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Astérisque, tome 322 (2008), p. 39-69

<http://www.numdam.org/item?id=AST_2008__322__39_0>

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322, 2008, p. 39–69

A SURVEY OF THE HYPOELLIPTIC LAPLACIAN

by

Jean-Michel Bismut

À Jean Pierre Bourguignon pour son soixantième anniversaire

Abstract. — The purpose of this paper is to review the construction of the hypoelliptic Laplacian, in the context of de Rham theory for smooth manifolds, and also the construction of the hypoelliptic Dirac operator in the context of complex Kähler manifolds.

***Résumé* (Compte-rendu sur le laplacien hypoelliptique).** — Le but de cet article est d'établir un compte-rendu de la construction du laplacien hypoelliptique dans le contexte de la théorie de de Rham des variétés lisses, ainsi que de la construction de l'opérateur de Dirac hypoelliptique dans le contexte des variétés kähleriennes complexes.

Introduction

The purpose of this survey is to review certain aspects of the construction of the hypoelliptic Laplacian, in de Rham and in Dolbeault theory. The hypoelliptic Laplacian was introduced in [**3**] in de Rham theory, and in [**5**] for Dirac operators. The crucial analytic foundations for the theory were developed by Lebeau and ourselves in [**8**].

One motivation given in [**3**] is to interpret the hypoelliptic Laplacian in de Rham theory as a semiclassical limit of the Witten deformation of the Hodge theory of the loop space of a Riemannian manifold, which is associated with the energy functional. This point of view remains formal, since the Hodge theory of the loop space of a manifold is not analytically well defined. The motivation for the construction of the hypoelliptic Dirac operator of [**5**] is to understand the effect of replacing the standard

2000 Mathematics Subject Classification. — 35H10, 58A14, 58J20.
Key words and phrases. — Hypoelliptic equations, Hodge theory, Index theory and related fixed point theorems.

The author is indebted to a referee for reading this paper very carefully.

L_2 metric on the loop space of a manifold by a H^1 metric. Again these considerations remain formal, although ultimately the hypoelliptic Dirac operator is well defined.

Whatever the motivations, and there are many others, some of which are explained in [**4, 6**], the conclusion is that a geometric Laplacian can be deformed into a family of hypoelliptic second order differential operators acting on the total space of the tangent or the cotangent bundle of the given manifold, which interpolates in the proper sense between the Laplacian and the generator of the geodesic flow. The existence of this deformation is counter-intuitive, since ellipticity is a stable property. However, the fact that the hypoelliptic Laplacian acts on a bigger space than the original elliptic Laplacian explains why ultimately it can be made to 'collapse' on the elliptic Laplacian.

Let us finally mention that up to lower order terms, the hypoelliptic Laplacian is the sum of a harmonic oscillator acting in the directions of the fibre, and of the vector field which generates the geodesic flow, these two operators being adequately scaled.

In this paper, first, we fully develop the theory in the case where the base manifold is the circle. The main point is that while in this case, the geometry is trivial, a complete understanding of the hypoelliptic Laplacian and of the interpolation property can be easily obtained via Fourier analysis on the circle and the spectral theory of the harmonic oscillator. The case of the circle is also useful, because the objects which appear there turn out to be at the same time the principal symbols of the geometric hypoelliptic operators, and because the circle is the model of a closed geodesic. The fact that the hypoelliptic Laplacian is self-adjoint with respect to a symmetric form of signature (∞, ∞) appears also naturally in that context.

The basic difference between the case of the circle and the geometric case is that the analysis of the hypoelliptic Laplacian is no longer explicit, and also that the convergence arguments, which are easy for the circle, are built on a functional analytic machinery described in detail in our work with Lebeau [**8**].

Also we describe the construction of the hypoelliptic Laplacian, in the de Rham case, and also for Kähler manifolds. We emphasize the role of the symmetric bilinear forms, at least in the de Rham case, because of the important spectral theoretic consequences which are derived in [**8**].

This paper is organized as follows. In section 1, we consider the case of the circle. Since the hypoelliptic Laplacian is ultimately obtained as a Hodge Laplacian with respect to an exotic bilinear form on the de Rham or the Dolbeault complex, this point of view is systematically emphasized in this simple case too.

In section 2, we recall classical results on the Hodge theory of a compact manifold, and on the Witten deformation of classical Hodge theory which is associated with a smooth function. Also we show that if (M, ω) is a symplectic manifold, there is a symplectic Witten Laplacian, which turns out to be the Lie derivative operator associated with the corresponding Hamiltonian vector field. This point of view is further developed in [**3**], where the hypoelliptic Laplacian in de Rham theory is obtained by linearly interpolating between the Riemannian metric of the base manifold, and the symplectic form of its cotangent bundle.

In section 3, we explain the construction of the hypoelliptic Laplacian in de Rham theory. We also give the main arguments of [**3**] in favour of the fact that the hypoelliptic Laplacian interpolates between the Hodge Laplacian and the geodesic flow.

In section 4, we give the construction of the hypoelliptic Dirac operator of [**5**] in the context of Kähler manifolds, and we give the arguments showing that this operator should indeed be a deformation of the classical elliptic Dirac operator.

As we already said, the analytic justifications which make that the whole construction ultimately exists as a mathematical theory are developed in detail in our work with Lebeau [**8**]. Also applications to Ray-Singer torsion [**19**] and Quillen metrics [**17**] are given in [**8**] and [**5**].

1. The case of the circle

The purpose of this section is to construct the hypoelliptic Laplacian in the case where the base manifold X is just S^1. In this case, all the objects are simple and natural. Besides, the operators which are obtained in this case can be viewed as the symbols of the operators which are obtained later in the geometric case.

This section is organized as follows. In subsection 1.1, we recall elementary properties of elliptic and hypoelliptic operators.

In subsection 1.2, we introduce the Kolmogorov operator on $S^1 \times \mathbf{R}$, which is a simple case of an operator verifying Hörmander's hypoellipticity theorem [**14**], and at the same time, coincides, up to important lower order terms, with the hypoelliptic Laplacian. Formal conjugation arguments are used to relate the hypoelliptic Laplacian to the elliptic Laplacian on S^1. The fact that the hypoelliptic Laplacian interpolates in the proper sense between the Laplacian and the generator of the geodesic flow can be exhibited by hand. One obtains this way a proof of Poisson's formula by interpolation.

In subsection 1.3, we show that our hypoelliptic Laplacian is a Hodge Laplacian with respect to an exotic bilinear form on the space of compactly supported differential forms on $S^1 \times \mathbf{R}$. This result will be used in section 3 to construct the geometric hypoelliptic Laplacian in the context of de Rham theory.

1.1. Elliptic and hypoelliptic operators. — Let X be a compact manifold. Let $\mathcal{X}^*$ be the total space of T^*X. Then X embeds in $\mathcal{X}^*$ as the zero section of T^*X.

Let E and F be two complex vector bundles on X. If P is a pseudodifferential operator of order m mapping $C^{\infty}(X,E)$ into $C^{\infty}(X,F)$, its principal symbol $\sigma_P(x,\xi)$ is a smooth map on $\mathcal{X}^* \setminus X$ with values in $\mathrm{Hom}(E,F)$, which is homogeneous of order m in the variable ξ. The operator P is said to be elliptic if $\sigma_P(x,\xi)$ is invertible on $\mathcal{X}^* \setminus X$.

If X is equipped with a Riemannian metric, if Δ^X is the Laplace-Beltrami operator acting on $C^{\infty}(X,\mathbf{R})$, then $-\Delta^X$ is an elliptic operator of order 2, and its principal symbol is $|\xi|^2$. The standard example is the operator $-\frac{\partial^2}{\partial x^2}$ acting on S^1.

Ellipticity is a stable property. Indeed a small deformation of an elliptic operator is still elliptic. This should make all the more surprising the fact that certain elliptic operators can be deformed into hypoelliptic operators. This is only possible because the deformed operators act on a different space than the original operator. Besides elliptic operators of order m act on Sobolev spaces, and decrease the Sobolev index by m. As an example, the operator $-\Delta^X$ decreases the Sobolev index by 2, and any pseudoinverse of $-\Delta^X$ (an inverse up to regularizing operators) increases the Sobolev index by 2. In particular if u is a scalar distribution on X such that $-\Delta^X u \in H^s$, then $u \in H^{s+2}$.

Hypoellipticity is a weaker property. A pseudodifferential operator P is said to be hypoelliptic if when u is a distribution such that Pu is C^{∞} on some open set, then u is also C^{∞} on this open set. For example the parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta^X$ on $\mathbf{R} \times X$ is hypoelliptic.

1.2. The Kolmogorov operator and Hörmander's theorem. — Consider the operator A on $\mathbf{R} \times \mathbf{R}^2$ introduced by Kolmogorov [**15**],

$$A = \frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial x}. \tag{1.1}$$

In [**15**], Kolmogorov computed the fundamental solution of (1.1), as a time dependent Gaussian kernel in the variables (x,y), from which the hypoellipticity of A follows.

The hypoellipticity of A prompted Hörmander [**14**] to develop his theory of hypoelliptic second order differential operators which we now briefly describe. Indeed if $X_0, \dots, X_m$ are smooth vector fields on $\mathbf{R}^n$, consider the differential operator

$$M = -\frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0. \tag{1.2}$$

Let $\mathcal{E}(x) \subset \mathbf{R}^n$ be the vector space spanned at x by $X_0, \dots, X_m$ and their Lie brackets. Hörmander's theorem asserts that if at each x, $\mathcal{E}(x) = \mathbf{R}^n$, then M is a hypoelliptic operator.

The fact that A is hypoelliptic is a consequence of Hörmander's theorem since the Lie bracket $\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial t} - y\frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial x}$ is enough to make the Hörmander distribution associated with $\frac{\partial}{\partial y}, \frac{\partial}{\partial t} - y\frac{\partial}{\partial x}$ span $\mathbf{R}^3$.

More generally, consider the operator A_n on $\mathbf{R}^{2n+1}$ which is given by

$$A_n = \frac{\partial}{\partial t} - \frac{1}{2}\Delta^V - \nabla_y. \tag{1.3}$$

In (1.3), Δ^V denotes the Laplacian in the variables $y_1, \ldots, y_n$, and ∇_y denotes differentiation on the variables $x^1, \ldots, x^n$ in the direction y, i.e., $\nabla_y = \sum_1^n y^i \frac{\partial}{\partial x^i}$. In this case, the n Lie brackets $\left[\frac{\partial}{\partial y^i}, \nabla_y\right] = \frac{\partial}{\partial x^i}$ are necessary to make the Hörmander distribution span $\mathbf{R}^{2n+1}$.

The parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}$ is the model of the geometric parabolic operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta^X$. Let us now describe the model of its hypoelliptic deformation.

Let L be the operator on $\mathbf{R}^3$,

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - y\frac{\partial}{\partial x}. \tag{1.4}$$

Clearly,

$$L = A + \frac{1}{2}\left(y^2 - 1\right). \tag{1.5}$$

The term which is added to A in the right-hand side of (1.5) has no effect on hypoellipticity, which is by definition a local property. On the other hand, the operator H given by

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) \tag{1.6}$$

is the harmonic oscillator, which has discrete spectrum and compact resolvent. From this point of view, the operator L is significantly different from the operator A in (1.1).

As in (1.3), we may as well define the operator L_n on $\mathbf{R}^{2n+1}$, which is given by

$$L_n = \frac{\partial}{\partial t} + \frac{1}{2}\left(-\Delta + |y|^2 - n\right) - \nabla_y. \tag{1.7}$$

To make the notation simpler, we now proceed with the case $n = 1$. Also we disregard for the moment the variable t, which can be included in everything which follows. For $b > 0$, set

$$\mathcal{L}_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) - \frac{1}{b}y\frac{\partial}{\partial x}. \tag{1.8}$$

Clearly,

$$\mathcal{L}_b = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + \left(y - b\frac{\partial}{\partial x}\right)^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.9}$$

In the sequel, it will be convenient to assume that $x \in S^1 = \mathbf{R}/\mathbf{Z}$, and that $y \in \mathbf{R}$ lies in TS^1 or T^*S^1.

Let us formally make the translation $y \to y + b\frac{\partial}{\partial x}$. Equivalently let U_b be the formal operator,

$$U_b = \exp\left(b\frac{\partial^2}{\partial x \partial y}\right). \tag{1.10}$$

Set

$$\mathcal{M}_b = U_b \mathcal{L}_b U_b^{-1}. \tag{1.11}$$

Then $\mathcal{M}_b$ is given by the operator,

$$\mathcal{M}_b = \frac{1}{2b^2}\left(-\Delta^V + y^2 - 1\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.12}$$

We can write the operator $\mathcal{M}_b$ in the form,

$$\mathcal{M}_b = \frac{H}{b^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.13}$$

Before we proceed, let us observe that conjugation by U_b has transformed the hypoelliptic operator $\mathcal{L}_b$ into the elliptic operator $\mathcal{M}_b$, in which the variables x, y have been uncoupled.

Since the spectrum of H is equal to $\mathbf{N}$, the spectrum of $\mathcal{M}_b$ is given by

$$\mathrm{Sp}\left(\mathcal{M}_b\right) = \frac{\mathbf{N}}{b^2} + \left\{2\pi^2 k^2, k \in \mathbf{Z}\right\}. \tag{1.14}$$

Therefore when $b \to 0$, the finite part of the spectrum of $\mathcal{M}_b$ converges to the spectrum of $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$, and as $b \to +\infty$, $\mathrm{Sp}\left(\mathcal{M}_b\right)$ while staying real, accumulates near 0. Also 0 is a simple eigenvalue of $\mathcal{M}_b$.

Before we explain how the spectrum of $\mathcal{M}_b$ relates to the spectrum of $\mathcal{L}_b$, let us first explain how to eliminate the nonzero eigenvalues of H. Let $\Lambda^{\cdot}\left(\mathbf{R}^*\right)$ be the exterior algebra of $\mathbf{R}$, which is spanned by $1, dy$. Let N be the number operator on $\Lambda^{\cdot}\left(\mathbf{R}^*\right)$, which acts like 0 on 0-forms, and 1 on 1-forms. Set

$$\mathcal{O} = H + N. \tag{1.15}$$

Let $\mathrm{Tr_s}$ be our notation for the supertrace. Indeed let $V = V_+ \oplus V_-$ be a $\mathbf{Z}_2$-graded Hilbert space, and let $\tau = \pm 1$ be the endomorphism of V which defines the $\mathbf{Z}_2$-grading. If $A \in \mathcal{L}(V)$ is trace class, then

$$\mathrm{Tr_s}\left[A\right] = \mathrm{Tr}\left[\tau A\right]. \tag{1.16}$$

Here we use the $\mathbf{Z}_2$-grading associated with the grading of $\Lambda^{\cdot}\left(\mathbf{R}^*\right)$. Then one has the easy identity,

$$\mathrm{Tr_s}\left[\exp\left(-t\mathcal{O}\right)\right] = 1. \tag{1.17}$$

Put

$$\mathcal{M}_b' = \mathcal{M}_b + \frac{N}{b^2}. \tag{1.18}$$

Of course (1.14) remains valid for $\mathcal{M}_b'$, and 0 is still a simple eigenvalue of $\mathcal{M}_b'$. From (1.17), (1.18), we get

$$\mathrm{Tr_s}\left[\exp\left(-t\mathcal{M}_b'\right)\right] = \mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]. \tag{1.19}$$

The remarkable fact in (1.19) is that it does not depend on $b > 0$. We already saw that as $b \to 0$, the spectrum of $\mathcal{M}_b'$ converges to the spectrum of $-\frac{1}{2}\frac{\partial^2}{\partial x^2}$. The question is now to know how to use (1.19) with $b \to +\infty$.

Using hypoellipticity, it is not difficult to show that $\mathcal{L}_b$ has a smooth heat kernel, and that for $t > 0$, $\exp(-t\mathcal{L}_b)$ is trace class.

We claim that

$$\mathrm{Tr}\left[\exp\left(-t\mathcal{L}_b\right)\right] = \mathrm{Tr}\left[\exp\left(-t\mathcal{M}_b\right)\right]. \tag{1.20}$$

One could try using the conjugation by the operator U_b which was described above to get (1.20). However, the operator U_b is poorly defined, and does not act on any natural function space.

However, we can use Fourier series to diagonalize the operator $\frac{\partial}{\partial x}$, and try obtaining an analogue of (1.20) for each eigenvalue $2i\pi k, k \in \mathbf{Z}$, from which (1.20) would follow by summation. This can indeed be done. In fact the eigenvectors of the harmonic oscillator H are given by $P_n(y)\exp\left(-y^2/2\right), n \in \mathbf{N}$, where the P_n are the Hermite polynomials. Now the complex translations $y \to y + 2i\pi bk, k \in \mathbf{Z}$ maps these eigenvectors into well defined elements of L_2. It is not difficult to conclude that the consequences of the above conjugation by U_b are correct, and that (1.20) holds.

Set

$$\mathcal{L}_b' = \mathcal{L}_b + \frac{N}{b^2}. \tag{1.21}$$

By (1.8) and (1.21), we get

$$\mathcal{L}_b' = \frac{1}{2b^2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) + \frac{N}{b^2} - \frac{1}{b}y\frac{\partial}{\partial x}. \tag{1.22}$$

Using (1.18)-(1.21), we obtain,

$$\mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b'\right)\right] = \mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]. \tag{1.23}$$

Instead of (1.23), one can replace (1.23) by a pointwise equality in the x variable of the integral of the corresponding kernels in the y variable, simply by using the Fourier series argument we just gave. However, this will not be used in the sequel.

Now we will make $b \to +\infty$ in equation (1.23). For $b > 0$, let K_b be the map

$$K_b s(x,y) = s(x, by). \tag{1.24}$$

Set

$$\mathfrak{L}_b = K_b \mathcal{L}'_b K_b^{-1}. \tag{1.25}$$

By (1.22), we get

$$\mathfrak{L}_b = \frac{y^2}{2} - y\frac{\partial}{\partial x} - \frac{1}{2b^4}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2}+N\right). \tag{1.26}$$

By (1.26), we find that as $b \to +\infty$,

$$\mathfrak{L}_b = \frac{y^2}{2} - y\frac{\partial}{\partial x} + \mathcal{O}\left(1/b^2\right). \tag{1.27}$$

Equation (1.27) indicates that up to the translation by $\frac{y^2}{2}$, the leading term in the asymptotics of $\mathfrak{L}_b$ is the generator of the geodesic flow.

We briefly show how the above can be used to give a proof of the Poisson formula. Indeed (1.26), (1.27) already indicates that $\mathrm{Tr_s}\left[\exp\left(-t\mathfrak{L}_b\right)\right]$ concentrates along the closed geodesics in S^1 parametrized by $[0,t]$, which start and end at x and have speed y. This means that $y = k/t, k \in \mathbf{Z}$. Let $R_{k,b}$ be the map

$$R_{k,b} s(x,y) = s\left(x, k/t + y/b^2\right). \tag{1.28}$$

Set

$$\mathfrak{L}'_{k,b} = R_{k,b}\mathfrak{L}_b R_{k,b}^{-1}. \tag{1.29}$$

By (1.26), we get

$$\mathfrak{L}'_{k,b} = \frac{1}{2}\left(k/t + y/b^2\right)^2 - \left(k/t + y/b^2\right)\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2}+N\right). \tag{1.30}$$

Now observe that the term $k/t\frac{\partial}{\partial x}$ can be disregarded, because, once it is multiplied by t, it exponentiates to the identity. We still use the notation $\mathfrak{L}'_{k,b}$ for the operator in which this term has been deleted. Let S_b be the map $s(x,y) \to s\left(b^2 x, y\right)$. Note that this map is only defined for $x \in \mathbf{R}$. Put

$$\mathfrak{L}''_{k,b} = S_b^{-1}\mathfrak{L}'_{k,b} S_b. \tag{1.31}$$

By (1.30), we obtain,

$$\mathfrak{L}''_{k,b} = \frac{1}{2}\left(k/t + y/b^2\right)^2 - y\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{b^2}\left(-\frac{1}{2}+N\right). \tag{1.32}$$

The effect of the above change of variables is that for every $k \in \mathbf{Z}$, we should evaluate the asymptotics as $b \to +\infty$ of $I_{k,b,t}$ given by

$$I_{k,b,t} = b^2 \int_{\mathbf{R}} \mathrm{Tr_s}\left[\exp\left(-t\mathfrak{L}''_{k,b}\right)\left((0,y),(0,y)\right)\right] dy. \tag{1.33}$$

In (1.33), $\exp\left(-t\mathfrak{L}''_{k,b}\right)((x,y),(x',y'))$ denotes the smooth kernel on $\mathbf{R}^2$ which is associated with the operator $\exp\left(-t\mathfrak{L}''_{k,b}\right)$. As to the factor b^2, it appears because of S_b.

Clearly,

$$b^2\mathrm{Tr_s}\left[\exp\left(-tN/b^2\right)\right] = b^2\left(1-e^{-t/b^2}\right), \tag{1.34}$$

so that as $b \to +\infty$,

$$b^2\mathrm{Tr_s}\left[\exp\left(-tN/b^2\right)\right] \to t. \tag{1.35}$$

Put

$$\mathfrak{N} = -y\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial y^2}. \tag{1.36}$$

By (1.33)-(1.36), we find that as $b \to +\infty$,

$$I_{k,b,t} \to I_{k,+\infty,t} = t\exp\left(-k^2/2t\right)\int_{\mathbf{R}}\mathrm{Tr}\left[\exp\left(-t\mathfrak{N}\right)\left(\left(0,y\right),\left(0,y\right)\right)\right]dy. \tag{1.37}$$

Now one verifies easily that

$$\int_{\mathbf{R}}\mathrm{Tr}\left[\exp\left(-t\mathfrak{N}\right)\left(\left(0,y\right),\left(0,y\right)\right)\right]dy = \frac{t^{-3/2}}{\sqrt{2\pi}}. \tag{1.38}$$

By (1.37), (1.38), we obtain,

$$I_{k,+\infty,t} = \frac{\exp\left(-k^2/2t\right)}{\sqrt{2\pi t}}, \tag{1.39}$$

which is exactly the contribution of $k \in \mathbf{Z}$ to $\mathrm{Tr}\left[\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)\right]$.

The same sort of argument can also be used to evaluate the full heat kernel for $\exp\left(\frac{t}{2}\frac{\partial^2}{\partial x^2}\right)$ on S^1.

The operator $\mathcal{L}'_b$ is the prototype of a hypoelliptic Laplacian. We have shown by elementary arguments how and in what sense it interpolates between the standard Laplacian and the generator of the geodesic flow. The remarkable fact is that the full spectrum of the Laplacian can be recovered from the spectrum of its hypoelliptic deformation, and the heat kernel on S^1 can also be obtained by this procedure.

Later, we will describe the deformation of the Laplacian of a manifold to a hypoelliptic Laplacian, that is in a geometric context. However, when taking the obvious n-dimensional extension of what we just did, the above exactly describes the deformation of the associated principal symbols. Needless to say, the proper geometric context cannot be described just via the principal symbol, the full symbol is obviously needed. This ultimately means that there is not only one hypoelliptic Laplacian, there are as many as possible geometric deformations which one can possibly envision. This will be illustrated in the sequel in the two main classes of examples, which correspond to deformations of de Rham Hodge theory, and of Dolbeault Hodge theory. Moreover it will not be possible to make geometric sense of a conjugation like the one in (1.11), because the considered vector fields will not commute.

Finally, it is instructive to observe that we made two kinds of translations on the variable y. One type of translations has been to replace y by $y + 2i\pi bk$ for $k \in \mathbf{Z}$, or equivalently to change y into $y + b\frac{\partial}{\partial x}$. This imaginary translation has allowed us to relate the hypoelliptic operator $\mathcal{L}'_b$ to the elliptic operator $\mathcal{M}_b$. The other kind of translation has been the real translation $y \to y + bk/t$, to connect the operator $\mathcal{L}'_b$ with the geodesic flow. It should then be clear that the possibility to make at the same time translations on y in the imaginary and in the real directions is critical in explaining the fact that $\mathcal{L}'_b$ interpolates between the Laplacian of S^1 and the geodesic flow of S^1.

1.3. The hypoelliptic Laplacian as a Hodge Laplacian. — Now we will explain in what sense the operator $\mathcal{L}'_b$ is a Laplacian of Hodge type.

Let $d^{S^1 \times \mathbf{R}}$ be the de Rham operator on $S^1 \times \mathbf{R}$. Then

$$d^{S^1 \times \mathbf{R}} = dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}. \tag{1.40}$$

The standard adjoint $d^{S^1 \times \mathbf{R}*}$ of $d^{S^1 \times \mathbf{R}}$ is given by

$$d^{S^1 \times \mathbf{R}*} = -i_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} - i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}. \tag{1.41}$$

Set

$$\mathcal{H}(y) = \frac{y^2}{2}. \tag{1.42}$$

Let $d_T^{S^1 \times \mathbf{R}}$ be the Witten twist of $d^{S^1 \times \mathbf{R}}$, i.e.

$$d_T^{S^1 \times \mathbf{R}} = e^{-T\mathcal{H}} d^{S^1 \times \mathbf{R}} e^{T\mathcal{H}}. \tag{1.43}$$

Then

$$d_T^{S^1 \times \mathbf{R}} = d^{S^1 \times \mathbf{R}} + Tydy \wedge. \tag{1.44}$$

Let $d_T^{S^1 \times \mathbf{R}*}$ be the usual adjoint of $d_T^{S^1 \times \mathbf{R}}$, i.e.,

$$d_T^{S^1 \times \mathbf{R}*} = e^{T\mathcal{H}} d^{S^1 \times \mathbf{R}*} e^{-T\mathcal{H}}. \tag{1.45}$$

Equivalently,

$$d_T^{S^1 \times \mathbf{R}*} = d^{S^1 \times \mathbf{R}*} + Tyi_{\frac{\partial}{\partial y}}. \tag{1.46}$$

Let $\square_T^{S^1 \times \mathbf{R}}$ be the corresponding Witten Laplacian [**20**], i.e.,

$$\square_T^{S^1 \times \mathbf{R}} = \left[d_T^{S^1 \times \mathbf{R}}, d_T^{S^1 \times \mathbf{R}*}\right]. \tag{1.47}$$

In (1.47), $[]$ is our notation for a supercommutator, which, in this case, is an anti-commutator. Then

$$\frac{1}{2}\square_T^{S^1\times\mathbf{R}} = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + T^2y^2 - T\right) + TN - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.48}$$

By (1.48), we get

$$K_{\sqrt{T}^{-1}}\frac{1}{2}\square_T^{S^1\times\mathbf{R}}K_{\sqrt{T}} = \frac{T}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2 - 1\right) + TN - \frac{1}{2}\frac{\partial^2}{\partial x^2}. \tag{1.49}$$

If $T = 1/b^2$, by comparing (1.12), (1.18) and (1.49), we get

$$K_{\sqrt{T}}^{-1}\frac{1}{2}\square_T^{S^1\times\mathbf{R}}K_{\sqrt{T}} = \mathcal{M}'_b. \tag{1.50}$$

Equation (1.50) suggests what should be done to write $\mathcal{L}'_b$ as a Hodge like Laplacian. Set

$$\mathcal{L}''_b = K_b^{-1}\mathcal{L}'_bK_b. \tag{1.51}$$

By (1.22), we obtain,

$$\mathcal{L}''_b = \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + \frac{y^2}{b^4} - \frac{1}{b^2}\right) + \frac{N}{b^2} - \frac{y}{b^2}\frac{\partial}{\partial x}. \tag{1.52}$$

Recall that U_b has been defined in (1.10). Set

$$\underline{d}_b^{S^1\times\mathbf{R}} = U_{b^2}^{-1}d_{1/b^2}^{S^1\times\mathbf{R}}U_{b^2}, \qquad \underline{d}_b^{S^1\times\mathbf{R}*} = U_{b^2}^{-1}d_{1/b^2}^{S^1\times\mathbf{R}*}U_{b^2}. \tag{1.53}$$

By (1.40), (1.41), (1.44), (1.45), we get

$$\begin{aligned} \underline{d}_b^{S^1\times\mathbf{R}} &= (dx - dy)\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + \frac{1}{b^2}ydy, \\ \underline{d}_b^{S^1\times\mathbf{R}*} &= -i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}\frac{\partial}{\partial x} - i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} + \frac{1}{b^2}yi_{\frac{\partial}{\partial y}}. \end{aligned} \tag{1.54}$$

Then (1.11), (1.18), (1.21), (1.47), (1.50) or an easy direct computation show that

$$\frac{1}{2}\left[\underline{d}_b^{S^1\times\mathbf{R}}, \underline{d}_b^{S^1\times\mathbf{R}*}\right] = \mathcal{L}''_b. \tag{1.55}$$

Let $\Omega^{\cdot c}\left(S^1\times\mathbf{R}\right)$ be the vector space of smooth forms on $S^1\times\mathbf{R}$ with compact support. Let r be the map $(x,y)\to(x,-y)$. Let h be the symmetric bilinear form on $\Omega^{\cdot c}\left(S^1\times\mathbf{R}\right)$,

$$h\left(s,s'\right) = \int_{S^1\times\mathbf{R}}\left\langle r^*s, s'\right\rangle dxdy. \tag{1.56}$$

In (1.56), $\langle\rangle$ is the obvious scalar product on $\Lambda^{\cdot}\left(T^*\left(S^1\times\mathbf{R}\right)\right)$. Then (1.54) shows that $\underline{d}_b^{S^1\times\mathbf{R}*}$ is the formal adjoint of $\underline{d}_b^{S^1\times\mathbf{R}}$ with respect to h.

Still $\underline{d}_b^{S^1\times\mathbf{R}}$ has no obvious relation to the de Rham operator. However, observe that

$$\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right)=dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+\frac{1}{b^2}ydy, \tag{1.57}$$

which we can rewrite in the form,

$$\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right)=d_{1/b^2}^{S^1\times\mathbf{R}}. \tag{1.58}$$

Set

$$\overline{d}_b^{S^1\times\mathbf{R}*}=\exp\left(dyi_{\frac{\partial}{\partial x}}\right)\underline{d}_b^{S^1\times\mathbf{R}*}\exp\left(-dyi_{\frac{\partial}{\partial x}}\right). \tag{1.59}$$

By (1.54),

$$\overline{d}_b^{S^1\times\mathbf{R}*}=-i_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}-i_{\frac{\partial}{\partial y}-\frac{\partial}{\partial x}}\frac{\partial}{\partial y}+\frac{y}{b^2}i_{\frac{\partial}{\partial y}-\frac{\partial}{\partial x}}. \tag{1.60}$$

By (1.52), (1.55), and (1.58)-(1.60), we get

$$\frac{1}{2}\left[d_{1/b^2}^{S^1\times\mathbf{R}},\overline{d}_b^{S^1\times\mathbf{R}*}\right]=\frac{1}{2}\left(-\frac{\partial^2}{\partial y^2}+\frac{y^2}{b^4}-\frac{1}{b^2}\right)+\frac{1}{b^2}\left(N-dyi_{\frac{\partial}{\partial x}}\right)-\frac{y}{b^2}\frac{\partial}{\partial x}. \tag{1.61}$$

Equations (1.54) and (1.61) should give ample matter to think about. First we consider (1.54). Observe that

$$\left(dx-i_{\frac{\partial}{\partial x}}\right)^2=-1, \qquad \left(dy+i_{\frac{\partial}{\partial y}}\right)^2=1, \tag{1.62}$$

which shows that

$$\left(dx-dy-i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}\right)^2=0. \tag{1.63}$$

Equation (1.63) exactly says the operator $dx-dy-i_{\frac{\partial}{\partial x}+\frac{\partial}{\partial y}}$ is nilpotent. This in turn explains why there is no term $\frac{\partial^2}{\partial x^2}$ in the right-hand side of (1.55).

Now we concentrate on the pair $\left(d_{1/b^2}^{S^1\times\mathbf{R}},\overline{d}_b^{S^1\times\mathbf{R}*}\right)$. From (1.57), (1.58), we will obtain an analogue of the bilinear form h. Indeed let $\mathfrak{h}^{T(S^1\times\mathbf{R})}$ be the bilinear form on $T\left(S^1\times\mathbf{R}\right)=\mathbf{R}\oplus\mathbf{R}$ which is given by the matrix,

$$\mathfrak{h}^{T(S^1\times\mathbf{R})}=\begin{pmatrix}1&1\\1&0\end{pmatrix}. \tag{1.64}$$

The corresponding bilinear form on $T^*\left(S^1\times\mathbf{R}\right)$, which we denote by $\mathfrak{h}^{T^*(S^1\times\mathbf{R})}$, is given by

$$\mathfrak{h}^{T^*(S^1\times\mathbf{R})}=\begin{pmatrix}0&1\\1&-1\end{pmatrix}. \tag{1.65}$$

Then $\mathfrak{h}^{T^*(S^1\times\mathbf{R})}$ induces a corresponding symmetric bilinear form $\mathfrak{h}^{\Lambda^\cdot(T^*(S^1\times\mathbf{R}))}$ on $\Lambda^\cdot\left(T^*\left(S^1\times\mathbf{R}\right)\right)$.

Let $\mathfrak{h}$ be the symmetric bilinear form on $\Omega^{\cdot c}\left(S^1 \times \mathbf{R}\right)$ which is given by

$$\mathfrak{h}\left(s, s'\right) = \int_{S^1 \times \mathbf{R}} \mathfrak{h}^{\Lambda^{\cdot}\left(T^*\left(S^1 \times \mathbf{R}\right)\right)}\left(s\left(x, -y\right), s'\left(x, y\right)\right) dx dy. \tag{1.66}$$

Observe that in (1.66), the map r is made only to act on the function $s\left(x, y\right)$ without acting on the form part of s. Then $\overline{d}_b^{S^1 \times \mathbf{R} *}$ is the formal adjoint of $d_{1/b^2}^{S^1 \times \mathbf{R}}$ with respect to $\mathfrak{h}$.

The bilinear forms h and $\mathfrak{h}$ are symmetric, but they are non local, in the sense their construction involves the antipodal map r. Consider instead the matrix ϕ acting on $T\left(S^1 \times \mathbf{R}\right)$,

$$\phi = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.67}$$

and the corresponding bilinear form η on $T\left(S^1 \times \mathbf{R}\right)$,

$$\eta\left(U, V\right) = \left\langle U, \phi V\right\rangle. \tag{1.68}$$

Then

$$\phi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.69}$$

If we identify $T\left(S^1 \times \mathbf{R}\right)$ and $T^*\left(S^1 \times \mathbf{R}\right)$ by ϕ, the corresponding bilinear form η^* on $T^*\left(S^1 \times \mathbf{R}\right)$ is given by

$$\eta^*\left(s, s'\right) = \left\langle \phi^{-1} s, s'\right\rangle. \tag{1.70}$$

Then η^* induces a nondegenerate bilinear form on $\Lambda^{\cdot}\left(T^*\left(S^1 \times \mathbf{R}\right)\right)$. If $s, s' \in \Omega^{\cdot c}\left(S^1 \times \mathbf{R}\right)$, set

$$\eta\left(s, s'\right) = \int_{S^1 \times \mathbf{R}} \eta^*\left(s, s'\right) dx dy. \tag{1.71}$$

Then one verifies that

$$\eta\left(s, d_{1/b^2}^{S^1 \times \mathbf{R}} s'\right) = \eta\left(\overline{d}_b^{S^1 \times \mathbf{R} *} s, s'\right). \tag{1.72}$$

Let ω be the symplectic form on $S^1 \times \mathbf{R}$,

$$\omega = dy \wedge dx. \tag{1.73}$$

Then observe that if $U, V \in T\left(S^1 \times \mathbf{R}\right)$,

$$\eta\left(U, V\right) = \left\langle \pi_* U, \pi_* V\right\rangle + \omega\left(U, V\right). \tag{1.74}$$

2. Hodge theory and the Witten Laplacian

In this section, we briefly recall elementary results in Hodge theory. Also we describe the Witten deformation of the classical Hodge Laplacian. Finally, we show that on a symplectic manifold, up to a constant, the symplectic Witten Laplacian is the Lie derivative operator associated with the corresponding Hamiltonian vector field.

This section is organized as follows. In subsection 2.1, we recall known results on Hodge theory and on the Witten Laplacian.

In subsection 2.2, we give a formula for the symplectic Witten Laplacian.

2.1. Classical Hodge theory and the Witten Laplacian. — Let X be a compact Riemannian manifold of dimension n, let g^{TX} be the metric on TX, and let dv_X be the associated volume form. The metric g^{TX} induces a corresponding scalar product $\langle\rangle_{\Lambda^\cdot(T^*X)}$ on $\Lambda^\cdot(T^*X)$.

Let $\left(\Omega^\cdot(X), d^X\right)$ be the de Rham complex on X. Let $\langle\rangle_{\Omega^\cdot(X)}$ be the scalar product on $\Omega^\cdot(X)$ associated with g^{TX}, i.e.,

$$\langle s, s'\rangle_{\Omega^\cdot(X)} = \int_X \langle s, s'\rangle_{\Lambda^\cdot(T^*X)}\, dv_X. \tag{2.1}$$

Let d^{X*} be the formal adjoint of d^X with respect to $\langle\rangle_{\Omega^\cdot(X)}$.

The Hodge Laplacian $\square^X$ is given by

$$\square^X = \left[d^X, d^{X*}\right]. \tag{2.2}$$

The Hodge Laplacian $\square^X$ is a second order elliptic self-adjoint nonnegative operator, whose principal symbol is $|\xi|^2$. If Δ^X is the Laplace-Beltrami operator, the restriction of $\square^X$ to smooth functions coincides with $-\Delta^X$.

Let $\mathcal{H} = \ker \square^X$ be the finite dimensional vector space of the harmonic forms. The basic result of Hodge theory asserts that

$$\mathcal{H} \simeq H^\cdot(X, \mathbf{R}). \tag{2.3}$$

Now we briefly describe the Witten deformation [**20**] of the above Hodge Laplacian. Its purpose is to provide an interpolation between classical Hodge theory and Morse theory. Let $f : X \to \mathbf{R}$ be a smooth function. For $T \geq 0$, as in (1.43), set

$$d_T^X = e^{-Tf} d^X e^{Tf}. \tag{2.4}$$

The complex $\left(\Omega^\cdot(X), d_T^X\right)$ is canonically isomorphic to the complex $\left(\Omega^\cdot(X), d^X\right)$.

Let d_T^{X*} be the formal adjoint of d_T^X with respect to $\langle\rangle_{\Omega^\cdot(X)}$, so that

$$d_T^{X*} = e^{Tf} d^{X*} e^{-Tf}. \tag{2.5}$$

The corresponding Witten Laplacian $\square_T^X$ is given by

$$\square_T^X = \left[d_T^X, d_T^{X*}\right]. \tag{2.6}$$

The Laplacian $\square_T^X$ has exactly the same properties as $\square^X$. In particular if

$$\mathcal{H}_T = \ker \square_T^X, \tag{2.7}$$

then

$$\mathcal{H}_T \simeq H^{\cdot}(X, \mathbf{R}). \tag{2.8}$$

Of course, for $T = 0$, $\square_T^X$ coincides with $\square^X$, so that $\square_T^X$ is a deformation of $\square^X$.
Clearly,

$$d_T^X = d^X + T df\wedge, \qquad d_T^{X*} = d^{X*} + T i_{\nabla f}. \tag{2.9}$$

Let $e_1, \dots, e_n$ be an orthonormal basis of TX, let $e^1, \dots, e^n$ be the corresponding dual basis of T^*X. From (2.9) we deduce that

$$\square_T^X = \square^X + T^2 |df|^2 + T\left(2\left\langle \nabla_{e_i}^{TX} \nabla f, e_j\right\rangle e^i i_{e_j} - \Delta^{TX} f\right). \tag{2.10}$$

An essentially equivalent construction is to keep d^X fixed, and instead to consider the adjoint of d^X with respect to the L_2 scalar product in (2.1), in which the volume form dv_X has been replaced by $e^{-2Tf} dv_X$. The adjoint of d^X is just d_{2T}^{X*} and the associated Laplacian is given by $e^{Tf} \square_T^X e^{-Tf}$.

Assume f to be a Morse function. Using (2.10), Witten showed in [**20**] that as $T \to +\infty$, most of the spectrum of $\square_T^X$ tends to $+\infty$, and the remaining finite eigenvalues tend to 0. Some of these are exactly 0, and correspond to the harmonic forms, and others are asymptotically small, decaying to 0 like $e^{-cT}, c > 0$. Let $F_T^{\cdot}$ be the direct sum of eigenforms of $\square_T^X$ for eigenvalues ≤ 1. Witten showed that as $T \to +\infty$, $F_T^{\cdot}$ localizes near the critical points of f. More precisely, F_T^i localizes near the critical points of index i. Also Witten conjectured that as $T \to +\infty$, the complex $\left(F_T^{\cdot}, d_T^X\right)$ approximates in the proper sense a complex constructed from the 'instantons' which connect the critical points. These instantons are integral curves of the gradient field ∇f. When ∇f is Morse-Smale, this complex was identified to be the Morse-Smale complex associated with ∇f. In [**12**], Helffer and Sjöstrand proved the conjecture of Witten. For another proof we refer to [**10**].

The Witten deformation was used in Bismut-Zhang [**9, 10**] to give a new proof of the Cheeger-Müller theorem [**11, 16**].

One of the main motivations given in [**3**] for the introduction of the hypoelliptic Laplacian has been an attempt to extend the construction of the Witten Laplacian to the loop space LX of X. On LX there are many natural functionals like the energy. If the Witten Laplacian associated with the energy existed, it would interpolate between the Hodge Laplacian $\square^{LX}$ of LX and the Morse theory of the energy functional, whose critical points are precisely the closed geodesics. The hypoelliptic Laplacian provides a semiclassical version of this interpolation. For a review of these aspects of the hypoelliptic Laplacian, we refer the reader to [**4, 6**].

2.2. The symplectic Witten Laplacian. — Let (M,ω) be a symplectic manifold of dimension n. The nondegenerate bilinear form ω on TM induces an isomorphism $\phi : TM \to T^*M$, so that

$$\omega(U,V) = \langle U, \phi V\rangle . \tag{2.11}$$

Let ω^* be the nondegenerate bilinear form on T^*M which corresponds to ω via the canonical isomorphism ϕ. We still denote by ω^* the associated bilinear form on $\Lambda^{\cdot}(T^*M)$. The form ω determines a volume form dv_M on M.

If $\alpha \in \Lambda^{\cdot}(T^*M)$, set

$$L\alpha = \omega \wedge \alpha. \tag{2.12}$$

Let Λ be the adjoint of L with respect to ω^*, so that

$$\omega^*(\Lambda s, s') = \omega^*(s, Ls') . \tag{2.13}$$

The operators L, Λ are the well-known Lefschetz operators. Let N be the number operator of $\Lambda^{\cdot}(T^*M)$, i.e. the operator which acts by multiplication by k on $\Lambda^k(T^*M)$. Set

$$H = \frac{1}{2}(N - n/2) \tag{2.14}$$

Then we have the well-known commutation relations

$$[H,L] = L, \qquad [H,\Lambda] = -\Lambda, \qquad [L,\Lambda] = 2H. \tag{2.15}$$

Let $\overline{d}^M$ be the formal adjoint of d^M with respect to the bilinear form associated with the symplectic form ω as in (1.71), (1.72), with η^* replaced by ω^* in (1.71).

Now we state the simple result in [**3**, Theorem 2.2].

Proposition 2.1. — *The following identities hold:*

$$\overline{d}^M = -\left[d^M, \Lambda\right], \qquad d^M = -\left[\overline{d}^M, L\right], \qquad \left[d^M, \overline{d}^M\right] = 0. \tag{2.16}$$

Proof. — Using Darboux's theorem, we may as well assume that locally, the form ω has constant coefficients. Then (2.16) is elementary linear algebra. In particular the last identity is just a reflection of the fact that ω vanishes on the diagonal. □

Let $\mathcal{H} : M \to \mathbf{R}$ be a smooth function. Let $d^M_{\mathcal{H}}$ be the twisted de Rham operator

$$d^M_{\mathcal{H}} = e^{-\mathcal{H}} d^M e^{-\mathcal{H}}, \tag{2.17}$$

and let $\overline{d}^M_{\mathcal{H}}$ be its symplectic adjoint, i.e.,

$$\overline{d}^M_{\mathcal{H}} = e^{\mathcal{H}} \overline{d}^M e^{-\mathcal{H}}. \tag{2.18}$$

Then $\left[d^M_{\mathcal{H}}, \overline{d}^M_{\mathcal{H}}\right]$ is the symplectic Witten Laplacian associated with $\mathcal{H}$.

Let $Y^{\mathcal{H}}$ be the Hamiltonian vector field associated with $\mathcal{H}$, so that

$$d\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0. \tag{2.19}$$

Let $L_{Y^{\mathcal{H}}}$ be the Lie derivative operator associated with $Y^{\mathcal{H}}$.

Now we state a simple formula established in [**6**, eq. (2.34)].

Proposition 2.2. — *The following identity holds:*

$$\left[d^M_{\mathcal{H}}, \overline{d}^M_{\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}. \tag{2.20}$$

Proof. — One verifies easily that

$$\overline{d}^M_{2\mathcal{H}} = \overline{d}^M - 2i_{Y^{\mathcal{H}}}, \tag{2.21}$$

so that using (2.16), we get

$$\left[d^M, \overline{d}^M_{2\mathcal{H}}\right] = -2L_{Y^{\mathcal{H}}}. \tag{2.22}$$

By conjugating (2.22) by $e^{-\mathcal{H}}$ and using the fact that $Y^{\mathcal{H}}$ preserves $\mathcal{H}$, we get (2.22). □

Proposition 2.2 is quite interesting. Indeed remember that our ultimate goal is to interpolate between the Hodge Laplacian $\square^X$ of the Riemannian manifold X and the generator $L_{Y^{\mathcal{H}}}$ of the geodesic flow on the total space $\mathcal{X}^*$ of the cotangent bundle of X. However, (2.20) indicates that $L_{Y^{\mathcal{H}}}$ is itself a symplectic Witten Laplacian. One possible construction of the hypoelliptic Laplacian consists in linearly interpolating between the scalar product of TX and the symplectic form of $\mathcal{X}^*$. This point of view is explained in detail in [**3**, section 2.12].We also refer to equations (1.74) and (3.5) for a hint on how to do this.

3. The hypoelliptic Laplacian in de Rham theory

The purpose of this section is to construct the hypoelliptic Laplacian in de Rham theory. This operator, which acts on the total space $\mathcal{X}^*$ of the cotangent bundle of a Riemannian manifold X, depends on a parameter $b > 0$. Also we give arguments showing that it should interpolate between the standard Hodge Laplacian of X and the generator of the geodesic flow on $\mathcal{X}^*$.

This section is organized as follows. In subsection 3.1, we give a formula for the operator $d^{\mathcal{X}^*}$.

In subsection 3.2, we introduce a symmetric bilinear form on $\Omega^{\cdot c}(\mathcal{X}^*)$, and we obtain the formal adjoint $\overline{d}^{\mathcal{X}^*}$ of $d^{\mathcal{X}^*}$ with respect to this form.

In subsection 3.3, given a Hamiltonian function $\mathcal{H}$ on $\mathcal{X}^*$, we obtain corresponding symmetric bilinear forms, and we construct the adjoint of the Witten twist $d^{\mathcal{X}^*}_{\mathcal{H}}$.

In subsection 3.4, we discuss the self-adjointness of our first order differential operators.

In subsection 3.5, we give Weitzenböck formulas for our new Hodge like Laplacians.

In subsection 3.6, when the function $\mathcal{H}$ is proportional to $|p|^2/2$, we show that our new Laplacians are hypoelliptic.

In subsection 3.7, we show that $b \to 0$, the hypoelliptic Laplacian should converge in the proper sense to the classical Hodge Laplacian of X.

Finally, in subsection 3.8, we give arguments showing that as $b \to +\infty$, the hypoelliptic Laplacian converges to the generator of the geodesic flow.

3.1. The de Rham operator on $\mathcal{X}^*$. — Let X be a compact Riemannian manifold of dimension n, let $\mathcal{X}, \mathcal{X}^*$ be the total spaces of the vector bundles TX, T^*X over X, and let π denote the projection from $\mathcal{X}$ or $\mathcal{X}^*$ on X. The metric g^{TX} induces an identification of the fibres TX and T^*X, and a corresponding isomorphism of $\mathcal{X}$ and $\mathcal{X}^*$.

Let ∇^{TX} be the Levi-Civita connection on TX, and let R^{TX} be its curvature. Let ∇^{T^*X} be the corresponding connection on T^*X, and let R^{T^*X} be its curvature.

Let

$$T\mathcal{X}^* = \pi^* \left(TX \oplus T^*X\right) \tag{3.1}$$

be the splitting of $\mathcal{X}^*$ which is associated with the connection ∇^{T^*X}. In (3.1), TX corresponds to the horizontal part of $T\mathcal{X}^*$, and T^*X to the tangent bundle to the fibres T^*X.

From (3.1), we get the isomorphism,

$$\Lambda^{\cdot}\left(T^*\mathcal{X}^*\right) = \pi^* \left(\Lambda^{\cdot}\left(T^*X\right) \widehat{\otimes} \widehat{\Lambda}^{\cdot}\left(TX\right)\right). \tag{3.2}$$

In (3.2), $\widehat{\Lambda}^{\cdot}(TX)$ is our notation for the exterior algebra of the fibre, the hat permitting us to distinguish $\widehat{\Lambda}^{\cdot}(TX)$ from the exterior algebra $\Lambda^{\cdot}(TX)$. Of course $\Lambda^{\cdot}(TX)$ and $\widehat{\Lambda}^{\cdot}(TX)$ are canonically isomorphic. Let $\nabla^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}$ be the connection on $\Lambda^{\cdot}(T^*\mathcal{X}^*)$ induced by ∇^{TX}.

Let $\left(\Omega^{\cdot}(\mathcal{X}^*), d^{\mathcal{X}^*}\right)$ be the de Rham complex of $\mathcal{X}^*$. Let $\mathbf{I}$ be the vector bundle on X of smooth sections of $\Lambda^{\cdot}(TX)$ along the fibre T^*X. By (3.2), we get

$$\Omega^{\cdot}\left(\mathcal{X}^*\right) = \Omega^{\cdot}\left(X, \mathbf{I}\right). \tag{3.3}$$

Let p be the tautological section of the fibre π^*T^*X over $\mathcal{X}^*$. Using (3.2), we may write $d^{\mathcal{X}^*}$ in the form,

$$d^{\mathcal{X}^*} = d^V + \nabla^{\mathbf{I}} + i_{\widehat{R^{T^*X}p}}. \tag{3.4}$$

In (3.4), d^V is the de Rham operator along the fibre T^*X, $\nabla^{\mathbf{I}}$ is the obvious connection on $\mathbf{I}$, and $i_{\widehat{R^{T^*X}p}}$ is the interior multiplication by the vertical vector $\widehat{R^{T^*X}p}$. Of course

R^{T^*X} is viewed as a 2-form on X, so that ultimately $i_{\widehat{R^{T^*X}p}}$ increases the total degree by 1.

3.2. A bilinear form on $\Omega^{\cdot c}(\mathcal{X}^*)$. — Now we inspire ourselves from the arguments which were given in subsection 1.3. Let $\Omega^{\cdot c}(\mathcal{X}^*)$ be the vector space of smooth forms on $\mathcal{X}^*$ which have compact support. Let ω be the symplectic form of $\mathcal{X}^*$. On $T\mathcal{X}^*$, let η be the nondegenerate bilinear form,

$$\eta(U,V) = \langle \pi_* U, \pi_* V\rangle + \omega(U,V). \tag{3.5}$$

The isomorphism $\phi : T\mathcal{X}^* \to T^*\mathcal{X}^*$ associated to η is given by

$$\phi = \begin{pmatrix} g^{TX} & -1|_{T^*X} \\ 1|_{TX} & 0 \end{pmatrix}. \tag{3.6}$$

Equation (3.5) should be compared with equation (1.74), and equation (3.6) should be compared with equation (1.67).

The volume form on $\mathcal{X}^*$ associated to η is exactly the symplectic volume form $dv_{\mathcal{X}^*}$. Let $\overline{d}^{\mathcal{X}^*}$ be the formal adjoint of $d^{\mathcal{X}^*}$ with respect to the bilinear form η on $\Omega^{\cdot c}(\mathcal{X}^*)$, which one obtains as in (1.71) from (3.5), (3.6). Of course, we use the same conventions as in subsection 1.3 to define the formal adjoint, and we use in particular equation (1.72).

Let $e_1, \dots, e_n$ be an orthonormal basis of TX, let $e^1, \dots, e^n$ be the corresponding dual basis of T^*X. Recall that $T\mathcal{X}^* = \pi^*(TX \oplus T^*X)$. We denote by $\widehat{e}^1, \dots, \widehat{e}^n$ the basis of the vertical fibre T^*X in $T\mathcal{X}^*$, and by $\widehat{e}_1, \dots, \widehat{e}_n$ the corresponding dual basis.

Set

$$R^{T^*X}p\wedge = \frac{1}{2} i_{\widehat{e^i}} i_{\widehat{e^j}} R^{T^*X}(e_i, e_j)\, p \wedge. \tag{3.7}$$

In (3.7), $R^{T^*X}(e_i, e_j)\, p$ is viewed as a section of T^*X, which lifts to a 1-form on $\mathcal{X}^*$. Therefore $R^{T^*X}p$ decreases the total degree by 1.

We now have the result established in [**3**, Proposition 2.10].

Proposition 3.1. — *The following identity holds:*

$$\overline{d}^{\mathcal{X}^*} = -i_{\widehat{e^i}} \nabla^{T\mathcal{X}^*}_{e_i} + i_{e_i} \nabla_{\widehat{e^i}} + R^{T^*X}p \wedge - i_{\widehat{e^i}} \nabla_{\widehat{e^i}}. \tag{3.8}$$

3.3. A Hamiltonian function. — Let $\mathcal{H} : \mathcal{X}^* \to \mathbf{R}$ be a smooth function. Let $Y^{\mathcal{H}}$ be the associated Hamiltonian vector field, so that $d\mathcal{H} + i_{Y^{\mathcal{H}}}\omega = 0$.We denote by $\widehat{\nabla^V \mathcal{H}}$ the fibrewise gradient field of $\mathcal{H}$.

Definition 3.2. — Set

$$d^{\mathcal{X}^*}_{\mathcal{H}} = e^{-\mathcal{H}} d^{\mathcal{X}^*} e^{\mathcal{H}}, \qquad \overline{d}^{\mathcal{X}^*}_{\mathcal{H}} = e^{\mathcal{H}} \overline{d}^{\mathcal{X}^*} e^{-\mathcal{H}}. \tag{3.9}$$

Observe that $\overline{d}^{\mathcal{X}^*}_{\mathcal{H}}$ is the adjoint of $d^{\mathcal{X}^*}_{\mathcal{H}}$ with respect to η. Also, if $s, s' \in \Omega^{\cdot c}(\mathcal{X}^*)$, put

$$\eta_{\mathcal{H}}(s, s') = \int_{\mathcal{X}^*} \eta^*(s, s') e^{-2\mathcal{H}} dv_{\mathcal{X}^*}. \tag{3.10}$$

Then $\overline{d}^{\mathcal{X}^*}_{2\mathcal{H}}$ is the adjoint of $d^{\mathcal{X}^*}$ with respect to $\eta_{\mathcal{H}}$.

Definition 3.3. — Set

$$A_{\mathcal{H}} = \frac{1}{2}\left(\overline{d}^{\mathcal{X}^*}_{2\mathcal{H}} + d^{\mathcal{X}^*}\right), \qquad \mathfrak{A}_{\mathcal{H}} = \frac{1}{2}\left(\overline{d}^{\mathcal{X}^*}_{\mathcal{H}} + d^{\mathcal{X}^*}_{\mathcal{H}}\right). \tag{3.11}$$

Then

$$\mathfrak{A}_{\mathcal{H}} = e^{-\mathcal{H}} A_{\mathcal{H}} e^{\mathcal{H}}. \tag{3.12}$$

Moreover,

$$A^2_{\mathcal{H}} = \frac{1}{4}\left[d^{\mathcal{X}^*}, \overline{d}^{\mathcal{X}^*}_{2\mathcal{H}}\right]. \tag{3.13}$$

We have the result established in [**3**, Proposition 2.18].

Proposition 3.4. — *The following identities hold:*

$$\begin{aligned} A_{\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e^i}}\right) \nabla^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}_{e_i} + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e^i}}\right) \nabla_{\widehat{e^i}} + \frac{1}{2}\left(R^{T^*X} p \wedge + i_{\widehat{R^{T^*X} p}}\right) \\ &\quad + i_{\widehat{e^i}} \nabla_{e_i} \mathcal{H} + i_{\widehat{e^i} - e_i} \nabla_{\widehat{e^i}} \mathcal{H}, \\ \mathfrak{A}_{\mathcal{H}} &= \frac{1}{2}\left(e^i - i_{\widehat{e^i}}\right) \nabla^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}_{e_i} + \frac{1}{2}\left(\widehat{e}_i + i_{e_i - \widehat{e^i}}\right) \nabla_{\widehat{e^i}} + \frac{1}{2}\left(R^{T^*X} p \wedge + i_{\widehat{R^{T^*X} p}}\right) \\ &\quad + \frac{1}{2}\left(e^i + i_{\widehat{e^i}}\right) \nabla_{e_i} \mathcal{H} + \frac{1}{2}\left(\widehat{e}_i + i_{\widehat{e^i} - e_i}\right) \nabla_{\widehat{e^i}} \mathcal{H}. \end{aligned} \tag{3.14}$$

Set

$$\mu_0 = \widehat{e}_i \wedge i_{e_i}. \tag{3.15}$$

Put

$$\mathfrak{A}'_{\mathcal{H}} = e^{-\mu_0} \mathfrak{A}_{\mathcal{H}} e^{\mu_0}. \tag{3.16}$$

The proper interpretation for (3.16) can be guessed from (1.57)-(1.59). The operator $\mathfrak{A}'_{\mathcal{H}}$ will also be considered in the sequel.

3.4. A self-adjointness property. — The bilinear form $\eta_{\mathcal{H}}$ on $\Omega^{\cdot}(\mathcal{X}^*)$ is in general not symmetric. However, we will here follow the arguments in (1.64)-(1.67).

Let $\mathfrak{h}^{T\mathcal{X}^*}$ be the bilinear form on $T\mathcal{X}^* = \pi^*(TX \oplus T^*X)$ which is given by

$$\mathfrak{h}^{T\mathcal{X}^*} = \begin{pmatrix} g^{TX} & 1|_{T^*X} \\ 1|_{TX} & 0 \end{pmatrix}. \tag{3.17}$$

Let $\mathfrak{p} : T\mathcal{X}^* \to T^*X$ be the projection with respect to the above splitting of $T\mathcal{X}^*$. If $U \in T\mathcal{X}^*$, then

$$\mathfrak{h}^{T\mathcal{X}^*}(U,U) = \langle \pi_* U, \pi_* U \rangle + 2 \langle \pi_* U, \mathfrak{p} U \rangle. \tag{3.18}$$

Then the volume form on $\mathcal{X}^*$ which is attached to $\mathfrak{h}^{T\mathcal{X}^*}$ is the symplectic volume form $dv_{\mathcal{X}^*}$. Let $\mathfrak{h}^{\Lambda^{\cdot}(T^*\mathcal{X})}$ be the corresponding symmetric form on $\Lambda^{\cdot}(T^*\mathcal{X}^*)$.

Let $r : (x,p) \to (x,-p)$ be the obvious involution of $\mathcal{X}^*$.

Definition 3.5. — Let $\mathfrak{h}$ be the symmetric form on $\Omega^{\cdot c}(\mathcal{X}^*)$,

$$\mathfrak{h}(s,s') = \int_{\mathcal{X}^*} \mathfrak{h}^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}(s \circ r, s')\, dv_{\mathcal{X}^*}. \tag{3.19}$$

As in (1.66), in (3.19), the change of variable $p \to -p$ is not made on the form part of s. Set

$$\mathfrak{h}_{\mathcal{H}}(s,s') = \mathfrak{h}\left(e^{-2\mathcal{H}} s, s'\right). \tag{3.20}$$

If $\mathcal{H}$ is r-invariant, then $\mathfrak{h}_{\mathcal{H}}$ is a symmetric form.

Let g^{TT^*X} be the natural metric on $T\mathcal{X}^*$ which is associated with the splitting of $T\mathcal{X}^*$, and let g be the scalar product on $\Omega^{\cdot c}(\mathcal{X}^*)$ associated to g^{TT^*X}. Let h be the symmetric form on $\Omega^{\cdot c}(\mathcal{X}^*)$,

$$h(s,s') = g(r^* s, s'). \tag{3.21}$$

The symmetric forms in (3.19) and (3.21) have signature (∞,∞). If $\mathcal{H}$ is r-invariant, the same property holds for the symmetric form in (3.20) .

We state a result established in [**3**, Theorems 2.21 and 2.30].

Theorem 3.6. — *If $\mathcal{H}$ is r-invariant, $A_{\mathcal{H}}$ is $\mathfrak{h}_{\mathcal{H}}$-symmetric, $\mathfrak{A}_{\mathcal{H}}$ is $\mathfrak{h}$-symmetric, and $\mathfrak{A}'_{\mathcal{H}}$ is h-symmetric.*

3.5. The Weitzenböck formula. — We give the Weitzenböck formula established in [**3**, Theorem 3.3].

Theorem 3.7. — *The following identities hold:*

$$A^2_{\mathcal{H}} = \frac{1}{4}\left(-\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e^k}} i_{\widehat{e^\ell}} + 2L_{\widehat{\nabla^V\mathcal{H}}}\right) - \frac{1}{2}L_{Y^{\mathcal{H}}}, \tag{3.22}$$

$$\mathfrak{A}^2_{\mathcal{H}} = \frac{1}{4}\Bigg(-\Delta^V - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e^k}} i_{\widehat{e^\ell}} + \left|\nabla^V\mathcal{H}\right|^2$$
$$- \Delta^V\mathcal{H} + 2\left(\nabla_{\widehat{e^i}}\nabla_{\widehat{e^j}}\mathcal{H}\right)\widehat{e}_i i_{\widehat{e^j}} + 2\left(\nabla_{\widehat{e^i}}\nabla_{e_j}\mathcal{H}\right)e^j i_{\widehat{e^i}}\Bigg) - \frac{1}{2}L_{Y^{\mathcal{H}}}.$$

3.6. The hypoelliptic Laplacian. — Let N the operator counting the degree in $\widehat{\Lambda}^{\cdot}(TX)$. For $c \in \mathbf{R}$, set

$$\mathcal{H}^c = c\frac{|p|^2}{2}. \tag{3.23}$$

Let $u \in \mathbf{R}$ be an extra variable. The following result was established in [**3**, Theorems 3.4 and 3.6].

Theorem 3.8. — *The following identities hold:*

$$A^2_{\mathcal{H}^c} = \frac{1}{4}\left(-\Delta^V + 2cL_{\widehat{p}} - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e^k}} i_{\widehat{e^\ell}}\right) - \frac{1}{2}L_{Y^{\mathcal{H}^c}},$$
$$\mathfrak{A}^2_{\mathcal{H}^c} = \frac{1}{4}\left(-\Delta^V + c^2|p|^2 + c(2N-n) - \frac{1}{2}\left\langle R^{TX}\left(e_i,e_j\right)e_k,e_l\right\rangle e^i e^j i_{\widehat{e^k}} i_{\widehat{e^\ell}}\right)$$
$$- \frac{1}{2}L_{Y^{\mathcal{H}^c}}. \tag{3.24}$$

For $c \neq 0$, the operators $\frac{\partial}{\partial u} - A^2_{\mathcal{H}^c}$, $\frac{\partial}{\partial u} - \mathfrak{A}^2_{\mathcal{H}^c}$ are hypoelliptic.

Remark 3.9. — Of course (3.24) follows from theorem 3.7. Hypoellipticity follows from Hörmander [**14**]. Any of the operators in theorem 3.8 is called a hypoelliptic Laplacian.

3.7. An interpolation property: the limit $b \to 0$ and classical Hodge theory. — In the sequel, we take $b > 0$, and we set $\mathcal{H} = |p|^2/2, c = 1/b^2$.

For $b > 0$, let K_b be the map $s(x,p) \to s(x,bp)$. By [**3**, Theorem 3.8], we get

$$K_b 2\mathfrak{A}'^2_{\mathcal{H}^c} K_b^{-1} = \frac{\alpha}{b^2} + \frac{\beta}{b} + \gamma, \tag{3.25}$$

with α, β given by

$$\alpha = \frac{1}{2}\left(-\Delta^V + |p|^2 - n\right) + N, \qquad \beta = -\nabla^{\Lambda^{\cdot}(T^*\mathcal{X}^*)}_{Y^{\mathcal{H}}}. \tag{3.26}$$

Observe that α is a standard self-adjoint harmonic oscillator. Also $\ker\alpha$ is spanned by the function $\exp\left(-|p|^2/2\right)$.

We identify $\Omega^{\cdot}(X)$ to $\ker \alpha$ by the map $s \to \pi^* s \exp\left(-|p|^2/2\right)/\pi^{n/4}$. Let P be the standard L_2 -projector from $\Omega^{\cdot}(\mathcal{X}^*)$ on $\ker \alpha$. Note that β maps $\ker \alpha$ into its L_2 orthogonal.

Assume for the moment that α, β are endomorphisms of a finite dimensional vector space E, that α is semisimple, so that

$$E = \ker \alpha \oplus \operatorname{Im} \alpha. \tag{3.27}$$

Let Q be the projector from E on $\ker \alpha$ with respect to the splitting (3.27). We also assume that β maps $\ker \alpha$ into $\operatorname{Im} \alpha$.

Let $u \in \operatorname{End}(E)$. We write u as a matrix with respect to the splitting (3.27).

$$u = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{3.28}$$

Suppose u to be invertible. Now we give a matrix expression for the inverse u^{-1} of u under the assumption that D is invertible. We will assume implicitly that other matrix expressions are invertible as well. We have the formula,

$$u^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -\left(A - BD^{-1}C\right)^{-1} BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1} BD^{-1} \end{bmatrix}. \tag{3.29}$$

Let α^{-1} be the inverse of α restricted to $\operatorname{Im} \alpha$. By (3.29), when $\lambda \in \mathbf{C}$, we get

$$\begin{aligned} &\left(\lambda - \frac{\alpha}{b^2} - \frac{\beta}{b} - \gamma\right)^{-1} \\ &\qquad = \begin{bmatrix} \left(\lambda - Q\gamma Q + Q\beta\alpha^{-1}\beta Q\right)^{-1} + \mathcal{O}(b) & \mathcal{O}(b) \\ \mathcal{O}(b) & \mathcal{O}\left(b^2\right) \end{bmatrix}. \end{aligned} \tag{3.30}$$

By (3.30) we find that as $b \to 0$,

$$\left(\lambda - \frac{\alpha}{b^2} - \frac{\beta}{b} - \gamma\right)^{-1} = Q\left(\lambda - Q\left(\gamma - \beta\alpha^{-1}\beta\right)Q\right)^{-1} Q + \mathcal{O}(b). \tag{3.31}$$

The operator appearing in the limit $b \to 0$ is $Q\left(\gamma - \beta\alpha^{-1}\beta\right)Q$ acting on $\ker \alpha$.

Passing from the above finite dimensional argument to an infinite dimensional considered in (3.25) is a wild jump. However, this is the sort of situation one encounters typically in adiabatic limit problems in the theory of Quillen metrics [**1**, **7**]. The major difference is that the operators considered in these references are elliptic and self-adjoint, which is not the case here.

We have given enough motivation for studying the operator $P\left(\gamma - \beta\alpha^{-1}\beta\right)P$ in the context of (3.25).

In [**3**, Theorem 3.14], the following result is established.

Theorem 3.10. — *The following identity holds:*

$$P\left(\gamma - \beta\alpha^{-1}\beta\right)P = \frac{1}{2}\square^X. \tag{3.32}$$

Remark 3.11. — Theorem 3.10 gives an argument in favour of the fact that $A^2_{\mathcal{H}^c}$ is a deformation of $\square^X/4$.

In joint work with Lebeau [**8**], the hard analysis involved in the convergence as $b \to 0$ of the resolvent of $2\mathfrak{A}'^2_{\mathcal{H}^c}$ to the resolvent of $\frac{1}{2}\square^X$ is carried out in detail. The convergence is also valid for the traces of the corresponding heat kernels, as well as for the spectrum of these operators.

3.8. An interpolation property: the limit $b \to +\infty$ and the geodesic flow. — We still take $\mathcal{H} = |p|^2/2, c = 1/b^2$. Let r_b be the map $(x,p) \to (x,bp)$. Using (3.22), we get

$$r^*_{b^2}2\mathfrak{A}^2_{\mathcal{H}^c}r^*_{1/b^2} = \frac{1}{2}|p|^2 - L_{Y^{\mathcal{H}}} + \mathcal{O}(1/b). \tag{3.33}$$

The dynamics associated to the right-hand side of (3.33) is the geodesic flow.

From (3.33), we deduce that when $b \to +\infty$, the trace of an operator like $\exp\left(-A^2_{\mathcal{H}^c}\right)$ should localize around closed geodesics.

4. The hypoelliptic Dirac operator

The purpose of this section is to briefly develop the construction of the hypoelliptic Dirac operator obtained in [**5**] in the case of Kähler manifolds. This deformation of the classical elliptic Dirac operator is not a generalization of what was done in section 3.

This section is organized as follows. In subsection 4.1, we discuss another method to obtain a Laplacian which looks like the hypoelliptic Laplacian of section 3.

In subsection 4.2, we construct the hypoelliptic Dirac operator, which depends again on a parameter $b > 0$.

In subsection 4.3, by squaring our Dirac operator, we get our new hypoelliptic Laplacian.

In subsection 4.4, we give arguments in favour of the fact that as $b \to 0$, our hypoelliptic Laplacian converges in the proper sense to the classical elliptic Hodge Dolbeault Laplacian of X.

4.1. Another approach to hypoellipticity. — Let $\left(X, g^{TX}\right)$ be a compact Riemannian manifold, let $\mathcal{X}$ be the total space of TX. The generic element of $\mathcal{X}$ will be denoted (x,Y). We will now try to give another approach to the construction of a second order hypoelliptic operator on $\mathcal{X}$.

Let $Y^{\mathcal{H}}$ be the generator of the geodesic flow over $\mathcal{X}$, and let $L_{Y^{\mathcal{H}}}$ be the corresponding Lie derivative operator. Then

$$L_{Y^{\mathcal{H}}} = \left[d^{\mathcal{X}}, i_{Y^{\mathcal{H}}}\right]. \tag{4.1}$$

On the other hand, one would would like to obtain as a square of a Dirac operator an operator $\mathcal{L}$ looking like the sum of a harmonic oscillator in the variable Y and of $\nabla_{Y^{\mathcal{H}}}$, i.e.,

$$\mathcal{L} = \frac{1}{2}\left(-\Delta^V + |Y|^2 - n\right) + \nabla_{Y^{\mathcal{H}}}. \tag{4.2}$$

We still write $d^{\mathcal{X}}$ as in (3.4), i.e.,

$$d^{\mathcal{X}} = d^V + \nabla^{\mathbf{I}} + i_{\widehat{R^{TX}Y}}. \tag{4.3}$$

Equation (4.3) expresses $d^{\mathcal{X}}$ as a superconnection on $\mathbf{I}$ in the sense of Quillen [**18**].

For $\nabla_{Y^{\mathcal{H}}}$ to appear in (4.2), one should think of replacing $d^{\mathcal{X}}$ by $d^{\mathcal{X}} + i_{Y^{\mathcal{H}}}$. However, how to obtain the full operator $\mathcal{L}$ is not clear, not to speak of the possibility of producing a deformation of the classical elliptic Dirac operator or of its square.

4.2. The hypoelliptic Dirac operator. — To explain the construction of the hypoelliptic deformation of the Dirac operator which is carried out in [**5**], we will work in the context of complex Kähler manifolds.

Let $\left(X, g^{TX}\right)$ be a compact complex Kähler manifold of real dimension n. Let TX be the holomorphic tangent bundle to X, and let $T_{\mathbf{R}}X$ be the corresponding real tangent bundle. Let $\left(E, g^E\right)$ be a holomorphic Hermitian vector bundle on X. We denote by ∇^{TX}, ∇^E the holomorphic Hermitian connections on TX, E, and by R^{TX}, R^E their curvatures. Let $\nabla^{\Lambda^\cdot\left(\overline{T^*X}\otimes E\right)}$ be the corresponding connection on $\Lambda^\cdot\left(\overline{T^*X}\right)\otimes E$.

Let $\left(\Omega^{(0,\cdot)}(X,E), \overline{\partial}^X\right)$ be the Dolbeault complex of smooth antiholomorphic forms on X with coefficients in E. The cohomology of this complex is denoted $H^{(0,\cdot)}(X,E)$.

Let $\langle\rangle$ be the L_2 Hermitian product on $\Omega^{(0,\cdot)}(X,E)$ which is associated with g^{TX}, g^E. Let $\overline{\partial}^{X*}$ be the formal adjoint of $\overline{\partial}^X$ with respect to $\langle\rangle$. Set

$$D^X = \overline{\partial}^X + \overline{\partial}^{X*}. \tag{4.4}$$

If $u \in TX$, let $u^* \in \overline{T^*X}$ corresponding to u by g^{TX}. Recall that $\Lambda^\cdot\left(\overline{T^*X}\right)$ is a $\left(T_{\mathbf{R}}X, g^{T_{\mathbf{R}}X}\right)$ Clifford algebra. Namely if $u \in TX$, set

$$c(u) = \sqrt{2}u^*\wedge, \qquad c(\overline{u}) = -\sqrt{2}i_{\overline{u}}. \tag{4.5}$$

We extend the definition of c to $T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C}$ by linearity. If $U, V \in T_{\mathbf{R}}X$, then

$$c(U)c(V) + c(V)c(U) = -2\langle U, V\rangle. \tag{4.6}$$

By [**13**], $\sqrt{2}D^X$ is a Dirac operator. Namely, if $e_1, \dots, e_n$ is an orthonormal basis of $T_{\mathbf{R}}X$, then

$$\sqrt{2}D^X = c(e_i)\nabla_{e_i}^{\Lambda^{\cdot}(\overline{T^*X})\otimes E}. \tag{4.7}$$

Let $\pi : \mathcal{X} \to X$ be the total space of TX, with fibre $\widehat{TX}$. The hat on $\widehat{TX}$ will allow us to distinguish the fibre $\widehat{TX}$ from the tangent bundle to X. Then $\mathcal{X}$ is a also a complex manifold. Let $i : X \to \mathcal{X}$ be the embedding of X into $\mathcal{X}$ as the zero section of $\widehat{TX}$. Using the connection ∇^{TX}, we have the identification of smooth vector bundles,

$$T\mathcal{X} \simeq \pi^*\left(TX \oplus \widehat{TX}\right). \tag{4.8}$$

From (4.8), we get the smooth identification,

$$\Lambda^{\cdot}\left(\overline{T^*\mathcal{X}}\right) = \pi^*\left(\Lambda^{\cdot}\left(\overline{T^*X}\right)\widehat{\otimes}\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)\right). \tag{4.9}$$

Set

$$F = \pi^*\left(\Lambda^{\cdot}(T^*X)\otimes E\right). \tag{4.10}$$

In (4.10), $\Lambda^{\cdot}(T^*X)$ is the holomorphic exterior algebra of the base X. However, since TX and $\widehat{TX}$ are isomorphic, $\Lambda^{\cdot}(T^*X)$ will also be considered as the holomorphic exterior algebra of the fibre $\widehat{TX}$.

Let $\left(\Omega^{(0,\cdot)}(\mathcal{X}, F), \overline{\partial}^{\mathcal{X}}\right)$ be the Dolbeault complex of smooth antiholomorphic forms on $\mathcal{X}$ with coefficients in F.

Let $\mathbf{I}$ be the vector bundle on X of the smooth sections of $\pi^*\left(\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)\otimes E\right)$ along the fibre $\widehat{TX}$. By proceeding as in (3.4) and using (4.9), we get

$$\overline{\partial}^{\mathcal{X}} = \nabla^{\mathbf{I}\prime\prime} + \overline{\partial}^{V}. \tag{4.11}$$

In (4.11), $\overline{\partial}^V$ is the Dolbeault operator along the fibre $\widehat{TX}$, and $\nabla^{\mathbf{I}\prime\prime}$ is the horizontal part of $\overline{\partial}^{\mathcal{X}}$. Note that contrary to what happens in (3.4), there is no extra term in (4.11). Writing $\overline{\partial}^{\mathcal{X}}$ in the form (4.11) emphasizes the fact that $\overline{\partial}^{\mathcal{X}}$ can also be viewed as a holomorphic superconnection on $\mathbf{I}$.

Let y be the tautological holomorphic section of $\pi^*\widehat{TX}$ over $\mathcal{X}$, and let $Y = y + \overline{y}$ be the corresponding section of $\pi^*\widehat{T_{\mathbf{R}}X}$. Of course $\widehat{TX}$ and TX are canonically isomorphic. In particular the operator i_y acts on $\pi^*\Lambda^{\cdot}(T^*X)$. The Koszul complex $\left(\mathcal{O}_{\mathcal{X}}\pi^*\Lambda^{\cdot}(T^*X), i_y\right)$ provides a resolution of the sheaf $i_*\mathcal{O}_X$. More generally the Koszul complex $\left(\mathcal{O}_{\mathcal{X}}(F), i_y\right)$ provides a resolution of $i_*\mathcal{O}_X(E)$.

Observe that

$$\left(\overline{\partial}^{\mathcal{X}} + i_y\right)^2 = 0. \tag{4.12}$$

Equation (4.12) reflects the fact that $\left(\Omega^{\cdot}(\mathcal{X}), \overline{\partial}^{\mathcal{X}} + i_y\right)$ is the Dolbeault resolution of the Koszul complex we just considered.

For $b > 0$, set

$$A''_b = \overline{\partial}^{\mathcal{X}} + i_y/b^2. \tag{4.13}$$

By (4.11), (4.13), we get

$$A''_b = \nabla^{\mathbf{I}''} + \overline{\partial}^V + i_y/b^2. \tag{4.14}$$

Then A''_b can be viewed as an operator acting on $\Omega^{(0,\cdot)}(\mathcal{X}, F)$. By (4.12),

$$A''^2_b = 0. \tag{4.15}$$

Let $\overline{\partial}^{V*}$ be the fibrewise formal adjoint of $\overline{\partial}^V$. Now we will take the 'adjoint' of A''_b partly in the sense of superconnections. Namely set

$$A'_b = \nabla^{\mathbf{I}'} + \overline{\partial}^{V*} + i_{\overline{y}}/b^2. \tag{4.16}$$

Then A'_b also acts on $\Omega^{(0,\cdot)}(\mathcal{X}, F)$. Indeed $\nabla^{\mathbf{I}'}$ increases the degree in $\Lambda^{\cdot}(T^*X)$ by 1, and $i_{\overline{y}}$ decreases the degree in $\Lambda^{\cdot}\left(\overline{T^*X}\right)$ by 1. Moreover,

$$A'^2_b = 0. \tag{4.17}$$

Set

$$A_b = A''_b + A'_b. \tag{4.18}$$

When making instead $y = 0$, we will denote by A'', A', A the corresponding operators. In particular, when identifying $Y \in \widehat{T_{\mathbf{R}}X}$ to the corresponding section of $T_{\mathbf{R}}X$, we get

$$A_b = A + i_Y/b^2. \tag{4.19}$$

Also A is a superconnection on $\mathbf{I}$.

Observe that the principal symbol of A or of A_b is exactly $i\xi^H \wedge +ic\left(\xi^V\right)/\sqrt{2}$, where ξ^H, ξ^V are the horizontal and vertical components of $\xi \in T^*_{\mathbf{R}}\mathcal{X}$. In particular the principal symbol of A^2_b is just $\left|\xi^V\right|^2/2$. Adding i_Y has no effect on the principle symbol of A^2_b. However,

$$A^2_b = A^2 + \left[A, i_Y/b^2\right]. \tag{4.20}$$

Now in $[A, i_Y]$ appears the critical operator $\nabla^{\mathbf{I}}_Y$, which makes the operator A^2_b hypoelliptic.

The operator A^2_b is still not the right one, since it does not contain a positive multiple of $|Y|^2/2$, which is needed to produce a harmonic oscillator in the fibre direction.

So we slightly modify the above construction. Let ω^X be the Kähler form associated with the metric g^{TX}. If J is the complex structure of $T_{\mathbf{R}}X$, if $U, V \in T_{\mathbf{R}}X$, then

$$\omega^X(U, V) = \langle U, JV\rangle. \tag{4.21}$$

We will view ω^X as a section of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\overline{T^*X}\right)$.

Put

$$B_b'' = A_b'', \qquad B_b' = e^{i\omega^X} A_b' e^{-i\omega^X}, \qquad B_b = B_b'' + B_b'. \tag{4.22}$$

Since ω^X is closed, we get the formula,

$$B_b' = A_b' + \overline{y}^* \wedge /b^2. \tag{4.23}$$

Of course, we still have

$$B_b''^2 = 0, B_b'^2 = 0. \tag{4.24}$$

However, the effect of the addition of $\overline{y}^* \wedge /b^2$ in (4.23) is precisely to produce the desired $|Y|^2/2b^4$ in B_b^2. We will give a formula for a conjugate of the operator B_b^2.

4.3. The hypoelliptic Laplacian in Dolbeault theory. — If $\widehat{U} \in \widehat{T_{\mathbf{R}}X}$, we define $c\left(\widehat{U}\right)$ as in (4.5). Then $c\left(\widehat{U}\right)$ acts on $\Lambda^{\cdot}\left(\widehat{T^*X}\right)$. If $u \in TX$, set

$$\widehat{c}'(u) = \sqrt{2} i_u, \qquad \widehat{c}'(\overline{u}) = \sqrt{2}\left(\overline{u}^* \wedge + i_{\overline{u}}\right). \tag{4.25}$$

We extend $\widehat{c}'$ by linearity into a linear map from $T_{\mathbf{R}}X \otimes_{\mathbf{R}} \mathbf{C}$ into $\operatorname{End}\left(\Lambda^{\cdot}\left(T_{\mathbf{R}}^*X\right)\right) \otimes_{\mathbf{R}} \mathbf{C}$, which is such that if $U, V \in T_{\mathbf{R}}X$,

$$\widehat{c}'(U)\widehat{c}'(V) + \widehat{c}'(V)\widehat{c}'(U) = 2\langle U, V\rangle. \tag{4.26}$$

Of course, if $\widehat{U} \in \widehat{T_{\mathbf{R}}X}, V \in T_{\mathbf{R}}X$,

$$\left[c\left(\widehat{U}\right), \widehat{c}'(V)\right] = 0. \tag{4.27}$$

The curvature R^E is a section of $\Lambda^2\left(T_{\mathbf{R}}^*X\right) \otimes \operatorname{End}(E)$, and R^{TX} a section of $\Lambda^2\left(T_{\mathbf{R}}^*X\right) \otimes \operatorname{End}(TX)$. The following result was established in [**5**, Theorem 3.8].

Theorem 4.1. — *The following identity holds:*

$$\begin{aligned} A_b^2 = \frac{1}{2}\left(-\Delta^V + \frac{|Y|^2}{b^4} + \frac{1}{b^2} c(\widehat{e}_i)\widehat{c}'(e_i)\right) - \nabla_{\widehat{R^{TX}Y}} + \frac{1}{4}\left\langle R^{TX} e_i, e_j\right\rangle c(\widehat{e}_i) c(\widehat{e}_j) \\ + \frac{1}{2}\operatorname{Tr}\left[R^{TX}\right] + \frac{1}{b^2}\nabla_Y^F + R^E. \end{aligned} \tag{4.28}$$

Let L be the operator $\alpha \to \omega^X \wedge \alpha$, and let Λ be its adjoint as in subsection 2.2. Set

$$C_b = \exp(i\Lambda) B_b \exp(-i\Lambda). \tag{4.29}$$

The operators $\nabla^{\mathbf{I}''}, \nabla^{\mathbf{I}'}$ increase the horizontal degree by 1. Let $\nabla^{\mathbf{I}''*}, \nabla^{\mathbf{I}'*}$ be their formal adjoints in the classical L_2 sense. These operators decrease the horizontal degree by 1.

Now we have the result of [**5**, Theorem 3.6].

Theorem 4.2. — *The following identity holds:*

$$C_b = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*} + \overline{\partial}^V + i_y/b^2 + \overline{\partial}^{V*} + \overline{y}^* \wedge /b^2. \tag{4.30}$$

Remark 4.3. — Using (4.30), the fact that the horizontal part of the principal symbol of C_b^2 is nilpotent follows from well-known identities in Kähler geometry.

4.4. The limit as $b \to 0$. — Let K_b be the map $s(x, Y) \to s(x, bY)$. Set

$$D_b = K_b C_b K_b^{-1}. \tag{4.31}$$

By (4.30), we get

$$D_b = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*} + \frac{1}{b}\left(\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge\right). \tag{4.32}$$

Let $\widehat{\omega}^{\mathcal{X},V}$ be the Kähler form of the fibre $\widehat{TX}$. Since $\Lambda^{\cdot}\left(\widehat{T^*X}\right)$ has been identified to $\Lambda^{\cdot}(T^*X)$, $\widehat{\omega}^{\mathcal{X},V}$ will be viewed as a section of $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)$.

By [**2**, Proposition 1.5 and Theorem 1.6], the fibrewise kernel of the operator $\overline{\partial}^V + i_y + \overline{\partial}^{V*} + \overline{y}^* \wedge$ is 1-dimensional and spanned by $\beta = \exp\left(i\widehat{\omega}^{\mathcal{X},V} - |Y|^2/2\right)$.

We will embed $\Omega^{(0,\cdot)}(X, E)$ into $\Omega^{(0,\cdot)}(\mathcal{X}, F)$ by the embedding $\alpha \to \pi^*\alpha \wedge \beta$. Let P be the orthogonal projection operator from $\Omega^{(0,\cdot)}(\mathcal{X}, F)$ on $\Omega^{(0,\cdot)}(X, E)$.

Set

$$E = \nabla^{\mathbf{I}''} + \nabla^{\mathbf{I}'} + \nabla^{\mathbf{I}''*} - \nabla^{\mathbf{I}'*}. \tag{4.33}$$

Let us pretend for the moment $\Omega^{(0,\cdot)}(\mathcal{X}, F)$ to be finite dimensional. Elementary linear algebra shows that under the proper conditions, as $b \to 0$,

$$\left(\lambda - D_b^{-1}\right)^{-1} \to P\left(\lambda - PEP\right)^{-1} P. \tag{4.34}$$

The critical result which was established in [**5**, Theorem 3.12] is as follows.

Theorem 4.4. — *The following identity holds:*

$$PEP = \overline{\partial}^X + \overline{\partial}^{X*}. \tag{4.35}$$

Proof. — Let $N^{\Lambda^{\cdot}(T^*X)}, N^{\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)}$ be the number operators of $\Lambda^{\cdot}(T^*X), \Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)$. Set

$$\mathcal{N} = N^{\Lambda^{\cdot}(T^*X)} - N^{\Lambda^{\cdot}\left(\overline{\widehat{T^*X}}\right)}. \tag{4.36}$$

Then $\Omega^{(0,\cdot)}(X, E)$ is of degree 0 with respect to $\mathcal{N}$. The operators $\nabla^{\mathbf{I}'}, \nabla^{\mathbf{I}'*}$ are of degree $+1$ and -1 with respect to $\mathcal{N}$, so that they disappear under the compression by P. The proof of our theorem is completed. □

Remark 4.5. — Theorem 4.4 is the main algebraic argument which justifies that when $b \to 0$, the operator D_b is indeed a deformation of the Dirac operator D^X. This result is intimately related with theorem 3.10. Indeed as explained in [**3**, Proposition 2.41] there is a corresponding version of theorem 4.4 in the context of de Rham theory. Conversely, by squaring (4.32), we see that the operator D_b^2 can be written in the form (3.25). In [**5**, Theorem 1.14], an analogue of Theorem 3.10 is proved. One of the proofs consists simply into squaring (4.32) and identifying properly the various terms.

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