published.

2. Notations

We denote by $\mathcal{X}(\mathbb{C}^2,0)$ the group of germs of complex analytic vector fields in a neighborhood of $0 \in \mathbb{C}^2$. The elements of $\mathcal{X}(\mathbb{C}^2,0)$ which are singular at $0 \in \mathbb{C}^2$ can be interpreted as derivations of the maximal ideal of the ring $\mathbb{C}\{x,y\}$. We denote by $\hat{\mathcal{X}}(\mathbb{C}^2,0)$ the group of derivations of the maximal ideal $\hat{\mathfrak{m}}$ of the ring $\mathbb{C}[[x,y]]$. An element $\hat{X} \in \hat{\mathcal{X}}(\mathbb{C}^2,0)$ can be expressed in the more conventional form

$$\hat{X} = \hat{X}(x)\frac{\partial}{\partial x} + \hat{X}(y)\frac{\partial}{\partial y}.$$

Let $\mathrm{Diff}\,(\mathbb{C}^2,0)$ be the group of germs of complex analytic diffeomorphisms in a neighborhood of $0 \in \mathbb{C}^2$. We define $\mathrm{Fix}(\varphi)$ the *fixed points set* of $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$. We denote by $\mathrm{Diff}_1(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ whose elements are tangent to the identity, i.e. given $\varphi \in \mathrm{Diff}\,(\mathbb{C}^2,0)$ then it belongs to $\mathrm{Diff}_1(\mathbb{C}^2,0)$ if $j^1\varphi \equiv Id$. Let $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2,0)$ be the formal completion of $\mathrm{Diff}\,(\mathbb{C}^2,0)$.

We denote by $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ the subgroup of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ of unfoldings of tangent to the identity diffeomorphisms. More precisely an element $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form

$$\varphi(x,y) = (x, f(x,y))$$

where $(\partial f/\partial y)(0,0) = 1$. We say that $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is *non-degenerate* if $y \circ \varphi(0,y) \not\equiv y$. The linear part $j^1\varphi$ of an element φ of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is of the form $(x, y+ax)$ for some $a \in \mathbb{C}$. The linear unipotent isomorphism $(x, y+ax)$ is the exponential of the linear nilpotent vector field $ax\partial/\partial y$. Indeed φ is the exponential of a unique nilpotent formal vector field $\hat{X}$. More precisely φ can be interpreted as an operator $g \to g \circ \varphi$ acting on $\hat{\mathfrak{m}}$. The operator φ is the exponential of an operator $\hat{X}$ acting on $\hat{\mathfrak{m}}$ as a derivation and such that $j^1 \hat{X} = ax\partial/\partial y$. We say that $\hat{X}$ is the *infinitesimal generator* of φ. We denote $\log \varphi = \hat{X}$.

Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$; we say that φ is *multi-parabolic* if $(\partial f/\partial y)_{|\mathrm{Fix}(\varphi)} \equiv 1$. We denote by $\mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ the set of multi-parabolic unfoldings and we call its elements MP-diffeomorphisms.

Given $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ we denote $\varphi \sim_{\mathrm{an}} \eta$ if φ and η are conjugated by some $\sigma \in \mathrm{Diff}(\mathbb{C}^2, 0)$ such that $x \circ \sigma = x$ and $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$. If we replace $\mathrm{Diff}(\mathbb{C}^2, 0)$ with the group of germs of homeomorphisms we obtain the equivalence $\varphi \sim_{\mathrm{top}} \eta$. By replacing $\mathrm{Diff}(\mathbb{C}^2, 0)$ with $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ we obtain $\varphi \sim_{for} \eta$. In the formal setting $\sigma_{|\mathrm{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$ means that $y \circ \sigma - y$ belongs to the ideal of $\overline{\mathrm{Fix}(\varphi) \setminus \{x = 0\}}$ in the ring $\mathbb{C}[[x, y]]$ (supposed $x \circ \sigma = x$).

3. Real flows

Our goal is describing the dynamics of $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Instead of trying a direct approach we consider a continuous dynamical system similar to φ. More precisely we choose a germ of holomorphic vector field $X_\varphi = g(x, y)\partial/\partial y$ satisfying the *proximity condition*, namely

$$y \circ \varphi - y \circ \exp(X_\varphi) \in (y \circ \varphi - y)^2.$$

Such a choice of X_φ is possible **[11]** even if not unique. Supposed $\varphi = \exp(X_\varphi)$ then the orbits of φ would be contained in the trajectories of the real flow $\Re(X_\varphi)$ of X_φ. Anyway $\Re(X_\varphi)$ provides a continuous "model" for the iterates of φ.

The proximity condition implies that $\mathrm{Fix}(\varphi) = SingX_\varphi$ and that φ is formally conjugated to $\exp(X_\varphi)$ **[11]**.

Definition 3.1. — *We say that φ satisfies the ϵ-property if there exist open neighborhoods $V \subset W$ of $(0,0)$ such that for all $(x, y) \in V$ and $j \in \mathbb{Z}$ satisfying*

$$\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\exp(X_\varphi)^k(x, y)\} \subset V$$

then $\cup_{k \in [\min(j,0), \max(j,0)] \cap \mathbb{Z}} \{\varphi^k(x, y)\} \subset W$ and

$$\varphi^j(x, y) \in \exp(B(0, \epsilon)X_\varphi)(\exp(X_\varphi)^j(x, y)).$$

We say that φ satisfies the stability property if it satisfies the ϵ-property for any $\epsilon > 0$ small enough.

Theorem 3.1 **(ϵ-theorem or stability theorem).** — **[10]** *Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Then φ satisfies the stability property for every choice of X_φ satisfying the proximity property.*

The ϵ-theorem implies that in the multi-parabolic case the orbits of φ and $\exp(X_\varphi)$ remain close independently of the number of iterations. Moreover, the dynamics of φ is roughly speaking the dynamics of $\exp(X_\varphi)$ plus some small "noise". Analyzing the noise is not trivial since not every MP-diffeomorphism is topologically conjugated to the exponential of a holomorphic vector field as we will see later on. The stability property is crucial [10] to provide a complete system of topological invariants for the MP-diffeomorphisms. The situation in the general case is different since

Theorem 3.2. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0) \setminus \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$. Then φ holds the ϵ-property for some choice of X_φ if and only if $\log \varphi \in \mathcal{X}(\mathbb{C}^2,0)$.*

Proof. — Denote $\varphi(x,y) = (x, f(x,y))$. Since $\varphi \notin \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ there exists an irreducible component γ_0 of $\mathrm{Fix}(\varphi)$ such that $(\partial(y \circ \varphi)/\partial y)_{|\gamma_0} \not\equiv 1$. Up to replace φ with $(x, f(x^k,y))$ and γ_0 with one of the irreducible components of $(x^k,y)^{-1}(\gamma_0)$ for some k in $\mathbb{N}$ we can choose γ_0 of the form $y = h(x)$. Up to the change of coordinates $(x, y - h(x))$ we can suppose $\gamma_0 \equiv \{y = 0\}$. Denote $L(w) = (\partial(y \circ \varphi)/\partial y)(w,0)$.

Let $w \in \mathbb{C}$; denote φ_w the germ of $\varphi_{|x=w}$ in the neighborhood of $(w,0)$. Suppose $L(w) \in \mathbb{S}^1 \setminus \{1\}$, then the ϵ-property implies that the sequence $\{\varphi_w^j\}$ is normal in some neighborhood of $(w,0)$. Therefore φ_w is analytically linearizable for any $w \in L^{-1}(\mathbb{S}^1 \setminus \{1\})$ and then for any w in a pointed neighborhood of 0 in $\mathbb{C}$.

The infinitesimal generator $\log \varphi$ is of the form $\hat{f}(x,y)\partial/\partial y$ for some $\hat{f} \in \mathbb{C}[[x,y]]$. We have $\hat{f} = \sum_{j \geq 0} f_j(x) y^j$. Indeed $\hat{f}$ is transversally formal along γ_0, i.e. there exists a neighborhood $V \subset \mathbb{C}$ of 0 such that $f_j \in \mathcal{O}(V)$ for any $j \geq 0$. This is a consequence of φ_w being linearizable for any $w \in L^{-1}(e^{2\pi i \mathbb{Q}} \setminus \{1\})$ [11].

Consider a path $\eta \subset V \setminus \{0\}$ turning once around 0 and transversal to $L^{-1}(\mathbb{S}^1)$. Moreover we can suppose that whenever $w \in \eta \cap L^{-1}(\mathbb{S}^1)$ then $L(w)$ is a Bruno number. Denote by $\sigma(w,y)$ the element of $\mathrm{Diff}_1(\mathbb{C},0)$ linearizing φ_w. By the choice of η then σ is continuous in $\eta \times W_0$ for some neighborhood W_0 of $0 \in \mathbb{C}$. As a consequence $\hat{f}$ is a continuous function in $\eta \times W$ for some neighborhood W of $0 \in \mathbb{C}$. Then there exists $C \in \mathbb{R}^+$ such that

$$|f_j(w)| \leq C^j \quad \text{for all } (w,j) \in \eta \times \mathbb{N}.$$

The modulus maximum principle implies that $\log \varphi \in \mathcal{X}(\mathbb{C}^2,0)$. □

The previous theorem implies that the dynamics of a generic $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ is unstable. Then the stable dynamics of a vector field is not a good model of the dynamics of φ. As a consequence the study of real flows of holomorphic vector fields is no good to classify topologically generic elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.

In spite of the previous discussion real flows are useful to provide a complete system of analytic invariants for non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$. Why this? This phenomenon is linked to the rigidity of analytic structures. Roughly speaking by doing cuts in the domain of definition of φ we can find subsets S such that the dynamics of $\varphi_{|S}$ is stable and close to the dynamics of $\exp(X_\varphi)_{|S}$. Moreover, the analytic class of φ is determined by the analytic classes of $\varphi_{|S}$ for good choices of S (here the rigidity in the analytic world plays a role). The cuts in the domain of definition allows us to avoid the instability related to resonances, small divisors and renormalized return maps. This point of view is developed in section 5 to obtain the theorem of analytic classification.

Theorem 3.3. — *Consider non-degenerate elements* φ, η *of* $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. *Suppose that* $\text{Fix}(\varphi) = \text{Fix}(\eta)$. *Then* $\varphi \sim_{\text{an}} \eta$ *if and only if there exists* $r \in \mathbb{R}^+$ *such that for any* w *in a pointed neighborhood of* 0 *the restrictions* $\varphi_{|x=w}$ *and* $\eta_{|x=w}$ *are conjugated by an injective holomorphic mapping defined in* $B(0, r)$ *and fixing the points in* $\text{Fix}(\varphi) \cap \{x = w\}$.

In the previous theorem the mappings conjugating the restrictions of φ and η to $x = w$ do not depend a priori continuously on w. Even so we can obtain an analytic conjugation because the spaces of orbits associated to the cuts $\varphi_{|S}$ are rigid (see section 5). The theorem is representative of a more general property: a non-degenerate element φ of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ inherits the rigidity properties associated to $\varphi_{|x=0}$ and its space of orbits.

Resuming, we can use real flows of holomorphic vector fields to catch the stable structures contained in the dynamics of $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. We will use such information to classify topologically stable elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$ and to classify analytically non-degenerate elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$.

$$\text{topological stability} \rightarrow \text{topological classification}$$

$$\text{analytic substability} + \text{rigidity of analytic structures} \rightarrow \text{analytic classification.}$$

4. Topological classification of MP-diffeomorphisms

Consider $\varphi \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$. Because of the stability property its dynamics by iteration is close to the dynamics of $\exp(X_\varphi)$. The latter one is embedded in the dynamics of the vector field $\Re(X_\varphi)$. The subsections 4.1, 4.2 and 4.4 are intended to describe briefly the dynamics of $\Re(X_\varphi)_{|x=w}$ for w in a neighborhood of $0 \in \mathbb{C}$.

In subsection 4.5 we introduce the tools to describe the instability phenomena attached to $\Re(X_\varphi)$, we also explain that the dynamics of $\Re(X_\varphi)_{|x=w}$ is simple for generic values of w. That could make us think that the dynamics of $\Re(X_\varphi)_{|x=w}$ does not depend on w. In subsection 4.6 we show that this is not the case if $\sharp(\text{Fix}(\varphi) \cap \{x = w\}) > 1$ for $w \neq 0$.

The subsection 4.7 is intended to describe the properties of the sets Z corresponding to unstable parameters. The limit of the dynamics of $\Re(X_\varphi)_{|x=w}$ when $w \in Z$ and $w \to 0$ is more complicated than the dynamics of $\Re(X_\varphi)_{|x=0}$. In order to do this, given an analytic curve $\gamma \subset Z$ and a sequence of points $\gamma \times \mathbb{C} \ni (w_n, y_n) \to (0, y_0) \neq (0, 0)$, we study the limits of the trajectories of $\Re(X_\varphi)$ passing through points (w_n, y_n) when $n \to \infty$. There are choices of γ and (w_n, y_n) such that the limit is bigger than the closure of the trajectory of $\Re(X_\varphi)$ passing through $(0, y_0)$. In subsection 4.8 we describe the evolution of the limits of trajectories with respect to γ. The existence of big limits (or long trajectories) is invariant by topological conjugation, their study provides the first topological invariants both for vector fields (subsection 4.9) and diffeomorphisms (subsection 4.10). These invariants are of formal type, they only depend on the formal class of the vector field or diffeomorphism. A second type of invariant, namely the analytic class of $\varphi_{|x=0}$, can be obtained by studying the evolution of the big limits with respect to curves γ. Finally the theorem of topological classification is presented in subsection 4.12.

4.1. Flower Type Vector Fields. — Consider the real flow $\Re(\xi)$ of a complex analytic vector field ξ defined in an open subset V of $\mathbb{C}$. Let $P \in V$ be a singular point of $\xi = a(y)\partial/\partial y$; the point P is parabolic if $a'(P) = 0$. Throughout this subsection we will consider a complex analytic vector field ξ defined in a neighborhood of $\overline{\mathbb{D}}$. By definition the vector field $\Re(\xi)$ is of *flower type* if $Sing\xi \cap \partial\mathbb{D} = \varnothing$ and all the singularities are parabolic.

We define $\Gamma_\xi[Q]$ the trajectory of $\Re(\xi)$ in $\mathbb{D}$ passing through Q. We also define the positive and negative trajectories $\Gamma_{\xi,+}[Q]$ and $\Gamma_{\xi,-}[Q]$ obtained by restraining $\Gamma_\xi[Q]$ to non-negative and non-positive times respectively. The trajectory $\Gamma_{\xi,+}[Q]$ is defined for times in some interval $[0,a)$ for some $a \in \mathbb{R}^+ \cup \{\infty\}$. Whenever $a < \infty$ we have $\Gamma_\xi[Q](a) \in \partial\mathbb{D}$; we denote $\omega_\xi(Q) = \infty$. Otherwise we denote by $\omega_\xi(Q)$ the omega limit of $\Gamma_\xi[Q]$. We can define the mapping α_ξ in an analogous way.

Remark 4.1. — *If $\omega_\xi(Q)$ contains a singular point $P \in Sing\xi$ then $\omega_\xi(Q) = \{P\}$ since singular points are parabolic.*

4.2. The dynamical Rolle property. — We say that a flower type vector field $\Re(\xi)$ satisfies the *dynamical Rolle property* if there is no connected transversal I such that $\Gamma_\xi[Q]$ cuts I for two different values of time. Our definition implies that any vector field having cycles can not hold the Rolle condition. Anyway, the definition coincides with the usual one if all the cycles are isolated. We also call *no return property* the dynamical Rolle property.

Proposition 4.1. — *Let $\Re(\xi)$ be a flower type vector field. Then $\Re(\xi)$ satisfies the dynamical Rolle property.*

Proof. — Suppose that $\Re(\xi)$ does not hold the dynamical Rolle property. There exist a trajectory $\gamma : [0,t] \to \mathbb{D}$ of $\Re(\xi)$ and a connected transversal $T \subset \mathbb{D}$ such that $\gamma^{-1}(T) = \{0,t\}$. We can suppose that T is closed and $\partial T = \{\gamma(0), \gamma(t)\}$.

Consider the bounded connected component B of $\mathbb{C} \setminus (\gamma[0,t] \cup T)$. We can suppose that B is invariant by the positive iteration of $\Re(\xi)$ by changing ξ with $-\xi$ if necessary. Choose a point $Q \in B$, the set $\omega_\xi(Q)$ is either a singular point or a cycle. In the latter case the cycle is limiting a bounded domain containing a singular point of ξ. Thus we obtain $Sing\xi \cap B \neq \varnothing$.

The mapping $\exp(s\xi)_{|B} : B \to B$ is well-defined for any $s \in \mathbb{R}^+$. Moreover $\exp(s\xi)_{|B}$ is tangent to the identity at the points in $Sing\xi \cap B$ by the flower type character. By Cartan's theorem we have $\exp(s\xi)_{|B} \equiv Id$ for any $s \in \mathbb{R}^+$. We obtain a contradiction since B is not contained in $Sing\xi$. □

4.3. Multi-parabolic vector fields

Definition 4.1. — *We denote by $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ the set of vector fields of the form $g(x,y)\partial/\partial y \in \mathcal{X}(\mathbb{C}^2,0)$ with $g(0,0) = (\partial g/\partial y)(0,0) = 0$. We say that an element $g(x,y)\partial/\partial y$ of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ is non-degenerate if $g(0,y) \not\equiv 0$.*

Definition 4.2. — *We say that $X = g(x,y)\partial/\partial y \in \mathcal{X}(\mathbb{C}^2,0)$ is a multi-parabolic vector field if $g(0,0) = 0$ and $(\partial g/\partial y)_{|SingX} \equiv 0$. We denote by $\mathcal{X}_{MP}(\mathbb{C}^2,0)$ the set of multi-parabolic vector fields.*

Definition 4.3. — *A vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ is of the form $x^{m(X)}h(x,y)\partial/\partial y$ with $h(0,y) \not\equiv 0$. The radical ideal $\sqrt{(h)}$ is generated by some $k \in \mathbb{C}\{x,y\}$; we denote by $N(X)$ the order $\nu(k(0,y))$ at $y=0$. Indeed $N(X)$ is the cardinal of $\{x = w\} \cap SingX$ for $w \neq 0$. We define $\nu(X)$ as $\nu(h(0,y)) - 1$. We can define $m(\varphi) = m(X_\varphi)$, $N(\varphi) = N(X_\varphi)$ and $\nu(\varphi) = \nu(X_\varphi)$.*

Clearly a vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$ is nilpotent. Moreover given a nilpotent germ of vector field $Y \in \mathcal{X}(\mathbb{C}^2,0)$ we have that $\exp(Y)$ belongs to $\mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ (resp. $\mathrm{Diff}_{p1}(\mathbb{C}^2,0)$) if and only if $Y \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$ (resp. $Y \in \mathcal{X}_{p1}(\mathbb{C}^2,0)$).

Remark 4.2. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. The multi-parabolic character excludes unstable behavior. Indeed $\Re(X)_{|x=w}$ enjoys the no-return property for any w in a neighborhood of 0 (prop. 4.1).*

In the case $N=1$, $m=0$ both multi-parabolic vector fields and diffeomorphisms are topological products. The problem of topological classification can be reduced to the setting of tangent to the identity diffeomorphisms in one variable.

Proposition 4.2. — *Let $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2,0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$. Suppose that $N(\varphi) = 1$ and $m(\varphi) = 0$. Then we have $\varphi \sim_{\mathrm{top}} \eta$.*

The case $N=1$, $m>0$ is a sort of singular "reparametrization" of the previous one since $X = x^{m(X)}Y$ with $m(Y) = 0$ and x is constant in $x = w$. We can use the one variable techniques to provide a complete system of topological invariants. Analogously the case $N = 0$ (and then $m > 0$) is also simple since it is a reparametrization of a regular case. Indeed proposition 4.2 is also valid for $N = 0$. From now on we focus on the most interesting case, namely $N > 1$. We will suppose the condition of non-degeneracy $m = 0$, it is not necessary but the presentation is simpler.

4.4. The graph. — Consider $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. Let us fix a domain of definition $U_\epsilon = B(0,\delta) \times B(0,\epsilon)$ for some $\epsilon, \delta > 0$ small enough. We denote $\Gamma_X^\epsilon[Q]$ the trajectory of $\Re(X)$ in U_ϵ passing through Q. Analogously to subsection 4.2 we can define α_X^ϵ and ω_X^ϵ.

We can associate an oriented graph $\mathcal{G}_{X,w}^\epsilon$ to $\Re(X)_{|\{w\}\times B(0,\epsilon)}$. The vertices are the elements of $SingX \cap \{x = w\}$. Consider the space $Sp(w)$ of trajectories γ of $\Re(X)_{|\{w\}\times B(0,\epsilon)}$ such that $(\alpha,\omega)_X^\epsilon(\gamma) \in SingX \times SingX$. The edges of $\mathcal{G}_{X,w}^\epsilon$ are the

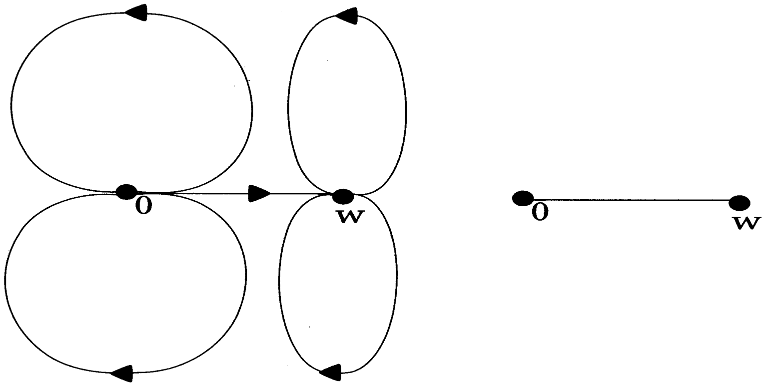

FIGURE 1. $\mathcal{G}_{X,w}$ and $\mathcal{N}G_{X,w}$ for $X = y^2(y-x)^2\partial/\partial y$ and $w \in \mathbb{R}^+$

isotopy classes of those trajectories in $Sp(w)$. We define the unoriented graph $\mathcal{N}G^{\epsilon}_{X,w}$ obtained from $\mathcal{G}^{\epsilon}_{X,w}$ by removing the reflexive edges and the orientation.

Proposition 4.3. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. Then $\mathcal{N}G^{\epsilon}_{X,w}$ has no cycles.*

The proposition shares the spirit of the no-return property and they have analogous proofs. The lack of connectedness of $\mathcal{N}G^{\epsilon}_{X,w}$ is related to the existence of "long" trajectories.

Proposition 4.4. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. Then $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty,\infty) \cap \{x = w\} = \emptyset$ implies that $\mathcal{N}G^{\epsilon}_{X,w}$ is connected.*

Proof. — Let $G_1, \ldots, G_l$ be the set of connected components of $\mathcal{N}G^{\epsilon}_{X,w}$. Denote by S_j the set of vertexes of G_j for any $j \in \{1,\ldots,l\}$. We define

$$F_j = ((\alpha^{\epsilon}_X)^{-1}(S_j) \cup (\omega^{\epsilon}_X)^{-1}(S_j)) \cap ((\{w\} \times B(0,\epsilon)) \setminus SingX) \ \ \forall j \in \{1,\ldots,l\}.$$

By the open character of parabolic fixed points we obtain that F_j is an open set for $j \in \{1,\ldots,l\}$. Moreover, we have $F_j \cap F_k = \emptyset$ for $j \neq k$ since G_j and G_k are different connected components of $\mathcal{N}G^{\epsilon}_{X,w}$. Then $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty,\infty) \cap \{x = w\} = \emptyset$ implies

$$(\{w\} \times B(0,\epsilon)) \setminus SingX = \cup_{j=1}^{l} F_j.$$

The set $(\{w\} \times B(0,\epsilon)) \setminus SingX$ is connected, thus we get $l = 1$. □

Remark 4.3. — *It can be proved that $(\alpha^{\epsilon}_X, \omega^{\epsilon}_X)^{-1}(\infty,\infty) \cap \{x = w\} = \emptyset$ and the connectedness of $\mathcal{N}G^{\epsilon}_{X,w}$ are equivalent for any w in some neighborhood of* 0.

4.5. Quantitative analysis of trajectories. — Let $X = g(x,y)\partial/\partial y \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$.

Definition 4.4. — *Let $\mathrm{Res}_X(x_0,y_0)$ be the residue of $dy/g(x_0,y)$ at $y = y_0$.*

Suppose that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = w\} \neq \emptyset$ for infinitely many points w in every neighborhood of 0. Then there exist a sequence $\{w_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} w_n = 0$ and trajectories $\gamma_n : [0,t_n] \to \{w_n\} \times B(0,\epsilon)$ of $\Re(X)$ such that

$$\gamma_n[0,t_n] \cap (\{w_n\} \times \partial B(0,\epsilon)) = \{\gamma_n(0), \gamma_n(t_n)\}.$$

Let $\eta_n : [0,a_n] \to \{w_n\} \times \partial B(0,\epsilon)$ be the arc going from $\gamma_n(0)$ to $\gamma_n(t_n)$ in counter clock-wise sense. We denote by $C_n(w_n)$ the connected component of the set

$$(\{w_n\} \times B(0,\epsilon)) \setminus \gamma_n(0,t_n)$$

such that $\eta_n[0,a_n] \subset \overline{C_n(w)}$. We define $E_n(w_n) = C_n(w_n) \cap SingX$. Given a set $E_n(w_n)$ we can define $E_n(w) \subset SingX \cap \{x = w\}$ for any w in a neighborhood of 0 by continuous extension of $E_n(w_n)$. Any set build in this way is called a *continuous set of singular points.*

By taking a subsequence we can suppose that the limits $\lim_{n\to\infty} \gamma_n(0)$ and $\lim_{n\to\infty} \gamma_n(t_n)$ exist. We can also suppose that there exists a continuous set of singular points E such that $E \equiv E_n$ for any $n \in \mathbb{N}$. Consider a holomorphic function ψ_0 defined in a neighborhood of $\lim_{n\to\infty} \gamma_n(0)$ and such that $X(\psi_0) \equiv 1$. Denote by ψ_∂ the analytic continuation of ψ_0 along the arc going from $\lim_{n\to\infty} \gamma_n(0)$ to $\lim_{n\to\infty} \gamma_n(t_n)$ in counter clock-wise sense. Finally we define $\psi_I(w_n,y)$ the analytic continuation of $(\psi_0)_{|x=w_n}$ along $\gamma_n[0,t_n]$. We have $\psi_I(\gamma_n(t_n)) - \psi_0(\gamma_n(0)) = t_n$ for any $n \in \mathbb{N}$ by definition.

The theorem of the residues implies

$$\psi_I = \psi_\partial - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P).$$

Therefore we obtain

$$\psi_\partial(\gamma_n(t_n)) - \psi_0(\gamma_n(0)) - 2\pi i \sum_{P\in E(w_n)} \mathrm{Res}_X(P) = t_n.$$

The quantity $\psi_\partial(\gamma_n(t_n)) - \psi_0(\gamma_n(0))$ is uniformly bounded by a constant depending only on X. The function $x \to 2\pi i \sum_{P\in E(x^k)} \mathrm{Res}_X(P)$ is meromorphic for some $k \in \mathbb{N}$ [**11**]. In spite of that, it is not a holomorphic function since otherwise the sequence $\{t_n\}$ is bounded and this implies $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = 0\} \neq \emptyset$.

Proposition 4.5. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2,0)$. The graph $\mathcal{N}G_{X,w}^\epsilon$ is connected for generic values of w.*

Proof. — A point w such that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty,\infty) \cap \{x = w\} \neq \emptyset$ is contained in some curve

$$\beta_{E,C} \equiv \{-2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) \in \mathbb{R}^+ + iC\}$$

for some continuous set E of singular points and some $C \in \mathbb{R}$. Moreover there exists a constant $D \in \mathbb{R}^+$ depending only on X and such that $C \in [-D,D]$. Denote $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{C}$ the mapping defined by $\pi(r,\lambda) = r\lambda$. The set of tangent directions $\overline{\pi^{-1}(\beta_{E,C} \setminus \{0\})} \cap (\{0\} \times \mathbb{S}^1)$ of $\beta_{E,C}$ at 0 is finite and it does not depend on

C. There are finitely many choices of continuous sets of singular points. Thus there exists a finite set $D_X \subset \mathbb{S}^1$ such that for any closed arc $ac \subset \mathbb{S}^1 \setminus D_X$ there exists $b > 0$ such that $(\alpha_X^\epsilon, \omega_X^\epsilon)^{-1}(\infty, \infty) \cap \{x = w\} = \varnothing$ for any $w \in (0, b)ac$. □

4.6. Splitting of the dynamics. — The dynamics of $\Re(X)_{|\{w\} \times B(0,\epsilon)}$ is not constant with respect to w if $N(X) > 1$ (see def. 4.3). Otherwise the graph $\mathcal{N}G_{X,w}^\epsilon$ is connected for any w in a neighborhood of 0 and it depends continuously on w. The next proposition states that this is not the case; indeed the graph can be disassembled.

Proposition 4.6. — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $N(X) > 1$. There are no permanent edges in $\mathcal{N}G_{X,w}^\epsilon$.*

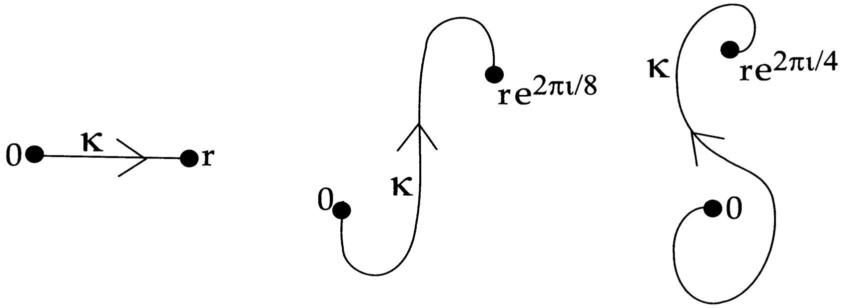

FIGURE 2. $X = y^2(y-x)^2\partial/\partial y$. Parameters $\theta = 0, 1/8, 1/4$

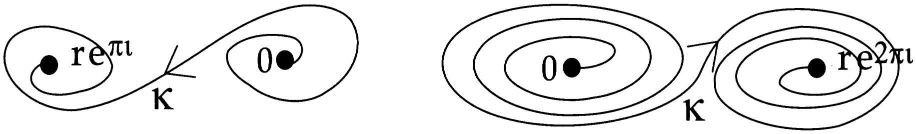

FIGURE 3. Parameters $\theta = 1/2$ and $\theta = 1$

Proposition 4.6 is proved by analyzing the dependence of the edge with respect to the parameter [**10**]. Let us consider the example $X = y^2(y-x)^2\partial/\partial y$. The edge $0 \to w$ belongs to $\mathcal{G}_{X,w}^\epsilon$ for any $w \in \mathbb{R}^+$. Suppose that $0 \to w$ is a permanent edge. Every trajectory $\gamma_w(-\infty, \infty)$ of $\Re(X)_{|\{w\} \times B(0,\epsilon)}$ representing the edge $0 \to w$ has well-defined tangents $\gamma_w(-\infty)$ at 0 and $\gamma_w(\infty)$ at w. Moreover the homotopy class $\kappa(w)$ of $\gamma_w[-\infty, \infty]$ in the space obtained by doing the real blow-up of $\mathbb{C}$ at the points 0 and w does not depend on the choice of γ_w. Fix $r > 0$ small enough and let us consider $x = re^{2\pi i\theta}$ when θ goes from 0 to 1. The point $\gamma_{re^{2\pi i\theta}}(-\infty)$

is the direction given by $e^{-4\pi i\theta}\mathbb{R}^+$ at $y=0$. The point $\gamma_{re^{2\pi i\theta}}(\infty)$ is the direction given by $re^{2\pi i\theta}+e^{\pi i-4\pi i\theta}\mathbb{R}^+$ at $y=re^{2\pi i\theta}$. Then both $\gamma_{re^{2\pi i\theta}}(-\infty)$ and $\gamma_{re^{2\pi i\theta}}(\infty)$ are turning in clock-wise sense whereas the singular point $y=re^{2\pi i\theta}$ turns around $y=0$ in counter clock-wise sense (see figures (2) and (3)). This phenomenon, i.e. tangents and singular points turning in opposite senses, forces the absurd inequality $\kappa(e^{2\pi i}r)\neq\kappa(r)$.

4.7. Long Trajectories. — The next two subsections are devoted to make clear that the limit of the dynamics of $\Re(X)_{|x=w}$ when $w\to 0$ is more complex than the dynamics of the limit $\Re(X)_{|x=0}$.

We already saw that the points w in $(\alpha_X^\epsilon,\omega_X^\epsilon)^{-1}(\infty,\infty)\cap\{x=w\}\neq\emptyset$ are in curves of the form $-2\pi i\sum_{P\in E(x)}\text{Res}_X(P)\in\mathbb{R}^+ + iC$ for some continuous set E of singular points and some $C\in\mathbb{R}$ (subsection 4.5). Then it is interesting to consider limits of trajectories along branches of analytic curves contained in the parameter space.

Let $X\in\mathcal{X}_{MP}(\mathbb{C}^2,0)$. Consider a branch of analytic curve β and a point y_0 in $\overline{B(0,\epsilon)}\setminus\{0\}$ such that $\omega_X^\epsilon(0,y_0)=(0,0)$. We are interested on describing the limit of $\Gamma_{X,+}^\epsilon[w,y_0]$ when $w\in\beta$ and $w\to 0$. Let $y_1\in\overline{B(0,\epsilon)}\setminus\{0\}$ be a point satisfying that there exists a mapping $y_1(w):\beta\to\mathbb{C}$ such that

- $(0,y_1)=\lim_{w\in\beta,w\to 0}(w,y_1(w))$ and $(0,y_1)\notin\cap_{\eta>0}\Gamma_{X,+}^{\epsilon+\eta}[0,y_0]$.
- For any $\eta>0$ there exists $\upsilon(\eta)\in\mathbb{R}^+$ such that $(w,y_1(w))\in\Gamma_{X,+}^{\epsilon+\eta}[w,y_0]$ for any $w\in B(0,\upsilon(\eta))\cap\beta$.

The set of points satisfying the previous conditions will be denoted by $L_{\beta,y_0}^{+,\epsilon}(X)$ (or just $L_{\beta,y_0}^{+,\epsilon}$ if X is implicit); it is the positive Long Limit (or just L-limit) associated to y_0, ϵ and β. We can define $L_{\beta,y_0}^{-,\epsilon}$ by replacing in the definition the positive trajectories with the negative ones.

Remark 4.4. — *We have $L_{\beta,y_0}^{+,\epsilon}(X)=L_{\beta,y_1}^{+,\epsilon}(X)$ if $(0,y_0)$ and $(0,y_1)$ are in the same trajectory of $\Re(X)_{|\{0\}\times\overline{B(0,\epsilon)}}$. Every connected component of a L-limit $L_{\beta,y_0}^{+,\epsilon}(X)$ is a trajectory of $\Re(X)_{|\{0\}\times\overline{B(0,\epsilon)}}$.*

Proposition 4.7 **(Existence of the L-limit).** — *Let $X\in\mathcal{X}_{MP}(\mathbb{C}^2,0)$. Suppose $N(X)>1$ (def. 4.3). Consider $y_0\in B(0,\epsilon)\setminus\{0\}$ such that $\omega_X^\epsilon(0,y_0)=(0,0)$. Then there exists a branch of analytic curve β such that $L_{\beta,y_0}^{+,\epsilon}(X)\neq\emptyset$.*

The proof of the previous proposition is based on the splitting of the dynamics phenomenon. Analogously as in the proposition 4.6 it can be proved that the trajectory $\Gamma_{X,+}^\epsilon[x,y_0]$ does not depend continuously on x. In particular the set $S=\{x\in B(0,\delta):\omega_X^\epsilon(x,y_0)\in SingX\}$ is not a neighborhood of the origin. The quantitative analysis in subsection 4.5 can be used to prove that ∂S is a union of branches of analytic curves. It is enough to choose $\beta\subset\partial S$.

Lemma 4.1. — *The number of connected components of $L_{\beta,y_0}^{+,\epsilon}(X)$ is finite.*

The lemma is a consequence of the dynamical Rolle property. Indeed a connected transversal to $\Re(X)_{|\{0\}\times B(0,\epsilon)}$ can not intersect two connected components of $L^{+,\epsilon}_{\beta,y_0}(X)$.

The Long limits admit a quantitative analysis. Given $y_1 \in L^{+,\epsilon}_{\beta,y_0}(X)$ there exist a continuous section $(x, y_1(x))$ for $x \in \beta \cup \{0\}$, a continuous set E_1 of singular points and a continuous function $T_1 : \beta \to \mathbb{R}^+$ such that

- $y_1(0) = y_1$ and $\lim_{x\in\beta,x\to 0} T_1(x) = \infty$.
- $(x, y_1(x)) = \exp(T_1(x)X)(x, y_0)$
- $T_1(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P\in E_1(x)} \mathrm{Res}_X(P)$ for any $x \in \beta$.

Last property is obtained as in subsection 4.5 for trajectories of $\Re(X)_{\{w\}\times B(0,\epsilon)}$ dividing $\{w\} \times B(0,\epsilon)$. Indeed Long Trajectories can be interpreted as dividing trajectories just by reducing the domain of definition.

The connected components of $L^{+,\epsilon}_{\beta,y_0}$ are ordered by the time of the flow. Let $(0, y_1), (0, y_2) \in L^{+,\epsilon}_{\beta,y_0}$. Denote by γ_j the trajectory of $\Re(X)_{|\{0\}\times \overline{B(0,\epsilon)}}$ containing $(0, y_j)$ for $j \in \{1, 2\}$. Consider data $(x, y_j(x))$, E_j and T_j as above. By the non-oscillating property for analytic curves we have three possibilities:

- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) = \infty$. We define $\gamma_1 < \gamma_2$.
- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) = -\infty$. We define $\gamma_1 > \gamma_2$.
- $\lim_{x\in\beta,x\to 0} T_2(x) - T_1(x) \in \mathbb{R}$. We have $\gamma_1 = \gamma_2$.

4.8. Evolution of the Long Trajectories. — In this subsection we study the dependence of $L^{+,\epsilon}_{\eta,y_0}(X)$ with respect to η. Moreover we introduce the openness principle. Namely, if there exists a Long Trajectory joining y_0 and y_1, i.e. $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$, then there exists a Long Trajectory joining y_0 and y_2 for any y_2 in a neighborhood of y_1.

Suppose $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$. There exist a continuous section $(x, y_1(x))$ for x in $\eta \cup \{0\}$, a continuous set E of singular points and a function $T : \eta \to \mathbb{R}^+$ such that

$$T(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P)$$

for any $x \in \eta$ (see subsection 4.7). Let $v \in \mathbb{R}$; denote by $\eta(v)$ the curve given by

$$iv + \psi_\partial(0, y_1) - \psi_0(0, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) \in \mathbb{R}^+$$

and whose tangent at $x = 0$ coincides with the tangent of η. The point y_1 belongs to $L^{+,\epsilon}_{\eta(0),y_0}(X)$. Moreover, we have

$$\lim_{x\in\eta(v),x\to 0} \psi_\partial \circ \exp(ivX)(0, y_1) - \psi_0(0, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P) = \infty + i0.$$

The previous equality is key to prove the openness principle.

***Proposition 4.8* (Openness principle).** — *Let $X \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ and $y_1 \in L^{+,\epsilon}_{\eta,y_0}(X)$. Then $\exp((\rho + iv)X)(0, y_1)$ belongs to $L^{+,\epsilon(\rho,v)}_{\eta(v),y_0}(X)$ for all $\rho, v \in \mathbb{R}$ in a small neighborhood of 0 and some $\epsilon(\rho, v) > 0$ such that $\lim_{(\rho,v)\to 0} \epsilon(\rho, v) = \epsilon$.*

Let us remind the reader that we are studying multi-parabolic vector fields as models of MP-diffeomorphisms. Therefore we are interested on the limit of the discrete trajectories $\{\exp(X)^n(x, y_0)\}_{n\in\mathbb{N}}$ when $x \to 0$. Denote

$$F_{\rho,v}(x) = \rho + iv + \psi_\partial(0, y_1) - \psi_0(0, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_X(P).$$

Consider $\rho, v \in \mathbb{R}$ in a small neighborhood of 0. We have

$$\exp((\rho + iv)X)(0, y_1) = \lim_{n\to\infty} \exp(F_{\rho,v}(x_n^{\rho,v})X)(x_n^{\rho,v}, y_0)$$

where $\{x_n^{\rho,v}\}$ is a sequence of points in $\eta(v) \cap F_{\rho,v}^{-1}(\mathbb{N})$ with $\lim_{n\to\infty} x_n^{\rho,v} = 0$. Thus the orbits $\{\exp(X)^n(x, y_0)\}$ accumulate on $\Gamma_X^\epsilon[\exp(ivX)(0, y_1)]$ when $x \in \eta(v)$ and $x \to 0$.

Resuming, the limit of the orbit $\{\exp(X)^n(x, y_0)\}_{n\in\mathbb{N}}$ when $x \in \eta$, $L^{+,\epsilon}_{\eta,y_0}(X) = \varnothing$ and $x \to 0$ is the discrete orbit $\{\exp(X)^n(0, y_0)\}_{n\in\mathbb{N}}$. Supposed $y_1 \in L^{+,\epsilon}_{\eta(0),y_0}(X)$ we have that $\{\exp(X)^n(x, y_0)\}_{n\in\mathbb{N}}$ generates the real flow $\Re(X)$ of X through $(0, y_1)$ when $x \in \eta(0)$ and $x \to 0$. Moreover $\{\exp(X)^n(x, y_0)\}_{n\in\mathbb{N}}$ generates the complex flow of X through $(0, y_1)$ when $x \in \cup_{v\in\mathbb{R}}\eta(v)$ and $x \to 0$ by the openness principle.

4.9. Formal type topological invariants. — In this subsection we introduce topological invariants obtained by analyzing how much time a multi-parabolic vector field spends in travelling along a Long Trajectory.

Consider X_1 and X_2 in $\mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $SingX_1 = SingX_2$. Suppose that $\exp(X_1)\sim_{\mathrm{top}}\exp(X_2)$ by a germ of homeomorphism σ. We have

$$\sigma(B(0, \delta) \times B(0, \epsilon_1)) \subset B(0, \delta) \times B(0, \epsilon_2)$$

for some $\epsilon_1, \epsilon_2 > 0$ small enough. Given $y_1 \in L^{+,\epsilon_1}_{\beta,y_0}(X_1)$ consider a continuous section $y_1(x) : \beta \cup \{0\} \to \mathbb{C}$, a continuous set of singular points E and a function $T : \beta \to \mathbb{R}$ such that

$$T(x) = \psi_{\partial,1}(x, y_1(x)) - \psi_{0,1}(x, y_0) - 2\pi i \sum_{P\in E(x)} \mathrm{Res}_{X_1}(P)$$

for any $x \in \beta$ (see subsection 4.7). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $\beta\cap T^{-1}(\mathbb{N})$ such that $\lim_{n\to\infty} x_n = 0$. Since $\exp(X_1)$ is topologically conjugated to $\exp(X_2)$ we obtain

$$T(x_n) = \psi_{\partial,2} \circ \sigma(x_n, y_1(x_n)) - \psi_{0,2}(x_n, y_0) - 2\pi i \sum_{P\in E(x_n)} \mathrm{Res}_{X_2}(P)$$

for any $n \in \mathbb{N}$. Denote $G(x) = \sum_{P\in E(x)} \mathrm{Res}_{X_1}(P) - \sum_{P\in E(x)} \mathrm{Res}_{X_2}(P)$. Suppose for simplicity that X_1 is non-degenerate. Hence the previous formula implies that the sequence $G(x_n)$ is bounded. Since $G(x)$ is a meromorphic function up to a ramification

$x \to x^k$ then $G(x)$ is a bounded function in a neighborhood of $x = 0$. Thus the principal parts of $\sum_{P \in E(x)} \mathrm{Res}_{X_1}(P)$ and $\sum_{P \in E(x)} \mathrm{Res}_{X_2}(P)$ coincide. By analogous techniques and the splitting of the dynamics phenomena we obtain the sufficient condition in the next theorem.

We say that the extended principal parts Ext.ppal.(X_1) and Ext.ppal.(X_2) of X_1 and X_2 respectively coincide if the function

$$x \to x^{m(X_1)k}(\mathrm{Res}_{X_1}(x^k, h(x)) - \mathrm{Res}_{X_2}(x^k, h(x)))$$

is holomorphic and vanishes at 0 for all continuous sections $(x^k, h(x))$ of $SingX_1$.

Theorem 4.1. — *Let* $X_1, X_2 \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ *with* $SingX_1 = SingX_2$. *Suppose that* $(N(X_1), m(X_1)) \neq (1, 0)$ *(see def. 4.3). Then* $\exp(X_1) \sim_{\mathrm{top}} \exp(X_2)$ *if and only if* Ext.ppal.$(X_1) =$ Ext.ppal.(X_2). *Moreover, we have*

$$\exp(X_1) \sim_{\mathrm{top}} \exp(X_2) \Leftrightarrow \Re(X_1) \sim_{\mathrm{top}} \Re(X_2).$$

The extended principal part is an invariant of formal type since $X_1 \sim_{for} X_2$ implies Ext.ppal.$(X_1) =$ Ext.ppal.(X_2).

4.10. Multi-parabolic diffeomorphisms. — The stability theorem 3.1 implies that MP-vector fields provide good models for MP-diffeomorphisms. The proof is based on dividing a neighborhood of $(0,0)$ in regions in which we use different techniques to show that the orbits of $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ remain close to the orbits of $\exp(X_\varphi)$. A finiteness argument, based on the dynamical Rolle property, is required to prove that the trajectories of $\Re(X_\varphi)$ can not visit infinitely many regions.

Definition 4.5. — *Let* $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. *We define* $\mathrm{Res}_\varphi(P) = \mathrm{Res}_{X_\varphi}(P)$ *(see def. 4.4) for* $P \in \mathrm{Fix}(\varphi)$. *Denote* Ext.ppal.$(\varphi) =$ Ext.ppal.(X_φ). *The definitions do not depend on the choice of* X_φ.

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. Suppose that $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ by a homeomorphism σ. Given $y_1 \in L_{\beta, y_0}^{+,\epsilon}(X_{\varphi_1})$ we have a continuous section $y_1(x) : \beta \cup \{0\} \to \mathbb{C}$, a continuous set of singular points E and a function $T : \beta \to \mathbb{R}^+$ such that

$$T(x) = \psi_\partial(x, y_1(x)) - \psi_0(x, y_0) - 2\pi i \sum_{P \in E(x)} \mathrm{Res}_{\varphi_1}(P).$$

The stability theorem implies that σ almost conjugates $\exp(X_{\varphi_1})$ and $\exp(X_{\varphi_2})$. Then we can use arguments analogous to those in the previous section to prove that the principal parts of $\sum_{P \in E(x)} \mathrm{Res}_{\varphi_1}(P)$ and $\sum_{P \in E(x)} \mathrm{Res}_{\varphi_2}(P)$ coincide. Indeed we can push forward the analogy to obtain

Theorem 4.2. — *Let* $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ *with* $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. *Suppose that* $(N(\varphi_1), m(\varphi_1)) \neq (1, 0)$. *Then* $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ *implies* Ext.ppal.$(\varphi_1) =$ Ext.ppal.(φ_2).

The extended principal part is, as for vector fields, an invariant of formal type since $\varphi_1 \sim_{for} \varphi_2$ implies Ext.ppal.$(\varphi_1) =$ Ext.ppal.(φ_2).

4.11. Analytic type topological invariants. — The extended principal part is a complete system of topological invariants for MP-vector fields. In spite of this more topological invariants are required in order to classify MP-diffeomorphisms.

Let $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. Denote $X = X_\varphi$. Given $y_1 \in B(0,\epsilon) \setminus \{0\}$ such that $\alpha_X^\epsilon(0, y_1) = (0,0)$ (the situation where $\omega_X^\epsilon(0, y_1) = (0,0)$ is analogous) there exists $L_{\eta(0),y_1}^{-,\epsilon} \neq \emptyset$ by prop. 4.7. The diffeomorphism $\varphi_{|x=0}$ is embedded in a natural way in a complex flow Y defined in the repelling petal $V \subset \{0\} \times B(0,\epsilon)$ containing $(0, y_1)$. The following theorem is the analogue of proposition 4.8 for MP-diffeomorphisms.

Theorem 4.3 **(Openness principle).** — *Let* $\varphi \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ *with* $N(\varphi) > 1$. *Consider* $y_1 \in B(0,\epsilon) \setminus \{0\}$ *such that* $\alpha_X^\epsilon(0, y_1) = (0,0)$. *There exists* $y_0 \in B(0,\epsilon) \setminus \{0\}$ *such that for any* $\rho + iv$ *in a neighborhood of* $0 \in \mathbb{C}$ *there exist a branch of analytic curve* $\eta(v)$ *and sequences* $\{T_n^{\rho,v}\}_{n\in\mathbb{N}}$ *of natural numbers and* $\{x_n^{\rho,v}\}_{n\in\mathbb{N}}$ *of points in* $\eta(v)$ *satisfying*

- $\lim_{n\to\infty} T_n^{\rho,v} = \infty$ *and* $\lim_{n\to\infty} x_n^{\rho,v} = 0$.
- $\lim_{n\to\infty} \varphi^{T_n^{\rho,v}}(x_n^{\rho,v}, y_0) = \exp((\rho + iv)Y)(0, y_1)$.

Moreover $\eta(v)$ *depends continuously on* v.

Let $\varphi_1, \varphi_2 \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi_1) = \mathrm{Fix}(\varphi_2)$. Suppose that $\varphi_1 \sim_{\mathrm{top}} \varphi_2$ by a homeomorphism σ. The openness theorem implies that given $y_1 \in B(0,\epsilon) \setminus \{0\}$ the restriction $\sigma_{|x=0}$ is determined in a neighborhood of $(0, y_1)$ by $\sigma(0, y_0)$ for some $y_0 \in B(0,\epsilon) \setminus \{0\}$. The germ of $\sigma_{|x=0}$ in the neighborhood of $(0, y_1)$ only depends on the set of analytic data φ_1, φ_2 and $\sigma(0, y_0)$. This property allows to prove that $\sigma_{|x=0}$ is analytic in the neighborhood of $(0, y_1)$ for any $y_1 \in B(0,\epsilon) \setminus \{0\}$. By Riemann's theorem we obtain

Theorem 4.4. — *Let* $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. *Suppose that we have* $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ *and* $(N, m)(\varphi) \neq (1, 0)$ *(see def. 4.3). Then*

$$\varphi \sim_{\mathrm{top}} \eta \implies \varphi_{|x=0} \sim_{\mathrm{an}} \eta_{|x=0}.$$

This theorem provides topological invariants of analytic type.

4.12. Theorem of topological classification. — The formal and analytic type topological invariants compose a complete system of topological invariants.

Theorem 4.5. — *Let* $\varphi, \eta \in \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$. *Suppose that we have* $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$ *and* $(N, m)(\varphi) \neq (1, 0)$. *Then*

$$\varphi \sim_{\mathrm{top}} \eta \Leftrightarrow \begin{cases} \mathrm{Ext.ppal.}(\varphi) = \mathrm{Ext.ppal.}(\eta) \\ \varphi_{|x=0} \sim_{\mathrm{an}} \eta_{|x=0} \end{cases}$$

The last theorem and the proposition 4.2 provide a complete system of topological invariants for the multi-parabolic diffeomorphisms. The discussion so far has been intended to introduce the invariants and the ideas in the proof of the implication $\Rightarrow$ in the previous theorem.

Remark 4.5. — *The topological invariants are related to the study of the unstable phenomena, more precisely the Long Trajectories. As a consequence there are no topological invariants related to the behavior of a MP-diffeomorphism in a neighborhood of its fixed points.*

Remark 4.6. — *The theorem of topological classification relates formal, analytic and topological invariants.*

The proof of the implication $\Leftarrow$ in theorem 4.5 can not be reduced to the case of vector fields since $\varphi \sim_{\text{top}} \exp(Y)$ implies $\varphi_{|x=0} \sim_{\text{an}} \exp(Y)_{|x=0}$. Thus φ is not topologically conjugated to the exponential of a vector field whenever $\varphi_{|x=0}$ is not embedded in an analytic flow, i.e in the generic situation.

We have that

$$\text{Ext.ppal.}(X_1) = \text{Ext.ppal.}(X_2) \implies \Re(X_1) \sim_{\text{top}} \Re(X_2)$$

for $X_1, X_2 \in \mathcal{X}_{MP}(\mathbb{C}^2, 0)$ with $SingX_1 = SingX_2$. What can we say for multi-parabolic diffeomorphisms? In other words how far are $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ from being topologically conjugated if $\text{Ext.ppal.}(\varphi) = \text{Ext.ppal.}(\eta)$?

Let $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\text{Fix}(\varphi) = \text{Fix}(\eta)$. We say that σ is a *tangential conjugation* if there exists $\epsilon > 0$ such that

- $x \circ \sigma \equiv x$ and $\sigma_{|\text{Fix}(\varphi) \setminus \{x=0\}} \equiv Id$.
- σ, σ^{-1} are homeomorphisms defined in $((B(0,\delta) \setminus \{0\}) \times B(0,\epsilon)) \cup \{(0,0)\}$.

Roughly speaking a tangential conjugation is not very different from a germ of homeomorphism, but we allow some noise when $x \to 0$.

Theorem 4.6. — *Let* $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ *with* $\text{Fix}(\varphi) = \text{Fix}(\eta)$. *Then the equality* $\text{Ext.ppal.}(\varphi) = \text{Ext.ppal.}(\eta)$ *implies the existence of a tangential* σ_T *conjugating* φ *and* η.

The choice of σ_T is not unique but it is possible to build σ_T in such a way [**10**] that σ_T is a germ of diffeomorphism if and only if $\varphi_{|x=0} \sim_{\text{an}} \eta_{|x=0}$. Thus the property $\varphi_{|x=0} \sim_{\text{an}} \eta_{|x=0}$ can be interpreted as a condition on elimination of noise for tangential conjugations.

Remark 4.7. — *Let* $\varphi, \eta \in \text{Diff}_{MP}(\mathbb{C}^2, 0)$ *with* $\text{Fix}(\varphi) = \text{Fix}(\eta)$. *Supposed* $\varphi \sim_{\text{top}} \eta$ *then the topological conjugation* σ *can be chosen to be* C^∞ *outside* $\text{Fix}(\varphi) \cup \{x = 0\}$. *It is known (Martinet-Ramis* [**6**], *Ahern-Rosay* [**1**] *and Rey* [**8**] *for optimal results) that in general* σ *can not be chosen* C^{ν_0} *for some* $\nu_0 \in \mathbb{N}$. *The MP-diffeomorphisms*

$$\varphi = \exp\left(\frac{y^3(y-x)^2}{1 + xy^2(y-x)^2} \frac{\partial}{\partial y}\right) \text{ and } \eta = \exp\left(y^3(y-x)^2 \frac{\partial}{\partial y}\right)$$

are topologically conjugated. It can be proved that the topological conjugation can not be chosen C^1 *in* $\{0\} \times (B(0,\epsilon) \setminus \{0\})$. *The fixed points set is singular with respect to the* C^ν *conjugation as well as the dynamical singular set* $x = 0$.

Finally let us stress some of the key points in our approach of the dynamics of multi-parabolic diffeomorphisms:

- The limit of the dynamics is more complex than the dynamics of the limit in the dynamically interesting cases ($N > 1$). Indeed the limit of discrete dynamics in the generic lines $x = w$ with $w \neq 0$ generate a complex flow-like structure in the limit line $x = 0$. A consequence of this property is theorem 4.4.
- The topological classification theorem relies on a qualitative and quantitative description of the dynamics.
- A multi-parabolic diffeomorphism φ supports non-trivial dynamics whenever $N(\varphi) > 1$. The existence of the L-limits is an example of this statement. Moreover, generically φ is not topologically conjugated to the exponential of a vector field.

5. Analytic classification of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$

We provide in this section an analytic classification of the elements of φ in $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ (theorems 5.3 and 5.4). Moreover, we sketch the proof of theorem 3.3. Both goals are fulfilled in subsection 5.9. We introduce an extension of the Ecalle-Voronin invariants (subsection 5.8) based on an extension of the Fatou coordinates of $\varphi_{|x=0}$ to the nearby parameters (subsection 5.7). The philosophy of this approach is explained in subsection 5.2.

The subsections 5.3, 5.4, 5.5 and 5.6 can be considered as a setup for the construction of the Fatou coordinates. The proof of theorem 3.3 requires quantitative estimates for the Fatou coordinates in the neighborhood of the fixed points. We introduce the notion of infinitesimal stability (subsection 5.4) allowing to obtain the desired estimates. This is a stronger version of a qualitative version of stability, namely topological stability provided in subsection 5.5. The definition of infinitesimal stability is obtained after splitting conveniently a neighborhood of the origin (subsection 5.3). Finally in subsection 5.6 we introduce the sets in which the extensions of the Fatou coordinates are defined.

5.1. Relation between the analytic and formal classifications. — Generically the analytic classification of elements in $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ can be reduced to a formal classification problem. Anyway our approach has an advantage, it allows to take profit of the rigidity properties of elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$.

Definition 5.1. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Consider an irreducible component γ of $\mathrm{Fix}(\varphi)$. We say that γ is parabolic if $(\partial(y \circ \varphi)/\partial y)_{|\gamma} \equiv 1$.*

Theorem 5.1. — *Let φ, η be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0) \setminus \mathrm{Diff}_{MP}(\mathbb{C}^2, 0)$ with $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\eta)$. Suppose that $\varphi \sim_{for} \eta$ by $\hat{\sigma} \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$. Suppose also that $y \circ \hat{\sigma}$ is transversally formal along a non-parabolic irreducible component γ of $\mathrm{Fix}(\varphi)$. Then σ is analytic.*

Proof. — Analogously to the proof of th. 3.2 we can suppose that $\gamma \equiv \{y = 0\}$. Denote $L(x) = (\partial(y \circ \varphi)/\partial y)(x, 0)$. Since $\varphi \sim_{for} \eta$ we have $L(x) \equiv (\partial(y \circ \eta)/\partial y)(x, 0)$ [**11**]. By hypothesis there exists a neighborhood V of $0 \in \mathbb{C}$ such that the series $y \circ \hat{\sigma}$ is of the form

$$y \circ \hat{\sigma} = \sum_{j=1}^{\infty} \sigma_j(x) y^j \text{ with } \sigma_j \in \mathcal{O}(V) \text{ for all } j \in \mathbb{N}.$$

Consider a path $\kappa \subset V \setminus \{0\}$ turning once around 0 and transversal to $L^{-1}(\mathbb{S}^1)$. Moreover we can suppose that whenever $w \in \kappa \cap L^{-1}(\mathbb{S}^1)$ then $L(w)$ is a Bruno number. The choice of κ implies that $\hat{\sigma}$ is continuous in $\kappa \times B(0, \upsilon)$ for some $\upsilon > 0$. Then there exists $C \in \mathbb{R}^+$ such that

$$|\sigma_j(x)| \leq C^j \text{ for all } (x, j) \in \kappa \times \mathbb{N}.$$

The modulus maximum principle implies that $\hat{\sigma} \in \text{Diff}(\mathbb{C}^2, 0)$. □

There is a downside in interpreting the analytic invariants of a non-degenerate $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$ as formal invariants along a non-parabolic component of $\text{Fix}(\varphi)$. Indeed this system of invariants does not provide as much information on the nature of φ as the one we are going to introduce. For instance theorem 3.3 relies on the fact that $\varphi_{|\{w\} \times B(0,\epsilon)}$ inherits the rigidity properties of tangent to the identity diffeomorphisms in one variable for $w \neq 0$. This property is of very difficult translation in the formal setting.

5.2. Analytic classification and extension of Fatou coordinates. — We provide a complete system of analytic invariants for non-degenerate $\varphi \in \text{Diff}_{p1}(\mathbb{C}^2, 0)$. The system of invariants is based on building extensions of the Fatou coordinates of $\varphi_{|x=0}$ to the nearby parameters. This topic is is a classical subject of study (Lavaurs [**3**], Shishikura [**13**], Oudkerk [**7**], Mardesic-Roussarie-Rousseau [**5**]).

Definition 5.2. — *A Fatou coordinate for a diffeomorphism* $\varphi \in \text{Diff}(\mathbb{C}^n, 0)$ *is a complex valued function* ψ^{φ} *such that* $\psi^{\varphi} \circ \varphi = \psi^{\varphi} + 1$.

Definition 5.3. — *Consider a non-degenerate element* φ *of* $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. *Denote* $\nu(\varphi) = \nu(y \circ \varphi(0, y)) - 1$ *where* $\nu(y \circ \varphi(0, y))$ *is the order of* $y \circ \varphi(0, y)$ *at* 0.

The methods to obtain extensions of the Fatou coordinates are based on choosing a system of transversals to the dynamics of $\varphi_{|x=w}$ depending continuously on w. Even if the dynamics of φ is discrete it is possible to make sense of the previous statement. The approach consists in considering coordinates (x, z) in which φ is very similar to $(x, z + 1)$. Then every real line not parallel to $\mathbb{R}$ provides a choice of transversal to $\varphi_{|x=w}$. Indeed we always choose transversals $T_w \subset \{x = w\}$ homeomorphic to $\mathbb{R}$. Given a parametrization $T_w(t) : (-\infty, \infty) \to \{x = w\}$ we demand

$$\lim_{t \to -\infty} T_w(t) \in \text{Fix}(\varphi) \ni \lim_{t \to \infty} T_w(t).$$

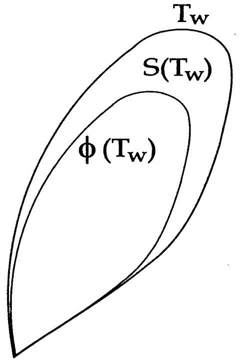

FIGURE 4.

The transversal T_w and its image $\varphi(T_w)$ enclose a strip $S(T_w)$. The space of orbits of $\varphi_{|S(T_w)}$ is biholomorphic to $\mathbb{C}^*$. Given a biholomorphism ρ_{T_w} conjugating the space of orbits of $\varphi_{|S(T_w)}$ and $\mathbb{C}^*$ the function $\psi^{\varphi}_{T_w} = (1/2\pi i)\ln \rho_{T_w}$ is a Fatou coordinate of φ in the fundamental domain $S(T_w)$. The Lavaurs vector field is by definition the unique holomorphic vector field $X^{\varphi}_{T_w}$ in $S(T_w)$ such that $X^{\varphi}_{T_w}(\psi^{\varphi}_{T_w}) \equiv 1$. Clearly it satisfies $\varphi \equiv \exp(X^{\varphi}_T)$.

Suppose that $\varphi_{|x=0}$ is generic, then $\varphi_{|x=0}$ is of codimension 1, i.e. $\nu(\varphi) = 1$, and $N(\varphi) = 2$. In Mardesic-Roussarie-Rousseau's paper [**5**] the diffeomorphism φ is "prepared". More precisely, up to an analytic change of coordinates, they suppose that the multipliers of the fixed points of φ be equal to the multipliers of $\exp(\frac{y^2-x}{1+a(x)y}\frac{\partial}{\partial y})$ for some $a \in \mathbb{C}\{x\}$. The change of coordinates at $x = w$ approaching the dynamics of φ to $(x, z+1)$ is obtained by lifting $\varphi_{|x=w}$ to the universal covering of $\mathbb{P}^1(\mathbb{C}) \setminus (\mathrm{Fix}(\varphi) \cap \{x = w\})$. The covering transformation $\pi_w : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \setminus (\mathrm{Fix}(\varphi) \cap \{x = w\})$ can be explicitly written and be made to depend holomorphically on w. Therefore a continuous system of transversals can be build. This method is only valid if $\varphi_{|x=0}$ is of codimension 1.

The Oudkerk's approach is different. He obtains Fatou coordinates for unfoldings of every $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ independently of the codimension of ϕ. Given a point $(w, y_0) \notin \mathrm{Fix}(\varphi)$ we consider the change of coordinates $\Delta(z) = \exp(zX)(w, y_0)$ defined from a neighborhood of 0 to a neighborhood of (w, y_0) in $x = w$. We have

$$\Delta^* X = \partial/\partial z \quad \text{and} \quad \Delta^{-1} \circ \exp(X) \circ \Delta = z + 1.$$

In order to find transversals to the dynamics of φ we choose a vector field X such that $\varphi \sim \exp(X)$. Then we fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and consider as transversals trajectories of the form $\Gamma^{\epsilon}_{\mu X}[x, y_0(x)]$ where y_0 is continuous. In order to obtain holomorphic extensions of the Fatou coordinates for x in a domain $D \subset \mathbb{C}^*$ then $\omega^{\epsilon}_{\mu X}(x, y_0(x))$ and $\alpha^{\epsilon}_{\mu X}(x, y_0(x))$ have to be continuous sections of $\mathrm{Fix}(\varphi)$ for x in D. In other words the trajectory $\Gamma^{\epsilon}_{\mu X}[x, y_0(x)]$ has a stable behavior for $x \in D$. Indeed given $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$

the trajectories of $\Re(\mu X)$ provide a good choice of system of transversals for the parameters in which $w \to \Re(\mu X)_{|x=w}$ is stable.

More precisely, consider a direction $e^{2\pi i\theta}\mathbb{R}^+$ ($\theta \in \mathbb{R}$) in the parameter space. We choose $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ such that $\exists v > 0$ satisfying that $\Re(\mu X)_{|\{w\}\times B(0,\epsilon)}$ is stable with respect to w for $w \in e^{2\pi i(\theta - v, \theta + v)}(0,\delta)$. We can obtain a continuous extension of the Fatou coordinates in $x \in e^{2\pi i(\theta - v,\theta+v)}(0,\delta) \cup \{0\}$ which is holomorphic for $x \in e^{2\pi i(\theta-v,\theta+v)}(0,\delta)$.

Our approach is of analytic type and takes profit of the properties of the unfolding. We choose $X = X_\varphi$ as the vector field such that $\varphi \sim \exp(X)$, i.e. we ask X to fulfill the proximity condition. In this way the model $\exp(X)$ reflects better the nature of φ in the neighborhood of the fixed points. The approach of Lavaurs, Shishikura, Oudkerk, Mardesic-Roussarie-Rousseau is of topological type; they implicitly use a notion of continuous stability. We replace it with a concept of infinitesimal stability for $\Re(\mu X)_{|x=w}$. Our approach provides:

- Asymptotic developments of the Lavaurs vector field X_T^φ until the first non-zero term in the neighborhood of the fixed points.
- Accurate estimates for the domains of definition of $\exp(cX_T^\varphi)$ for $c \in \mathbb{C}$.
- Canonical normalizing conditions for the Fatou coordinates.

These improvements allow us to:

- Identify the Taylor's series expansion of the analytic mappings conjugating φ, $\zeta \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$.
- Study the dependance of the domain of definition of a conjugation with respect to the parameter.
- Give a geometrical interpretation of our complete system of analytic invariants (theorem 3.3).

5.3. Dynamical splitting. — We provide a division of the domain of definition of a multi-parabolic vector field. The division is a key tool to handle the concept of infinitesimal stability that we will introduce later on.

Remark 5.1. — *Let $X = g(x,y)\partial/\partial y$ be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$. For the sake of simplicity we suppose that all the irreducible components of SingX are parameterized by x. In the non-degenerate case, this property can always be obtained by replacing X with $g(x^k,y)\partial/\partial y$ for some $k \in \mathbb{N}$. Then X is of the form*

$$X = u(x,y)(y - g_1(x))^{n_1} \dots (y - g_p(x))^{n_p}\partial/\partial y$$

where u is a unit in $\mathbb{C}\{x,y\}$ and $n_1 + \dots + n_p \geq 2$.

We say that $\overline{U}_\epsilon = B(0,\delta) \times \overline{B}(0,\epsilon)$ is a *seed*. If $p = 1$ we do not divide $\overline{U}_\epsilon$ and we call $\overline{U}_\epsilon$ a terminal seed. We say that $\overline{U}_\epsilon$ is a product-like set (or also an exterior set). Denote by $\partial_e\overline{U}_\epsilon$ its exterior boundary $B(0,\delta) \times \partial B(0,\epsilon)$. Denote

$$L = \{(\partial g_1/\partial x)(0), \dots, (\partial g_p/\partial x)(0)\}.$$

Suppose $p > 1$. Consider a set $\overline{U}_\epsilon \cap \{(x,y) \in \mathbb{C}^2 : |y| \geq C|x|\}$ with $C >> 1$. Since $C >> \max L$ an observer in $|y| \geq C|x|$ can not distinguish the singular points. Thus we obtain

$$X = u(x,y)y^{\nu(X)+1}\left(1 - \frac{g_1(x)}{y}\right)^{n_1} \dots \left(1 - \frac{g_p(x)}{y}\right)^{n_p} \frac{\partial}{\partial y} \sim u(0,0)y^{\nu(X)+1}\frac{\partial}{\partial y}$$

in $\overline{U}_\epsilon \cap \{(x,y) \in \mathbb{C}^2 : |y| \geq C|x|\}$. Indeed it can be proved that the dynamics of $\Re(X)_{|\overline{U}_\epsilon \cap \{|y| \geq C|x|\}}$ is a topological product. The set $\overline{U}_\epsilon \cap \{|y| \geq C|x|\}$ is called exterior set or also product-like set associated to the seed $\overline{U}_\epsilon$.

Next we want to study the dynamics of $\Re(X)$ in $|y| \leq C|x|$. We consider the change of coordinates $x = x$, $y = xt$. The set $\{(x,y) \in \mathbb{C}^2 : |y| \leq C|x|\}$ and the vector field X become $\{(x,t) \in \mathbb{C}^2 : |t| \leq C\}$ and

$$X = u(x,xt)x^{\nu(X)}\left(t - \frac{g_1(x)}{x}\right)^{n_1} \dots \left(t - \frac{g_p(x)}{x}\right)^{n_p} \frac{\partial}{\partial t}$$

respectively in the new coordinates. Consider the polynomial vector field

$$Y(\lambda) = u(0,0)\lambda^{\nu(X)}\left(t - \frac{\partial g_1}{\partial x}(0)\right)^{n_1} \dots \left(t - \frac{\partial g_p}{\partial x}(0)\right)^{n_p} \frac{\partial}{\partial t}$$

for $\lambda \in \mathbb{S}^1$. We say that Y is the polynomial vector field associated to X and the seed $\overline{U}_\epsilon$. Consider a domain

$$\mathcal{C} = \{(x,t) \in \mathbb{C}^2 : (x,t) \in B(0,\delta) \times [\overline{B(0,C)} \setminus \cup_{t_0 \in L} B(t_0, a(t_0))]\}$$

where $0 < a(t_0) << 1$ for any $t_0 \in L$. We say that $\mathcal{C}$ is a *compact-like set* associated to X and the seed $\overline{U}_\epsilon$. We say also that Y is associated to $\mathcal{C}$. We denote by $\partial_e \mathcal{C}$ and $\mathcal{C}^\circ$ the exterior boundary $|t| = C$ and the interior $\{|t| < C\} \setminus \cup_{t_0 \in L}\{|t - t_0| \leq a(t_0)\}$ of $\mathcal{C}$ respectively. A set of the form $\{(x,t) \in \mathbb{C}^2 : (x,t) \in B(0,\delta) \times \overline{B(t_0, a(t_0))}\}$ for some $t_0 \in L$ is called a seed that is a son of $\overline{U}_\epsilon$. We have

$$\frac{u(x,xt)x^{\nu(X)}(t - g_1(x)/x)^{n_1} \dots (t - g_p(x)/x)^{n_p}}{u(0,0)x^{\nu(X)}(t - (\partial g_1/\partial x)(0))^{n_1} \dots (t - (\partial g_p/\partial x)(0))^{n_p}} \to 1$$

uniformly in $\mathcal{C}$ when $x \to 0$. Therefore we obtain that

$$\Re(\mu X)_{|\mathcal{C} \cap \{x = r\lambda\}} \sim \Re(\mu Y(\lambda)) \text{ for } r \in \mathbb{R}^+ \text{ and } (\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1.$$

The last property provides a justification for the term compact-like since the behavior of $\Re(\mu X)_{|\mathcal{C} \cap \{x = r\lambda\}}$ is determined by a polynomial vector field in one variable and a parameter (λ, μ) belonging to the compact set $\mathbb{S}^1 \times \mathbb{S}^1$. We can repeat the process with the seeds $|t - t_0| \leq a(t_0)$ for $j \in \{1, \dots, p\}$. We say that $\mathcal{B}$ is a *basic set* if $\mathcal{B}$ is either a product-like set or a compact-like set.

EXAMPLE: Consider $X = y(y - x^2)(y - x)\partial/\partial y$. Denote $t = y/x$. The vector field X has the form $x^2 t(t - x)(t - 1)\partial/\partial t$ in coordinates (x,t). The polynomial vector field $Y(\lambda)$ associated to the seed $\mathcal{S} = \overline{U}_\epsilon$ is equal to $\lambda^2 t^2 (t-1)\partial/\partial t$

The product and compact-like sets associated to $\mathcal{S}$ are $\mathcal{E} = \mathcal{S} \cap \{|y| \geq C|x|\}$ and $\mathcal{C} = \{|y| \leq C|x|\} \setminus (\{|t| < a_0\} \cup \{|t - 1] < a_1\})$ respectively. The sons of $\mathcal{S}$ are the

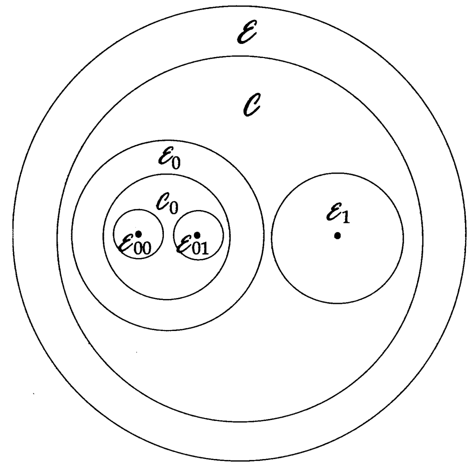

FIGURE 5. Splitting for $X = y(y-x^2)(y-x)\partial/\partial y$ in a line $x = x_0$

seeds $\mathcal{S}_0 = \{|t| \leq a_0\}$ and $\mathcal{E}_1 = \mathcal{S}_1 = \{|t-1| \leq a_1\}$. The seed $\mathcal{S}_1$ is terminal since it only contains one irreducible component of $SingX$.

Denote $w = t/x$. We have $X = x^3 w(w-1)(xw-1)\partial/\partial w$ in coordinates (x, w). Thus $-\lambda^3 w(w-1)\partial/\partial w$ is the polynomial vector field $Y_0(\lambda)$ associated to $\mathcal{S}_0$. The seed $\mathcal{S}_0$ contains a product-like set $\mathcal{E}_0 = \mathcal{S}_0 \cap \{|t| \geq C_0|x|\}$ for $C_0 >> 1$, a compact-like set $\mathcal{C}_0 = \{|w| \leq C_0\} \setminus (\{|w| < b_0\} \cup \{|w-1| < b_1\})$ and two terminal seeds $\mathcal{E}_{00} = \mathcal{S}_{00} = \{|w| \leq b_0\}$ and $\mathcal{E}_{01} = \mathcal{S}_{01} = \{|w-1| \leq b_1\}$ for some $0 < b_0, b_1 << 1$.

5.4. Infinitesimal stability. — The dynamics of $\Re(\mu X)$ in a product-like set is a topological product. As a consequence it is stable with respect to the parameter x. Then the stability is a property depending of the behavior of $\Re(\mu X)$ in the compact-like sets.

Many of the results in this subsection on polynomial vector fields either come from [**2**] or have been found independently in [**2**].

Definition 5.4. — *Given a polynomial vector field $Y = P(t)\partial/\partial t \in \mathcal{X}(\mathbb{C}, 0)$ we define $\nu(Y) = \deg(P(t)) - 1$. Let $t_0 \in SingY$, we define $\mathrm{Res}_Y(t_0)$ as the residue of the differential form $dt/P(t)$ at $t = t_0$.*

Consider a polynomial vector field $Y = P(t)\partial/\partial t \in \mathcal{X}(\mathbb{C}, 0)$ such that $\nu(Y) \geq 1$. We want to characterize the behavior of Y in the neighborhood of ∞. We define the set $Tr_{\to\infty}(Y)$ of trajectories $\gamma : (c, d) \to \mathbb{C}$ of $\Re(Y)$ such that $c \in \mathbb{R} \cup \{-\infty\}$, $d \in \mathbb{R}$ and $\lim_{\zeta \to d} \gamma(\zeta) = \infty$. In an analogous way we define $Tr_{\leftarrow\infty}(Y) = Tr_{\to\infty}(-Y)$. We define $Tr_\infty(Y) = Tr_{\leftarrow\infty}(Y) \cup Tr_{\to\infty}(Y)$.

We consider a change of coordinates $z = 1/t$. We have

$$Y = \frac{-1}{z^{\nu(Y)-1}}(z^{\nu(Y)+1}P(1/z))\frac{\partial}{\partial z}$$

where $z^{\nu(Y)+1}P(1/z)$ is a unit in the neighborhood of $z = 0$. Thus the meromorphic vector field Y is analytically conjugated to $1/(\nu(Y)z^{\nu(Y)-1})\partial/\partial z = (z^{\nu(Y)})^*(\partial/\partial z)$ in a neighborhood of $t = \infty$. We have $Tr_{\to\infty}(\partial/\partial z) = \mathbb{R}^-$ and $Tr_{\leftarrow\infty}(\partial/\partial z) = \mathbb{R}^+$. Hence the set $Tr_\infty(Y)$ has $2\nu(Y)$ trajectories in the neighborhood of ∞, there is exactly one of them which is tangent to $\arg(w) = -\arg(C)/\nu(Y) + k\pi/\nu(Y)$ for $0 \leq k < 2\nu(Y)$ where $C = (z^{\nu(Y)+1}P(1/z))(0)$. The even values of k correspond to $Tr_{\to\infty}(Y)$.

We say that $\Re(Y)$ has *homoclinic trajectories* if $Tr_{\to\infty}(Y) \cap Tr_{\leftarrow\infty}(Y) \neq \emptyset$. In other words there exists a trajectory $\gamma : (c_{-1}, c_1) \to \mathbb{C}$ of $\Re(Y)$ such that $c_{-1}, c_1 \in \mathbb{R}$ and $\lim_{\zeta\to c_s} \gamma(\zeta) = \infty$ for any $s \in \{-1, 1\}$. The notion of homoclinic trajectory has been introduced in [**2**] for the study of deformations of elements of $\mathrm{Diff}_1(\mathbb{C}, 0)$. We say that $\Re(Y)$ is stable if the dynamics of $\Re(\mu Y)$ is stable in a neighborhood of $\mu = 1$ in $\mathbb{S}^1$.

Theorem 5.2. — [**2**] *Let $Y \in \mathcal{X}(\mathbb{C}, 0)$ be a polynomial vector field such that $\nu(Y) \geq 1$. Then $\Re(Y)$ is stable if and only if $\Re(Y)$ has no homoclinic trajectories.*

Definition 5.5. — *We denote by $\mathcal{X}_\infty(\mathbb{C}, 0)$ the set of polynomial vector fields Y in $\mathcal{X}(\mathbb{C}, 0)$ such that $\nu(Y) \geq 1$ and $\sum_{P\in E} \mathrm{Res}_Y(P) \notin i\mathbb{R}^*$ for any subset E of $SingY$.*

Proposition 5.1. — *Let $Y \in \mathcal{X}_\infty(\mathbb{C}, 0)$. Then $\Re(Y)$ is stable. Moreover given Q in $\mathbb{C} \setminus Tr_{\to\infty}(Y)$ we have that $\omega_Y(Q)$ is a finite singular point. In particular $\Re(Y)$ has no cycles.*

Proof. — Let $\gamma : (a, b) \to \mathbb{C}$ be a homoclinic trajectory for $\Re(Y)$. We suppose that (a, b) is a maximal domain of definition of the trajectory. The set $\mathbb{P}^1(\mathbb{C}) \setminus (\gamma \cup \{\infty\})$ has two connected components C_1 and C_2. Denote $E = C_1 \cap SingY$. By the theorem of the residues we have

$$b - a = \mp 2\pi i \sum_{P\in E} \mathrm{Res}_Y(P).$$

We obtain a contradiction.

Given $Q \in \mathbb{C} \setminus Tr_{\to\infty}(Y)$ suppose that $P \in \omega_Y(Q) \cap (\mathbb{C} \setminus SingY)$. Consider a transversal T to $\Re(Y)$ passing through P. Let $\{t_n\}_{n\in\mathbb{N}}$ be the increasing sequence of points in $\{t \in \mathbb{R}^+ : \Gamma_{Y,+}[Q](t) \in T\}$. There exists $\eta > 0$ such that $t_{n+1} - t_n > \eta$ for any $n \in \mathbb{N}$. Denote by γ_n the curve composed by $\Gamma_{Y,+}[Q][t_n, t_{n+1}]$ and the segment S_n of T such that $\partial S_n = \{\Gamma_{Y,+}[Q](t_n), \Gamma_{Y,+}[Q](t_{n+1})\}$. Let C_n be the bounded connected component of $\mathbb{C} \setminus \gamma_n$. Denote $E_n = C_n \cap SingY$. We obtain

$$\mp 2\pi i \sum_{P\in E_n} \mathrm{Res}_Y(P) = (t_{n+1} - t_n) + a_n$$

where $a_n \in \mathbb{C}$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$. Thus there exists a set $E \subset SingY$ such that $\mp 2\pi i \sum_{P\in E} \mathrm{Res}_Y(P) \in [\eta, \infty)$. That is a contradiction. □

Let $Y \in \mathcal{X}(\mathbb{C}, 0)$ be a polynomial vector field such that $\nu(Y) \geq 1$. We denote by $\mathcal{U}(Y)$ the complementary of the set $\{\kappa \in \mathbb{S}^1 : \Re(\kappa Y) \text{ is stable}\}$. It is clear that κY belongs to $\mathcal{X}_\infty(\mathbb{C}, 0)$ for any $\kappa \in \mathbb{S}^1$ outside a finite set. Thus we obtain

Proposition 5.2. — *Let $Y \in \mathcal{X}(\mathbb{C}, 0)$ be a polynomial vector field such that $\nu(Y) \geq 1$. Then $\mathcal{U}(Y)$ is a finite set.*

Let $Y_1(\lambda) = \lambda^{m_1} Y_1(1), \dots, Y_l(\lambda) = \lambda^{m_l} Y_l(1)$ the polynomial vector fields associated to a non-degenerate $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$. We define

$$\mathcal{U}(X) = \cup_{j=1}^{l} \{\lambda \in \mathbb{S}^1 : \lambda^{m_j} \in \mathcal{U}(Y_j(1))\}.$$

Clearly $\mathcal{U}(X)$ is a finite set. Denote by $\mathcal{S}_1, \dots, \mathcal{S}_l$ the seeds associated to the vector fields $Y_1, \dots, Y_l$ respectively. Consider the compact-like set $\mathcal{C}_j$ associated to $\mathcal{S}_j$. We say that $\Re(\mu X)$ is *infinitesimally stable* at the direction $x \in \lambda\mathbb{R}^+$ if $\Re(\mu Y_j(\lambda))$ is stable for any $j \in \{1, \dots, l\}$. This property is equivalent to $\lambda \notin \mathcal{U}(\mu X)$.

The set $\mathcal{C}_j$ is of the form $\{(x,t) \in B(0,\delta) \times [\overline{B(0,C)} \setminus \cup_{j=1}^{k} B(t_j, a_j)]\}$ for some coordinates (x,t). Denote $x = r\lambda$ ($r \in \mathbb{R}^+ \cup \{0\}$ and $\lambda \in \mathbb{S}^1$). We have

$$\Re(\mu X)(r\lambda, y) \sim \Re(\mu Y_j(\lambda)) = \Re(\lambda^{m_j} \mu Y_j(1))$$

in $\mathcal{C}_j$ for $j \in \{1, \dots, l\}$. Suppose $\lambda \notin \mathcal{U}(\mu X)$. Since $\Re(\lambda^{m_j} \mu Y_j(1))$ is stable in the neighborhood of $\{\infty\} \cup SingY_j(1)$ and $C >> 1$, $a_j << 1$ then the stability of $\Re(\lambda^{m_j} \mu Y_j(1))$ implies the stability of $\Re(\mu X)_{|\mathcal{C}_j}$ in the neighborhood of the direction $x \in \lambda\mathbb{R}^+$. This discussion leads us to

Proposition 5.3. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Suppose that $\lambda \in \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then $\Re(\mu X)_{|\mathcal{C}}$ is stable in the neighborhood of $x \in \lambda\mathbb{R}^+$ for any compact-like set $\mathcal{C}$ associated to X.*

5.5. Topological stability. — The infinitesimal stability is a pretty strong property. For most of the discussion in the paper a weaker property, namely topological stability, is sufficient. Anyway, the quantitative estimations on the constructions rely on the use of infinitesimal stability as well as the theorems depending on those estimations (for instance theorem 3.3). In this subsection we define topological stability and show that it is implied by its infinitesimal counterpart.

Infinitesimal stability is incompatible with the existence of centers.

Proposition 5.4. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. The vector field $X_{|x=x_0}$ has no center at y_0 for any $(x_0, y_0) \in ([0,\delta)K \times B(0,\epsilon)) \cap SingX$.*

Proof. — We have to prove that the multiplicator ι of $X_{|x=x_0}$ at $y = y_0$ does not belong to $i\mathbb{R}^*$. We can suppose that $\iota \neq 0$. The point (x_0, y_0) belongs to a curve $y = h(x)$ of $SingX$. Consider the non-terminal seed $\mathcal{S}$ such that (x_0, y_0) belongs to a terminal seed $\mathcal{S}_0 = \{(x,t) \in B(0,\delta) \times \overline{B(t_0, a(t_0))}\}$ which is a son of $\mathcal{S}$. Let $\lambda^m Y(1)$ be the polynomial vector field associated to $\mathcal{S}$. The infinitesimal stability implies $\text{Res}(Y(1), t_0)/(\lambda^m \mu) \notin i\mathbb{R}^*$ for $\lambda \in K$. The multiplicator of X at $(x, h(x))$ is of the form $x^m \mu / \text{Res}(Y(1), t_0) + O(x^{m+1})$. Since we have

$$\frac{x^m \mu}{\text{Res}(Y(1), t_0)} + O(x^{m+1}) = |x|^m \left(\frac{\lambda^m \mu}{\text{Res}(Y(1), t_0)} + O(x) \right)$$

with $\lambda = x/|x|$ then (x_0, y_0) is not a center if x_0 is in a neighborhood of 0. □

Infinitesimal stability also implies that the limit of the dynamics is not more complicated than the dynamics of the limit. More precisely, the limit of a sequence of trajectories of $\Re(\mu X)_{|x=x_n}$ when $x_n \to 0$ is a priori only invariant by the flow and it can contain several trajectories of $\Re(\mu X)_{|x=0}$. Next proposition shows that this is not the case.

Proposition 5.5. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. There do not exist sequences (x_n, y_n^0), (x_n, y_n^1) of points in $(0, \delta)K \times B(0, \epsilon)$ such that*

- $\lim_{n\to\infty}(x_n, y_n^k)$ *exists and belongs to* $\{0\} \times (B(0, \epsilon) \setminus \{0\})$ $\forall k \in \{0, 1\}$.
- (x_n, y_n^1) *is in the positive trajectory of* $\Re(\mu X)_{|\{x_n\}\times B(0,\epsilon)}$ *passing through* (x_n, y_n^0) *for any* $n \in \mathbb{N}$.
- $\lim_{n\to\infty}(x_n, y_n^1)$ *is not in the positive trajectory of* $\Re(\mu X)_{|\{0\}\times \overline{B(0,\epsilon)}}$ *passing through* $\lim_{n\to\infty}(x_n, y_n^0)$.

Sketch of proof. — Suppose that the result is not true. Consider $t_n \in \mathbb{R}^+$ such that $(x_n, y_n^1) = \Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0](t_n)$ for $n \in \mathbb{N}$. The points (x_n, y_n^0) and (x_n, y_n^1) are in the product-like set $\mathcal{E}$ associated to the seed $\overline{U}_\epsilon$ for $n >> 0$. The trajectory $\Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0][0, t_n]$ is not contained in $\mathcal{E}$, otherwise $\lim_{n\to\infty}(x_n, y_n^1)$ is in the positive trajectory of $\Re(\mu X)_{|\{0\}\times\overline{B}(0,\epsilon)}$ passing through $\lim_{n\to\infty}(x_n, y_n^0)$. Therefore there exists a sequence $0 < t_n^1 < \ldots < t_n^k < t_n$ such that given $t \in [0, t_n]$ the point $\Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0](t)$ belongs to the boundary of a basic set if $t \in \{t_n^1, \ldots, t_n^k\}$. The property

$$Pr(\mathcal{B}, n) = \left(\exists\ 0 < s < k \ s.t.\ \Gamma^{\epsilon}_{\mu X,+}[x_n, y_n^0]\{t_n^s, t_n^{s+1}\} \subset \partial_e \mathcal{B}\right)$$

holds true for some basic set $\mathcal{B}$ and any $n >> 0$. The set $\mathcal{B}$ can not be product-like, it is necessarily a compact-like set

$$\{(x, t) \in B(0, \delta) \times [\overline{B}(0, C) \setminus \cup_{t_0 \in L} B(t_0, a(t_0))]\}.$$

Let Y be the polynomial vector field associated to $\mathcal{B}$. If the property $Pr(\mathcal{B}, n)$ holds true for all $C > 0$ and $n > n(C)$ then there exists a homoclinic trajectory of $\Re Y(\lambda)$ for any $\lambda \in K$ being an accumulation point of the sequence $x_n/|x_n|$. Then the absence of homoclinic trajectories implies that there exists a choice of the dynamical splitting leading us to a contradiction. □

Definition 5.6. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let μ in $\mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We say that $\Re(\mu X)$ is topologically stable at K if it satisfies the theses in propositions 5.4 and 5.5.*

The result on next corollary is a consequence of the topological stability property.

Corollary 5.1. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then $\mathcal{N}G^{\epsilon}_{\mu X,w}$ is connected and has no cycles for any $w \in (0, \delta)K$.*

Proof. — The proposition 5.5 implies that $(\alpha^{\epsilon}_{\mu X}, \omega^{\epsilon}_{\mu X})^{-1}(\infty,\infty) \cap \{x = w\} = \emptyset$ for any $w \in (0,\delta)K$. We can argue as in proposition 4.4 to prove that $\mathcal{N}G^{\epsilon}_{\mu X,w}$ is connected since the proof is based on the openness of basins of attraction and repulsion of singular points. This is guaranteed in this context by prop. 5.4.

Suppose there exists a cycle in $\mathcal{N}G^{\epsilon}_{\mu X,w}$. Then there exists a bounded region T in $x = w$ bounded by singular points and trajectories of $\Re(\mu X)$ representing the edges of the cycle. We claim that the mapping $\omega^{\epsilon}_{\mu X} : T \setminus SingX \to SingX$ is well-defined. The absence of centers (prop. 5.4) implies that there are no cycles. We obtain that every ω limit contains a singular point and then it is a singular point since the no-center property implies that a pointed neighborhood of a singular point is a union of open basins of attraction and repulsion. The mapping $\omega^{\epsilon}_{\mu X} : T \setminus SingX \to SingX$ is not constant, otherwise all the trajectories of $\Re(\mu X)$ in ∂T represent the same edge. This provides a contradiction since the topological stability implies that $\omega^{\epsilon}_{\mu X} : T \setminus SingX \to SingX$ is locally constant. □

The corollary 5.1 is a "topological" regularity property. The infinitesimal stability is a deeper condition and provides regularity in infinitesimal neighborhoods of the singular points. For instance we can associate graphs to every seed $\mathcal{S}$ just by considering the dynamics of $\Re(\mu X)_{|\mathcal{S}}$. The graphs $\mathcal{N}G^{\mathcal{S}}_{\mu X,w}$ obtained are connected and with no cycles for compact connected sets $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$ and $w \in (0,\delta(K))K$.

5.6. Defining regions. — Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2,0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$.

Definition 5.7. — *We define* $\mathrm{Reg}^{\epsilon}(\mu X, K)$ *as the set of connected components of*

$$(\alpha^{\epsilon}_{X}, \omega^{\epsilon}_{X})^{-1}(SingX \times SingX) \cap ([[0,\delta)K \times B(0,\epsilon)] \setminus SingX).$$

We call regions the elements of $\mathrm{Reg}^{\epsilon}(\mu X, K)$. *Given a region* $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$ *and a point* $w \in B(0,\delta)$ *we denote by* $R(w)$ *the set* $R \cap (\{w\} \times B(0,\epsilon))$.

Given a non-degenerate element $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ and $X = X_{\varphi}$ the regions in $\mathrm{Reg}^{\epsilon}(\mu X, K)$ are the sets where we define extensions of the Fatou coordinates of $\varphi_{|x=0}$. We take profit that the orbit space of $\varphi_{|R^{\circ}}$ is biholomorphic to the set $(0,\delta)K^{\circ} \times \mathbb{C}^*$ for any $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$.

Definition 5.8. — *Denote by* $\mathrm{orb}(\varphi, E)$ *the space of orbits of* $\varphi_{|E}$.

Definition 5.9. — *Given* $R \in \mathrm{Reg}^{\epsilon}(\mu X, K)$ *we denote* $R \in \mathrm{Reg}^{\epsilon}_1(\mu X, K)$ *if we have* $(\alpha^{\epsilon}_{X})_{|R} \equiv (\omega^{\epsilon}_{X})_{|R}$, *otherwise we denote* $R \in \mathrm{Reg}^{\epsilon}_2(\mu X, K)$.

As a consequence of the stability a region $R \in \mathrm{Reg}^{\epsilon}_1(\mu X, K)$ is homeomorphic to $[0,\delta)K \times R(0)$ by a mapping $(x, h(x,y))$. Therefore $\mathrm{orb}(\varphi, R)$ is homeomorphic to $[0,\delta)K \times \mathbb{C}^*$ and $\mathrm{orb}(\varphi, R^{\circ})$ is biholomorphic to $(0,\delta)K^{\circ} \times \mathbb{C}^*$ (see the regions R_2 and R_5 in picture (6)). A region $R \in \mathrm{Reg}^{\epsilon}_2(\mu X, K)$ satisfies that $R(x)$ has one connected component for any x in $(0,\delta)K$ whereas $R(0)$ has two connected components.

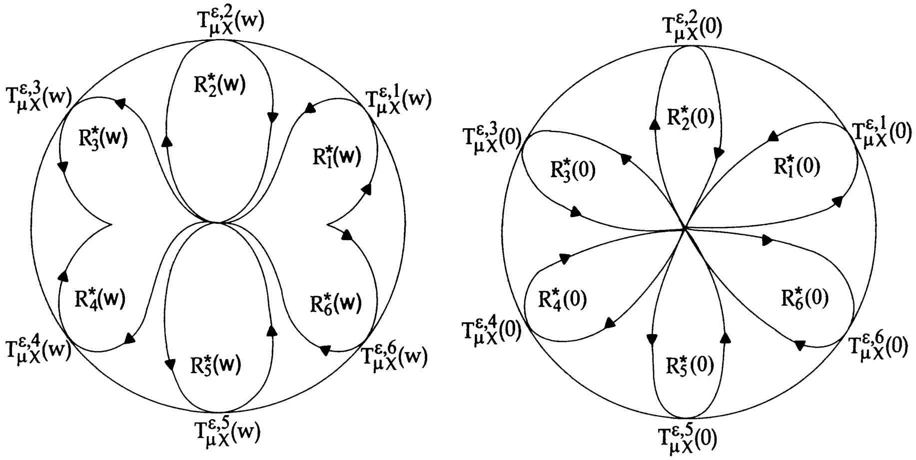

FIGURE 6. Regions for $\mu X = y^2(y-x)(y+x)\partial/\partial y$ and $w \in \mathbb{R}^+$

Moreover $\text{orb}(\varphi, R^\circ)$ is biholomorphic to $(0,\delta)K^\circ \times \mathbb{C}^*$ by stability and $\text{orb}(\varphi, R(0))$ is biholomorphic to the union of two disjoint copies of $\mathbb{C}^*$ (see the regions $R_1 = R_6$ and $R_3 = R_4$ in picture (6)).

Denote by $T^\epsilon_{\mu X}$ the set of tangent points between $\Re(\mu X)$ and $\partial_e U_\epsilon$. Denote $T^\epsilon_{\mu X} \cap (\{w\} \times \mathbb{C})$ by $T^\epsilon_{\mu X}(w)$. As the picture (6) suggests we have

Lemma 5.1. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Given $R \in \text{Reg}^\epsilon_k(\mu X, K)$ we have $\sharp(\overline{R(x)} \cap T^\epsilon_{\mu X}(x)) = k$ for any $x \in [0,\delta)K$.*

The cardinal of $\text{Reg}^\epsilon_2(\mu X, K)$ coincides with the number of edges in $\mathcal{N}G^\epsilon_{\mu X, w}$ for $w \in (0,\delta)K$. The corollary 5.1 implies

Lemma 5.2. — *Let $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then we have $\sharp\text{Reg}^\epsilon_2(\mu X, K) = N(X) - 1$.*

Let $X = u(x,y)(y - g_1(x))^{n_1} \dots (y - g_p(x))^{n_p}\partial/\partial y$ where $u \in \mathbb{C}\{x,y\}$ is a unit. The number of regions of $\text{Reg}^\epsilon(\mu X, K)$ in $(\alpha^\epsilon_X, \omega^\epsilon_X)^{-1}(\{y = g_j(x)\} \times \{y = g_j(x)\})$ is equal to $2(n_j - 1)$ for any $j \in \{1, \dots, l\}$. This leads us to

Lemma 5.3. — *Let X be a non-degenerate element of $\mathcal{X}_{p1}(\mathbb{C}^2, 0)$ and let $\mu \in \mathbb{S}^1$. Consider a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Then we have*

$$\sharp\text{Reg}^\epsilon_1(\mu X, K) + \sharp\text{Reg}^\epsilon_2(\mu X, K) = 2\nu(X) - N(X) + 1$$

and $\sharp\text{Reg}^\epsilon_1(\mu X, K) + 2\sharp\text{Reg}^\epsilon_2(\mu X, K) = 2\nu(X)$.

There are $2\nu(X)$ continuous sections $T^{\epsilon,1}_X, \dots, T^{\epsilon,2\nu(X)}_X$ of the set T^ϵ_X. We will always suppose that $T^{\epsilon,1}_X, \dots, T^{\epsilon,2\nu(X)}_X, T^{\epsilon,2\nu(X)+1}_X = T^{\epsilon,1}_X$ are ordered in counter clock-wise sense. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$. There exists a unique $T^{\epsilon,j}_{\mu X}(x) \in T^{\epsilon}_{\mu X}(x)$ in the arc of $\partial U_\epsilon(x)$ going in counter clock-wise sense from $T^{\epsilon,j}_X(x)$ to $T^{\epsilon,j+1}_X(x)$. We define $R_j^{\mu X,K}$ (or just R_j when μX and K are implicit) as the element of $\text{Reg}^\epsilon(\mu X, K)$ such that $T^{\epsilon,j}_{\mu X}(x) \in \partial R_j(x)$ for any $x \in [0,\delta)K$.

Definition 5.10. — *We define R_j^* as the union of $R_j \setminus R_j(0)$ and the connected component of $R_j(0)$ whose closure contains $T^{\epsilon,j}_{\mu X}(0)$. Given $w \in B(0,\delta)$ we denote by $R_j^*(w)$ the set $R_j^* \cap (\{w\} \times \mathbb{C})$.*

The set $\text{orb}(\varphi, R_j^*)$ is homeomorphic to $[0,\delta)K \times \mathbb{C}^*$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. We denote by $\text{Reg}^\epsilon_*(\mu X, K)$ the set $\{R_1^*, \dots, R^*_{2\nu(X)}\}$ (see picture (6) where we have $R_1 = R_6$ and $R_3 = R_4$).

We want to define an integral ψ_j^X of the time form of X (i.e. $X(\psi_j^X) = 1$) in an element R_j^* of $\text{Reg}^\epsilon_*(\mu X, K)$ for $j \in \mathbb{Z}$. An integral of the time form then transforms the vector field X in the regular vector field $\partial/\partial y$. The set $R_j^*(x)$ is simply connected for any $x \in [0,\delta)K$. We fix a holomorphic integral ψ_1^X of the time form of X in a neighborhood of $T^{\epsilon,1}_X(0)$. We can extend ψ_1^X to a neighborhood of $T^{\epsilon,1}_{\mu X}(0)$ by doing analytic continuation along the arc going from $T^{\epsilon,1}_X(0)$ to $T^{\epsilon,1}_{\mu X}(0)$ in counter clock-wise sense. We want to define a holomorphic integral ψ_j^X of the time form of X in a neighborhood of $T^{\epsilon,j}_{\mu X}(0)$ for $j \in \mathbb{Z}$; then we extend analytically ψ_j^X to $R_j^*(x)$ for any $x \in [0,\delta)K$. We obtain that ψ_j^X is continuous in R_j^* and holomorphic in $(R_j^*)^\circ$.

Given ψ_j^X we denote by $\psi_j^X(T^{\epsilon,j+1}_{\mu X}(x))$ the result of evaluating at $T^{\epsilon,j+1}_{\mu X}(x)$ the analytic extension of ψ_j^X along the arc joining $T^{\epsilon,j}_{\mu X}(0)$ and $T^{\epsilon,j+1}_{\mu X}(0)$ in $\partial_e U_\epsilon(0)$ in counter clock-wise sense. We require two conditions to the sequence $\{\psi_j^X\}_{j\in\mathbb{Z}}$, namely

- $\psi^X_{j+2\nu(X)} \equiv \psi_j^X$ for any $j \in \mathbb{Z}$.
- $c(x) \equiv \psi^X_{j+1}(T^{\epsilon,j+1}_{\mu X}(x)) - \psi^X_j(T^{\epsilon,j+1}_{\mu X}(x))$ is independent of $j \in \mathbb{Z}$.

We define the function $\zeta_X(x) = -\pi i \nu(X)^{-1} \sum_{P \in Sing X \cap (\{x\} \times B(0,\epsilon))} \text{Res}_X(P)$. It is holomorphic in a neighborhood of 0. It turns out that the previous conditions imply $c \equiv \zeta_X$. Moreover the choice of ψ_1^X determines $\{\psi_j^X\}_{j\in\mathbb{Z}}$.

5.7. Extension of the Fatou coordinates.

— Fix a non-degenerate element φ of $\text{Diff}_{p1}(\mathbb{C}^2,0)$. Denote $X_\varphi = f(x,y)\partial/\partial y$ and $X = X_\varphi$. There exists a unique Δ_φ in the ideal (f) of the ring $\mathbb{C}\{x,y\}$ such that

$$\varphi(x,y) = (\exp(tX)(t,x,y)) \circ (1 + \Delta_\varphi(x,y), x, y).$$

The previous formula is a consequence of the implicit function theorem.

Consider $\mu \in \mathbb{S}^1$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We obtain

$$\psi_j^X \circ \varphi = \psi_j^X + 1 + \Delta_\varphi \quad \text{in} \quad R_j^* \in \text{Reg}^\epsilon_*(\mu X, K).$$

Hence ψ_j^X is "almost" a Fatou coordinate of φ in R_j^*. In this subsection we sketch how to obtain a Fatou coordinate ψ_j^φ of φ in R_j^* by slight deformation of ψ_j^X.

Fix $j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})$. Consider a trajectory γ of $\Re(\mu X)$ passing through a point $T_{\mu X}^{\epsilon,j}(w)$ for some $w \in [0,\delta)K$. The strip $\exp([0,1]X)(\gamma)$ is a fundamental domain of $\exp(X)$ in $R_j^*(w)$. Denote by $S_j(w)$ the strip enclosed by γ and $\varphi(\gamma)$. Then $S_j(w)$ is a fundamental domain of φ in $R_j^*(w)$. We can build a C^∞ diffeomorphism σ_γ from a

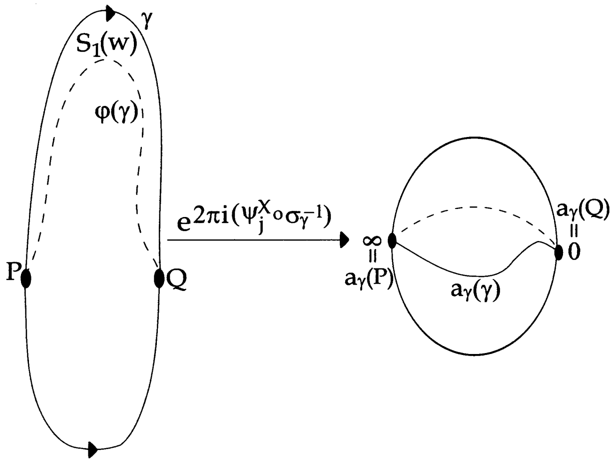

FIGURE 7. Constructing Fatou coordinates

neighborhood of $\exp([0,1]X)(\gamma)$ in $x = w$ to a neighborhood of $S_j(w)$ in $x = w$ and such that

- $\sigma_\gamma \circ \exp(X) = \varphi \circ \sigma_\gamma$ and $\sigma(\exp([0,1]X)(\gamma)) = S_j(w)$.
- $\psi_j^X \circ \sigma_\gamma - \psi_j^X = O(f)$.

The function $\psi_j^X \circ \sigma_\gamma^{-1}$ is a C^∞ Fatou coordinate of φ in the neighborhood of $S_j(w)$, i.e. $(\psi_j^X \circ \sigma_\gamma^{-1}) \circ \varphi = \psi_j^X \circ \sigma_\gamma^{-1} + 1$. We want to deform $\psi_j^X \circ \sigma_\gamma^{-1}$ slightly in order to obtain a holomorphic Fatou coordinate. Consider the diffeomorphism

$$a_\gamma : \operatorname{orb}(\varphi, S_j(w)) \overset{e^{2\pi i(\psi_j^X \circ \sigma_\gamma^{-1})}}{\longrightarrow} \mathbb{C}^*.$$

The complex dilatation $\kappa_\gamma(z) = \kappa_{a_\gamma^{-1}}(z) = (\partial a_\gamma^{-1}/\partial \bar{z})/(\partial a_\gamma^{-1}/\partial z)$ of a_γ^{-1} satisfies $|\kappa_\gamma|(z) = O(f(w,y))$ where $\psi_j^X \circ \sigma_\gamma^{-1}(w,y) = (\ln z)/(2\pi i)$. We can suppose that $\sup_{\mathbb{C}^*} |\kappa| \leq 1/2$ since $f(0,0) = 0$. Thus there exists a unique quasi-conformal mapping $d : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\kappa_d = \kappa_\gamma$, $d(0) = 0$, $d(\infty) = \infty$ and $d(1) = 1$. By choice $d \circ a_\gamma$ is a biholomorphism from $\operatorname{orb}(\varphi, S_j(w))$ to $\mathbb{C}^*$; there exists a Fatou coordinate $\tilde{\psi}_j^\varphi$ in $S_j(w)$ given by the formula $\tilde{\psi}_j^\varphi = (\ln(d \circ a_\gamma))/(2\pi i)$. The Fatou coordinate can be extended to a holomorphic function defined in $R_j^*(w)$ by using the formula $\tilde{\psi}_j^\varphi \circ \varphi \equiv \tilde{\psi}_j^\varphi + 1$. The function $\tilde{\psi}_j^\varphi + b$ is also a Fatou coordinate of φ in $R_j^*(w)$ for any $b \in \mathbb{C}$. The set $\{\tilde{\psi}_j^\varphi + b\}_{b \in \mathbb{C}}$ does not depend on the choices of ψ_j^X and σ_γ.

We would like to have

$$(1) \qquad \lim_{P\in S_j(w),\ \mathrm{Img}(\psi_j^X(P))\to\pm\infty} \tilde{\psi}_j^{\varphi}(P) - \psi_j^X(P) = c_{\pm}^{\gamma} \in \mathbb{C}.$$

In other words we want $\tilde{\psi}_j^{\varphi} - \psi_j^X$ to have a continuous extension to $\overline{S_j(w)} \cap \mathrm{Fix}(\varphi)$. The importance of the previous condition can be understood in terms of the Lavaurs vector fields.

Definition 5.11. — *There exists a unique vector field X_j^{φ} defined in $R_j^*(w)$ and such that $X_j^{\varphi}(\tilde{\psi}_j^{\varphi}) \equiv 1$. Therefore we have $\varphi \equiv \exp(X_j^{\varphi})$ in $R_j^*(w)$. Moreover, since X_j^{φ} is unique then it is continuous in R_j and holomorphic in R_j°. The vector field X_j^{φ} is the Lavaurs vector field associated to φ in R_j.*

The vector field $X_{|x=w}$ is of the form $a(y-y_0)^r(1+O(y-y_0))\partial/\partial y$ in the neighborhood of a point (w,y_0) in $\overline{S_j(w)}\cap\mathrm{Fix}(\varphi)$ for some $a \in \mathbb{C}^*$ and $r \in \mathbb{N}$. The equation (1) implies that $(X_j^{\varphi})_{|x=w}$ is of the form $a(y-y_0)^r(1+h(y))\partial/\partial y$ where h is a continuous function defined in $R_j^*(w) \cup (\overline{S_j(w)}\cap\mathrm{Fix}(\varphi))$ such that $h(y_0)=0$. The Lavaurs vector field coincides with X until the first non-zero term. In absence of the equation (1) we can only say that X_j^{φ} can be extended continuously to $\overline{S_j(w)} \cap \mathrm{Fix}(\varphi)$ as a singular vector field. The analytic invariants can be expressed in terms of the Lavaurs vector fields; as a consequence a better knowledge of their behavior provides more accurate results. An example is given by the "quantitative" theorem 3.3.

Equation (1) holds if and only if the quasi-conformal mapping $d : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is conformal at 0 and ∞. To get the conformality at 0 is enough to prove that

$$\frac{1}{\pi}\int_{|z|<r} \frac{1}{1-|\kappa_\gamma(z)|}\frac{|\kappa_\gamma(z)|}{|z|^2}d\sigma < \infty$$

for any $r \in \mathbb{R}^+$ (theorem 6.1 in page 232 of [**4**]). Such a property can be obtained if

$$(2) \qquad |\kappa_\gamma| = O\left(\frac{1}{(1+|\ln z|)^{1+1/\nu(\varphi)}}\right) \Leftarrow f_{|S_j(w)} = O\left(\frac{1}{(1+|\psi_j^X|)^{1+1/\nu(\varphi)}}\right).$$

Denote $S_j = \cup_{x\in[0,\delta)K} S_j(x)$. Now suppose that there exists $y_1 \in B(0,\epsilon)\setminus\{0\}$ such that $(0,y_1) \notin S_j(0)$ but $(0,y_1)$ belongs to $\overline{S_j}$ (of course such a behavior is ruled out because of stability but that is precisely the point, to justify the choice of infinitesimal stability). Then typically $f(0,y_1) \neq 0$ whereas

$$\lim_{(x,y)\in S_j,\ (x,y)\to(0,y_1)} \psi_j^X(x,y) = \infty$$

since $(0,y_1) \notin S_j(0)$. In this context the estimate in the right hand side of condition (2) does not hold and then we can not get the good estimates for Fatou coordinates and Lavaurs vector fields in the neighborhood of the fixed points. The problem is associated to the fact that $\lim_{x\in(0,\delta)K,x\to0}\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ is bigger than $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(0)]$.

In order to obtain condition (2) is not enough to ask for $\overline{S_j} \cap \{x=0\} \subset \overline{S_j(0)}$ (see picture (8)). The condition (2) is of quantitative type and it can be broken if we

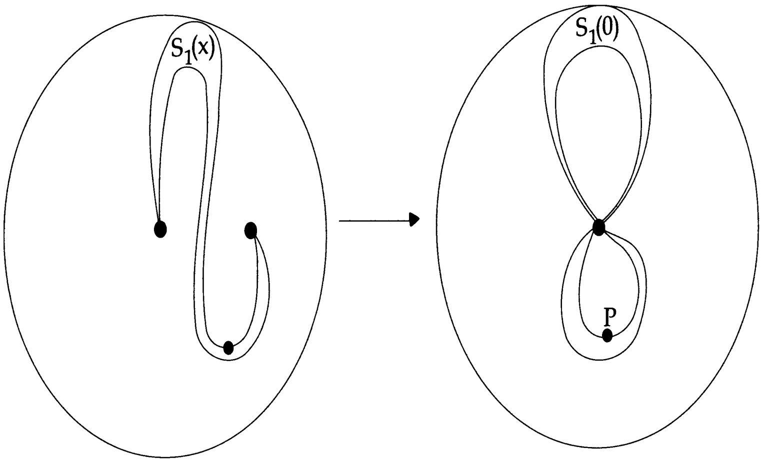

FIGURE 8. The point P is in $\overline{S_j} \setminus \overline{S_j(0)}$

have instable phenomena even in an infinitesimal scale, for instance if $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ does not converge "fast enough" to $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(0)]$ when $x \in (0,\delta)K$ and $x \to 0$. An erratic behavior of $\Gamma_{\mu X}[T_{\mu X}^{\epsilon,j}(x)]$ makes difficult to obtain similar Fatou coordinates for φ and X. An example of bad convergence would be provided by the existence of a sequence $(w_n, y_n) \in S_j$ tending to $(0,0)$ and with $1/(1+|\psi_j^X(w_n,y_n)|)^{1+1/\nu(\varphi)}$ tending faster to 0 than $|f(w_n,y_n)|$. (see picture (9), it is somehow an infinitesimal version of picture (8)). All these pathologies are excluded as a consequence of the infinitesimal stability. The infinitesimal stability plays here a somehow analogous role to the Rolle property in the multi-parabolic case. It is a property of non-wandering type, forcing the trajectories of $\Re(\mu X)$ to go "fast" towards the fixed points. We bound the "complexity" of trajectories of $\Re(\mu X)$. Our quantitative study makes all these ideas rigorous.

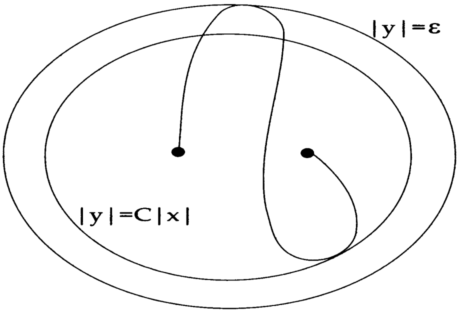

FIGURE 9. There is no trajectory splitting $y = |C|x$

Proposition 5.6. — *Let φ be a non-degenerate element of* $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. *Consider* $X_\varphi = f\partial/\partial y$, $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ *and a compact connected set* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. *Then we have*

$$\Delta_\varphi = O(f) = O(y \circ \varphi - y) = O\left(\frac{1}{(1 + |\psi_j^X|)^{1+\frac{1}{\nu(\varphi)}}}\right)$$

in S_j *for any* $R_j^* \in \mathrm{Reg}_*^\epsilon(\mu X, K)$.

Fix $w \in [0, \delta)K$. By the previous discussion there exists a Fatou coordinate $\tilde{\psi}_j^\varphi$ of φ in $R_j^* \in \mathrm{Reg}_*^\epsilon(\mu X, K)$ satisfying the condition (1). It is unique if we require

$$\lim_{P \in S_j(w),\ \mathrm{Img}(\psi_j^X(P)) \to +\infty} \tilde{\psi}_j^\varphi(P) - \psi_j^X(P) = 0. \tag{3}$$

The uniqueness implies that the Fatou coordinate $\tilde{\psi}_j^\varphi$ is continuous in R_j^* and holomorphic in R_j°. In other words the uniqueness forces the holomorphic dependance with respect to w. A stronger property can be proved, namely the function $\tilde{\psi}_j^\varphi - \psi_j^X$ admits a continuous extension to the fixed points.

Proposition 5.7. — *Let φ be a non-degenerate element of* $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. *Consider* $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ *and a compact connected set* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. *Then* $\tilde{\psi}_j^\varphi - \psi_j^X$ *is continuous in* $R_j \cup (\mathrm{Fix}(\varphi) \cap \overline{R_j})$ *and holomorphic in its interior for any* R_j *in* $\mathrm{Reg}^\epsilon(\mu X, K)$.

With respect to the previous proposition let us remark that a priori $\tilde{\psi}_j^\varphi - \psi_j^X$ is defined only in R_j^* and not in the whole R_j when $R_j \in \mathrm{Reg}_2(\mu X, K)$. But then there exists a unique $k \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}) \setminus \{j\}$ such that $R_j \setminus R_j(0) = R_k \setminus R_k(0)$. The function $(\tilde{\psi}_j^\varphi - \psi_j^X) - (\tilde{\psi}_k^\varphi - \psi_k^X)$ depends only on x and by condition (3) is identically 0. As a consequence the function $\tilde{\psi}_j^\varphi - \psi_j^X$ is well-defined in $R_j = R_j^* \cup R_k^*$. The next result is the analogous of proposition 5.7 for Lavaurs vector fields.

Corollary 5.2. — *Let* $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ *be non-degenerate. Consider* $X_\varphi = f\partial/\partial y$. *Let* $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ *and consider a compact connected set* $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. *Then* X_j^φ *is of the form* $f(x, y)(1 + q(x, y))\partial/\partial y$ *where q is continuous in* $R_j \cup (\mathrm{Fix}(\varphi) \cap \overline{R_j})$ *and holomorphic in its interior for any* $R_j \in \mathrm{Reg}^\epsilon(\mu X, K)$. *Moreover we have* $q_{|\mathrm{Fix}(\varphi) \cap \overline{R_j}} \equiv 0$.

Fix an irreducible component $\tau = \{y = h(x)\}$ of $\mathrm{Fix}(\varphi)$. Consider a system of integrals of the time form $\{\psi_j^X\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$ of X constructed as described in subsection 5.6. There exists a unique family $\{b_j\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$ of functions being continuous in $[0, \delta)K$ and holomorphic in $(0, \delta)K^\circ$ such that the family

$$\{\psi_j^\varphi\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})} \stackrel{\mathrm{def}}{=} \{\tilde{\psi}_j^\varphi + b_j\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$$

satisfies

- $(\psi_{j+1}^{\varphi} - \psi_j^{\varphi})_{|\overline{R_j} \cap \overline{R_{j+1}} \cap \text{Fix}(\varphi)} \equiv \zeta_X(x)$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.
- $(\psi_j^{\varphi} - \psi_j^{X})_{|\tau} \equiv 0$ for any $j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$.

We say that τ is a *privileged curve* and that $\{\psi_j^{\varphi}\}_{j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})}$ is a *privileged system of Fatou coordinates* with respect to $\{\psi_j^{X}\}_{j \in \mathbb{Z}/(2\nu(X)\mathbb{Z})}$ and τ.

5.8. Extension of the Ecalle-Voronin invariants. — Fix a non-degenerate element φ of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$. Denote $X = X_{\varphi}$. Consider $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We denote $D(\varphi) = \mathbb{Z}/(2\nu(\varphi)\mathbb{Z})$. We define

$$D_{-1}(\varphi) = \{j \in \mathbb{Z}/(2\nu(\varphi)\mathbb{Z}) : \Re(X) \text{ points at } T_{\mu X}^{\epsilon,j}(0) \text{ towards } R_j\}.$$

We denote $D_1(\varphi) = D(\varphi) \setminus D_{-1}(\varphi)$.

The mapping (x, ψ_j^{φ}) conjugates φ and $(x, z+1)$ in R_j^* for any $j \in D(\varphi)$. The set $R_j^*(0)$ is contained in an attracting petal (resp. an repelling petal) P_j of $\varphi_{|x=0}$ if $j \in D_{-1}(\varphi)$ (resp. $j \in D_1(\varphi)$). For instance in the repelling case the set P_j is composed by the points $y \in B(0, \epsilon)$ satisfying that there exists $k \in \mathbb{N}$ such that $\varphi^{-k}(0, y) \in R_j^*(0)$ and $\varphi^{-l}(0, y) \in \{0\} \times B(0, \epsilon)$ for any $0 < l < k$. We have that $P_j \cap P_k \neq \emptyset$ implies $k \in \{j-1, j, j+1\}$.

We can interpret $\varphi_{|x=0}$ as the diffeomorphism $z+1$ defined in the manifold with charts $\psi_j^{\varphi}(P_j)$ for $j \in D(\varphi)$ and whose changes of charts commute with $z+1$ and are of the form $\psi_{j+1}^{\varphi}(0, y) \circ (\psi_j^{\varphi}(0, y))^{-1}$ for $j \in D(\varphi)$. Moreover, the function $\psi_{j+1}^{\varphi}(0, y) \circ (\psi_j^{\varphi}(0, y))^{-1}$ is defined in $\psi_j^{\varphi}(\{0\} \times (P_j \cap P_{j+1})) \sim \{s\text{Img}z > I\}$ for all $s \in \{-1, 1\}$, $j \in D_s(\varphi)$ and some $I \in \mathbb{R}^+$.

The map $e^{2\pi i \psi_j^{\varphi}}(0, y)$ is a biholomorphism from $\text{orb}(\varphi, R_j^*(0)) = \text{orb}(\varphi_{|x=0}, P_j)$ to $\mathbb{C}^*$. Thus $\text{orb}(\varphi, x=0)$ is the union of $\text{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ for $j \in D(\varphi)$. We have that $\text{orb}(\varphi, R_j^*(0)) \cap \text{orb}(\varphi, R_{j+1}^*(0)) \neq \emptyset$ in $\text{orb}(\varphi, x=0)$ since $P_j \cap P_{j+1} \neq \emptyset$ for $j \in D(\varphi)$. Suppose $j \in D_1(\varphi)$; then the germs of $\text{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ and $\text{orb}(\varphi, R_{j+1}^*(0)) \sim \mathbb{C}^*$ at 0 are identified by a germ $\Upsilon_j \in \text{Diff}(\mathbb{C}, 0)$ defined in a domain $B(0, e^{-2\pi I})$. For $j \in D_{-1}(\varphi)$ the germs of $\text{orb}(\varphi, R_j^*(0)) \sim \mathbb{C}^*$ and $\text{orb}(\varphi, R_{j+1}^*(0)) \sim \mathbb{C}^*$ at ∞ are identified by a germ $\Upsilon_j \in \text{Diff}(\mathbb{C}, \infty)$ defined in a domain $\{|z| > e^{2\pi I}\}$. The space of orbits $\text{orb}(\varphi, x=0)$ is a string of spheres glued by the system of changes of charts $\{\Upsilon_j\}_{j \in D(\varphi)}$. Indeed the string of spheres provides a complete system of analytic invariants for the classification of elements of $\text{Diff}_1(\mathbb{C}, 0)$. This is the Martinet-Ramis [**6**] presentation of the Ecalle-Voronin invariants. We will apply the same program to non-degenerate elements of $\text{Diff}_{p1}(\mathbb{C}^2, 0)$.

Let $R_j \in \text{Reg}_2^{\epsilon}(\mu X, K)$. There exists a seed $\mathcal{S}_j$ containing $\{\alpha_X^{\epsilon}(R_j), \omega_X^{\epsilon}(R_j)\}$ but such that $\alpha_X^{\epsilon}(R_j(x))$ and $\omega_X^{\epsilon}(R_j(x))$ are in different seeds among the sons of $\mathcal{S}_j$ for any $x \in (0, \delta)K$. Consider the compact-like set $\mathcal{C}_j$ associated to $\mathcal{S}_j$. We define $R_j^{\natural} = \{Q \in R_j : \Gamma_{\mu X}^{\epsilon}[Q] \subset \mathcal{C}_j\}$ and $R_j^{\flat}$ as the connected component of $R_j^* \setminus R_j^{\natural}$ whose closure contains $T_{\mu X}^{\epsilon,j}(x)$ for any $x \in [0, \delta)K$. We define $R_j^{\flat} = R_j$ for $R_j \in \text{Reg}_1^{\epsilon}(\mu X, K)$. Denote by $R_{j,j+1}$ the connected component of the set $([0, \delta)K \times B(0, \epsilon)) \setminus (R_j \cup$

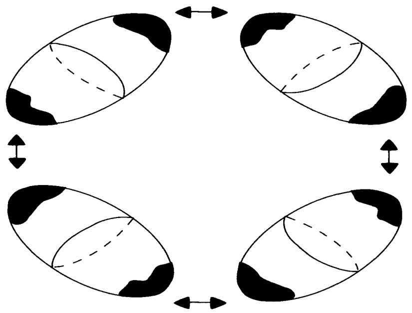

FIGURE 10. Space of orbits of $\varphi_{|x=0}$ with identifications in black

$R_{j+1} \cup \mathrm{Fix}(\varphi))$ such that $\overline{R_{j,j+1}(x)}$ contains the arc in $\partial U_{\epsilon}(x)$ going from $T_{\mu X}^{\epsilon,j}(x)$ to $T_{\mu X}^{\epsilon,j+1}(x)$ in counter clock-wise sense for any point $x \in [0,\delta)K$.

The restriction of $\Re(X)$ to $R_j^* \cup R_{j+1}^* \cup R_{j,j+1}$ satisfies the no-return property (see subsection 4.2) if and only if $R_j \neq R_{j+1}$. This is the reason because we introduced the sets $R_j^\flat$ since the restriction of $\Re(X)$ to $R_j^\flat \cup R_{j+1}^\flat \cup R_{j,j+1}$ satisfies the no-return property for any $j \in D(\varphi)$. Thus $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ are embedded in $\mathrm{orb}(\varphi, R_j^\flat \cup R_{j+1}^\flat \cup R_{j,j+1})$ and their intersection is not empty. Define the change of charts $\varpi_{\varphi,\mu,K}^j(x,z) = e^{2\pi i \psi_{j+1}^\varphi} \circ (x, e^{2\pi i \psi_j^\varphi})^{\circ(-1)}$ between $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ for any $j \in D(\varphi)$. We have

$$\mathrm{orb}(\varphi, R_j^*) \cap \mathrm{orb}(\varphi, R_{j+1}^*) \sim [0,\delta)K \times \{0 < |z|^s < h\}$$

or more precisely $\varpi_{\varphi,\mu,K}^j$ is defined in $[0,\delta)K \times \{0 < |z|^s < h\}$ for all $s \in \{-1,1\}$, $j \in D_s(\varphi)$ and some $h > 0$.

The value of $\psi_{j+1}^\varphi - \psi_j^\varphi$ at the fixed point $(\overline{R_{j,j+1}} \cap \mathrm{Fix}(\varphi)) \cap \{x = w\}$ is equal to $\zeta_X(w)$ for any $w \in [0,\delta)K$. The previous discussion implies:

Proposition 5.8. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1,1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_\varphi)$. Then there exists $h \in \mathbb{R}^+$ such that for all $s \in \{-1,1\}$ and $j \in D_s(\varphi)$ we have*

- $\varpi_{\varphi,\mu,K}^j \in C^0([0,\delta)K \times \{|z|^s < h\}) \cap \mathcal{O}((0,\delta)K^\circ \times \{|z|^s < h\})$.
- $\varpi_{\varphi,\mu,K}^j(0,z)$ *does not depend on μ and K.*
- $\varpi_{\varphi,\mu,K}^j$ *is of the form* $e^{2\pi i \zeta_{X_\varphi}(x)} z \left(1 + \sum_{l=1}^{\infty} b_{j,l,\mu,K}^\varphi(x) z^{sl}\right)$.
- $b_{j,l,\mu,K}^\varphi \in C^0([0,\delta)K) \cap \mathcal{O}((0,\delta)K^\circ)$ *for any $l \in \mathbb{N}$.*

We define the *μ-space of orbits* of φ at K as the variety obtained by taking an atlas composed of charts $\mathrm{orb}(\varphi, R_j^*) \sim [0,\delta)K \times \mathbb{C}^*$ for $j \in D(\varphi)$ and the changes of charts $(x, \varpi_{\varphi,\mu,K}^j)$ identifying subsets of $\mathrm{orb}(\varphi, R_j^*)$ and $\mathrm{orb}(\varphi, R_{j+1}^*)$ for any j in $D(\varphi)$. The μ-space and the space of orbits of φ coincide for $x = 0$. They are different for $x = w \in (0,\delta)K$ since the μ-space of orbits does not contain the identifications

$$\mathrm{orb}(\varphi, R_j^* \setminus R_j^*(0)) \sim \mathrm{orb}(\varphi, R_k^* \setminus R_k^*(0))$$

for $R_j = R_k \in \mathrm{Reg}_2^\epsilon(\mu X, K)$ with $R_j^* \neq R_k^*$. We have that $\psi_k^\varphi - \psi_j^\varphi \equiv \psi_k^X - \psi_j^X$ is a pure meromorphic function in $\mathbb{C}\{x\}[x^{-1}]$. As a consequence we obtain

$$(x, e^{2\pi i \psi_k^\varphi}) \circ (x, e^{2\pi i \psi_j^\varphi})^{\circ(-1)} = (x, e^{2\pi i(\psi_k^X - \psi_j^X)(x)} z).$$

The μ-space of orbits of φ at K is the space of orbits of φ restricted to

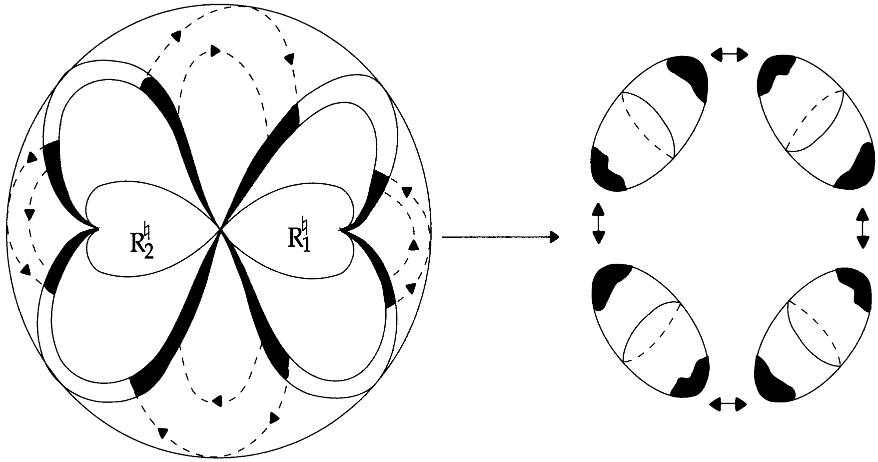

FIGURE 11. The μ-space of orbits restricted to $x = w$

$$B_{\varphi,\mu,K} \stackrel{\mathrm{def}}{=} ([0,\delta)K \times B(0,\epsilon)) \setminus ((\cup_{R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)} R_j^\natural) \cup \mathrm{Fix}(\varphi)).$$

Moreover the restriction of φ to $B_{\varphi,\mu,K}$ satisfies the no-return property. Thus the μ-space of orbits depend nicely on the parameter x. By removing $\cup_{R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)} R_j^\natural$ we dismiss the complexity related to small divisors problems. Of course we can recover any dynamical information attached to the system by adding the "singular" identifications $(x, e^{2\pi i(\psi_k^X - \psi_j^X)(x)} z)$ for any $R_j \in \mathrm{Reg}_2^\epsilon(\mu X, K)$ where $R_k = R_j$ and $R_k^* \neq R_j^*$.

Remark 5.2. — *Given a direction $x \in \lambda\mathbb{R}^+$ with $\lambda \in \mathbb{S}^1$ the set*

$$\{\mu \in \mathbb{S}^1 : \lambda \in \mathcal{U}(\mu X)\}$$

is finite. The choices of μ in the same connected component of

$$e^{(0,\pi i)} \setminus \{\iota \in \mathbb{S}^1 : \lambda \in \mathcal{U}(\iota X)\}$$

provide equivalent definitions of the μ-space of orbits in the neighborhood of the direction $x \in \lambda\mathbb{R}^+$. Thus there are finitely many choices of the μ-space of orbits.

The diffeomorphism φ is of the form $(x, z+1)$ in a manifold with changes of charts

$$(x, \xi^{j}_{\varphi,\mu,K}) = (x, \psi^{\varphi}_{j+1}) \circ (x, \psi^{\varphi}_{j})^{-1}$$

for $j \in D(\varphi)$. Since

$$\varpi^{j}_{\varphi,\mu,K} = e^{2\pi i z} \circ \xi^{j}_{\varphi,\mu,K} \circ \left(x, \frac{\ln z}{2\pi i}\right)$$

then proposition 5.8 implies

Proposition 5.9. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_{\varphi})$. Then there exists $I \in \mathbb{R}^+$ such that for all $s \in \{-1, 1\}$ and $j \in D_s(\varphi)$ we have*

- $\xi^{j}_{\varphi,\mu,K}(x, z+1) = \xi^{j}_{\varphi,\mu,K}(x, z) + 1$.
- $\xi^{j}_{\varphi,\mu,K} \in C^0([0,\delta)K \times \{s\mathrm{Im}z > I\}) \cap \mathcal{O}((0,\delta)K^{\circ} \times \{s\mathrm{Im}z > I\})$.
- $\xi^{j}_{\varphi,\mu,K}(0, z)$ *does not depend on* μ *and* K.
- $\xi^{j}_{\varphi,\mu,K}$ *is of the form* $z + \zeta_{X_{\varphi}}(x) + \sum_{l=1}^{\infty} a^{\varphi}_{j,l,\mu,K}(x) e^{2\pi i s l z}$.
- $a^{\varphi}_{j,l,\mu,K} \in C^0([0,\delta)K) \cap \mathcal{O}((0,\delta)K^{\circ})$ *for any* $l \in \mathbb{N}$.

5.9. Analytic conjugacy. — Fix a non-degenerate element φ of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Denote $X = X_{\varphi}$. Consider $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. The μ-space of orbits of φ at K is a rigid object since it is composed of copies of $[0,\delta)K \times \mathbb{C}^*$ and the set of biholomorphisms of $\mathbb{C}^*$ fixing 0 and ∞ is isomorphic to $\mathbb{C}^*$. The rigidity implies that the μ-space of orbits determines the analytic class of conjugacy of φ. We obtain a complete system of analytic invariants for non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Roughly speaking it is a fibered version of the Ecalle-Voronin system of invariants.

Let φ, η be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose that φ and η are formally conjugated. Then there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ satisfying $\hat{\sigma} \circ \varphi = \eta \circ \hat{\sigma}$ and whose action on $\mathrm{Fix}(\varphi)$ is analytic **[9]** **[11]**. More precisely there exists an element σ of $\mathrm{Diff}\,(\mathbb{C}^2, 0)$ such that

(a) $x \circ \hat{\sigma} \equiv x \circ \sigma$.
(b) $y \circ \hat{\sigma} - y \circ \sigma$ belongs to the ideal associated to the analytic set $\mathrm{Fix}(\varphi)$.

Indeed the mapping σ satisfies **[9]** **[11]**

(1) $x \circ \sigma \in \mathbb{C}\{x\}$
(2) $(y \circ \eta - y) \circ \sigma = (y \circ \varphi - y)$
(3) $\mathrm{Res}(X_{\eta}, \sigma(P)) = \mathrm{Res}(X_{\varphi}, P)$ for any $P \in \mathrm{Fix}(\varphi)$.

Moreover, given σ holding conditions (1), (2) and (3) there exists $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ conjugating φ and η and such that σ and $\hat{\sigma}$ satisfy the conditions (a) and (b). Then up to identify such σ and replacing η with $\sigma^{-1} \circ \eta \circ \sigma$ we can suppose

(A) $(y \circ \eta - y) = (y \circ \varphi - y)$
(B) $\mathrm{Res}(X_{\eta}, P) = \mathrm{Res}(X_{\varphi}, P)$ for any $P \in \mathrm{Fix}(\varphi)$ (see def. 4.4).

The ideal $(X_\varphi(y)) = (y \circ \varphi - y)$ and the function $\mathrm{Res}(X_\varphi) : \mathrm{Fix}(\varphi) \to \mathbb{C}$ compose a complete system of analytic invariants of X_φ. As a consequence if the conditions (A) and (B) are satisfied there exists $\tilde{\sigma} = (x, \sigma_1(x,y)) \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ such that $\tilde{\sigma}_{|\mathrm{Fix}(\varphi)} \equiv Id$ and and $\tilde{\sigma}_* X_\varphi = X_\eta$. Then up to replace η with $\tilde{\sigma}^{-1} \circ \eta \circ \tilde{\sigma}$ we can suppose that $X_\varphi \equiv X_\eta$.

Definition 5.12. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. We say that φ and η have common normal form* $\exp(X)$ *if there exists a vector field $X \in \mathcal{X}_{p1}(\mathbb{C}^2, 0)$ satisfying the proximity condition for both φ and η. We say that a formal conjugation $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ between φ and η is normalized if $\hat{\sigma}_{|\mathrm{Fix}(\varphi)} \equiv Id$. We denote by $\hat{Z}(\varphi)$ the group of normalized elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ commuting with φ.*

The previous discussion implies that we can suppose that when dealing with formal conjugacy problems we can reduce the problem to diffeomorphisms φ, η with common normal form and the equivalence $\sim_{\mathrm{an}}$ instead of analytic classification.

Let φ, η be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common normal form. Suppose that $\varphi \sim_{for} \eta$ by $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$. The set of normalized elements of $\widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ conjugating φ and η is equal to $\hat{Z}(\eta) \circ \hat{\sigma}$. Then it makes sense to analyze the nature of $\hat{Z}(\eta)$.

Proposition 5.10. — *Let φ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose $N(\varphi) = 1$. Then we obtain that $\hat{Z}(\varphi)$ is the semi-direct product of a finite group of order $\nu(\varphi)$ and $\{\exp(\hat{c}(x) \log \varphi) : \hat{c} \in \mathbb{C}[[x]]\}$. Moreover $\hat{Z}(\varphi)$ is commutative.*

Proposition 5.11. — *Let φ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose $N(\varphi) > 1$. Then we obtain $\hat{Z}(\varphi) = \{\exp(\hat{c}(x) \log \varphi) : \hat{c} \in \mathbb{C}[[x]]\}$. In particular $\hat{Z}(\varphi)$ is commutative.*

Let φ, η be non-degenerate elements of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ with common normal form $\exp(X)$. Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. We choose a system $\{\psi_j^X\}_{j \in D(\varphi)}$ as described in subsection 5.6. By fixing a privileged curve τ in $\mathrm{Fix}(\varphi)$ we can construct $\{\psi_j^\varphi\}_{j \in D(\varphi)}$ and $\{\psi_j^\eta\}_{j \in D(\varphi)}$ (subsection 5.7). We can define the mapping

$$\sigma_j(\varphi, \eta) = (x, \psi_j^\eta)^{-1} \circ (x, \psi_j^\varphi)$$

defined in $R_j \in \mathrm{Reg}^\epsilon(\mu X, K)$ for any $j \in D(\varphi)$ and some $\epsilon > 0$ independent of j. As usual the diffeomorphism $\sigma_j(\varphi, \eta)$ admits an extension to a bigger domain by using the formula $\sigma_j(\varphi, \eta) \circ \varphi = \eta \circ \sigma_j(\varphi, \eta)$. Suppose that $\varphi \sim_{\mathrm{an}} \eta$ by $\sigma \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$. Supposed $N(\varphi) > 1$, it satisfies

$$\sigma_{|R_j} = \exp(c(x) X_j^\eta) \circ \sigma_j(\varphi, \eta)$$

for any $j \in D(\varphi)$ and some $c \in \mathbb{C}\{x\}$ independent of j and K. In the case $N(\varphi) = 1$ the group $\hat{Z}(\eta)$ contains a finite group; thus σ satisfies

$$\sigma_{|R_j} = \exp(c(x) X_j^\eta) \circ [(x, \psi_{j+l}^\eta)^{-1} \circ (x, \psi_j^\eta)] \circ \sigma_j(\varphi, \eta)$$

for any $j \in D(\varphi)$ and some $c \in \mathbb{C}\{x\}$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ independent of j and K. The latter equation can be rewritten in the form

$$\sigma_{|R_j} = (x, \psi_{j+l}^{\eta})^{-1} \circ (x, \psi_j^{\varphi} + c(x))$$

since $\exp(c(x)X_j^{\eta}) = (x, \psi_j^{\eta})^{-1} \circ (x, \psi_j^{\eta} + c(x))$. Then we obtain $\varphi \sim_{\mathrm{an}} \eta$ if the equations

$$(4) \qquad (x, \psi_{j+l}^{\eta})^{-1} \circ (x, \psi_j^{\varphi} + c(x)) = (x, \psi_{j+1+l}^{\eta})^{-1} \circ (x, \psi_{j+1}^{\varphi} + c(x)) \ \ \forall j \in D(\varphi)$$

hold true in the intersections of their domains of definition for some $c \in \mathbb{C}\{x\}$ and l in $\mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ (note that $l = 0$ if $N(\varphi) > 1$). The equation (4) is equivalent to

$$\xi_{\eta,\mu,K}^{j+l}(x, z + c(x)) \equiv \xi_{\varphi,\mu,K}^{j}(x, z) + c(x) \ \ \forall j \in D(\varphi).$$

As a consequence we obtain

Theorem 5.3. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate with common normal form* $\exp(X)$. *Suppose $N(\varphi) = 1$. Then $\varphi \sim_{\mathrm{an}} \eta$ if and only if*

$$\xi_{\eta,i,\mathbb{S}^1}^{j+l}(x, z + c(x)) \equiv \xi_{\varphi,i,\mathbb{S}^1}^{j}(x, z) + c(x) \ \ \forall j \in D(\varphi)$$

for some $c \in \mathbb{C}\{x\}$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$.

Theorem 5.4. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate with common normal form* $\exp(X)$. *Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$. Suppose $N(\varphi) > 1$. Then $\varphi \sim_{\mathrm{an}} \eta$ if and only if there exists $c \in \mathbb{C}\{x\}$ such that*

$$\xi_{\eta,\mu,K}^{j}(x, z + c(x)) \equiv \xi_{\varphi,\mu,K}^{j}(x, z) + c(x)$$

for any $j \in D(\varphi)$.

Corollary 5.3. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate. Fix $\mu \in \mathbb{S}^1 \setminus \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X_{\varphi})$. Then $\log \varphi \in \mathcal{X}(\mathbb{C}^2, 0)$ if and only if $\xi_{\varphi,\mu,K}^{j}(x, z) \equiv z + \zeta_{X_{\varphi}}(x)$ for any $j \in D(\varphi)$.*

Proof of the corollary. — The infinitesimal generator $\log \varphi$ of φ is analytic if and only if $\varphi \sim_{\mathrm{an}} \exp(X_{\varphi})$. Indeed if $\log \varphi$ is analytic then $\log \varphi$ and X_{φ} satisfy conditions (A) and (B). Thus we obtain $\log \varphi \sim_{\mathrm{an}} X_{\varphi}$ and then $\varphi \sim_{\mathrm{an}} \exp(X_{\varphi})$. The corollary is now a consequence of $\xi_{\exp(X_{\varphi}),\mu,K}^{j}(x, z) \equiv z + \zeta_{X_{\varphi}}(x)$ for any $j \in D(\varphi)$ and theorems 5.3 and 5.4. □

The theorem 5.3 is a parametrized version of the Ecalle-Voronin theorem of analytic classification. The proof of the implication $\Rightarrow$ in theorem 5.4 is simple. The implication $\Leftarrow$ is trickier since a priori the conjugating mapping σ is defined in a set $[0, \delta)K \times B(0, \epsilon)$. Anyway, given a compact connected $J \subset \mathcal{U}(\mu' X)$ the existence of σ implies that

$$\xi_{\eta,\mu',J}^{j}(x, z + c(x)) \equiv \xi_{\varphi,\mu',J}^{j} + c(x)$$

in $\{x \in [0, \delta)(K \cap J)\}$ for any $j \in D(\varphi)$. By analytic continuation we can extend the previous equalities to $\{x \in [0, \delta)J\}$. We then construct a conjugation σ_J between φ and η and defined in a set $[0, \delta)J \times B(0, \epsilon)$. The mappings σ and σ_J coincide, we have extended σ to $[0,\delta)(K \cup J) \times B(0,\epsilon)$. By iterating this process we obtain that σ belongs to Diff $(\mathbb{C}^2, 0)$.

We discuss next why theorem 3.3 holds true. Let $r \in \mathbb{R}^+$ such that there exists an injective κ_w defined in $B(0,r)$, with $(\kappa_w)_{|\mathrm{Fix}(\varphi)\cap\{x=w\}} \equiv Id$ and conjugating $\varphi_{|x=w}$ and $\eta_{|x=w}$ for any $w \neq 0$. The existence of κ_w for $w \neq 0$ implies that φ and η satisfy conditions (A) and (B). As a consequence we can suppose that φ and η have a common normal form $\exp(X)$. It can be proved that the mappings κ_w have a moderated behavior. More precisely, by considering a smaller $r > 0$ if necessary we can suppose that there exists $R > 0$ such that $\kappa_w(B(0,r)) \subset B(0,R)$ for any $w \neq 0$. Fix $\mu \in \mathbb{S}^1 \backslash \{-1, 1\}$ and a compact connected set $K \subset \mathbb{S}^1 \backslash \mathcal{U}(\mu X_\varphi)$. Given $w \in (0,\delta)K$ then κ_w satisfies

$$(\kappa_w)_{|R_j(w)}(y) = \exp(c(w)X_j^\eta) \circ \left([(x, \psi_{j+l(w)}^\eta)^{-1} \circ (x, \psi_j^\eta)] \circ \sigma_j(\varphi,\eta)(w,y)\right)$$

for some $c(w) \in \mathbb{C}$, $l(w) \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ and any $j \in D(\varphi)$. Indeed we have

$$\xi_{\eta,\mu,K}^{j+l(w)}(w, z + c(w)) \equiv \xi_{\varphi,\mu,K}^{j}(w,z) + c(w) \quad \forall (w,j) \in (0,\delta)K \times D(\varphi).$$

We can suppose that both $\log\varphi$ and $\log\eta$ are non-analytic (divergent), otherwise both of them are analytic by corollary 5.3 and theorems 5.3 and 5.4. In such a case $\log\varphi$ and $\log\eta$ satisfy conditions (A) and (B) and then $\varphi \sim_{\mathrm{an}} \eta$. The mappings

$$[(x, \psi_{j+l(w)}^\eta)^{-1} \circ (x, \psi_j^\eta)] \circ \sigma_j(\varphi,\eta)(w,y)$$

are moderated, i.e. their images are contained in $B(0,R)$ for all $w \in (0,\delta)K$ and $j \in D(\varphi)$ (maybe by considering a smaller $r > 0$). Moreover since X is very similar to X_j^φ in R_j (prop. 5.7 and cor. 5.2) then

$$\exp(c(w)X)(\{w\} \times B(0,r)) \subset \{w\} \times B(0,R) \tag{5}$$

for any $w \in (0,\delta)K$. We obtain the expression (5) for any $w \neq 0$ by considering other compact sets. Now it is not complicated to prove that the previous condition implies that c is bounded in a pointed neighborhood of 0. The existence of κ_w for $w \neq 0$ also implies the existence of a function $d(x)$ and $l \in \mathbb{Z}/(\nu(\varphi)\mathbb{Z})$ such that

$$\xi_{\eta,\mu,K}^{j+l}(x, z + d(x)) \equiv \xi_{\varphi,\mu,K}^{j}(x,z) + d(x) \quad \forall j \in D(\varphi).$$

The function d is defined in the universal covering of $B(0,\delta) \setminus T$ where T is a closed set such that $T \setminus \{0\}$ is discrete. It turns out that since η is not analytically trivial the function $\mathrm{Im}(d)$ is well-defined in $B(0,\delta) \setminus T$. Moreover the boundness of c implies that $\mathrm{Im}(d)$ is bounded in $B(0,\delta) \setminus T$. By using the last property we deduce that d can be extended to a neighborhood of the origin to obtain an element of $\mathbb{C}\{x\}$. Theorems 5.3 and 5.4 imply that $\varphi \sim_{\mathrm{an}} \eta$.

6. Compactification of the Ecalle-Voronin invariants

Let φ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Suppose that there exists a parabolic component $\gamma = \{y = h(x)\}$ of $\mathrm{Fix}(\varphi)$ (see def. 5.1). Given $w \in B(0,\delta)$

we can consider the germ $\varphi_{(w,h(w))}$ of $\varphi_{|x=w}$ in the neighborhood of $y = h(w)$. This section is devoted to sketch the proof of the following theorem:

Theorem 6.1. — *Let $\varphi, \eta \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ be non-degenerate with common normal form $\exp(X)$. Suppose that there exists a parabolic irreducible component γ of $\mathrm{Fix}(\varphi)$. Suppose also that φ_0 is highly non-trivial and $N(\varphi) = 2$. Then $\varphi \sim_{an} \eta$ if and only if $\varphi_Q \overset{an_1}{\sim} \eta_Q$ for any Q in a pointed neighborhood of 0 in γ.*

Definition 6.1. — *We denote $\phi \overset{an_1}{\sim} \tau$ for $\phi, \tau \in \mathrm{Diff}\,(\mathbb{C}, 0)$ if there exists a mapping in $\mathrm{Diff}_1(\mathbb{C}, 0)$ conjugating ϕ and τ.*

Definition 6.2. — *We say that $\phi \in \mathrm{Diff}_1(\mathbb{C}, 0)$ is highly non-trivial if no change of charts associated to ϕ is a translation. Given $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$ the highly non-triviality of $\varphi_{|x=0}$ is equivalent to $\xi^j_{\varphi,\mu,K}(0, z) \not\equiv z + \zeta_{X_\varphi}(0)$ for any $j \in D(\varphi)$. This condition is generic and independent of the choices of $\mu \in \mathbb{S}^1$ and $K \subset \mathbb{S}^1 \setminus \mathcal{U}(\mu X)$.*

Definition 6.3. — *Let φ be a non-degenerate element of $\mathrm{Diff}_{p1}(\mathbb{C}^2, 0)$. Given an irreducible component γ of $\mathrm{Fix}(\varphi)$ we define $\nu_\varphi(\gamma)$ as the only natural number such that $y \circ \varphi - y \in I(\gamma)^{\nu_\varphi(\gamma)+1} \setminus I(\gamma)^{\nu_\varphi(\gamma)+2}$ where $I(\gamma)$ is the ideal of γ.*

Remark 6.1. — *There is a version of theorem 6.1 for the case $N(\varphi) = 1$. It is much less interesting since in that case the number of changes of charts associated to φ_Q for $Q \in \gamma \setminus \{(0, 0)\}$ and to φ coincide. Roughly speaking, the analytic invariants of φ are the union of the Ecalle-Voroning invariants of φ_Q for $Q \in \gamma \setminus \{(0, 0)\}$.*

Remark 6.2. — *The case $N(\varphi) = 2$ is generic among the diffeomorphisms satisfying $N(\varphi) > 1$.*

The implication $\Rightarrow$ in theorem 6.1 is trivial. We focus on the implication $\Leftarrow$. Suppose without lack of generality that $\gamma = \{y = 0\}$. Consider a compact connected set $K' \subset \mathbb{S}^1 \setminus \mathcal{U}(iX)$. Consider also a compact connected set $K' \subset K \subset \mathbb{S}^1$ such that there exists a continuous function $\mu_K : K \to \mathbb{S}^1 \setminus \{-1, 1\}$ satisfying that $\lambda \notin \mathcal{U}(\mu_K(\lambda)X)$ for any $\lambda \in K$. We also require the condition $\mu_K^{-1}(i) \cap K' \neq \varnothing$.

We can order the petals $P_1^\varphi(w), \ldots, P_{2\nu_\varphi(\gamma)}^\varphi(w)$ of $\varphi_{(w,h(w))}$ in counter clock-wise sense. These petals are open sets and they depend continuously on w. More precisely the sets $P_1^\varphi, \ldots, P_{2\nu_\varphi(\gamma)}^\varphi$ are open in $\mathbb{C}^2$. Each petal $P_j^\varphi \cap ([0, \delta)K' \times B(0, \epsilon))$ with $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ contains a region $R_{\upsilon(K,j)}$ of $\mathrm{Reg}_1^\epsilon(iX, K')$. We choose $\psi^\varphi_{\upsilon(K,j)}$ as a Fatou coordinate of φ in $R_{\upsilon(K,j)}$, it can be extended to $P_j^\varphi \cap ([0, \delta)K' \times B(0, \epsilon))$ by using $\psi^\varphi_{\upsilon(K,j)} \circ \varphi = \psi^\varphi_{\upsilon(K,j)} + 1$. Moreover, the Fatou coordinate $\psi^\varphi_{\upsilon(K,j)}$ can be extended continuously to $P_j^\varphi \cap ([0, \delta)K \times B(0, \epsilon))$ and is holomorphic in the set $P_j^\varphi \cap ((0, \delta)K^\circ \times B(0, \epsilon))$. This is a consequence of the fact that μ-spaces of orbits and Fatou coordinates in $[0, \delta)\lambda \times B(0, \epsilon)$ for $\lambda \in \mathbb{S}^1$ do not depend on μ but on the connected component of $\{\mu_0 \in \mathbb{S}^1 : \lambda \notin \mathcal{U}(\mu_0 X)\}$ containing μ (see remark 5.2). Define $d(K, j)$ as the smallest natural number such that

$$\upsilon(K, j + 1) - \upsilon(K, j) = d(K, j) + 2\nu(\varphi)\mathbb{Z}.$$

Suppose that $v(K,j) \in D_s(\varphi)$; we have that the change of charts

$$\xi^j_{\varphi,\gamma,K}(x,z) = \psi^{\varphi}_{v(K,j+1)} \circ (x, \psi^{\varphi}_{v(K,j)})^{-1}(x,z)$$

associated to the petals P^{φ}_j and P^{φ}_{j+1} is of the form

$$\xi^j_{\varphi,\gamma,K}(x,z) = z + \zeta_{K,j}(x) + \sum_{l=1}^{\infty} a^{\varphi}_{j,l,\gamma,K}(x) e^{2\pi i s l z} \tag{6}$$

where $\zeta_{K,j}(x)$ belongs to $\mathbb{C}\{x\}[x^{-1}]$ and does not depend on φ but only on X_{φ} (indeed it does not exactly depend on X_{φ} but on $\mathrm{Fix}(\varphi)$ and Res_{φ}). In the case $d(K,j) = 1$ we have $\zeta_{K,j}(x) \equiv \zeta_{X_{\varphi}}(x)$ and $\xi^{v(K,j)}_{\varphi,i,K} \equiv \xi^j_{\varphi,\gamma,K}$.

6.1. Strategy of the proof of theorem 6.1. — . It is easy to construct a conjugation σ defined in a neighborhood of $\gamma \setminus (T \cup \{(0,0)\})$ where T is a discrete set in $\gamma \setminus \{(0,0)\}$. The mapping σ is of the form

$$\sigma_{|P^{\varphi}_j \cap ([0,\delta)K \times B(0,\epsilon))} = \exp(c(x) X^{\eta}_{v(K,j)}) \circ [(x, \psi^{\eta}_{v(K,j)})^{-1} \circ (x, \psi^{\varphi}_{v(K,j)})]$$

The function c is holomorphic in the universal covering of $\gamma \setminus (T \cup \{(0,0)\})$ and it does not depend on the choices of j and K. Suppose that we can prove that $c(x)$ is a holomorphic function defined in the neighborhood of 0, this is the key step. The map $\sigma_{|P^{\varphi}_j}$ has an asymptotic development $\hat{\sigma}$ at γ. Moreover $\hat{\sigma}$ is the only formal diffeomorphism conjugating φ and η of the form

$$\exp(c(x) \log \eta) \circ (x, y + \hat{a}(x,y))$$

where $\hat{a} \in (y)^{\nu_{\varphi}(\gamma)+2} \subset \mathbb{C}[[x,y]]$ and $\log \eta$ is the infinitesimal generator of η. Now σ is defined in a neighborhood of $\gamma \setminus \{(0,0)\}$ and $\hat{\sigma}$ is its asymptotic development at γ. We deduce that $\hat{\sigma} \equiv \sigma$. As a consequence $y \circ \hat{\sigma}$ converges in a neighborhood of $\gamma \setminus \{(0,0)\}$. By the modulus maximum principle (see proof of theorem 3.2) we obtain that $\hat{\sigma}$ belongs to $\mathrm{Diff}(\mathbb{C}^2, 0)$.

A priori controlling c when $x \to 0$ is difficult since c is expected to behave as an essential singularity. We will use the extension of the Ecalle-Voronin invariants provided in subsection 5.8 in sectors $[0,\delta)K \times B(0,\epsilon)$ to extend to $x = 0$ some of the changes of charts $\xi^j_{\varphi,\gamma,K}$ (namely the ones satisfying $v(K,j+1) = v(K,j)+1$). Then, the highly non-triviality of φ_0 implies that $c(x)$ is a holomorphic function in the neighborhood of 0.

6.2. Geometrical motivation. — The vector field X is of the form

$$X = u(x,y) y^{\nu_{\varphi}(\gamma)+1} (y - q(x))^n \partial/\partial y$$

where u is a unit of $\mathbb{C}\{x,y\}$. Denote by κ the order of the series q at 0. The vector field X has associated κ polynomial vector fields $\lambda^{m_1} Y_1(1), \ldots, \lambda^{m_{\kappa}} Y_{\kappa}(1)$. We have

$$Y_k(\lambda) = \lambda^{k(\nu_{\varphi}(\gamma)+n)} u(0,0) t^{\nu_{\varphi}(\gamma)+n+1} \partial/\partial t$$

for $1 \leq k < \kappa$. The vector field $Y_k(1)$ has a unique singular point at 0 such that $\mathrm{Res}(Y_k(1), 0) = 0$ for any $1 \leq k < \kappa$. As a consequence we obtain that

$$\mathcal{U}_k(\mu X) = \{\lambda_0 \in \mathbb{S}^1 : \mu \in \mathcal{U}(Y_k(\lambda_0))\} \text{ and } \mathcal{U}_k^{\lambda}(X) = \{\mu_0 \in \mathbb{S}^1 : \lambda \in \mathcal{U}_k(\mu_0 X)\}$$

are empty for all $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$ and $1 \leq k < \kappa$. The dynamics of $\Re(\mu Y_k(\lambda))$ is the dynamics of a Fatou flower for all $1 \leq k < \kappa$ and $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$. Then the compact-like set to which $Y_k(\lambda)$ is associated behaves as a product-like set when dealing with the dynamics of $\Re(\mu X)$ for $\mu \in \mathbb{S}^1$ and $k < \kappa$. The dynamics of $\Re(\mu X)_{|[0,\delta)\lambda \times B(0,\epsilon)}$ is determined by the dynamics of $\Re(\mu Y_\kappa(\lambda))$. The vector field $Y_\kappa(\lambda)$ is of the form

$$Y_\kappa(\lambda) = \lambda^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0) t^{\nu_\varphi(\gamma)+1} (t - t_0)^n \partial/\partial t$$

where t_0 is the coefficient of x^κ in $q(x)$. The set $\mathcal{U}_\kappa(\mu X)$ has $2m_\kappa = 2\kappa(\nu_\varphi(\gamma) + n)$ points whereas $\mathcal{U}_\kappa^\lambda(X)$ has 2 points for any $(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1$.

Consider a direction $\lambda_0 \notin \mathcal{U}(iX)$ and $\lambda_1 = e^{i\pi/m_\kappa}\lambda_0$. We have $\lambda_1 \notin \mathcal{U}(iX)$ and $i\lambda_0^{m_\kappa} Y_\kappa(1) = -i\lambda_1^{m_\kappa} Y_\kappa(1)$. As a consequence the vector fields $\Re(iX)_{|[0,\delta)\lambda_0 \times B(0,\epsilon)}$ and $\Re(iX)_{|[0,\delta)\lambda_1 \times B(0,\epsilon)}$ are expected to be qualitatively analogous. Let us be more precise. Denote $\lambda_s = \lambda_0 e^{is\pi/m_\kappa}$ for $s \in [0,1]$. The set of tangent points between $\Re(iY_\kappa(\lambda_s))$ and $B(0,\rho)$ for $\rho >> 0$ is composed of $2(\nu_\varphi(\gamma)+n)$ points. They are very close to the points of the set

$$T_s \stackrel{\text{def}}{=} \begin{cases} \mathrm{Im}(\lambda_s^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0) t^{\nu_\varphi(\gamma)+n}) = 0 \\ |t| = \rho. \end{cases}$$

Moreover, we obtain $T_s = e^{-i\pi s/(\nu_\varphi(\gamma)+n)} T_0$. A point $t^0 \in T_0$ corresponds by continuous extension to the point $e^{-i\pi s/(\nu_\varphi(\gamma)+n)} t^0 \in T_s$. We have $T_1 = -T_0 = T_0$ but it turns out that $e^{-i\pi/(\nu_\varphi(\gamma)+n)} t^0$ is not equal to t^0, indeed t^0 is the point following $e^{-i\pi/(\nu_\varphi(\gamma)+n)} t^0$ in T_0 when we consider the points of T_0 ordered in counter clock-wise sense. The dynamics of $\Re(iX)_{|\{\delta_0\lambda_0\} \times \overline{B(0,\epsilon)}}$ is topologically equivalent by a mapping H_{δ_0} to the dynamics of $\Re(iX)_{|\{\delta_0\lambda_1\} \times \overline{B(0,\epsilon)}}$ for $\delta_0 \in (0,\delta)$ but the role of the points of T_{iX}^ϵ is not preserved by H_{δ_0}. The previous discussion implies that $H_{\delta_0}(T_{iX}^{\epsilon,j}(\delta_0\lambda_0)) = T_{iX}^{\epsilon,j+1}(\delta_0\lambda_1)$ for $j \in D(\varphi)$. Every petal of $\exp(Y_\kappa(\lambda_s))$ at $t = 0$ contains the germ of a line $\mathbb{R}^+\omega$ for some $\omega \in \mathbb{S}^1$ in the set

$$L_s \stackrel{\text{def}}{=} \{\omega \in \mathbb{S}^1 : \lambda_s^{\kappa(\nu_\varphi(\gamma)+n)} u(0,0)(-t_0)^n \omega^{\nu_\varphi(\gamma)} \in \mathbb{R}\}.$$

We get $L_s = e^{-i\pi s/\nu_\varphi(\gamma)} L_0$ and $L_1 = e^{-i\pi/\nu_\varphi(\gamma)} L_0$. Therefore we obtain

$$H_{\delta_0}(P_j^\varphi(\delta_0\lambda_0)) = P_{j+1}^\varphi(\delta_0\lambda_1) \ \ \forall j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z}).$$

The dynamics of $\Re(iX)_{|\{\delta_0\lambda_0\} \times \overline{B(0,\epsilon)}}$ and $\Re(iX)_{|\{\delta_0\lambda_1\} \times \overline{B(0,\epsilon)}}$ in an example are represented in picture (12).

Denote $\tau = i\tau_0$ for $\tau_0 = \mathrm{Res}(Y_\kappa(1), 0)/|\mathrm{Res}(Y_\kappa(1), 0)|$. We obtain

$$\{(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda \in \mathcal{U}(\mu X)\} = \{(\lambda, \mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda^{m_\kappa}\mu \in \{-\tau, \tau\}\}.$$

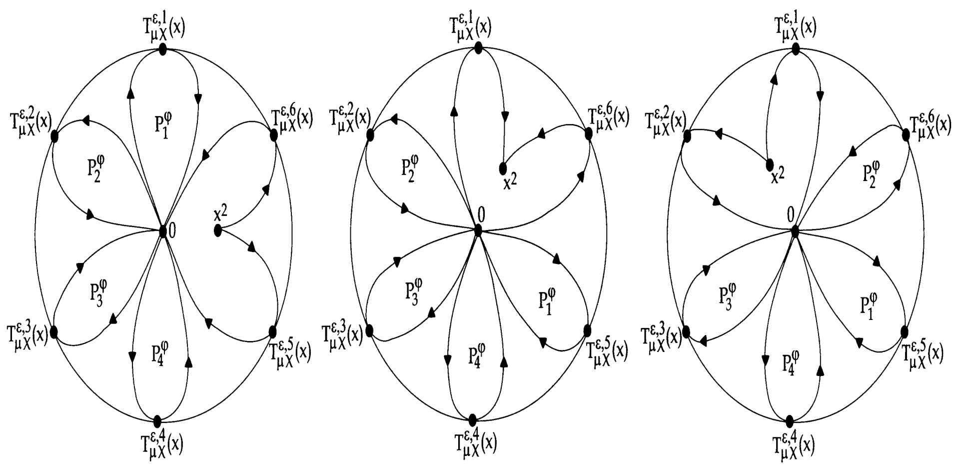

FIGURE 12. $\Re(iX)_{|\{\delta_0\lambda\}\times\overline{B(0,\epsilon)}}$ for $X = -iy^3(y-x^2)\partial/\partial y$ and $\lambda \in \{1, e^{i\pi/6}, e^{i\pi/3}\}$

We choose $\lambda_0 \in \mathcal{U}_\kappa(X)$. We define $K'_j = \{\lambda_0 e^{i\pi j/m_\kappa}\}$ for $j \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. We have $(\cup_{j=0}^{2m_\kappa-1} K'_j) \cap \mathcal{U}(iX) = \varnothing$. We can define K_j to be any compact connected set containing K'_j and contained in the open set $V_j = \lambda_0 e^{i\pi j/m_\kappa} e^{i\pi(-1,1)/m_\kappa}$ of $\mathbb{S}^1$. The continuous function $\mu_j = \mu_{K_j}$ is given by

$$\mu_j(\lambda_0 e^{i\pi j/m_\kappa} e^{i\pi s/m_k}) = e^{i\pi(1-s)/2} \text{ for } s \in (-1,1)$$

for $j \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. We have $\mu_j(V_j) \subset \mathbb{S}^1 \setminus \{-1,1\}$ and $\lambda \notin \mathcal{U}(\mu_j(\lambda)X)$ for any $\lambda \in V_j$ by construction. Every point λ in $\mathbb{S}^1 \setminus \cup_{l=0}^{2m_\kappa-1} K'_l = \mathbb{S}^1 \setminus \mathcal{U}(X)$ is contained exactly in two open sets V_k and V_{k+1} (among all the sets of the form V_l) for some k in $\mathbb{Z}/(2m_\kappa\mathbb{Z})$. Since $\mathcal{U}^\lambda_\kappa(X)$ contains exactly one point in $e^{i(0,\pi)}$ then $\{\xi^j_{\varphi,i,K_k}\}_{j\in D(\varphi)}$ and $\{\xi^j_{\varphi,i,K_{k+1}}\}_{j\in D(\varphi)}$ are the two systems of changes of charts associated to the direction $x \in \lambda\mathbb{R}^+$. A point $\lambda_0 e^{i\pi k/m_\kappa}$ belongs to V_k and does not belong to V_l for $l \in \mathbb{Z}/(2m_\kappa\mathbb{Z}) \setminus \{k\}$. The set $\mathcal{U}^{\lambda_0 e^{i\pi k/m_\kappa}}_\kappa(X)$ does not contain points $e^{i(0,\pi)}$. Indeed $\{\xi^j_{\varphi,i,K_k}\}_{j\in D(\varphi)}$ is the unique system of changes of charts associated to the direction $x \in \lambda_0 e^{i\pi k/m_\kappa}\mathbb{R}^+$.

Let us focus on the example $X = -iy^3(y-x^2)\partial/\partial y$ with $\gamma = \{y=0\}$. We have

$$\{(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda \in \mathcal{U}(\mu X)\} = \{(\lambda,\mu) \in \mathbb{S}^1 \times \mathbb{S}^1 : \lambda^6\mu \in \{-1,1\}\}.$$

We define $\lambda_0 = 1$. Consider the notations in the picture (12). Given $k \in \{3,7,11\}$ there exists a function $\xi^1_{\varphi,\gamma,k}$ defined in the union of the domains of definition of $\xi^1_{\varphi,\gamma,K_{k-1}}$, $\xi^1_{\varphi,\gamma,K_k}$ and $\xi^1_{\varphi,\gamma,K_{k+1}}$. We obtain

- $(\xi^1_{\varphi,\gamma,11})_{|\{x\in[0,\delta)K_k\}} = \xi^1_{\varphi,\gamma,K_k} \equiv \xi^1_{\varphi,i,K_k}$ for $k \in \{10,11,0\}$.
- $(\xi^1_{\varphi,\gamma,3})_{|\{x\in[0,\delta)K_k\}} = \xi^1_{\varphi,\gamma,K_k} \equiv \xi^5_{\varphi,i,K_k}$ for $k \in \{2,3,4\}$.
- $(\xi^1_{\varphi,\gamma,7})_{|\{x\in[0,\delta)K_k\}} = \xi^1_{\varphi,\gamma,K_k} \equiv \xi^3_{\varphi,i,K_k}$ for $k \in \{6,7,8\}$.

The union of the domains of definition of $\xi^1_{\varphi,\gamma,11}$, $\xi^1_{\varphi,\gamma,3}$ and $\xi^1_{\varphi,\gamma,7}$ is of the form

$$\{x \in [0,\delta)\,([K_2 \cup K_3 \cup K_4] \cup [K_6 \cup K_7 \cup K_8] \cup [K_{10} \cup K_{11} \cup K_0])\} \times \{\mathrm{Im} z < -I\}$$

for some $I \in \mathbb{R}^+$. Analogously given $k \in \{0,4,8\}$ there exists a function $\xi^2_{\varphi,\gamma,k}$ defined in the union of the domains of definition of $\xi^1_{\varphi,\gamma,K_{k-1}}$, $\xi^1_{\varphi,\gamma,K_k}$ and $\xi^1_{\varphi,\gamma,K_{k+1}}$. We have

- $(\xi^2_{\varphi,\gamma,0})_{|\{x\in[0,\delta)K_k\}} = \xi^2_{\varphi,\gamma,K_k} \equiv \xi^2_{\varphi,i,K_k}$ for $k \in \{11,0,1\}$.
- $(\xi^2_{\varphi,\gamma,4})_{|\{x\in[0,\delta)K_k\}} = \xi^2_{\varphi,\gamma,K_k} \equiv \xi^6_{\varphi,i,K_k}$ for $k \in \{3,4,5\}$.
- $(\xi^2_{\varphi,\gamma,8})_{|\{x\in[0,\delta)K_k\}} = \xi^2_{\varphi,\gamma,K_k} \equiv \xi^4_{\varphi,i,K_k}$ for $k \in \{7,8,9\}$.

The union of the domains of definition of $\xi^1_{\varphi,\gamma,0}$, $\xi^1_{\varphi,\gamma,4}$ and $\xi^1_{\varphi,\gamma,8}$ is of the form

$$\{x \in [0,\delta)\,([K_3 \cup K_4 \cup K_5] \cup [K_7 \cup K_8 \cup K_9] \cup [K_{11} \cup K_0 \cup K_1])\} \times \{\mathrm{Im} z > I\}$$

for some $I \in \mathbb{R}^+$. Let us remark that $\cup_{k=0}^{11} K_k = \mathbb{S}^1$. Moreover, in every sector $\{x \in [0,\delta)K_k\}$ either the change of charts between the Fatou coordinates of P_1^φ and P_2^φ or the change of charts between the Fatou coordinates of P_2^φ and P_3^φ admits a continuous extension to $x = 0$. This kind of argument fact is key to prove theorem 6.1.

6.3. Proof of theorem 6.1.

— The first part of the proof is a technical lemma.

Lemma 6.1. — *Let $\varphi \in \mathrm{Diff}_{p1}(\mathbb{C}^2,0)$ be non-degenerate. Suppose that there exists a parabolic irreducible component γ of $\mathrm{Fix}(\varphi)$. Then $\log\varphi$ belongs to $\mathcal{X}(\mathbb{C}^2,0)$ if and only if $\log\varphi_Q$ is analytic for any $Q \in \gamma$.*

Proof. — The implication $\Rightarrow$ is trivial. Let us prove the implication $\Leftarrow$. We can suppose that $\gamma = \{y = 0\}$ without lack of generality. The infinitesimal generator $\log\varphi$ is of the form $\hat{f}(x,y)\partial/\partial y$ for some $\hat{f} \in \mathbb{C}[[x,y]]$. By hypothesis the function $\hat{f}$ is analytic in a neighborhood of $\gamma \setminus \{(0,0)\}$. This implies $\hat{f} \in \mathbb{C}\{x,y\}$ by the modulus maximum principle (see proof of theorem 3.2). □

Consider the compact sets K_k for $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$ defined in subsection 6.2. We can define (see equation 6)

$$E_k^\varphi = \{l \in \mathbb{N} : \exists j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z}) : a^\varphi_{j,l,\gamma,K_k} \not\equiv 0\}.$$

We claim that E_k^φ does not depend on k. We have

$$\xi^j_{\varphi,\gamma,K_{k+1}}(x, z + c_{j,k}(x)) \equiv \xi^j_{\varphi,\gamma,K_k}(x,z) + d_{j,k}(x)$$

for all $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ and $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$ and some functions $c_{j,k}$ and $d_{j,k}$ defined in $(0,\delta)(K_k \cap K_{k+1})$. Thus we obtain $E_k^\varphi = E_{k+1}^\varphi$ for any $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. The highly non-triviality of φ_0 implies $\log\varphi \notin \mathcal{X}(\mathbb{C}^2,0)$. The sets E_k^φ are not empty. We define $g(\varphi) = \gcd(E_1^\varphi) = \ldots = \gcd(E_{2m_\kappa}^\varphi)$.

Since $\varphi_Q \overset{an_1}{\sim} \eta_Q$ for $Q \in \gamma \setminus \{(0,0)\}$ there exists a function $c' : B(0,\delta) \setminus \{0\} \to \mathbb{C}$ such that

$$\xi^j_{\eta,\gamma,K_k}(x, z + c'(x)) = \xi^j_{\varphi,\gamma,K_k}(x,z) + c'(x) \tag{7}$$

for all $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ and $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. A priori the function c' is not even continuous.

Fix $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. Each $j \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ except one satisfies that $d(K_k, j) = 1$. Choose $j_0 \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ such that $d(K_k, j_0) = 1$. Since φ_0 is highly non-trivial then we obtain $\xi_{\varphi,i,K_k}^{v(K_k,j_0)}(0,z) \not\equiv z + \zeta_X(0)$. We deduce the existence of $l \in \mathbb{N}$ such that $a_{j_0,l,\gamma,K_k}^{\varphi}(0) \neq 0$. Denote by s the point of $\{-1,1\}$ such that $v(K_k, j_0) \in D_s(\varphi)$. By equation (7) we obtain

$$e^{-2\pi i s l c'(x)} = \frac{a_{j_0,l,\gamma,K_k}^{\eta}}{a_{j_0,l,\gamma,K_k}^{\varphi}}(x)$$

for any $x \in (0,\delta)K_k$. As a consequence the function $e^{-2\pi i s l c'(x)}$ is continuous in $[0,\delta)K_k$, it is holomorphic in $(0,\delta)K_k^\circ$ and $e^{-2\pi i s l c'(x)}((0,\delta)K_k) \subset \mathbb{C}^*$. By definition of $g(\varphi)$ there exists $l_1, \ldots, l_q$ in E_k^φ and integer numbers $p_1, \ldots, p_q$ such that $g(\varphi) = p_1 l_1 + \ldots p_q l_q$. Choose j_e such that $a_{j_e,l_e,\gamma,K_k}^{\varphi} \not\equiv 0$ for $e \in \{1, \ldots, q\}$. Denote by s_e the element of $\{-1,1\}$ such that $v(K_k, j_e) \in D_{s_e}(\varphi)$. We obtain

$$e^{-2\pi i s g(\varphi) c'(x)} = (e^{-2\pi i s_1 l_1 c'(x)})^{s_1 s p_1} \ldots (e^{-2\pi i s_q l_q c'(x)})^{s_q s p_q}$$

and then $e^{-2\pi i s g(\varphi) c'(x)}$ is well-defined in $(0,\delta)K_k$ since we have

$$e^{-2\pi i s g(\varphi) c'(x)} = \left(\frac{a_{j_1,l_1,\gamma,K_k}^{\eta}}{a_{j_1,l_1,\gamma,K_k}^{\varphi}}(x)\right)^{s_1 s p_1} \cdots \left(\frac{a_{j_q,l_q,\gamma,K_k}^{\eta}}{a_{j_q,l_q,\gamma,K_k}^{\varphi}}(x)\right)^{s_q s p_q}.$$

The equality $(e^{-2\pi i s g(\varphi) c'(x)})^l = (e^{-2\pi i s l c'(x)})^{g(\varphi)}$ implies that $e^{-2\pi i s g(\varphi) c'(x)}$ is a complex-valued function that is continuous in $[0,\delta)K_k$ and holomorphic in $(0,\delta)K_k^\circ$. Moreover, it satisfies $e^{-2\pi i s g(\varphi) c'(x)}((0,\delta)K_k) \subset \mathbb{C}^*$.

The previous argument can be repeated for every K_k with $k \in \mathbb{Z}/(2m_\kappa\mathbb{Z})$. As a consequence $e^{2\pi i g(\varphi) c'(x)}$ is a meromorphic function in a neighborhood of 0 such that $e^{-2\pi i s g(\varphi) c'(0)} \in \mathbb{C}$. Since we can choose $j_0' \in \mathbb{Z}/(2\nu_\varphi(\gamma)\mathbb{Z})$ such that $d(K_{k+1}, j_0') = 1$ and $v(K_{k+1}, j_0') \in D_{-s}(\varphi)$ then we obtain also $e^{2\pi i s g(\varphi) c'(0)} \in \mathbb{C}$. As a consequence $e^{2\pi i g(\varphi) c'(x)}$ is a unit of $\mathbb{C}\{x,y\}$ and there exists a function $c \in \mathbb{C}\{x\}$ such that

$$e^{2\pi i g(\varphi) c(x)} \equiv e^{2\pi i g(\varphi) c'(x)}.$$

The function c is the function we are looking for. □

References

[1] P. Ahern & J.-P. Rosay – "Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables", *Trans. Amer. Math. Soc.* **347** (1995), p. 543–572.

[2] A. Douady, F. Estrada & P. Sentenac – "Champs de vecteurs polynômiaux sur $\mathbb{C}$", *To appear in the Proceedings of Boldifest.*

[3] P. Lavaurs – "Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques", Ph.D. Thesis, Université de Paris-Sud, 1989.

[4] O. Lehto & K. I. Virtanen – *Quasiconformal mappings in the plane*, second ed., Die Grund. Math. Wiss., vol. 126, Springer, 1973.

[5] P. Mardešić, R. Roussarie & C. Rousseau – "Modulus of analytic classification for unfoldings of generic parabolic diffeomorphisms", *Mosc. Math. J.* **4** (2004), p. 455–502, 535.

[6] J. Martinet & J.-P. Ramis – "Problèmes de modules pour des équations différentielles non linéaires du premier ordre", *Publ. Math. I.H.É.S.* **55** (1982), p. 63–164.

[7] R. Oudkerk – "The parabolic implosion for $f_0(z) = z + z^{\nu+1} + O(z^{\nu+2})$", Ph.D. Thesis, University of Warwick, 1999.

[8] J. Rey – "Difféomorphismes résonnants de $(\mathbb{C}^1, 0)$", Ph.D. Thesis, Université Paul Sabatier, 1996.

[9] J. Ribón – "Germes de difféomorphisme de $(\mathbb{C}^2, 0)$ tangents à l'identité qui préservent la fibration de hopf", Ph.D. Thesis, Universidad de Valladolid, 2001.

[10] ———, "Topological classification of families of diffeomorphisms without small divisors", *To appear in Memoirs of the AMS* .

[11] J. Ribón – "Formal classification of unfoldings of parabolic diffeomorphisms", *Ergodic Theory Dynam. Systems* **28** (2008), p. 1323–1365.

[12] ———, "Modulus of analytic classification for unfoldings of resonant diffeomorphisms", *Mosc. Math. J.* **8** (2008).

[13] M. Shishikura – "Bifurcation of parabolic fixed points", in *The Mandelbrot set, theme and variations*, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, 2000, p. 325–363.

J. Ribón, UFF, Instituto de Matemática, Valonguinho, Rua Mário Santos Braga, s/n, Centro, Niterói, RJ - Brasil 24020-140 • *E-mail :* `jfribon@impa.br`