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SEMI-CLASSICAL MEASURES AND ENTROPY
[after Nalini Anantharaman and Stéphane Nonnenmacher]

by **Yves COLIN de VERDIÈRE**

INTRODUCTION

This report is about recent progress on semi-classical localization of eigenfunctions for quantum systems whose classical limit is hyperbolic (Anosov systems); the main example is the Laplace operator on a compact Riemannian manifold with strictly negative curvature whose classical limit is the geodesic flow; the quantizations of hyperbolic cat maps, called "quantum cat maps", are other nice examples. All this is part of the field called "quantum chaos". The new results are:

- Examples of eigenfunctions for the cat maps with a strong localization ("scarring") effect due to S. de Bièvre, F. Faure and S. Nonnenmacher [**17, 16**].
- Uniform distribution of Hecke eigenfunctions in the case of arithmetic Riemann surfaces by E. Lindenstrauss [**26**].
- General lower bounds on the entropy of semi-classical measures due to N. Anantharaman [**1**] and improved by N. Anantharaman–S. Nonnenmacher [**3**] and N. Anantharaman–H. Koch–S. Nonnenmacher [**2**]. This lower bound is sharp with respect to the cat maps examples.

We will mainly focus on this last result.

1. THE 2 BASIC EXAMPLES

1.1. Cat maps

We start with a matrix $A \in SL_2(\mathbb{Z})$ which is assumed to be hyperbolic: the eigenvalues $\lambda_\pm$ of A satisfy $0 < |\lambda_-| < 1 < |\lambda_+|$. The action of A onto $\mathbb{R}^2$ defines a symplectic action U of A on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by considering action on points mod $\mathbb{Z}^2$.

Such a map is a simple example of a chaotic map. It has been observed since a long time that such a map can be quantized: for each integer N, we consider the Hilbert space $\mathcal{H}_N$ of dimension N of Schwartz distributions f which are periodic of period one and of which Fourier coefficients are periodic of period N: if $f(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}$, we have, for all $k \in \mathbb{Z}$, $a_{k+N} = a_k$. Using the metaplectic representation applied to A, we get a natural unitary action $\hat{U}_N$ onto the space $\mathcal{H}_N$. We are mainly interested in the eigenfunctions of $\hat{U}_N$. The semi-classical parameter is $\hbar = 1/N$ and the classical limit corresponds to large values of N. A good reference is [**8**].

1.2. The Laplace operators

On a smooth compact connected Riemannian manifold (X, g) without boundary, we consider the Laplace operator Δ given in local coordinates by

$$\Delta = -|g|^{-1}\partial_i g^{ij}|g|\partial_j$$

with $|g| = \det(g_{ij})$. The Laplace operator Δ is essentially self-adjoint on $L^2(X)$ with domain the smooth functions and has a compact resolvent. The spectrum is discrete and denoted by

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

with an orthonormal basis of eigenfunctions φ_k satisfying $\Delta\varphi_k = \lambda_k\varphi_k$. It is useful to introduce an effective Planck constant (the semi-classical small parameter) $\hbar := \lambda_k^{-\frac{1}{2}}$. We will rewrite the eigenfunction equation $\hbar^2\Delta\varphi = \varphi$. The semi-classical limit $\hbar \to 0$ corresponds to the high frequency limit for the periodic solutions $u(x,t) = \exp(i\sqrt{\lambda_k}t)\varphi_k$ of the wave equation $u_{tt} + \Delta u = 0$. Instead of the wave evolution, we will use the Schrödinger evolution which is given by

$$\frac{\hbar}{i}u_t = -\frac{\hbar^2}{2}\Delta u \ ,$$

and introduce the unitary dynamics defined by the 1-parameter group

$$\hat{U}^t = \exp(-it\hbar\Delta/2), \ t \in \mathbb{R}.$$

For the basic definitions, one can read [**5**].

1.3. The geodesic flow

If (X, g) is a Riemannian manifold and $v \in T_xX$ a tangent vector at the point $x \in X$, we define, for $t \in \mathbb{R}$, $G^t(x,v) = (y,w)$ as follows: if $\gamma(t)$ is the geodesic which satisfies $\gamma(0) = x$, $\dot{\gamma}(0) = v$, we put $y := \gamma(t)$ and $w := \dot{\gamma}(t)$. By using the identification of the tangent bundle with the cotangent bundle induced by the metric g (which is also the Legendre transform of the Lagrangian $\frac{1}{2}g_{ij}(x)v_iv_j$), we get a flow $(G^t)^\star$ on $T^\star X$ which preserves the unit cotangent bundle denoted by Z. We denote by U^t the restriction of $(G^t)^\star$ to Z. The *Liouville measure* dL on Z is the Riemannian measure normalized as a probability measure. The Liouville measure dL is invariant by the geodesic flow.

2. CLASSICAL CHAOS

Good textbooks on the classical chaos are [**21, 28, 10**].

2.1. Classical Hamiltonian systems

We consider a closed phase space Z which is the torus $\mathbb{R}^2/\mathbb{Z}^2$ in the case of the cat map and the unit cotangent bundle in the case of the Laplace operator. On Z, we have the Liouville measure dL which is normalized as a probability measure. Moreover, we have a measure preserving smooth dynamics on Z which is the action of U in the cat map example and the geodesic flow in the Riemannian case. We will denote this action by U^t where t belongs to $\mathbb{Z}$ or to $\mathbb{R}$.

2.2. Ergodicity

DEFINITION 2.1. — *The dynamical system* (Z, U^t, dL) *is* ergodic *if every measurable set which is invariant by* U^t *is of measure* 0 *or* 1.

As a consequence, we get the celebrated *Birkhoff ergodic Theorem:*

THEOREM 2.2. — *If* (Z, U^t, dL) *is ergodic, for every* $f \in L^1(Z, dL)$ *and almost every* $z \in Z$*:*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(U^t z)dt = \int_Z f dL \ .$$

The cat map is ergodic and the geodesic flow of every closed Riemannian manifold with < 0 sectional curvature is ergodic too.

2.3. Mixing

A much stronger property is the *mixing property* which says that we have a correlation decay:

DEFINITION 2.3. — *The dynamical system* U^t *is* mixing *if for every* f, $g \in L^2(Z, dL)$ *with* $\int_Z f dL = 0$, *we have*

$$\lim_{t\to\infty} \int_Z f(U^t(z))g(z)dL = 0 \ .$$

Cat maps as well as geodesic flows on manifolds with < 0 curvature are mixing. Mixing systems are ergodic.

2.4. Liapounov exponent

Chaotic systems are often presented as (deterministic) dynamical systems which are very sensitive to initial conditions.

DEFINITION 2.4. — *The global Liapounov exponent* Λ_+ *of the smooth dynamical system* (Z, U^t) *is defined as the lower bounds of the* Λ*'s for which the differential* dU^t *of the dynamics satisfies*

$$\|dU^t(z)\| = O(e^{\Lambda t}) \ ,$$

for $t \to +\infty$*, uniformly w.r. to* z*.*

For cat maps given by A, $\Lambda_+ = \log |\lambda_+|$. If X is a Riemannian manifold of sectional curvature -1, $\Lambda_+ = 1$.

2.5. K-S entropy

Kolmogorov and Sinaï start from the work of Shannon in information theory in order to introduce an entropy $h_{\mathrm{KS}}(\mu)$ for a dynamical system with an invariant probability measure μ. The definition of the entropy uses partitions of the phase space and how they are refined by the dynamics:

DEFINITION 2.5. — *If* $\mathcal{P} = \{\Omega_j | j = 1, \cdots, N\}$ *is a finite measurable partition of* Z*, we define the* entropy $h(\mathcal{P}) := -\sum \mu(\Omega_j) \log \mu(\Omega_j)$.

In terms of information theory, it is the average information you get by knowing in which of the Ω_j's the point z lies. Let $\mathcal{P}^{\vee N}$ be the partition whose sets are

$$\Omega_{j_1, j_2, \cdots, j_N} = \{z \in Z \text{ so that, for } l = 1, \cdots, N+1, \ U^{l-1}(z) \in \Omega_{j_l}\} \ .$$

If we define $\mathcal{P}_1 \vee \mathcal{P}_2$ as the partition whose elements are the intersections of one element of the partition $\mathcal{P}_1$ and one element of the partition $\mathcal{P}_2$, we get from the properties of the log function:

$$h(\mathcal{P}_1 \vee \mathcal{P}_2) \leq h(\mathcal{P}_1) + h(\mathcal{P}_2) \ .$$

Let us define $\mathcal{P}_1 = \mathcal{P}^{\vee n}$ and $\mathcal{P}_2 = U^{-n}(\mathcal{P}^{\vee m})$. Using the *invariance*[(1)] *of* μ *by* U, we get $h(\mathcal{P}_2) = h(\mathcal{P}^{\vee m})$. From $\mathcal{P}^{\vee(n+m)} = \mathcal{P}_1 \vee \mathcal{P}_2$, we get the *sub-additivity* of the sequence $N \to h(\mathcal{P}^{\vee N})$.

We define

$$h_{\mathrm{KS}}(\mathcal{P}) := \lim_{N \to \infty} h(\mathcal{P}^{\vee N})/N \ ,$$

and $h_{\mathrm{KS}}(\mu) = \sup_{\mathcal{P}} h_{\mathrm{KS}}(\mathcal{P})$.

In the case of an hyperbolic dynamics, the entropy is reached by a partition whose all sets have small enough diameters.

(1) The invariance of μ is used in a crucial way here and, as we will see, it is one of the problem we have to solve when passing to the quantum case.

Useful remarks are:

- In the case of an hyperbolic dynamics, the entropy is reached by a partition whose all sets have small enough diameters.
- The entropy h_{KS} is an affine function on the convex set of invariant probability measures.
- The entropy is lower semi-continuous for the weak topology on the set of invariant probability measures.

A more intuitive definition was provided by the work of Brin and Katok. Let us choose some point $z \in Z$ and some $\epsilon > 0$. We define

$$d_t(z, z') = \sup_{0 \le l \le t} d(U^l(z), U^l(z')) .$$

THEOREM 2.6. — *If μ is a probability measure on Z which is invariant by U^t, the Kolmogorov-Sinaï entropy $h_{KS}(\mu)$ is given by*

$$h_{KS}(\mu) = \int_Z h_\mu(z) d\mu$$

with

$$h_\mu(z) = \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{|\log(\mu(\{z' | d_t(z, z') \le \epsilon\})|}{t} .$$

2.6. Hyperbolicity

Cat maps as well as geodesic flows on manifolds with < 0 curvature are *hyperbolic systems* in the sense of *Anosov.* They are the smooth dynamical systems which have the strongest chaotic properties. Let us give the definitions for flows:

DEFINITION 2.7. — *A smooth dynamical system (Z, U^t) generated by the vector field V is* Anosov *if there is a continuous splitting*

$$TZ = E_+ \oplus E_- \oplus \mathbb{R}V$$

so that, if dU^t is the differential of U^t, the splitting is preserved by dU^t, and, if dU^t_+ (resp. dU^t_-) is the restriction of dU^t to E_+ (resp. E_-), there exist $C > 0$ and $k > 0$ so that:

$$\forall t \ge 0, \ \|dU^t_+\| \le Ce^{-kt},$$

$$\forall t \le 0, \ \|dU^t_-\| \le Ce^{kt}.$$

The bundle E_+ (resp. E_-) is called the stable *(resp.* unstable*) bundle.*

Remark 2.8. — The stable and the unstable bundles are *integrable.* Each leaf is smooth: a stable leaf consists of points z which have asymptotic trajectories as $t \to +\infty$. However, in general, the stable bundle and the unstable bundle are *not smooth, but only Hölder continuous.*

We define then the unstable Jacobian $J_u(z)$ as the absolute value of the Jacobian determinant of $dU^1_-(z)$ w.r. to some Riemannian metric on Z. We have the following nice result which is a combination of results by Ruelle, Pesin [**28**] and Ledrappier-Young [**25**]:

THEOREM 2.9. — *If the dynamical system (Z, U^t) is Anosov and dL is an invariant absolutely continuous measure, for every invariant probability measure μ, we have:*

$$h_{\mathrm{KS}}(\mu) \leq \int_Z \log(J_u(z))d\mu \ .$$

Moreover, with equality if and only if $\mu = dL$ [(2)].

3. TIME SCALES IN SEMI-CLASSICS

Good introductions to semi-classical analysis are [**13, 14**].

3.1. Ehrenfest time

Due to Heisenberg uncertainty principle, the wave packets in quantum mechanics cannot be localized into sets of "size" [(3)] less than $\hbar$.

The Ehrenfest time is the time it takes for a cell of size $\hbar$ to be expanded to the whole phase space, more precisely:

DEFINITION 3.1. — *The* Ehrenfest time T_{E} *is defined by*

$$T_{\mathrm{E}} := \frac{|\log \hbar|}{\Lambda_+} \ .$$

Many estimates in semi-classics, which are well known for fixed finite time, can be extended uniformly to times which are of the order of a suitable fraction of T_{E}. For example Egorov Theorem [**9**] and the semi-classical trace formula [**15**].

(2) The Jacobian $J_u(z)$ depends on the choice of a metric on Z, but the previous integral does not.
(3) In fact Heisenberg principle would give a diameter of the order $\sqrt{\hbar}$, but it will only change the Ehrenfest time by a factor 2.

3.2. Heisenberg time

The Heisenberg time is the time needed to resolve the spectrum from the observation of a wave at some point $x_0 \in X$: we have $u(x_0,t) = \sum a_j \exp(-itE_j/\hbar)$ and we can get approximate values of the E_j's only by knowing $u(x_0,t)$ on a window of time larger than the Heisenberg time.

This time is of the order of $\hbar/\delta E$ where δE is the (mean) spacing of eigenvalues. Using Weyl's law, δE is of the order $\hbar^d$ where d is the dimension of the configuration space.

DEFINITION 3.2. — *The Heisenberg time is*

$$T_{\mathrm{H}} := \frac{\hbar}{\delta E} \ .$$

This time is usually of the order of $\hbar^{-(d-1)}$ which is much larger than the Ehrenfest time.

Asymptotic calculations of the eigenmodes need a knowledge of the quantum dynamics until the Heisenberg time. It is possible to do that (at the moment) only for integrable systems for which the Ehrenfest time is $+\infty$. Gutzwiller type trace formulae are valid up to Ehrenfest times and are not quantization rules except for integrable systems for which they are equivalent, via the Bohr-Sommerfeld rules, to the Poisson summation formula.

4. THE SCHNIRELMAN ERGODIC THEOREM

4.1. Quasi-modes

DEFINITION 4.1. — *If $f(\hbar)$ is a function satisfying* $\lim_{\hbar\to 0} f(\hbar) = 0$, *a sequence of L^2 normalized smooth functions φ_k is said to be an $f-$*quasi-mode *if* $\|\hbar^2\Delta\varphi_k - \varphi_k\|_2 = O(\hbar f(\hbar))$.

If φ_k is an f-quasi-mode for the Laplace operator, $\exp(-it/\hbar)\varphi_k$ is a good approximation to $\hat{U}^t\varphi_k$ on a time interval of the order of $f(\hbar)^{-1}$.

4.2. Wigner measures and semi-classical measures

To any function $a \in C_o^\infty(T^\star\mathbb{R}^d)$, we can associate a pseudo-differential operator which is given by:

$$\mathrm{Op}_\hbar(a)u(x) := \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\langle x-y|\xi\rangle/\hbar}a(x,\xi)u(y)|dyd\xi| \ .$$

We call such a recipe $a \to \mathrm{Op}_\hbar(a)$ a *quantization.* Using partitions of unity, we can get a similar quantization on any closed manifold. In particular, if $a = a(x)$ is a function on X, $\mathrm{Op}_\hbar(a)$ is the multiplication by a.

For a family of functions $f_\hbar$ of L^2 norms $\equiv 1$, we define the *Wigner measures* as the Schwartz distributions defined on the manifold by

$$\int_{T^\star X} a dW_\hbar := \langle \mathrm{Op}_\hbar(a) f_\hbar | f_\hbar \rangle \ .$$

They are also called the microlocal lifts of $|f_\hbar|^2|dx|$ because they project onto such measures by the canonical projection from $T^\star X$ onto X.

THEOREM 4.2. — *If $f_\hbar$ is a sequence of $o(1)$ quasi-modes (see Definition 4.1), all weak limits (as Schwartz distributions) of $dW_\hbar$ are probability measures on Z which are invariant by the geodesic flow.*

Remark 4.3. — It is possible to choose the quantization so that for any $a \geq 0$, $\mathrm{Op}_\hbar(a)$ is a positive symmetric operator. The Wigner measures dW_h depend on the chosen quantization, but the asymptotic behavior as $\hbar \to 0$ does not.

DEFINITION 4.4. — *Any such limit measure is called a* semi-classical measure.

Such measures were also introduced as a general tool in the study of partial differential equations by P. Gérard [**18**] and L. Tartar [**35**].

Remark 4.5. — If μ is the semi-classical measure of a sequence φ_{k_j}, the measures $|\varphi_{k_j}|^2|dx|$ on X converge to the projection of μ on X.

4.3. Localized eigenfunctions and scars

It has been well known since 40 years [**4, 31**], that it is possible to build f-quasi-modes, with $f(\hbar) = \hbar^N$, associated to any generic stable closed geodesic γ. The associated semi-classical measure is the average on γ. Typical eigenfunctions of integrable systems have semi-classical measures which are Lebesgue measures on Lagrangian tori. If $V(x)$ is a double well potential with a local maximum at $x = x_0$, the Dirac measure $\delta(x_0, 0)$ (the unstable equilibrium point) is also a semi-classical measure. An example, with a Laplace operator, of a sequence of eigenfunctions, for which the semi-classical measure is the average on an unstable closed geodesic, is described in [**12**].

Sequences of eigenfunctions can be very large at some places and can still have a uniform measure as a semi-classical measure: from the point of view of numerical calculations, it is impossible to see the difference. The numerical observations of such abnormally large eigenfunctions started with the work of S.W. McDonald & A.N. Kaufman [**29, 30**] in the case of the stadium billiard. They were called *scars*

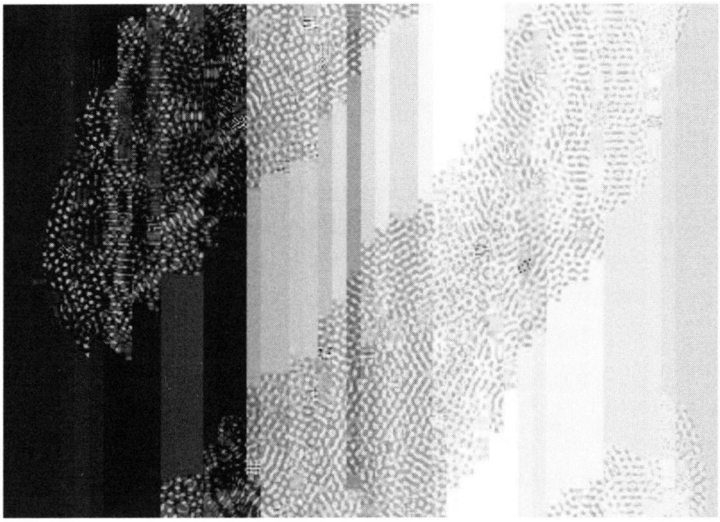

FIGURE 1. Scars for the stadium billiard: the intensity of some eigenfunctions is larger around some specific closed geodesics.

by E. Heller [**20**] which gave the following "definition": *a quantum eigenstate of a classically chaotic system has a scar of a periodic orbit if its density on the classical invariant manifolds near the periodic orbit differs significantly from the classical expected density.* A typical problem related to scars is to get upper bounds of the L^∞ norms of the eigenfunctions. Some people called *strong scarring* the fact that the limit of the Wigner measures is not the Liouville measure.

4.4. The (micro-)local Weyl law

We consider some average of Wigner measures as follows:

$$dm := (2\pi\hbar)^{-d} \sum_{\hbar^2\lambda_k \leq 1} dW_{\varphi_k} \ .$$

The micro-local version of Weyl law, of which the local Weyl law (and hence the usual Weyl law) is a consequence if we integrate a function $a = a(x)$, is:

THEOREM 4.6. — *As $\hbar \to 0^+$, the measure dm converges weakly to the Liouville measure on the unit ball bundle $B_1^\star X$.*

This result is an easy consequence of the functional calculus of pseudo-differential operators by looking at asymptotic of traces of $\Phi(\hbar^2\Delta)$.

4.5. The Schnirelman Theorem

The beginning of this story is the celebrated Schnirelman Theorem [**34, 36, 11**] and, for the case of manifold with boundary (a billiard), [**19, 37**]:

THEOREM 4.7. — *Let X be a closed Riemannian manifold whose geodesic flow is* ergodic. *Let (φ_k, λ_k) be an eigendecomposition of the Laplace operator. There exists a density one sub-sequence (λ_{k_j}) of the eigenvalues sequence*[(4)] *so that the sequence $dW_{\varphi_{k_j}}$ weakly converges to the Liouville measure on the unit cotangent bundle.*

Since more than twenty years, the existence of atypical sub-sequences has been considered as an important problem. In particular, Rudnick and Sarnak [**32**] formulated the so-called *Quantum unique ergodicity conjecture* (QUE): there are no exceptional sub-sequences at least for the case of < 0 curvature.

4.6. Arithmetic case

Recently, E. Lindenstrauss [**26**] proved the QUE for a Hecke eigenbasis of *arithmetic* Riemann surfaces with constant curvature. His proof uses sophisticated results in ergodic theory of M. Ratner.

5. LOCALIZED STATES FOR THE CAT MAP

The only counter-example to QUE is for linear cat maps (see [**7, 17, 16**]). The basic fact is that the quantum cat map $\hat{U}_N$ is a unitary periodic operator (i.e. there exists a non zero integer $T(N)$ so that $\hat{U}_N^{T(N)} = e^{iT(N)\alpha_n}\mathrm{Id}$) in sharp contrast with the classical cat map which is chaotic! The smallest positive period $T_0(N)$ is the period of the permutation induced by the linear map A on $(\mathbb{Z}/N\mathbb{Z})^2$. The period $T_0(N)$ satisfies

$$2T_{\mathrm{E}} = \frac{2|\log \hbar|}{\Lambda_+} \leq T_0(N) \leq 3N \ .$$

We will choose a sequence N_k so that the periods are close to $2T_{\mathrm{E}}$. Let us denote $T_k := T_0(N_k)$. For such sequences, we have $T_H \sim T_{\mathrm{E}}$.

(4) The sub-sequence λ_{k_j} of the sequence λ_k is of density 1 if

$$\lim_{\lambda \to +\infty} \frac{\#\{j | \lambda_{k_j} \leq \lambda\}}{\#\{k | \lambda_k \leq \lambda\}} = 1 \ .$$

THEOREM 5.1. — *Let $\varphi \in \mathcal{H}_{N_k}$ be a coherent state located at the origin of the torus. The state*

$$\psi := \sum_{l=-T_k/2}^{T_k/2-1} e^{-il\alpha_N} \hat{U}_{N_k}^l \varphi$$

is an eigenstate of $\hat{U}_{N_k}$ with eigenvalue $e^{i\alpha_N}$ and the associated semi-classical measure is $\mu = \frac{1}{2}(\delta(0) + dL)$. The entropy of μ is $\log(\lambda_+)/2$.

The idea of the proof is as follows: we split the state ψ into 2 parts: $\psi = \psi_{\rm loc} + \psi_{\rm equi}$, where $\psi_{\rm loc} = \sum_{|l|\leq T_k/4} e^{-il\alpha_N} \hat{U}_{N_k}^l \varphi$ while $\psi_{\rm equi}$ is the remaining part of that sum. The state $\psi_{\rm loc}$ stays localized because all components involve times less than $T_{\rm E}/2$, the part $\psi_{\rm equi}$ is equidistributed.

6. LOWER BOUNDS ON THE ENTROPY: THE A-N THEOREM

N. Anantharaman and S. Nonnenmacher in [**3**] and, with H. Koch, in [**2**] were improving a previous result of N. Anantharaman [**1**] as follows:

THEOREM 6.1. — *Let (X, g) be a smooth closed Riemannian manifold of dimension d with strictly negative sectional curvature. Let μ be any semi-classical measure (a weak limit of a sequence of Wigner measures) for an $o(|\log\hbar|^{-1})$-quasi-mode of the Laplace operator. We have the following lower bound for the entropy of μ:*

$$h_{\rm KS}(\mu) \geq \int_Z \log J_u(z) d\mu - \frac{1}{2}(d-1)\Lambda_+ \ .$$

If the curvature is $\equiv -1$, it gives

$$h_{\rm KS}(\mu) \geq \frac{d-1}{2} \ .$$

If the curvature varies a lot, the lower bound can be negative. In [**1**], it was proved that

THEOREM 6.2. — *If X is a closed Riemannian manifold with strictly negative curvature, then, for any semi-classical measure μ, the entropy $h_{\rm KS}(\mu)$ is strictly positive.*

In particular, convex combinations of averages on closed geodesics are not semi-classical measures.

This cannot be obtained by local considerations around the closed geodesic as shown in the paper [**12**].

The analog of Theorem 6.1 for linear cat maps on the 2-torus is the lower bound

$$h_{\rm KS}(\mu) \geq \frac{1}{2}\Lambda_+$$

which is a sharp bound w.r. to the example discussed in Section 5.

It is interesting to compare the previous results to the following one [**24**]:

THEOREM 6.3. — *Let X be a closed 2D Riemannian manifold with < 0 curvature and μ a probability measure on the unit cotangent bundle Z invariant by the geodesic flow for which $h_{\rm KS}(\mu) > \frac{1}{2}\int_Z \log J_u(z)d\mu$; then the projection of μ onto X is absolutely continuous w.r. to the Lebesgue measure.*

7. ABOUT THE PROOF OF THE A-N THEOREM

We will not give the full proof, but only the key points avoiding the most technical parts for which we refer to the original papers [**1, 3, 2**]. Moreover, we will assume that $\varphi_\hbar$ is an eigenfunction, not only a quasi-mode.

7.1. Heuristics

Let us start with a partition $\mathcal{P} = \{P_1, \cdots, P_M\}$ of Z and a sequence $\varphi_\hbar$ of eigenfunctions with a semi-classical measure μ on Z. Let p_j be the characteristic function of P_j. In order to get an estimate of the exponential decay of $C_n := \mu\left(p_{j_n}\circ U^{(n-1)}.\cdots .p_{j_2}\circ U^1.p_{j_1}\right)$ (and hence a lower bound of the entropy), we replace the partition of unity p_j by a smooth one and try to evaluate the quantum analog Q_n of C_n defined by

$$Q_n := \langle \hat{U}^{-(n-1)}\pi_{j_n}\hat{U}^{n-1}\circ\cdots\circ\hat{U}^{-1}\pi_{j_2}\hat{U}^1\circ\pi_{j_1}\varphi_\hbar|\varphi_\hbar\rangle ,$$

where the π_j's are pseudo-differential operators of symbol p_j. Indeed, for fixed n, the expression Q_n converges to C_n as $\hbar\to 0$ due to the Egorov Theorem: $\hat{U}^{-j}\pi_j\hat{U}^j$ is a pseudo-differential operator of principal symbols $p_j\circ U^j$. N. Anantharaman already got a nice decay estimate for Q_n in [**1**]. The problem is that the decay estimates involve the expected classical exponential decay with an extra negative power of $\hbar$: the exponential decay of Q_n starts only for n of the order of $|\log\hbar|$, more precisely the Ehrenfest time T_E. But the Egorov Theorem is only valid for time of the order of $T_E/2$! So we need to play with that: first, we introduce a quantum entropy and then, using the Egorov Theorem for a time $T_E/2$, we get a subadditivity estimate for it which allows to recover a nice estimate for a fixed time. We can then take the limit $\hbar\to 0$. The main 3 parts are:

- The *Quantum* part: abstract quantum entropy estimates (Section 7.2)
- The *Classical* part: decay estimates for Q_n (Sections 7.4, 7.5)
- The *Semi-Classical* part: subadditivity (Section 7.6).

7.2. Entropic uncertainty principle

The way to get a lower bound for the entropy from upper estimates is by an adaptation of the entropic uncertainty principle conjectured by Kraus in [**23**] and proved by Maassen and Uffink [**27**]. This principle states that, if a unitary matrix has "small" entries, then any of its eigenvectors must have a "large" Shannon entropy.

Let $(\mathcal{H}, \|.\|)$ be a complex Hilbert space.

DEFINITION 7.1. — *A* quantum partition of unity *is a family* $\pi = (\pi_k)_{k=1,\ldots,N}$ *of linear operators* $\pi_k : \mathcal{H} \to \mathcal{H}$ *which satisfies*

$$\sum_{k=1}^{N} \pi_k^\star \pi_k = Id \ . \tag{1}$$

In other words, for all $\psi \in \mathcal{H}$, we have

$$\|\psi\|^2 = \sum_{k=1}^{N} \|\psi_k\|^2 \qquad \text{where we set } \psi_k = \pi_k \psi \quad \text{for } k = 1, \ldots, N \, .$$

DEFINITION 7.2. — *Let us give a family* $\alpha = (\alpha_k)_{k=1,\ldots,N}$ *of positive real numbers; if* $\|\psi\| = 1$, *we define the* entropy *of* ψ *with respect to the partition* π *by:*

$$h_\pi(\psi) = -\sum_k \|\psi_k\|^2 \log(\|\psi_k\|^2) \ ,$$

and the pressure *w.r.t. the sequence* α *by:*

$$p_{\pi,\alpha}(\psi) = -\sum_k \|\psi_k\|^2 \log(\alpha_k^2 \|\psi_k\|^2) \ .$$

THEOREM 7.3. — *Let* $\mathcal{O}$ *be a bounded operator and* $\hat{U}$ *an isometry on* $\mathcal{H}$ *and let us give* 2 *quantum partitions of unity* $\pi = (\pi_k)_{1\leq k\leq N}$ *and* $\tau = (\tau_j)_{1\leq j\leq N}$ *and* 2 *sequences of positive numbers* $\alpha = (\alpha_k)$, $\beta = (\beta_j)$. *Define* $A = \max|\alpha_k|$ *and* $B = \max|\beta_j|$ *and*

$$c^{\pi,\alpha;\tau,\beta}(\hat{U}) := \max_{j,k} \ \alpha_j \beta_k \|\tau_j \hat{U} \pi_k^\star\| \, .$$

Then, for any normalized $\psi \in \mathcal{H}$ *satisfying*

$$\|(\mathrm{Id} - \mathcal{O})\pi_k \psi\| \leq \epsilon \ ,$$

the pressures satisfy

$$p_{\tau,\beta}\big(\hat{U}\psi\big) + p_{\pi,\alpha}\big(\psi\big) \geq -2\log\big(c^{\pi,\alpha;\tau,\beta}(\hat{U}) + NAB\epsilon\big) \, .$$

In particular, if ψ *is an eigenvector of* $\hat{U}$, *we have*

$$p_{\pi,\alpha}\big(\psi\big) + p_{\tau,\beta}\big(\psi\big) \geq -2\log\big(c^{\pi,\alpha}(\hat{U}) + NAB\epsilon\big) \ .$$

Remark 7.4. — The result of [**27**] corresponds to the case where $\mathcal{H}$ is an N-dimensional Hilbert space, $\alpha_j = \beta_k = 1$, and the operators $\pi_j = \tau_k$ are the orthogonal projectors on an orthonormal basis of $\mathcal{H}$. In this case, Theorem 7.3 reads

$$h_\pi(\hat{U}\psi) + h_\pi(\psi) \geq -2 \log c(\hat{U}) ,$$

where $c(\hat{U})$ is the supremum of all matrix elements of $\hat{U}$ in the orthonormal basis associated to π.

The proof of Theorem 7.3 uses quite standard arguments of interpolation close to the Riesz-Thorin Theorem. It is given in Section 6 of [**3**].

7.3. Pseudo-differential partitions of unity

DEFINITION 7.5. — *A* semi-classical partition of unity *on the unit cotangent bundle $Z = T_1^\star X$, associated to a finite open covering $(\Omega_l)_{2\leq l\leq M}$ of Z, is a family of pseudo-differential operators $\pi_1, \cdots, \pi_l, \cdots, \pi_M$ which satisfies $\pi_l = \mathrm{Op}_\hbar(q_l)$ with*

- *$q_1 \equiv 0$ near Z and $q_1 \equiv 1$ outside a compact set;*
- *for $l > 1$, $q_l \in C_o^\infty(\Omega_l)$ (in fact, the q_l's are symbols, i.e. they have a full asymptotic expansion into powers of $\hbar$), and*

$$\sum_{l=1}^{M} \pi_l^\star \pi_l = \mathrm{Id} .$$

Remark 7.6. — The existence of such partitions of unity can be shown in two steps: first do it up to $0(\hbar^\infty)$, then find an explicit formula removing the $0(\hbar^\infty)$ part: if $\sum_{l=1}^M \tilde{\pi}_l^\star \tilde{\pi}_l = \mathrm{Id} + T$ with $T = O(\hbar^\infty)$, take $\pi_l = \tilde{\pi}_l(\mathrm{Id} + T)^{-1/2}$.

We plan to apply Theorem 7.3 to the following objects:

- $\mathcal{H} = L^2(X)$;
- $N = M^n$;
- $\mathcal{O} = \chi_\hbar(\hbar^2\Delta - 1)$ with $\chi_\hbar(E) = \chi_1(E/\hbar^{1-\delta})$ and $\chi_1 \in C_o^\infty(\mathbb{R})$ equal to 1 near 0;
- the following partition with $N = M^n$ elements:

DEFINITION 7.7. — *For any sequence $\vec{\epsilon} = (\epsilon_1, \cdots, \epsilon_n) \in \{1, \cdots, M\}^n$, we define:*

- *for any operator A, $A(l) = \hat{U}^{-l} A \hat{U}^l$;*
- *the pseudo-differential operators*

$$\Pi_{\vec{\epsilon}} := \pi_{\epsilon_n}(n-1)\pi_{\epsilon_{n-1}}(n-2)\cdots\pi_{\epsilon_1} ;$$

- *the coarse-grained unstable Jacobian*

$$J_u^{\vec{\epsilon}} := \Pi_{l=0}^n \sup_{z\in\Omega_{\epsilon_l}} J_u(z) .$$

We will use the following quantum partitions of unity of $\mathcal{H}$:

$$\mathcal{P}^{\vee n} = \{\Pi^\star_{\vec{\epsilon}} \mid |\vec{\epsilon}| = n\}$$

and

$$\mathcal{T}^{\vee n} = \{\Pi_{\vec{\epsilon}} \mid |\vec{\epsilon}| = n\} \ ,$$

and the weights:

$$\alpha_{\vec{\epsilon}} = \beta_{\vec{\epsilon}} = (J_u^{\vec{\epsilon}})^{\frac{1}{2}} \ .$$

7.4. Statement of the the main estimate

We need the main estimate:

THEOREM 7.8. — *Let us assume that the pseudo-differential partition of unity $(\pi_l)_{1\leq l\leq M}$ is given. Let us give some constant $C > 0$ and some $\delta > 0$ small enough. There exist a constant $c > 0$ independent of δ and a constant $C_\delta > 0$, so that, for any $n = |\vec{\epsilon}| \leq C|\log\hbar|$:*

- *if X is a closed $d-$manifold with < 0 sectional curvature,*

$$\|\Pi_{\vec{\epsilon}}\mathcal{O}\| \leq C_\delta \hbar^{-\frac{d-1}{2}-c\delta}(J_u^{\vec{\epsilon}})^{-\frac{1}{2}} \ ;$$

- *for an hyperbolic quantum map on a $2d-$torus, the same estimate holds with $(d-1)/2$ replaced by $d/2$.*

The previous estimates will be useful, because $\Pi_{\vec{\epsilon}'}\hat{U}^n\Pi_{\vec{\epsilon}} = \hat{U}^n\Pi_{\vec{\epsilon}',\vec{\epsilon}}$, in order to apply Theorem 7.3.

7.5. Proof of the main estimate

The proof of Theorem 7.8 is highly technical using a lot of careful estimates (19 pages in [**3**]!) and starts with the following identity:

$$\|\Pi_{\vec{\epsilon}}\| = \|\pi_{\epsilon_n}\hat{U}\pi_{\epsilon_{n-1}}\hat{U}\cdots\pi_{\epsilon_1}\| \ .$$

Let us give some ideas which make that we "believe" that such an estimate holds!

7.5.1. *The linear hyperbolic map case.* — In order to see the plausibility of such an estimate in the case of the linear cat map, I will show a similar one for the quite simple case where $\hat{U}$ is the quantization of the linear map $U : T^\star\mathbb{R} \to T^\star\mathbb{R}$ given by $U(x,\xi) = (\lambda^{-1}x, \lambda\xi)$ with $\lambda > 1$.

$$\hat{U}f(x) = \lambda^{\frac{1}{2}}f(\lambda x) \ .$$

Let us assume that $\text{Supp}(f) \subset [-1,+1]$ and $\text{Supp}(\hat{g}) \subset [-1,+1]$ where $\hat{g}(\xi) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}}\exp(-ix\xi/\hbar)g(x)dx$. We want to get an estimate for

$$B(f,g) := \langle\hat{U}^n f|g\rangle = \lambda^{-n/2}\int f(x)g(x/\lambda^n)dx$$

in terms of the L^2 norms of f and g. Now we have the trivial inequality $\|g\|_{L^\infty} \leq C\hbar^{-1/2}\|\hat{g}\|_{L^2}$ and we can conclude

$$|B(f,g)| \leq C\lambda^{-n/2}\hbar^{-1/2}\|f\|_{L^2}\|g\|_{L^2} .$$

Note that in this rather trivial case the estimate holds without any restriction on n. We see also that there is a bad negative power of $\hbar$ which cannot be removed!

7.5.2. *The case of (non-linear) hyperbolic map.* — A more geometric argument, in the case of an hyperbolic map, is as follows:

- decompose any semi-classical state of the form $f_1 = \pi_{\epsilon_1}(f)$ as a superposition of Lagrangian (WKB) states e_η associated to a smooth Lagrangian foliation of Ω_{ϵ_1} with leaves L_η transversal to the stable and the unstable foliations:
$$f_1(x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d}\hat{f}(\eta)e_\eta(x)d\eta$$
where $\hat{f}$ belongs to a bounded set of $C_o^\infty(\mathbb{R}^d)$ independently of $\hbar$;
- let us consider the part L_η^n of L_u which satisfies, for $k = 1,\cdots,n-1$, $U^k(L_\eta^n) \subset \Omega_{\epsilon_{k+1}}$. Then, if L_η^n is non empty for $n\to\infty$, there exists a point $z_0 \in \Omega_{\epsilon_1}$ so that $U^k(z_0) \in \Omega_{k-1}$ for all k and all such points are on the same stable leaf. As $n\to\infty$, the manifolds $U^{n-1}(L_\eta^n)$ smoothly converge to the intersection of the unstable manifold of $U^n(z_0)$ with Ω_{ϵ_n}, which is smooth;
- we can then get that the state $\Pi_{\vec{\epsilon}}(e_\eta)$ is close to a Lagrangian state associated to an unstable leaf and symbol $\sim (J_u^n(z_0))^{-\frac{1}{2}}$;
- a nice estimate for $K(\eta,\eta') = \langle\Pi_{\vec{\epsilon}}(e_\eta)|e_{\eta'}\rangle$ is provided from the fact that both functions are WKB states associated to transversal Lagrangian manifolds. We can use the symbolic calculus which gives the estimates $K(y,y') = O(\hbar^{-d/2}(J_u^{\vec{\epsilon}}(z_0))^{-\frac{1}{2}})$.

7.5.3. *The case of an Anosov flow.* — The case of a Riemannian manifold presents new difficulties related to the localization near Z introduced with the operator $\mathcal{O}$: in order to get $(d-1)/2$, we need a kind of semi-classical reduction. We take $\mathcal{O} = P_{[1-\hbar^{1-b},1+\hbar^{1-b}]}$ where P_I is the spectral projector of $\hbar^2\Delta$ on the interval I.

7.6. Large time Egorov Theorem and sub-additivity

We have seen in Section 2.5 that the sub-additivity of $h(\mathcal{P}^N)$ is a consequence of the invariance of the measure μ. Here we have only an approximate invariance due to the Egorov Theorem.

The usual Egorov Theorem is:

THEOREM 7.9. — *Let us give $a \in C_o^\infty(T^\star X)$ and t fixed, then, if $A = \mathrm{Op}_\hbar(a)$ and $A(t) = \hat{U}^{-t} A \hat{U}(t)$, the operator $A(t)$ is a pseudo-differential operator of principal symbol $a \circ U^t$.*

In particular

$$\|A(t) - \mathrm{Op}_\hbar(a \circ U^t)\|_{L^2 \to L^2} = O(\hbar) \ .$$

In order to prove the sub-additivity of quantum entropy, we will need the following weak (and easy) version of the main result of [**9**]:

THEOREM 7.10. — *Let γ satisfy $0 < \gamma < 1$ and $a \in C_o^\infty(T^\star X)$. We have, for $|t| \le (1-\gamma) T_{\mathrm{E}}/2$:*

$$\|\hat{U}^{-t} \mathrm{Op}_\hbar(a) \hat{U}^t - \mathrm{Op}_\hbar(a \circ U^t)\|_{L^2 \to L^2} = O(|t| \hbar^{(1+\gamma)/2)})$$

and the:

COROLLARY 7.11. — *For any $A = \mathrm{Op}_\hbar(a)$, $B = \mathrm{Op}_\hbar(a)$ with $a, b \in C_o^\infty(T^\star X)$, we have, for $|t| \le (1-\gamma) T_{\mathrm{E}}/2$:*

$$\| \left[A(t), B \right] \| = O\left(\hbar^\gamma\right) \ .$$

COROLLARY 7.12. — *For any $A = \mathrm{Op}_\hbar(a)$ with $a \in C_o^\infty(T^\star X)$, we have, for $|t| \le (1-\gamma) T_{\mathrm{E}}$:*

$$\|[A, A(t)]\| = O\left(\hbar^\gamma\right) \ .$$

This is because $\|[A, A(2t)]\| = \|[A(-t), A(t)]\|$.

For large times t, the function $a \circ U^t$ becomes less and less smooth due to the exponential divergence of trajectories. More precisely, we have

$$\|\partial_z^\alpha (a \circ U^t)\| = O(e^{\Lambda_+ |\alpha t|}) \ .$$

It implies that for $|t| \le (1-\gamma) T_{\mathrm{E}}/2$, the function $a \circ U^t$ is in some symbol class Σ_ϵ with $\epsilon < \frac{1}{2}$ which is the limit for a nice pseudo-differential calculus. Here $b \in \Sigma_\epsilon$ means $\|\partial_z^\alpha b\| = O(\hbar^{-\epsilon|\alpha|})$.

We will apply the results of Section 7.2 to the quantum partition $\Pi_{\vec{\epsilon}}$ with all $\vec{\epsilon}$ of length n. We have the following approximate sub-additivity:

THEOREM 7.13. — *Let us choose a family of normalized Laplace eigenfunctions $\Delta \varphi_\hbar = \hbar^{-2} \varphi_\hbar$. Let us denote by p_n the pressure of $\varphi_\hbar$ associated to the partition $\mathcal{P}^{\vee n}$ and the weights $\alpha_{\vec{\epsilon}} = (J_u^{\vec{\epsilon}})^{\frac{1}{2}}$. We have, for any n_0 fixed and $n_0 + m \le (1-\delta') T_{\mathrm{E}}$:*

$$p_{n_0+m} \le p_{n_0} + p_m + O_{n_0}(1) \ .$$

The previous theorem will give nice lower bounds of the pressure for fixed n_0 while the bound given in Theorem 7.8 is interesting only for n of the size of $|\log \hbar|$ due to the negative powers of $\hbar$.

7.7. The scheme of the proof

The proof of Theorem 6.1 involves the following steps:

7.7.1. *Applying the quantum uncertainty principle.* — We apply the quantum uncertainty principle (Theorem 7.3) to the following data:

- $\mathcal{H} := L^2(X)$, $N = M^n$ with $n \sim (1-\delta')T_E$;
- the partitions $\mathcal{P}^{\vee n}$ and $\mathcal{T}^{\vee n}$ defined in Section 7.4 and the associated weights $\alpha_{\vec{\epsilon}}$; we will denote by p_n (resp. q_n) the corresponding pressures;
- the sequence of eigenfunctions $\varphi_\hbar$ satisfies $\hbar^2\Delta\varphi_\hbar = \varphi_\hbar$ and has the semi-classical measure μ.

Using Theorem 7.8 in order to estimate the coefficients $c^{\mathcal{P}^{\vee n},\alpha_n;\mathcal{T}^{\vee n},\alpha_n}$, we get the following inequality:

$$\frac{p_n+q_n}{2} \geq -\left(\frac{d-1}{2} - c\delta\right)|\log\hbar| - O_\delta(1)\ .$$

It is not possible to use this inequality for fixed n because $\log\hbar$ tends to $-\infty$ as $\hbar\to 0$. For $n\sim(1-\delta')T_E$, the previous inequality gives:

$$\frac{p_n+q_n}{2n} \geq -\left(\frac{d-1}{2} - c\delta\right)\frac{\Lambda_+}{1-\delta'} - O_\delta(1)\ . \tag{2}$$

7.7.2. *Using sub-additivity.* — Before taking the semi-classical limit, we apply Theorem 7.13, in order to get the inequality (2) modulo $O(n_0^{-1})$ for $n=n_0$ fixed.

7.7.3. *Taking the semi-classical limit.* — We take now the semi-classical limit in inequality (2) using Egorov Theorem. Let us define $q_{\vec{\epsilon}} = q_{\epsilon_1}.q_{\epsilon_2}\circ U\cdots .q_{\epsilon_{n_0}}\circ U^{n_0-1}$ and denote by μ the semi-classical measure of a sequence $\varphi_\hbar$. We get

$$n_0^{-1}\left(-\sum_{|\vec{\epsilon}|=n_0}\mu(q_{\vec{\epsilon}}^2)\log\mu(q_{\vec{\epsilon}}^2) - \sum_{|\vec{\epsilon}|=n_0}\mu(q_{\vec{\epsilon}}^2)\log J_u^{\vec{\epsilon}}\right) \geq -\left(\frac{d-1}{2}-c\delta\right)\frac{\Lambda_+}{1-\delta'} - O_\delta\left(\frac{1}{n_0}\right).$$

The second sum in the lefthandside can be simplified using the multiplicative property of $J_u^{\vec{\epsilon}}$ and the fact that μ is invariant by U. We get

$$n_0^{-1}\left(-\sum_{|\vec{\epsilon}|=n_0}\mu(q_{\vec{\epsilon}}^2)\log\mu(q_{\vec{\epsilon}}^2)\right) - \sum_{l=1}^{M}\mu(q_l^2)J_u^{\{l\}} \geq -\left(\frac{d-1}{2}-c\delta\right)\frac{\Lambda_+}{1-\delta'} - O_\delta\left(\frac{1}{n_0}\right).$$

7.7.4. *Smoothing the initial partition.* — If the q_l's were the characteristic functions of a partition of Z, we would have finished the proof. We start with a generating partition whose boundaries are of μ measure 0 and we can apply a smoothing argument.

8. EQUIPARTITION BY TIME EVOLUTIONS

Here, I will describe a very nice related result by R. Schubert [**33**]. Similar results for cat maps were already proved in [**8**]. Let us consider again the case of a d-dimensional closed Riemannian manifold X with < 0 curvature. Let us define $\varphi_0(x) = \hbar^{-d/2}$
$\chi((x - x_0)/\hbar)\,\eta(x)$ with $\chi \in C_o^\infty(\mathbb{R}^d \setminus \{0\})$, $\eta \in C_o^\infty(X)$, $\eta \equiv 1$ near x_0.

THEOREM 8.1. — *If $\varphi(t)$ is the solution of the wave equation $\varphi_{tt} + \Delta\varphi = 0$ on X at time t with Cauchy data $\varphi(0) = \varphi_0$, $\varphi_t(0) = 0$, we have*

$$\left| \int_{T^\star X} a dW_{\varphi(t)} - \left(\int_{T^\star X} a dL \right) \|\varphi(0)\|_{L^2}^2 \right| = O(\hbar \exp(t\Lambda_+)) + o_{t\to\infty}(1) \ .$$

This implies that for $0 \ll t \leq T_E$, the weak limit of the Wigner measure of $\varphi(t)$ is the Liouville measure times the square of the L^2 norm of φ_0.

The proof of this result uses the large time Egorov Theorem (see Section 7.6) and the mixing property (+ a little bit of hyperbolicity).

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