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HOMOGENEOUS COMMUTING VECTOR FIELDS ON $\mathbb{C}^2$

by

Alcides Lins Neto

Abstract. — In the main result of this paper we give a method to construct all pairs of homogeneous commuting vector fields on $\mathbb{C}^2$ of the same degree $d \geq 2$ (Theorem 1). As an application, we classify, up to linear transformations of $\mathbb{C}^2$, all pairs of commuting homogeneous vector fields on $\mathbb{C}^2$, when $d = 2$ and $d = 3$ (corollaries 1 and 2). We obtain also necessary conditions in the cases of quasi-homogeneous vector fields and when the degrees are different (theorem 2).

***Résumé* (Champs de vecteurs homogènes commutants dans $\mathbb{C}^2$).** — Dans le résultat principal de ce papier on donne une méthode de construction de tous les paires de champs de vecteurs homogènes de même degré $d \geq 2$ qui commutent (théorème 1). Comme application, on classifie les paires de champs de vecteurs homogènes commutantes dans $\mathbb{C}^2$ de degrés $d = 2$ et $d = 3$ (corollaires 1 et 2). Nous obtenons aussi des conditions nécessaires dans les cas quasi-homogènes et quand les degrés sont différents (théorème 2).

1. Introduction

A. Guillot in his thesis and in [**3**], gave a non-trivial example of a pair of commuting homogeneous vector fields of degree two on $\mathbb{C}^3$. The example is non-trivial in the sense that it cannot to be reduced to two vector fields in separated variables, like in the pair $X := P(x, y)\partial_x + Q(x, y)\partial_y$ and $Y := R(z)\partial_z$. This suggested me the problem of classification of pairs of polynomial commuting vector fields on $\mathbb{C}^n$. This problem, in this generality, seems very difficult, even for $n = 2$. Even the restricted problem of classification of pairs of commuting vector fields, homogeneous of degree d, seems very dificult for $n \geq 3$ and $d \geq 2$ (see problem 3). However, for $n = 2$ and $d \geq 2$ it is possible to give a complete classification, as we will see in this paper.

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Let X and Y be two homogeneous commuting vector fields on $\mathbb{C}^2$, where $dg(X) = k$ and $dg(Y) = \ell$, and $R = x\,\partial_x + y\,\partial_y$ be the radial vector field.

Definition 1.1. — *We will say that X and Y are* colinear *if $X \wedge Y = 0$. In this case, we will use the notation $X//Y$. When $dg(X) = dg(Y)$, we will consider the 1-parameter family $(Z_\lambda)_{\lambda \in \mathbb{P}^1}$ given by $Z_\lambda = X + \lambda.Y$ if $\lambda \in \mathbb{C}$ and $Z_\infty = Y$. It will be called the* pencil generated *by X and Y. The pencil will be called* trivial, *if $Y = \lambda.X$ for some $\lambda \in \mathbb{C}$. Otherwise, it will be called* non-trivial.

From now on, we will set:

$$\begin{cases} X \wedge Y = f\,\partial_x \wedge \partial_y \\ R \wedge X = g\,\partial_x \wedge \partial_y \\ R \wedge Y = h\,\partial_x \wedge \partial_y \end{cases} . \tag{1}$$

Since $dg(X) = k$ and $dg(Y) = \ell$, the polynomials f, g and h are homogeneous and $dg(f) = k + \ell$, $dg(g) = k + 1$, $dg(h) = \ell + 1$. Moreover, $f \not\equiv 0$ iff X and Y are non-colinear.

Our main result concerns the case where $k = \ell \geq 2$. In this case, if $g, h \not\equiv 0$, we will consider the meromorphic function $\phi = g/h$ as a holomorphic function $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$:

$$\phi[x : y] = \frac{g(x,y)}{h(x,y)} .$$

Theorem 1. — *Let $(Z_\lambda)_\lambda$ be a non-trivial pencil of homogeneous commuting vector fields of degree $d \geq 2$ on $\mathbb{C}^2$. Let X and Y be two generators of the pencil and f, g, h and ϕ be as before. If the pencil is colinear then $X = \alpha.R$ and $Y = \beta.R$, where α and β are homogeneous polynomials of degree $d - 1$. If the pencil is non-colinear then:*

(a) *$f, g, h \not\equiv 0$.*
(b) *f/g (resp. f/h) is a non-constant meromorphic first integral of X (resp. Y).*
(c) *Let s be the (topological) degree of $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$. Then $1 \leq s \leq d - 1$.*
(d) *The decompositions of f, g and h into irreducible linear factors are of the form:*

$$\begin{cases} f = \Pi_{j=1}^{r} f_j^{2k_j + m_j} \\ g = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_i \\ h = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} h_i \end{cases} \tag{2}$$

where $s + \sum_{j=1}^{r} k_j = d + 1$ and $\sum_{j=1}^{r} m_j = 2s - 2$. Moreover, we can choose the generators X and Y in such a way that $g_1, \dots, g_s, h_1, \dots, h_s$ are two by two relatively primes.

(e) *Considering the direction $(f_j = 0) \subset \mathbb{C}^2$ as a point $p_j \in \mathbf{P}^1$, then*

$$m_j = \mathrm{mult}(\phi, p_j) - 1 \ , \ j = 1, \dots, r \ , \tag{3}$$

where $\mathrm{mult}(\phi, p)$ *denotes the ramification index of ϕ at $p \in \mathbf{P}^1$.*

(f) *The generators X and Y can be choosen as:*

$$(4)\qquad \begin{cases} X = g.[\sum_{j=1}^{r} (k_j + m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{s} \frac{1}{g_i}(g_{ix}\partial_y - g_{iy}\partial_x)] \\ Y = h.[\sum_{j=1}^{r} (k_j + m_j)\frac{1}{f_j}(f_{jx}\partial_y - f_{jy}\partial_x) - \sum_{i=1}^{t} \frac{1}{h_i}(h_{ix}\partial_y - h_{iy}\partial_x)] \end{cases}$$

Conversely, given a non-constant map $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree $s \geq 1$ and a divisor D on $\mathbf{P}^1$ of the form

$$(5)\qquad D = \sum_{p \in \mathbf{P}^1} (2k(p) + \mathrm{mult}(\phi, p) - 1).[p] \ ,$$

where $k(p) \geq min(1, \mathrm{mult}(\phi, p) - 1)$ and $\sum_p k(p) < +\infty$, there exists an unique pencil $(Z_\lambda)_\lambda$ of homogeneous commuting vector fields of degree $d = \sum_p k(p) + s - 1$ with generators X and Y given by (4), and the $f_{j's}$, $g_{i's}$ and $h_{i's}$ given in the following way: let $\{p_1 = [a_1 : b_1], \ldots, p_r = [a_r : b_r]\} = \{p \in \mathbf{P}^1 \mid 2k(p) + \mathrm{mult}(\phi, p) - 1 > 0\}$. Set $k_j = k(p_j)$, $m_j = \mathrm{mult}(\phi, p_j) - 1$ and $f_j(x, y) = a_j\, y - b_j\, x$. Set $\phi[x : y] = G_1(x, y)/H_1(x, y)$, where G_1 and H_1 are homogeneous polynomials of degree s. Then the $g_{i's}$ and $h_{i's}$ are the linear factors of G_1 and H_1, respectively.

Definition 1.2. — *Let X, Y, $g = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_i$ and $h = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{t} h_i$ be as in theorem 1. We call $(f_j = 0)$, $j = 1, \ldots, r$, the fixed directions of the pencil.*

Given $\lambda \in \mathbb{C}$, the polynomial $g_\lambda = g + \lambda.h$ plays the same role for the vector field $Z_\lambda = X + \lambda.Y$ than g and h for X and Y. Its decomposition into irreducible factors is of the form

$$g_\lambda = \Pi_{j=1}^{r} f_j^{k_j}.\Pi_{i=1}^{s} g_{i,\lambda} \ .$$

Definition 1.3. — *The directions given by $(g_{i,\lambda} = 0)$ are called the movable directions of the pencil.*

In particular, the number s of movable directions coincides with the degree of the map $\phi = g/h\colon \mathbf{P}^1 \to \mathbf{P}^1$.

As an application of Theorem 1, we obtain the classification of the pencils of homogeneous commuting vector fields of degrees two and three.

Corollary 1. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree two vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators X and Y of the pencil can be written as:*

(a) *$X = g.R$ and $Y = h.R$, where g and h are homogeneous polynomials of degree one and $R = x.\partial_x + y.\partial_y$.*

(b) *$X = x^2\partial_x$ and $Y = y^2\partial_y$. In this case, the pencil has two fixed directions.*

(c) *$X = y^2\partial_x$ and $Y = 2xy\partial_x + y^2\partial_y$. In this case, the pencil has one fixed direction.*

Corollary 2. — *Let $(Z_\lambda)_\lambda$ be a pencil of commuting homogeneous of degree three vector fields on $\mathbb{C}^2$. Then, after a linear change of variables on $\mathbb{C}^2$, the generators X and Y of the pencil can be written as:*

(a) *$X = g.R$ and $Y = h.R$, where g and h are homogeneous polynomials of degree two and $R = x.\partial_x + y.\partial_y$.*
(b) *$X = y^3\partial_x$ and $Y = 3xy^2\partial_x + y^3\partial_y$. In this case, the pencil has one movable and one fixed direction.*
(c) *$X = x^2y\partial_x$ and $Y = xy^2\partial_x - y^3\partial_y$. In this case, the pencil has one movable and two fixed directions.*
(d) *$X = (2x^2y + x^3)\partial_x - x^2y\partial_y$ and $Y = -xy^2\partial_x + (2xy^2 + y^3)\partial_y$. In this case, the pencil has one movable and three fixed directions.*
(e) *$X = x^3\partial_x$ and $Y = y^3\partial_y$. In this case, the pencil has two movable and two fixed directions.*

Some of the preliminary results that we will use in the proof of Theorem 1 are also valid for quasi-homogeneous vector fields.

Definition 1.4. — *Let S be a linear diagonalizable vector field on $\mathbb{C}^n$ such that all eigenvalues of S are relatively primes natural numbers. We say that a holomorphic vector field $X \not\equiv 0$ is quasi-homogeneous with respect to S if $[S, X] = mX$, $m \in \mathbb{C}$.*

It is not difficult to prove that, in this case, we have the following:

(I) $m \in \mathbb{N} \cup \{0\}$.
(II) X is a polynomial vector field.

Our next result concerns two commuting vector fields which are quasi-homogeneous with respect to the same linear vector field S. Let X and Y be two commuting vector fields on $\mathbb{C}^2$, quasi-homogeneous with respect to the same vector field S with eigenvalues $p, q \in \mathbb{N}$ (relatively primes), where $[S, X] = mX$ and $[S, Y] = nY$. Since S is diagonalizable, after a linear change of variables, we can assume that $S = p\,x\partial_x + q\,y\partial_y$. Set $X \wedge Y = f\,\partial_x \wedge \partial_y$, $S \wedge X = g\,\partial_x \wedge \partial_y$ and $S \wedge Y = h\,\partial_x \wedge \partial_y$. We will always assume that $X, Y \not\equiv 0$

Remark 1.1. — *We would like to observe that f, g and h are quasi-homogeneous with respect to S, that is, we have $S(f) = (m + n + tr(S))f$, $S(g) = (m + tr(S))g$ and $S(h) = (n + tr(S))h$, where $tr(S) = p + q$. It is known that in this case, any irreducible factor of f, g or h, is the equation of an orbit of S, that is, x, y or a polynomial of the form $y^p - c\,x^q$, where $c \neq 0$.*

Theorem 2. — *In the above situation, suppose that $f, h \not\equiv 0$ and $n \neq 0$. Then:*

(a) *$g \not\equiv 0$ and f/g is a non-constant meromorphic first integral of X.*
(b) *Suppose that $m, n \neq 0$. Then f, g and h satisfy the two equivalent relations below:*

$$mn\,f^2\,dx \wedge dy = f\,dg \wedge dh + g\,dh \wedge df + h\,df \wedge dg \tag{6}$$

$$(m-n)\frac{df}{f} + n\frac{dh}{h} - m\frac{dg}{g} = \frac{m\,n\,f}{gh}(qy\,dx - px\,dy) \tag{7}$$

(c) *Suppose that* $m, n \neq 0$. *Then any irreducible factor of* f *divides* g *and* h. *Conversely, if* $p = gcd(g, h)$ *then any irreducible factor of the* p *divides* f. *Moreover, the decompositions of* f, g *and* h *into irreducible factors, are of the form*

$$\begin{cases} f = \Pi_{j=1}^{r} f_j^{\ell_j} \\ g = \Pi_{j=1}^{r} f_j^{m_j} . \Pi_{i=1}^{s} g_i^{a_i} \\ h = \Pi_{j=1}^{r} f_j^{n_j} . \Pi_{i=1}^{t} h_i^{b_i} \end{cases} \tag{8}$$

where $r > 0$, $m_j, n_j > 0$, $\ell_j \geq m_j + n_j - 1$, *for all* j, *and any two polynomials in the set* $\{f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t\}$ *are relatively primes.*

(d) *Suppose that* f, g *and* h *are as in (8). Then vector fields* X *and* Y *can be written as*

$$\begin{cases} X = \frac{1}{n} g . [\sum_{j=1}^{r} (\ell_j - m_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^{s} a_i \frac{1}{g_i} (g_{ix} \partial_y - g_{iy} \partial_x)] \\ Y = \frac{1}{m} h . [\sum_{j=1}^{r} (\ell_j - n_j) \frac{1}{f_j} (f_{jx} \partial_y - f_{jy} \partial_x) - \sum_{i=1}^{t} b_i \frac{1}{h_i} (h_{ix} \partial_y - h_{iy} \partial_x)] \end{cases} \tag{9}$$

As an application, we have the following result:

Corollary 3. — *Let* X *and* Y *be germs of holomorphic commuting vector fields at* $0 \in \mathbb{C}^2$. *Let*

$$X = \sum_{j=d}^{\infty} X_j$$

be the Taylor series of X *at* $0 \in \mathbb{C}^2$, *where* X_j *is homogeneous of degree* $j \geq d$. *Assume that* $d \geq 2$ *and that the vector field* X_d *has no meromorphic first integral and that* 0 *is an isolated singularity of* X_d. *Then* $Y = \lambda . X$, *where* $\lambda \in \mathbb{C}$.

We would like to recall a well-known criterion for a homogeneous vector field of degree d on $\mathbb{C}^2$, say X_d, to have a meromorphic first integral (see [**1**]). Since the radial vector field $R = x\,\partial_x + y\,\partial_y$ has the meromorphic first integral y/x, we can assume that $R \wedge X_d = g\,\partial_x \wedge \partial_y \not\equiv 0$. Let $\omega = i_{X_d}(dx \wedge dy)$, where i denotes the interior product. Then the form $\omega_1 = \omega/g$ is closed. In this case, if $g = \Pi_{j=1}^{r} g_j^{k_j}$ is the decomposition of g into linear irreducible factors, then we have

$$\omega_1 = \sum_{j=1}^{r} \lambda_j \frac{dg_j}{g_j} + d(h/g_1^{k_1-1} \cdots g_r^{k_r-1}) \ ,$$

where $\lambda_j \in \mathbb{C}$, for all $1 \leq j \leq r$ and h is homogeneous of degree $d + 1 - r = dg(X_d) + 1 - r = dg(g/g_1 \cdots g_r)$. In this case, X_d has a meromorphic first integral if, and only if, either $\lambda_1 = \cdots = \lambda_r = 0$, or $\lambda_j \neq 0$ for some $j \in \{1, \ldots, r\}$, $h \equiv 0$ and $[\lambda_1 : \cdots : \lambda_r] = [m_1 : \cdots : m_r]$, where $m_1, \ldots, m_r \in \mathbb{Z}$. In particular, we obtain that the set of homogeneous vector fields of degree $d \geq 1$ with a meromorphic first integral is a countable union of Zariski closed sets.

Let us state some natural problems related to the above results.

Problem 1. — *Classify the pencils of commuting homogeneous vector fields of degree* $d \geq 2$ *on* $\mathbb{C}^n$, $n \geq 3$.

Problem 1 seems dificult even in dimension three.

Problem 2. — *Let $\mathcal{X}_2$ be the set of germs at $0 \in \mathbb{C}^2$ of holomorphic vector fields. Given $X \in \mathcal{X}_2$, $X \neq 0$, to determine the set*

$$C(X) = \{Y \mid [X,Y] = 0\} .$$

Under which conditions is $C(X)$ of finite dimension?

Problem 3. — *Classify all pairs of commuting polynomial vector fields on $\mathbb{C}^2$.*

Observe that problem 3 has the following relation with the so called Jacobian conjecture: let f and g be two polynomials on $\mathbb{C}^2$ such that $f_x.g_y - f_y.g_x \equiv 1$. Then their hamiltonians $X = f_y\,\partial_x - f_x\,\partial_y$ and $Y = g_y\,\partial_x - g_x\partial_y$ commute. By this reason, problem 3 seems very difficult.

2. Preliminary results

In this section we prove some general results that will be used in the next sections. Let S, X and Y be holomorphic vector fields defined in some domain U of $\mathbb{C}^2$. Assume that:

(I) $[S,X] = m.X$, $[S,Y] = n.Y$ and $[X,Y] = 0$, where $m,n \in \mathbb{C}$.
(II) $X \wedge Y = f.\partial_x \wedge \partial_y$, $S \wedge X = g.\partial_x \wedge \partial_y$ and $S \wedge Y = h.\partial_x \wedge \partial_y$, where $f,g,h \not\equiv 0$.

We consider also the holomorphic 1-forms $\omega = i_X(dx \wedge dy)$ and $\eta = i_Y(dx \wedge dy)$, where i denotes the interior product.

Lemma 2.1. — *In the above situation we have:*

(a) *The meromorphic functions f/g and f/h are first integrals of X and Y, respectively. Moreover, f/g (resp. f/h) is constant if, and only if, $n = 0$ (resp. $m = 0$).*
(b) *If $n \neq 0$ (resp. $m \neq 0$) then*

$$\omega = \frac{g}{n}\left[\frac{dg}{g} - \frac{df}{f}\right] \ (\textit{resp. } \eta = \frac{h}{m}\left[\frac{dh}{h} - \frac{df}{f}\right]) . \tag{10}$$

(c) *The polynomials f, g and h satisfy the relation:*

$$mn\, f^2\, dx \wedge dy = f\, dg \wedge dh + g\, dh \wedge df + h\, df \wedge dg . \tag{11}$$

Proof. — Let us prove (a). Assume that $n \neq 0$. First of all, note that

$$L_X(S \wedge X) = [X,S] \wedge X + S \wedge [X,X] = -m.X \wedge X = 0$$

and simillarly $L_X(X \wedge Y) = 0$, where L denotes the Lie derivative. Since $X \wedge Y = (f/g).S \wedge Y$, we get

$$\begin{aligned} 0 &= L_X(X \wedge Y) = L_X((f/g).S \wedge X) \\ &= X(f/g).S \wedge X + (f/g).L_X(S \wedge X) = X(f/g).S \wedge X \implies \\ &\implies X(f/g) = 0 . \end{aligned}$$

Therefore, f/g is a first integral of X. It remains to prove that f/g is a constant if, and only if $n = 0$. Since $L_S(X \wedge Y) = (m+n)\, X \wedge Y$ and $L_S(S \wedge X) = m\, S \wedge X$, we get

$$\begin{aligned}(m+n)\, X \wedge Y &= L_S((f/g).S \wedge X)\\ &= S(f/g).S \wedge X + (f/g).L_S(S \wedge X)\\ &= (S(f/g) + m.(f/g))\, S \wedge X\end{aligned}$$

which implies that $S(f/g) = n.(f/g)$. Hence, if f/g is a constant then $n = 0$.

Conversely, if $n = 0$ then $S(f/g) = 0$ and f/g is a first integral of S and X simultaniously. If f/g was not constant then the vector fields X and S would be colinear in the non-empty open subset of U defined by $d(f/g) \neq 0$. This would imply that $S \wedge X \equiv 0$, and so $g \equiv 0$, a contradiction. Therefore, f/g is a constant.

Now, let $\omega = i_X(dx \wedge dy)$ and suppose that $n \neq 0$. Since f/g is a non-constant first integral of X, we get $\omega \wedge d(f/g) = 0$, which implies that

$$\omega = k\left(\frac{dg}{g} - \frac{df}{f}\right) ,$$

where k is meromorphic on U. On the other hand, we have

$$\begin{aligned}g &= -i_S(i_X(dx \wedge dy)) = -i_S(\omega)\\ &= k\left(\frac{S(f)}{f} - \frac{S(g)}{g}\right) = k\,\frac{S(f/g)}{f/g} = n.k \implies k = g/n .\end{aligned}$$

This proves (10).

Let us prove (c). Note first that $\omega \wedge \eta = f.dx \wedge dy$. We leave the proof of this fact to the reader. If $n = 0$ (or $m = 0$) then (11) follows from $f/g = c \neq 0$ (or $f/h = c \neq 0$), where c is a constant. We leave the proof to the reader in this case. On the other hand, if $m, n \neq 0$ then

$$\begin{aligned}&f.dx \wedge dy = \omega \wedge \eta\\ &\quad = \frac{g}{n}\left[\frac{dg}{g} - \frac{df}{f}\right] \wedge \frac{h}{m}\left[\frac{dh}{h} - \frac{df}{f}\right] = \frac{g.h}{m.n}\left[\frac{dh \wedge df}{h.f} + \frac{df \wedge dg}{f.g} + \frac{dg \wedge dh}{g.h}\right] ,\end{aligned}$$

which implies (11). □

In the next result we prove a kind of converse of (11).

Lemma 2.2. — *Let f, g and h be holomorphic functions on a domain $U \subset \mathbb{C}^2$. Suppose that f/g and f/h are non-constant meromorphic functions on U. Define meromorphic vector fields X and Y by $i_X(dx \wedge dy) = g[\frac{dg}{g} - \frac{df}{f}]$ and $i_Y(dx \wedge dy) = h[\frac{dh}{h} - \frac{df}{f}]$. Suppose that*

$$f\, dg \wedge dh + g\, dh \wedge df + h\, df \wedge dg = \lambda\, f^2\, dx \wedge dy ,$$

where $\lambda \neq 0$. Then $[X, Y] = 0$.

Proof. — The idea is to prove that $d(f/g) \wedge d(f/h) \not\equiv 0$ and $[X,Y](f/g) = [X,Y](f/h) = 0$. This will imply that f/g and f/h are two independent meromorphic first integrals of $[X,Y]$, and so $[X,Y]=0$. □

Proof of $d(f/g) \wedge d(f/h) \not\equiv 0$. — Note that

$$d(f/g)\wedge d(f/h) = \frac{f}{g^2h^2}[f\,dg\wedge dh + h\,df\wedge dg + g\,dh\wedge df] = \lambda.\frac{f^3}{g^2h^2}\,dx\wedge dy \neq 0 \implies$$

$$\implies d(f/g)\wedge d(f/h) \neq 0.$$ □

Proof of $[X,Y]=0$. — We have

$$[X,Y](f/g) = X(Y(f/g)) - Y(X(f/g)) = X(Y(f/g)) ,$$

because $X(f/g)=0$. On the other hand, a straightforward computation shows that

$$Y(f/g)\,dx\wedge dy = d(f/g)\wedge\eta , \tag{12}$$

where $\eta = i_Y(dx\wedge dy)$. Since $\eta = h[\frac{dh}{h} - \frac{df}{f}] = -\frac{h^2}{f}d(f/h)$, we get from (12) that

$$d(f/g)\wedge\eta = -\frac{h^2}{f}d(f/g)\wedge d(f/h) = -\frac{\lambda f^2}{g^2}\,dx\wedge dy \implies Y(f/g) = -\lambda\,(f/g)^2 \implies$$

$\implies X(Y(f/g)) = 0$. In a similar way, we get $[X,Y](f/h)=0$. □

3. Proofs

Proof of Theorem 2. — Assume that $n\neq 0$, $f,h\not\equiv 0$ and $g\equiv 0$. Since S has an isolated singularity at $0\in\mathbb{C}^2$ and $S\wedge X = g.\partial_x\wedge\partial_y = 0$, we get $X=\psi.S$, where $\psi\neq 0$ is a polynomial. It follows that

$$0 = [Y,X] = [Y,\psi.S] = Y(\psi).S - \psi.[S,Y] = Y(\psi).S - n.\psi.Y \implies Y(\psi)\not\equiv 0$$

and $S\wedge Y = 0$, which implies $h\equiv 0$, a contradiction. Hence, $g\not\equiv 0$. It follows from lemma 2.1 that f/g is a non-constant meromorphic first integral of X. This proves (a) of theorem 2.

Lemma 2.1 implies also that f, g and h satisfy relation (6). Let us prove that (6) is equivalent to (7). We will use the following fact: let μ be a 2-form in $\mathbb{C}^2$ such that $L_S(\mu) = \lambda.\mu$, where $\lambda\in\mathbb{C}$. Then

$$d(i_S(\mu)) = L_S(\mu) = \lambda.\mu \tag{13}$$

Set $\mu = f\,dg\wedge dh + g\,dh\wedge df + h\,df\wedge dg$ and $\mu_1 = mn\,f^2\,dx\wedge dy$. We have seen in remark 1.1 that $S(f) = (m+n+tr(S)).f$, $S(g) = (m+tr(S)).g$ and $S(h) = (n+tr(S)).h$. As the reader can check, this implies that $L_S(\mu) = \lambda.\mu$ and $L_S(\mu_1) = \lambda.\mu_1$, where $\lambda = 2m+2n+3tr(S)\neq 0$.

On the other hand, we have

$$\begin{cases} i_S(\mu_1) = mn\,f^2(px\,dy - qy\,dx) \\ i_S(\mu) = -n\,fg\,dh + m\,fh\,dg + (n-m)\,gh\,df \end{cases}$$

as the reader can check. If we assume (6), we have $\mu_1 = \mu$, so that $i_S(\mu) = i_S(\mu_1)$ and

$$mn\, f^2(px\, dy - qy\, dx) = -n\, fg\, dh + m\, fh\, dg + (n-m)\, gh\, df \implies (7)\ .$$

If we assume (7), then we have

$$(7) \implies i_S(\mu_1 - \mu) = 0 \overset{(13)}{\implies} \lambda\,(\mu_1 - \mu) = d(i_S(\mu_1 - \mu)) = 0 \implies (6)\ .$$

This proves (b) of theorem 2.

Let us prove (c). We will use (7) in the form

$$(14) \qquad (m-n)\, g.h\, df + n\, f.g\, dh - m\, f.h\, dg = m\, n\, f^2\, (q\, y\, dx - p\, x\, dy)\ .$$

It follows from (14) that, if k is an irreducible factor of both polynomials g and h, then k divides f^2, and so it divides f.

Let us prove that any factor of f is a factor of both polynomials g and h. Here we use that f/g is a first integral of X. This implies that

$$(15) \qquad f.X(g) = g.X(f)\ .$$

Recall that any irreducible factor of f or g is the equation of an orbit of S (remark 1.1). Let $f = \Pi_{j=1}^r f_j^{\ell_j}$ $(r, \ell_j > 0)$, be the decomposition of f into irreducible factors and set $F = \Pi_j f_j$. It follows from (15) that

(16)

$$F.X(g) = F\,\frac{X(f)}{f}\, g = g.k \text{ ,where } k = F\,\frac{X(f)}{f} = \sum_{j=1}^{r} \ell_j . f_1 \cdots f_{j-1}.X(f_j).f_{j+1} \cdots f_r\ .$$

On the other hand, (16) implies that for any $j = 1, \ldots, r$, f_j divides g or $X(f_j)$. If f_j divides g, we are done. If f_j divides $X(f_j)$ then $(f_j = 0)$ is invariant for X. Since $(f_j = 0)$ is also invariant for S, it is a common orbit of X and S. This implies that f_j divides $S \wedge X$, and so it divides g. Similarly, any irreducible factor of f divides h.

Now, we can assume that the decompositions of f, g and h into irreducible factors are as in (8):

$$\begin{cases} f = \Pi_{j=1}^r f_j^{\ell_j} \\ g = \Pi_{j=1}^r f_j^{m_j} . \Pi_{i=1}^s g_i^{a_i} \\ h = \Pi_{j=1}^r f_j^{n_j} . \Pi_{i=1}^t h_i^{b_i} \end{cases}$$

where $\ell_j, m_j, n_j > 0$ and any two polynomials in the set

$$\{f_1, \ldots, f_r, g_1, \ldots, g_s, h_1, \ldots, h_t\}$$

are relatively primes. Let us prove that $\ell_j \geq m_j + n_j - 1$. As the reader can check, it follows from (14) that $f_j^{m_j+n_j+\ell_j-1}$ divides f^2. This implies that $m_j+n_j+\ell_j-1 \leq 2\ell_j$, and we are done.

It remains to prove (d). Let $\omega = i_X(dx \wedge dy)$. We have seen in lemma 2.1 that

$$\omega = \frac{g}{n}\left[\frac{dg}{g} - \frac{df}{f}\right] = \frac{g}{n}\left[\sum_{i=1}^{s} a_i\,\frac{dg_i}{g_i} - \sum_{j=1}^{r} (\ell_j - m_j)\,\frac{df_j}{f_j}\right]$$

As the reader can check, this implies that X is like in (9). Similarly, Y is also as in (9). ☐

Proof of Corollary 3. — Let $X = \sum_{j=d}^{\infty} X_j$ and $Y \not\equiv 0$ be germs of holomorphic vector fields at $0 \in \mathbb{C}^2$ such that $[X,Y] = 0$. Assume that $d \geq 2$ and X_d has an isolated singularity at $0 \in \mathbb{C}^2$ and no meromorphic first integral. Set $Y = \sum_{i=r}^{\infty} Y_j$, where Y_j is homogeneous of degree j, $r \geq 0$, and $Y_r \neq 0$. We have $[R, X_d] = m\,X_d$, $[R, Y_r] = n\,Y_r$, where $m = d-1 \neq 0$ and $n = r-1$. Note also that $[X_d, Y_r] = 0$.

Claim 3.1. — *We have $r = d$ and $Y_d = \lambda.X_d$, where $\lambda \neq 0$.*

Proof. — As before, set $X_d \wedge Y_r = f.\partial_x \wedge \partial_y$, $R \wedge X_d = g.\partial_x \wedge \partial_y$ and $R \wedge Y_r = h.\partial_x \wedge \partial_y$. Observe that $g \not\equiv 0$. Indeed, if $g \equiv 0$ then $R \wedge X_d = 0$. Since 0 is an isolated singularity of R, it follows from De Rham's division theorem (cf. [4]) that $X_d = \phi.R$, where ϕ is a homogeneous polynomial of degree $d-1 > 0$. But, this implies that $sing(X_d) \supset (\phi = 0)$, and so 0 is not an isolated singularity of X_d.

Suppose by contradiction that $r \neq d$. Let us prove that in this case we have $f, h \not\equiv 0$. Suppose by contradiction that $f \equiv 0$. This implies that $X_d \wedge Y_r \equiv 0$. Since X_d has an isolated singularity at $0 \in \mathbb{C}^2$, it follows from De Rham's division theorem that $Y_r = \phi.X_d$, where ϕ is a homogeneous polynomial of degree $r-d > 0$. Therefore,

$$0 = [X_d, Y_r] = [X_d, \phi.X_d] = X_d(\phi).X_d \implies X_d(\phi) = 0 \implies$$

that ϕ is a non-constant first integral of X_d, a contradiction. Hence, $f \not\equiv 0$. Suppose by contradiction that $h \equiv 0$. This implies that $R \wedge Y_r \equiv 0$, so that $Y_r = \phi.R$, where $\phi \neq 0$ is a homogeneous polynomial of degree $k = r-1$. From this we get

$$0 = [X_d, Y_r] = [X_d, \phi.R] = X_d(\phi).R + \phi.[X_d, R] = X_d(\phi).R - (d-1).\phi.X_d \implies$$

$$X_d(\phi).R = (d-1).\phi.X_d \ .$$

If $\phi \neq 0$ is a constant then $d = 1$, a contradiction. If ϕ is not a constant then $X_d(\phi) \neq 0$, for otherwise ϕ would be a non-constant first integral of X_d. In this case, we get $R \wedge X_d = 0$, and so $g \equiv 0$, a contradiction. Hence, $f, g, h \not\equiv 0$. Now, we can apply (a) of lemma 2.1.

If $r \neq 1$ then $n = r-1 \neq 0$ and f/g is a non-constant meromorphic first integral of X_d, a contradiction. If $r = 1$ then $n = 0$ and (a) of lemma 2.1 implies that $f = c.g$, where $c \in \mathbb{C}$. Therefore,

$$0 = (f - cg)\,\partial_x \wedge \partial_y = X_d \wedge (Y_1 + c.R) \implies Y_1 = -c.R \neq 0 \ ,$$

by the division theorem and the fact that $d = dg(X_d) > 1$. But, this implies that $0 = [X_d, Y_1] = c(d-1).X_d \neq 0$, a contradiction. Hence, $r = d$.

Now, $r = d$ implies that $n = m = d-1 > 0$ and $f \equiv 0$, for otherwise, f/g would be a non-constant meromorphic first integral of X_d. It follows that $X_d \wedge Y_d = 0$, and so $Y_d = \lambda.X_d$, where $\lambda \neq 0$ is a constant. This proves the claim. ☐

Let us finish the proof of corollary 3. Let $Z = Y - \lambda.X$. Then $[X, Z] = 0$. If $Z \not\equiv 0$, then we could write $Z = \sum_{j=r}^{\infty} Z_j$, where $r > d$, Z_j is homogeneous of degree j and $Z_r \neq 0$. But, this contradicts claim 3.1 and proves the corollary. □

Proof of Theorem 1. — Let $(Z_\lambda)_{\lambda \in \mathbf{P}^1}$ be a non-trivial pencil of homogeneous of degree $d \geq 2$ commuting vector fields on $\mathbb{C}^2$. Fix two generators of the pencil, X and Y, and set as before $X \wedge Y = f.\partial_x \wedge \partial_y$, $R \wedge X = g.\partial_x \wedge \partial_y$ and $R \wedge Y = h.\partial_x \wedge \partial_y$.

Suppose first that the pencil is colinear, that is, $f \equiv 0$. In this case, we can write $X = \alpha.Z$, where α is the greatest common divisor of the components of X and Z has an isolated singularity at $0 \in \mathbb{C}^2$. Since $Y \wedge X = 0$, we get $Y \wedge Z = 0$, and so $Y = \beta.Z$, where β is a homogeneous polynomial with $dg(\beta) = dg(\alpha)$, by De Rham's division theorem. Now,

$$0 = [X, Y] = [\alpha.Z, \beta.Z] = (\alpha\, Z(\beta) - \beta\, Z(\alpha)).Z \implies Z(\beta/\alpha) = 0 \; .$$

Since the pencil is non-trivial, β/α is non-constant. On the other hand, we can write $\frac{\beta(x,y)}{\alpha(x,y)} = \phi(y/x)$, where $\phi(t) = \frac{\beta(1,t)}{\alpha(1,t)}$, because α and β are homogeneous of the same degree. Therefore,

$$0 = Z(\phi(y/x)) = \phi'(y/x).Z(y/x) \implies Z(y/x) = 0 \; ,$$

because $\phi' \not\equiv 0$. This implies that $y\, Z(x) = x\, Z(y)$. If we set $Z = A\, \partial_x + B\, \partial_y$, then we get $y\, A = x\, B$, and so $A = \lambda.x$ and $B = \lambda.y$, where λ is a homogeneous polynomial. Since 0 is an isolated singularity of Z, it follows that λ is a constant. Hence, $X = \alpha_1.R$ and $Y = \beta_1.R$, where $\alpha_1 = \lambda.\alpha$ and $\beta_1 = \lambda.\beta$ are homogeneous polynomials of degree $d - 1$. This proves the first part of theorem 1.

Suppose now that the pencil is non-colinear. In this case, we have $f \not\equiv 0$. Let us prove that $g, h \not\equiv 0$. If $g \equiv 0$, for instance, then $X = \phi.R$, where $\phi \neq 0$ is a homogeneous polynomial of degree $m = n = d - 1 > 0$, by the division theorem. Therefore,

$$0 = [Y, \phi.R] = Y(\phi).R - m.\phi.Y \; .$$

Since $m.\phi.Y \neq 0$, the above relation implies that Y and R are colinear. Hence, $X//Y$, a contradiction. This proves (a) of theorem 1.

Since $m = n \neq 0$, it follows from (a) of theorem 2 that f/g and f/h are non-constant meromorphic first integrals of X and Y, respectively, which proves (b) of theorem 1. Recall that f, g and h are homogeneous polynomials, where $dg(f) = 2d$, $dg(g) = dg(h) = d + 1$.

It follows from (c) of theorem 2 that we can write the decomposition of f, g and h into irreducible linear factors as $f = \Pi_{j=1}^{r} f_j^{\ell_j}$, $g = \Pi_{j=1}^{r} f_j^{m_j}.\Pi_{i=1}^{a} g_i^{a_i}$ and $h = \Pi_{j=1}^{r} f_j^{n_j}.\Pi_{i=1}^{b} h_i^{b_i}$, where $r > 0$, $m_j, n_j > 0$, $\ell_j \geq m_j + n_j - 1$ and any two polynomials of the set $\{f_1, \ldots, f_r, g_1, \ldots, g_a, h_1, \ldots, h_b\}$ are relatively primes. Set $k_j = min(m_j, n_j)$.

Claim 3.2. — *The generators of the pencil can be choosen in such a way that:*

(a) $m_j = n_j = k_j$ *for all* $j = 1, \ldots, r$.

(b) *$a = b$ and $a_i = b_i = 1$ for all $i = 1, \dots, a$.*

Proof. — Set $X_\lambda = X + \lambda.Y$ and $R \wedge X_\lambda = g_\lambda.\partial_x \wedge \partial_y$,where $g_\lambda = g + \lambda.h$. It follows from Bertini's theorem that for a generic set of $\lambda \in \mathbb{C}$ the decomposition of g_λ into linear irreducible factors is of the form:

$$g_\lambda = \Pi_{j=1}^r f_j^{k_j}.\Pi_{i=1}^s g_{i\lambda} \ , \tag{17}$$

where $s + \sum_j k_j = d + 1$ and any two polynomials in the set $\{f_1, \dots, f_r, g_{1\lambda}, \dots, g_{s\lambda}\}$ are relatively primes. Now, it is sufficient to take $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ such that g_{λ_1} and g_{λ_2} are as in (17). Set $X_1 = X_{\lambda_1}$, $Y_1 = X_{\lambda_2}$, $g = g_{\lambda_1}$ and $h = g_{\lambda_2}$. Then X_1 and Y_1 are generators of the pencil with the properties required in claim 3.2. □

From now on, we will suppose that the generators X and Y of the pencil satisfy claim 3.2. Let us prove that the decomposition of f into irreducible linear factors is of the form

$$f = \Pi_{j=1}^r f_j^{2k_j + m_j} \ , \text{ where } m_j \geq 0. \tag{18}$$

Since $m = n = d - 1 > 0$, relation (14) implies that

$$g\,dh - h\,dg = m\,f(y\,dx - x\,dy) \ , \ m \neq 0.$$

Set $g = \psi.G_1$ and $h = \psi.H_1$, where $\psi = \Pi_{j=1}^r f_j^{k_j}$. As the reader can check, we have

$$g\,dh - h\,dg = \psi^2.(G_1\,dH_1 - H_1\,dG_1) = m\,f(y\,dx - x\,dy) \implies \psi^2 \,|\, f \ .$$

Hence, the decomposition of f is like in (18) and we get

$$G_1\,dH_1 - H_1\,dG_1 = m\,\Pi_{j=1}^r f_j^{m_j}(y\,dx - x\,dy) \ .$$

Now, consider the map ϕ: $\mathbf{P}^1 \to \mathbf{P}^1$ given by

$$\phi[x : y] = \frac{g(x, y)}{h(x, y)} = \frac{G_1(x, y)}{H_1(x, y)} \ .$$

Since G_1 and H_1 are relatively primes, the degree of ϕ is $s = dg(G_1) = dg(H_1)$. Let $\{p_1, \dots, p_t\} \subset \mathbf{P}^1$ be the critical set of ϕ and $\phi(p_j) = c_j \in \mathbf{P}^1$. If $c_j \neq \infty$ set $K_j = G_1 - c_j.H_1$, and if $c_j = \infty$ set $K_j = H_1$. Suppose that p_j is a critical point with $\text{mult}(\phi, p_j) = \ell_j \geq 2$. This implies that we can write $K_j = \psi_j^{\ell_j}.A$, where ψ_j is a linear polynomial, A a homogeneous polynomial and ψ_j does not divide A. We claim that $\psi_j^{\ell_j - 1} \,|\, \Pi_i f_i^{m_i}$. Indeed, if $c_j \neq \infty$, we get

$$K_j\,dH_1 - H_1\,dK_j = G_1\,dH_1 - H_1\,dG_1 = m\,\Pi_{i=1}^r f_i^{m_i}(y\,dx - x\,dy) \ . \tag{19}$$

Since $\psi_j^{\ell_j - 1}$ divides $K_j\,dH_1 - H_1\,dK_j$, relation (19) implies the claim. If $c_j = \infty$ then $\psi_j^{\ell_j - 1}$ divides $G_1\,dH_1 - H_1\,dG_1$ and we get also the claim. Therefore, $\psi_j = \lambda_j.f_{i(j)}$, $\lambda_j \in \mathbb{C}^*$, for some $i(j) \in \{1, \dots, r\}$ and $\ell_j - 1 \leq m_{i(j)}$. In particular, we get $t \leq r$. By reordering the $f_{i's}$, if necessary, we can suppose without lost of generality that $i(j) = j$, $j = 1, \dots, t$. Set $\ell_j = 1$ for $t < j \leq r$. With these conventions, we have $m_j - (\ell_j - 1) \geq 0$ for all $j = 1, \dots, r$.

Let us prove that $m_j = \ell_j - 1$ for all $j = 1, \dots, r$. Recall that $s + \sum_i k_i = d + 1$. Since $f = \Pi_i f_i^{2k_i + m_i}$ and $dg(f) = 2\,d$, we get

$$\sum_i m_i = dg(\Pi_i f_i^{m_i}) = 2\,d - 2\sum_i k_i = 2\,d - 2\,(d+1-s) = 2\,s - 2\ .$$

On the other hand, it follows from Riemann-Hurwitz formula (cf. [2]) and $m_i - (\ell_i - 1) \geq 0$ that

$$\sum_i (\ell_i - 1) = 2\,s - 2 = \sum_i m_i \implies 0 \leq \sum_{i=1}^{m} [m_i - (\ell_i - 1)] = 0 \implies m_i = \ell_i - 1\ ,\ \forall i\ .$$

This proves (d) and (e) of theorem 1. Note that (f) follows from (d) of theorem 2.

Let us prove that $1 \leq s \leq d-1$ and $1 \leq r \leq d$. First of all note that

$$k_j \geq 1 \implies 2r \leq \sum_{j=1}^{r} (2k_j + m_j) = 2d \implies 1 \leq r \leq d\ .$$

Moreover,

$$s = d + 1 - \sum_{j=1}^{r} k_j \implies s \leq d + 1 - r \leq d \implies 0 \leq s \leq d\ .$$

Suppose by contradiction that $s = 0$. This implies that the map ϕ is constant, and so $g = \lambda.h$, where $\lambda \in \mathbb{C}^*$. It follows that

$$R \wedge (X - \lambda.Y) = 0 \implies X - \lambda.Y = \psi.R\ ,$$

where ψ is homogeneous of degree $d - 1$. Therefore, the first part of theorem implies that X and Y are colinear with the radial vector field, a contradiction. Hence, $s \geq 1$. It remains to prove that $s \leq d - 1$. Suppose by contradiction that $s = d$. In this case, we get $g = f_1.g_1 \cdots g_d$, $h = f_1.h_1 \cdots h_d$ and $f = f_1^{2d}$. It follows that the map $\phi = (g_1 \cdots g_d)/(h_1 \cdots h_d)$ has degree $d \geq 2$ and just one ramification point, $(f_1 = 0)$, with multiplicity $2d - 1$. However, this is not possible, because this would imply that

$$\text{mult}(\phi, (f_1 = 0)) = 2d - 1 > d\ .$$

It remains to prove that in the converse construction the vector fields X and Y defined by (9) in theorem 1 commute. But, this is a consequence of lemma 2.2 and the fact that f, g and h satisfy (b) of Theorem 2. This finishes the proof of Theorem 1. ☐

Proof of Corollary 1. — Let X_1 and Y_1 be generators of a pencil of commuting of degree two homogeneous vector fields on $\mathbb{C}^2$. As before, define f_1, g_1 and h_1 by $X_1 \wedge Y_1 = f_1\,\partial_x \wedge \partial_y$, $R \wedge X_1 = g_1\,\partial_x \wedge \partial_y$ and $R \wedge Y_1 = h_1\,\partial_x \wedge \partial_y$, respectively. If $g_1 \equiv h_1 \equiv 0$ then X_1 and Y_1 are multiple of the radial vector field, and so we are in case (a) of corollary 1. If not, then $f_1, g_1, h_1 \not\equiv 0$, by (a) of theorem 1. Moreover, the rational map $\phi = g_1/h_1$ has degree $s = 1$, by (c) of theorem 1. Therefore, the pencil has one movable direction and one or two fixed directions, because g_1 has degree $d + 1 = 3$.

Suppose that it has two fixed directions. In this case, we can suppose that they are $(x = 0)$ and $(y = 0)$. This implies that $g_1 = x.y.g_2$, $h_1 = x.y.h_2$ and $f_1 = x^2.y^2$, where g_2 and h_2 correspond to the movable direction. Since g_2 and h_2 are relatively primes, there exist $(a,b),(c,d)$ such that $a\,g_2 + b\,h_2 = x$ and $c\,g_2 + d\,h_2 = y$. If we set $g := x^2.y = x.y(a\,g_2 + b\,h_2)$ and $h := x.y^2 = x.y(c\,g_2 + d\,h_2)$, then we can apply lemma 2.2 to $f = x^2.y^2$, g and h. We get the first integrals $f/g = (x^2.y^2)/(x^2.y) = y$, $f/h = (x^2.y^2)/(x.y^2) = x$, the forms $\omega := g\,\frac{d(f/g)}{f/g} = x^2\,dy$, $\eta := h\,\frac{d(f/h)}{f/h} = y^2\,dx$, and the vector fields $X = x^2\,\partial_x$, $Y = y^2\,\partial_y$. So, we are in case (b) of corollary 1.

Suppose that it has one fixed direction. We can suppose that it is $(y = 0)$. In this case, we have $g_1 = y^2.g_2$, $h_1 = y^2.h_2$ and $f = y^4$. Consider linear combinations $a\,g_2 + b\,h_2 = x$ and $c\,g_2 + d\,h_2 = y$. So, we have just to apply lemma 2.2 to the polynomials $f = y^4$, $g = x.y^2$ and $h = y^3$. By doing this, we obtain case (c) of corollary 1, as the reader can check. □

Proof of Corollary 2. — Let f, g and h be as in theorem 1. If $g \equiv h \equiv 0$ then we are in case (a) of corollary 2. If not, then $f,g,h \not\equiv 0$ and $\phi = g/h$ has degree s, where $s \in \{1,2\}$.

Let us consider the case where $s = 2$. Let $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ be a map of degree two. It follows from Riemann-Hurwitz formula that $\sum_p(\text{mult}(\phi,p) - 1) = 2\,s - 2 = 2$, and so the map must have two ramification points, both of multiplicity two. After composing the map in both sides with Moëbius transformations, we can suppose that $\phi[x : y] = y^2/x^2$. This implies that $(x = 0)$ and $(y = 0)$ are fixed directions of the pencil, so that $x.y$ divides g and h. Since $dg(g) = dg(h) = 4$ and $s = 2$, we get $g = x.y.g_1.g_2$ and $h = x.y.h_1.h_2$, and so $k_1 = k_2 = 1$ in (2) of theorem 1. Since $dg(f) = 6$ and $\text{mult}(\phi,(x = 0)) = \text{mult}(\phi,(y = 0)) = 2$, we must have $m_1 = m_2 = 1$ and $f = x^3.y^3$. In this case, we have

$$\phi = \frac{g}{h} = \frac{(g/x.y)}{(h/x.y)} = \frac{y^2}{x^2} \quad\Longrightarrow\quad g = x.y^3 \text{ and } h = x^3.y\ .$$

So, when we apply lemma 2.2, we get $f/g = x^2$, $f/h = y^2$, $\omega = 2y^3\,dx$ and $\eta = 2x^3\,dy$. Hence, we can set $X = x^3\,\partial_x$ and $Y = y^3\,\partial_y$. In this case we get case (e) of corollary 2.

Suppose now that $s = 1$. In this case, we have just one movable direction and the map ϕ has no ramification points, which implies that $m_j = 0$ for all $j = 1,\dots,r$. This implies that $f = \Pi_{j=1}^r f_j^{2k_j}$. Since $dg(f) = 6$, we have three possibilities: (1). $r = 1$ and $k_1 = 3$. (2). $r = 2$, $k_1 = 1$ and $k_2 = 2$. (3). $r = 3$ and $k_1 = k_2 = k_3 = 1$.

CASE (1). In this case, we have just one fixed direction f_1. After a linear change of variables in $\mathbb{C}^2$, we can suppose that it is $f_1 = y$. This implies that $f = y^6$, $g = y^3.g_1$ and $h = y^3.h_1$. Since g_1 and h_1 are relatively primes, there exist $a,b,c,d \in \mathbb{C}$ such that $a.d - b.c \neq 0$ and $a.g_1 + b.h_1 = x$ and $c.g_1 + d.h_1 = y$. Therefore, we can apply the construction of lemma 2.2 to $f = y^6$, $g = y^4$ and $h = x.y^3$. This gives the first integrals $f/g = y^2$ and $f/h = y^3/x$. Moreover,

$$\begin{cases} \omega = i_X(dx \wedge dy) = 2\,y^4\,\frac{dy}{y} = 2\,y^3\,dy \implies X = 2\,y^3\,\partial_x \\ \eta = i_Y(dx \wedge dy) = x.y^3\,(3\,\frac{dy}{y} - \frac{dx}{x}) = 3\,xy^2\,dy - y^3\,dx \implies Y = 3\,xy^2\partial_x + y^3\,\partial_y. \end{cases}$$

Therefore, we get case (b) of corollary 2.

CASE (2). In this case, we have two fixed directions, that we can suppose to be $f_1 = x$ and $f_2 = y$. Since $k_1 = 1$ and $k_2 = 2$, we get $g = x.y^2.g_1$, $h = x.y^2.h_1$ and $f = x^2.y^4$. After taking linear combinations, we can suppose that $g = x^2.y^2$ and $h = x.y^3$. This gives the first integrals y^2 and $x.y$ and so $\omega = 2\,x^2\,y\,dy$ and $\eta = xy^2\,dy + y^3\,dx$ and we are in case (c).

CASE (3). In this case, we have three fixed directions. After a linear change of variables we can suppose that they are $f_1 = x$, $f_2 = y$ and $f_3 = x + y$. This gives $g = x\,y\,(x + y).g_1$, $h = x\,y\,(x + y).h_1$ and $f = x^2\,y^2\,(x + y)^2$. After taking linear combinations of g_1 and h_1, we can suppose that $g = x^2\,y\,(x + y)$ and $h = x\,y^2\,(x + y)$. Therefore we get the first integrals are $f/g = y\,(x + y)$, $f/h = x\,(x + y)$ and

$$\begin{cases} \omega = x^2\,y\,(x + y)\,[\frac{dy}{y} + \frac{dx+dy}{x+y}] = x^2\,y\,dx + (2\,x^2\,y + x^3)\,dy \\ \quad \implies X = (2\,x\,y^2 + x^3)\,\partial_x - x^2\,y\,\partial_y \\ \eta = x\,y^2\,(x + y)\,[\frac{dx}{x} + \frac{dx+dy}{x+y}] = (2\,x\,y^2 + y^3)\,dx + x\,y^2\,dy \\ \quad \implies Y = -x\,y^2\,\partial_x + (2\,x\,y^2 + y^3)\,\partial_y. \end{cases}$$

Therefore, we are in case (d) of corollary 2. □

References

[1] D. CERVEAU & J.-F. MATTEI – “Formes intégrables holomorphes singulières”, *Astérisque* **97** (1982).

[2] H. M. FARKAS & I. KRA – *Riemann surfaces*, Springer, 1980.

[3] A. GUILLOT – “Sur les exemples de Lins Neto de feuilletages algébriques”, *C. R. Math. Acad. Sci. Paris* **334** (2002), p. 747–750.

[4] G. DE RHAM – “Sur la division de formes et de courants par une forme linéaire”, *Comment. Math. Helv.* **28** (1954), p. 346–352.

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