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SLOPE FILTRATIONS FOR RELATIVE FROBENIUS

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Abstract. — The slope filtration theorem gives a partial analogue of the eigenspace decomposition of a linear transformation, for a Frobenius-semilinear endomorphism of a finite free module over the Robba ring (the ring of germs of rigid analytic functions on an unspecified open annulus of outer radius 1) over a discretely valued field. In this paper, we give a third-generation proof of this theorem, which both introduces some new simplifications (particularly the use of faithfully flat descent, to recover the theorem from a classification theorem of Dieudonné-Manin type) and extends the result to allow an arbitrary action on coefficients (previously the action on coefficients had to itself be a lift of an absolute Frobenius). This extension is relevant to a study of (ϕ, Γ)-modules associated to families of p-adic Galois representations, as initiated by Berger and Colmez.

Résumé **(Filtrations de pentes pour le Frobenius relatif).** — Le théorème de filtration par les pentes donne un analogue partiel de la décomposition en espaces propres d'une transformation linéaire, pour un endomorphisme semilinéaire (pour Frobenius) d'un module libre de type fini sur l'anneau de Robba (l'anneau des germes de fonctions analytiques rigides sur une couronne ouverte non précisée de rayon externe 1) sur un corps à valuation discrète. Dans cet article, nous donnons une preuve de troisième génération de ce théorème, qui introduit quelques simplifications nouvelles (en particulier, l'emploi de la descente fidèlement plate, pour obtenir le théorème à partir d'un théorème de classification de type Dieudonné-Manin). Nous étendons aussi le résultat pour permettre une action arbitraire sur les coefficients (auparavant, cette action devait être un relèvement d'un Frobenius absolu). Cette extension est utile pour l'étude des (ϕ, Γ)-modules associés à des familles de représentations galoisiennes p-adiques; Berger et Colmez ont commencé cette étude.

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Introduction

This paper describes a third-generation proof of the slope filtration theorem for Frobenius modules over the Robba ring (Theorem 1.7.1 herein). This proof is more expedient than what one finds in our original paper [**21**] or its sequel [**22**]. In addition, we generalize the slope filtration theorem by allowing for ring endomorphisms which do not act as Frobenius lifts on scalars, only on the series variable. This is intended as a prelude to a theory of Frobenius modules in families; we will not develop such a theory here, but see the next section for reasons one might want to do so, from the realm of p-adic Hodge theory. (Note that [**22**] itself generalizes [**21**] in a different direction, replacing the power series rings by somewhat more general objects; we do not treat that generalization here.)

For an alternate perspective on this theorem and some related results in p-adic differential equations and p-adic Hodge theory, we also recommend Colmez's Bourbaki notes [**11**].

0.1. Context. — The slope filtration theorem [**21**, Theorem 6.10] (also exposed in [**22**]) gives a partial classification of Frobenius-semilinear transformations on finite free modules over the Robba ring (a certain ring of univariate formal Laurent series with p-adic coefficients). It is loosely analogous to the eigenspace decomposition of a linear transformation in ordinary linear algebra; it is also closely related to Manin's classification of rational Dieudonné modules.

The slope filtration theorem was originally introduced in the context of Berthelot's rigid cohomology, a p-adic Weil cohomology for varieties in characteristic p. There, one obtains a analogue of the ℓ-adic local monodromy theorem, originally conjectured by Crew [**14**]; this analogue can be used to establish various structural results such as finiteness of cohomology [**23**] and purity in the sense of Deligne [**24**].

The effect of the slope filtration theorem on p-adic Hodge theory has perhaps been even more acute: it enables one to study p-adic Galois representations via their associated (ϕ, Γ)-modules over the Robba ring. This point of view has been put forth chiefly by Berger with striking consequences: he has proved Fontaine's conjecture that de Rham representations are potentially semistable [**4**], and given an alternate proof of the Colmez-Fontaine theorem on admissibility of filtered (ϕ, N)-modules [**5**]. (A useful variant of the latter argument has been given by Kisin [**27**].) More recently Colmez [**13**] used this viewpoint to define a class of *trianguline representations* of a p-adic Galois group; these play an important role in the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [**12**]. The trianguline representations are also important in the theory of p-adic modular forms, as most local Galois representations attached to overconvergent p-adic modular forms (namely, those of noncritical slope) are trianguline. The p-adic local Langlands correspondence in turn has touched off a flurry of activity, which this introduction is not the right place to summarize; we merely note the resolution of Serre's conjecture by Khare-Wintenberger [**25, 26**], and progress on the Fontaine-Mazur conjecture by Kisin [**28**] and Emerton (in preparation).

In both rigid cohomology and p-adic Hodge theory, one is led to study Frobenius modules in families, i.e., over the Robba ring with coefficients not in a p-adic field but in, say, an affinoid algebra. In either situation, the first step to studying Frobenius modules in families is to pass from a family to a generic point, which on rings amounts to replacing an integral affinoid algebra with a complete field containing it. In the rigid cohomology version of this argument, the resulting field is itself acted on by Frobenius, so the slope filtration theorem as presented in [**21, 22**] is immediately applicable; indeed, the key technique in [**23**] is to extend the application of the local monodromy theorem on the generic point to a large enough subspace of the base space. However, in the p-adic Hodge theory version, one might like to allow "Frobenius" to act in some fashion on the base of the family other than simply a lift of the p-power map; in fact, one natural situation is where the base is not moved at all.

One goal of this paper, and in fact the principal reason for its existence, is to generalize the slope filtration theorem to modules over the Robba ring with an action of a "relative Frobenius", which may do whatever one wishes to coefficients as long as it acts like a Frobenius lift on the series parameter. We hope this will lead to some study of p-adic Hodge theory in families; some of the corresponding analysis in equal characteristics has been initiated by Hartl [**17**], using an equal-characteristic analogue of the slope filtration theorem based on the work of Hartl and Pink [**18**]. In mixed characteristics, Hartl [**16**] has set up part of a corresponding theory, which addresses a conjecture of Rapoport and Zink [**40**] from their work on period spaces for p-divisible groups; results are presently quite fragmentary, but a good theory of (ϕ, Γ)-modules in families may help. Another potential application would be to analysis of the local geometry of the Coleman-Mazur eigencurve [**10**], which parametrizes the Galois representations attached to certain p-adic modular forms, or of higher-dimensional "eigenvarieties" associated to automorphic representations on groups besides GL_2. An initial step in this direction has already been taken by Bellaïche-Chenevier [**3**], who study deformations of trianguline representations; however, this involves only a zero-dimensional base, so they can already apply the usual slope filtration theory after a restriction of scalars. For other questions, e.g., properness, one would want to consider positive-dimensional bases like a punctured disc. In this direction, Berger and Colmez have introduced a theory of étale (ϕ, Γ)-modules associated to p-adic Galois representations in families [**6**], which relativizes some of the results of Cherbonnier-Colmez [**9**] and Berger [**5**] for a single p-adic Galois representation.

0.2. About the results. — For the sake of introduction, we give here a very brief description of what the original slope filtration theorem says, how the main result of this paper extends it, and what novelties in the argument are introduced in this paper. Start with a complete discretely valued field K of mixed characteristics $(0,p)$. Let $\mathscr{R}$ be the ring of formal Laurent series $\sum_{n\in\mathbb{Z}} c_n u^n$ convergent on some annulus with outer radius 1 (but whose inner radius may depend on which series is being considered). Let $\phi_K : K \to K$ be an endomorphism lifting the absolute q-power Frobenius on the residue field of K, for some power q of p, and define a map $\phi : \mathscr{R} \to \mathscr{R}$ by the formula $\phi(\sum c_n u^n) = \sum \phi_K(c_n)\phi(u)^n$, where $\phi(u) - u^q$ has all coefficients of norm less than 1. Let M be a finite free $\mathscr{R}$-module equipped with a ϕ-semilinear map $F : M \to M$ which takes any basis of M to another basis of M (it is enough to check for a single basis). Then [**21**, Theorem 6.10] asserts that M admits an exhaustive filtration whose successive quotients are each pure of some slope (i.e., some power of F times some scalar acts on some basis via an invertible matrix over the subring of $\mathscr{R}$ of series with *integral* coefficients), and the slopes increase as you go up the filtration; moreover, those requirements uniquely characterize the filtration.

As noted earlier, the slope filtration should be thought of as analogous to what one might get from a linear transformation over K by grouping eigenspaces, interpreting the slope of an eigenspace as the valuation of its eigenvalue. One can in fact deduce an analogous such result for semilinear transformations over K, which also follows from the Dieudonné-Manin classification theorem. One might then expect that the slope filtration can be generalized so as to allow *any* isometric action on K, not just a Frobenius lift; that is what is established in this paper (Theorem 1.7.1).

As promised earlier in this introduction, one happy side effect of this generalization is the introduction of some technical simplifications. We give a development of the theory of slopes which does not depend on already having established the Dieudonné-Manin-style classification; this follows up on a suggestion made in [**22**]. We give a much simplified version of the descent argument that deduces the filtration theorem from the DM classification, based on the idea of replacing the Galois descent used previously with faithfully flat descent; this avoids the use of comparison between generic and special Newton polygons, and of some intricate approximation arguments. (In particular, there is no longer any need to deal with finite extensions of the Robba ring, which allows for some notational and expository simplifications.) That substitution creates some flexibility in what we may take as the "extended Robba ring" for the DM classification; here we use a ring made from generalized power series, some of whose properties are a bit more transparent than for the corresponding "big rings" in [**21**] and [**22**].

0.3. Structure of the paper. — The structure of this paper is a bit unusual, as we have attempted to make the paper more friendly to the novice reader by fronting some of the key assertions and pushing back more technical aspects. (This assertion applies both to the paper as a whole, and to Sections 2 and 3 individually.) The consequence is that the logical structure is a bit loopy: results are stated, and sometimes used, before having been proved. However, we hope that it is not too hard to see that there are indeed no vicious circles in the reasoning.

In Section 1, we introduce the Robba ring, the category of ϕ-modules, the notions of degree and slope, the subcategories of pure ϕ-modules of various slopes, and the statement of the filtration theorem.

In Section 2, we introduce an extended Robba ring (whose elements are modeled on Hahn-Mal'cev-Neumann generalized power series rather than ordinary power series), state a classification theorem for ϕ-modules over the extended Robba ring, then perform the calculations required to prove this theorem.

In Section 3, we deduce the slope filtration theorem from the classification theorem over the extended Robba ring. The key tool here is an invocation of faithfully flat descent for modules.

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1. Statement of the filtration theorem

1.1. The Robba ring

Definition 1.1.1. — Let K be a field complete for a discrete valuation, with residue field k; let $\mathfrak{o}_K$ denote the valuation subring of K and let $\mathfrak{m}_K$ denote the maximal ideal of $\mathfrak{o}_K$. (We need not make any restriction on the characteristics of K, k.) Write $|\cdot|$ for some fixed norm corresponding to the valuation (the normalization does not matter). For $r > 0$, let $\mathscr{R}^r$ be the ring of rigid analytic functions on the annulus $e^{-r} \leq |t| < 1$ (these are just Laurent series in the variable t convergent on this region), and let $\mathscr{R}$ be the union of the $\mathscr{R}^r$. The ring $\mathscr{R}$ is called the *Robba ring* over K. It follows from the work of Lazard [**29**] that $\mathscr{R}$ is a *Bézout domain*, that is, an integral domain in which every finitely generated ideal is principal.

Remark 1.1.2. — Any Bézout domain R enjoys a number of nice properties generalizing properties of principal ideal domains, including the following. Some of these are actually properties of *Prüfer domains*, in which every finitely generated ideal is projective; these generalize Dedekind domains to the non-noetherian setting.

- Any finite locally free R-module is free [**21**, Proposition 2.5].
- Any torsion-free R-module is flat; this holds for any Prüfer domain [**8**, VII Proposition 4.2].
- Any finitely presented projective R-module is free [**14**, Proposition 4.8].
- If M is a finite free R-module and N is a submodule of M which is saturated, i.e., $N = M \cap (N \otimes_R \operatorname{Frac} R)$, then N and M/N are both free [**14**, Proposition 4.8], [**21**, Lemma 2.4].

Definition 1.1.3. — Let $\mathscr{R}^{\mathrm{int}}$ be the subring of $\mathscr{R}$ consisting of series with coefficients in $\mathfrak{o}_K$; this ring is a discrete valuation ring with residue field $k((t))$, which is not complete but is henselian [21, Lemma 3.9]. Let $\mathscr{R}^{\mathrm{bd}}$ be the subring of $\mathscr{R}$ consisting of series with bounded coefficients; it is the fraction field of $\mathscr{R}^{\mathrm{int}}$.

Remark 1.1.4. — Note that for $x \in \mathscr{R}$, one has $x \in \mathscr{R}^{\mathrm{int}}$ if and only if there exists an integer n such that the function $t^n x$ is bounded by 1 on some annulus $e^{-r} \leq |t| < 1$.

Remark 1.1.5. — Lazard's work [**29**] includes a theory of Newton polygons for elements of $\mathscr{R}$, using which one can read off numerous structural properties. One key example is that the units in $\mathscr{R}$ are precisely the nonzero elements of $\mathscr{R}^{\mathrm{bd}}$ [**21**, Corollary 3.23].

Remark 1.1.6. — One can also define the Robba ring even if the valuation on K is not discrete, but its properties are very different. For instance, $\mathscr{R}^{\mathrm{bd}}$ is no longer the fraction field of $\mathscr{R}^{\mathrm{int}}$. This makes even the formulation of a slope theory over such K, let alone any proofs, somewhat more delicate than the approach we take here.

1.2. Frobenius lifts on the Robba ring

Definition 1.2.1. — Fix an integer $q > 1$. (To see why we forbid $q = 1$, see Remark 1.7.9.) A *relative (q-power) Frobenius lift* on the Robba ring is a homomorphism $\phi : \mathscr{R} \to \mathscr{R}$ of the form $\sum_i c_i t^i \mapsto \sum_i \phi_K(c_i) u^i$, where ϕ_K is an isometric field endomorphism of K and $u \in \mathscr{R}^{\mathrm{int}}$ is such that $u - t^q$ is in the maximal ideal of $\mathscr{R}^{\mathrm{int}}$. If k has characteristic $p > 0$ and q is a power of p, we define an *absolute (q-power) Frobenius lift* as a relative Frobenius lift in which ϕ_K is itself a q-power Frobenius lift.

Remark 1.2.2. — The treatments in [**21, 22**] only allow absolute Frobenius lifts, and the approaches do not carry over easily to the general case because of the use of Galois descent at some key moments. See the introduction for discussion of why one needs the relative case.

Definition 1.2.3. — For $r > 0$, let $|\cdot|_r$ denote the supremum norm on the circle $|t| = e^{-r}$, as applied to elements of $\mathscr{R}^r$; one easily verifies that

$$\left|\sum_{i\in\mathbb{Z}} c_i t^i\right|_r = \sup_i\{|c_i|e^{-ri}\}.$$

We extend the definition to vectors by taking the maximum over entries.

Remark 1.2.4. — Note that for f analytic on the entire open unit disc (i.e., represented by an ordinary power series rather than a Laurent series), we have $|f|_r \leq |f|_s$ whenever $0 < s \leq r$; in other words, the supremum of f over the entire disc $|t| \leq e^{-s}$ occurs on the circle $|t| = e^{-s}$. In fancier language, the circle $|t| = e^{-s}$ is the *Shilov boundary* of the disc $|t| \leq e^{-s}$, as in [**7**, Corollary 2.4.5].

Remark 1.2.5. — Let ϕ be a relative Frobenius lift; then for some $r_0 > 0$, we have $|\phi(t)/t^q - 1|_{r_0/q} < 1$. It follows that for $r \in (0, r_0)$ and $f \in \mathscr{R}^r$, $\phi(f) \in \mathscr{R}^{r/q}$ and $|f|_r = |\phi(f)|_{r/q}$. In geometric terms, ϕ induces a surjective map from the annulus $e^{-r/q} < |t| < 1$ to the annulus $e^{-r} < |t| < 1$. (Compare [**21**, Lemma 3.7].)

The following is both a typical example of how to make calculations on Robba rings and a crucial ingredient in what follows.

Proposition 1.2.6. — *Let ϕ be a relative Frobenius lift, and let A be an $n \times n$ matrix over $\mathscr{R}^{\mathrm{int}}$. Then the map $\mathbf{v} \mapsto \mathbf{v} - A\phi(\mathbf{v})$ on column vectors induces a bijection on $(\mathscr{R}/\mathscr{R}^{\mathrm{bd}})^n$.*

Proof. — The problem is unaffected if we replace $\mathbf{v}, A$ by $t^m\mathbf{v}, (t^m/\phi(t^m))A$, so by Remark 1.1.4, we may reduce to the case where the entries of A are bounded by 1 on some annulus with outer radius 1. Choose r_0 as in Remark 1.2.5. To check injectivity, we must argue that if $\mathbf{w} = \mathbf{v} - A\phi(\mathbf{v})$ is bounded, then so is $\mathbf{v}$. Choose $r \in (0, r_0)$ such that $A, \mathbf{w}, \phi(\mathbf{v})$ have entries which are defined on the annulus $e^{-r} \leq |t| < 1$, and the entries of A are bounded by 1 there. Choose $c > 0$ such that $|\mathbf{w}|_s \leq c$ for $0 < s \leq r$, and such that $|\phi(\mathbf{v})|_s \leq c$ for $r/q \leq s \leq r$. (The latter is possible because every analytic function on a closed annulus is bounded.) Then $|\mathbf{v}|_s = |\mathbf{w} + A\phi(\mathbf{v})|_s \leq c$ for $r/q \leq s \leq r$, so $|\phi(\mathbf{v})|_s \leq c$ for $r/q^2 \leq s \leq r/q$. Repeating the argument, we see that $|\mathbf{v}|_s \leq c$ for $0 < s \leq r$, proving the claim. (Compare [**22**, Lemma 3.3.3].)

To check surjectivity, take $\mathbf{w} \in \mathscr{R}^n$. Choose $r \in (0, r_0)$ such that $A, \mathbf{w}$ have entries which are defined on the annulus $e^{-r} \leq |t| < 1$, and the entries of A are bounded by 1 there. Define the sequence $\{\mathbf{w}_l\}_{l=0}^{\infty}$ as follows. Start with $\mathbf{w}_0 = \mathbf{w}$. Given $\mathbf{w}_l$, write $\mathbf{w}_l = \sum_{i\in\mathbb{Z}} \mathbf{w}_{l,i}t^i$, put $\mathbf{w}_l^+ = \sum_{i>0} \mathbf{w}_{l,i}t^i$ and $\mathbf{w}_l^- = \mathbf{w}_l - \mathbf{w}_l^+$, and put $\mathbf{w}_{l+1} = A\phi(\mathbf{w}_l^+)$. Since the entries of $t^{-1}\mathbf{w}_l^+$ are analytic on the entire open unit disc, by Remark 1.2.4 we have

$$|\mathbf{w}_l^+|_r \leq e^{-r+r/q}|\mathbf{w}_l^+|_{r/q} \leq e^{-r+r/q}|\mathbf{w}_l|_{r/q};$$

consequently, $|\mathbf{w}_{l+1}|_{r/q} \leq e^{-r+r/q}|\mathbf{w}_l|_{r/q}$. Thus the sequence $\mathbf{w}_l^+$ converges to zero under $|\cdot|_{r/q}$, and hence also under $|\cdot|_s$ for $s \geq r/q$ by Remark 1.2.4. On the other hand, for $0 < s \leq r/q$, applying Remark 1.2.4 after substituting $t \mapsto t^{-1}$ gives

$$|\mathbf{w}_l^-|_s \leq |\mathbf{w}_l^-|_{r/q} \leq |\mathbf{w}_l|_{r/q}.$$

Now set $\mathbf{v} = \sum_{l=0}^{\infty} \mathbf{w}_l^+$; then $\mathbf{v}$ has entries analytic on the closed disc of radius $e^{-r/q}$, and $\mathbf{w} - \mathbf{v} + A\phi(\mathbf{v}) = \sum_{l=0}^{\infty} \mathbf{w}_l^-$ is bounded on $e^{-r/q} \leq |t| < 1$. Since $\phi(\mathbf{v})$ is analytic on the closed disc of radius e^{-r/q^2}, we can write $\mathbf{v} = \mathbf{w} + A\phi(\mathbf{v}) - \sum_{l=0}^{\infty} \mathbf{w}_l^-$ and thus extend $\mathbf{v}$ across the annulus $e^{-r/q} \leq |t| \leq e^{-r/q^2}$; by induction, $\mathbf{v}$ extends to the entire open unit disc. This proves the desired surjectivity. □

One can also prove the following, as in [**22**, Lemma 5.4.1].

Proposition 1.2.7. — *Let $\mathscr{E}$ denote the $\mathfrak{m}_K$-adic completion of $\mathscr{R}^{\mathrm{bd}}$. Let ϕ be a relative Frobenius lift on $\mathscr{R}$, and let A be an $n \times n$ matrix over $\mathscr{R}^{\mathrm{int}}$. If $\mathbf{v} \in \mathscr{E}^n$ is a column vector such that $A\mathbf{v} = \phi(\mathbf{v})$, then $\mathbf{v} \in (\mathscr{R}^{\mathrm{bd}})^n$.*

Proof. — This will follow later from Proposition 2.5.8; we will not use it in the interim. □

Remark 1.2.8. — In the case where A is invertible, Proposition 1.2.7 was proved independently by Cherbonnier (unpublished, but see [**9**, Théorème III.1.1]) and Tsuzuki [**41**, Proposition 4.1.1]. Tsuzuki's underlying argument can be used even when A is not invertible; see [**41**, Proposition 2.2.2].

Remark 1.2.9. — It should be possible to carry everything in this paper over to the case where one only assumes $\phi(t) = \sum_i c_i t^i$ such that $c_q \in \mathfrak{o}_K^*$ and $c_i \in \mathfrak{m}_K$ for $i < q$. (For instance, in the theory of (ϕ, Γ)-modules, the composition of the usual ϕ with any nontrivial $\gamma \in \Gamma$ would have this property.) The proof of Proposition 1.2.6 extends to this setting, but the embedding of $\mathscr{R}$ into the extended Robba ring $\tilde{\mathscr{R}}$ of Section 2 must be modified, as accordingly must the projection construction of Section 3.

1.3. ϕ-modules

Definition 1.3.1. — Define a *ϕ-(ring/field)* to be a ring/field R equipped with an endomorphism ϕ; we say R is *inversive* if ϕ is bijective. Define a *(strict) ϕ-module* over a ϕ-ring R to be a finite free R-module M equipped with an isomorphism $\phi^*M \to M$, which we also think of as a semilinear ϕ-action on M; the semilinearity means that for $r \in R$ and $m \in M$, $\phi(rm) = \phi(r)\phi(m)$. Note that the category of ϕ-modules admits tensor products, symmetric and exterior powers, and duals.

Remark 1.3.2. — The definition of ϕ-module used here is somewhat more restrictive than one sees in other contexts, hence the optional modifier "strict". For instance, in some cases one allows modules which are projective but not free, or worse. In other cases, one allows the ϕ-action to take kernel and cokernel in some ϕ-stable Serre category of R-modules; we will do this ourselves shortly.

Remark 1.3.3. — It will be convenient for us to describe ϕ-modules in terms of bases and matrices. If M is a ϕ-module and $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a basis of M, we can completely describe the ϕ-action on M by specifying the invertible $n \times n$ matrix A which satisfies $\phi(\mathbf{e}_j) = \sum_i A_{ij}\mathbf{e}_i$. Note that the semilinearity skews conjugation: if $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is another basis and the change of basis matrix U is defined by $\mathbf{e}'_j = \sum_i U_{ij}\mathbf{e}_i$, then the ϕ-action on the new basis is via the matrix $U^{-1}A\phi(U)$.

It is also useful to think of ϕ-modules as modules for a twisted polynomial ring.

Definition 1.3.4. — For R a ϕ-ring, define the *twisted polynomial ring* $R\{T\}$ to be the set of finite formal sums $\sum_{i=0}^{\infty} a_i T^i$ with $a_i \in R$, equipped with the noncommutative ring structure in which $Ta = \phi(a)T$ for $a \in R$. If R is a field, then all left ideals of $R\{T\}$ are principal, by the division algorithm [**36**, Theorem 6]. If R is inversive, one may similarly define a *twisted Laurent polynomial ring* $R\{T^{\pm}\}$.

Remark 1.3.5. — In general, a ϕ-module over R can be interpreted as a left $R\{T\}$-module which is finite free over R, but one must remember the condition that ϕ carries some basis to another basis. On the other hand, if R is inversive, then the data of a ϕ-module over R is equivalent to the data of a left $R\{T^{\pm}\}$-module which is finite free over R. If R is an inversive ϕ-field, then irreducible ϕ-modules over R all have the form $R\{T^{\pm}\}/R\{T^{\pm}\}P$ for some irreducible twisted polynomial P.

When talking about pure slopes, it will be helpful to switch from working with ϕ to working with a power of ϕ; the following definition facilitates this switch.

Definition 1.3.6. — View ϕ-modules as left modules for the twisted polynomial ring $R\{T\}$. For a a positive integer, define the *a-pushforward functor* $[a]_*$ from ϕ-modules to ϕ^a-modules to be the restriction along the inclusion $R\{T^a\} \to R\{T\}$. Define the *a-pullback functor* $[a]^*$ from ϕ^a-modules to ϕ-modules to be the extension of scalars functor

$$M \mapsto R\{T\} \otimes_{R\{T^a\}} M.$$

The following are easily verified (as in [**22**, §3.2]):

- The functors $[a]^*$ and $[a]_*$ form an adjoint pair.
- The functors $[a]_*$ and $[a]^*$ are exact and commute with duals; consequently, $[a]_*$ and $[a]^*$ also form an adjoint pair (i.e., in the other order).
- The functor $[a]_*$ commutes with tensor products over R (but $[a]^*$ does not).

- If M is a ϕ-module and N is a ϕ^a-module, then $M \otimes [a]^* N \cong [a]^*([a]_* M \otimes N)$.
- If M is a ϕ-module, then $\mathrm{rank}([a]_* M) = \mathrm{rank}(M)$.
- If N is a ϕ^a-module, then $\mathrm{rank}([a]^* N) = a\,\mathrm{rank}(N)$.
- If N is a ϕ^a-module, then $[a]_*[a]^* N \cong N \oplus \phi^*(N) \oplus \cdots \oplus (\phi^{a-1})^*(N)$.

Definition 1.3.7. — For M a ϕ-module, put

$$H^0(M) = \ker(\phi - 1 : M \to M), \qquad H^1(M) = \mathrm{coker}(\phi - 1 : M \to M).$$

One easily checks that in the category of ϕ-modules over R,

$$\mathrm{Hom}(M, N) \cong H^0(M^\vee \otimes N), \qquad \mathrm{Ext}(M, N) \cong H^1(M^\vee \otimes N).$$

Moreover, for N a ϕ^a-module, there are natural bijections

$$H^i(N) \cong H^i([a]^* N) \qquad (i = 0, 1).$$

Remark 1.3.8. — Beware that although the pullback/pushforward terminology was inspired by a related construction in [**18**], the two do not agree in that context.

1.4. Degrees, slopes, and stability. — For the rest of this section, we will put ourselves in the following situation. Note that Hypothesis 1.4.1 has a weak form and a strong form; we will assume only the weak form unless otherwise specified. (Thanks to Peter Schneider for suggesting this dichotomy.)

Hypothesis 1.4.1. — Let $R^{\mathrm{int}} \subseteq R^{\mathrm{bd}} \subseteq R$ be inclusions of Bézout domains such that $R^* \subset R^{\mathrm{bd}}$. Let ϕ be an endomorphism of R acting also on R^{bd} and R^{int}. Let $w : R^{\mathrm{bd}} \to \mathbb{Z} \cup \{+\infty\}$ be a ϕ-equivariant valuation such that $w(R^*) = \mathbb{Z}$ and $R^{\mathrm{int}} = \{r \in R^{\mathrm{bd}} : w(r) \geq 0\}$. Suppose in addition that for any $n \times n$ matrix A over R^{int}, the map $\mathbf{v} \mapsto \mathbf{v} - A\phi(\mathbf{v})$ on column vectors induces an injection (weak form) or bijection (strong form) on $(R/R^{\mathrm{bd}})^n$. Note that the analogous hypothesis for ϕ^a also holds, since one can identify the kernel and cokernel of $\mathbf{v} \mapsto \mathbf{v} - A\phi^a(\mathbf{v})$ on $(R/R^{\mathrm{bd}})^n$ with the kernel and cokernel of

$$(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{a-1}) \mapsto (\mathbf{v}_0 - A\phi(\mathbf{v}_{a-1}), \mathbf{v}_1 - \phi(\mathbf{v}_0), \dots, \mathbf{v}_{a-1} - \phi(\mathbf{v}_{a-2}))$$

on $(R/R^{\mathrm{bd}})^{na}$. (Compare the last remark in Definition 1.3.7.)

Example 1.4.2. — For our purposes, the principal example of strong Hypothesis 1.4.1 is as follows. We take $R, R^{\mathrm{bd}}, R^{\mathrm{int}} = \mathscr{R}, \mathscr{R}^{\mathrm{bd}}, \mathscr{R}^{\mathrm{int}}$ to be the Robba ring and variants over K; note that $\mathscr{R}^{\mathrm{bd}} = \mathscr{R}^* \cup \{0\}$. We take ϕ to be a relative Frobenius lift, and w to be the valuation on $\mathscr{R}^{\mathrm{bd}}$ for which $\mathscr{R}^{\mathrm{int}}$ is the valuation subring. The last condition in strong Hypothesis 1.4.1 holds by virtue of Proposition 1.2.6. We will construct a variation of this example, the extended Robba ring $\tilde{\mathscr{R}}$, in Section 2; using the axiomatic approach avoids some repetition.

Example 1.4.3. — Besides the Robba ring, additional examples of strong Hypothesis 1.4.1 are also possible. Here is one from the work of Hartl and Pink [**18**]: take $\mathbb{C}$ to be the completed algebraic closure of a local field of equal characteristic p, R to be the Laurent series over $\mathbb{C}$ convergent on the punctured open unit disc, R^{bd} to be the series which are meromorphic at zero, ϕ to be the map $\sum c_i t^i \mapsto \sum c_i^q t^i$ for q a power of p, and w to be the order of vanishing at 0. See Remark 1.7.6 and Question 1.7.7 for further discussion around this example.

Definition 1.4.4. — For M a ϕ-module over R of rank n, the top exterior power $\wedge^n M$ has rank 1 over R; let $\mathbf{v}$ be a generator, and write $\phi(\mathbf{v}) = r\mathbf{v}$ for some $r \in R^*$. Define the *degree* of M by setting $\deg(M) = w(r)$; note that this does not depend on the choice of the generator by virtue of the ϕ-equivariance of w. If M is nonzero, define the *slope* of M by setting $\mu(M) = \deg(M)/\operatorname{rank}(M)$.

Remark 1.4.5. — Keeping in mind that degree is analogous to the valuation of the determinant (of a linear transformation on a finite dimensional vector space over a valued field), the following formal properties are easily verified (as in [**22**, §3.4]).

- If $0 \to M_1 \to M \to M_2 \to 0$ is exact, then $\deg(M) = \deg(M_1) + \deg(M_2)$; hence $\mu(M)$ is a weighted average of $\mu(M_1)$ and $\mu(M_2)$.
- We have $\mu(M_1 \otimes M_2) = \mu(M_1) + \mu(M_2)$.
- We have $\mu(\wedge^i M) = i\mu(M)$.
- We have $\deg(M^\vee) = -\deg(M)$ and $\mu(M^\vee) = -\mu(M)$.
- If M is a ϕ-module, then $\mu([a]_* M) = a\mu(M)$.
- If N is a ϕ^a-module, then $\mu([a]^* N) = a^{-1}\mu(N)$.

By analogy with the theory of vector bundles, we make the following definition.

Definition 1.4.6. — We say a ϕ-module M is *(module-)semistable* if for any nontrivial ϕ-submodule N, we have $\mu(N) \geq \mu(M)$. We say M is *(module-)stable* if for any proper nontrivial ϕ-submodule N, we have $\mu(N) > \mu(M)$. Note that both properties are preserved under *twisting* (tensoring with a rank 1 module).

Remark 1.4.7. — In [**22**], the terms "stable" and "semistable" were used without the "module" modifier; here we will usually retain the modifier in statements and drop it in proofs. The modifier is meant to emphasize the difference between this notion of semistability and the concept of a "semistable (ϕ, Γ)-module" in the sense of p-adic Hodge theory, meaning one which appears to come from a semistable Galois representation. In the end, over the Robba ring the term "module-semistable" will be shown to be synonymous with "pure", so the terminological overload will cease to be a problem.

Remark 1.4.8. — Those familiar with stability of vector bundles (or with [**18**]) will notice that our definitions differ from the usual convention by an overall minus sign. The sign convention here (which is also the one used in [**21, 22**]) seems to be more consistent with usage in the theory of crystalline cohomology.

Proposition 1.4.9. — *Any ϕ-module of rank 1 is module-stable.*

Proof. — This is a consequence of the assumptions built into weak Hypothesis 1.4.1. Namely, by twisting, it suffices to show that the trivial ϕ-module $M \cong R$ is stable. If N is a nonzero ϕ-submodule of M, we may write $N = Rx$ for some $x \in M$ such that $\lambda = \phi(x)/x \in R^*$, and by definition $\mu(N) = w(\lambda)$. If $\mu(N) \leq 0$, then $x - \lambda^{-1}\phi(x) = 0$ implies $x \in R^{\mathrm{bd}}$ by weak Hypothesis 1.4.1; hence $N = M$ and $\mu(N) = w(\phi(x)) - w(x) = 0$. In other words, $\mu(N) > 0$ unless $N = M$, as desired. □

Corollary 1.4.10. — *If $N \subseteq M$ is an inclusion of ϕ-modules of the same rank, then $\mu(N) \geq \mu(M)$, with equality if and only if $N = M$.*

Proof. — Put $n = \operatorname{rank} M$ and apply Proposition 1.4.9 to the inclusion $\wedge^n N \subseteq \wedge^n M$. □

Lemma 1.4.11. — *Let M be a ϕ-module over R. Then the slopes of nonzero ϕ-submodules of M are bounded below.*

Proof. — We proceed by induction on $\operatorname{rank}(M)$. By Corollary 1.4.10, the slopes of ϕ-submodules of M of full rank are bounded below by $\mu(M)$. If M has no nontrivial ϕ-submodules of lower rank, then there is nothing more to check. Otherwise, let N be a saturated ϕ-submodule of lower rank; then by hypothesis, the slopes of nonzero ϕ-submodules of both N and M/N are bounded below. If now P is any nonzero ϕ-submodule of M, then the sequence

$$0 \to N \cap P \to P \to P/(N \cap P) \to 0$$

is exact. If both factors are nonzero, we have $\mu(N \cap P) \geq \mu(N)$ and $\mu(P/(N \cap P)) \geq \mu(M/N)$, and $\mu(P)$ is a weighted average of $\mu(N \cap P)$ and $\mu(P/(N \cap P))$, so it is bounded below. If one factor vanishes, then $\mu(P)$ simply equals the slope of the other factor, so the same conclusion holds. □

Lemma 1.4.12. — *Let M be a nonzero ϕ-module over R. Then there is a largest ϕ-submodule of M of least slope, which is module-semistable.*

Proof. — The fact that there is a least slope s holds by Lemma 1.4.11 and the fact that the denominators of slopes are bounded above by the rank of M; clearly any ϕ-submodule of slope s must be semistable. If N_1 and N_2 are two such submodules, then the kernel of the surjection $N_1 \oplus N_2 \to N_1 + N_2$ must have slope at least s, so $\mu(N_1 + N_2) \leq s$. On the other hand, $\mu(N_1 + N_2) \geq s$ because $N_1 + N_2 \subseteq M$, so $\mu(N_1 + N_2) = s$. Hence the ϕ-submodules of M of slope s are closed under sum, yielding the existence of a largest such submodule. □

Corollary 1.4.13. — *Let M be a ϕ-module over R. Then for any positive integer a, M is module-semistable if and only if $[a]_* M$ is module-semistable.*

Proof. — If $[a]_* M$ is semistable, evidently M is too. Conversely, if $[a]_* M$ is not semistable, then its largest ϕ^a-submodule of least slope is a ϕ^a-submodule M_1 of lower rank. By the uniqueness in Lemma 1.4.12, M_1 must in fact be preserved by ϕ, so M is not semistable either. □

Definition 1.4.14. — Let M be a ϕ-module over R. A *module-semistable filtration* of M is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated ϕ-submodules such that each quotient M_i/M_{i-1} is module-semistable. A *Harder-Narasimhan (HN) filtration* is a module-semistable filtration in which

$$\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1}).$$

Proposition 1.4.15. — *Every ϕ-module over R admits a unique HN filtration, whose first step is the submodule defined in Lemma 1.4.12.*

Proof. — This is a formal consequence of Lemma 1.4.12; see [**22**, Proposition 4.2.5]. □

Definition 1.4.16. — Define the *slope multiset* of a module-semistable filtration of a ϕ-module of M as the multiset in which each slope of a successive quotient occurs with multiplicity equal to the rank of that quotient. These assemble into the lower boundary of a convex region in the xy-plane as follows: start at $(0,0)$, then take each slope s in increasing order and append to the polygon a segment with slope s and width equal to the multiplicity of s. The result is called the *slope polygon* of the filtration; for the HN filtration, we call the result the *HN polygon*.

Proposition 1.4.17. — *The HN polygon lies on or above the slope polygon of any module-semistable filtration, with the same endpoint.*

Proof. — This is a formal consequence of the definition of an HN filtration: see [**22**, Proposition 3.5.4]. □

Proposition 1.4.18. — *Let M_1, M_2 be ϕ-modules over R such that each slope of the HN polygon of M_1 is less than each slope of the HN polygon of M_2. Then* $\mathrm{Hom}(M_1, M_2) = 0$.

Proof. — Choose $f \in \mathrm{Hom}(M_1, M_2)$. Let N_1 be the first step in the HN filtration of M_1; then either $f(N_1) = 0$, or $\mu(f(N_1)) \leq \mu(N_1)$. The latter is impossible because $\mu(f(N_1))$ is no less than the least slope of M_2, whereas $\mu(N_1)$ is no greater than the greatest slope of M_1. Hence f factors through M_1/N_1; repeating, we obtain $f = 0$. □

1.5. Étale ϕ-modules

Definition 1.5.1. — A ϕ-module M over R or R^{bd} is said to be *étale* (or *unit-root*) if it can be obtained by base extension from a (strict) ϕ-module over R^{int}; that is, M must admit an R^{int}-lattice N such that ϕ induces an isomorphism $\phi^* N \to N$. We call such an N an *étale lattice* of M. Note that N is not in general unique; for instance, it may be rescaled. Note also that the dual of an étale ϕ-module is again étale.

Remark 1.5.2. — The term "unit-root" is standard in applications to crystalline cohomology, where it refers to the process of extracting the unit roots (roots of valuation 0) of a p-adic polynomial. By contrast, the term "étale" is standard in applications to p-adic Hodge theory.

One of the basic results about étale ϕ-modules is that in a certain sense, they do not lose information when base-changed from R^{bd} to R. This can be deduced from a slightly more general result, which we already used once (to justify that the Robba ring satisfies Hypothesis 1.4.1) and will use again shortly (in the proof of Theorem 1.6.10).

Definition 1.5.3. — Define an *isogeny ϕ-module* over R^{int} to be a finite free R^{int}-module M equipped with an injection $\phi^* M \to M$ whose cokernel is killed by some power of a uniformizer of R^{int}. Such an object becomes a strict ϕ-module upon tensoring with R^{bd} or R.

Proposition 1.5.4. — *Let M be an isogeny ϕ-module over R^{int}. Then the natural maps $H^i(M \otimes R^{\mathrm{bd}}) \to H^i(M \otimes R)$ for $i = 0$ (under weak Hypothesis 1.4.1) or $i = 0, 1$ (under strong Hypothesis 1.4.1) are bijective.*

Proof. — This is an immediate consequence of the final clause of Hypothesis 1.4.1. □

Proposition 1.5.5. — *The base change functor from étale ϕ-modules over R^{bd} to étale ϕ-modules over R is an equivalence of categories.*

Proof. — The essential surjectivity holds by definition, so we need only check full faithfulness. That is, for any étale ϕ-modules M_1, M_2 over R^{bd}, we must check that the natural map

$$H^0(M_1^\vee \otimes M_2) \to H^0(M_1^\vee \otimes M_2 \otimes R)$$

is a bijection; this follows from Proposition 1.5.4. □

Proposition 1.5.6. — *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of ϕ-modules over R. If any two of M_1, M_2, M are étale (except possibly M_1, M_2 in the case of weak Hypothesis 1.4.1), then so is the third.*

Proof. — First, suppose that M and M_2 are étale. By Proposition 1.5.5, the ϕ-modules M, M_2 and the morphism $M \to M_2$ all descend to R^{bd}. By Lemma 1.5.7 below, we can then produce an étale lattice in M_1 by taking the kernel of the map from an étale lattice of M to M_2.

Next, suppose that M and M_1 are étale. We then dualize to obtain a second exact sequence in which $M^\vee$ and $M_1^\vee$ are étale. By the previous paragraph, $M_2^\vee$ is then étale, as then is M_2.

Finally, suppose that M_1 and M_2 are étale and that strong Hypothesis 1.4.1 holds. By applying Proposition 1.5.4, M_1, M_2, and the exact sequence $0 \to M_1 \to M \to M_2 \to 0$ all descend to R^{bd}; by rescaling appropriately, we can descend the sequence to R^{int}. We can then produce an étale lattice in M by lifting an étale lattice from M_2, then adding an étale lattice from M_1. □

Lemma 1.5.7. — *Let M be an étale ϕ-module over R^{bd}. Then any finitely generated ϕ-stable R^{int}-submodule of M is a ϕ-module over R^{int}.*

Proof. — Let M_0 be an étale lattice of M, and let N be a finitely generated ϕ-stable R^{int}-submodule of M; by rescaling, we may assume $N \subseteq M_0$. Then N is already an isogeny ϕ-module, and it suffices to check that $\deg(N) = 0$; we may do this after replacing M by $\wedge^{\mathrm{rank}(N)} M$, i.e., we may assume $\mathrm{rank}(N) = 1$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis of M_0, let $\mathbf{v} = \sum_{i=1}^n c_i \mathbf{e}_i$ be a generator of N, and put $\phi(\mathbf{v}) = \sum_{i=1}^n d_i \mathbf{e}_i$. Then $\deg(N) = \min_i\{w(d_i)\} - \min_i\{w(c_i)\}$, but this difference is zero because M_0 is an étale lattice. □

We can also show that étale ϕ-modules are module-semistable, but it will be convenient to do that more generally for pure ϕ-modules in the next subsection.

1.6. Pure ϕ-modules. —

Definition 1.6.1. — Let M be a ϕ-module over R^{bd} or R of slope $s = c/d$, where c, d are coprime integers with $d > 0$. We say M is *pure* (or *isoclinic*, or sometimes *isocline*) of slope s if for some ϕ-module N of rank 1 and degree $-c$, $([d]_* M) \otimes N$ is étale (the same then holds for any such N). It will follow from Lemma 1.6.3 below that it is equivalent to impose this condition for any one pair $c, d \in \mathbb{Z}$ with $s = c/d$ and $d > 0$. Note that:

- any ϕ-module of rank 1 is pure;
- a ϕ-module is pure of slope 0 if and only if it is étale;
- the dual of a pure ϕ-module of slope s is itself pure of slope $-s$.

Remark 1.6.2. — This definition is not that of [**22**, Definition 6.3.1], but it is equivalent to it by [**22**, Proposition 6.3.5]. It has the advantage that it can be stated without reference to any sort of Dieudonné-Manin classification; the downside is that one must expend a bit of effort to check some natural-looking properties, as we do below.

Lemma 1.6.3. — *Let M be a ϕ-module over R^{bd} or R, and let a be a positive integer. Then M is pure of some slope s if and only if $[a]_*M$ is pure of slope as.*

Proof. — We first check the case where $s = 0$. If M is étale, then clearly $[a]_*M$ is too. Conversely, if $[a]_*M$ is étale, then ϕ induces isomorphisms $(\phi^{i+1})^*[a]_*M \to (\phi^i)^*[a]_*M$ over R; by Proposition 1.5.5, these isomorphisms descend to R^{bd}. That is, we may reduce to working over R^{bd}. In this case, let N_0 be an étale lattice of $[a]_*M$. Let N be the R^{int}-span of $N_0, \phi(N_0), \dots, \phi^{a-1}(N_0)$; then N is an étale lattice of M. Hence M is étale.

In the general case, write $s = c/d$ in lowest terms, and put $b = \gcd(a, d)$; then in lowest terms, $as = (ac/b)/(d/b)$. Let N be a ϕ^d-module of rank 1 and degree $-c$; then $[a/b]_*N$ has rank 1 and degree $-ac/b$. The following are equivalent:

- M is pure of slope s;
- $([d]_*M) \otimes N$ is étale (definition);
- $[a/b]_*(([d]_*M) \otimes N) \cong ([ad/b]_*M) \otimes ([a/b]_*N) \cong ([d/b]_*([a]_*M)) \otimes ([a/b]_*N)$ is étale (by above);
- $[a]_*M$ is pure of slope as (definition).

This yields the claim. □

Corollary 1.6.4. — *If M_1, M_2 are pure ϕ-modules of slopes s_1, s_2, then $M_1 \otimes M_2$ is pure of slope $s_1 + s_2$.*

Proof. — By Lemma 1.6.3, we may reduce to the case where $s_1, s_2 \in \mathbb{Z}$. By twisting, we may then reduce to the case where $s_1 = s_2 = 0$. In this case the result follows from the fact that ϕ-modules over R^{int} admit tensor products. □

We can thus generalize Propositions 1.5.5 and 1.5.6 as follows.

Theorem 1.6.5. — *For any rational number s, the base change functor from pure ϕ-modules of slope s over R^{bd} to pure ϕ-modules of slope s over R is an equivalence of categories.*

Proof. — If M_1, M_2 are pure of slope s, then $M_1^\vee \otimes M_2$ is étale. Hence the proof of Proposition 1.5.5 goes through unchanged. □

Theorem 1.6.6. — *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of ϕ-modules over R. If any two of M_1, M_2, M are pure of slope s (except possibly M_1, M_2 in the case of weak Hypothesis 1.4.1), then so is the third.*

Proof. — By Lemma 1.6.3, we may apply $[a]_*$ to reduce to the case where $s \in \mathbb{Z}$; by twisting, we may force $s = 0$. The result now follows from Proposition 1.5.6. □

Remark 1.6.7. — In a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ over R, the fact that M is pure of slope s does not by itself imply the same for M_1 and M_2, unless the sequence splits (see Corollary 1.6.11). For example, if M is pure of rank 2 and slope 0, it can happen that M_1 is pure of rank 1 and slope 1, while M_2 is pure of rank 1 and slope -1. This sort of example arises naturally from p-adic Hodge theory, as in the theory of trianguline representations introduced by Colmez [**13**].

Lemma 1.6.8. — *Let M be a pure ϕ-module over R of positive slope. Then* $H^0(M) = 0$.

Proof. — By replacing M with $[a]_*M$ for $a = \mathrm{rank}(M)$, we can reduce to the case where $\mu(M) \in \mathbb{Z}_{>0}$. By Theorem 1.6.5, there exists a pure ϕ-module M_0 over R^{bd} with $M \cong M_0 \otimes R$. By Proposition 1.5.4, we have $H^0(M_0) = H^0(M)$.

Choose a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of M_0 such that the matrix A defined by $\phi(\mathbf{e}_j) = \sum_i A_{ij}\mathbf{e}_i$ has all entries of valuation at least $\mu(M)$. If $\mathbf{v} = \sum c_i \mathbf{e}_i \in H^0(M)$ is nonzero, then $c_i = \sum_j A_{ij}\phi(c_j)$ implies that $\min_i\{w(c_i)\} > \min_j\{w(c_j)\}$, contradiction. Hence $H^0(M) = 0$. □

Corollary 1.6.9. — *If M and N are pure ϕ-modules over R with $\mu(M) < \mu(N)$, then* $\mathrm{Hom}(M, N) = 0$.

Proof. — The conditions ensure that $M^\vee \otimes N$ is pure of positive slope; by Lemma 1.6.8, $\mathrm{Hom}(M, N) = H^0(M^\vee \otimes N) = 0$. □

Theorem 1.6.10. — *Let M be a pure ϕ-module over R of slope s.*

(a) *M is module-semistable.*
(b) *If N is a ϕ-submodule of M with $\mu(N) = s$, then N is saturated, and both N and M/N are pure of slope s.*

Proof. — For (a), let N be a ϕ-submodule of M; we wish to show that $\mu(N) \geq s$. By replacing M by $\wedge^{\mathrm{rank}(N)} M$, we may assume that $\mathrm{rank}(N) = 1$. By Lemma 1.6.3, we may assume further that $s \in \mathbb{Z}$. By twisting, we may assume further that N is trivial, so that $H^0(M) \neq 0$. To avoid contradicting Lemma 1.6.8, we must then have $s \leq 0 = \mu(N)$, yielding semistability.

For (b), by applying $[a]_*$ and twisting, we may again reduce to the case $s = 0$. Let M_0 be an étale lattice in M; by Lemma 1.5.7, the kernel of $M_0 \to M/N$ is a ϕ-module over R^{int}, so the image is as well. Let P be the R-span of this image; it is an étale ϕ-submodule of M/N of the same rank. Since $\mu(N) = \mu(M) = 0$, we also have $\mu(M/N) = 0$, so $M/N = P$ by Corollary 1.4.10. Hence M/N is étale; the same logic applied after dualizing implies that $N^\vee$ is étale, as then is N. □

Corollary 1.6.11. — *If M_1, M_2 are ϕ-modules, then $M = M_1 \oplus M_2$ is pure of slope s if and only if both M_1 and M_2 are pure of slope s.*

Proof. — If M_1 and M_2 are pure of the same slope, then visibly so is M. Conversely, if M is pure of slope s, then M is semistable by Theorem 1.6.10(a), so the ϕ-submodules M_1 and M_2 each have slope at least s. Since $\mu(M)$ is a weighted average of $\mu(M_1)$ and $\mu(M_2)$, we must in fact have $\mu(M_1) = \mu(M_2) = s$; by Theorem 1.6.10(b), M_1 and M_2 are both pure of slope s. □

Corollary 1.6.12. — *Let M be a ϕ^a-module over R. Then M is pure of some slope s if and only if $[a]^*M$ is pure of slope s/a.*

Proof. — By Lemma 1.6.3, $[a]^*M$ is pure of slope s/a if and only if $[a]_*[a]^*M$ is pure of slope s. If M is pure of slope s, then so are $(\phi^i)^*M$ for $i = 0, \dots, a-1$; since

$$[a]_*[a]^*M \cong \oplus_{i=0}^{a-1} (\phi^i)^*M \tag{1.6.12.1}$$

by Definition 1.3.6, $[a]_*[a]^*M$ is pure of slope s.

Conversely, if $[a]_*[a]^*M$ is pure of slope s, then (1.6.12.1) shows that M is a direct summand of $[a]_*[a]^*M$, and hence is pure by Corollary 1.6.11. □

1.7. The slope filtration theorem. — So far all of our work has been formal modulo the assumption of an appropriate analogue of Proposition 1.2.6. We now restrict attention from general rings R as in strong Hypothesis 1.4.1 to the Robba ring $\mathscr{R}$ (as in Example 1.4.2), where one can make the description of ϕ-modules much more precise.

We have already described a natural filtration on ϕ-modules over $\mathscr{R}$, namely the Harder-Narasimhan filtration. The trouble is that the construction is so formal that one cannot deduce any useful properties about the resulting filtration or its associated slopes; for instance, it is not clear that module-semistability is preserved by tensor product. (The fact that the analogous statement is true for vector bundles on smooth varieties in characteristic 0 is highly nontrivial: it reduces to the case of tensoring two semistable vector bundles of slope 0 on curves [**33**], in which case it follows from an analytic classification of stable bundles due to Narasimhan-Seshadri [**34, 35**].) The slope filtration theorem, which is the main result of this paper, asserts that in fact the steps of the Harder-Narasimhan filtration are much more structured than one might have otherwise predicted.

***Theorem 1.7.1* (Slope filtration theorem).** — *Every module-semistable ϕ-module over the Robba ring $\mathscr{R}$ is pure. In particular, every ϕ-module M over $\mathscr{R}$ admits a unique* *filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ *by saturated* ϕ*-modules whose successive quotients are pure with* $\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1})$.

This theorem is stated as a forward reference, as its proof will occupy most of the rest of the paper; here we give only a top-level summary.

Proof of Theorem 1.7.1. — The proof of Theorem 1.7.1 will be obtained by constructing (in Subsection 2.2) an extended Robba ring $\tilde{\mathscr{R}}$ which also satisfies strong Hypothesis 1.4.1, and then establishing the following facts.

- If M is a semistable ϕ-module over $\mathscr{R}$, then $M \otimes \tilde{\mathscr{R}}$ is also semistable (Theorem 3.1.2).
- If $\tilde{M}$ is a semistable ϕ-module over $\tilde{\mathscr{R}}$, then $\tilde{M}$ is pure (Theorem 2.1.8).
- If M is a ϕ-module over $\mathscr{R}$ and $M \otimes \tilde{\mathscr{R}}$ is pure, then M is pure (Theorem 3.1.3).

These together yield the claim. □

Remark 1.7.2. — Theorem 1.7.1 implies that the tensor product of module-semistable ϕ-modules is pure (by Corollary 1.6.4) and hence module-semistable (by Theorem 1.6.10). This formally implies that the slopes of ϕ-modules behave like valuations of eigenvalues, or like Deligne's weights in étale cohomology. That is, if M has slopes $c_1, \dots, c_m$ and M' has slopes $c'_1, \dots, c'_n$, both counted with multiplicity, then:

- the slopes of $M \oplus M'$ are $c_1, \dots, c_m, c'_1, \dots, c'_n$;
- the slopes of $M \otimes M'$ are $c_i c'_j$ for $i = 1, \dots, m$ and $j = 1, \dots, n$;
- the slopes of $\wedge^d M$ are $c_{i_1} + \cdots + c_{i_d}$ for $1 \leq i_1 < \cdots < i_d \leq m$;
- the slopes of $[a]_* M$ are $ac_1, \dots, ac_m$;
- the slopes of $M(b)$ are $c_1 + b, \dots, c_m + b$;
- the slopes of $[a]^* M$ are $c_1/a, \dots, c_m/a$, each repeated a times.

In some sense, the slope filtration theorem is thus playing a role in this theory analogous to Deligne's analysis of determinantal weights in his second proof of the Weil conjectures [**15**].

Remark 1.7.3. — The uniqueness in Theorem 1.7.1 means that the slope filtration inherits any additional group action on the original ϕ-module. In particular, if M is a (ϕ, Γ)-module, then the steps of the slope filtration are (ϕ, Γ)-submodules of M. As shown by Berger [**5**, Théorème V.2.1], this leads to a proof of the Colmez-Fontaine theorem that (ϕ, N)-modules over a p-adic field which are *weakly admissible*, in the sense of satisfying a necessary numerical criterion, indeed arise from Galois representations via p-adic Hodge theory. (See also the variant of Berger's argument given by Kisin [**27**].)

Remark 1.7.4. — The étale (ϕ, Γ)-modules attached to Galois representations of a p-adic field were originally defined by Fontaine over the p-adic completion of $\mathscr{R}^{\mathrm{bd}}$; the fact that they can be descended to $\mathscr{R}^{\mathrm{bd}}$ is a theorem of Cherbonnier and Colmez [**9**, Corollaire III.5.2]. The fact that the descent is unique follows from the fact that the base change from étale ϕ-modules over $\mathscr{R}^{\mathrm{bd}}$ to its completion is fully faithful, which in turn follows from Proposition 1.2.7.

Remark 1.7.5. — In the context of p-adic differential equations and rigid cohomology, Theorem 1.7.1 arises with M carrying the extra structure of a connection $\nabla : M \to M \otimes \Omega^1_{\mathscr{R}/K}$ compatible with the ϕ-action; that is, M is a *(ϕ, ∇)-module.* One can see that the steps of the slope filtration are (ϕ, ∇)-submodules by using Corollary 1.6.9 as follows. The map ∇ induces a homomorphism $M_1 \to (M/M_1) \otimes \Omega^1_{\mathscr{R}/K}$ of ϕ-modules. Since $\Omega^1_{\mathscr{R}/K}$ is a rank 1 ϕ-module of nonnegative slope (the slope is actually positive, but we don't need this here), each slope of $(M/M_1) \otimes \Omega^1_{\mathscr{R}/K}$ is strictly greater than $\mu(M_1)$. Repeated application of Corollary 1.6.9 yields the claim.

Given that the slope filtration is a filtration by (ϕ, ∇)-submodules, one may prove the local monodromy theorem for p-adic differential equations as in [**21**], by showing each successive quotient in the slope filtration becomes trivial as a ∇-module after tensoring with a suitable finite unramified extension of $\mathscr{R}^{\mathrm{int}}$. This reduces easily to the étale case, which is a theorem of Tsuzuki [**42**, Theorem 4.2.6]. Beware, however, that this last step only applies for ϕ_K absolute; in particular, this approach cannot be used to prove [**6**, Proposition 6.2.1].

Remark 1.7.6. — By [**18**, Theorem 11.1], the conclusion of Theorem 1.7.1 also holds in the situation of Example 1.4.3; indeed, what one obtains is an analogue of the classification of ϕ-modules over the extended Robba ring $\tilde{\mathscr{R}}$ to be introduced in Section 2. That result is not covered by this paper, though (as [**18**] already points out) there are very strong parallels between the ensuing calculations. However, Theorem 1.7.1 itself does address a related situation: if we take $K = k((z))$ with k of characteristic $p > 0$, and ϕ_K to be a power of the absolute Frobenius, then $\mathscr{R}$ consists of Laurent series in t over z which converge for $|z|^c < |t| < 1$ for some $c > 0$. Since the valuation on k is trivial, it is equivalent to require convergence when $0 < |z| < |t|^{1/c}$; that is, we are considering series in z over $k((t))$ convergent on some punctured disc around the origin. In this case (assuming q is a power of p), Theorem 1.7.1 is a result of Hartl [**17**, Theorem 1.7.7].

It would be interesting to know about the following q-analogue of Remark 1.7.6; it may be related to the formal classification of linear difference operators [**38**], in much the same way that the construction of the canonical lattice of an irregular meromorphic connection [31] reduces to the formal classification of linear differential operators [30].

Question 1.7.7. — *Let K be a complete field, either archimedean or nonarchimedean. Take R to be the ring of germs of analytic functions over K on punctured discs around the origin, R^{bd} to be the germs meromorphic at zero, w to be the order of vanishing at zero, and ϕ to be the map $\sum c_i t^i \mapsto \sum c_i q^i t^i$ for some $q \in K^*$ with $|q| < 1$. Does the analogue of Theorem 1.7.1 hold in this setting?*

Remark 1.7.8. — The conclusion of Theorem 1.7.1 also holds for ϕ-modules over K itself; this is a straightforward consequence of Proposition 2.4.5. In addition, if ϕ is bijective on K, then it is easy to check that $H^1(M) = 0$ for M pure of nonzero slope, so the slope filtration splits uniquely. This gives a semilinear analogue of the eigenspace decomposition of a vector space equipped with a linear transformation. If k is algebraically closed of characteristic $p > 0$ and ϕ is an absolute Frobenius lift, this recovers the Dieudonné-Manin classification of rational Dieudonné modules [32].

Remark 1.7.9. — The conclusion of Theorem 1.7.1 does not hold for ϕ equal to the identity map on $\mathscr{R}$. In fact, the conclusion is equivalent to the condition that the characteristic polynomial of ϕ have all coefficients in $\mathscr{R}^{\mathrm{bd}}$, whereas the definition of a ϕ-module only forces the determinant to belong to $\mathscr{R}^{\mathrm{bd}}$.

2. Classification over an extended Robba ring

In this section and the next, we give a proof of Theorem 1.7.1. Although somewhat simplified in some technical aspects, the argument follows the same arc as in [21] and [22], with two basic stages. In the first stage, performed in this section, we show that ϕ-modules over a suitable overring of $\mathscr{R}$ admit a very simple classification (analogous to the Dieudonné-Manin classification alluded to in Remark 1.7.8), and in particular admit a slope filtration. In the second stage, we show that the slope filtration descends back to $\mathscr{R}$.

On a first reading, we recommend reading only Subsection 2.1 for an overview, then returning later for the technical details in the rest of the section.

2.1. Overview

Hypothesis 2.1.1. — Throughout this section, assume that ϕ is a relative Frobenius lift on $\mathscr{R}$ such that ϕ_K is an *automorphism* of K. Also assume that any étale ϕ-module over K is trivial; this is equivalent to asking that any ϕ-module over the residue field k be trivial. It also implies that H^1 vanishes for any étale ϕ-module over K or any ϕ-module over k. Using Definition 1.3.7, we deduce the same conclusions with ϕ replaced by ϕ^a for any positive integer a.

Remark 2.1.2. — In the absolute Frobenius case, Hypothesis 2.1.1 can be satisfied by taking k to be algebraically closed. In general, one must work a bit harder; see Proposition 3.2.4.

We will define (Definition 2.2.4) an *extended Robba ring* $\tilde{\mathscr{R}}$ which has the following properties:

- $\tilde{\mathscr{R}}$ is a Bézout domain containing $\mathscr{R}$, and admits an automorphism ϕ extending the given Frobenius lift on $\mathscr{R}$ (see Remark 2.2.5 and Proposition 2.2.6).
- The units in $\tilde{\mathscr{R}}$ are the nonzero elements of a subfield $\tilde{\mathscr{R}}^{\mathrm{bd}}$, which is the fraction field of a discrete valuation ring $\tilde{\mathscr{R}}^{\mathrm{int}}$ for which $\tilde{\mathscr{R}}^{\mathrm{int}} \cap \mathscr{R} = \mathscr{R}^{\mathrm{int}}$ (see Remark 2.2.5).
- The strong form of Hypothesis 1.4.1 holds for $R = \tilde{\mathscr{R}}$ (see Proposition 2.2.8).

The classification of ϕ-modules over $\tilde{\mathscr{R}}$ rests on a sequence of structural results, which we state in roughly increasing order of difficulty; their proofs occupy the remainder of this section.

Proposition 2.1.3. — *Let M, N be pure ϕ-modules over $\tilde{\mathscr{R}}$ obtained by base change from K, with $\mu(M) > \mu(N)$. Then* $\mathrm{Hom}(M, N) \neq 0$.

Proof. — See Subsection 2.2. □

Notation 2.1.4. — Choose a uniformizer π of K, and let $\tilde{\mathscr{R}}(1)$ be the ϕ-module of rank 1 and degree 1 on which ϕ acts on some generator via multiplication by π. We use $\tilde{\mathscr{R}}(1)$ as a twisting sheaf, writing $M(n) = M \otimes \tilde{\mathscr{R}}(1)^{\otimes n}$.

Proposition 2.1.5. — *Let M be a nonzero ϕ-module over $\tilde{\mathscr{R}}$. Then for all sufficiently large integers n, $H^0(M(-n)) \neq 0$ and $H^1(M(-n)) = 0$.*

Proof. — See Subsection 2.3. □

Proposition 2.1.6. — *For any rational number s, the base change functor from pure ϕ-modules of slope s over K to pure ϕ-modules of slope s over $\tilde{\mathscr{R}}$ is an equivalence of categories.*

Proof. — See Subsection 2.5. □

Proposition 2.1.7. — *Let n be a positive integer, let N' be a pure ϕ^n-module over $\tilde{\mathscr{R}}$ of rank* 1 *and degree* 1, *let P be a pure ϕ-module over $\tilde{\mathscr{R}}$ of rank* 1 *and degree* -1, *and suppose*

$$0 \to [n]^* N' \to M \to P \to 0$$

is a short exact sequence of ϕ-modules. Then $H^0(M) \neq 0$.

Proof. — See Subsection 2.6. □

These assemble to give the following classification theorem.

Theorem 2.1.8. — *Any module-semistable ϕ-module over $\tilde{\mathscr{R}}$ is pure. Consequently, the successive quotients of the HN filtration of a ϕ-module over $\tilde{\mathscr{R}}$ are all pure.*

Remark 2.1.9. — Before proving Theorem 2.1.8, we make an observation which figures prominently in the argument. If one knows Theorem 2.1.8 for ϕ-modules of rank $\leq n$, it follows from Propositions 2.1.3 and 2.1.6 (and the assumption that étale ϕ-modules over K are trivial, as built into Hypothesis 2.1.1) that for M a pure ϕ-module over $\tilde{\mathscr{R}}$ and N an *arbitrary* ϕ-module over $\tilde{\mathscr{R}}$ with $\operatorname{rank}(N) \leq n$ and $\mu(M) \geq \mu(N)$, we have $\operatorname{Hom}(M, N) \neq 0$; in particular, if $\operatorname{rank}(M) = 1$, we would have an injection of M into N. This is because the first step of the HN filtration of N always has slope $\leq \mu(N)$.

Proof of Theorem 2.1.8. — We proceed by induction on rank, the case of rank 1 being evident. Assume that $n \geq 1$ and that for every positive integer a, every semistable ϕ^a-module of rank $\leq n$ is pure. Suppose that M is a semistable ϕ^a-module of rank $n+1$ over $\tilde{\mathscr{R}}$; we wish to show that M is pure. We may reduce to the case where $\mu(M) \in \mathbb{Z}$ by applying $[d]_*$ and invoking Corollary 1.4.13 (to see that semistability is preserved) and Lemma 1.6.3 (to see that purity is reflected); we may then twist to ensure $\mu(M) = 0$. For ease of notation, we will assume hereafter that M is a ϕ-module (at the expense of replacing ϕ by a power, which does not disturb Hypothesis 2.1.1).

Put $M' = [n]_* M$; then M' is semistable by Corollary 1.4.13 again. By Proposition 2.1.5, there exists a nonnegative integer c such that M' admits a pure ϕ^n-submodule N' of rank 1 and slope c; choose c as small as possible. Suppose that $c \geq 2$; since $\mu(M'/N') < 0 \leq c-2$, we may apply Remark 2.1.9 to produce a ϕ^n-submodule of M'/N' isomorphic to $\tilde{\mathscr{R}}(c-2)$. Let Q' be the inverse image of that submodule in M'; applying Proposition 2.1.7 (in the case $n = 1$) to the exact sequence

$$0 \to N'(1-c) \to Q'(1-c) \to \tilde{\mathscr{R}}(-1) \to 0,$$

we see that $H^0(Q'(1-c)) \neq 0$ and hence $H^0(M'(1-c)) \neq 0$, contradicting the minimality of c.

Suppose that $c = 1$. Put $N = [n]^* N'$; then N is pure of slope $1/n$ by Corollary 1.6.12. The adjunction between $[n]^*$ and $[n]_*$ converts the inclusion $N' \hookrightarrow M'$ into a nonzero map $f : N \to M$. Since N is semistable by Theorem 1.6.10, $\mu(f(N)) \leq 1/n$; moreover, the denominator of $\mu(f(N))$ is at most $\operatorname{rank}(f(N)) \leq n$. Consequently, either $\mu(f(N)) \leq 0$, in which case Remark 2.1.9 implies that $H^0(f(N)) \neq 0$; or $\mu(f(N)) = 1/n$, in which case f must be injective and we have an exact sequence

$$0 \to N \to M \to P \to 0$$

with P pure of rank 1 and slope -1, to which we apply Proposition 2.1.7 to deduce that $H^0(M) \neq 0$. In either case, we contradict the minimality of c.

We deduce that $c = 0$, i.e., M' admits a nontrivial étale ϕ-submodule N'; the quotient M'/N' is also semistable, hence pure by the induction hypothesis. By Theorem 1.6.6, M' is pure, as then is M by Lemma 1.6.3. This completes the proof. □

Remark 2.1.10. — In the proof of Theorem 2.1.8, the passage from M to M' is made in order to simplify the statement of Proposition 2.1.7. One can do some extra work to prove a version of Proposition 2.1.7 in which $[n]^*N'$ is replaced by any pure ϕ-module of rank n and degree 1; however, the internal improvement is immaterial in the end, as even this stronger form of Proposition 2.1.7 is itself an immediate consequence of Theorem 2.1.8.

2.2. The extended Robba ring. — We now go back and construct the extended Robba ring $\tilde{\mathscr{R}}$.

Definition 2.2.1. — Let R be a ring and let G be a totally ordered abelian group. The ring of *Hahn series* (or *Mal'cev-Neumann series*, or *generalized power series*) over R with value group G is the set of functions $f : G \to R$ with well-ordered support, with pointwise addition and multiplication given by convolution; it is a standard calculation [**37**, Chapter 13] to verify that these operations give a well-defined ring, which is a field if R is. We typically represent elements of this ring as formal series $\sum_{g\in G} r_g u^g$ in some dummy variable u with powers indexed by $g \in G$, and the ring is correspondingly denoted $R((u^G))$. For $G \subseteq \mathbb{R}$, we view $R((u^G))$ as being equipped with the u-adic valuation v sending $\sum_g r_g u^g$ to the smallest g for which $r_g \neq 0$ (i.e., the least element of the support).

Lemma 2.2.2. — *Let $\phi : R((u^{\mathbb{Q}})) \to R((u^{\mathbb{Q}}))$ be an automorphism of the form $\sum_i a_i u^i \mapsto \sum_i \phi_R(a_i) u^{qi}$, with ϕ_R an automorphism of R. Then the map $1-\phi$ is bijective on the set of series with zero constant term.*

Proof. — If $x \in R((u^{\mathbb{Q}}))$ and $v(x) < 0$, then $v(x - \phi(x)) = qv(x)$, whereas if $v(x) > 0$, then $v(x - \phi(x)) = v(x)$. This proves injectivity.

Given $x \in R((u^{\mathbb{Q}}))$, write $x = \sum_i x_i u^i$, and put

$$y_+ = \sum_{j=0}^{\infty} \sum_{i>0} \phi_R^j(x_i) u^{iq^j}$$

$$y_- = \sum_{i<0} \left(\sum_{j=0}^{\infty} -\phi_R^{-j-1}(x_{iq^{j+1}}) \right) u^i.$$

Since both sums give well-defined elements of $R((u^{\mathbb{Q}}))$ (in the definition of y_-, the sum over j is finite for each i), we may put $y = y_+ + y_-$, which has zero constant term and satisfies $y - \phi(y) = x - x_0$. This proves surjectivity. □

Corollary 2.2.3. — *With k as in Hypothesis 2.1.1, for any $c \in k^*$, the map $1 - c\phi$ on $k((u^{\mathbb{Q}}))$ is surjective.*

Proof. — By Hypothesis 2.1.1, there exists $a \in k^*$ such that $\phi(a) = ca$, so we can always write

$$(1 - c\phi)(x) = a^{-1}(ax - \phi(ax)).$$

It thus suffices to check the case $c = 1$; this follows from Lemma 2.2.2 and the fact that $1 - \phi$ is surjective on k, which again is a consequence of Hypothesis 2.1.1. □

Corresponding to the extension from power series to generalized power series, we define an enlargement of the Robba ring. We first construct the ring, then the embedding of the original Robba ring into it.

Definition 2.2.4. — For $r > 0$, let $\tilde{\mathscr{R}}^r$ be the set of formal sums $\sum_{i \in \mathbb{Q}} a_i u^i$ with $a_i \in K$, satisfying the following conditions.

- For each $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geq c$ is well-ordered.
- We have $|a_i| e^{-ri} \to 0$ as $i \to -\infty$.
- For all $s > 0$, we have $|a_i| e^{-si} \to 0$ as $i \to +\infty$.

Then $\tilde{\mathscr{R}}^r$ can be shown to form a ring. We call the union $\tilde{\mathscr{R}} = \tilde{\mathscr{R}}_K = \cup_r \tilde{\mathscr{R}}^r$ the *extended Robba ring* over K. Let $\tilde{\mathscr{R}}^{\mathrm{bd}}$ and $\tilde{\mathscr{R}}^{\mathrm{int}}$ be the subrings of $\tilde{\mathscr{R}}$ consisting of series with bounded and integral coefficients, respectively. We equip $\tilde{\mathscr{R}}^r$ with the norm

$$\left| \sum_i a_i u^i \right|_r = \sup_i \{ |a_i| e^{-ri} \}$$

and $\tilde{\mathscr{R}}$ with the automorphism

$$\phi\left(\sum_i a_i u^i \right) = \sum_i \phi_K(a_i) u^{qi}.$$

Remark 2.2.5. — The ring $\tilde{\mathscr{R}}$ can be viewed as an example of an "analytic ring" in the sense of [**22**, §2.4], by taking ϕ_K to be an absolute Frobenius lift on K. Thus the results of [**22**, Chapter 2] apply to show that $\tilde{\mathscr{R}}$ shares many of the nice properties of $\mathscr{R}$, as follows.

- The ring $\tilde{\mathscr{R}}$ is a Bézout domain [**22**, Theorem 2.9.6].
- The ring $\tilde{\mathscr{R}}^{\mathrm{int}}$ is a henselian discrete valuation ring, and its fraction field is $\tilde{\mathscr{R}}^{\mathrm{bd}}$ [**22**, Lemma 2.1.12].
- The units of $\tilde{\mathscr{R}}$ are the nonzero elements of $\tilde{\mathscr{R}}^{\mathrm{bd}}$ [**22**, Lemma 2.4.7].

Proposition 2.2.6. — *There exists a ϕ-equivariant embedding $\psi : \mathscr{R} \hookrightarrow \tilde{\mathscr{R}}$ such that for any r_0 as in Remark 1.2.5 and any $r \in (0, r_0)$, $\mathscr{R}^r$ maps to $\tilde{\mathscr{R}}^r$ preserving $|\cdot|_r$.*

Proof. — We inductively construct homomorphisms $\psi_l : \mathcal{R} \to \tilde{\mathcal{R}}$, each of the form $\psi_l(\sum c_i t^i) = \sum c_i u_l^i$ for some $u_l \in \tilde{\mathcal{R}}^{\mathrm{int}}$ with $|u_l|_r = |t|_r$ for $r \in (0, r_0)$, satisfying

$$\psi_l(\phi(x)) \equiv \phi(\psi_l(x)) \pmod{\pi^l} \qquad (x \in \mathcal{R}^{\mathrm{int}}),$$

starting with $u_1 = u$. Given ψ_l, we may repeatedly invoke Corollary 2.2.3 (if $q \neq 0$ in k) or the fact that ϕ is surjective on $\tilde{\mathcal{R}}^{\mathrm{int}}$ (if $q = 0$ in k) to construct $\Delta \in \tilde{\mathcal{R}}^{\mathrm{int}}$ with

$$\phi(\pi^l \Delta/u) - q(\pi^l \Delta/u) = (\psi_l(\phi(t)) - \phi(u_l))/u^q. \tag{2.2.6.1}$$

For any $r \in (0, r_0)$,

$$|\psi_l(\phi(t))|_{r/q}, |\phi(u_l)|_{r/q} \leq |t^q|_r = |u^q|_r$$

and so the right side of (2.2.6.1) has (r/q)-norm at most 1. From this plus either the proof of Lemma 2.2.2 (if $q \neq 0$ in k) or direct inspection (if $q = 0$ in k), we deduce that $|\pi^l \Delta/u|_r \leq 1$. We may thus set $u_{l+1} = u_l + \pi^l \Delta$ to construct ψ_{l+1}; this has the desired effect because

$$\psi_{l+1}(\phi(t)) \equiv \psi_l(\phi(t)) + q\pi^l \Delta u^{q-1} \pmod{\pi^{l+1}}.$$

The property $|u_l|_r = |t|_r$ implies that each ψ_l carries $\mathcal{R}^r$ to $\tilde{\mathcal{R}}^r$ preserving $|\cdot|_r$. By continuity, we obtain a map ψ with the same property, as desired. □

Lemma 2.2.7. — *The fixed elements of $\tilde{\mathcal{R}}$ under ϕ all belong to K.*

Proof. — For $x = \sum_i a_i u^i \in \tilde{\mathcal{R}}$, we have $\phi(x) = \sum_i \phi_K(a_i) u^{qi}$. If $\phi(x) = x$ and $a_i \neq 0$ for some $i \neq 0$, then $|a_{iq^n}| = |a_i|$ for all $n \in \mathbb{Z}$; but this contradicts the fact that for any $c > 0$, the set of $i \in \mathbb{Q}$ with $|a_i| \geq c$ is well-ordered. Hence $a_i = 0$ for all $i \neq 0$, proving the claim. □

We now notice that strong Hypothesis 1.4.1 holds for $\tilde{\mathcal{R}}$.

Proposition 2.2.8. — *Let A be an $n \times n$ matrix over $\tilde{\mathcal{R}}^{\mathrm{int}}$. Then the map $\mathbf{v} \mapsto \mathbf{v} - A\phi(\mathbf{v})$ on column vectors induces a bijection on $(\tilde{\mathcal{R}}/\tilde{\mathcal{R}}^{\mathrm{bd}})^n$.*

Proof. — The proof proceeds as in Proposition 1.2.6, using the definition of $|\cdot|_r$ given in Definition 2.2.4. □

Remark 2.2.9. — As a reminder, here are some key properties of $\tilde{\mathcal{R}}$ which we will use going forward.

- Given a relative Frobenius lift ϕ on $\mathcal{R}$, we can define an action of ϕ on $\tilde{\mathcal{R}}$ and an equivariant embedding $\psi : \mathcal{R} \hookrightarrow \tilde{\mathcal{R}}$ which preserves $|\cdot|_r$ for $r \in (0, r_0)$ (Proposition 2.2.6).
- The map ϕ is bijective on $\tilde{\mathcal{R}}$.
- The map $1 - \phi$ is bijective on $\tilde{\mathcal{R}}^{\mathrm{int}}/\mathfrak{o}_K$ (easy consequence of Lemma 2.2.2).
- There is a natural direct limit topology, restricting to the direct limit of Fréchet topologies on $\mathcal{R}$, under which $\tilde{\mathcal{R}}$ is complete.

In [21] and [22], the role of our $\tilde{\mathscr{R}}$ is played by the ring $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}$, which is constructed to be minimal for the above properties; that ring coincides with the ring denoted $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{an}}$ (as in [4, §II]) or more commonly $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ (as in [11]). We opt here for the ring $\tilde{\mathscr{R}}$ instead in hopes that the construction using generalized power series makes the analogy to $\mathscr{R}$ a bit more apparent.

To conclude this section, we prove Proposition 2.1.3: if M, N are pure ϕ-modules over $\tilde{\mathscr{R}}$ obtained by base change from K, with $\mu(M) > \mu(N)$, then $\mathrm{Hom}(M, N) \neq 0$.

Proof of Proposition 2.1.3. — It is equivalent to show that if M is pure with $\mu(M) < 0$, obtained by base change from K, then $H^0(M) \neq 0$. Write $M = M_0 \otimes_K \tilde{\mathscr{R}}$ for M_0 a pure ϕ-module over K. Take any nonzero $\mathbf{w} \in M_0$ and any $i > 0$; the sum

$$\mathbf{v} = \sum_{n \in \mathbb{Z}} \phi^n(u^i \mathbf{w})$$

will converge to a nonzero element of $H^0(M)$. (Compare [22, Proposition 3.3.4(c2)].) □

2.3. Construction of fixed vectors. — We next treat Proposition 2.1.5: if M is a nonzero ϕ-module over $\tilde{\mathscr{R}}$, then for all sufficiently large integers n, $H^0(M(-n)) \neq 0$ and $H^1(M(-n)) = 0$. (Also compare [18, Theorem 4.1].)

Proof of Proposition 2.1.5. — We follow [22, Proposition 4.2.2]. View M as a space of column vectors with the action of ϕ given by multiplication by the matrix A times the componentwise action. Choose $r > 0$ so that A and A^{-1} have entries in $\tilde{\mathscr{R}}^{qr}$.

For $d \in \mathbb{Q}_{>0}$ to be specified below, define the "splitting functions" f_d^+, f_d^- as follows: given $x = \sum a_i u^i$, put

$$f_d^+(x) = \sum_{i \geq d} a_i u^i, \qquad f_d^-(x) = \sum_{i < d} a_i u^i,$$

then extend to vectors componentwise. For $\mathbf{w}$ a vector, we write $\mathbf{w}^{\pm}$ for $f_d^{\pm}(\mathbf{w})$.

Define the map $g : M \to M$ by

$$g(\mathbf{w}) = \pi^{-n} A \phi(\mathbf{w}^+) + \phi^{-1}(\pi^n A^{-1} \mathbf{w}^-)$$

and note that

$$|g(\mathbf{w})|_r \leq \max\{|\pi|^{-n} |A|_r e^{-rd(q-1)}, |\pi|^n |A^{-1}|_{qr} e^{-rd(q^{-1}-1)}\} |\mathbf{w}|_r. \tag{2.3.0.1}$$

If we can choose d such that the two quantities in the maximum in (2.3.0.1) are both strictly less than 1, then g will be contractive towards zero. This happens if

$$d \in \left(\frac{n(-\log|\pi|) + \log|A|_r}{r(q-1)}, \frac{qn(-\log|\pi|) - q\log|A^{-1}|_{qr}}{r(q-1)} \right); \tag{2.3.0.2}$$

for n sufficiently large the interval is nonempty. (Note that consistently with Proposition 2.1.3, if M is étale over K we can take any $n > 0$.)

Fix n, d satisfying (2.3.0.2). Given $\mathbf{w}$ with entries in $\tilde{\mathscr{R}}^r$, we define the sequence $\mathbf{w}_0 = \mathbf{w}, \mathbf{w}_{l+1} = g(\mathbf{w}_l)$, then set

$$\mathbf{v} = \sum_{l=0}^{\infty} (\mathbf{w}_l^+ - \phi^{-1}(\pi^n A^{-1} \mathbf{w}_l^-)), \tag{2.3.0.3}$$

so that $|\mathbf{v}|_r \leq |\mathbf{w}|_r$ and $\mathbf{v} - \pi^{-n} A\phi(\mathbf{v}) = \mathbf{w}$. We only know *a priori* that the sum defining $\mathbf{v}$ converges under $|\cdot|_r$, but using the equation $\mathbf{v} = \pi^{-n} A\phi(\mathbf{v}) + \mathbf{w}$, we may deduce that the sum converges under $|\cdot|_{r/q}$, $|\cdot|_{r/q^2}$, and so on. Hence $\mathbf{v}$ has entries in $\tilde{\mathscr{R}}^r$, yielding $H^1(M(-n)) = 0$.

To deduce $H^0(M(-n)) \neq 0$, we modify the previous construction slightly. Put $\mathbf{w} = (u^d, 0, \dots, 0)$ and construct $\mathbf{v}$ as in (2.3.0.3). Then put $\mathbf{w}'_0 = \mathbf{w}$, $\mathbf{w}'_1 = \phi^{-1}(\pi^n A^{-1} \mathbf{w}'_0)$, and $\mathbf{w}'_{l+1} = g(\mathbf{w}'_l)$ for $l \geq 1$. (That is, at the first step, transfer the boundary term u^d from the plus part to the minus part.) If we now define

$$\mathbf{v}' = -\phi^{-1}(\pi^n A^{-1} \mathbf{w}'_0) + \sum_{l=1}^{\infty} ((\mathbf{w}'_l)^+ - \phi^{-1}(\pi^n A^{-1} (\mathbf{w}'_l)^-)),$$

we obtain $\mathbf{v}' - \pi^{-n} A\phi(\mathbf{v}') = \mathbf{w}$ as before. However, $|\mathbf{v}|_r = |u^d|_r$ whereas $|\mathbf{v}'|_r < |u^d|_r$, so $\mathbf{v} - \mathbf{v}'$ is a nonzero element of $H^0(M(-n))$, as desired. □

2.4. Twisted polynomials and their Newton polygons. — Before continuing, we need to analogize, to the realm of twisted polynomials over $k((u^{\mathbb{Q}}))$, some facts about polynomials over valued fields and their Newton polygons. With a bit of care, we can obtain at the same time some results over K which we will need later (see Proposition 3.2.4).

Notation 2.4.1. — Throughout this subsection only, fix a real number $s \geq 1$, and let F be a field equipped with an automorphism $\phi = \phi_F$ and a valuation v_F with the properties that F is complete under v_F and $v_F(\phi_F(x)) = s v_F(x)$ for all $x \in F$. Let $\mathfrak{o}_F$ and $\mathfrak{m}_F$ denote the valuation subring of F and the maximal ideal of $\mathfrak{o}_F$, respectively.

Definition 2.4.2. — For $i \in \mathbb{Z}$, write $[i] = \sum_{j=0}^{i-1} s^j$, so that $[0] = 0$, $[1] = 1$, and $[i + j] = [i] + s^i[j]$. For $r \in \mathbb{R}$ and $P(T) \in F\{T^{\pm}\}$, write $P(T) = \sum_{i \in \mathbb{Z}} a_i T^i$, and write

$$v_r(P) = \min_i \{v_F(a_i) + r[i]\}.$$

Define the *homogeneous Newton polygon* of P as the lower convex hull of the set

$$\{(-[i], v_F(a_i)) : i \in \mathbb{Z}\};$$

we refer to the slopes of this polygon as the *(Newton) slopes of P*.

Lemma 2.4.3. — *For $P(T) \in F\{T\}$ and $Q(T) \in F\{T^{-1}\}$ such that $v_r(Q) \geq 0$, we have $v_r(PQ) \geq v_r(P) + v_r(Q)$.*

Proof. — Write $P(T) = \sum_{i \geq 0} a_i T^i$ and $Q(T) = \sum_{j \leq 0} b_j T^j$. We have

$$(PQ)(T) = \sum_k \sum_{i+j=k} a_i \phi^i(b_j) T^k,$$

and

$$v_F(a_i \phi^i(b_j)) + [i+j]r = v_F(a_i) + [i]r + s^i(v_F(b_j) + [j]r). \tag{2.4.3.1}$$

The right side of (2.4.3.1) is at least $v_r(P) + s^i v_r(Q)$. Since $i \geq 0$ and $s \geq 1$, if $v_r(Q) \geq 0$, then the right side of (2.4.3.1) is at least $v_r(P) + v_r(Q)$. This yields the claim. □

Proposition 2.4.4. — *Let $r_0 \in \mathbb{R}$ be a real number, and suppose that $P(T) \in F\{T\}$ and $Q(T) \in F\{T^{-1}\}$ are such that P has constant term 1 and all slopes $\leq r_0$, and Q has constant term 1 and all slopes $\geq r_0$. Then the slopes of PQ are obtained by taking the union (with multiplicities) of the sets of slopes of P and Q.*

Proof. — The conditions on the slopes of P and Q imply that

$$\begin{aligned} r \geq r_0 &\implies v_r(P) = 0, v_r(Q) \leq 0 \\ r \leq r_0 &\implies v_r(P) \leq 0, v_r(Q) = 0. \end{aligned}$$

It thus suffices to check that

$$v_r(PQ) = \begin{cases} v_r(Q) & r > r_0 \\ 0 & r = r_0 \\ v_r(P) & r < r_0. \end{cases}$$

Retain notation as in Lemma 2.4.3. If $r \geq r_0$, take the smallest j that minimizes $v_F(b_j) + [j]r$; then (2.4.3.1) equals $v_r(Q)$ for $i = 0$ but not for any other pair i, j with the same sum. If $r \leq r_0$, take the largest i that minimizes $v_F(a_i) + [i]r$; then (2.4.3.1) equals $v_r(P)$ for $j = 0$ but not for any other pair i, j with the same sum. This yields the desired result. □

Proposition 2.4.5. — *Let $r \in \mathbb{R}$ be a real number, and suppose that $R \in F\{T^{\pm}\}$ satisfies $v_r(R-1) > 0$. Then there exist $c \in F$, $P(T) \in F\{T\}$, $Q(T) \in F\{T^{-1}\}$ such that $v_F(c-1) > 0$, P has constant term 1 and all slopes $< r$, Q has constant term 1 and all slopes $> r$, and $cPQ = R$.*

Proof. — Put $c_0 = P_0 = Q_0 = 1$. Given c_i, P_i, Q_i, write $R - c_iP_iQ_i = \sum_j r_jT^j$, and put

$$\begin{aligned} c_{i+1} &= c_i + r_0 \\ P_{i+1} &= P_i + \sum_{j>0} r_jT^j \\ Q_{i+1} &= Q_i + \sum_{j<0} r_jT^j. \end{aligned}$$

Suppose that $\min\{v(c-1), v_r(P_i - 1), v_r(Q_i - 1)\} \geq v_r(R-1)$. By Lemma 2.4.3, $v_r(R - c_iP_iQ_i) \geq v_r(R-1)$, and

$$v_r(R - c_{i+1}P_{i+1}Q_{i+1}) \geq v_r(R - c_iP_iQ_i) + v_r(R-1).$$

It follows that c_i, P_i, Q_i converge to limits c, P, Q with the desired properties. □

Corollary 2.4.6. — *If $R(T) \in F\{T^{\pm}\}$ is irreducible, then it has only one slope.*

2.5. Classification of pure ϕ-modules. — We next classify the ϕ-modules over $k((u^{\mathbb{Q}}))$, then classify the pure ϕ-modules over $\tilde{\mathscr{R}}$ (Proposition 2.1.6).

Notation 2.5.1. — Throughout this subsection only, write $F = k((u^{\mathbb{Q}}))$; note that this is consistent with Notation 2.4.1 if we put $s = q$, take v_F to be the u-adic valuation, and take ϕ_F of the form $\sum c_iu^i \mapsto \sum \phi_k(c_i)u^{qi}$ for some automorphism ϕ_k of k.

Lemma 2.5.2. — *Let $P(T) \in F\{T\}$ be a twisted polynomial over F with all Newton slopes equal to 0. Then there exists $x \in \mathfrak{o}_F^*$ such that $P(\phi)(x) = 0$.*

Proof. — We may assume that P has constant term 1. Since ϕ-modules over k are trivial (by Hypothesis 2.1.1), we can find $z \in \mathfrak{o}_F^*$ with $P(\phi)(z) \in \mathfrak{m}_F$. Since $(P-1)(\phi)$ is contractive towards 0 on $\mathfrak{m}_F$, we can find $y \in \mathfrak{m}_F$ such that $P(\phi)(y) = P(\phi)(z)$. Put $x = z - y$; then $P(\phi)(x) = 0$. □

Lemma 2.5.3. — *Let $P(T) \in F\{T\}$ be a monic twisted polynomial over F with all Newton slopes equal to 0. Then $P(T)$ factors as a product $\prod_j(T - a_j)$ for some $a_j \in \mathfrak{o}_F^*$.*

Proof. — By Lemma 2.5.2, there exists $x \in \mathfrak{o}_F^*$ such that $P(\phi)(x) = 0$. By the division algorithm for twisted polynomials, $P(T)$ is right divisible by $T - a$ for $a = \phi(x)/x$; the claim then follows by induction. □

Lemma 2.5.4. — *Every irreducible ϕ-module over F is trivial.*

Proof. — Let V be an irreducible ϕ-module over F; we can then write V as $F\{T^{\pm}\}/F\{T^{\pm}\}P$ for some monic irreducible twisted polynomial $P(T)$. By Corollary 2.4.6, P has only one slope, which we can force to be 0 by rescaling. By Lemma 2.5.3, P must equal $T - a$ for some $a \in \mathfrak{o}_F^*$. But the equation $\phi(x) = ax$ has a solution $x \in \mathfrak{o}_F^*$ by Lemma 2.5.2, yielding the triviality of V. □

Proposition 2.5.5. — *Every ϕ-module over $F = k((u^{\mathbb{Q}}))$ is trivial.*

Proof. — Any ϕ-module over F can be written as a successive extension of irreducibles, which are all trivial by Lemma 2.5.4. By Corollary 2.2.3, the extensions between trivial ϕ-modules all split, yielding the claim. □

Definition 2.5.6. — For $P(T) = \sum_i a_i T^i \in F\{T^{\pm}\}$ nonzero and $z \in F$, define the *inhomogeneous Newton polygon* of the pair (P, z) as the lower convex hull of the set

$$\{(-q^i, v_F(a_i)) : i \in \mathbb{Z}\} \cup \{(0, v_F(z))\};$$

note that any slope of this polygon not involving the point $(0, v_F(z))$ is equal to $q-1$ times a slope of the homogeneous Newton polygon.

Proposition 2.5.7. — *Given $P(T) \in F\{T^{\pm}\}$ nonzero and $z \in F$, for each $r \in \mathbb{R}$ occurring as a slope of the inhomogeneous Newton polygon of (P, z), there exists $x \in F$ with $v_F(x) = r$ such that $P(\phi)(x) = z$.*

Proof. — By applying Proposition 2.4.5, we may reduce to the case where P has a single homogeneous Newton slope; by twisting, we may force that slope to be 0. By Lemma 2.5.3, we may reduce to the case $P(T) = T - a$ for $a \in \mathfrak{o}_F^*$. By Lemma 2.5.2, we may assume that $a = 1$; in this case, the claim follows from Corollary 2.2.3. □

Before proving Proposition 2.1.6, we need one more calculation, which includes Proposition 1.2.7 (see also Remark 1.2.8).

Proposition 2.5.8. — *Let $\tilde{\mathscr{E}}$ denote the $\mathfrak{m}_K$-adic completion of $\tilde{\mathscr{R}}^{\mathrm{bd}}$. Let A be an $n \times n$ matrix over $\tilde{\mathscr{R}}^{\mathrm{int}}$. If $\mathbf{v} \in \tilde{\mathscr{E}}^n$ is a column vector such that $A\mathbf{v} = \phi(\mathbf{v})$, then $\mathbf{v} \in (\tilde{\mathscr{R}}^{\mathrm{bd}})^n$.*

Proof. — By rescaling by a factor of u (as in the proof of Proposition 1.2.6), we may reduce to the case where the entries of A are bounded by 1 under $|\cdot|_r$; we may also assume $\mathbf{v}$ has entries in the completion of $\tilde{\mathscr{R}}^{\mathrm{int}}$. Write $\mathbf{v} = \sum_{j=1}^n \sum_{i \in \mathbb{Q}} c_{ij} u^i \mathbf{e}_j$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors; it suffices to show that $|c_{ij} u^i|_r \leq 1$ for all i, j, as then $\mathbf{v}$ will have entries in $\tilde{\mathscr{R}}^s$ for any $s \in (0, r)$.

Suppose the contrary; note that $|c_{ij}| \leq 1$ for all i, j by our normalization of $\mathbf{v}$, so any pair i, j with $|c_{ij} u^i|_r > 1$ must have $i < 0$, and hence

$$|\phi^{-1}(c_{ij} u^i)|_r = |c_{ij} u^{i/q}|_r < |c_{ij} u^i|_r. \tag{2.5.8.1}$$

Let h be the maximum of $|c_{ij}|$ over all pairs i,j with $|c_{ij}u^i|_r > 1$. Then there is a pair (i_0, j_0) with $|c_{i_0,j_0}| = h$ which maximizes $|c_{i_0,j_0}u^{i_0}|_r$. However, if we expand $A\mathbf{v} = \sum_{j=1}^n \sum_{i\in\mathbb{Q}} d_{ij}u^i\mathbf{e}_j$, then for each pair i,j with $|d_{ij}| = h$, we have $|\phi^{-1}(d_{ij}u^i)|_r < |c_{i_0,j_0}u^{i_0}|_r$ by (2.5.8.1). This contradicts the equality $\mathbf{v} = \phi^{-1}(A\mathbf{v})$, proving the claim. □

We now prove Proposition 2.1.6: the categories of pure ϕ-modules over K and over $\tilde{\mathscr{R}}$ of a given slope s are equivalent.

Proof of Proposition 2.1.6. — We first check full faithfulness. By Lemma 1.6.3 and twisting, it suffices to check this for $s = 0$; that is, we must check that given an étale ϕ-module M_0 over K, we must have $H^0(M_0) \cong H^0(M_0 \otimes_K \tilde{\mathscr{R}})$. By Hypothesis 2.1.1, we may assume that M_0 is trivial; then Lemma 2.2.7 yields the claim.

We next check essential surjectivity; we may proceed as in the proof of Theorem 1.6.5 to reduce to the case $s = 0$. Let M be an étale ϕ-module over $\tilde{\mathscr{R}}$, and choose an étale lattice M_0 of M. By repeated application of Proposition 2.5.5, after tensoring with the $\mathfrak{m}_K$-adic completion of $\tilde{\mathscr{R}}^{\mathrm{int}}$, we can find a basis of M_0 fixed by ϕ. By Proposition 2.5.8, this basis is in fact contained in M_0 itself, yielding the claim. □

2.6. The local calculation. — We now perform the explicit calculation that proves Proposition 2.1.7, thus completing the proof of Theorem 2.1.8. To avoid notational overload, we elide a few routine calculations that can be found in [**21**]. (Also compare [**18**, §9,10].)

Definition 2.6.1. — Let $\tilde{\mathscr{R}}^{\mathrm{tr}}$ (for "truncated") denote the set of elements of $\tilde{\mathscr{R}}$ whose support is bounded below. This forms a subring of $\tilde{\mathscr{R}}$ carrying a u-adic valuation v. Note that a unit in $\tilde{\mathscr{R}}^{\mathrm{tr}}$ is precisely an element $x = \sum_i a_i u^i$ for which the support of x has a least element j, and for which $|a_i| \leq |a_j|$ for all $i \in \mathbb{Q}$; in particular, such elements belong to $\tilde{\mathscr{R}}^{\mathrm{bd}}$, so we can apply the valuation w to them.

Lemma 2.6.2. — *Let P be a ϕ-module over K of rank 1 and degree $n > 0$, and fix a generator $\mathbf{v}$ of P.*

(a) *For any $x \in \tilde{\mathscr{R}}^{\mathrm{tr}}$ with support in $[0, +\infty)$, the class of $x\mathbf{v}$ in $H^1(P \otimes \tilde{\mathscr{R}})$ vanishes.*
(b) *Each class in $H^1(P \otimes \tilde{\mathscr{R}})$ has a representative of the form $\sum_{j=0}^{n-1} u_j\mathbf{v}$, where for each j, either $u_j = 0$, or $u_j \in (\tilde{\mathscr{R}}^{\mathrm{tr}})^*$, $w(u_j) = j$, and $v(u_j) < 0$.*

Proof. — For (a), we first use Hypothesis 2.1.1 to eliminate constant terms, then note that if x has no constant term, the sum $\sum_{i=0}^{\infty} \phi^i(x\mathbf{v})$ converges and its limit $\mathbf{w}$ satisfies $\mathbf{w} - \phi(\mathbf{w}) = x\mathbf{v}$. We deduce (b) from (a) plus a direct calculation; see also [**21**, Lemmas 4.13 and 4.14] or [**22**, Lemma 4.3.2]. □

We now prove Proposition 2.1.7: if N' is a pure ϕ^n-module over $\tilde{\mathscr{R}}$ of rank 1 and degree 1, P is a pure ϕ-module over $\tilde{\mathscr{R}}$ of rank 1 and degree -1, and

$$0 \to [n]^* N' \to M \to P \to 0 \qquad (2.6.2.1)$$

is a short exact sequence of ϕ-modules, then $H^0(M) \neq 0$.

Proof of Proposition 2.1.7. — The snake lemma gives an exact sequence

$$H^0(M) \to H^0(P) \to H^1([n]^* N'),$$

where the second map is pairing with the class $\alpha \in H^1(P^\vee \otimes [n]^* N')$ corresponding to the extension (2.6.2.1); it suffices to show that this second map has nonzero kernel.

Note that $P^\vee \otimes [n]^* N' \cong [n]^*([n]_* P^\vee \otimes N')$ as in Definition 1.3.6, so we may view α as an element of $H^1([n]^*([n]_* P^\vee \otimes N')) \cong H^1([n]_* P^\vee \otimes N')$. Similarly, we may view the pairing with α as the composition of the map $H^0(P) \to H^0([n]_* P)$ with the map $H^0([n]_* P) \to H^1(N')$ given by pairing with the class in $H^1([n]_* P^\vee \otimes N')$. If the class vanishes, there is nothing to check, so we may assume that it does not vanish.

By Proposition 2.1.6, P and N' are obtained by base change from certain ϕ- and ϕ^n-modules P_0 and N'_0, respectively, over K; choose generators $\mathbf{v}$ and $\mathbf{w}$ of P_0 and N'_0, and define $\lambda, \mu \in K^*$ by $\phi(\mathbf{v}) = \lambda \mathbf{v}$ and $\phi^n(\mathbf{w}) = \mu \mathbf{w}$. Put $Q_0 = [n]_* P_0^\vee \otimes N'_0$ and $Q = [n]_* P^\vee \otimes N' \cong Q_0 \otimes_K \tilde{\mathscr{R}}$; let $\mathbf{x}$ be the generator $\mathbf{v}^\vee \otimes \mathbf{w}$ of Q_0 (where $\mathbf{v}^\vee$ is the generator of $P^\vee$ dual to $\mathbf{v}$). By Lemma 2.6.2, we can then represent the class $\alpha \in H^1(Q)$ by a nonzero element of Q of the form $\sum_{j=0}^n u_j \mathbf{x}$, where each u_j is either zero or a unit in $\tilde{\mathscr{R}}^{\mathrm{tr}}$ with $w(u_j) = j$ and $v(u_j) < 0$.

We now follow [**21**, Lemma 4.12]. For $j \in \{0, \dots, n\}$ such that $u_j \neq 0$, $l \in \mathbb{Z}$, and $m \in (0, +\infty)$, define

$$e(j, l, m) = (v(u_j) + m q^{-l}) q^{-n(j+l)}.$$

For fixed j and m, $e(j, l, m)$ approaches 0 from below as $l \to +\infty$, and tends to $+\infty$ as $l \to -\infty$. Hence the minimum $h(m) = \min_{j,l}\{e(j, l, m)\}$ is well-defined; we observe that h is a continuous, piecewise linear, and increasing map from $(0, +\infty)$ to $(-\infty, 0)$, and that $h(qm) = q^{-n} h(m)$ because $e(j, l+1, qm) = q^{-n} e(j, l, m)$. Another interpretation is that the lower convex hull of the set H of points

$$(-q^{-nj-(n+1)l}, q^{-nj-nl} v(u_j)) \qquad (j = 0, \dots, n; \quad l \in \mathbb{Z})$$

has all slopes positive, and all segments finite.

Pick $r \in (0, +\infty)$ at which h changes slope; that is, r is a slope of the lower convex hull of H. Let S denote the set of ordered pairs (j, l) for which $e(j, l, r) < q^{-n} h(r)$; this set is finite. Let T be the set of ordered pairs (j, l) for which $e(j, l, r) < 0$; this set (which contains S) is infinite, but the values of l for pairs $(j, l) \in T$ are bounded below. For each pair (j, l), put $s(j, l) = \lfloor \log_{q^n}(h(r)/e(j, l, r)) \rfloor$. Then the following properties hold.

(a) For $(j,l) \in T$, $s(j,l) \geq 0$.
(b) For $(j,l) \in T$, $e(j,l,r)q^{ns(j,l)} \in [h(r), q^{-n}h(r))$.
(c) We have $(j,l) \in S$ if and only if $(j,l) \in T$ and $s(j,l) = 0$.
(d) For any $c > 0$, there are only finitely many pairs $(j,l) \in T$ with $s(j,l) \leq c$.

Define the twisted powers $\lambda^{\{m\}}$ and $\mu^{\{m\}}$ of λ and μ by the two-way recurrences

$$\lambda^{\{0\}} = 1, \qquad \lambda^{\{m+1\}} = \phi(\lambda^{\{m\}})\lambda$$
$$\mu^{\{0\}} = 1, \qquad \mu^{\{m+1\}} = \phi^n(\mu^{\{m\}})\mu.$$

For $c \in \mathbb{R}$, let U_c be the set of $z \in \tilde{\mathscr{R}}^{\mathrm{tr}} \cap \tilde{\mathscr{R}}^{\mathrm{int}}$ with $v(z) \geq c$. Then the function

$$R(z) = \sum_{(j,l)\in T} \mu^{\{-j-l+s(j,l)\}}\phi^{-nj-nl+ns(j,l)}(u_j\lambda^{\{-l\}}\phi^{-l}(z))$$

carries U_r into $U_{h(r)}$ by a direct calculation. Modulo π, we have

$$R(z) \equiv \sum_{(j,l)\in S} \mu^{\{-j-l\}}\phi^{-nj-nl}(u_j\lambda^{\{-l\}}\phi^{-l}(z)); \tag{2.6.2.2}$$

note that the values $-nj-(n+1)l$ are distinct for all $(j,l) \in S$, since j only runs over $\{0,\dots,n\}$. Write the reduction modulo π of the right side of (2.6.2.2) as $Q(\phi)(z)$ for some twisted Laurent polynomial $Q(T) \in F\{T^{\pm}\}$ with $F = k((u^{\mathbb{Q}}))$. By Proposition 2.5.7 applied repeatedly, we can construct a nonzero $z \in U_r$ such that $R(z) = 0$.

One now calculates using Lemma 2.6.2(a) (see [**21**, Lemma 4.12] for the full calculation) that the element

$$\sum_{l\in\mathbb{Z}} \phi^{-l}(z\mathbf{v}) = \sum_{l\in\mathbb{Z}} \lambda^{\{-l\}}\phi^{-l}(z)\mathbf{v} \in H^0(P)$$

pairs to zero with the class of α. This yields the desired result. □

3. Descending the slope filtration

As noted at the beginning of the previous section, the proof of the slope filtration theorem (Theorem 1.7.1) consists of two stages, the first of which (classifying ϕ-modules over the overring $\tilde{\mathscr{R}}$ of $\mathscr{R}$) has been accomplished in the previous section. In this section, we explain how to descend the resulting slope filtration from $\tilde{\mathscr{R}}$ back to $\mathscr{R}$.

As was done in the previous section, we recommend on a first reading to read only the overview (Subsection 3.1), then return later for the technical details.

3.1. Overview

Definition 3.1.1. — We now revert to allowing K to be an arbitrary field as in Definition 1.1.1. Choose a complete extension L of K with the same value group, admitting an extension ϕ to an automorphism, such that every étale ϕ-module over L is trivial. More precisely, form such an L by first taking the completed direct limit of $K \xrightarrow{\phi} K \xrightarrow{\phi} \cdots$ and then applying Proposition 3.2.4 below. Under these conditions, we can embed $\mathscr{R}_K$ into $\mathscr{R}_L$, and then embed $\mathscr{R}_L$ into $\tilde{\mathscr{R}}_L$ as in Proposition 2.2.6.

Recall that we are trying to prove Theorem 1.7.1, which states that every module-semistable ϕ-module over $\mathscr{R}$ is pure. As noted earlier, this result follows from Theorem 2.1.8 (which asserts that module-semistable ϕ-modules over $\tilde{\mathscr{R}}_L$ are pure) plus the following assertions.

Theorem 3.1.2. — *Let M be a module-semistable ϕ-module over $\mathscr{R}$. Then $M \otimes \tilde{\mathscr{R}}_L$ is module-semistable.*

Theorem 3.1.3. — *Let M be a ϕ-module over $\mathscr{R}$ such that $M \otimes \tilde{\mathscr{R}}_L$ is pure. Then M is pure.*

The proofs of Theorems 3.1.2 and 3.1.3 amount to faithfully flat descent: Theorem 3.1.2 relies on the fact that the first step of the HN filtration of $M \otimes \tilde{\mathscr{R}}_L$ descends to $\mathscr{R}$, while Theorem 3.1.3 depends on the fact that the pure ϕ-module over $\tilde{\mathscr{R}}_L^{\mathrm{bd}}$ obtained by descending $M \otimes \tilde{\mathscr{R}}_L$ itself descends to $\mathscr{R}^{\mathrm{bd}}$. The rest of this section will be occupied with setting up the descent formalism, then making the calculations that allow the use of faithfully flat descent.

3.2. Splitting étale ϕ-modules. — We now construct the field L demanded by Definition 3.1.1.

Definition 3.2.1. — Suppose that ϕ_K is bijective. By an *admissible extension* of K, we will mean a field L containing K, complete for a nonarchimedean absolute value extending the one on K with the same value group, and equipped with an isometric field automorphism ϕ_L extending ϕ_K.

Lemma 3.2.2. — *For any $z \in K^*$, there exists an admissible extension L of K such that the equation $\phi(x) - x = z$ has a solution $x \in L$.*

Proof. — Let L be the completion of the rational function field $K(x)$ for the Gauss norm with $|x| = |z|$. Extend ϕ_K to an automorphism ϕ_L of L by setting $\phi_L(x) = x + z$. □

Lemma 3.2.3. — *Let $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ be a twisted polynomial over $\mathfrak{o}_K$ with $|a_0| = 1$. Then there exists an admissible extension L of K such that the equation $P(\phi)(x) = 0$ has a solution $x \in \mathfrak{o}_L^*$.*

Proof. — Let L be the completion of the rational function field $K(y_0, \dots, y_{n-1})$ under the Gauss norm normalized with $|y_0| = \cdots = |y_{n-1}| = 1$. Extend ϕ_K to an automorphism ϕ_L of L by setting $\phi_L(y_i) = y_{i+1}$ for $i = 0, \dots, n-2$ and $\phi_L(y_{n-1}) = -a_{n-1}y_{n-1} - \cdots - a_0y_0$, then take $x = y_0$. □

Proposition 3.2.4. — *There exists a complete extension L of K with the same value group, equipped with an extension of ϕ_K, such that any étale ϕ-module over L is trivial.*

Proof. — It suffices to construct L trivializing a single irreducible étale ϕ-module M over K, as we can construct the desired field by transfinitely iterating this construction and completing at all limit stages.

Since M is irreducible, we must have $M \cong K\{T^{\pm}\}/K\{T^{\pm}\}P(T)$ for some irreducible monic twisted polynomial $P(T)$. If we write $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$, then $|a_0| = 1$ because $\deg(M) = 0$. By Corollary 2.4.6 (in the case $s = 1$), P can only have one Newton slope, which must be 0; hence $P(T)$ has coefficients in $\mathfrak{o}_K$. We can then apply Lemma 3.2.3 to construct L over which the equation $P(\phi)(x) = 0$ has a solution $x \in \mathfrak{o}_L^*$; that solution gives rise to a nontrivial ϕ-submodule of M.

Repeating the construction, we obtain a field over which M becomes a successive extension of trivial étale ϕ-modules of rank 1. By repeated use of Lemma 3.2.2, we can split this filtration by passing to a suitably large L. This yields the claim. □

Remark 3.2.5. — Note that the field L constructed above is not a Picard-Vessiot extension of K in the sense of the Galois theory of difference fields; this Galois theory is a bit complicated because it cannot be carried out within the category of fields, as examples like the difference equation $\phi(x) = -x$ show. See [**39**, Chapter 1] for more discussion of this point, and a development of difference Galois theory in a restricted setting; see also [**2**] for a more general development. (Thanks to Michael Singer for pointing out this reference.)

3.3. The use of faithfully flat descent. — In this subsection, we set up faithfully flat descent and illustrate how we will use it to prove Theorems 3.1.2 and 3.1.3.

Definition 3.3.1. — Let $R \to S$ be a faithfully flat morphism of rings equipped with compatible endomorphisms ϕ. Let M be a ϕ-module over R, put $M_S = M \otimes_R S$, and let N_S be a ϕ-submodule of M_S. We say that N_S *descends to* R if there exists a ϕ-submodule N of M such that the image of $N \otimes_R S$ in M_S coincides with N_S. We say a filtration descends to R if each term does so.

Proposition 3.3.2. — *Let $R \to S$ be a faithfully flat morphism of domains equipped with compatible endomorphisms ϕ. Put $S_2 = S \otimes_R S$ and define $i_1, i_2 : S \to S_2$ by $i_1(s) = s \otimes 1$ and $i_2(s) = 1 \otimes s$. Let M be a ϕ-module over R, put $M_S = M \otimes_R S$, and let N_S be a ϕ-submodule of M_S. Then N_S descends to R if and only if $N \otimes_{i_1} S_2 = N \otimes_{i_2} S_2$ within $M \otimes_R S_2$; moreover, if this occurs, then there is a* unique *ϕ-submodule N of M such that $N_S = N \otimes_R S$ within M_S.*

Proof. — The equality $N \otimes_{i_1} S_2 = N \otimes_{i_2} S_2$ implies that the effective descent datum obtained from M induces a descent datum on N (the cocycle condition can be checked on M). We may thus apply faithfully flat descent for modules [**1**, Exposé VIII, Corollaire 1.3] to conclude. □

We use faithfully flat descent as follows.

Definition 3.3.3. — Define

$$\begin{aligned} \mathscr{S} &= \tilde{\mathscr{R}}_L \otimes_{\mathscr{R}} \tilde{\mathscr{R}}_L \\ \mathscr{S}^{\mathrm{bd}} &= \tilde{\mathscr{R}}_L^{\mathrm{bd}} \otimes_{\mathscr{R}^{\mathrm{bd}}} \tilde{\mathscr{R}}_L^{\mathrm{bd}} \\ \mathscr{S}^{\mathrm{int}} &= \tilde{\mathscr{R}}_L^{\mathrm{int}} \otimes_{\mathscr{R}^{\mathrm{int}}} \tilde{\mathscr{R}}_L^{\mathrm{int}}. \end{aligned}$$

We will show later that $\mathscr{R} \to \tilde{\mathscr{R}}_L$, $\mathscr{R}^{\mathrm{bd}} \to \tilde{\mathscr{R}}_L^{\mathrm{bd}}$ are faithfully flat and that $\mathscr{S}^{\mathrm{bd}} \to \mathscr{S}$ is injective (Remark 3.5.3).

The following weak analogue of Proposition 1.2.6 will be proved in Subsection 3.5.

Proposition 3.3.4. — *Let A be an $n \times n$ matrix over $\mathscr{S}^{\mathrm{int}}$, and let $\mathbf{v}$ be a column vector over $\mathscr{S}$ such that $\mathbf{v} = A\phi(\mathbf{v})$. Then $\mathbf{v}$ has entries in $\mathscr{S}^{\mathrm{bd}}$.*

We now demonstrate how Proposition 3.3.4 can be used to establish the theorems asserted at the start of this section.

Proof of Theorem 3.1.2. — Suppose that $M \otimes \tilde{\mathscr{R}}_L$ is not semistable. Let $0 = M_{L,0} \subset M_{L,1} \subset \cdots \subset M_{L,l} = M_L$ denote the HN filtration of $M_L = M \otimes \tilde{\mathscr{R}}_L$. We will show that $M_{L,1} \otimes_{i_2} S_2 \subseteq M_{L,j} \otimes_{i_1} S_2$ for $j = l, l-1, \ldots, 1$ by descending induction; the base case $j = l$ is trivial.

Given that $M_{L,1} \otimes_{i_2} S_2 \subset M_{L,j} \otimes_{i_1} S_2$ for some $j > 1$, we get a homomorphism

$$M_{L,1} \otimes_{i_2} S_2 \to (M_{L,j}/M_{L,j-1}) \otimes_{i_1} S_2.$$

Since $M_{L,1}$ and $M_{L,j}/M_{L,j-1}$ are pure and $\mu(M_{L,1}) < \mu(M_{L,j}/M_{L,j-1})$, this homomorphism is forced to vanish: otherwise, by Proposition 3.3.4 the morphism would be defined over $\mathscr{S}^{\mathrm{bd}}$, but in that case it would have to preserve slopes because $\mathscr{S}^{\mathrm{bd}}$ carries an $\mathfrak{m}_K$-adic valuation. Hence $M_{L,1} \otimes_{i_2} S_2 \subseteq M_{L,j-1} \otimes_{i_1} S_2$, completing the induction.

The induction shows that $M_{L,1}$ satisfies the condition for faithfully flat descent (Proposition 3.3.2), so it descends to $\mathscr{R}$. Hence M cannot be semistable either. $\square$

Proof of Theorem 3.1.3. — By applying $[a]_*$ (invoking Lemma 1.6.3) and twisting, we may reduce to the case $\mu(M) = 0$, so $M \otimes \tilde{\mathscr{R}}_L$ is étale. Choose a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of an étale lattice of $M \otimes \tilde{\mathscr{R}}_L$, so that the matrix A defined by $\phi(\mathbf{v}_j) = \sum_i A_{ij} \mathbf{v}_i$ is invertible over $\tilde{\mathscr{R}}_L^{\mathrm{int}}$.

There exists an invertible change-of-basis matrix U over $\mathscr{S}$ such that

$$\mathbf{v}_j \otimes_{i_1} 1 = \sum_i U_{ij}(\mathbf{v}_i \otimes_{i_2} 1).$$

Upon applying ϕ to both sides, we deduce that $U(A \otimes_{i_1} 1) = (A \otimes_{i_2} 1)\phi(U)$. By Proposition 3.3.4, U has entries in $\mathscr{S}^{\mathrm{bd}}$, as does its inverse by the same argument with M replaced by $M^\vee$. Hence by Proposition 3.3.2, M descends to $\mathscr{R}^{\mathrm{bd}}$; let N be the resulting ϕ-module over $\mathscr{R}^{\mathrm{bd}}$.

Choose any basis of N and let P be the $\mathscr{R}^{\mathrm{int}}$-span of the images of the basis elements under powers of ϕ. By computing in terms of $\mathbf{v}_1, \dots, \mathbf{v}_n$, we see that P is bounded, hence is a ϕ-stable $\mathscr{R}^{\mathrm{int}}$-lattice in M. By Lemma 1.5.7, $P \otimes \tilde{\mathscr{R}}_L^{\mathrm{int}}$ is a ϕ-module, as then must be P. Thus M is étale, as desired. $\square$

It now remains to prove the faithful flatness results and to make the calculation to check Proposition 3.3.4; these occupy the remainder of the chapter.

3.4. Interlude: tensoring over Bézout domains. — In order to use faithfully flat descent for our purposes, it will help to gather a few facts about tensoring over Bézout domains.

Proposition 3.4.1. — *Let $R \hookrightarrow S$ be an inclusion of domains with R Bézout. Then S is faithfully flat over R if and only if $S^* \cap R = R^*$.*

Proof. — Recall that S is flat (resp. faithful) over R if and only if for each finitely generated proper ideal I of R, the multiplication map $I \otimes S \to S$ is injective (resp. not surjective). Since R is Bézout, I admits a single generator $r \notin R^*$, and $I \otimes S = rR \otimes S \cong rS$, so the map $I \otimes S \to S$ is injective, and it is surjective if and only if $r \in S^*$. This yields the claim. $\square$

Lemma 3.4.2. — *Let M, N be modules over a Bézout domain R. Given a presentation $\sum_{i=1}^n y_i \otimes z_i$ of $x \in M \otimes_R N$ and elements $u_1, \dots, u_n \in R$ generating the unit ideal, there exists another presentation $\sum_{j=1}^n y'_j \otimes z'_j$ of x with $y'_1 = \sum_{i=1}^n u_i y_i$.*

Proof. — By [**21**, Lemma 2.3], we can construct an invertible matrix U over R with $U_{i1} = u_i$ for $i = 1, \dots, n$. Then

$$\begin{aligned}\sum_i y_i \otimes z_i &= \sum_{i,j,l} U_{ij}(U^{-1})_{jl} y_i \otimes z_l \\ &= \sum_j \left(\sum_i U_{ij} y_i\right) \otimes \left(\sum_l (U^{-1})_{jl} z_l\right),\end{aligned}$$

so we may take $y'_j = \sum_{i=1}^n U_{ij} y_i$ and $z'_j = \sum_{l=1}^n (U^{-1})_{jl} z_l$. □

Corollary 3.4.3. — *Let M, N be modules over a Bézout domain R. If $\sum_{i=1}^n y_i \otimes z_i$ is a presentation of some $x \in M \otimes_R N$ with n minimal, then $y_1, \dots, y_n$ are linearly independent over R.*

Proof. — If on the contrary $y_1, \dots, y_n$ are linearly dependent over R, then we can find $u_1, \dots, u_n \in R$ such that $u_1 y_1 + \cdots + u_n y_n = 0$. By the Bézout property, $u_1, \dots, u_n$ generate a principal ideal, so we can divide through by a generator to reduce to the case where $u_1, \dots, u_n$ generate the unit ideal. Applying Lemma 3.4.2 now yields a contradiction to the minimality of n. □

3.5. Projections.

— The key to the descent argument is the construction of a certain projection from $\tilde{\mathscr{R}}_L$ back to $\mathscr{R}$, sectioning the inclusion going the other way that was constructed by Proposition 2.2.6. We now construct this projection, then use it to resolve all the outstanding statements needed to complete the proof of Theorem 1.7.1.

Definition 3.5.1. — Let ℓ be the residue field of L, fix a basis $\overline{B}$ of ℓ over k containing 1, lift $\overline{B}$ to a subset B of $\mathfrak{o}_L$ containing 1, and fix a uniformizer π of K. Then as in [**20**, Proposition 4.1], one sees that every element $x \in \tilde{\mathscr{R}}_L^{\mathrm{int}}/\mathfrak{m}_K^n \tilde{\mathscr{R}}_L^{\mathrm{int}}$ can be written uniquely as a formal sum

$$\sum_{\alpha \in [0,1) \cap \mathbb{Q}} \sum_{b \in B} x_{\alpha,b} u^\alpha b \qquad (x_{\alpha,b} \in \mathscr{R}^{\mathrm{int}}/\mathfrak{m}_K^n \mathscr{R}^{\mathrm{int}})$$

in which:

- for each $\alpha \in [0,1) \cap \mathbb{Q}$, there are only finitely many b for which $x_{\alpha,b} \neq 0$;
- if we write S_c for the set of $\alpha \in [0,1) \cap \mathbb{Q}$ for which the t-adic valuation of any $x_{\alpha,b}$ (which is well-defined because $x_{\alpha,b}$ is truncated modulo π^n) is less than c, then S_c is well-ordered for all c and empty for sufficiently small c.

Given x thusly presented, write $f(x) = x_{0,1}$; then again as in [**20**, Proposition 4.1], one checks that for r_0 as in Remark 1.2.5 and $r \in (0, r_0)$, f induces a continuous map $\tilde{\mathscr{R}}_L^r \to \mathscr{R}^r$ with the property that for $x \in \tilde{\mathscr{R}}_L^r$,

$$|x|_r = \sup_{\alpha \in [0,1) \cap \mathbb{Q}, a \in L^*} \{|a|^{-1} e^{-\alpha r} |f(a u^{-\alpha} x)|_r\}. \tag{3.5.1.1}$$

(Compare also [**19**, Proposition 8.1] and [**22**, Lemma 2.2.19].)

Proposition 3.5.2. — *The multiplication map $\tilde{\mathscr{R}}_L^{\mathrm{bd}} \otimes_{\mathscr{R}^{\mathrm{bd}}} \mathscr{R} \to \tilde{\mathscr{R}}_L$ is injective.*

Proof. — Suppose the contrary; choose $x \neq 0$ in the kernel of the multiplication map, and choose a presentation $x = \sum_{i=1}^n y_i \otimes z_i$ with n minimal. Then $z_1, \dots, z_n$ are linearly independent over $\mathscr{R}^{\mathrm{bd}}$ by Corollary 3.4.3. On the other hand, as a corollary of (3.5.1.1), we may choose $\alpha \in [0,1) \cap \mathbb{Q}$ and $a \in L^*$ such that $f(au^{-\alpha} y_1) \neq 0$; we then obtain the nontrivial dependence relation $0 = \sum_{i=1}^n f(au^{-\alpha} y_i) z_i$, contradiction. $\square$

Remark 3.5.3. — We now have a number of faithfully flat inclusions. For one, $\mathscr{R}^{\mathrm{bd}} \to \mathscr{R}$ is faithfully flat by Proposition 3.4.1 and the fact that $\mathscr{R}^* = (\mathscr{R}^{\mathrm{bd}})^*$ (Remark 1.1.5). For another, $\mathscr{R} \to \tilde{\mathscr{R}}_L$ is faithfully flat by Proposition 3.4.1 and the fact that $\tilde{\mathscr{R}}_L^* = (\tilde{\mathscr{R}}_L^{\mathrm{bd}})^*$ (Remark 2.2.5); similarly, $\mathscr{R}^{\mathrm{bd}} \to \tilde{\mathscr{R}}_L^{\mathrm{bd}}$ is faithfully flat. Putting these together and using Proposition 3.5.2 yields injections

$$\tilde{\mathscr{R}}_L^{\mathrm{bd}} \otimes_{\mathscr{R}^{\mathrm{bd}}} \tilde{\mathscr{R}}_L^{\mathrm{bd}} \hookrightarrow \tilde{\mathscr{R}}_L^{\mathrm{bd}} \otimes_{\mathscr{R}^{\mathrm{bd}}} \tilde{\mathscr{R}}_L \cong (\tilde{\mathscr{R}}_L^{\mathrm{bd}} \otimes_{\mathscr{R}^{\mathrm{bd}}} \mathscr{R}) \otimes_{\mathscr{R}} \tilde{\mathscr{R}}_L \hookrightarrow \tilde{\mathscr{R}}_L \otimes_{\mathscr{R}} \tilde{\mathscr{R}}_L;$$

that is, $\mathscr{S}^{\mathrm{bd}} \to \mathscr{S}$ is injective.

In order to calculate on $\mathscr{S}$, we use the following two-variable analogue of (3.5.1.1).

Lemma 3.5.4. — *For $x \in \mathscr{S}$, we have $x \in \mathscr{S}^{\mathrm{bd}}$ if and only if for some $r > 0$, the quantities*

$$|ab|^{-1} e^{-\alpha s - \beta s} |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})x)|_s \tag{3.5.4.1}$$

are bounded over all $s \in (0, r]$, all $a, b \in L^*$, and all $\alpha, \beta \in [0,1) \cap \mathbb{Q}$.

Proof. — If $x \in \mathscr{S}^{\mathrm{bd}}$, then we can bound the quantity (3.5.4.1) by bounding each term in a presentation of x. Conversely, suppose the quantity (3.5.4.1) is bounded. Choose a presentation $x = \sum_{i=1}^n y_i \otimes z_i$ with $y_i, z_i \in \tilde{\mathscr{R}}_L$ and n minimal. We proceed by induction on n; we may assume $x \neq 0$. Then $y_1 \neq 0$, so we can choose a, α with $f(au^{-\alpha} y_1) \neq 0$.

By (3.5.1.1), $\sum_{i=1}^n f(au^{-\alpha} y_i) z_i \in \tilde{\mathscr{R}}_L^{\mathrm{bd}}$; in particular, the ideal generated by the $f(au^{-\alpha} y_i)$ in $\mathscr{R}$ extends to the unit ideal in $\tilde{\mathscr{R}}_L$. Since the ideal in $\mathscr{R}$ is finitely generated, it is principal, and since $\tilde{\mathscr{R}}_L^* = (\tilde{\mathscr{R}}_L^{\mathrm{bd}})^*$, the generator in $\mathscr{R}$ must already be a unit. That is, the $f(au^{-\alpha} y_i)$ generate the unit ideal in $\mathscr{R}$; by Lemma 3.4.2, we can choose another presentation $x = \sum_{i=1}^n y_i' \otimes z_i'$ with $z_1' = \sum_{i=1}^n f(au^{-\alpha} y_i) z_i \in \tilde{\mathscr{R}}_L^{\mathrm{bd}}$. We must have $z_1' \neq 0$ to avoid contradicting the minimality of n.

Pick b, β so that $f(bu^{-\beta} z_1')$ is nonzero and hence is a unit in $\mathscr{R}$ (since it must lie in $\mathscr{R}^{\mathrm{bd}}$). Put $c_i = f(bu^{-\beta} z_i')/f(bu^{-\beta} z_1')$ for $i = 2, \dots, n$, then set

$$y_i'' = \begin{cases} y_1' + c_2 y_2' + \cdots + c_n y_n' & i = 1 \\ y_i' & i > 1, \end{cases} \qquad z_i'' = \begin{cases} z_i' & i = 1 \\ z_i' - c_i z_1' & i > 1, \end{cases}$$

so that $x = \sum_{i=1}^n y_i'' \otimes z_i''$. Then $f(bu^{-\beta}z_i'') = 0$ for $i = 2, \dots, n$, so $y_1'' f(bu^{-\beta}z_1'') = \sum_{i=1}^n y_i'' f(bu^{-\beta}z_i'') \in \tilde{\mathscr{R}}_L^{\mathrm{bd}}$ by (3.5.1.1). Since already $f(bu^{-\beta}z_1'') \in \mathscr{R}^{\mathrm{bd}}$, we have $y_1'' \in \tilde{\mathscr{R}}_L^{\mathrm{bd}}$. Applying the induction hypothesis to $x - y_1'' \otimes z_1'' = \sum_{i=2}^n y_i'' \otimes z_i''$ yields the claim. □

Proof of Proposition 3.3.4. — For each entry $\mathbf{v}_i$ of $\mathbf{v}$, choose a presentation $\sum_j y_{ij} \otimes z_{ij}$ with $y_{ij}, z_{ij} \in \tilde{\mathscr{R}}_L$. As in the proof of Proposition 1.2.6, after possibly rescaling by a power of u, we may choose $r \in (0, r_0)$ such that each term in a presentation of A has entries in $\tilde{\mathscr{R}}_L^r$ and is bounded by 1 on the annulus $e^{-r} \leq |u| < 1$; we may also ensure that $y_{ij}, z_{ij} \in \tilde{\mathscr{R}}_L^r$ for all i, j. Choose $c > 0$ such that for $s \in [r/q, r]$ and all i, j, $|y_{ij}|_s \leq c$ and $|z_{ij}|_s \leq c$ (possible because we are picking s in a closed interval); then for all nonnegative integers m, we have $|\phi^m(y_{ij})|_{s/q^m} \leq c$ and $|\phi^m(z_{ij})|_{s/q^m} \leq c$. From the equation

$$\mathbf{v} = A\phi(A) \cdots \phi^{m-1}(A)\phi^m(\mathbf{v}),$$

we deduce that for all $\alpha, \beta \in [0, 1)$ and all $a, b \in L^*$,

$$|ab|^{-1} e^{-\alpha s - \beta s} |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})\mathbf{v})|_s \leq c$$

for all $s \in [r/q^{m+1}, r/q^m]$; by varying m, we get the same conclusion for all $s \in (0, r]$. By Lemma 3.5.4, $\mathbf{v}$ has entries in $\mathscr{S}^{\mathrm{bd}}$, as desired. □

References

[1] *Revêtements étales et groupe fondamental (SGA 1)* – Documents Mathématiques, vol. 3, Société Mathématique de France, 2003.

[2] Y. André – "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup. (4)* **5** (2001), p. 685–739.

[3] J. Bellaïche & G. Chenevier – "p-adic families of Galois representations and higher rank Selmer groups", preprint arXiv:math.NT/0602340, 2007.

[4] L. Berger – "Représentations p-adiques et équations différentielles", *Invent. Math.* **148** (2002), p. 219–284.

[5] ______, "Équations différentielles p-adiques et (ϕ, N)-modules filtrés", preprint arXiv:math.NT/0406601, 2004.

[6] L. Berger & P. Colmez – "Familles de représentations de de Rham et monodromie p-adique", in this volume.

[7] V. G. Berkovich – *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, 1990.

[8] H. Cartan & S. Eilenberg – *Homological algebra*, Princeton University Press, 1956.

[9] F. Cherbonnier & P. Colmez – "Représentations p-adiques surconvergentes", *Invent. Math.* **133** (1998), p. 581–611.

[10] R. Coleman & B. Mazur – "The eigencurve", in *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, London Math. Soc. Lecture Note Ser., vol. 254, Cambridge Univ. Press, 1998, p. 1–113.

[11] P. Colmez – "Les conjectures de monodromie p-adiques", *Astérisque* **290** (2003), p. 53–101, Séminaire Bourbaki, vol. 2001/2002, exposé n° 897.

[12] ———, "La série principale unitaire de $\mathrm{GL}_2(\mathbb{Q}_p)$", preprint `http://www.math.jussieu.fr/~colmez/`, 2007.

[13] ———, "Répresentations triangulines de dimension 2", in this volume.

[14] R. Crew – "Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve", *Ann. Sci. École Norm. Sup. (4)* **31** (1998), p. 717–763.

[15] P. Deligne – "La conjecture de Weil. II", *Publ. Math. I.H.É.S.* **52** (1980), p. 137–252.

[16] U. Hartl – "On a conjecture of Rapoport and Zink", preprint arXiv:math.NT/0605254, 2006.

[17] ———, "Period spaces for Hodge structures in equal characteristic", preprint arXiv:math.NT/0511686, 2006.

[18] U. Hartl & R. Pink – "Vector bundles with a Frobenius structure on the punctured unit disc", *Compos. Math.* **140** (2004), p. 689–716.

[19] A. J. de Jong – "Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic", *Invent. Math.* **134** (1998), p. 301–333.

[20] K. S. Kedlaya – "Full faithfulness for overconvergent F-isocrystals", in *Geometric aspects of Dwork theory. Vol. I, II*, Walter de Gruyter, 2004, p. 819–835.

[21] ———, "A p-adic local monodromy theorem", *Ann. of Math. (2)* **160** (2004), p. 93–184.

[22] ———, "Slope filtrations revisited", *Doc. Math.* **10** (2005), p. 447–525.

[23] ———, "Finiteness of rigid cohomology with coefficients", *Duke Math. J.* **134** (2006), p. 15–97.

[24] ———, "Fourier transforms and p-adic 'Weil II'", *Compos. Math.* **142** (2006), p. 1426–1450.

[25] C. Khare & J.-P. Wintenberger – "Serre's modularity conjecture I", preprint `http://www.math.utah.edu/~shekhar`, 2006.

[26] ———, "Serre's modularity conjecture II", preprint `http://www.math.utah.edu/~shekhar`, 2006.

[27] M. Kisin – "Crystalline representations and F-crystals", in *Algebraic geometry and number theory*, Progr. Math., vol. 253, Birkhäuser, 2006, p. 459–496.

[28] ———, "The Fontaine-Mazur conjecture for GL_2", preprint `http://math.uchicago.edu/~kisin`, 2006.

[29] M. Lazard – "Les zéros des fonctions analytiques d'une variable sur un corps valué complet", *Publ. Math. I.H.É.S.* **14** (1962), p. 47–75.

[30] A. H. M. Levelt – "Jordan decomposition for a class of singular differential operators", *Ark. Mat.* **13** (1975), p. 1–27.

[31] B. Malgrange – "Connexions méromorphes. II. Le réseau canonique", *Invent. Math.* **124** (1996), p. 367–387.

[32] J. I. Manin – "Theory of commutative formal groups over fields of finite characteristic", *Uspehi Mat. Nauk* **18** (1963), p. 3–90.

[33] V. B. Mehta & A. Ramanathan – "Semistable sheaves on projective varieties and their restriction to curves", *Math. Ann.* **258** (1981/82), p. 213–224.

[34] M. S. Narasimhan & C. S. Seshadri – "Stable bundles and unitary bundles on a compact Riemann surface", *Proc. Nat. Acad. Sci. U.S.A.* **52** (1964), p. 207–211.

[35] ———, "Stable and unitary vector bundles on a compact Riemann surface", *Ann. of Math. (2)* **82** (1965), p. 540–567.

[36] O. Ore – "Theory of non-commutative polynomials", *Ann. of Math. (2)* **34** (1933), p. 480–508.

[37] D. S. Passman – *The algebraic structure of group rings*, Robert E. Krieger Publishing Co. Inc., 1985, Reprint of the 1977 original.

[38] C. Praagman – "The formal classification of linear difference operators", *Nederl. Akad. Wetensch. Indag. Math.* **45** (1983), p. 249–261.

[39] M. van der Put & M. F. Singer – *Galois theory of difference equations*, Lecture Notes in Mathematics, vol. 1666, Springer, 1997, errata at `http://www4.ncsu.edu/~singer/papers/errata.ps`.

[40] M. Rapoport & T. Zink – *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, 1996.

[41] N. Tsuzuki – "The overconvergence of morphisms of étale ϕ-∇-spaces on a local field", *Compositio Math.* **103** (1996), p. 227–239.

[42] ______, "Finite local monodromy of overconvergent unit-root F-isocrystals on a curve", *Amer. J. Math.* **120** (1998), p. 1165–1190.

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