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NEW RESULTS AND PROBLEMS ON KÄHLER-RICCI FLOW

by

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Abstract. — In this paper, I give a brief tour on a program of studying the Kähler-Ricci flow with surgery and its interaction with the classification of projective manifolds. The Kähler-Ricci flow may develops singularity at finite time. It is important to understand how to extend the Kähler-Ricci flow across the singular time, that is, construct solution of the Kähler-Ricci flow with surgery. The first task of this paper is to describe a procedure of constructing global solutions for the Kähler-Ricci flow with surgery. This procedure is rather canonical. I will discuss properties of such solutions with surgery and their geometric implications. I will also discuss their asymptotic limits at time infinity. The results discussed here were mainly from my joint works with Z. Zhang, J. Song *et al.* Some open problems will be also discussed. The paper is mostly expository.

***Résumé* (Nouveaux problèmes et résultats sur le flot de Kähler-Ricci).** — Dans cet article, nous donnons un aperçu rapide d'un programme d'études sur le flot de Kähler-Ricci avec chirurgie et son interaction avec la classification des variétés projectives. Le flot de Kähler-Ricci peut développer des singularités en un temps fini. Il est important de comprendre comment étendre le flot de Kähler-Ricci à travers le temps singulier, c'est-à-dire, comment construire une solution du flot de Kähler-Ricci avec chirurgie. La première tâche de cette article consiste à décrire une procédure de construction de solutions globales pour le flot de Kähler-Ricci avec chirurgie. Cette procédure est plutôt canonique. Nous allons discuter les propriétés de telles solutions avec chirurgie et leurs implications géométriques. Nous allons également discuter leurs limites asymptotiques au temps infini. Les résultats présentés ici proviennent principalement de travaux communs avec Z. Zhang, J. Song *et al.* Nous allons également présenter certains problèmes ouverts. L'article est plutôt explicatif.

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1. Introduction

Let X be an n-dimensional compact Kähler manifold. We denote a Kähler metric by its Kähler form ω, in local complex coordinates $z^1, \ldots, z^n$,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where we use the standard convention for summation and $(g_{i\bar{j}})$ is the positive Hermitian matrix valued function given by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right).$$

The Ricci flow was introduced by R. Hamilton. It has a nice property: If the initial metric is Kählerian, so do any metrics which evolve along the Ricci flow. This can be proved by either using the uniqueness of its local solutions or applying the maximum principle in an appropriate way. Thus we can consider the following Kähler-Ricci flow

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t), \qquad \tilde{\omega}_0 = \omega_0, \tag{1.1}$$

where ω_0 is any given Kähler metric and $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates above,

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

This paper is essentially expository. In this paper, I will discuss some new results and open problems in recent study of the Kähler-Ricci flow. They were mainly from my joint works with Z. Zhang, J. Song et al. I will also describe briefly a program of studying the singularity formation of the Kähler-Ricci flow and how it interacts with the classification of projective manifolds. The results and problems discussed here arise from our long efforts in pursuing this program (cf. [**28**], [**30**], [**20**], [**22**], [**31**], [**6**] etc.).

2. A sharp local existence for Kähler-Ricci flow

By the local existence of Ricci flow, given any initial Kähler metric ω_0, there is a unique solution $\tilde{\omega}_t$ of (1.1) ($t \in [0, T)$) for some $T > 0$. The following theorem was proved in [**30**] (also see [**2**] (1)) and characterizes the maximal T for which the solution $\tilde{\omega}_t$ exists for $t < T$.

(1) In this cited paper, the authors claimed a proof of a related result under certain extra technical assumptions.

Theorem 2.1. — *Let X be a compact Kähler manifold. Then for any initial Kähler metric ω_0, the flow* (1.1) *has a maximal solution $\tilde{\omega}_t$ on $X \times [0, T_{\max})$, where*

$$T_{\max} = \sup\{t \mid [\omega_0] - t\, c_1(X) > 0\,^{(2)}\}.$$

In particular, if the canonical class K_X is numerically effective, then (1.1) *has a global solution $\tilde{\omega}_t$ for all $t > 0$. Here, $c_1(X)$ denotes the 2π mutiple of the first Chern class.*

In [**1**], Cao proved this theorem in the case that $c_1(X)$ is definite and proportional to the initial Kähler class. In the case that K_X is nef, i.e., numerically effective, and the initial metric ω_0 is sufficiently positive, H. Tsuji proved in [**32**] the above theorem, that is, (1.1) has a global solution $\tilde{\omega}_t$.

Now let us sketch a proof of the above theorem following the arguments in the proof of Proposition 1.1 in [**30**]. (3)

For any small $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that $T_\epsilon + \epsilon < T_{\max}$ and a real closed $(1,1)$ form ψ_ϵ such that $[\psi_\epsilon] = c_1(X)$ and $\omega_0 - (T_\epsilon + \epsilon)\,\psi_\epsilon \geq 0$. Choose a smooth volume form Ω_ϵ such that $\mathrm{Ric}(\Omega_\epsilon) = \psi_\epsilon$. This Ω_ϵ is unique up to multiplication by a positive constant.

Set $\omega_t = \omega_0 - t\psi_\epsilon$ for $t \in [0, T_\epsilon]$. One can easily show that $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ satisfies (1.1) if u satisfies

$$\frac{\partial u}{\partial t} = \log \frac{\tilde{\omega}_t^n}{\Omega_\epsilon}, \qquad u(0,\cdot) = 0. \tag{2.1}$$

We shall show the solution for (2.1) exists for $t \in [0, T_\epsilon]$.

First observe that ω_t is a Kähler metric for $t \in [0, T_\epsilon]$ with uniformly bounded geometry.

By the standard theory, u exists for small $t > 0$. In order to prove that u exists for $t \in [0, T_\epsilon]$, we only need to get uniform estimates of u whenever it exists for $t \in [0, T_\epsilon]$.

Applying the Maximum Principle to (2.1), we can easily have $|u| \leq C_\epsilon$. (4) In fact, the upper bound is independent of ϵ.

Taking derivative of (2.1) with respect to t, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_{\tilde{\omega}_t}\left(\frac{\partial u}{\partial t}\right) - \langle \tilde{\omega}_t, \psi_\epsilon \rangle,$$

where Δ_ω denotes the Laplacian of a Kähler metric ω and $\langle \omega, F \rangle$ means the trace of F with respect to ω for a real $(1,1)$-form F.

It follows

$$\frac{\partial}{\partial t}\left(t\frac{\partial u}{\partial t} - u\right) = \Delta_{\tilde{\omega}_t}\left(t\frac{\partial u}{\partial t} - u\right) + n - \langle \tilde{\omega}_t, \omega_0 \rangle. \tag{2.2}$$

(2) This means that $[\omega_0] - tc_1(X) > 0$ represents a Kähler class.

(3) The flow equation in [**30**] is not the same as, but equivalent to (1.1).

(4) The constant C, C_ϵ may differ at various places. A subscript indicates the dependence on another constant.

Noticing $\langle \tilde{\omega}_t, \omega_0\rangle > 0$ and applying the Maximum Principle, we see that the maximum of $t\frac{\partial u}{\partial t} - u - nt$ is non-increasing, so we have that

$$t\frac{\partial u}{\partial t} - u - nt \leqslant 0.$$

Now we combine it with local existence for small time and the uniform upper bound for u to conclude that

$$\frac{\partial u}{\partial t} \leq C.$$

On the other hand, we have

$$\begin{aligned}(2.3)\quad & \frac{\partial}{\partial t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right) \\ &\qquad = \Delta_{\tilde{\omega}_t}\left((T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u\right) - n + \langle \tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle.\end{aligned}$$

Since $\langle \tilde{\omega}_t, \omega_0 - (T_\epsilon + \epsilon)\psi_\epsilon\rangle \geq 0$, by the Maximum Principle, we see that minimum of $(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt$ is non-decreasing. It follows

$$(T_\epsilon + \epsilon - t)\frac{\partial u}{\partial t} + u + nt \geqslant (T_\epsilon + \epsilon)\min_{t=0}\frac{\partial u}{\partial t} = -C_\epsilon,$$

from this we can conclude

$$\frac{\partial u}{\partial t} > -C_\epsilon.$$

Now we have gotten all the C^0-estimates needed. By using the Maximum principle and the standard arguments, one can derive the second and higher order estimates for u (cf. [**30**] for more details). Then one obtains the existence of solution for (2.1) for $t \in [0, T_\epsilon]$.

The desired existence of the solution for (1.1) can be proved by considering the relations between all the equations as (2.1) for different ϵ's as follows:

Consider (2.1) for some $\delta > 0$. Assume $\psi_\delta = \psi_\epsilon + \sqrt{-1}\partial\bar{\partial}f$ for some smooth real function f over X. Since $\mathrm{Ric}_{\Omega_\epsilon} = \psi_\epsilon$, we have $\mathrm{Ric}_{e^{-f}\Omega_\epsilon} = \psi_\delta$. Thus we can take $\Omega_\delta = e^{-f}\Omega_\epsilon$. Now the new "$\omega_t$" is

$$\eta_t = \omega_0 - t\psi_\delta = \omega_t - t\sqrt{-1}\partial\bar{\partial}f.$$

The equation (2.1) for δ is

$$\frac{\partial v}{\partial t} = \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^f\Omega_\epsilon}, \qquad v(0,\cdot) = 0.$$

Define $\tilde{u} = v - tf$. Then

$$\begin{aligned}(2.4)\qquad \frac{\partial \tilde{u}}{\partial t} = \frac{\partial v}{\partial t} - f &= \log\frac{(\eta_t + \sqrt{-1}\partial\bar{\partial}v)^n}{e^{-f}\Omega_\epsilon} + f \\ &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}\tilde{u})^n}{\Omega_\epsilon}.\end{aligned}$$

Noticing that $\tilde{u}(0,\cdot) = v(0,\cdot) = 0$, from the uniqueness of the solution for (2.1), we conclude that $\tilde{u}$ coincides with u.

This actually gives the explicit relation between solutions of (2.1) associated to different ϵ's and would allow us to glue together all these solutions for (2.1) to get a maximal solution of (1.1) until the time $T_{\max}$. Thus Theorem 2.1 is proved.

Remark 2.2. — Note that ω_t depends on ϵ and may not be a Kähler metric for t sufficiently close to $T_{\max}$ The above arguments also show that the solution u of (2.1) extends to all $t < T_{\max}$ even if ω_t is a Kähler metric when t is sufficiently close to $T_{\max}$.

Next we need to examine behavior of $\tilde{\omega}_t$ as t tends to $T_{\max}$.

3. Finite-time singularity

In this section, we assume that $T = T_{\max} < \infty$, that is, the Kähler-Ricci flow develops singularity at finite time T. We want to examine the limiting behavior of $\tilde{\omega}_t$ as t tends to T. We shall adopt the notations in the last section.

First we observe

Lemma 3.1. — *Let ψ be any smooth (1,1)-form ψ representing $c_1(X)$. Then there is a smooth solution, say $\tilde{u}_t$, for* (2.1) *with $\psi = \psi_\epsilon$ satisfying:*

(1) $\tilde{\omega}_t = \omega_0 - t\psi - \sqrt{-1}\partial\bar{\partial}\tilde{u}_t$*;*

(2) *For any sequence $t_i \to T$, a subsequence of $\tilde{\omega}_{t_i}$ converges to a positive current $\tilde{\omega}_T$ weakly.* (5)

(3) *If $\overline{\lim}_{t\to T} \sup_X \tilde{u}_t$ is not $-\infty$, then $\tilde{u}_t$ converges to a unique $\tilde{u}_T$ in any L^p-topology as t tends to T for any $p > 1$. In particular, $\tilde{\omega}_t$ converges to a unique positive current $\tilde{\omega}_T$ weakly as t tends to T in this case.*

Proof. — (1) follows directly from the remark at the end of last section.

For (2), we notice that $\tilde{\omega}_t \geq 0$ and

$$\int_X \tilde{\omega}_t \wedge \omega_0^{n-1} = ([\omega_0] - tc_1(X))[\omega_0]^{n-1}(X),$$

so there is a $\alpha > 0$ such that (cf. [**23**])

$$\int_X e^{-\alpha(\tilde{u}_t - \sup_X \tilde{u}_t)}\omega_0^n \leq C'.$$

In particular, for any $p \geq 1$, $v_t = \tilde{u}_t - \sup_X \tilde{u}_t - 1$ has uniformly bounded L^p norm.

(5) $\tilde{\omega}_T$ can be 0.

Furthermore, for any $\delta \in (0,1)$, we have

$$C' \geq \int_X (-v_t)^{-\delta}(\omega_t - \tilde{\omega}_t) \wedge \omega_0^{n-1} \geq \frac{4\delta}{(1-\delta)^2} \int_X |\nabla(-v_t)^{\frac{1-\delta}{2}}|^2 \omega_0^n.$$

Choose $\delta = 1/3$. By the Sobolev embedding theorem, for any sequence t_i with $\lim t_i = T$, there is a subsequence, again denoted by t_i for simplicity, such that $(1 + \sup_X \tilde{u}_t - \tilde{u}_{t_i})^{\frac{1}{3}}$ converges to some function $(-v)^{\frac{1}{3}}$ in L^2-norm. Since v_t have uniform L^p-norm for any $p > 1$, v_i converges to v in the L^p-topology. Then (2) follows.

Now we prove (3). First recall that by the Maximum Principle, we have proved in last section

$$\tilde{u}_t \leq C \quad \text{and} \quad t\,\frac{\partial \tilde{u}_t}{\partial t} - \tilde{u}_t - nt \leq 0.$$

Here C is a uniform constant. It follows that $t^{-1}\tilde{u}_t - n \log t$ is non-increasing, consequently $\tilde{u}_t$ converges to a unique function $\tilde{u}_T$, which may take $-\infty$ as values. as t tends to T. By our assumption, $\tilde{u}_T$ is not identically $-\infty$, so $\sup_X \tilde{u}_t$ is uniformly bounded. It follows that the above v coincides with $\tilde{u}_T - \sup_X \tilde{u}_T$. So we have proved (3). □

Let $\tilde{\omega}_T$ be a limiting positive current at the finite-time singularity from the above lemma. A natural question is: *How regular is this limiting $\tilde{\omega}_T$?* It is reasonable to expect that $\tilde{\omega}_T$ is bounded and smooth on a Zariski open subset of X. We also expect that it has controlled behavior along its subvariety of singularity in a suitable sense.

We conjecture that *the limiting current $\tilde{\omega}_T$ is independent of the choice of the sequence $\{t_i\}$*. But we can not prove it in full generality yet. The following lemma gives a sufficient condition for this to be true.

Lemma 3.2. — *If there is a representative ψ of $c_1(X)$ such that $\omega_0 - T\psi \geq 0$ as a (1,1)-form. Then the limiting potential $\tilde{u}_T$ is unique and bounded. If, in addition, $\int_X (\omega_0 - T\psi)^n > 0$, then $\tilde{u}_T$ is continuous.*

Proof. — Set $\omega_t = \omega_0 - t\psi$, then $\omega_t \geq 0$ for any $t \in [0,T]$. We have derived in last section

$$\frac{\partial}{\partial t}\left((T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t\right) = \Delta_{\tilde{\omega}_t}\left((T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t\right) - n + \langle \tilde{\omega}_t, \omega_T \rangle. \tag{3.1}$$

By using the Maximum principle, we can deduce from this equation that the auxiliary function

$$(T-t)\frac{\partial \tilde{u}_t}{\partial t} + \tilde{u}_t + nt$$

is non-decreasing. Since $\frac{\partial \tilde{u}_t}{\partial t}$ is bounded form above, $\tilde{u}_t$ is bounded from below. Then the uniqueness follows from Lemma 3.1.

The continuity follows from the extension of S. Kolodziej's work [**17**] by Z. Zhang in [**35**] or [**9**] (also see [**11**], [**8**]) since

$$\lim_{t\to T}\int_X \tilde\omega_t^n = \lim_{t\to T}\int_X \omega_t^n = \int_X \omega_T^n > 0. \qquad \square$$

Now let me discuss some special cases:

First we assume that $\tilde\omega_T = 0$, that is, $c_1(X) = \frac{1}{T}[\omega_0]$ is positive.[6] Set

$$\omega(s) = e^{\frac{s}{T}}\tilde\omega_{T(1-e^{-\frac{s}{T}})}.$$

Here s goes from 0 to ∞. Then we have

$$\frac{\partial\omega(s)}{\partial s} = -\left(\mathrm{Ric}(\omega(s)) - \frac{1}{T}\omega(s)\right). \tag{3.2}$$

This $\omega(s)$ is a global solution for the renormalized Ricci flow. A challenging problem is to show the convergence of $\omega(s)$ as s goes to ∞. A folklore conjecture claims that *there is a family of diffeomorphisms $\phi(s) : X \mapsto X$ such that $\phi(s)^*\omega(s)$ converges to a Kähler-Ricci soliton on a variety with possible singularity of codimension 2* (cf. [**14**], [**27**], [**18**]). In the case that $X = S^2$, $\omega(s)$ converges to the standard metric on S^2 as shown in [**13**] and [**7**] (also see [**3**]). In the case that ω_0 has non-negative bisectional curvature, it was proved in [**4**], [**5**] that $\omega(s)$ converges to the unique Kähler-Einstein metric on X. Perelman proved that the scalar curvature and the diameter of $\omega(s)$ are uniformly bounded along (3.2) (cf. [**19**]). It follows that the above conjecture holds if one can bound the Ricci curvature of (3.2) [**19**]. The following theorem was first claimed by Perelman and proved in [**31**].

Theorem 3.3. — *Assume that X has no non-trivial holomorphic fields. If X admits a Kähler-Einstein metric and $c_1(X) = \frac{1}{T}[\omega_0]$, then $\omega(s)$ converges to a Kähler-Einstein metric.*

In [**31**], the above theorem was also extended to the case that X admits only a Kähler-Ricci soliton. The proof of the above theorem was proved by using one of Perelman's estimates and exploring the properness of the K-energy.

Next we consider $X = X_1 \times X_2$ with both $c_1(X_1)$ and $c_1(X_2)$ definite. For simplicity, we assume that $H_2(X_i,\mathbb{Z}) = \mathbb{Z}$ with generator represented by a Kähler form β_i for $i = 1,2$. Then the initial Kähler class $[\omega_0] = \mu_1[\beta_1] + \mu_2[\beta_2]$ with $\mu_1,\mu_2 > 0$. We further assume that $c_1(X_i) = m_i\beta_i$ with $m_1 > \max(0,m_2)$. Then the flow (1.1) develops singularity at $T = \mu_1/m_1$. First we assume that ω_0 is a product metric $\omega_{01} + \omega_{02}$, where ω_{0i} is a Kähler metric on X_i, then the flow becomes a product flow $\tilde\omega_t = \tilde\omega_{t1} + \tilde\omega_{t2}$, where $\tilde\omega_{ti}$ solves (1.1) on X_i with initial metric ω_{0i}. Then $\tilde\omega_{t1}$ converges

[6] One can easily show that the limiting current is unique in this case, in fact, it is always zero.

to 0 as t tends to T, while $\tilde{\omega}_{t2}$ exists on $X_2 \times [0, T+\epsilon]$ for some $\epsilon > 0$. Hence, the flow $\tilde{\omega}_t$ collapses to $\tilde{\omega}_{t2}$ on X_2 at T and continues beyond T.

Now if ω_0 is a general Kähler metric in $\mu_1[\beta_1] + \mu_2[\beta_2]$, then there is a smooth function θ such that $\omega_0 = \omega_{01} + \omega_{02} + \sqrt{-1}\partial\bar{\partial}\theta$. By the Maximum Principle, the solution $\tilde{\omega}_t$ of (1.1) with initial metric ω_0 is equal to $\tilde{\omega}_{t1} + \tilde{\omega}_{t2} + \sqrt{-1}\partial\bar{\partial}\theta_t$ with θ_t uniformly bounded. This implies that modulo a bounded potential function, $\tilde{\omega}_t$ collapses to a current on X_2 at T. I believe that this collapsing occurs in the L^∞-topology.

In our next example, we assume that X is a projective manifold with Kodaira dimension ≥ 0 and ω_0 is rational. Then $T = T_{\max}$ is rational and consequently, $m[\omega_0]$ is the first Chern class of a line bundle L and $a = mT$ is an integer for some $m > 0$. Clearly, $L + aK_X$ is nef. Since the Kodaira dimension is non-negative, for m sufficiently large, aK_X admits a holomorphic section S. It follows that S^kS' is a global section of $k(L+aK_X)$ for any section S' of kL, so $\dim H^0(X, k(L+aK_X)) \geq ck^n$ for some $c > 0$. It follows that $(L+aK_X)^n > 0$, i.e., it is big. By a result of Kawamata [**16**], $L + aK_X$ is semi-positive, i.e., there is a $k > 0$ such that any basis of $H^0(X, k(L+aK_X))$ maps X onto a subvariety in some $\mathbb{C}P^N$. In particular, there is a ψ representing $c_1(X)$ such that $\omega_0 - T\psi$ is a semi-positive smooth form. In this case, we can say more about the limiting behavior of $\tilde{\omega}_t$ as $t \to T$.

The following lemma can be found in [**15**].

Lemma 3.4. — *Let E be a divisor in a projective manifold X. If E is nef. and big, then there is an effective divisor D such that $E - \epsilon D > 0$ for any sufficiently small $\epsilon > 0$.*

The proof follows essentially from the openness of the big cone of X which clearly contains the positive cone and the fact that E is in the closure of the positive cone. In fact one can choose D to be big. (7)

Applying the above lemma to $L + aK_X$, there is a Hermitian metric h_ϵ on D such that for any small $\epsilon > 0$,

$$\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log h_\epsilon > 0.$$

Let σ be a defining holomorphic section for D. Then we have

$$\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0,$$

where $|\cdot|$ denotes the norm induced by h_ϵ. (8)

The following theorem was essentially proved in [**30**]. (9)

(7) Even if $[\omega_0]$ is irrational, the arguments for proving the above lemma still work.

(8) For simplicity, if there is no possible confusion, we will drop the subscript ϵ in the norm later.

(9) In [**30**], K_X is assumed to be big. It is clear from the arguments in the proof that this assumption was not used.

Theorem 3.5. — *Let X, $L+aK_X$ be as above. Then the solution $\tilde\omega_t$ of* (1.1) *converges to a unique current $\tilde\omega_T$ as $t\to T$ satisfying:*

(1) *$\tilde\omega_T$ represents the cohomology class of $L+aK_X$;*

(2) *$\tilde\omega_T$ is a smooth Kähler metric outside a subvariety $B_T\subset X$ along which $c_1(L+aK_X)$ vanishes;*

(3) *$\tilde\omega_t$ converges to $\tilde\omega_T$ on any compact subset outside B_T in the C^∞-topology.*

Proof. — We will outline a proof of this theorem following [**30**].

Since $L+aK_X$ is semi-positive and big, by Lemma 3.2, we know that the limiting current $\tilde\omega_T$ exists with locally continuous potential and satisfies (1). It suffices to prove (2).

Let σ be a defining section of D. Then $\log|\sigma|^2$ is a well-defined function outside $D\subset X$.

First we need a second order estimate. Set

$$\omega_{t,\epsilon}=\omega_t+\epsilon\sqrt{-1}\partial\bar\partial\log|\sigma|^2.$$

Then for any $t\in[0,T+\delta]$, where $\delta=\delta(\epsilon)$ may depend on ϵ, [10] $\omega_{t,\epsilon}$ is a smooth Kähler metric, in particular, there is a bound on their curvature which is uniform in $t\in[0,T+\delta]$ but may depend on ϵ.

In order to derive the second order estimate, we need a lower bound on $\frac{\partial\tilde u_t}{\partial t}$ for any $t\in[0,T]$. Using the same arguments in deriving (3.1), we get

$$\frac{\partial w_t}{\partial t}=\Delta_{\tilde\omega_t}w_t-n+\langle\tilde\omega_t,\omega_{T+\delta,\epsilon}\rangle,\tag{3.3}$$

where

$$w_t=(T+\delta-t)\frac{\partial\tilde u_t}{\partial t}+\tilde u_t-\epsilon\log|\sigma|^2.$$

Since $\langle\tilde\omega_t,\omega_{T+\delta,\epsilon}\rangle\geq 0$, by the Maximum Principle, we can show that the minimum of w_t+nt is non-decreasing. Since $\tilde u_t$ is bounded for $t\in[0,T]$, we conclude from this

$$\frac{\partial\tilde u_t}{\partial t}\geq\frac{\epsilon}{\delta}\log|\sigma|^2-C_\delta,\tag{3.4}$$

where C_δ is a uniform constant which may depend on δ.

Now we write

$$\tilde\omega_t=\omega_{t,\epsilon}+\sqrt{-1}\partial\bar\partial(\tilde u_t-\epsilon\log|\sigma|^2).$$

Note that the function $v_t=\tilde u_t-\epsilon\log|\sigma|^2$ is defined only outside D.

On $X\setminus D$, we can rewrite (2.1) as

$$(\omega_{t,\epsilon}+\sqrt{-1}\partial\bar\partial v_t)^n=e^{\frac{\partial\tilde u_t}{\partial t}}\Omega.$$

Note that $\mathrm{Ric}(\Omega)=\psi$.

[10] One can show that $\delta\geq b\epsilon$ for some $b>0$.

As in [**34**], [**1**] and [**32**], using the bound on $\frac{\partial \tilde{u}_t}{\partial t}$ and the curvature of $\omega_{t,\epsilon}$, one can deduce

$$
\begin{aligned}
& e^{Cv_t}(\Delta_{\tilde{\omega}_t} - \frac{\partial}{\partial t})\left(e^{-Cv_t}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle\right) \\
& > -C' + (C\frac{\partial u}{\partial t} - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}} \\
& > -C' + (\frac{C\epsilon}{\delta}\log|\sigma|^2 - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}}.
\end{aligned} \tag{3.5}
$$

Here C, C' etc. are constants which may depend on ϵ. For instance, we need to choose C such that $C + \inf_M Rm(\omega_{t,\epsilon}) \geq 1$ for $t \in [0, T]$, where $Rm(\omega')$ denotes the bisectional curvature tensor of ω'.

Clearly, $e^{-C(u-\epsilon\log|\sigma|^2)}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle$ attains its maximum in $X \setminus \{\sigma = 0\}$. At such a maximum point, we have

$$
\begin{aligned}
0 &> -C' + (C''\log|\sigma|^2 - C')\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{n}{n-1}} \\
&= -C' + C'\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle\left(\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle^{\frac{1}{n-1}} + C''\log|\sigma|^2 - C'\right).
\end{aligned}
$$

Here $C'' = C\epsilon/\delta$. Since $|\sigma|$ is bounded, it follows from this

$$\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle \leq (C' - C''\log|\sigma|^2)^{n-1}.$$

Hence, at this maximum point,

$$e^{-Cv_t}\langle \omega_{t,\epsilon}, \tilde{\omega}_t\rangle \leq (C' - C''\log|\sigma|^2)^{n-1}e^{-Cv_t} \leq C_1(1 - \log|\sigma|^2)|\sigma|^{C\epsilon}.$$

Here we have used that fact that $\tilde{u}_t$ is uniformly bounded and C_1 is a constant which depends on ϵ.

Then we can easily deduce the second order estimate:

$$\langle \omega_0, \tilde{\omega}_t\rangle \leqslant C_2|\sigma|^{-C\epsilon}. \tag{3.6}$$

Observe that our lower bound estimate on $\frac{\partial \tilde{u}_t}{\partial t}$ implies the volume estimate:

$$\tilde{\omega}_t^n > C_3|\sigma|^{2\epsilon}{\omega_0}^n.$$

It follows that $\tilde{\omega}_t$ defines a Kähler metric on $X \setminus \{\sigma = 0\}$. Furthermore, we have a uniform bound on $\frac{\partial \tilde{u}_t}{\partial t}$ on any given compact subset outside D.

The higher order derivative estimates for $\tilde{u}_T$ outside $\{\sigma = 0\}$ follow from the standard theory on Monge-Ampere equations ([**10**] etc.) or Calabi's third order estimates as shown in [**34**].

We have shown that $\tilde{u}_T = \lim_{t\to T}\tilde{u}_t$ exists. The above shows that $\tilde{u}_T$ is smooth and defines a smooth Kähler metric $\tilde{\omega}_T$ outside D. Moreover, we have

$$(\omega_T + \sqrt{-1}\partial\bar{\partial}\tilde{u}_T)^n = e^{\frac{\partial \tilde{u}_t}{\partial t}|_T}\Omega, \quad \text{on } X \setminus \{\sigma = 0\}. \tag{3.7}$$

Notice that D may not be unique. We can choose any D's in the above discussions so long as it satisfies Lemma 3.4. Since the limit $\tilde{u}_T$ is unique, $\tilde{u}_T$ is smooth and gives rise to a Kähler metric outside the intersection B_T of all such D's. Thus this theorem follows. □

Theorem 3.5 tells us that the solution $\tilde{\omega}_t$ extends to a Kähler metric $\tilde{\omega}_T$ outside the subvariety $B_T \subset X$. However, this limiting $\tilde{\omega}_T$ does have singularity along B_T. This singular behavior can be caused by the metric's either blowing-up or failing to be non-degenerate along B_T. In order to extend the Ricci flow across T, we need to study how $\tilde{\omega}_T$ behaves along B_T. Here is what we expect (also see [**20**])

Conjecture 3.6. — *Let X_1 be the metric completion of $X \setminus B_T$ with respect to the distance d_T on $X \setminus B_T$ induced by $\tilde{\omega}_T$. Then X_1 is a projective variety which can be obtained from X by flips or algebraic surgeries of certain "standard" type. Moreover, $(L_0 + aK_X)|_{X\setminus B_T}$ extends to an ample line bundle over X_1.*

If X has the Kodaira dimension $-\infty$ and $[\omega_0]$ is again rational, then $[\omega_0] - Tc_1(M)$ is still rational and nef, but it is not big anymore. If $[\omega_0] - Tc_1(X) \neq 0$ and the well-known Abundance Conjecture holds, then for k sufficiently large, any basis of $H^0(X, k(L_0 + aK_X))$ maps to a subvariety $Y \subset \mathbb{C}P^N$ for some $N > 0$. By Lemma 3.2, the limit $\tilde{u}_T$ exists and clearly descends to a bounded function on Y. It follows that $\tilde{\omega}_T$ descends to a positive current on Y, denoted by $\tilde{\omega}_T$ again for simplicity. We expect

Conjecture 3.7. — *The limit $\tilde{u}_T$ is continuous and $\tilde{\omega}_T$ is a smooth Kähler metric outside a subvariety B'_T of Y. If Y_1 denotes the metric completion of $Y \setminus B'_T$ with respect to the distance induced by $\tilde{\omega}_T$, then Y_1 is a projective variety and $(L_0 + aK_X)|_{Y\setminus B'_T}$ extends to an ample line bundle over Y_1.*

More generally, I believe that even if X is only a Kähler manifold (not necessarily projective) or ω_0 may be irrational, what we have shown and conjectured in the above still hold with slight modification. But it is harder to prove them.

4. Extending Kähler-Ricci flow across singular time

In this section, we discuss how to extend the Kähler-Ricci flow $\tilde{\omega}_t$ across the singular time T, assuming that we have solved the two conjectures proposed at the end of last section. Then we have a projective variety X_T, which can be either X_1 or Y_1 as above, and a limit $\tilde{\omega}_T$ on X_T which is smooth outside a subvariety B. A natural question is *how to continue the Kähler-Ricci flow on X_T starting at $\tilde{\omega}_T$*. There are two difficulties:

1. X_T may not be smooth;
2. Even if X_T is smooth, $\tilde{\omega}_T$ or its potential $\tilde{u}_T$ may not be smooth.

Hence, we need a local existence theorem for (1.1) when the underlying space may be singular or initial Kähler potential is non-smooth.

First we assume that X_T is smooth. We have shown that the limiting current $\tilde{\omega}_T$ has a bounded Kähler potential $\tilde{u}_T$. Then, it follows from the theory of complex Monge-Ampere equations that $\tilde{\omega}_T^k$, where $k = \dim_{\mathbb{C}} X_T$, is well-defined as a measure. So it makes sense to consider the Kähler-Ricci flow (2.1) with a weak initial value $\tilde{u}_T$. *Is there a smooth solution* $\varphi(t)$ *of* (2.1) *for* $t > 0$ *such that* $\lim_{t\to 0}\varphi(t) = \tilde{u}_T$? A partial answer to this question was provided in the following theorem.

Theorem 4.1. — [**6**] *Let* X *be a compact Kähler manifold and* ω_t *be a smooth family of Kähler metrics* ($t \in [0, t_0]$)*. Assume that* ψ_0 *is any bounded function satisfying: There are smooth functions* ψ_ϵ ($\epsilon > 0$) *such that*

(1) $\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_\epsilon > 0$;

(2) $\lim_{\epsilon\to 0}\psi_\epsilon = \psi_0$;

(3) *The volume form* $\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_0$ *is* $L^p(M,\omega)$ *for some* $p \geq 3$.

Then there is a unique smooth solution $\varphi(t)$ *of* (2.1), *and consequently, a solution* $\omega(t)$ *of* (1.1), *for* $t \in (0, t_0]$ *such that* $\lim_{t\to 0}\varphi(t) = \psi_0$ *and* $\omega(t)^n$ *converges to* $(\omega_0 + \sqrt{-1}\partial\overline{\partial}\psi_0)^n$ *strongly in the* L^2*-topology.*

If the Kodaira dimension of X is non-negative, then $L_0 + aK_X$ is nef and big on X_T and $\dim_{\mathbb{C}} X_T = n$. According to Conjecture 3.6, if X_T is smooth, then $\tilde{\omega}_T$ extends to be a Kähler class on X_T. Since $\frac{\partial \tilde{u}_t}{\partial t}$ is uniformly bounded from above for $t \in (0, T)$, we can show that the assumptions in the above theorem are satisfied. Then one can extend (1.1) across T and continue the flow on X_T until $T_2 > T$ when $[\omega_T] - (t-T)c_1(X_T)$ fails to be a Kähler class. If T_2 is finite, one can proceed as we did for $\tilde{\omega}_t$ at T.

However, in general, the resulting variety X_T from the surgery at T may not be smooth.[11] Nevertheless, we expect

Conjecture 4.2. — *The algebraic variety* X_T *given above has only mild singularity on which we can still run the Kähler-Ricci flow.*

There is an approach in [**21**] to this conjecture: One can try to run the Kähler-Ricci flow on a resolution $\tilde{X}_T$ of X_T with the initial value being the the pull-back of ω_T to $\tilde{X}_T$, which may be a degenerate Kähler metric vanishing along the exceptional divisor E.

Assuming that one can affirm the above three conjectures. When (1.1) runs into a finite-time singularity at T, one can apply the solutions to the above conjectures to

[11] It will be interesting to construct an explicit example of such a singular X_T, even though I believe it does exist.

extend (1.1) across T and evolve the Kähler metrics along the flow on X_T until we run into another finite-time singularity at $T_2 > T$. So we can get a solution $(X_t, \tilde{\omega}_t)$ with surgery for (1.1) for $t \in [0, T_2)$ satisfying:

(1) For $t \in [0, T)$, $X_t = X$ and $\tilde{\omega}_t$ is a standard solution of (1.1) with initial Kähler metric ω_0;

(2) For $t \in [T, T_2)$, $X_t = X_T$ and $\tilde{\omega}_t$ is a solution of (1.1) on X_T such that the potential $\tilde{u}_t$ of $\tilde{\omega}_t$ converges to the potential $\tilde{u}_T$ of $\tilde{\omega}_T$ in the L^∞-topology as t tends to T.

As usual, we call T a surgery time. One repeats the above process to continue the flow beyond T_2 and so on. Thus one can construct a global solution $(X_t, \tilde{\omega}_t)$ with surgery of (1.1)$(t \geq 0)$. We expect that this process ends after finitely many finite-time singularities, that is,

Conjecture 4.3. — *There are only finitely many surgery times $T_0 = 0 < T_1 < T_2 < \cdots < T_N < \infty$ such that $X_t = X_{T_i}$ and $\tilde{\omega}_t$ is a solution of* (1.1) *on X_{T_i} for $t \in [T_i, T_{i+1})$ $(i = 0, 1, \ldots, N-1)$ or $t \in [T_N, \infty)$. Furthermore, for $t \geq T_N$, either $X_t = \varnothing$ or K_{X_t} is nef and consequently,* (1.1) *has a global solution.*

There are two possibilities for $t > T_N$. In the first case, $X_t = \varnothing$, i.e., (1.1) becomes extinct at T_N. At each T_i $(i = 1, \ldots, N)$, we do surgery along some "rational" components along which $c_1(X)$ integrates positively. In particular, X_{T_i} is birational to X_t for $t < T_i$. Thus we have

Conjecture 4.4. — *The Kähler-Ricci flow* (1.1) *becomes extinct at finite time if and only if X is birational to a Fano manifold.*[12]

We will leave the second case to the next section. Note that X_{T_N} has nef canonical bundle if it is non-empty.

5. Asymptotic behavior of Kähler-Ricci flow

In last two sections, we have discussed results and speculations on singularity formation of the Kähler-Ricci flow at finite time. We also conjectured that there is always a global solution $(X_t, \tilde{\omega}_t)$ with surgery of (1.1) with only finitely many surgery times. This generalized solution with surgery becomes an usual solution $\tilde{\omega}_t$ of (1.1) on a variety with nef canonical bundle when t is sufficiently large. In this section, we study the asymptotic behavior of $\tilde{\omega}_t$ as t goes to ∞. For simplicity, we assume that X is a compact Kähler manifold with K_X nef. The general case can be dealt with in the same approach as we did for Conjecture 4.3 in case of possible singular varieties.

(12) To be safer, we may need to include some algebraic manifolds which are Fano-like if such manifolds ever exist.

It is known that (1.1) has a global solution $\tilde{\omega}_t$ for any given initial metric. Set $t = e^s - 1$ and $\tilde{\omega}(s) = e^{-s}\tilde{\omega}_t$, then $\tilde{\omega}(s)$ is a solution of the following normalized Kähler-Ricci flow:

$$\frac{\partial \tilde{\omega}(s)}{\partial s} = -\mathrm{Ric}(\tilde{\omega}(s)) - \tilde{\omega}(s), \qquad \tilde{\omega}(0) = \omega_0. \tag{5.1}$$

The advantage of doing this is that $[\tilde{\omega}(s)] = e^{-s}[\omega_0] - (1-e^{-s})c_1(X)$, which converges to $-c_1(X)$ as $s \to \infty$.

We also assume that there is a (1,1)-form $\psi \geq 0$ representing $-c_1(X)$. This is of course the case if K_X is semi-positive or equivalently, for m sufficiently large, $H^0(X, K_X^m)$ is free of base points. The Abundance conjecture in algebraic geometry claims that it is true for any X with K_X nef.

Since $H^0(X, K_X^m)$ is base-point free, any basis of it induces a holomorphic map $\phi : X \mapsto \mathbb{C}P^N$ for some $N > 0$ so that $\phi^*\mathcal{O}_{\mathbb{C}P^N}(1) = K_X^m$. The dimension of ϕ's image is just the Kodaira dimension $\kappa = \kappa(X)$ of X.

If $\kappa(X) = 0$, then $c_1(X) = 0$ and by the result in [**1**], the global solution $\tilde{\omega}_t$ of (1.1) converges to a Calabi-Yau metric on X.

If $\kappa(X) = \dim X = n$, then X is minimal and of general type. It follows from [**32**] and [**30**] that $\tilde{\omega}(s)$ converges to the unique (possibly singular along a subvariety) Kähler-Einstein metric with scalar curvature $-n$ on X as s tends to ∞.

The more tricky cases are for those X with $1 \leq \kappa(X) \leq n-1$. If X is such a manifold, one can not expect the existence of any Kähler-Einstein metrics (even with possibly singular along a subvariety) on X since $K_X^n = 0$. Hence, the first problem is to find what limiting metrics for $\tilde{\omega}(s)$ one supposes to have as s tends to ∞. To solve this problem, we introduced a class of new canonical metrics which we call generalized Kähler-Einstein metrics in [**20**] (13) and [**22**]. Let us briefly describe them.

Since we assume that K_X is semi-ample, the canonical ring

$$R(X) = \oplus_{m \geq 0} H^0(X, K_X^m)$$

is finitely generated, so there is a canonical model X_{can} of X (possibly singular). Let $\pi : X \mapsto X_{\mathrm{can}}$ be the natural map from X onto its canonical model X_{can}. Then generic fibers of π are Calabi-Yau manifolds of dimension $n - \kappa$, and consequently, there is a holomorphic map $f : X^0_{\mathrm{can}} \mapsto \mathcal{M}_{CY}$ which assigns $p \in X^0_{\mathrm{can}}$ to the fiber $\pi^{-1}(p)$ in the moduli $\mathcal{M}_{CY}$, where X^0_{can} consists of all p such that $\pi^{-1}(p)$ is smooth.

The moduli $\mathcal{M}_{CY}$ admits a canonical metric, the Weil-Petersson metric. Let us recall its definition. Let $\mathcal{X} \to \mathcal{M}_{CY}$ be a universal family of Calabi-Yau manifolds. Let $(U; t_1, \ldots, t_\ell)$ be a local holomorphic coordinate chart of $\mathcal{M}_{CY}$, where $\ell = \dim \mathcal{M}$.

(13) [**20**] is mainly for complex surfaces, but the part of introducing limiting metrics works for any dimensions.

Then each $\frac{\partial}{\partial t_i}$ corresponds to an element $\iota(\frac{\partial}{\partial t_i}) \in H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$ through the Kodaira-Spencer map ι. The Weil-Petersson metric is defined by the L^2-inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T_{\mathcal{X}_t})$. In the case of Calabi-Yau manifolds, as shown in [**24**], it has the following simple expression: Let Ψ be a nonzero holomorphic $(n-\kappa, 0)$-form on the fibre $\mathcal{X}_t$ and $\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})$ be the contraction of Ψ and $\frac{\partial}{\partial t_i}$. Then the Weil-Petersson metric is given by

$$(5.2) \qquad \left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j}\right)_{\omega_{WP}} = \frac{\int_{\mathcal{X}_t} \Psi \lrcorner \iota(\frac{\partial}{\partial t_i}) \wedge \overline{\Psi \lrcorner \iota(\frac{\partial}{\partial t_i})}}{\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi}}.$$

Now we can introduce the generalized Kähler-Einstein metrics.

Definition 5.1. — Let X, X_{can} etc. be as above. A closed positive $(1,1)$-current ω on X_{can} is called a generalized Kähler-Einstein metric if it satisfies the following.

1. $f^*\omega \in -c_1(X)$;
2. ω is smooth on X^0_{can}; (14)
3. $\mathrm{Ric}(\omega) = -\sqrt{-1}\,\partial\bar{\partial}\log \omega^\kappa$ lifts to a well-defined current on X and on X^0_{can}

$$(5.3) \qquad \mathrm{Ric}(\omega) = -\omega + f^*\omega_{WP}.$$

If $\kappa = n$, then it is just the equation for Kähler-Einstein metrics with negative scalar curvature.

Remark 5.2. — More generally, one can consider the generalized Kähler-Einstein equation:

$$\mathrm{Ric}(\omega) = -\lambda\,\omega + f^*\omega_{WP},$$

where λ is a constant.

In [**22**], the following theorem was proved.

Theorem 5.3. — *Let X be an n-dimensional projective manifold with semi-ample canonical bundle K_X. Suppose that $0 < \kappa(X) \leq n$. There exists a unique generalized Kähler-Einstein metric on X_{can}.*

To prove this theorem, we reduce (5.3) to a complex Monge-Ampere equation as in the proof of the Aubin-Yau theorem.

First we introduce a function which will appear in such a complex Monge-Ampere equation.

(14) One can establish an extra property: $(\pi^*\omega)^\kappa \wedge \Theta$ extends to a continuous function on X, where Θ is the (n-κ, n-κ)-form which restricts to polarized flat volume form on each smooth fiber (see [**22**], p15).

Since K_X is semi-ample, there is a semi-ample form $\pi^*\chi$ representing $-c_1(X)$, where χ is defined in the following way: X_{can} can be embedded into some projective space $\mathbb{C}P^N$ by using any basis of $H^0(X, K_X^m)$ for a sufficiently large m, then

$$\chi = \frac{1}{m}\omega_{FS}|_{X_{\mathrm{can}}}.$$

Let Ω be a volume form on X satisfying:

$$\sqrt{-1}\,\partial\bar\partial \log \Omega = \chi.$$

We push forward Ω to get a current $\pi_*\Omega$, where $\pi : X \to X_{\mathrm{can}}$ as above, as follows: For any continuous function ψ on X_{can}

$$\int_{X_{\mathrm{can}}} \psi\, \pi_*\Omega = \int_X (\pi^*\psi)\, \Omega.$$

It is easy to see that for any $x \in X^0_{\mathrm{can}}$, we have

$$\pi_*\Omega(x) = \int_{\pi^{-1}(x)} \Omega.$$

Definition 5.4. — We define a function F on X_{can} by

$$F\,\chi^\kappa = \pi_*\Omega. \tag{5.4}$$

There is another way of defining F: Choose any Kähler class β on X, by using the Hodge theory, one can find a flat relative volume form Θ on $X^0 = \pi^{-1}(X^0_{\mathrm{can}})$ in the cohomology class $\beta^{n-\kappa}$, this means a $(n-\kappa, n-\kappa)$-form Θ in $\beta^{n-\kappa}$ whose restriction to each fiber $\pi^{-1}(x)$ for $x \in X^0_{\mathrm{can}}$ is flat, that is,

$$\partial\bar\partial \log \Theta|_{\pi^{-1}(x)} = 0.$$

This is possible because $c_1(X)$ vanishes along each smooth fiber. One can show

$$c\,\pi^*F = \left(\frac{\Omega}{\Theta \wedge \pi^*\chi^\kappa}\right), \tag{5.5}$$

where c is a constant determined by

$$c\int_{\pi^{-1}(x)} \beta^{n-\kappa} = 1,$$

where x is any point in X^0_{can}. For simplicity, assume that $c = 1$. In particular, it follows that $\Theta \wedge \pi^*\chi^\kappa$ can be extended to X as a current. Furthermore, one can show (see [**24**])

$$f^*\omega_{WP} = \sqrt{-1}\partial\bar\partial \log(\Theta \wedge \chi^\kappa) - \sqrt{-1}\partial\bar\partial \log \chi^\kappa.$$

The function F may not extend smoothly to X_{can}, but we have some controls on it along the subvariety $X_{\mathrm{can}} \setminus X^0_{\mathrm{can}}$.

Lemma 5.5. — *F is smooth on X^0_{can} and is in $L^{1+\epsilon}(X_{\mathrm{can}})$ for some $\epsilon > 0$, where the L^p-norm is defined by using the metric corresponding to χ.*

To prove it, we notice

$$\int_{X_{\mathrm{can}}} F^{1+\epsilon}\chi^\kappa = \int_X \pi^*F^{1+\epsilon}\pi^*\chi^\kappa \wedge \Theta = \int_X \pi^*F^\epsilon\Omega.$$

Furthermore, one can show that if $\iota : Y \to X_{\mathrm{can}}$ is any resolution of X_{can}, then ι^*F has at worst pole singularities on Y. The proof is a bit technical and we refer the readers to [**22**] for details. Consequently, π^*F^ϵ is integrable for sufficiently small $\epsilon > 0$ (see [**22**], Proposition 3.2).

Consider

$$(\chi + \sqrt{-1}\partial\bar\partial\varphi)^\kappa = Fe^\varphi\chi^\kappa. \tag{5.6}$$

If φ is a bounded solution for (5.6), then $\omega = \chi + \sqrt{-1}\partial\bar\partial\varphi$ is a generalized Kähler-Einstein metric. To see this, we first observe that $[\pi^*\omega] = [\pi^*\chi] = -c_1(X)$. Next we observe

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\,\partial\bar\partial\log\omega^\kappa = -\sqrt{-1}\partial\bar\partial\log\chi^\kappa - \sqrt{-1}\partial\bar\partial\log F - \sqrt{-1}\partial\bar\partial\varphi$$

is a well-defined current on X_{can}. A direct computation shows

$$\begin{aligned}
& \sqrt{-1}\partial\bar\partial\log\chi^\kappa + \sqrt{-1}\partial\bar\partial\log F + \sqrt{-1}\partial\bar\partial\varphi \\
= \;& \sqrt{-1}\partial\bar\partial\log\chi^\kappa + \sqrt{-1}\partial\bar\partial\log\left(\frac{\Omega}{\Theta\wedge\chi^\kappa}\right) + \omega - \chi \\
= \;& \omega + \sqrt{-1}\left(-\partial\bar\partial\log(\Theta\wedge\chi^\kappa) + \partial\bar\partial\log\chi^\kappa\right) \\
= \;& \omega - f^*\omega_{WP}.
\end{aligned}$$

Therefore

$$\mathrm{Ric}(\omega) = -\omega + f^*\omega_{WP}.$$

Thus, in order to prove Theorem 5.3, we only need to prove the following

Theorem 5.6. — *There exists a unique solution $\varphi \in C^0(X_{\mathrm{can}}) \cap C^\infty(X^0_{\mathrm{can}})$ for (5.6) with $\chi + \sqrt{-1}\partial\bar\partial\varphi \geq 0$.*

This is proved by using the continuity method and establishing an a priori C^3-estimate for solutions of (5.6). We refer the readers to [**22**] for its proof.

We would like to point out that $\pi^*\omega^\kappa \wedge \Theta = \Omega\, e^{f^*\varphi}$ is continuous since both $\pi^*\varphi$ and Ω are continuous on X.

Now we can discuss the limit of $\tilde\omega(s)$ in (5.1) as s tends to ∞. The following theorem was proved in [**22**] (also see [**20**] for complex surfaces).

Theorem 5.7. — *Let X be a projective manifold with semi-ample canonical bundle K_X. So X admits an algebraic fibration $\pi : X \to X_{\mathrm{can}}$ over its canonical model X_{can}. Suppose $0 < \dim X_{\mathrm{can}} = \kappa < \dim X = n$. Then for any initial Kähler metric ω_0, the solution $\tilde\omega(s)$ for (5.1) converges to $\pi^*\omega_{\mathrm{can}}$ as currents, where ω_{can} is the* *unique generalized Kähler-Einstein metric on X_{can}. Moreover, for any compact subset $K \subset X^0_{\mathrm{can}}$, there is a constant C_K such that*

$$(5.7) \qquad ||R(\tilde{\omega}(s))||_{L^\infty(\pi^{-1}(K))} + e^{(n-\kappa)s} \sup_{x\in K} ||\tilde{\omega}(s)^{n-\kappa}|_{\pi^{-1}(x)}||_{L^\infty(\pi^{-1}(x))} \leq C_K,$$

where $R(\tilde{\omega}(s))$ denotes the scalar curvature of $\tilde{\omega}(s)$.

If $n = 2$, then the above implies the convergence in the $C^{1,\alpha}$-topology for any $\alpha \in (0,1)$ on any compact subset in X^0_{can}. We believe that *the same can be proved in any dimensions.* Moreover, we also expect

Conjecture 5.8. — *The solution $\tilde{\omega}(s)$ converges to the unique limit $\pi^*\omega_{GKE}$ in the Gromov-Hausdorff topology and the convergence is in the smooth topology in $\pi^{-1}(X^0_{\mathrm{can}})$.*

This is even open for complex surfaces.

In the above, we assume that X has semi-ample K_X. This is indeed true if the Abundance conjecture holds. If K_X is nef, (5.1) still has a global solution $\tilde{\omega}(s)$. Clearly, it will be extremely interesting to study the asymptotic behavior of $\tilde{\omega}(s)$ without assuming the Abundance Conjecture, namely, give a differential geometric proof of the convergence of $\tilde{\omega}(s)$. The success of such a direct approach will yield many deep applications to studying the structures of Kähler manifolds.

To solve the above conjecture or succeed in the above direct approach, we may need to develop a theory of compactness for Kähler metrics with bounded scalar curvature. For Kähler surfaces, a compactness theorem of this sort was proved in [**29**]. Also note that the scalar curvature is uniformly bounded along (5.1) on any compact projective manifold with big and nef canonical bundle (see [**36**]).

6. The case of algebraic surfaces

In this section, we will carry out the program described above for complex surfaces. Basically, all the results in this section are taken from [**30**] (for surfaces of general type) and [**20**] (for elliptic surfaces). We just make a few simple observations in order to deduce the program from those previous works.

Let X be a compact algebraic surface.

As before, let $\tilde{\omega}_t$ be a maximal solution of (1.1) on $X \times [0,T]$. If $T < \infty$, then $[\omega_0] - Tc_1(X)$ is nef. There are three possibilities:

1. If $[\omega_0] - Tc_1(X) = 0$, then X is a Del-Pezzo surface and $\tilde{\omega}(s) = (1 - \frac{t}{T})^{-1}\tilde{\omega}_t$, where $s = -T\log(1 - \frac{t}{T})$, converges to a Kähler-Ricci soliton as $s \to \infty$ or equivalently, $t \to T$ (cf. [**26**], [**31**], [**33**]).

2. If $[\omega_0]-Tc_1(X) \neq 0$ but $([\omega_0]-Tc_1(X))^2 = 0$, then there is a fibration $\pi : X \mapsto \Sigma$ with rational curves as fibers (possibly with finitely many singular fibers) such that $[\omega_0] - Tc_1(X) = \pi^*[\omega_\Sigma]$ for some Kähler metric ω_Σ on Σ. It follows that as $t \to T$, $\tilde{\omega}_t$ converges to a positive current of the form $\pi^*(\omega_\Sigma + \sqrt{-1}\partial\bar{\partial}u_T)$ for some bounded function u_T on Σ. To extend (1.1) across T, one needs to solve (2.1) on Σ with u_T as the initial value. This is the same as solving the following for $t \geq T$,
$$\frac{\partial u}{\partial t} = \log\left(\frac{\omega_\Sigma - (t-T)\psi_\Sigma + \sqrt{-1}\partial\bar{\partial}u}{\Omega_\Sigma}\right), \quad u(T,\cdot) = u_T, \tag{6.1}$$
where Ω_Σ is a volume form on Σ with $\mathrm{Ric}(\Omega_\Sigma) = \psi_\Sigma$. One can solve this flow by using the standard potential theory in complex dimension 1. Let $\tilde{\omega}_t$ be the resulting maximal solution of (6.1) ($t \geq T$). If the genus $g(\Sigma)$ of Σ is zero, then $\tilde{\omega}_t$ becomes extinct at some finite time $T_2 > T$ or after appropriate scaling, these metrics converge to the standard round metric on $\Sigma = S^2$ as $t \to T_2$. Hence, it verifies Conjecture 4.4 in case of algebraic surfaces. If $g(\Sigma) = 1$, then $\tilde{\omega}_t$ exists for all $t \geq T$ and converges to a flat metric as $t \to \infty$. If $g(\Sigma) > 1$, then $\tilde{\omega}_t$ exists for all $t \geq T$ and after scaling, converges to a hyperbolic metric as $t \to \infty$.
3. If $([\omega_0] - Tc_1(X))^2 > 0$, then $[\omega_0] - Tc_1(X)$ is semi-ample, so it can vanish only along a divisor. It is easy to see that for each irreducible component D of this divisor, $K_X \cdot D < 0$. Moreover, $D^2 < 0$. By the Adjunction Formula, D is a rational curve of self-intersection -1, so the divisor is made of finite disjoint (-1) rational curves and consequently, we can blow down them to get a new algebraic surface X_T. Moreover, the limit $\tilde{\omega}_T$ descends to a positive current with continuous potential and well-defined bounded volume form. By Theorem 4.1, one can extend (1.1) across T.

Notice that the extension $\tilde{\omega}_t$ for $t > T$ is smooth. Either K_{X_T} is nef and there is a global solution on X_T,or $\tilde{\omega}_t$ develops finite-time singularity at some $T_2 > T$. In the later case, one can repeat the above steps 1, 2 and 3. Since $H_2(X, \mathbb{Z})$ is finite, after finitely many surgeries, we will arrive at a minimal algebraic surface X_N, that is, K_{X_N} is nef. Then (1.1) has a global solution, denoted again by $\tilde{\omega}_t$, on X_N. Let us study its asymptotic behavior.

There are 3 possibilities according to the Kodaira dimension $\kappa(X)$ of X:

1. If $\kappa(X) = 0$, then $c_1(X)_{\mathbb{R}} = 0$ or a finite cover of X is either a K3 surface or an Abelian surface. In this case, the solution $\tilde{\omega}_t$ on X_N converges to a Ricci flat Kähler metric.

   In other two cases, we better use the normalized Kähler-Ricci flow (5.1) on X_N:
$$\frac{\partial\tilde{\omega}(s)}{\partial s} = -\mathrm{Ric}(\tilde{\omega}(s)) - \tilde{\omega}(s), \quad \tilde{\omega}(0) = \omega_0,$$

where $t = e^s - 1$ and $\tilde{\omega}(s) = e^{-s}\tilde{\omega}_t$.

2. If $\kappa(X) = 1$, then X_N is a minimal elliptic surface: $\pi : X_N \mapsto \Sigma$. It was proved in [**20**] that as $s \to \infty$, $\tilde{\omega}(s)$ converges to a positive current of the form $\pi^*(\tilde{\omega}_\infty)$ and the convergence is in the $C^{1,1}$-topology on any compact subset outside singular fibers $F_{p_1}, \ldots, F_{p_k}$, where $p_1, \ldots, p_k \in \Sigma$. Furthermore, $\tilde{\omega}_\infty$ satisfies the generalized Kähler-Einstein equation:
$$\mathrm{Ric}(\tilde{\omega}_\infty) = -\tilde{\omega}_\infty + f^*\omega_{WP}, \qquad \text{on } \Sigma \setminus \{p_1, \ldots .p_k\},$$
where f is the induced holomorphic map from $\Sigma \setminus \{p_1, \ldots .p_k\}$ into the moduli of elliptic curves.
3. If $\kappa(X) = 2$, then X_N is a surface of general type and its canonical model X_{can} is a Kähler orbifold with possibly finitely many rational double points and ample canonical bundle. By the version of the Aubin-Yau Theorem for orbifolds, there is an unique Kähler-Einstein metric $\tilde{\omega}_\infty$ on X_{can} with scalar curvature -2. It was proved in [**30**] that as $s \to \infty$, $\tilde{\omega}(s)$ converges to $\tilde{\omega}_\infty$ and converges in the C^∞-topology outside those rational curves over the rational double points.

This verifies that our program indeed works for algebraic surfaces except that we did not check if the blown-down surfaces coincide with the metric completions described in Conjecture 3.6.

Furthermore, it should be possible to extend all the above discussions to compact Kähler surfaces which may not be projective.

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