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ON LANGLANDS FUNCTORIALITY FROM CLASSICAL GROUPS TO GL_n

by

David Soudry

Abstract. — This article is a survey of the descent method of Ginzburg, Rallis and Soudry. This method constructs, for an irreducible, automorphic, cuspidal, self-conjugate representation τ on $\mathrm{GL}_n(\mathbb{A})$, an irreducible, automorphic, cuspidal, generic representation $\sigma(\tau)$, on a corresponding quasi-split classical group G, which lifts weakly to τ. This construction works well also for all representations of $\mathrm{GL}_n(\mathbb{A})$, which are in the so called "tempered" part of the expected image of Langlands functorial lift from G to GL_n.

***Résumé* (Sur la fonctorialité de Langlands des groupes classiques à GL_n).** — Cet article est une exposition de la méthode de descente de Ginzburg, Rallis et Soudry. Cette méthode construit, pour une représentation irréductible, automorphe et cuspidale τ telle que $\tau = \tau^*$, une représentation irréductible, automorphe, cuspidale et générique $\sigma(\tau)$ d'un groupe classique quasi-deployé G (qui dépend de GL_n et τ), telle que τ corresponde à $\sigma(\tau)$ par la correspondance fonctorielle faible (« weak lifting »). Cette construction est valable aussi pour toutes les représentations de $\mathrm{GL}_n(\mathbb{A})$ qui appartiennent à la partie dite « tempérée » de l'image de la correspondance fonctorielle de Langlands de G à GL_n.

Introduction

In these notes, I survey a long term work, joint with D. Ginzburg and S. Rallis, where we develop a descent method, which associates to a given irreducible automorphic representation τ of $\mathrm{GL}_n(\mathbb{A})$, an irreducible, automorphic, cuspidal, generic representation σ_τ on a given appropriate split classical group G, such that σ_ν lifts to τ_ν, for almost all places ν, where τ_ν is unramified. Of course, not every τ is obtained in such a way. We have to restrict ourselves to τ which lies in the expected

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(conjectural) image of the functorial lift from G to GL_n, restricted to cuspidal representations σ of $G(\mathbb{A})$. We restrict ourselves even more and consider only generic σ. This also applies to quasi-split unitary groups G. Here $\mathbb{A}$ denotes the adele ring of a number field F. Thus, for example, let E be a quadratic extension of F, and let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_{2n+1}(\mathbb{A}_E)$, such that its partial Asai L-function $L^S(\tau, \mathrm{Asai}, s)$ has a pole at $s = 1$. Then we construct an irreducible, automorphic, cuspidal, generic representation σ_τ of $U_{2n+1}(\mathbb{A})$, which lifts weakly (*i.e.* lifts at all places, where τ is unramified) to τ. Here, U_{2n+1} is the quasi-split unitary group in $2n+1$ variables, which corresponds to E. We regard it as an algebraic group over F. Note that σ_τ would probably be a generic member of "an L-packet which lifts to τ". Of course, σ_τ is a generic member of the near equivalence class which lifts to τ.

The basic ideas of our descent method (backward lift) can be found in [**GRS7, GRS8**]. A more detailed account appears in [**GRS1**], where we also start focusing on the descent from cuspidal τ on $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and $L(\tau, 1/2) \neq 0$, to ψ-generic cuspidal representations σ on the metaplectic cover of Sp_{2n}. We complete the study of this case (for non-cuspidal τ as well) in [**GRS2, GRS3, GRS4, GRS6**]. In [**GRS9**], we consider the lift from (split) SO_{2n+1} to GL_{2n}. I review this last case in Chapter 1 of these notes. Here we can prove more; namely, that the generic cuspidal representation σ_τ is unique up to isomorphism. This is achieved due to a "local converse theorem" for generic representations of $\mathrm{SO}_{2n+1}(k)$, over a p-adic field k, proved in [**Ji.So.1**]. In Chapter 2, I review integral representations for standard L-functions for $G \times \mathrm{GL}_m$ (valid only for generic representations). The integrals are of Rankin-Selberg or Shimura type. They are certain Gelfand-Graev, or Fourier-Jacobi coefficients applied to Eisenstein series or cusp forms. In Chapter 3, I review the descent from GL_n to G in general, and in Chapter 4, I illustrate various proofs through low rank examples.

This survey is the content of a minicourse that I gave at Centre Émile Borel, IHP, Paris, when I took part in the special semester in automorphic forms (Spring 2000). I thank the organizers H. Carayol, M. Harris, J. Tilouine, and M.-F. Vignéras for their invitation, and I thank my audience for their attention.

Frequently used notation

F – a number field.

$\mathbb{A} = \mathbb{A}_F$ – the adele ring of F.

F_ν – the completion of F at a place ν.

$\mathcal{O}_\nu$ – the ring of integers of F_ν, in case $\nu < \infty$.

$\mathcal{P}_\nu$ – the prime ideal of $\mathcal{O}_\nu$.

$q_\nu = |\mathcal{O}_\nu/\mathcal{P}_\nu|$.

$\mathrm{SO}_m(F) = \{g \in \mathrm{GL}_m(F) | {}^t g J g = J\}$, where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

Let $\mathbb{R}^+$ denote the group of positive real numbers. Let $i : \mathbb{R}^+ \to \mathbb{A}^*$ be defined by $i(r) = \{x_\nu\}$, where for all finite places ν, $x_\nu = 1$, and for each archimedean place ν, $x_\nu = r$. We denote $i(\mathbb{R}^+) = \mathbb{A}^+_\infty$. For an irreducible representation τ, ω_τ denotes its central character. Sometimes we denote by V_τ a vector space realization of τ. When τ is an automorphic cuspidal representation, we assume that τ comes together with a specific vector space realization of cusp forms, which we sometimes denote by τ as well. Finally, given representations $\tau_1, \dots, \tau_r$ of $\mathrm{GL}_{n_1}(F_\nu), \dots, \mathrm{GL}_{n_r}(F_\nu)$ respectively, we denote by $\tau_1 \times \cdots \times \tau_n$ the representation of $\mathrm{GL}_n(F_\nu)$, $n = n_1 + \cdots + n_r$, induced from the standard parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_{n_1}(F_\nu) \times \cdots \times \mathrm{GL}_{n_r}(F_\nu)$, and the representation $\tau_1 \otimes \cdots \otimes \tau_r$.

1. The weak lift from SO_{2n+1} to GL_{2n}

In this chapter we survey the results on the weak lift from SO_{2n+1} to GL_{2n}, obtained after applying our descent method (backward lift). Together with the existence of this weak lift for generic representations [**C.K.PS.S.**], we obtain a fairly nice description of this weak lift, which turns out to be not weak at all.

1.1. Some preliminaries. — Let $\sigma \cong \otimes \sigma_\nu$ be an irreducible, automorphic, cuspidal representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. For almost all ν, σ_ν is unramified and is completely determined by a semisimple conjugacy class $[a_\nu]$ in ${}^L\mathrm{SO}^\circ_{2n+1} = \mathrm{Sp}_{2n}(\mathbb{C})$, so that $L(\sigma_\nu, s) = \det(I_{2n} - q_\nu^{-s} a_\nu)^{-1}$. Let i be the embedding $\mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C})$. Then the conjugacy class $[i(a_\nu)]$ in $\mathrm{GL}_{2n}(\mathbb{C})$ determines an unramified representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$, such that $L(\tau_\nu, s) = L(\sigma_\nu, s)$. The unramified representation τ_ν is called the local Langlands lift of σ_ν. This notion (of local Langlands lift) is conjecturally defined at all finite places and is well defined at archimedean places. For an archimedean place ν, σ_ν is determined by its Langlands parameter, which is an admissible homomorphism $\varphi_\nu : W_\nu \to \mathrm{Sp}_{2n}(\mathbb{C})$ from the Weil group of F_ν. The local lift of σ_ν is the representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$, whose Langlands parameter is $i \circ \varphi_\nu : W_\nu \to \mathrm{GL}_{2n}(\mathbb{C})$. (For finite places ν, where σ_ν is not unramified, σ_ν is conjecturally parameterized by an admissible homomorphism from the Weil-Deligne group $\varphi_\nu : W_\nu \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$, and an irreducible representation τ_ν of $\mathrm{GL}_{2n}(F_\nu)$ would be a local lift of σ_ν, if τ_ν corresponds to the homomorphism $i \circ \varphi_\nu$, under the local Langlands reciprocity law for GL_{2n}, now proved by Harris-Taylor [**H.T.**] and by Henniart [**H**].) An irreducible, automorphic representation $\tau \cong \otimes \tau_\nu$ is a weak lift of σ, if for every archimedean place ν and for almost all finite places ν where σ_ν is unramified, τ_ν is the local lift of σ_ν. Using the converse theorem for GL_m [**C.PS.**] and L-functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ constructed and studied by Shahidi [**Sh1**], the existence of a weak lift from SO_{2n+1} to GL_{2n} was established for *globally generic* σ, by J. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi.

Theorem **([C.K.PS.S.])**. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$. *Then σ has a weak lift to* $\mathrm{GL}_{2n}(\mathbb{A})$.

Here we remark that a weak lift of σ is realized as an irreducible subquotient of the space of automorphic forms on $\mathrm{GL}_{2n}(\mathbb{A})$. Moreover, by the strong multiplicity one property for GL_{2n} [**J.S.**], all weak lifts of σ are constituents of one representation of $\mathrm{GL}_{2n}(\mathbb{A})$ of the form $\tau_1 \times \cdots \times \tau_r$, where τ_i are (irreducible, automorphic) cuspidal representations of $\mathrm{GL}_{m_i}(\mathbb{A})$, $m_1 + \cdots + m_r = 2n$ and the set $\{\tau_1, \ldots, \tau_r\}$ is uniquely determined. In particular, if σ has a cuspidal weak lift, then it is unique. We are going to describe the image of the above weak lift, starting with its cuspidal part.

1.2. The cuspidal part of the image. — Let σ be an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Assume that σ has a cuspidal weak lift τ on $\mathrm{GL}_{2n}(\mathbb{A})$. As we just remarked, τ is uniquely determined (even with multiplicity one). Clearly $\tau_\nu \cong \widehat{\tau}_\nu$ (and $\omega_{\tau_\nu} = 1$), for almost all ν. By the strong multiplicity one and multiplicity one properties for GL_{2n}, [**J.S.**], [**Sk**], we have $\tau = \widehat{\tau}$, *i.e.* τ is self-dual. (Similarly, $\omega_\tau = 1$). Let S be a finite set of places, including those at infinity, outside which σ and τ are unramified. We have

$$L^S(\sigma \times \tau, s) = L^S(\tau \times \tau, s) = L^S(\widehat{\tau} \times \tau, s),$$

and hence $L^S(\sigma \times \tau, s)$ has a pole at $s = 1$. Recall that

$$L^S(\tau \times \tau, s) = L^S(\tau, \mathrm{sym}^2, s) L^S(\tau, \Lambda^2, s).$$

By Langlands' conjectures, one expects τ to be "symplectic", and so the pole of $L^S(\tau \times \tau, s)$ at $s = 1$ should come from $L^S(\tau, \Lambda^2, s)$.

Theorem 1. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$. *Assume that σ has a cuspidal weak lift τ on* $\mathrm{GL}_{2n}(\mathbb{A})$. *Then $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$.*

Proof. — Let us express the pole at $s = 1$ of $L^S(\sigma \times \tau, s)$ through a Rankin-Selberg type integral which represents this L-function [**So1**], [**G.PS.R.**]. It has the form

$$(1.1) \qquad \mathcal{L}(\varphi_\sigma, f_{\tau,s}) = \int_{\mathrm{SO}_{2n+1}(F)\backslash \mathrm{SO}_{2n+1}(\mathbb{A})} \varphi_\sigma(g) E^\psi(f_{\tau,s}, g)\, dg,$$

where φ_σ is a cusp form in the space of σ, $E(f_{\tau,s}, \cdot)$ is an Eisenstein series on split $\mathrm{SO}_{4n}(\mathbb{A})$ corresponding to a K-finite holomorphic section $f_{\tau,s}$ in $\mathrm{Ind}_{P_{2n}(\mathbb{A})}^{\mathrm{SO}_{4n}(\mathbb{A})} \tau |\det \cdot|^{s-1/2}$, where P_{2n} is the Siegel parabolic subgroup of SO_{4n}. E^ψ denotes a Fourier coefficient along the subgroup

$$N_n = \left\{ u = \begin{pmatrix} z & y & e \\ & I_{2n+2} & y' \\ & & z^* \end{pmatrix} \in \mathrm{SO}_{4n} \;\middle|\; z \in Z_{n-1} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\},$$

with respect to the character

$$\chi_\psi : u \longmapsto \psi(z_{12} + z_{23} + \cdots + z_{n-2,n-1} + y_{n-1,n+1} - y_{n-1,n+2}).$$

Here ψ is a fixed nontrivial character of $F\backslash\mathbb{A}$. The stabilizer of χ_ψ inside $\begin{pmatrix} I_{n-1} & & \\ & \mathrm{SO}_{2n+2} & \\ & & I_{n-1}\end{pmatrix}$ is the subgroup of all $\begin{pmatrix} I_{n-1} & & \\ & g & \\ & & I_{n-1}\end{pmatrix}$, where g fixes the vector

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}$$

(inside F^{2n+2}). This defines (split) SO_{2n+1} and its embedding (over F) inside SO_{4n}, all implicit in the definition of $\mathcal{L}(\varphi_\sigma, f_{\tau,s})$. For a suitable choice of data,

$$\mathcal{L}(\varphi_\sigma, f_{\tau,s}) = \frac{L^S(\sigma\times\tau, s)}{L^S(\tau, \Lambda^2, 2s)} R(s), \tag{1.2}$$

where $R(s)$ is a meromorphic function, which can be made holomorphic and nonzero at a neighbourhood of a given point s_0. We consider $s_0 = 1$. Since τ is unitary, $L^S(\tau, \Lambda^2, 2s)$ is holomorphic at $s = 1$. We conclude from the last equation that $\mathcal{L}(\varphi_\sigma, f_{\tau,s})$, and hence $E(f_{\tau,s}, \cdot)$, has a pole at $s = 1$ (for some choice of data). This implies that the constant term of $E(f_{\tau,s}, I)$, along the radical of P_{2n}, has a pole at $s = 1$, for some decomposable section, and this has the form

$$f_{\tau,s}(I) + \prod_{\nu\in S'} M(f^{(\nu)}_{\tau,s}) \frac{L^{S'}(\tau, \Lambda^2, 2s-1)}{L^{S'}(\tau, \Lambda^2, 2s)}, \tag{1.3}$$

for some finite set of places S' containing S. By [**K**, Lemma 2.4], $M(f^{(\nu)}_{\tau,s})$ (the corresponding local intertwining operator at I) is holomorphic for $\mathrm{Re}(s) \geqslant 1$. We conclude that $L^{S'}(\tau, \Lambda^2, s)$ has a pole at $s = 1$. Since $L(\tau_\nu, \Lambda^2, s)$ is nonzero for (each s and) each ν, $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$. $\square$

Remarks

(1) For each place ν, $L(\tau_\nu, \Lambda^2, s)$ is holomorphic at $s = 1$. We thus may replace $L^S(\tau, \Lambda^2, s)$ by $L^{S'}(\tau, \Lambda^2, s)$, for any S' and even by $L(\tau, \Lambda^2, s)$.

(2) If σ is not (globally) generic, $\mathcal{L}(\varphi_\sigma, f_{\tau,s})$ is identically zero.

The argument in the last proof proves the second direction of the following proposition. (The first direction is easy and appears in [**GRS1**, p. 814].)

Proposition 2. — *Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_k(\mathbb{A})$, $k \geqslant 2$. Assume that the central character of τ is trivial on $\mathbb{A}^+_\infty$. Let $s_0 \in \mathbb{C}$ be such that $\mathrm{Re}(s_0) \geqslant 1$. Then $E(f_{\tau,s}, \cdot)$ (similarly constructed on $\mathrm{SO}_{2k}(\mathbb{A})$) has a pole at s_0 (as $f_{\tau,s}$ varies), if and only if k is even, $s_0 = 1$, and $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$.*

From this proposition we conclude

Theorem 3. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$, *and let τ be an irreducible, automorphic, cuspidal representation of* $\mathrm{GL}_k(\mathbb{A})$, $k \geqslant 2$, *such that* $\omega_\tau\big|_{\mathbb{A}_\infty^+} = 1$. *Then* $L^S(\sigma\times\tau, s)$ *is holomorphic for* $\mathrm{Re}(s) > 1$, *and if* $L^S(\sigma \times \tau, s)$ *has a pole at* s_0, *such that* $\mathrm{Re}(s_0) = 1$, *then k is even,* $s_0 = 1$ *and* $L^S(\tau, \Lambda^2, s)$ *has a pole at* $s = 1$. *(S, as usual, is a finite set of places, outside of which both σ and τ are unramified.) Finally, if τ is an automorphic character of* $\mathbb{A}^*$, *then* $L^S(\sigma \times \tau, s)$ *is entire.*

Proof. — As in the proof of Theorem 1, we can express $L^S(\sigma \times \tau, s)$ using global integrals (see [**G**], [**So1**], [**G.PS.R.**]). We will review them in more detail later. They involve the Eisenstein series $E(f_{\tau,s}, \cdot)$ on $\mathrm{SO}_{2k}(\mathbb{A})$ when $k \geqslant 2$, so that, as in Theorem 1, if $L^S(\sigma \times \tau, s)$ has a pole at s_0, $\mathrm{Re}(s_0) > 1$, then $E(f_{\tau,s}, \cdot)$ has a pole at s_0, and by Proposition 2, we get what we want. In case $k = 1$, the global integrals turn out to be entire, and then it is easy to conclude that $L^S(\sigma \times \tau, s)$ is entire as well. □

Let us start now with an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$. As we have seen in Theorem 1, this is a necessary condition for (a cuspidal) τ to lie in the image of the weak lift from $\mathrm{SO}_{2n+1}(\mathbb{A})$. If τ is a weak lift of a generic σ, then by (1.2) $\mathcal{L}(\varphi_\sigma, f_{\tau,s})$ has a pole at $s = 1$ (for suitable choice of data), and hence (see (1.1)) there is a non-trivial L^2–pairing between (the space of) σ and

$$\sigma_\psi(\tau) = \mathrm{Span}\{\mathrm{Res}_{s=1} E^{\psi^{-1}}(f_{\tau,s}, \cdot)\big|_{\mathrm{SO}_{2n+1}(\mathbb{A})}\}. \tag{1.4}$$

Now we note that $\sigma_\psi(\tau)$ can be defined as in (1.4) for any cuspidal τ, such that $L(\tau, \Lambda^2, s)$ has a pole at $s = 1$. $\sigma_\psi(\tau)$ is a space of automorphic functions on $\mathrm{SO}_{2n+1}(\mathbb{A})$. The descent map $\tau \mapsto \sigma_\psi(\tau)$ is the main vehicle, which will lead us to the description of the functorial lift from SO_{2n+1} to GL_{2n}. One of the main theorems is

Theorem 4. — *Let τ be an irreducible, automorphic, cuspidal representation of* $\mathrm{GL}_{2n}(\mathbb{A})$. *Assume that* $L(\tau, \Lambda^2, s)$ *has a pole at* $s = 1$. *Then* $\sigma_\psi(\tau)$ *is a nonzero, irreducible, automorphic, cuspidal, generic representation of* $\mathrm{SO}_{2n+1}(\mathbb{A})$, *which weakly lifts to τ. Every other such representation has a non-trivial L^2-pairing with* $\sigma_\psi(\tau)$.

Guidelines to the proof

(1) $\sigma_\psi(\tau)$ is cuspidal: put, for short $e_\tau(h) = \mathrm{Res}_{s=1} E(f_{\tau,s}, h)$. We have to show that all constant terms of $e_\tau^{\psi^{-1}}$, along unipotent radicals (of parabolic subgroups) in SO_{2n+1}, vanish. Consider then the constant term of e_τ along the unipotent radical of the standard parabolic subgroup of SO_{2n+1}, which preserves a p-dimensional isotropic

subspace, $1 \leqslant p \leqslant n$. This constant term (evaluated at $h = I$) equals [**GRS1**, Chapter 2]

$$(1.5) \qquad \sum_{\gamma \in Z_p(F)\backslash \mathrm{GL}_p(F)} \int_{\overline{\mathcal{L}}_p(\mathbb{A})} e_\tau^{(N_{n-p},\psi^{-1})}(\widehat{\gamma} x \beta) dx,$$

where Z_p is the standard maximal unipotent subgroup of GL_p, $\overline{\mathcal{L}}_p$ is a certain unipotent subgroup inside the Levi part of P_{2n}, β is a certain Weyl element of SO_{4n}, and $\widehat{\gamma} = \begin{pmatrix} \gamma & & \\ & I_{2(2n-p)} & \\ & & \gamma^* \end{pmatrix}$. $e_\tau^{(N_{n-p},\psi^{-1})}$ is the Fourier coefficient of e_τ along

$$N_{n-p} = \left\{ u = \begin{pmatrix} z & y & e \\ & I_{2(n-p)+2} & y' \\ & & z^* \end{pmatrix} \in \mathrm{SO}_{4n} \;\middle|\; z \in Z_{n+p-1} \right\},$$

with respect to the character

$$\chi_\psi^{(n-p)} : u \longmapsto \psi^{-1}\Big(\sum_{i=1}^{n+p-2} z_{i,i+1} \Big) \psi^{-1}(y_{n+p-1,n-p+1} - y_{n+p-1,n-p+2}).$$

As for the case $p = 0$, $\chi_\psi^{(n-p)}$ is fixed by $\mathrm{SO}_{2(n-p)+1}$, appropriately embedded in SO_{4n}, and we may consider

$$\sigma_\psi^{(n-p)}(\tau) = \mathrm{Span}\{ e_\tau^{(N_{n-p},\psi^{-1})}|_{\mathrm{SO}_{2(n-p)+1}(\mathbb{A})} \}.$$

The cuspidality of $\sigma_\psi(\tau)$ is implied by

$$(1.6) \qquad \sigma_\psi^{(k)}(\tau) = 0, \quad \forall 0 \leqslant k < n.$$

This is proved using just one place. First, note that the residues e_τ are square integrable. Next, take an irreducible summand π of the space of the residues e_τ. At a place ν, where π_ν is unramified, π_ν is the spherical constituent of $\mathrm{Ind}_{P_{2n}(F_\nu)}^{\mathrm{SO}_{4n}(F_\nu)} \tau_\nu |\det \cdot|^{1/2}$. One shows, using Bruhat theory, that the corresponding Jacquet modules vanish

$$(1.7) \qquad J_{N_k(F_\nu),\chi_{\psi_\nu}^{(k)}}(\pi_\nu) = 0, \quad \forall \quad 0 \leqslant k < n.$$

This depends only on the fact that (unramified) τ_ν is self-dual and $\omega_{\tau_\nu} = 1$.

(2) $\sigma_\psi(\tau)$ is nontrivial: this depends only on the fact that τ is (globally) generic. We can relate the ψ-Whittaker coefficient of $\sigma_\psi(\tau)$ to that of τ.

(3) Write $\sigma_\psi(\tau) = \oplus \sigma_i$ – a direct sum of irreducible (cuspidal) representations. Each summand σ_i weakly lifts to τ. This follows from the fact that at a place ν, where π_ν (as in (1.7)) and τ_ν are unramified, $J_{N_n(F_\nu),\chi_{\psi_\nu}}(\pi_\nu)$, which surjects on $\sigma_{i,\nu}$, shares its unramified constituent with that of $\mathrm{Ind}_{B(F_\nu)}^{\mathrm{SO}_{2n+1}(F_\nu)} \mu_{1,\nu} \otimes \cdots \otimes \mu_{n,\nu}$, where B is the Borel subgroup of SO_{2n+1}, and τ_ν is the unramified constituent of $\mu_{1,\nu} \times \cdots \times \mu_{n,\nu} \times \mu_{n,\nu}^{-1} \times \cdots \times \mu_{1,\nu}^{-1}$ on $\mathrm{GL}_{2n}(F_\nu)$ ($\mu_{i\nu}$ are unramified characters of F_ν^*).

(4) Decompose $\sigma_\psi(\tau)$ into a direct sum $\oplus\sigma_i$ of irreducible cuspidal representations. Each summand σ_i has a non-trivial L^2-pairing with $\sigma_\psi(\tau)$, and so by definition ((1.4)), $\mathcal{L}(\varphi_{\sigma_i}, f_{\tau,s}) \not\equiv 0$ (see (1.1)). By Remark (2), after the proof of Theorem 1, σ_i must be generic for all i.

Note that since σ_i is generic, it has a weak lift τ' on $\mathrm{GL}_{2n}(\mathbb{A})$ [**C.K.PS.S.**]. By the strong multiplicity one and multiplicity one properties for GL_{2n}, we must have $\tau' = \tau$. In particular, τ_ν is the local lift of $\sigma_{i,\nu}$ at infinite places as well.

(5) $\sigma_\psi(\tau)$ is multiplicity free: if σ_i and σ_j acting in subspaces $V_{\sigma_i}, V_{\sigma_j}$ are isomorphic summands, choose an isomorphism (of representations) $T : V_{\sigma_i} \to V_{\sigma_j}$, such that $T(\varphi) - \varphi$ has a zero ψ-Whittaker coefficient for all cusp forms $\varphi \in V_{\sigma_i}$. This follows from the uniqueness up to scalars of a Whittaker functional. The argument of (4) applied to σ_i' acting in $\{T(\varphi) - \varphi \mid \varphi \in V_{\sigma_i}\}$ shows that σ_i must be globally generic. This is a contradiction, unless $T = \mathrm{id}$.

(6) $\sigma_\psi(\tau)$ is irreducible: it follows from Cor. 4 in Sec. 6 of [**C.K.PS.S.**] that for any two summands σ_i, σ_j, and any place ν, we have an equality of local gamma factors:

$$\gamma(\sigma_{i,\nu} \times \eta, s, \psi_\nu) = \gamma(\sigma_{j,\nu} \times \eta, s, \psi_\nu),$$

for any irreducible representation η of $\mathrm{GL}_k(F_\nu)$, $k = 1, 2, \dots$ By the local converse theorem (for generic representation of $\mathrm{SO}_{2n+1}(F_\nu)$ of [**Ji.So.1**], we conclude that $\sigma_{i,\nu} \cong \sigma_{j,\nu}$, for all finite places ν. For archimedean ν, we already know that $\sigma_{i,\nu} \cong \sigma_{j,\nu}$ (both representations have the same Langlands parameter as τ_ν, for ν archimedean). We conclude that $\sigma_i \cong \sigma_j$, and by (5) $\sigma_i = \sigma_j$, and so $\sigma_\psi(\tau)$ has only one irreducible summand (appearing with multiplicity one) i.e $\sigma_\psi(\tau)$ is irreducible. $\square$

1.3. Description of the image in general, and endoscopy. — In general, an irreducible, automorphic, cuspidal, generic representation σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$ weakly lifts to an irreducible automorphic representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, which is a constituent of an induced representation of the form

$$\delta_1|\det\cdot|^{z_1} \times \cdots \times \delta_j|\det\cdot|^{z_j} \times \tau_1 \times \cdots \times \tau_\ell \times \widehat{\delta}_j|\det\cdot|^{-z_j} \times \cdots \times \widehat{\delta}_1|\det\cdot|^{-z_1},$$

where $\mathrm{Re}(z_1) \leqslant \cdots \leqslant \mathrm{Re}(z_j) \leqslant 0$, and each of the representations δ_i, τ_k is irreducible, automorphic, unitary, cuspidal, or an automorphic character of the idele group, so that their central characters are trivial on $\mathbb{A}_\infty^+$, and also $\tau_i = \widehat{\tau}_i$, for $i = 1, \dots, \ell$. We have (for appropriate S)

$$L^S(\sigma \times \widehat{\delta}_1, s) = \prod_{i=1}^{j} L^S(\delta_i \times \widehat{\delta}_1, s + z_i) L^S(\widehat{\delta}_i \times \widehat{\delta}_1, s - z_i) \prod_{i=1}^{\ell} L^S(\tau_i \times \widehat{\delta}_1, s).$$

This product has a pole at $s = 1 - z_1$. (It comes from $L^S(\delta_1 \times \widehat{\delta}_1, s + z_1)$. Note that $\mathrm{Re}(1 - z_1)$, $\mathrm{Re}(1 - z_1 \pm z_i) \geqslant 1$, so that the other factors in the product do not cancel this pole.) From Theorem 3, we conclude, in particular, that δ_1 is not a character of the idele group, $z_1 = 0$ and $\delta_1 = \widehat{\delta}_1$, but then $L^S(\sigma \times \delta_1, s)$ has a double pole at

$s = 1$, which is impossible. (The global integral which represents $\frac{L^S(\sigma\times\delta_1,s)}{L^S(\delta_1,\Lambda^2,2s)}$ involves the Eisenstein series on $\mathrm{SO}_{2k_1}(\mathbb{A})$, induced from δ_1 and the Siegel parabolic subgroup. This Eisenstein series can have at most simple poles for $\mathrm{Re}(s) \geqslant 1/2$.) We conclude that "there are no δ_i-s", and

$$\tau \cong \tau_1 \times \tau_2 \times \cdots \times \tau_\ell,$$

where τ_i are irreducible, self-dual, automorphic, cuspidal, such that (again by Theorem 3) $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$, and also $\tau_i \neq \tau_j$, for $1 \leqslant i \neq j \leqslant \ell$. (We just need to repeat the last argument.) Note that for any irreducible, automorphic, unitary representations $\tau_1, \ldots, \tau_\ell$ (on $\mathrm{GL}_{k_1}(\mathbb{A}), \ldots, \mathrm{GL}_{k_\ell}(\mathbb{A})$ respectively) the representation $\tau_1 \times \cdots \times \tau_\ell$ is irreducible. This proves

Theorem 5. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Then σ weakly lifts to a representation (on $\mathrm{GL}_{2n}(\mathbb{A})$) of the form $\tau = \tau_1 \times \cdots \times \tau_\ell$, where $\tau_1, \ldots, \tau_\ell$ are pairwise different irreducible, automorphic, cuspidal representations of $\mathrm{GL}_{2n_1}(\mathbb{A}), \ldots, \mathrm{GL}_{2n_\ell}(\mathbb{A})$, $n_1 + \cdots + n_\ell = n$, respectively, such that $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$, for $1 \leqslant i \leqslant \ell$.*

Conversely, let τ be an irreducible representation of $\mathrm{GL}_{2n}(\mathbb{A})$ of the form just described in Theorem 5. We can apply the same procedure as in Sec. 1.2 (case $\ell = 1$) and construct $\sigma_\psi(\tau)$ – an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, which lifts weakly to τ. For this, we consider the Eisenstein series on $\mathrm{SO}_{4n}(\mathbb{A})$ corresponding to a K-finite, holomorphic section $f_{\tau,\underline{s}}$ in $\mathrm{Ind}_{Q_{\mathbb{A}}}^{\mathrm{SO}_{4n}(\mathbb{A})} \tau_1|\det\cdot|^{s_1-1/2} \otimes \cdots \otimes \tau_\ell|\det\cdot|^{s_\ell-1/2}$, where $\underline{s} = (s_1, \ldots, s_\ell)$ and Q is the standard parabolic subgroup of SO_{4n}, whose Levi part is isomorphic to $\mathrm{GL}_{2n_1} \times \cdots \times \mathrm{GL}_{2n_\ell}$. Denote this Eisenstein series by $E(f_{\tau,\underline{s}}, h)$. As in [**GRS4**, Theorem 2.1], we can prove that the function

$$(s_1 - 1)(s_2 - 1) \cdot \cdots \cdot (s_\ell - 1) E(f_{\tau,\underline{s}}, h)$$

is holomorphic at $\underline{s} = (1, 1, \ldots, 1)$ and is not identically zero, as the section varies. Consider

$$\mathrm{Res}_{\underline{s}=\underline{1}}\, E(f_{\tau,\underline{s}}, h) = \lim_{\underline{s}\to\underline{1}} (s_1 - 1) \cdot \cdots \cdot (s_\ell - 1) E(f_{\tau,\underline{s}}, h),$$

where $\underline{1} = (1, \ldots, 1)$. These residues generate a square integrable automorphic representation of $\mathrm{SO}_{4n}(\mathbb{A})$. Consider, as in (1.4)

$$\sigma_\psi(\tau) = \mathrm{Span}\{\mathrm{Res}_{\underline{s}=\underline{1}}\, E^{\psi^{-1}}(f_{\tau,\underline{s}}, \cdot)\big|_{\mathrm{SO}_{2n+1}(\mathbb{A})}\}.$$

Theorem 6. — *Let $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_\ell$ be the irreducible representation of $\mathrm{GL}_{2n}(\mathbb{A})$, induced from $\tau_1 \otimes \cdots \otimes \tau_\ell$, where $\tau_1, \ldots, \tau_\ell$ are pairwise inequivalent irreducible, automorphic, cuspidal representations on $\mathrm{GL}_{2n_1}(\mathbb{A}), \ldots, \mathrm{GL}_{2n_\ell}(\mathbb{A})$ respectively, $n_1 + \cdots + n_\ell = n$, such that for each $1 \leqslant i \leqslant \ell$, $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$. Then $\sigma_\psi(\tau)$ is a nonzero, irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$,* *which weakly lifts to τ. Any other such representation has a non-trivial L^2-pairing with $\sigma_\psi(\tau)$.*

Proof. — The nontriviality of $\sigma_\psi(\tau)$ is shown exactly as in case $\ell = 1$. As we mentioned in the proof of Theorem 4, only the fact that τ is generic is important here. The cuspidality of $\sigma_\psi(\tau)$ is shown as in case $\ell = 1$, only we need also to use induction on ℓ. Let σ be an irreducible summand of $\sigma_\psi(\tau)$. Then

$$\int\limits_{\mathrm{SO}_{2n+1}(F)\backslash \mathrm{SO}_{2n+1}(\mathbb{A})} \varphi_\sigma(g)\,\mathrm{Res}_{\underline{s}=\underline{1}}\,E^\psi(f_{\tau,\underline{s}}, g)dg \not\equiv 0,$$

as the data φ_σ and $f_{\tau,\underline{s}}$ vary. In particular

$$\mathcal{L}(\varphi_\sigma, f_{\tau,\underline{s}}) = \int\limits_{\mathrm{SO}_{2n+1}(F)\backslash \mathrm{SO}_{2n+1}(\mathbb{A})} \varphi_\sigma(g)E^\psi(f_{\tau,\underline{s}}, g)dg \not\equiv 0.$$

As in (1.4), also in this case the integrals $\mathcal{L}(\varphi_\sigma, f_{\tau,\underline{s}})$ represent

$$\frac{\prod_{i=1}^{\ell} L^S(\sigma \times \tau_i, s_i)}{\prod_{1\leqslant i<j\leqslant \ell} L^S(\tau_i \times \tau_j, s_i + s_j)\prod_{i=1}^{\ell} L^S(\tau_i, \Lambda^2, 2s_i)},$$

for generic σ. Moreover, as in case $\ell = 1$, if σ is not (globally) generic, then the last two integrals above are identically zero. The rest of the proof is now exactly as in Theorem 4. In particular, the irreducibility of $\sigma_\psi(\tau)$ follows from the local converse theorem in [**Ji.So.1**]. □

As a corollary, we obtain that generic cuspidal representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$ satisfy the strong multiplicity one property.

Theorem 7. — *Let σ_1 and σ_2 be two irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Assume that $\sigma_{1,\nu} \cong \sigma_{2,\nu}$, for almost all places ν. Then $\sigma_1 \cong \sigma_2$.*

Proof. — Both σ_1 and σ_2 weakly lift to the same representation τ on $\mathrm{GL}_{2n}(\mathbb{A})$. τ has the form as in Theorem 6. By Theorem 6, σ_1 and σ_2 have non-trivial L^2-pairings with $\sigma_\psi(\tau)$. In particular $\sigma_1 \cong \sigma_\psi(\tau) \cong \sigma_2$. □

Example. — Consider the group $\mathrm{SO}_5(\mathbb{A}) \cong \mathrm{PGSp}_4(\mathbb{A})$. Every irreducible, automorphic, cuspidal, generic representation of $\mathrm{PGSp}_4(\mathbb{A})$ has a unique weak lift to $\mathrm{GL}_4(\mathbb{A})$. The image of this lift consists of all irreducible, automorphic, cuspidal representations τ of $\mathrm{GL}_4(\mathbb{A})$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and of all representations of the form $\tau_1 \times \tau_2$, where τ_1 and τ_2 are different, irreducible, automorphic, cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, each one having a trivial central character.

Remark. — In [**Ji.So.1, Ji.So.2**] a Langlands reciprocity law is established for generic representations of $\mathrm{SO}_{2n+1}(F_\nu)$ (ν finite). Theorem 6.3 of [**Ji.So.2**] says (in above notation) that if σ weakly lifts to τ, then at all places ν, σ_ν locally lifts to τ_ν in the sense that both σ_ν and τ_ν correspond to the same Langlands parameter (which is symplectic).

Finally, if σ (as before) does not lift to a cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$ then, as in Theorems 5,6, it lifts to a representation $\tau = \tau_1 \times \cdots \times \tau_\ell$, as in Theorem 6. By Theorem 4, each τ_i is the lift of $\sigma_i = \sigma_\psi(\tau_i)$ on $\mathrm{SO}_{2n_i+1}(\mathbb{A})$. Thus σ is the (generalized) endoscopic lift of $\sigma_1 \otimes \cdots \otimes \sigma_\ell$ on $\mathrm{SO}_{2n_1+1}(\mathbb{A}) \times \cdots \times \mathrm{SO}_{2n_\ell+1}(\mathbb{A})$. This lift is compatible with the L-group map

$$\mathrm{Sp}_{2n_1}(\mathbb{C}) \times \cdots \times \mathrm{Sp}_{2n_\ell}(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{C}).$$

Conversely, let $\sigma_1, \ldots, \sigma_\ell$ be irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2n_1+1}(\mathbb{A}), \ldots, \mathrm{SO}_{2n_\ell+1}(\mathbb{A})$ respectively. Consider the lifts τ_i of σ_i to $\mathrm{GL}_{2n_i}(\mathbb{A})$, $\tau_i = \tau_{i1} \times \cdots \times \tau_{i\ell_i}$, $i = 1, \ldots, \ell$. Denote $C_i = \{\tau_{ij}\}_{j=1}^{\ell_i}$. Clearly, if $C_i \cap C_{i'} = \varnothing$ for all $1 \leqslant i \neq i' \leqslant \ell$, then $\tau = \times_{i=1}^{\ell} \tau_i = \times_{i=1}^{\ell} \times_{j=1}^{\ell_i} \tau_{ij}$ lies in the image of the lift from $\mathrm{SO}_{2n+1}(\mathbb{A})$, and hence $\sigma_\psi(\tau)$ is an irreducible, automorphic, cuspidal, general representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$, which is the lift of $\sigma_1 \otimes \cdots \otimes \sigma_\ell$. Summarizing

Theorem 8. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n+1}(\mathbb{A})$. Assume that the lift of σ to $\mathrm{GL}_{2n}(\mathbb{A})$ is not cuspidal. Then there exist irreducible, automorphic, cuspidal, generic representations $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ on $\mathrm{SO}_{2n_1+1}(\mathbb{A}), \mathrm{SO}_{2n_2+1}(\mathbb{A}), \ldots, \mathrm{SO}_{2n_\ell+1}(\mathbb{A})$ respectively, $n_1 + \cdots + n_\ell = n$ such that σ is the lift of $\sigma_1 \otimes \cdots \otimes \sigma_\ell$. The set $\{\sigma_1, \sigma_2, \ldots, \sigma_\ell\}$ is unique up to permutation and up to isomorphism.*

Conversely, let $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ be irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2n_1+1}(\mathbb{A}), \ldots, \mathrm{SO}_{2n_\ell+1}(\mathbb{A})$ respectively, $n_1 + \cdots + n_\ell = n$. Consider the sets $\{C_i\}_{i=1}^{\ell}$ as above. If $C_i \cap C_j = \varnothing$ for all $1 \leqslant i \neq j \leqslant \ell$, then there is a unique up to isomorphism, irreducible, automorphic, cuspidal, general representation σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, which is a lift of $\sigma_1 \otimes \cdots \otimes \sigma_\ell$. Otherwise, cuspidal data on $\mathrm{SO}_{2n+1}(\mathbb{A})$ can be specified, so that $\sigma_1 \otimes \cdots \otimes \sigma_\ell$ lifts to a constituent of the corresponding induced representation.

Example. — Let $\sigma_1, \ldots, \sigma_n$ be pairwise different irreducible, automorphic, cuspidal representations of $\mathrm{PGL}_2(\mathbb{A})$. Then, up to isomorphism, there is a unique irreducible, automorphic, cuspidal, generic representation σ of $\mathrm{SO}_{2n+1}(\mathbb{A})$, which is the lift of $\sigma_1 \otimes \cdots \otimes \sigma_n$.

1.4. Base change. — Let us compose our descent map $\tau \mapsto \sigma_\psi(\tau)$ ("backward lift") with the base change lift for GL_{2n}. Let E/F be a cyclic extension of odd prime degree p. Let σ be an irreducible, automorphic, cuspidal, generic representation of

$SO_{2n+1}(\mathbb{A})$. Let τ be the lift of σ on $GL_{2n}(\mathbb{A})$. We would like to follow the diagram

$$\begin{tikzcd}
\sigma' = \sigma_\psi(\tau') \arrow[r, leftrightarrow] & \tau' = bc(\tau) \\
SO_{2n+1}(\mathbb{A}_E) & GL_{2n}(\mathbb{A}_E) \arrow[l] \\
SO_{2n+1}(\mathbb{A}_F) \arrow[r] \arrow[u, dotted] & GL_{2n}(\mathbb{A}_F) \arrow[u, "\text{base change}"'] \\
\sigma \simeq \sigma_\psi(\tau) \arrow[r, leftrightarrow] & \tau
\end{tikzcd}$$

Here $\tau' = bc(\tau)$ is the base change lift of τ [**A.C.**]. The top arrow of the diagram exists if we show that τ' lies in the image of the lift (restricted to generic representations) from $SO_{2n+1}(\mathbb{A}_E)$. The image is described in Theorems 5,6. This is indeed the case. For this, choose a nontrivial character η of $\mathbb{A}_F^*/F^* N_{E/F}\mathbb{A}_F^*$, and a generator ε of $\mathrm{Gal}(E/F)$. Starting with a generic σ on $SO_{2n+1}(\mathbb{A}_F)$, we know that its lift τ on $GL_{2n}(\mathbb{A}_F)$ has the form $\tau_1 \times \cdots \times \tau_\ell$ as in Theorem 5. Since $bc(\tau) = bc(\tau_1) \times \cdots \times bc(\tau_\ell)$, we have to analyze each representation $bc(\tau_i)$. There are two cases according to whether τ_i is isomorphic or not isomorphic to $\tau_i \otimes \eta$. If $\tau_i \neq \tau_i \otimes \eta$, then $bc(\tau_i) = \theta_i$ is cuspidal and ε-invariant. We have

$$L^S(\theta_i, \Lambda^2, s) = \prod_{k=0}^{p-1} L^S(\tau_i, \Lambda^2 \otimes \eta^k, s).$$

It is a theorem of Shahidi [**Sh2**] that each factor in the last product is nonzero at $s = 1$, and since $L^S(\tau_i, \Lambda^2, s)$ has a pole at $s = 1$, we conclude that $L^S(\theta_i, \Lambda^2, s)$ has a pole at $s = 1$. If $\tau_i = \tau_i \otimes \eta$, then $p | 2n_i$, and

$$bc(\tau_i) = \theta_i \times \theta_i^{\varepsilon} \times \cdots \times \theta_i^{\varepsilon^{p-1}},$$

where θ_i is cuspidal, such that $\theta_i \neq \theta_i^{\varepsilon}$. We have

$$\begin{aligned}
[L^S(\tau_i, \Lambda^2, s)]^p &= \prod_{k=0}^{p-1} L^S(\tau_i, \Lambda^2 \otimes \eta^k, s) = L^S(bc(\tau), \Lambda^2, s) \\
&= \prod_{\theta \leqslant j < k \leqslant \ell} L^S(\theta_i^{\varepsilon^j} \times \theta_i^{\varepsilon^k}, s) \prod_{j=0}^{p-1} L^S(\theta_i^{\varepsilon^j}, \Lambda^2, s).
\end{aligned}$$

We conclude that the last product has a pole of order p at $s = 1$. It is easy to see that θ_i is self-dual. (This follows from the self-duality of τ_i and the fact that p is odd.) In particular, $\theta_i^{\varepsilon^j} \neq \widehat{\theta_i^{\varepsilon^k}}$, for $0 \leqslant j < k \leqslant p$. We conclude that $\prod_{j=0}^{p-1} L^S(\theta_i^{\varepsilon^j}, \Lambda^2, s)$ has a pole of order p at $s = 1$, and hence $L^S(\theta_i^j, \Lambda^2, s)$ has a pole at $s = 1$, for $0 \leqslant j \leqslant p-1$. Finally, it is easy to see that in

$$bc(\tau) = bc(\tau_1) \times \cdots \times bc(\tau_\ell) = \mathop{\times}_{\tau_i \neq \tau_i \otimes \eta} \theta_i \times \Big(\mathop{\times}_{\tau_i = \tau_i \otimes \eta} \Big(\mathop{\times}_{j=0}^{p-1} \theta_i^{\varepsilon^j} \Big) \Big),$$

all factors are different. This shows (by Theorem 6) that $\tau' = bc(\tau)$ is in the image of the lift from $\mathrm{SO}_{2n+1}(\mathbb{A}_E)$. The representation $\sigma' = \sigma_\psi(\tau')$ is an irreducible, automorphic, cuspidal and generic, and it is a base change lift of σ. Summarizing

Theorem 9. — *Let E/F be a cyclic extension of odd prime degree. Then there is a base change lift from irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2n+1}(\mathbb{A}_F)$ to irreducible, automorphic, cuspidal, generic representations of $\mathrm{SO}_{2n+1}(\mathbb{A}_E)$.*

Conclusion. — The descent map (backward lift) $\tau \mapsto \sigma_\psi(\tau)$ is a very powerful tool. This chapter demonstrated the nice results obtained for SO_{2n+1} using the descent map. The ideas and methods are general and apply to other quasi-split classical groups G. The definition of $\sigma_\psi(\tau)$ (for appropriate τ) is intimately related to global integrals (of Rankin-Selberg type, or of Shimura type) representing the standard L-function for $G \times \mathrm{GL}_*$. These integrals are available, and we will survey them in the next chapter. These integrals suggest the construction of $\sigma_\psi(\tau)$, which arises as a natural object; it is constructed so that $L^S(\sigma_\psi(\tau) \times \tau, s)$ has a pole at $s = 1$. The representation $\sigma_\psi(\tau)$ is defined by taking certain Gelfand-Graev, or Fourier-Jacobi coefficients of the residue at 1 of a certain Eisenstein series induced from τ. The study of $\sigma_\psi(\tau)$ is now the study of these Gelfand-Graev, or Fourier-Jacobi coefficients of the residual Eisenstein series induced from τ. The three main problems concerning $\sigma_\psi(\tau)$ are the following (for appropriate τ, *i.e.* in the expected image of the lift from G to GL_N, for appropriate N.)

(1) Show that $\sigma_\psi(\tau) \neq 0$.
(2) Show that $\sigma_\psi(\tau)$ is cuspidal.
(3) Show that each summand of $\sigma_\psi(\tau)$ weakly lifts to τ.

In Chapters 4-6, we will indicate how to prove these properties through low rank examples. In this way we construct examples of generic cuspidal representations σ on G, which weakly lift to a given τ in the expected image. Similarly, we get examples of (generalized) endoscopy and base change. Once the existence of the weak lift from G to GL_N is established (and not much is missing for the proof by converse theorem to be completed) then our examples above give the general case.

Note added in proof. — Recently, the existence of the weak lift of cuspidal generic representations on G to GL_N has indeed been established. See [**C.K.PS.S.1**].

2. L-functions for $G \times \mathrm{GL}_k$, where G is a quasi-split classical group (generic representations)

In this chapter, we survey the global integrals (of Rankin-Selberg type, or of Shimura type) which represent the standard L-functions for generic representations on $G \times \mathrm{GL}_k$. Note that these L-functions were obtained by Shahidi [**Sh1**] using the

Langlands-Shahidi method. However, the integrals we present here relate the fact that $L^S(\sigma \times \tau, s)$ has a pole at $s = 1$, and the fact that σ has a nontrivial L^2-pairing with the descent applied to τ.

We'll first present the notions of certain Gelfand-Graev models and Fourier-Jacobi models, which enter in the definitions of the global integrals.

2.1. Gelfand-Graev models. — Let F be a field of characteristic different than 2. (Eventually we'll be interested in a number field F or in its completion in one of its places.) Let E be either F or a quadratic extension of F. Denote by $x \mapsto \overline{x}$ the nontrivial element of $\mathrm{Gal}(E/F)$ in case $[E : F] = 2$. If $E = F$, we agree that $\overline{x} = x$ on F. Let V be a finite dimensional vector space over E, equipped with a non-degenerate bilinear form $(\,,\,)$, which is either symmetric, or anti-symmetric in case $E = F$, and is Hermitian in case $[E : F] = 2$. Let $H = H(V)$ be the connected component of the isometry group of $(V, (\,,\,))$. We assume that H acts on V from the left.

Assume that

$$V = V_\ell^+ + W + V_\ell^-, \tag{2.1}$$

where $V_\ell^\pm$ are isotropic subspaces of dimension ℓ, which are in duality under $(\,,\,)$ (*i.e.* $(\,,\,)$ restricted to $V_\ell^+ \times V_\ell^-$ is non-degenerate), and $W = (V_\ell^+ + V_\ell^-)^\perp$. Let P_ℓ be the parabolic subgroup of H, which preserves V_ℓ^+. Write its Levi decomposition

$$P_\ell = M_\ell \ltimes \mathcal{U}_\ell.$$

Let us write the elements of H in matrix form, following the decomposition (2.1). Then (with evident notation)

$$M_\ell = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} \,\middle|\, g \in \mathrm{GL}(V_\ell^+), h \in H(W) \right\}, \tag{2.2}$$

$$\mathcal{U}_\ell = \left\{ u = \begin{pmatrix} I_{V_\ell^+} & y & x \\ & I_W & y' \\ & & I_{V_\ell^-} \end{pmatrix} \in H \right\}. \tag{2.3}$$

Fix nonzero vectors $w_0 \in W$, $v_0^- \in V_\ell^-$. Define for $u \in \mathcal{U}_\ell$ (written as in (2.3)) the following rational character

$$\chi_{w_0, v_0^-}(u) = (u \cdot w_0, v_0^-).$$

We have

$$\mathrm{Stab}_{M_\ell}(\chi_{w_0,v_0^-}) = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & g^* \end{pmatrix} \in H \,\middle|\, h \cdot w_0 = w_0, \quad g^* \cdot v_0^- = v_0^- \right\}. \tag{2.4}$$

Thus, if w_0 is anisotropic, then $h \cdot w_0 = w_0$ means that $h \in H(w_0^\perp \cap W)$, and if w_0 is isotropic, then $h \cdot w_0 = w_0$ means that h lies in the parabolic subgroup P_{W,w_0} of $H(W)$, which fixes the isotropic subspace $E \cdot w_0$ (and also $h \cdot w_0 = w_0$). Put in this case (*i.e.* $(w_0, w_0) = 0$)

$$P^1_{W,w_0} = \{h \in P_{W,w_0} \mid h \cdot w_0 = w_0\}.$$

The condition $g^*v_0^- = v_0^-$ in (2.4) means that g lies in the so called "mirabolic" subgroup of $\mathrm{GL}(V_\ell^+)$. Let us insert more coordinates. Choose a basis $\{v_1, \dots, v_\ell\}$ of V_ℓ^+ and a dual basis $\{v_{-\ell}, \dots, v_{-1}\}$ of V_ℓ^- (*i.e.* $(v_i, v_{-j}) = \delta_{ij}$, for $1 \leqslant i, j \leqslant \ell$). We assume that $v_0^- = v_{-\ell}$. We identify $\mathrm{GL}(V_\ell^\pm)$ with $\mathrm{GL}_\ell(E)$ using these bases. Note that for $g \in \mathrm{GL}_\ell(E)$, $g^* = w_\ell{}^t\overline{g}^{-1}w_\ell$, where $w_\ell = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$, and $g^*v_{-\ell} = v_{-\ell}$ means that $g \in \left(\frac{*}{0\cdots01}\right)$. Let Z_ℓ be the standard maximal unipotent subgroup of $\mathrm{GL}_\ell(E)$. Put

$$\widehat{Z}_\ell = \left\{\widehat{z} = \begin{pmatrix} z & & \\ & I_W & \\ & & z^* \end{pmatrix} \;\middle|\; z \in Z_\ell\right\},$$

$$L_{W,w_0} = \begin{cases} H(w_0^\perp \cap W), & (w_0, w_0) \neq 0 \\ P^1_{W,w_0}, & (w_0, w_0) = 0 \end{cases},$$

$$N_\ell = \widehat{Z}_\ell \mathcal{U}_\ell,$$

$$R_{\ell,w_0} = N_\ell L_{W,w_0}.$$

Fix a nontrivial character ψ of F. Put $\psi_E = \psi \circ tr_{E/F}$. Let ψ_{ℓ,w_0} be the following character of N_ℓ

$$\psi_{\ell,w_0}(\widehat{z} \cdot u) = \psi_{Z_\ell}(z)\psi_E(\chi_{w_0,v_\ell^-}(u)) = \psi_E\Big(\sum_{i=1}^{\ell-1} z_{i,i+1}\Big)\psi_E((u \cdot w_0, v_\ell^-)).$$

Assume now that w_0 is anisotropic. (This precludes symplectic groups H.) Let F be a local field, and let σ be an irreducible (smooth) representation of $H(w_0^\perp \cap W)$. We say that an irreducible (smooth) representation π of H has a Gelfand-Graev model with respect to $(R_{\ell,w_0}; \sigma, \psi)$ if

$$\mathrm{Hom}_{R_{\ell,w_0}}(\pi, \psi_{\ell,w_0} \otimes \widehat{\sigma}) \neq 0. \tag{2.5}$$

(ψ_{ℓ,w_0} may be viewed as a character of R_{ℓ,w_0} by trivial extension.)

Now assume that F is a global field, that ψ is a non-trivial character of $F\backslash\mathbb{A}$ ($\mathbb{A} = \mathbb{A}_F$), and that π is an automorphic representation of $H_\mathbb{A}$, acting in a space of automorphic forms V_π. Put, for $\varphi_\pi \in V_\pi$

$$\varphi_\pi^{\psi_{\ell,w_0}}(h) = \int\limits_{N_\ell(F)\backslash N_\ell(\mathbb{A})} \varphi_\pi(vh)\psi_{\ell,w_0}^{-1}(v)dv. \tag{2.6}$$

Note that $\varphi_\pi^{\psi_{\ell,w_0}}(\gamma h) = \varphi_\pi^{\psi_{\ell,w_0}}(h)$, for $\gamma \in H(w_0^\perp \cap W)_F$. We call the Fourier coefficient (2.6) the Gelfand-Graev coefficient of φ_π with respect ψ_{ℓ,w_0}.

Let σ be an automorphic representation of $H(w^\perp \cap W)_\mathbb{A}$ (acting in a space of automorphic forms V_σ). We say that π has a global Gelfand-Graev coefficient with respect to $(R_{\ell,w_0}; \sigma, \psi)$ if (the following integral converges absolutely and)

$$b(\varphi_\pi, \varphi_\sigma) = \int\limits_{H(w_0^\perp \cap W)_F\backslash H(w_0^\perp \cap W)_\mathbb{A}} \varphi_\pi^{\psi_{\ell,w_0}}(g)\varphi_\sigma(g)dg \not\equiv 0, \tag{2.7}$$

as φ_π varies in V_π and φ_σ varies in V_σ. The corresponding Gelfand-Graev model of π is the space of functions on $H_{\mathbb{A}}$ spanned by the functions $h \to b(\pi(h)\varphi_\pi, \varphi_\sigma)$, as φ_π varies in V_π and φ_σ varies in V_σ. In practice, one of (π, σ) will be cuspidal and the other will be "Eisensteinian".

2.2. Fourier-Jacobi models. — We continue with the previous notations. Assume that w_0 is isotropic and that $(\,,\,)$ is not symmetric (*i.e.* H is either symplectic or unitary). Write

$$W = Ew_0 + W' + Ew_{-0},$$

where w_{-0} is isotropic, $(w_0, w_{-0}) = 1$ and $W' = (Ew_0 + Ew_{-0})^\perp \cap W$. Put $v_{\ell+1} = w_0$, $v_{-(\ell+1)} = w_{-0}$, $V^+_{\ell+1} = \mathrm{Span}\{v_1, \ldots, v_\ell, v_{\ell+1}\}$, $V^-_{\ell+1} = \mathrm{Span}\{v_{-(\ell+1)}, v_{-\ell}, \ldots, v_{-1}\}$ and identify, as before, $\mathrm{GL}(V^\pm_{\ell+1})$ with $\mathrm{GL}_{\ell+1}(E)$. Using these coordinates, an element of $\mathcal{U}_\ell$ has the form

$$u = \begin{pmatrix} I_\ell & y & * & * & * \\ & 1 & 0 & 0 & * \\ & & I_{W'} & 0 & * \\ & & & 1 & y' \\ & & & & I_\ell \end{pmatrix},$$

and

$$\psi_{\ell,w_0}(u) = \psi_E(y_\ell).$$

Note also that an element of L_{W,w_0} has the form

$$\begin{pmatrix} I_\ell & & & & \\ & 1 & x & t & \\ & & g & x' & \\ & & & 1 & \\ & & & & I_\ell \end{pmatrix}, \quad g \in H(W').$$

The unipotent radical of L_{W,w_0} is isomorphic to the Heisenberg group of W', $\mathcal{H}_{W'} = W' \oplus F$. Note that $N_\ell \backslash N_{\ell+1} \cong \mathcal{H}_{W'}$. Fix an isomorphism $j : N_\ell \backslash N_{\ell+1} \to \mathcal{H}_{W'}$. Let F be a local field. Let ω_ψ be the Weil representation of $\mathcal{H}_{W'} \rtimes \widetilde{\mathrm{Sp}}(W')$. If H is a symplectic group, then $H(W') = \mathrm{Sp}(W')$. If H is a unitary group, then so is $H(W')$, and we embed $H(W')$ inside $\widetilde{\mathrm{Sp}}(W')$ (W' viewed over F). This requires a choice of a character γ of E^*, such that $\gamma\big|_{F^*} = \omega_{E/F}$ – the non-trivial quadratic character of F^*, associated to E. See [**Ge.Ro.**]. Denote, in this case, by $\omega_{\psi,\gamma}$ the restriction of ω_ψ to the image of $H(W')$. Put $\omega_{\psi,1} = \omega_\psi$ in case H is symplectic (thus denoting here $\gamma = 1$).

Let σ be an irreducible representation of $H(W')$, in case H is unitary, and of $H(W')^\varepsilon$, $\varepsilon = 0, 1$, in case H is symplectic, where

$$H(W')^\varepsilon = \begin{cases} \mathrm{Sp}(W'), & \varepsilon = 0 \\ \widetilde{\mathrm{Sp}}(W'), & \varepsilon = 1 \end{cases},$$

Then $\omega_{\psi,\gamma} \otimes \widehat{\sigma}$ is a representation of $\mathcal{H}_{W'} \rtimes H(W')$ in case H is unitary, and of $\mathcal{H}_{W'} \rtimes H(W')^{1-\varepsilon}$ in case H is symplectic. Let $R^\sim_{\ell,w_0}$ denote R_{ℓ,w_0} in case H is unitary, or $\varepsilon = 1$, and $N_{\ell+1} \cdot \widetilde{\mathrm{Sp}}(W')$ in case $\varepsilon = 0$. We view ψ_{ℓ,w_0} as a character of $R^\sim_{\ell,w_0}$ by trivial extension.

Let π be an irreducible representation of H in case H is unitary and of $H^{1-\varepsilon} = H(V)^{1-\varepsilon}$ in case H is symplectic. We say that π has a Fourier-Jacobi model with respect to $(R_{\ell,w_0}; \psi, \gamma, \sigma)$ if

$$\operatorname{Hom}_{R_\ell^\sim}(\pi, \psi_\ell \otimes (\omega_{\psi,\gamma} \otimes \widehat{\sigma})) \neq 0, \tag{2.8}$$

where we shorten the notation in this case: $\psi_\ell = \psi_{\ell,w_0}$, $R_\ell = R_{\ell,w_0}$. Here is a short table which summarizes the above cases.

π	$H(V)$–unitary	$\mathrm{Sp}(V)$	$\widetilde{\mathrm{Sp}}(V)$
σ	$H(W')$–unitary	$\widetilde{\mathrm{Sp}}(W')$	$\mathrm{Sp}(W')$
$R_\ell^\sim$	R_ℓ	R_ℓ	$N_{\ell+1} \cdot \widetilde{\mathrm{Sp}}(W')$

Note that $R_\ell \cong N_\ell \rtimes (\mathcal{H}_{W'} \rtimes H(W'))$ (using the isomorphism $j : N_\ell \backslash N_{\ell+1} \xrightarrow{\sim} \mathcal{H}_{W'}$).

Assume now that F is a global field and that ψ is a non-trivial character of $F\backslash\mathbb{A}$. Let ω_ψ be the Weil representation of $\widetilde{\mathrm{Sp}}(W')_{\mathbb{A}}$, and in case H is a unitary group, fix a character γ of $E^*\backslash\mathbb{A}_E^*$, such that $\gamma|_{\mathbb{A}_F^*} = {}^{\omega}E/F$, and denote by $\omega_{\psi,\gamma}$ the restriction of ω_ψ to the image of $H(W')_{\mathbb{A}}$ determined by (γ, ψ). Denote, as before, $\omega_{\psi,1} = \omega_\psi$ in the symplectic case. Denote, for a Schwartz function ϕ in a Schrodinger model of ω_ψ, by $\theta_{\psi,\gamma}^\phi$ the corresponding theta series.

Let π be an automorphic representation of $H_{\mathbb{A}}$, in case H is a unitary group, or of $H_{\mathbb{A}}^{1-\varepsilon}$, in case H is symplectic.

Put, for $\varphi_\pi \in V_\pi$

$$\varphi^{\psi_\ell,\gamma,\phi}(h) = \int\limits_{N_{\ell+1}(F)\backslash N_{\ell+1}(\mathbb{A})} \varphi_\pi(vh)\psi_\ell^{-1}(v)\theta_{\psi^{-1},\gamma^{-1}}^\phi(j(v)h)dv \tag{2.9}$$

Recall that ψ_ℓ is extended trivially to $N_{\ell+1}$ and j is the isomorphism $N_\ell\backslash N_{\ell+1} \xrightarrow{\sim} \mathcal{H}_{W'}$. (We keep denoting by j its composition with $N_{\ell+1} \to N_\ell\backslash N_{\ell+1}$.) Note that

$$\varphi_\pi^{\psi_\ell,\gamma,\phi}(rh) = \varphi_\pi^{\psi_\ell,\gamma,\phi}(h), \quad \forall\, r \in H(W')_F.$$

$\varphi_\pi^{\psi_\ell,\gamma,\phi}$ is called a Fourier Jacobi coefficient of φ_π with respect to $\omega_{\psi,\gamma}$ (and ϕ). Let σ be an automorphic representation of $H(W')_{\mathbb{A}}$ in case H is a unitary group, and of $H(W')_{\mathbb{A}}^\varepsilon$ in case H is symplectic. We say that π has a global Fourier-Jacobi model with respect to $(R_\ell; \psi, \gamma, \sigma)$ if (the following integral is absolutely convergent and)

$$\int\limits_{H(W')_F\backslash H(W')_{\mathbb{A}}} \varphi_\pi^{\psi_\ell,\gamma,\phi}(g)\varphi_\sigma(g)dg \not\equiv 0, \tag{2.10}$$

as φ_π and φ_σ vary in V_π and V_σ respectively. (In both cases, local, or global, representations of metaplectic covers are assumed to be genuine.) In practice, we will take one of (π, σ) to be cuspidal and the other to be "Eisensteinian".

In the following remark, we relate the above models to degenerate Whittaker models, as formulated in [**M.W.**]. It is meant just for completeness sake, and may be skipped at a first reading.

Remark. — The equivariance properties with respect to N_ℓ or $N_{\ell+1}$ of the models just introduced are special cases of the general set-up of degenerate Whittaker models. To relate to the terminology [**M.W.**], we have to choose a nilpotent element f in Lie (H), and a one parameter subgroup φ of H, such that

$$\mathrm{Ad}(\varphi(t))\cdot f = t^{-2}\cdot f, \qquad \forall\, t\in F^*. \tag{2.11}$$

We realize

$$\mathrm{Lie}(H) = \{A\in End_E(V) \mid (Av_1,v_2)+(v_1,Av_2)=0, \quad \forall\, v_1,v_2\in V\},$$

and write its elements in matrix form following (2.1). Consider again the rational character χ_{w_0,v_0^-} of $\mathcal{U}_\ell$. Clearly, there is a unique element $f_1(w_0)\in\mathrm{Hom}(V_\ell^+,W)$, such that

$$\chi_{w_0,v_0^-}\left(\exp\begin{pmatrix}0_\ell & y & x\\ & 0_W & y'\\ & & 0_\ell\end{pmatrix}\right) = \mathrm{tr}\left(\begin{pmatrix}0_\ell & & \\ f_1(w_0) & 0_W & \\ 0 & f_1(w_0)' & 0_\ell\end{pmatrix}\cdot\begin{pmatrix}0_\ell & y & x\\ & 0_W & y'\\ & & 0_\ell\end{pmatrix}\right) \tag{2.12}$$
$$(=2\,\mathrm{tr}(f_1(w_0)\circ y)).$$

Here, we think of y as an element of $\mathrm{Hom}_E(W,V_\ell^+)$ etc. Identifying $\mathrm{Hom}(V_\ell^+,W)$ and $W\times\cdots\times W$ (ℓ times), using the basis $\{v_1,\dots,v_\ell\}$, it is clear, by the choice $v_{-\ell}=v_0^-$, that $f_1(w_0)$ is identified with an ℓ-tuple of the form $(0,\dots,0,w_0^*)$. Let

$$f_{\ell,w_0} = \begin{pmatrix} z_\ell & 0 & 0\\ f_1(w_0) & 0 & 0\\ 0 & f_1(w_0)' & \tilde{z}_\ell\end{pmatrix}\in\mathrm{Lie}(H),$$

where $z_\ell = \frac12\begin{pmatrix}0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0\end{pmatrix}$ (and $f_1(w_0)=(0,\dots,0,w_0^*)$). Note that

$$2\,\mathrm{tr}\left(z_\ell\circ\begin{pmatrix}0 & x_1 & & & \\ & 0 & x_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & x_{\ell-1}\\ & & & & 0\end{pmatrix}\right) = x_1+\cdots+x_{\ell-1}. \tag{2.13}$$

From (2.12) and (2.13) we have, for $S\in\mathrm{Lie}(N_\ell)$,

$$\psi_{\ell,w_0}(\exp S) = \psi_E(\mathrm{tr}(f_{\ell,w_0}\cdot S)).$$

Next, we have to explain what was our choice of a one parameter subgroup φ of H. Let

$$\varphi_\ell(t) = \begin{pmatrix} a_\ell(t) & & \\ & I_W & \\ & & a_\ell(t)^*\end{pmatrix}\in H,$$

where

$$a_\ell(t) = \mathrm{diag}(t^{2\ell},t^{2\ell-2},t^{2\ell-4},\dots,t^2).$$

If w_0 is anisotropic, we choose $\varphi = \varphi_\ell$, and if w_0 is isotropic, we choose

$$\varphi(t) = \begin{pmatrix} ta_\ell(t) & & & & \\ & t & & & \\ & & I_{W'} & & \\ & & & t^{-1} & \\ & & & & t^{-1}a_\ell(t)^* \end{pmatrix}.$$

Note that (2.11) is satisfied. Now decompose

$$\mathrm{Lie}(H) = \oplus\ \mathrm{Lie}(H)_i,$$

where

$$\mathrm{Lie}(H)_i = \{S \in \mathrm{Lie}(H) \mid \mathrm{Ad}\,\varphi(t)\cdot S = t^i S,\ \forall\, t \in F^*\}.$$

Clearly, if w_0 is anisotropic, then

$$\mathrm{Lie}(N_\ell) = \bigoplus_{i\geqslant 2} \mathrm{Lie}(H)_i = \bigoplus_{i\geqslant 1} \mathrm{Lie}(H)_i.$$

If w_0 is isotropic, then

$$\mathrm{Lie}(N_\ell\cdot \mathrm{Center}(L_{W,w_0})) = \bigoplus_{i\geqslant 2} \mathrm{Lie}(H)_i,$$

and

$$\mathrm{Lie}(N_{\ell+1}) = \bigoplus_{i\geqslant 1} \mathrm{Lie}(H)_i.$$

(Note that $N_\ell\cdot$ Center$(L_{W,w_0}) = j^{-1}(\mathrm{Center}(\mathcal{H}_{W'}))$, where j is the composition of $N_{\ell+1}\to N_\ell\backslash N_{\ell+1}\xrightarrow{\sim}\mathcal{H}_{W'}$).

2.3. The global integrals: overview. — The general form of the global integrals is just an application of a global Gelfand-Graev model, or a global Fourier-Jacobi model to an Eisenstein series on $H_{\mathbb{A}}$, or on $H_{\mathbb{A}}^{1-\varepsilon}$, in case H is symplectic, induced from a cuspidal representation on a maximal parabolic subgroup of H. The global model is taken against a cuspidal representation σ on $H(w_0^\perp\cap W)_{\mathbb{A}}$, in case w_0 is anisotropic, on $H(W')_{\mathbb{A}}$, in case H is unitary, or on $H(W')_{\mathbb{A}}^{\varepsilon}$, in case H is symplectic. Thus, in (2.7) and in (2.9), π is an Eisenstein series induced from a cuspidal representation $\tau\otimes\sigma_0$ on a parabolic subgroup, whose Levi part is isomorphic to $\mathrm{GL}_k\times H(W_k)$, where $V = V_k^+ + W_k + V_k^-$, as in (2.1). With normalized Eisenstein series, these integrals represent $L^S(\sigma\times\tau, s)$, the partial standard L-function for $H(w_0^\perp\cap W)\times\mathrm{GL}_k$, (resp. $H(W')\times\mathrm{Res}_{E/F}\mathrm{GL}_k$, resp. $H(W')^\varepsilon\times\mathrm{GL}_k$) provided σ and σ_0 are related through an appropriate global Gelfand-Graev model (resp. Fourier-Jacobi model). For example, if W_k is a subspace of $w_0^\perp\cap W$, in case w_0 is anisotropic, or a subspace of W', in case w_0 is isotropic, then σ should have a global model with respect to a subgroup $R_{\ell',w_0'}\subset H(w_0^\perp\cap W)$ (resp. $H(W')$), whose reductive part is isomorphic to $H(W_k)$, on which we take σ_0. In this generality, the global integrals were studied in [**G.PS.R.**] for orthogonal groups H. Special cases were treated in [**Ge.Ro.**] (Fourier-Jacobi model for π cuspidal on $U_{2,1}$) and in [**N**] (Gelfand-Graev model for π cuspidal on $\mathrm{SO}_{3,2}$ (actually on GSp_4)). We will be interested here in the case where the bilinear form has maximal Witt index (*i.e.* $[\frac{1}{2}\dim_E V]$). Thus, if $E = F$, H is split and if

$[E:F]=2$, H is the quasi-split unitary group in $\dim_E V$ variables. In this case, we will apply the above global models to π – an Eisenstein series induced from the Siegel parabolic subgroup and a cuspidal representation τ on $\mathrm{Res}_{E/F}\mathrm{GL}_k$. We will choose w_0, (when anisotropic), such that $H(w_0^\perp \cap W)$ is quasi-split or split. Again, with normalized Eisenstein series, the integrals (2.7) and (2.9) represent $L^S(\sigma\times\tau,s)$. These cases were studied in [**So1, So2, So3**] (H – even orthogonal) and in [**GRS3**] (H symplectic or metaplectic). The case where $H(w_0^\perp\cap W)$ is of rank one less than k was studied in [**G.PS.**] (H orthogonal). The remaining cases are treated similarly and will appear in detail in future works. We will summarize them here. In these cases, except for $H=U_{2k+1}$, these integrals are identically zero, unless σ has a global Whittaker model (with respect to an appropriate character). Finally, we also consider the cases where π is cuspidal on H (resp. on $H^{1-\varepsilon}$, when H is symplectic) and σ is an Eisenstein series on $H(w_0^\perp\cap W)$, when $(w_0,w_0)\neq 0$ (resp. on $H(W')^{1-\varepsilon}$, when H is symplectic, or $H(W')$, when H is unitary and w_0 is isotropic). This Eisenstein series is induced from the Siegel parabolic subgroup and a cuspidal representation τ on $\mathrm{Res}_{E/F}\mathrm{GL}_n$ ($n=[\frac12\dim(w_0^\perp\cap W)]$, or $\frac12\dim(W')$. Again, in these cases, except when $H(w_0^\perp\cap W)$ or $H(W')$ are U_{2n+1}, the integrals (2.7), (2.9) are identically zero, unless π has a global Whittaker model (with respect to an appropriate character) and then they represent $L^S(\pi\times\tau,s)$ once the Eisenstein series is normalized. These cases were studied in [**G**] (H – split orthogonal) and in [**GRS3**] (H symplectic or metaplectic). The cases where $k=n$ were studied in [**G.PS.**], [**T**], [**W**]. The remaining cases (H unitary, $k>n$) are treated similarly and will appear in detail in future works. We will summarize them here.

From now on, we assume that $(\,,\,)$ has Witt index $\left[\frac12\dim_E V\right]$. We will denote $r=\dim_E V$, and $H=H_r$. We realize V as the column space E^r and represent $(\,,\,)$ in terms of the matrix $\begin{pmatrix} & & & 1\\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \varepsilon & & & \end{pmatrix}$ where $\varepsilon=\pm1$; $\varepsilon=-1$ is reserved just for symplectic groups. H_r is realized as a matrix group. We denote by P_r the Siegel parabolic subgroup of H_r. Its Levi part is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_{[r/2]}$.

2.4. The global integrals: Gelfand-Graev models. — It remains to specify w_0. We do this in the following table. Write ${}^tw_0=(0,{}^tw_0',0)$, where 0 denotes a zero row vector in ℓ coordinates. Recall that for

$$v=\begin{pmatrix} z & y & x\\ & I_W & y'\\ & & z^*\end{pmatrix}\in N_\ell \qquad (z\in Z_\ell),$$

$$\psi_{\ell,w_0}(v)=\psi_E(z)\psi_E(y_\ell\cdot w_0')=\psi_E\Big(\sum_{i=1}^{\ell-1} z_{i,i+1}+y_\ell\cdot w_0'\Big), \tag{2.14}$$

where y_ℓ denotes the last row of y. In the following table (2.15) we indicate the choice of ${}^tw_0'$. We also write ℓ in terms of $m+1=\dim_E W$ and $r=\dim_E V$.

	$H = H_r$	$\dim_E W = m+1$	ℓ	${}^t w'_0$	$\psi_E(y_\ell \cdot w'_0)$	$H(w_0^\perp \cap W) \cong H_m^{(\alpha)}$
(1)	SO_{2k}	$2n+2$	$k-n-1$	$(0,\ldots,0,1,-1,0,\ldots,0)$	$\psi(y_{\ell,n+1} - y_{\ell,n+2})$	SO_{2n+1}
(2)	U_{2k}	$2n+2$	$k-n-1$	$(0,\ldots,0,1-1,0,\ldots,0)$	$\psi_E(y_{\ell,n+1} - y_{\ell,n+2})$	U_{2n+1}
(3)	SO_{2k+1}	$2n+1$	$k-n$	$(0,\ldots,0,1,0,\alpha,0,\ldots,0)$	$\psi(y_{\ell,n} + \alpha y_{\ell,n+2})$	$\mathrm{SO}_{2n}^{\alpha}$
(3′)	SO_{2k+1}	$2n+1$	$k-n$	$(0,\ldots,0,1,0,\ldots,0)$	$\psi(y_{\ell,n+1})$	SO_{2n}
(4)	U_{2k+1}	$2n+1$	$k-n$	$(0,\ldots,0,1,0,\ldots,0)$	$\psi_E(y_{\ell,n+1})$	U_{2n}

Table (2.15)

Here, we also denote $H_m^{(-1)} = H_m$, so that in all cases except (3), $\alpha = -1$. In case (3), $\mathrm{SO}_{2n}^{(\alpha)} = H_{2n}^{(\alpha)}$, $(\alpha \in F^*)$, denotes the quasi-split orthogonal group with respect to the symmetric form, whose matrix is

$$\begin{pmatrix} & & w_{n-1} \\ & \begin{smallmatrix} 1 & 0 \\ 0 & -2\alpha \end{smallmatrix} & \\ w_{n-1} & & \end{pmatrix} \quad \left(w_{n-1} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \right).$$

Note that $\mathrm{SO}_{2n}^{(\alpha)} \cong \mathrm{SO}_{2n}$, if and only if $2\alpha \in (F^*)^2$. (In this case we may replace Case (3) by Case (3$'$).) We denote by $\psi_{\ell,\alpha}$ the character (2.14). Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_k(\mathbb{A}_E)$ $(k = [r/2])$. We consider now all cases except case (4). Denote

$$\rho_{\tau,s}^{H_r} = \mathrm{Ind}_{P_r(\mathbb{A}_F)}^{H_r(\mathbb{A}_F)} \tau |\det \cdot|_E^{s-1/2}.$$

Let $\xi_{\tau,s}$ be a holomorphic K-finite section for $\rho_{\tau,s}$ and denote by $E_{H_r}(\xi_{\tau,s}, h)$ the corresponding Eisenstein series on $H_r(\mathbb{A}_F)$. Let σ be an irreducible, automorphic, cuspidal representation of $H_m^{(\alpha)}(\mathbb{A}_F)$ ($\alpha = -1$ for all cases except (3)). Fix an F-isomorphism $H_m^{(\alpha)} \xrightarrow{\sim} H(w_0^\perp \cap W)$, and denote by $i_{m,r}$ its composition with the inclusion $H(w_0^\perp \cap W) \hookrightarrow H_r$. Define, for a cusp form φ_σ in the space of σ,

$$\mathcal{L}(\varphi_\sigma, \xi_{\tau,s}) = \int_{H_m^{(\alpha)}(F)\backslash H_m^{(\alpha)}(\mathbb{A}_F)} \varphi_\sigma(g) E_{H_r}^{\psi_{\ell,\alpha}}(\xi_{\tau,s}, i_{m,r}(g))dg. \tag{2.16}$$

These integrals converge absolutely and are meromorphic in s. For Re(s) large enough, the integral (2.16) equals an Eulerian integral which depends on the ψ-Whittaker coefficient of φ_σ. [For example, for $H_r = U_{2k}$ $(H_m^{(\alpha)} = U_{2n+1})$ and Re$(s) \gg 0$,

$$\mathcal{L}(\varphi_\sigma, \xi_{\tau,s}) = \int_{N_{\mathbb{A}_F}\backslash U_{2n+1}(\mathbb{A}_F)} W_{\varphi_\sigma}^{\psi}(g) \int_{M_{\ell\times(n+1)}(\mathbb{A}_E)\times h_\ell(\mathbb{A}_F)} \xi_{\tau,s}^{\psi^{-1}}\left(\beta_{k,n}\begin{pmatrix} I_\ell & x & 0 & y \\ & I_{n+1} & 0 & 0 \\ & & I_{n+1} & x' \\ & & & I_\ell \end{pmatrix} i_{m,r}(g)\right) \psi_E(x_{\ell,n+1})d(x,e)dg \tag{2.17}$$

where N is the standard maximal unipotent subgroup of U_{2n+1},

$$W_{\varphi_\sigma}^{\psi}(g) = \int_{N_F\backslash N_{\mathbb{A}_F}} \varphi_\sigma(ug)\psi_N^{-1}(u)du$$

is the ψ-Whittaker function of φ_σ $\left(\psi_N(u) = \psi_E\left(\sum_{i=1}^m u_{i,i+1}\right)\right)$; $\xi_{\tau,s}^{\psi^{-1}}(h)$ is the composition of $\xi_{\tau,s}$ with the ψ^{-1}-Whittaker coefficient on τ, *i.e.*

$$\xi_{\tau,s}^{\psi^{-1}}(h) = \int_{Z_k(E)\backslash Z_k(\mathbb{A}_E)} \xi_{\tau,s}\left(\left(\begin{smallmatrix} z & 0 \\ 0 & z^* \end{smallmatrix}\right)h\right)\psi_E(z)dz;$$

$\beta_{k,n}$ is the Weyl element

$$\begin{pmatrix} 0 & I_{n+1} & 0 & 0 \\ 0 & 0 & 0 & I_\ell \\ I_\ell & 0 & 0 & 0 \\ 0 & 0 & I_{n+1} & 0 \end{pmatrix}$$

and $h_\ell = \{A \in M_\ell(E) \mid {}^t\overline{(Aw_\ell)} + (Aw_\ell) = 0\}]$. The integrals (2.16) are identically zero, unless the ψ-Whittaker coefficient of φ_σ is nontrivial as a function on $H_m^{(\alpha)}(\mathbb{A}_F)$. Thus σ has to be globally ψ-generic. (This is not the condition one gets in case (4). This is why we exclude it now.) Assume then that σ is ψ-generic. For decomposable data, the Eulerian integral of (2.16) has the form

$$\mathcal{L}(\varphi_\sigma, \xi_{\tau,s}) = R(s)\frac{L^S(\sigma \times \tau, s)}{L^S(\tau, \delta, 2s)}. \tag{2.18}$$

Here S is a finite set of places of F, including the ones at infinity, outside which σ, τ and the components of φ_σ, $\xi_{\tau,s}$ are unramified. $R(s)$ is a finite product of "local integrals" (over S), where data can be chosen so that $R(s)$ is holomorphic and nonzero at a neighborhood of a given point s_0. $L^S(\tau, \delta, z)$ is the partial L-function which enters in the normalizing factor of $E_{H_r}(\xi_{\tau,s}, h)$. Let us summarize this in the following table

$L^S(\sigma \times \tau, s)$ for the group		$L^S(\tau, \delta, 2s)$
$\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$,	$k > n$	$L^S(\tau, \Lambda^2, 2s)$
$U_{2n+1} \times \mathrm{Res}_{E/F}(\mathrm{GL}_k)$,	$k > n$	$L^S(\tau, \mathrm{Asai}, 2s)$
$\mathrm{SO}_{2n}^{(\alpha)} \times \mathrm{GL}_k$,	$k \geqslant n$	$L^S(\tau, \mathrm{sym}^2, 2s)$

(2.19)

Next, we may take a cusp form on H and an Eisenstein series on $H_m^{(\alpha)}$. We go back to table (2.15) and assume now that case (2) is excluded, and also in case (3) we consider $\alpha = -1$ (and so we may replace (3) by (3′)). Let σ be an irreducible, automorphic, cuspidal representation of $H_r(\mathbb{A})$. Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and consider the Eisenstein series $E_{H_m}(\xi_{\tau,s}, g)$ corresponding to a K-finite holomorphic section $\xi_{\tau,s}$ for $\rho_{\tau,s}^{H_m}$. Define, for a cusp form φ_σ in the space of σ

$$\mathcal{L}(\varphi_\sigma, \xi_{\tau,s}) = \int_{H_m(F)\backslash H_m(\mathbb{A}_F)} \varphi_\sigma^{\psi_\ell,-1}(i_{m,r}(g))E_{H_m}(\xi_{\tau,s}, g)dg. \tag{2.20}$$

As before, for $\mathrm{Re}(s)$ large enough, the integral (2.20) equals an Eulerian integral which depends on the ψ-Whittaker coefficient of φ_σ. [For example, for $H_r = U_{2k+1}$

$(H_m = U_{2n})$ and $\mathrm{Re}(s) \gg 0$

(2.21) $\mathcal{L}(\varphi_\sigma, \xi_{\tau,s})$

$$= \int\limits_{N_{\mathbb{A}_F}\backslash U_{2n}(\mathbb{A}_F)} \int\limits_{M_{(k-n)\times n}(\mathbb{A}_E)} W^{\psi}_{\varphi_\sigma}\left(\begin{pmatrix} I_n & \\ x & I_{k-n}\end{pmatrix}^{\wedge} \cdot \widehat{w}_{n,k} i_{2n,2k+1}(g)\right) \xi^{\psi^{-1}}_{\tau,s}(g)dxdg,$$

where, for $g \in \mathrm{GL}_k(\mathbb{A}_E)$, we denote $\widehat{g} = \begin{pmatrix} g & & \\ & 1 & \\ & & g^* \end{pmatrix}$, $w_{n,k} = \begin{pmatrix} & I_n \\ I_{k-n} & \end{pmatrix}$. N denotes the standard maximal unipotent subgroup of U_{2n}. The remaining notation is as in (2.17).] As before, $\mathcal{L}(\varphi_\sigma, \xi_{\tau,s})$ is identically zero, unless the ψ-Whittaker coefficient of φ_σ is non-trivial (as a function on $H_r(\mathbb{A})_F$). Thus, assume that σ is globally ψ-generic, and then for decomposable data the Eulerian integral of (2.10) has the form

$$\mathcal{L}(\varphi_\sigma, \xi_{\tau,s}) = R(s)\frac{L^S(\sigma\times\tau, s)}{L^S(\tau,\delta,2s)}, \tag{2.22}$$

as in (2.18). $L^S(\tau,\delta,2s)$ is given by (2.19), switching roles of k and n. More precisely

$L^S(\sigma\times\tau,s)$ for the group		$L^S(\tau,\delta,2s)$
$\mathrm{SO}_{2k+1}\times\mathrm{GL}_n$,	$k \geqslant n$	$L^S(\tau,\Lambda^2,2s)$
$U_{2k+1}\times\mathrm{Res}_{E/F}(\mathrm{GL}_n)$,	$k\geqslant 2n$	$L^S(\tau,\mathrm{Asai},2s)$
$\mathrm{SO}_{2k}\times\mathrm{GL}_n$,	$k>n$	$L^S(\tau,\mathrm{sym}^2,2s)$

(2.23)

2.5. The global integrals: Fourier-Jacobi models. — We use the notation of Sec. 2.2, where w_0 was already chosen. We will denote by $H(W')^\sim$ the group $H(W')$ in case H is a unitary group, and in case H is symplectic $H(W')^\sim = H(W')^\varepsilon$, $\varepsilon = 0,1$. The cases we consider appear in the following table ($r = \dim_E V$)

	$H^\sim = H_r^\sim$	$\dim_E W = m+2$	ℓ	$H(W')\simeq H_m$	$H_n^\sim$
(1)	Sp_{2k}	$2n+2$	$k-n-1$	Sp_{2n}	$\widetilde{\mathrm{Sp}}_{2n}$
(2)	$\widetilde{\mathrm{Sp}}_{2k}$	$2n+2$	$k-n-1$	Sp_{2n}	Sp_{2n}
(3)	U_{2k}	$2n+2$	$k-n-1$	U_{2n}	U_{2n}

(2.24)

Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_k(\mathbb{A}_E)$. Denote by $\rho^{H^\sim_{2k}}_{\tau,s}$ the representation of $H_{2k}(\mathbb{A}_F)^\sim$ induced from $\tau|\det\cdot|^{s-1/2}$ on the Siegel parabolic subgroup in cases (1), (3). In case (2) we replace τ by $\gamma_\psi\cdot\tau$, where γ_ψ is the Weil factor. Let $\xi_{\tau,s}$ be a K-finite holomorphic section for $\rho_{\tau,s}$, and let $E_{H^\sim_{2k}}(\xi_{\tau,s},h)$ be the corresponding Eisenstein series. Let σ be an irreducible, automorphic, cuspidal representation of $H^\sim_{2n}(\mathbb{A}_F)$. Fix an F-embedding $j_{2n,2k} : H_{2n} \to H_{2k}$, so that the image of $j_{2n,2k}$ is $H(W')$. Define for a cusp form φ_σ in the space of σ

$$\mathcal{L}(\varphi_\sigma, \phi, \xi_{\tau,s}) = \int\limits_{H_{2n}(F)\backslash H_{2n}(\mathbb{A}_F)} \varphi_\sigma(g) E_{H_\tau^\sim}^{\psi_\ell,\gamma,\phi}\Big(\xi_{\tau,s}, j_{2n,2k}(g)\Big)dg. \tag{2.25}$$

As before, this integral equals an Eulerian integral, for $\mathrm{Re}(s) \gg 0$, and it depends on the ψ-Whittaker coefficient of φ_σ. [For example, for $H_{2k}^\sim = U_{2k}$ ($H_{2n}^\sim = U_{2n}$), we get, for $\mathrm{Re}(s) \gg 0$,

$$\begin{aligned}(2.26)\quad &\mathcal{L}(\varphi_\sigma, \phi, \xi_{\tau,s}) \\ &= \int\limits_{N_{\mathbb{A}_F}\backslash U_{2n}(\mathbb{A}_F)} W_{\varphi_\sigma}^{\psi}(g) \int\limits_{M_{(k-n)\times n}(\mathbb{A}_E)\times h_{k-n}(\mathbb{A}_F)} \xi_{\tau,s}^{\psi^{-1}}\left(\alpha_{k,n}\begin{pmatrix} I_{k-n} & x & 0 & y \\ & I_n & 0 & 0 \\ & & I_n & x' \\ & & & I_{k-n}\end{pmatrix} i_{2n,2k}(g)\right) \\ &\qquad \omega_{\psi^{-1},\gamma^{-1}}(x_{k-n}, 0; \mathrm{Im}(y_{k-n,1})g)\phi(\varepsilon_n)d(x,y)dg.\end{aligned}$$

Here we assume that $E = F[\sqrt{\rho}]$, and $W_{\varphi_\sigma}^{\psi}(g)$ is the Whittaker coefficient of φ_σ (at g) with respect to the non-degenerate character of $N_F\backslash N_{\mathbb{A}_F}$ given by

$$u \longmapsto \psi_E\Big(u_{12} + u_{23} + \cdots + u_{n-1,n} + \frac{u_{n,n+1}}{2\sqrt{\rho}}\Big); \quad \alpha_{k,n} = \begin{pmatrix} 0 & \frac{1}{\sqrt{\rho}}I_n & 0 & 0 \\ 0 & 0 & 0 & I_{k-n} \\ I_{k-n} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\rho}I_n & 0\end{pmatrix}.$$

We regard E^{2n} as a symplectic space over F with respect to the form $\langle v_1, v_2\rangle = -2\,\mathrm{Im}(v_1, v_2)$. ($Im$ denotes the "imaginary" part: $\mathrm{Im}(a + \sqrt{b}) = b$.) This defines $\omega_{\psi^{-1},\gamma^{-1}}$, realized in $S(\mathbb{A}_E^n)$. Finally, $\phi \in S(\mathbb{A}_E^n)$ and $\varepsilon_n = (0, \ldots, 0, 1)$. The rest of the notation is as in (2.17).] Thus, assume that σ is globally ψ-generic. For decomposable data the Eulerian integral of (2.25) has the form

$$\mathcal{L}(\varphi_\sigma, \phi, \xi_{\tau,s}) = R(s)L^S(\sigma, \tau, s) \tag{2.27}$$

where $L^S(\sigma, \tau, s)$ is given by the following table

	σ on $H_{2n}^\sim$	τ on	Eisenstein series $E(\xi_{\tau,s}, \cdot)$ on $H_{2k}^\sim$	$L^S(\sigma, \tau, s)$
(1)	$\widetilde{Sp}_{2n}$	GL_k	Sp_{2k}	$\frac{L_\psi^S(\sigma\times\tau,s)}{L^S(\tau,s+\frac{1}{2})L^S(\tau,\Lambda^2,2s)}$
(2)	Sp_{2n}	GL_k	$\widetilde{\mathrm{Sp}}_{2k}$	$\frac{L^S(\sigma\times\tau,s)}{L^S(\tau,\mathrm{sym}^2,2s)}$
(3)	U_{2n}	$\mathrm{Res}_{E/F}\,\mathrm{GL}_k$	U_{2k}	$\frac{L^S(\sigma\times\tau,\gamma^{-1},s)}{L^S(\tau,\mathrm{Asai},2s)}$

(2.27) $(k > n)$

In case (1) there is no canonical way to attach an L-function to $\sigma \times \tau$. At places ν where σ is unramified (and ψ normalized) we write the unramified characters corresponding to σ_ν in the form $\gamma_{\psi_\nu} \cdot \mu_\nu$, where μ_ν is an unramified character of F_ν^*. We write the parameter of σ_ν as a conjugacy class in $\mathrm{Sp}_{2n}(\mathbb{C})$ (constructed from the $\mu_\nu(p_\nu)^{\pm 1}$). Another choice $\gamma_{\psi_\nu a_\nu}$ would yield a different conjugacy class. This explains the dependence on ψ in $L_\psi^S(\sigma \times \tau, s)$. The function $R(s)$ in (2.26) can be chosen to have the same properties as in (2.18), (2.22).

Finally, as in the previous case (Gelfand-Graev models) we may reverse the roles of $H_r^{\sim}$ and $H_{2n}^{\sim}$. We go back to table (2.24) and consider now an irreducible, automorphic, cuspidal representation σ of $H_r^{\sim}(\mathbb{A}_F)$ and an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_n(\mathbb{A}_E)$. Consider the Eisenstein series $E_{H_{2n}^{\sim}}(\xi_{\tau,s},g)$ on $H_{2n}^{\sim}(\mathbb{A}_F)$ corresponding to a holomorphic K-finite section $\xi_{\tau,s}$ for $\rho_{\tau,s}^{H_{2n}^{\sim}}$. Define for a cusp form φ_σ in the space of σ,

$$\mathcal{L}(\varphi_\sigma,\phi,\xi_{\tau,s}) = \int\limits_{H_{2m}(F)\backslash H_{2n}(\mathbb{A}_F)} \varphi_\sigma^{\psi_\ell,\gamma,\phi}(j_{2n,r}(g))E_{H_{2n}^{\sim}}(\xi_{\tau,s},g)dg. \tag{2.28}$$

Again, for $\mathrm{Re}(s)$ large, the integral (2.28) equals an Eulerian integral which depends on the ψ-Whittaker function of φ_σ. [For example, for $H_r^{\sim}=U_{2k}$ $(H_{2n}^{\sim}=U_{2n})$ and σ on $U_{2k}(\mathbb{A}_F)$, and τ on $\mathrm{GL}_n(\mathbb{A}_F)$, we get for $\mathrm{Re}(s)\gg 0$

$$\begin{aligned}(2.29)\quad &\mathcal{L}(\varphi_\sigma,\phi,\xi_{\tau,s})\\ &= \int\limits_{N_{\mathbb{A}_F}\backslash U_{2n}(\mathbb{A}_F)} \int\limits_{M_{(k-n)\times n}(\mathbb{A}_E)} W_{\varphi_\sigma}^{\psi}\left(\begin{pmatrix} I_n & \\ x & I_{k-n}\end{pmatrix}^{\wedge}\widehat{w}_{n,k}j_{2n,r}(g)\right)\\ &\qquad\qquad \omega_{\psi^{-1},\gamma^{-1}}(g)\phi(x_{k-n})\xi_{\tau,s}^{\psi^{-1}}(g)dxdg\end{aligned}$$

Here $W_{\varphi_\sigma}^{\psi}$ is as in (2.16). For $g\in \mathrm{Res}_{E/F}\mathrm{GL}_k$, we denote $\widehat{g}=\begin{pmatrix} g & \\ & g^*\end{pmatrix}\in U_{2k}$. For $x\in M_{(k-n)\times n}$, x_{k-n} denotes the last row of x; $w_{n,k}=\begin{pmatrix} & I_n\\ I_{k-n} & \end{pmatrix}$. The rest of the notation is as before.]

Assume that σ is globally ψ-generic. Then for decomposable data

$$\mathcal{L}(\varphi_\sigma,\phi,\xi_{\tau,s}) = R(s)L^S(\sigma,\tau,s). \tag{2.30}$$

$L^S(\sigma,\tau,s)$ is given by the last column of table (2.27) (where we switch the roles of $E(\xi_{\tau,s},\cdot)$ and σ).

In (2.25), (2.28) the case $k=n$ is missing. Here, for a ψ-generic cuspidal representation σ on $H_{2n}^{\sim}(\mathbb{A}_F)$ and a cuspidal representation τ on $\mathrm{GL}_n(\mathbb{A}_E)$, we consider

$$\mathcal{L}(\varphi_\sigma,\phi,\xi_{\tau,s}) = \int\limits_{H_{2n}(F)\backslash H_{2n}(\mathbb{A}_F)} \varphi_\sigma(g)\theta_{\psi^{-1},\gamma^{-1}}^{\phi}(g)E_{H_{2n}^{\sim}}(\xi_{\tau,s},g)dg, \tag{2.31}$$

where, as before, for $H_{2n}=U_{2n}$, $H_{2n}^{\sim}=U_{2n}$, and for $H_{2n}=\mathrm{Sp}_{2n}$, if σ is on H_{2n}^{ε} then the Eisenstein series is on $H_{2n}^{1-\varepsilon}$, $\varepsilon=0,1$. For $\mathrm{Re}(s)\gg 0$, we obtain as in the previous cases (for decomposable data)

$$\mathcal{L}(\varphi_\sigma,\phi,\xi_{\tau,s}) = R(s)L^S(\sigma,\tau,s),$$

as in the last two cases.

3. On the weak lift from a quasi-split classical group to GL_N.

We construct examples of cuspidal generic representations on a given quasi-split classical group G, which weakly lift to automorphic representations on GL_N (appropriate N) in the expected image of this lift. The methods are those of Chapter 1, constructing a descent map (backward lift), as suggested by the global integrals reviewed in Chapter 2. We use the notation of Chapter 2.

3.1. The cuspidal part of the image of the weak lift from G to GL_N

Let G be a group of the form $H(w_0^\perp \cap W)$ or $H(W')^\sim$, as in table (2.15) (without case (4)), or table (2.24). (For the moment $\dim_E V$ is not so important.) Let N be the degree of the standard representation of ${}^LG^0$. The Langlands conjectures predict the existence of a functorial lift from irreducible, automorphic, cuspidal representations of $G_{\mathbb{A}_F}$ to irreducible automorphic representations of $\mathrm{GL}_N(\mathbb{A}_E)$. Let $\sigma \cong \otimes\sigma_\nu$ be such a representation, and assume that σ has a weak lift to an irreducible automorphic representation τ of $\mathrm{GL}_N(\mathbb{A}_E)$, where the notion of a weak lift is similar to the one explained in Sec. 1.1. It is clear that $\tau_\nu^* \cong \tau_\nu$, and $\omega_{\tau_\nu}\big|_{F_\nu^*} = 1$, for almost all ν, except in case (3) of (2.24), when 2α is not a square in F_ν^*, in which case, ω_{τ_ν} is the quadratic character associated to 2α. Here $\tau_\nu^* = \widehat{\tau}_\nu'$ and τ_ν' is the composition of τ_ν with the automorphism $x \mapsto \overline{x}$ of E_ν over F_ν. (If $E = F$, then $\overline{x} = x$, and $\tau_\nu^* = \widehat{\tau}_\nu$.) We conclude that $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, except in case (3) of (2.24), when 2α is not a square, in which case ω_τ is the quadratic character, associated to 2α. Let us assume that τ is cuspidal. Then by the strong multiplicity one and multiplicity one properties for GL_N, we conclude that $\tau^* = \tau$, and we also have that $L^S(\sigma\times\widehat{\tau}, s) = L^S(\tau\times\widehat{\tau}, s)$ has a simple pole at $s = 1$, for an appropriate finite set of places S. (In case G is metaplectic, we have to fix ψ, a nontrivial character of $F\backslash\mathbb{A}_F$ and consider $L_\psi^S(\sigma\times\widehat{\tau}, s)$ instead.) Assume further that σ is globally ψ-generic. Then we can use the global integrals of Sections 2.4, 2.5 to represent the partial L-function of σ twisted by $\widehat{\tau}$, and consider its pole at $s = 1$. Let H be the group in the first column of (2.15) or (2.24), which has a Siegel parabolic subgroup whose Levi part is isomorphic to GL_N. Now consider the integrals (2.16) or (2.25) which represent the above L-function. Note that if G is not a unitary group, then $\widehat{\tau} = \tau$, and we take the Eisenstein series on $H_{\mathbb{A}_F}$ corresponding to $\rho_{\tau,s}^H$. If $G = U_{2n+1}$, $\widehat{\tau} = \tau'$ and we take $\rho_{\tau',s}^H$. If $G = U_{2n}$ we take $\rho_{\tau'\otimes\gamma,s}^H$. For decomposable data the integrals above are of the forms (2.18) or (2.27) respectively, and we can choose $R(s)$ to be holomorphic and nonzero at $s = 1$. Looking at the quotients (2.18) in table (2.19) and in table (2.27), we see that the denominators are holomorphic and nonzero at $s = 1$. Since $L^S(\sigma\times\widehat{\tau}, s)$ (resp. $L_\psi^S(\sigma\times\tau, s)$ if G is metaplectic) has a pole at $s = 1$, we conclude that the global integral $\mathcal{L}(\varphi_\sigma, \xi_{\widehat{\tau},s})$ in (2.18), $\mathcal{L}(\varphi_\sigma, \phi, \xi_{\tau,s})$ in (2.27), cases (1), (2), and $\mathcal{L}(\varphi_\sigma, \phi, \xi_{\widehat{\tau}\otimes\gamma,s})$ in (2.27), case 3 has a pole at $s = 1$. This pole then comes from the Eisenstein series which appears in $\mathcal{L}(\varphi_\sigma, \dots)$. Therefore, we expect that the (partial) L-function $L^S(\tau, \beta, s)$ which appears in the normalizing factor of this Eisenstein series to have a pole at $s = 1$. The following table summarizes the various cases, when we take $N = N_{2n}$. (In table (3.1), $N_k = k$ in cases (1), (2), (4), (5), and $N_k = k+1$ in cases (3), (6).)

	G	$\mathrm{Res}_{E/F}\,\mathrm{GL}_{N_k}$	$H = H_{G,k}$	$L^S(\tau, \beta_H, s)$
(1)	SO_{2n+1}	GL_k	SO_{2k}	$L^S(\tau, \Lambda^2, 2s-1)$
(2)	SO_{2n}	GL_k	SO_{2k+1}	$L^S(\tau, \mathrm{sym}^2, 2s-1)$
(3)	U_{2n+1}	$\mathrm{Res}_{E/F}\,\mathrm{GL}_{k+1}$	U_{2k+2}	$L^S(\tau', \mathrm{Asai}, 2s-1)$
(4)	$\widetilde{\mathrm{Sp}}_{2n}$	GL_k	Sp_{2k}	$L^S(\tau, s-\frac{1}{2})L^S(\tau, \Lambda^2, 2s-1)$
(5)	U_{2n}	$\mathrm{Res}_{E/F}\,\mathrm{GL}_k$	U_{2k}	$L^S(\tau' \otimes \gamma, \mathrm{Asai}, 2s-1)$
(6)	Sp_{2n}	GL_{k+1}	$\widetilde{\mathrm{Sp}}_{2k+2}$	$L^S(\tau, \mathrm{sym}^2, 2s-1)$

(3.1)

Case (2) in table (3.1) includes both split and quasi-split even orthogonal groups. We now proceed exactly as in case (1), which was proved in Theorem 1. The constant term of the Eisenstein series mentioned before, evaluated at I, is the sum of the section evaluated at I and the corresponding intertwining operator, applied to the section, and evaluated at I. The first summand is holomorphic, and hence the pole at $s = 1$ occurs for the second summand, which for decomposable data, equals as in (1.3) to a finite product, over a finite set of places S of local intertwining operators times a quotient of of the form $\frac{L^S(\tau,\beta,s)}{L^S(\tau,\beta,s+\frac{1}{2})}$, except in case (4) of table (3.1) ($\beta = \beta_{H_{G,2n}}$), where it is $\frac{L^S(\tau,s-\frac{1}{2})L^S(\tau,\Lambda^2,2s-1)}{L^S(\tau,s+\frac{1}{2})L^S(\tau,\Lambda^2,2s)}$. In all cases, it is easy to see that the denominator of the last quotient is holomorphic and nonzero at $s = 1$. By [**K**, Lemma 2.4], the local intertwining operators above are holomorphic and nonzero for $\mathrm{Re}(s) \geqslant 1$. (Note that the standard module conjecture needed in loc. cit. is needed here just for $(\mathrm{Res}_{E/F}\,\mathrm{GL}_{2n})(F_\nu)$ or $(\mathrm{Res}_{E/F}\,\mathrm{GL}_{2n+1})(F_\nu)$, and hence is valid.) We conclude that $L^S(\tau, \beta, s)$ has a pole at $s = 1$. Summarizing

Theorem 10. — *Let σ be an irreducible, automorphic, cuspidal representation of $G_{\mathbb{A}_F}$. Assume that σ is globally ψ-generic, and that σ has a weak lift to an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_{N_{2n}}(\mathbb{A}_E)$, as in table (3.1). Then $\tau^* = \tau$, the partial L-function $L^S(\tau, \beta_{H_{G,2n}}, s)$ has a pole at $s = 1$, and $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, except in case $G = \mathrm{SO}_{2n}^\alpha$, when 2α is not a square, in which case ω_τ is the quadratic character asociated to 2α.*

We conclude in exactly the same way, using the global integrals of Sec. 2.4, 2.5, the analogs of Proposition 2 and Theorem 3.

Theorem 11. — *Let σ be an irreducible, automorphic, cuspidal representation of $G_{\mathbb{A}_F}$. Assume that σ is globally ψ-generic. Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_k(\mathbb{A}_E)$, $k \geqslant 2$, such that $\omega_\tau\big|_{(\mathbb{A}_F)^+_\infty} = 1$. Then $L^S(\sigma \times \tau, s)$*

(resp. $L^S_\psi(\sigma \times \tau, s)$ if G is metaplectic) is holomorphic for $\mathrm{Re}(s) > 1$ *and if it has a pole at s_0, such that* $\mathrm{Re}(s_0) = 1$*, then $s_0 = 1$ and $L^S(\tau, \beta_H, s)$ (table 3.1) has a pole at $s = 1$. The same assertions hold true, if τ is an automorphic unitary character of the idele group, which is trivial on $(\mathbb{A}_F)^+_\infty$, except in cases (1), (4). In case (1), we know that $L^S(\sigma \times \tau, s)$ is entire, and in case (4), the L function (with respect to ψ, where σ is globally ψ-generic) may have a pole for* $\mathrm{Re}(s) > 1$*, and then it must be at $s = 3/2$, and τ must be trivial.*

We remark that the last case of Theorem 11 occurs when σ is a theta lift with respect to ψ from a generic cuspidal representation of $\mathrm{SO}_{2n-1}(\mathbb{A})$.

Start now with an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_N(\mathbb{A}_E)$, $(N = N_{2n})$ such that $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, except in case τ is on $\mathrm{GL}_{2n}(\mathbb{A}_F)$, where we allow ω_τ to be either trivial or quadratic. If the quadratic character is associated to 2α, then in the following, the Gelfand-Graev coefficient is taken with respect to SO^α_{2n}. Assume that $L^S(\tau, \beta, s)(\beta = \beta_{H_{G,2n}})$ has a pole at $s = 1$ (notation of table (3.3)). By Theorem 10, these are necessary conditions that (cuspidal) τ needs to satisfy in order to be in the image of the weak lift from generic cuspidal representations on $G_{\mathbb{A}_F}$. (The second condition implies $\tau^* = \tau$.) If τ is a weak lift of σ (generic, cuspidal) on $G_{\mathbb{A}_F}$, then by (2.18), (2.27), $\mathcal{L}(\varphi_\sigma, \xi_{\hat{\tau},s})$ has a pole at $s = 1$ in cases (1)–(3) of Table (3.1), $\mathcal{L}(\varphi, \phi, \xi_{\hat{\tau},s})$ has a pole at $s = 1$ in cases (4),(6), and $\mathcal{L}(\varphi, \phi, \xi_{\tau'\otimes\gamma,s})$ has a pole at $s = 1$ in case (5) (as data vary). Thus, the Gelfand-Graev coefficient (resp. the Fourier-Jacobi coefficient) of the residue at $s = 1$ of the Eisenstein series which appear in the global integrals has a non-trivial $L^2(G_F\backslash G_{\mathbb{A}_F})$-pairing against σ. This leads us to define

$$\sigma_\psi(\tau) = \begin{cases} \mathrm{Span}\{\mathrm{Res}_{s=1} E_H^{\psi^{-1}_{n-1,1}}(\xi_{\tau,s}, \cdot)\big|_{G_{\mathbb{A}_F}}\}, & G = \mathrm{SO}_{2n+1} \\ \mathrm{Span}\{\mathrm{Res}_{s=1} E_H^{\psi^{-1}_{n,1}}(\xi_{\tau',s}, \cdot)\big|_{G_{\mathbb{A}_F}}\}, & G = \mathrm{SO}_{2n}, U_{2n+1} \\ \mathrm{Span}\{\mathrm{Res}_{s=1} E_H^{\psi^{-1}_{n-1},\gamma,\phi}(\xi_{\tau'\otimes\gamma,s}, \cdot)\big|_{G_{\mathbb{A}_F}}\}, & G = \widetilde{\mathrm{Sp}}_{2n}, U_{2n} \\ \mathrm{Span}\{\mathrm{Res}_{s=1} E_H^{\psi^{-1}_{n,1},\phi}(\xi_{\tau,s}, \cdot)\big|_{G_{\mathbb{A}_F}}\}, & G = \mathrm{Sp}_{2n} \end{cases}$$

(3.2)

Our main theorem is

Theorem 12. — *Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_N(\mathbb{A}_E)$, with central character, as above. Assume that $L^S(\tau, \beta, s)$ has a pole at $s = 1$. (We use the notation of table 3.1, with $N = N_{2n}, \beta = \beta_{H_{G,2n}}$). Assume also that $n \geqslant 2$, in case $G = \mathrm{SO}_{n,n}$. Then*

(1) $\sigma_\psi(\tau) \neq 0$

(2) *The representation $\sigma_\psi(\tau)$ of $G_{\mathbb{A}_F}$ is cuspidal.*

(3) *Let* σ *be an irreducible summand of* $\sigma_\psi(\tau)$. *Then* σ *is globally* ψ*-generic, and* σ_ν *lifts to* τ_ν*, for almost all finite places* ν. *(If* $G = \widetilde{\mathrm{Sp}}_{2n}$*,* σ_ν *lifts to* τ_ν *with respect to* ψ_ν*).*

(4) *Every irreducible, automorphic, cuspidal,* ψ*-generic representation* σ *of* $G_{\mathbb{A}_F}$*, which lifts weakly to* τ *has a nontrivial* L^2*-pairing with* $\sigma_\psi(\tau)$.

(5) $\sigma_\psi(\tau)$ *is a multiplicity free representation.*

Remark. — The guidelines to the proof are similar to those of Theorem 4, except that the proof of (1) in case G is even orthogonal or symplectic is not direct. In these cases, we show, once we fix ψ, that there is $\beta \in F^*$, such that $\sigma_{\psi,\beta}(\tau) \neq 0$, where $\sigma_{\psi,\beta}(\tau)$ is defined as in (3.2) only that the coefficient (Gelfand-Graev, or Fourier-Jacobi) of the residual Eisenstein series induced from τ is taken with respect $\psi^{-1}_{n,\beta}$, in case $G = \mathrm{SO}_{2n}$, and in case $G = \mathrm{Sp}_{2n}$, we take in (2.9) a residual Eisenstein series, induced from τ, on $\widetilde{Sp}_{4n}(\mathbb{A})$, instead of φ, and $\theta^\phi_{\psi^{-\beta}}$, instead of $\theta^\phi_{\psi^{-1}}$ ($\gamma = 1$). In the first case we obtain a non-trivial cuspidal representation $\sigma_{\psi,\beta}(\tau)$ of $H^{(\beta)}_{2n}(\mathbb{A})$ (see table (2.15)), for which the following Whittaker coefficient is nontrivial

$$\begin{pmatrix} z & x & y \\ & I_2 & \widetilde{x} \\ & & z^* \end{pmatrix} \longmapsto \psi(z_{12} + z_{23} + \cdots + z_{n-2,n-1} + x_{n-1,2}). \tag{3.3}$$

Here $z \in Z_{n-1}(\mathbb{A})$, and we write the elements of $H^{(\beta)}_{2n}$, with respect to $\begin{pmatrix} & & & w_{n-1} \\ & 1 & & \\ & & -2\beta & \\ w_{n-1} & & & \end{pmatrix}$.

Let σ be an irreducible summand of $\sigma_{\psi,\beta}(\tau)$, which is globally generic with respect to the character (3.3). Then it has a weak lift to τ, and hence, $\omega_\tau(t) = (2\beta, t)$ (Hilbert symbol). This implies that $\sigma_\psi(\tau)$ is non-trivial. In the second case, ($G = \mathrm{Sp}_{2n}$), $\sigma_{\psi,\beta}(\tau)$ is a (nontrivial) automorphic cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$, which is globally ψ^β-generic. Let σ be such an irreducible summand of $\sigma_{\psi,\beta}(\tau)$. Examining the unramified parameters of σ, we show that

$$L^S(\sigma, s) = \frac{L^S(\tau \times \chi_\beta, s)}{L^S(\chi_\beta, s)} L^S(1, s).$$

Here, $\chi_\beta = (\beta, t)$ (Hilbert symbol). If $\chi_\beta \neq 1$, this implies that $L^S(\sigma, s)$ has a pole at $s = 1$. By [**GRS5**], we conclude that σ is a theta lift (with respect to an appropriate character) of a generic cuspidal representation π on split $\mathrm{SO}_{2n}(\mathbb{A})$. We have

$$L^S(\tau, s) = L^S(\pi \times \chi_\beta, s) L^S(1, s),$$

and hence $L^S(\tau, s)$ has a pole at $s = 1$. This is impossible, and so $\chi_\beta = 1$, *i.e.* $\sigma_\psi(\tau)$ is nontrivial.

3.2. The image (in general) of the weak lift from G to GL_N. — Let σ be an irreducible, automorphic, cuspidal generic representation of $G_{\mathbb{A}_F}$. Assume that σ has a weak lift to GL_N, and that it lifts to an irreducible, automorphic representation τ, which as in Sec. 1.3, is a constituent of

$$\delta_1 |\det\cdot|^{z_1} \times \cdots \times \delta_j |\det\cdot|^{z_j} \times \tau_1 \times \cdots \times \tau_\ell \times \delta_j^* |\det\cdot|^{-z_j} \times \cdots \times \delta_1^* |\det\cdot|^{-z_1} \tag{3.4}$$

where $\mathrm{Re}(z_1) \leqslant \cdots \leqslant \mathrm{Re}(z_j) \leqslant 0$, the representations δ_i, τ_k are irreducible, automorphic and unitary, with central characters which are trivial on $(\mathbb{A}_F)^+_\infty$, and $\tau_i = \tau_i^*$, for $1 \leqslant i \leqslant \ell$. If δ_i (resp. τ_k) is on $\mathrm{GL}_r(\mathbb{A})$, $r > 1$, we assume it is cuspidal.

Consider $L^S(\sigma \times \widehat{\delta}_1, s)$. As in Sec. 1.3, we see that $L^S(\sigma \times \widehat{\delta}_1, s)$ has a pole at $s = 1 - z_1$. (If G is metaplectic, consider $L^S_\psi(\sigma \times \widehat{\delta}_1, s)$). By Theorem 11, except in case G is metaplectic, and $\delta_1 = 1$, we have $z_1 = 0$ and $L^S(\widehat{\delta}_1, \beta_{H_{G,r'}}, s)$ has a pole at $s = 1$. Here δ_1 is on $\mathrm{GL}_r(\mathbb{A}_E)$, and $r' = r$ in all cases of Table (3.1), except cases (3) and (6), where $r' = r - 1$. Note that since $L^S(\widehat{\delta}_1, \beta_{H_{G,r'}}, s)$ has a pole at $s = 1$, we must have $\delta_1 = \delta_1^*$. (For example, in case of a unitary group, and $\eta = \widehat{\delta}_1$,

$$L^S(\eta \otimes \eta', s) = L^S(\eta, \mathrm{Asai}, s) L^S(\eta \otimes \gamma, \mathrm{Asai}, s), \tag{3.5}$$

and since one of the factors on the r.h.s. of (3.5) has a pole at $s = 1$, $L^S(\eta \otimes \eta', s)$ has a pole at $s = 1$, which implies that $\widehat{\eta}' = \eta$, *i.e.* $\eta^* = \eta$). We conclude that $L^S(\sigma \times \widehat{\delta}_1, s)$ has a double pole at $s = 1$. This is impossible, and we conclude that (3.4) has the form

$$\tau_1 \times \cdots \times \tau_\ell,$$

and repeating the last argument, we conclude that $L^S(\tau_i, \beta_{H_{G,r_i'}}, s)$ has a pole at $s = 1$, for $i = 1, \ldots, \ell$, and also that $\tau_i \neq \tau_j$, for $1 \leqslant i \neq j \leqslant \ell$. Here τ_i is on $\mathrm{GL}_{r_i}(\mathbb{A}_E)$. Finally, in case G is metaplectic, we see from Theorem 11, that it is possible to have $\delta_1 = 1$, and $z_1 = -\frac{1}{2}$, and as we remarked before, in this case σ is a (ψ) theta lift from a cuspidal generic representation of $\mathrm{SO}_{2n-1}(\mathbb{A})$, so that by Section 1.5, the lift of σ to $\mathrm{GL}_{2n}(\mathbb{A})$ has the form $|\ |^{-1/2} \times \tau_1 \times \cdots \times \tau_\ell \times |\ |^{1/2}$, where τ_i are as before, each one with its exterior square L-function having a pole at $s = 1$. This proves

Theorem 13. — *Let σ be an irreducible, automorphic, cuspidal, generic representation of $G_{\mathbb{A}_F}$. Assume that σ lifts weakly to an irreducible automorphic representation τ of $\mathrm{GL}_{N_{2n}}(\mathbb{A}_E)$ as in Table (3.1). Then except in case (4), τ has the form $\tau_1 \times \cdots \times \tau_\ell$, where for $1 \leqslant i \leqslant \ell$, τ_i is an irreducible, automorphic, unitary representation of $\mathrm{GL}_{r_i}(\mathbb{A}_E)$, cuspidal in case $r_i > 1$, such that $\tau_i^* = \tau_i$, $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, except in case $G = \mathrm{SO}^\alpha_{2n}$, in which case $\omega_\tau = \chi_{2\alpha}$. The partial L-function $L^S(\tau_i, \beta_{H_{G,r_i'}}, s)$ has a pole at $s = 1$, and $\tau_i \neq \tau_j$, for all $1 \leqslant i \neq j \leqslant \ell$. In case (4), either τ has the form above, or it has the form $|\ |^{-1/2} \times \tau_1 \times \cdots \times \tau_\ell \times |\ |^{1/2}$, where the product of the τ_i is in the image of the lift from generic cuspidal representations from (split) $\mathrm{SO}_{2n-1}(\mathbb{A})$ to $\mathrm{GL}_{2n-2}(\mathbb{A})$.*

We consider the converse to Theorem 13, except the last case mentioned there. Let $\tau_1, \ldots, \tau_\ell$ be ℓ different irreducible, automorphic, unitary representations of $\mathrm{GL}_{r_1}(\mathbb{A}_E), \ldots, \mathrm{GL}_{r_\ell}(\mathbb{A}_E)$ respectively, and τ_i is cuspidal, if $r_i > 1$, and such that $r_1 + \cdots + r_\ell = N = N_{2n}$ (as in table (3.1), $\tau_i^* = \tau_i$, and $L^S(\tau_i, \beta_{H_{G,r_i'}}, s)$ has a pole at $s = 1$, for $i = 1, \ldots, \ell$. Let $\tau = \tau_1 \times \cdots \times \tau_\ell$. Assume also that $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, except in case N is even, and $E = F$, where we also allow ω_τ to be quadratic, and if it is, say, $\chi_{2\alpha}$, then, in the sequel, we'll take the Gelfand-Graev coefficient with respect to $\mathrm{SO}_{2n}^{\alpha}$. If τ is a lift at almost all finite places of an irreducible, automorphic, cuspidal, ψ-generic representation σ on $G_{\mathbb{A}_F}$, then by (2.18), (2.27), $\mathcal{L}(\varphi_\sigma, \xi_{\widehat{\tau}_i,s})$ has a pole at $s=1$ in cases (1)–(3) of Table (3.1), $\mathcal{L}(\varphi_\sigma, \phi, \xi_{\widehat{\tau}_i,s})$ has a pole at $s=1$ in cases (4),(6), and $\mathcal{L}(\varphi_\sigma, \phi, \xi_{\tau_i'\otimes\gamma,s})$ has a pole at $s=1$ in case (5), as data vary, and $i=1,\dots,\ell$. Consider the Eisenstein series on $H=H_{G,2n}$ (Table (3.1)) induced from $\tau_1 \mid \mid^{s_1-1/2} \times \cdots \times \tau_\ell \mid \mid^{s_\ell-1/2}$ and the standard parabolic subgroup of H, whose Levi part is isomorphic to $\mathrm{Res}_{E/F}\,\mathrm{GL}_{r_1} \times \cdots \times \mathrm{Res}_{E/F}\,\mathrm{GL}_{r_\ell}$. Denote it, for a K-finite holomorphic section $\xi_{\tau,\overline{s}}$ by $E_H(\xi_{\tau,\overline{s}},\cdot)$ where $\overline{s}=(s_1,\dots,s_\ell)$. We can show that $(s_1-1)\cdot\dots\cdot(s_\ell-1)E_H(\xi_{\tau,\overline{s}},\cdot)$ is holomorphic and nontrivial at $\overline{s}=1,\dots,1)$. Denote the value at $(1,\dots,1)$ by $\mathrm{Res}_{(1,\dots,1)}\,E_H(\xi_{\tau,\overline{s}},\cdot)$, and now define $\sigma_\psi(\tau)$ on $G_{\mathbb{A}_F}$ exactly as in (3.2). Our main theorem in its most general form is

Theorem 14. — *Fix the group G. Let $N=N_{2n}$ as in Table (3.1). Let $\tau=\tau_1\times\cdots\times\tau_\ell$ be the irreducible representation of $\mathrm{GL}_N(\mathbb{A}_E)$ induced from $\tau_1\otimes\cdots\otimes\tau_\ell$, where $\tau_1,\dots,\tau_\ell$ are pairwise inequivalent, irreducible, automorphic, unitary representations of $\mathrm{GL}_{r_1}(\mathbb{A}_E),\dots,\mathrm{GL}_{r_\ell}(\mathbb{A}_E)$ respectively, τ_i is cuspidal in case $r_i>1$, such that $r_1+\cdots+r_\ell=N$, $\tau_i^*=\tau_i$, and $L^S(\tau_i,\beta_{H_{G,r_i'}},s)$ has a pole at $s=1$, for $i=1,\dots,\ell$. Assume that the central character of τ is as above. Then*

(1) $\sigma_\psi(\tau)\neq 0$.

(2) *The representation $\sigma_\psi(\tau)$ of $G_{\mathbb{A}_F}$ is cuspidal.*

(3) *Let σ be an irreducible summand of $\sigma_\psi(\tau)$. Then σ is globally ψ-generic, and σ_ν lifts to τ_ν, for almost all finite places ν. (If $G=\widetilde{\mathrm{Sp}}_{2n}$, σ_ν lifts to τ_ν with respect to ψ_ν).*

(4) *Every irreducible, automorphic, cuspidal, ψ-generic representation σ of $G_{\mathbb{A}_F}$, which lifts to τ at almost all finite places, has a nontrivial L^2-pairing with $\sigma_\psi(\tau)$.*

(5) *$\sigma_\psi(\tau)$ is a multiplicity free representation.*

Assume, for simplicity that $\omega_{\tau_i}\big|_{\mathbb{A}_F^*}=1$, for each i in the last theorem. Then for each τ_i, we may apply Theorem 12 and consider the cuspidal ψ-generic representation $\sigma_\psi(\tau_i)$ on a corresponding group $G_i(\mathbb{A}_F)$. Let σ_i be an irreducible summand of $\sigma_\psi(\tau_i)$, $i=1,\dots,\ell$, and let σ be an irreducible summand of $\sigma_\psi(\tau)$ ($\sigma_1,\dots,\sigma_\ell$, σ are all ψ-generic). Then $\sigma_1\otimes\cdots\otimes\sigma_\ell$ (on $G_1(\mathbb{A}_F)\times\cdots\times G_\ell(\mathbb{A}_F)$) lifts at almost all finite places to σ. Both representations lift at almost all places to τ on $\mathrm{GL}_N(\mathbb{A}_E)$. These are examples of (generalized) endoscopy. The following table summarizes the various cases, where we stay a little vague in specifying central characters, and in specifying even orthogonal groups and base change lifts to even unitary groups. (So far, for simplicity, we constructed only lifts from U_{2n} to $\mathrm{Res}_{E/F}\,\mathrm{GL}_{2n}$, with central character, whose restriction to $\mathbb{A}_F$ is trivial.) Here, as above, σ_i is an irreducible summand of $\sigma_\psi(\tau_i)$.

$\tau_1 \otimes \cdots \otimes \tau_\ell$ on $\mathrm{GL}_{r_1}(\mathbb{A}_E) \times \cdots \times \mathrm{GL}_{r_\ell}(\mathbb{A}_E)$	pole condition for τ_i	$\sigma_1 \otimes \cdots \otimes \sigma_\ell$ on $G_1(\mathbb{A}_F) \times \cdots \times G_\ell(\mathbb{A}_F)$ $\longrightarrow$	σ on $G(\mathbb{A}_F)$
$\mathrm{GL}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2n_\ell}(\mathbb{A}_F)$	$\mathrm{Res}_{s=1} L^S(\tau_i, \Lambda^2, s) \neq 0$	$\mathrm{SO}_{2n_1+1}(\mathbb{A}_F) \times \cdots \times \mathrm{SO}_{2n_\ell+1}(\mathbb{A}_F)$	$\mathrm{SO}_{2(n_1+\cdots+n_1)+1}(\mathbb{A}_F)$
$\mathrm{GL}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2n_t}(\mathbb{A}_F)\times$ $\times\, \mathrm{GL}_{2m_1+1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2m_{2r+1}+1}(\mathbb{A}_F)$	$\mathrm{Res}_{s=1} L^S(\tau_i, \mathrm{sym}^2, s) \neq 0$	$\mathrm{SO}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{SO}_{2n_t}(\mathbb{A}_F)\times$ $\times\, \mathrm{Sp}_{2m_1}(\mathbb{A}_F) \times \cdots \times \mathrm{Sp}_{2m_{2r+1}}(\mathbb{A}_F)$	$\mathrm{Sp}_{2(n_1+\cdots+m_{2r+1}+r)}(\mathbb{A}_F)$
$\mathrm{GL}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2n_t}(\mathbb{A}_F)\times$ $\times\, \mathrm{GL}_{2m_1+1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2m_{2r}}(\mathbb{A}_F)$	$\mathrm{Res}_{s=1} L^S(\tau_i, \mathrm{sym}^2, s) \neq 0$	$\mathrm{SO}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{SO}_{2n_t}(\mathbb{A}_F)\times$ $\times\, \mathrm{Sp}_{2m_1}(\mathbb{A}_F) \times \cdots \times \mathrm{Sp}_{2m_{2r}}(\mathbb{A}_F)$	$\mathrm{SO}_{2(n_1+\cdots+m_{2r}+r)}(\mathbb{A}_F)$
$\mathrm{GL}_{n_1}(\mathbb{A}_E) \times \cdots \times \mathrm{GL}_{n_\ell}(\mathbb{A}_E)$	$\mathrm{Res}_{s=1} L^S(\tau_i', \mathrm{Asai}, s) \neq 0$, if $n_i \equiv 1 \pmod 2$ $\mathrm{Res}_{s=1} L^S(\tau_i' \otimes \gamma, \mathrm{Asai}, s) \neq 0$, if $n_i \equiv 0 \pmod 2$	$U_{n_1}(\mathbb{A}_F) \times \cdots \times U_{n_\ell}(\mathbb{A}_F)$	$U_{n_1+\cdots n_\ell}(\mathbb{A}_F)$
$\mathrm{GL}_{2n_1}(\mathbb{A}_F) \times \cdots \times \mathrm{GL}_{2n_\ell}(\mathbb{A}_F)$	$\mathrm{Res}_{s=1} L^S(\tau_i, s-\frac{1}{2}) L^S(\tau_i, \Lambda^2, 2s-1) \neq 0$	$\widetilde{\mathrm{Sp}}_{2n_1}(\mathbb{A}_F) \times \cdots \times \widetilde{\mathrm{Sp}}_{2n_\ell}(\mathbb{A}_F)$	$\widetilde{\mathrm{Sp}}_{2(n_1+\cdots+n_\ell)}(\mathbb{A}_F)$

(Table 3.6)

Example. — The functorial lift $U_3 \to \mathrm{Res}_{E/F}\,\mathrm{GL}_3$ is completely known from the work of Rogawski [**R**]. The cuspidal part of the image is the set of all irreducible, automorphic, cuspidal representations τ of $\mathrm{GL}_3(\mathbb{A}_E)$, such that $\tau^* = \tau$ and $\omega_\tau\big|_{\mathbb{A}_F^*=1}$. In this case, this is equivalent to $L^S(\tau', \mathrm{Asai}, s)$ having a pole at $s = 1$. In this case, using the multiplicity one property for cuspidal representations on $U_3(\mathbb{A}_F)$ [**R**] it follows that $\sigma_\psi(\tau)$ is an irreducible, automorphic, cuspidal, generic representation of $U_3(\mathbb{A}_F)$, which lifts to τ. $\sigma_\psi(\tau)$ is the generic member of the L-packet on $U_3(\mathbb{A}_F)$, parametrized by τ. The following representations occur in the non-cuspidal part of the image of the lift above, restricted to generic representations.

(1) $\mu_\eta \times \pi$, where η is an automorphic character of $U_1(\mathbb{A}_F)$ and μ_η is the character of $\mathbb{A}_E^*$ defined by $\mu_\eta(x) = \eta(x/\overline{x})$. The representation π is on $\mathrm{GL}_2(\mathbb{A}_E)$, and it is irreducible, automorphic, and cuspidal such that $\pi^* = \pi$, $\omega_\pi\big|_{\mathbb{A}_F^*} = 1$ and $L^S(\pi' \otimes \gamma, \mathrm{Asai}, s)$ has a pole at $s = 1$. The representation $\sigma_\psi(\mu_\eta \times \pi)$ is an irreducible, automorphic, cuspidal, generic representation of $U_3(\mathbb{A}_F)$, which lifts to $\mu_\eta \times \pi$.

(2) $\mu_{\eta_1} \times \mu_{\eta_2} \times \mu_{\eta_3}$, where $\{\eta_1, \eta_2, \eta_3\}$ are three different automorphic characters of $U_1(\mathbb{A}_F)$. The representation $\sigma_\psi(\mu_{\eta_1} \times \mu_{\eta_2} \times \mu_{\eta_3})$ is an irreducible, automorphic, cuspidal, generic representation of $U_3(\mathbb{A}_F)$, which lifts to $\mu_{\eta_1} \times \mu_{\eta_2} \times \eta_{\mu_3}$. See [**G.J.R.**], [**Ge.Ro.So1, Ge.Ro.So2, Ge.Ro.So3**].

In the remaining part of this paper, we will illustrate the proof of Theorem 12 through (low rank) examples.

4. Illustrations of Proofs in Low Rank Examples

4.1. An observation on unramified factors of residual Eisenstein series

Fix the group G. Let $N = N_{2n}$ as in Table (3.1). Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_N(\mathbb{A}_E)$, such that $\tau^* = \tau$, $\omega_\tau\big|_{\mathbb{A}_F^*} = 1$, and $L^S(\tau, \beta_{H_{G,2n}}, s)$ has a pole at $s = 1$. Consider the residue at $s = 1$ of the Eisenstein series on $H_{G,2n}(\mathbb{A}_F)$ induced from $\tau' \otimes \gamma \cdot |\det\cdot|^{s-1/2}$. Denote this residual representation by E_τ. (In all cases, except case (5) in Table (3.1), $\gamma = 1$. Also $\tau' = \tau$ in all cases except cases (3), (5).) We abuse notation and think of E_τ also as the space of automorphic forms spanned by the residues. So, for example, when we refer to a constant term of E_τ, we mean that we consider this constant term applied to all automorphic forms in (the space of) E_τ. It is easy to check that E_τ consists of square integrable automorphic forms. Indeed, E_τ is concentrated along the Siegel parabolic subgroup (*i.e.* all constant terms, with respect to unipotent radicals of standard parabolic subgroups, other than the Siegel parabolic subgroup, vanish on E_τ). The constant term of E_τ along the Siegel radical has one exponent, which is negative. Now use Jacquet's criterion for square integrability [**J**]. Consider an unramified factor π_ν at a place ν of (an irreducible summand of) E_τ. By our assumption on τ, we have $\tau_\nu^* \cong \tau_\nu$ and

$\omega_{\tau_\nu}\big|_{F_\nu^*} = 1$. Since τ_ν is unramified, we see that τ_ν is the unramified constituent of a representation of $\mathrm{GL}_N(E_\nu)$ induced from the Borel subgroup and an unramified character of the torus of the form

$$(4.1)\quad \begin{aligned} \mathrm{diag}(t_1,\dots,t_{2n}) &\longmapsto \mu_1\Big(\frac{t_1}{\bar t_{2n}}\Big)\cdots\mu_n\Big(\frac{t_n}{\bar t_{n+1}}\Big), && \text{if } N=2n\\ \mathrm{diag}(t_1,\dots,t_{2n+1}) &\longmapsto \mu_1\Big(\frac{t_n}{\bar t_{2n+1}}\Big)\cdots\mu_n\Big(\frac{t_n}{\bar t_{n+2}}\Big), && \text{if } N=2n+1.\end{aligned}$$

Recall that if $E=F$, $\bar t = t$, for $t\in E$. If $[E:F]=2$ and ν is a place which splits in F, then $E_\nu = F_\nu\oplus F_\nu$, $\overline{(a,b)} = (b,a)$ and the characters μ_i are given by pairs of characters of F_ν^*. Let Q be the standard parabolic subgroup of $H = H_{G,2n}$, whose Levi part is isomorphic to $(\mathrm{Res}_{E/F}\mathrm{GL}_2)^n$ in cases (1),(2),(4),(5) of Table (3.1), or to $(\mathrm{Res}_{E/F}\mathrm{GL}_2)^n\times H_0$ where $H_0=U_2$ in case (3) and $H_0=\mathrm{SL}_2$ in case (6). (In case (6) we should really take the inverse image in $\widetilde{\mathrm{Sp}}_{4n+2}$, at each place ν: $\widetilde{\mathrm{GL}_2(F_\nu)^n\times\mathrm{SL}_2(F_\nu)}$). Denote by $\pi_{\mu_1,\dots,\mu_n}$ the unramified constituent of the representation $\rho_{\mu_1,\dots,\mu_n}$ of $H(F_\nu)$ induced from $Q(F_\nu)$ and the character $(\mu_1\cdot\det)\otimes\cdots\otimes(\mu_n\cdot\det)$. (In cases (3) and (6) of Table (3.1), it is trivial on $H_0(F_\nu)$. In case (6) we also have to multiply by γ_ψ). Denote $\mu_j'(t)=\mu_j(\bar t)$. Denote by ω the simple Weyl reflection in O_{4n}, which flips the two middle coordinates in the diagonal subgroup.

Proposition 15. — *Using the notation above, let τ_ν be the unramified representation of $\mathrm{GL}_N(E_\nu)$, corresponding to the unramified character (4.1). Then $\pi_\nu\cong\pi_{\mu_1'\gamma_\nu,\dots,\mu_n'\gamma_\nu}$, except in case 1 of Table 3.1, with n odd, where we have $\pi_\nu\cong\pi^\omega_{\mu_1,\dots,\mu_n}$ (outer conjugation).*

Proof. — Denote by $\rho_{\tau_\nu'\otimes\gamma_\nu}$ the representation of $H(F_\nu)$ induced from the Siegel parabolic subgroup and $\tau_\nu'\otimes\gamma_\nu|\ |^{1/2}$. (We have to modify by γ_ψ in case (6).) Consider first cases (1),(4),(5) in Table (3.1). In case (1), assume for simplicity that n is even. Here $\rho_{\tau_\nu'\otimes\gamma_\nu}$ is induced from the following character of the Borel subgroup

$$(4.2)\quad \mathrm{diag}(t_1,\dots,t_{2n},\bar t_{2n}^{-1},\dots,\bar t_1^{-1}) \longmapsto \mu_1'\gamma_\nu\Big(\frac{t_1}{\bar t_{2n}}\Big)|t_1t_{2n}|^{1/2}\cdot\cdots\cdot\mu_n'\gamma_\nu\Big(\frac{t_n}{\bar t_{n+1}}\Big)|t_nt_{n+1}|^{1/2}$$

This character is conjugate, under a suitable Weyl element of H, to the character

$$(4.3)\quad \mathrm{diag}(t_1,\dots,t_{2n},\bar t_{2n}^{-1},\dots,\bar t_1^{-1}) \longmapsto \mu_1'\gamma_\nu(t_1t_{2n})\Big|\frac{t_1}{t_{2n}}\Big|^{1/2}\cdot\cdots\cdot\mu_n'\gamma_\nu(t_nt_{n+1})\Big|\frac{t_n}{t_{n+1}}\Big|^{1/2},$$

and this character is conjugate, under a suitable Weyl element of GL_N, to the character

$$(4.4)\quad \mathrm{diag}(t_1,\dots,t_{2n},\bar t_{2n}^{-1},\dots,\bar t_1^{-1}) \longmapsto \mu_1'\gamma_\nu(t_1t_2)\Big|\frac{t_1}{t_2}\Big|^{1/2}\cdot\cdots\cdot\mu_n'\gamma_n(t_{2n-1}t_{2n})\Big|\frac{t_{2n-1}}{t_{2n}}\Big|^{1/2}.$$

Thus π_ν is the unramified constituent of the representation $\eta_{\mu'_1\gamma_\nu,\dots,\mu'_n\gamma_\nu}$ induced from the character of the Borel subgroup defined by (4.4). Clearly $\eta_{\mu'_1\gamma_\nu,\dots,\mu'_n\gamma_\nu}$ maps onto $\rho_{\mu'_1\gamma_\nu,\dots,\mu'_n\gamma_\nu}$. Since the last representation is still unramified, we conclude that π_ν is the unramified constituent of $\rho_{\mu'_1\sigma_\nu,\dots,\mu'_n\gamma_\nu}$. (If n is odd in case (1), we get that $\pi_\nu \cong \pi^\omega_{\mu'_1\gamma_\nu,\dots,\mu'_n\gamma_\nu}$, where ω is as above.) In case (2) the proof is the same, only that in (4.2)–(4.4), the left hand side is $\mathrm{diag}(t_1,\dots,t_{2n},1,t_{2n}^{-1},\dots,t_1^{-1})$ and in the right hand side there is no change except that $\mu'_i = \mu_i$, $\gamma'_\nu = 1$. In case (4) the proof is the same, only that in (4.2)–(4.4) the l.h.s. is $\mathrm{diag}(t_1,\dots,t_{2n+1},\bar{t}_{2n+1}^{-1},\dots,\bar{t}_1^{-1})$. The r.h.s. of (4.2)–(4.4) remains the same. In case (6), the l.h.s of (4.2)–(4.4) is $\mathrm{diag}(t_1,\dots,t_{2n+1},t_{2n+1}^{-1},\dots,t_1^{-1})$, and in the r.h.s. we have to multiply by $\gamma_\psi(t_1\cdot\dots\cdot t_{2n+1})$ (and take $\mu'_i = \mu_i, \gamma_\nu = 1$). $\square$

4.2. Nonvanishing of $\sigma_\psi(\tau)$: Case $G = U_3$, $H = U_6$, τ – on $\mathrm{GL}_3(\mathbb{A}_E)$

Let τ be a irreducible, automorphic, cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_E)$, such that $\tau^* = \tau$, $\omega_\tau|_{\mathbb{A}_F^*} = 1$, and $L^S(\tau', \mathrm{Asai}, s)$ has a pole at $s = 1$. (Actually, the last condition is equivalent to the first two conditions). The proof that $\sigma_\psi(\tau) \neq 0$ consists of two steps. First, we introduce (in (4.8)) a unipotent group V of U_6, and a certain character ψ_V of $V_F\backslash V_{\mathbb{A}_F}$, and prove that the Fourier coefficient along V, with respect to ψ_V, is nontrivial on (the space of) E_τ (Proposition 16). To do so, we prove that this nontriviality is equivalent to the nontriviality of another Fourier coefficient on E_τ. This last Fourier coefficient is along a unipotent subgroup U, and with respect to a character ψ_U of $U_F\backslash U_{\mathbb{A}_F}$. The group U is almost the maximal unipotent subgroup of U_6. It "misses" just one root subgroup, namely the simple root which lies in the Siegel radical. The character ψ_U is the restriction to $U_{\mathbb{A}_F}$ of the standard nondegenerate character determined by ψ. Thus, the nontriviality of the (U, ψ_U) coefficient on E_τ follows from the fact that τ is (globally) generic. In the second step we show that the nontriviality of the (V, ψ_V) coefficient on E_τ is equivalent to the nonvanishing of $\sigma_\psi(\tau)$. We develop for these proofs (and for the sequel) a tool that we call, for lack of a better name, "exchanging roots". In practice, it enables us to conclude that an automorphic representation, realized in a given space of automorphic forms, has a nontrivial (V_1, ψ_{V_1}) Fourier coefficient, if and only if it has a nontrivial (V_2, ψ_{V_2}) Fourier coeffiecient, where the unipotent groups V_1, V_2 are generated by root subgroups, and the passage from V_1 to V_2 is by "deleting" a certain root subgroup, and "replacing it, in exchange", with another certain root subgroup (outside V_1). The characters ψ_{V_i} are equal on the subgroup generated by the roots common to V_1 and V_2, and extend trivially to "the rest of" V_i.

Let $H = U_6$, and let P be the Siegel parabolic subgroup. Let $\rho_{\tau',s} = \mathrm{Ind}_{P_{\mathbb{A}_F}}^{U_6(\mathbb{A}_F)} \tau'|\det\cdot|^{s-1/2}$, and consider for a holomorphic, K-finite section $\xi_{\tau',s}$ of $\rho_{\tau',s}$, the corresponding Eisenstein series $E(\xi_{\tau',s}, h)$ on $U_6(\mathbb{A}_F)$. We know that $E(\xi_{\tau',s}, h)$ has a simple pole at $s = 1$, as data vary. Recall that the space of $\sigma_\psi(\tau)$

is spanned by the $\psi_{1,1}^{-1}$ – Fourier coefficients of $\mathrm{Res}_{s=1} E(\xi_{\tau',s},\cdot)$ along N_1. Let us repeat the definitions in this case

$$N_1 = \left\{ u = \begin{pmatrix} 1 & y & z \\ & I_4 & y' \\ & & 1 \end{pmatrix} \in U_6 \right\}. \tag{4.5}$$

For $u \in N_1(\mathbb{A}_F)$ as in (4.5),

$$\psi_{1,1}(u) = \psi_E(y_2 - y_3). \tag{4.6}$$

The stabilizer of $\psi_{1,1}$ inside $\begin{pmatrix} 1 & & \\ & U_4 & \\ & & 1 \end{pmatrix}$ is

$$L = \left\{ \begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix} \in U_6 \;\middle|\; h \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

We fix an F-isomorphism $i : U_3 \xrightarrow{\sim} L$. The representation $\sigma_\psi(\tau)$ of $U_3(\mathbb{A}_F)$ acts in the space of automorphic functions spanned by

$$g \longmapsto \int\limits_{N_1(F)\backslash N_1(\mathbb{A}_F)} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, ui(g))\psi_{1,1}^{-1}(u)du. \tag{4.7}$$

In this section we show that (4.7) is not identically zero. Consider the following subgroup of U_6

$$V = \left\{ v = \begin{pmatrix} I_2 & a & b \\ & I_2 & a' \\ & & I_2 \end{pmatrix} \in U_6 \right\}, \tag{4.8}$$

and the following character of $V_F\backslash V_{\mathbb{A}_F}$

$$\psi_V(v) = \psi_E(a_{11} - a_{22}).$$

Let us denote by E_τ the residual representation of $U_6(\mathbb{A}_F)$ acting in $\mathrm{Span}\{\mathrm{Res}_{s=1} E(\xi_{\tau',s}\cdot)\}$.

Proposition 16. — *The Fourier coefficient of E_τ with respect ψ_V along $V_F\backslash V_{\mathbb{A}_F}$ is nontrivial, i.e.*

$$\int\limits_{V_F\backslash V_{\mathbb{A}_F}} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, v)\psi_V^{-1}(v)dv \not\equiv 0.$$

Proof. — Let

$$w = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & 1 & 0 & 0 & 0 & \\ & 0 & 0 & 1 & 0 & \\ & & & & & 1 \end{pmatrix}$$

Write v in (4.8) in the form

$$v = \begin{pmatrix} 1 & 0 & a & b & * & * \\ & 1 & c & d & * & * \\ & & 1 & 0 & -\bar{d} & -\bar{b} \\ & & & 1 & -\bar{c} & -\bar{a} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}. \tag{4.9}$$

Then

$$wvw^{-1} = \begin{pmatrix} 1 & a & * & 0 & b & * \\ & 1 & -\bar{d} & & 0 & -\bar{b} \\ & & 1 & & & 0 \\ 0 & c & * & 1 & d & * \\ & 0 & -\bar{c} & & 1 & -\bar{a} \\ & & 0 & & & 1 \end{pmatrix} \tag{4.10}$$

(zeroes elsewhere). Let $V' = wVw^{-1}$. Then by (4.10), the elements of V' have the form

$$v' = \begin{pmatrix} z & x \\ y & z' \end{pmatrix} \in U_6, \tag{4.11}$$

where z is upper unipotent, x, y are upper nilpotent (such that $x_{23} = \bar{x}_{12}, y_{23} = -\bar{y}_{12}$). The conjugation (4.10) takes the character ψ_V to the character $\psi_{V'}$ of $V'_F \backslash V'_{\mathbb{A}_F}$, defined by

$$\psi_{V'}(v') = \psi_E(z_{12} + z_{23})$$

(v' is of the form (4.11)). Since $\mathrm{Res}_{s=1} E(\xi_{\tau',s}, w \cdot v) = \mathrm{Res}_{s=1} E(\xi_{\tau',s}, v)$ and

$$\mathrm{Res}_{s=1} E(\xi_{\tau',s}, whw^{-1}) = E_\tau(w^{-1})\Big(\mathrm{Res}_{s=1} E(\xi_{\tau',s}, \cdot)\Big)(h),$$

what we have to prove is equivalent to

$$\int\limits_{V'_F \backslash V'_{\mathbb{A}_F}} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, v')\psi_{V'}^{-1}(v')dv' \not\equiv 0. \tag{4.12}$$

We will now "exchange roots" in V' in (4.12), in the sense that (4.12) is equivalent to

$$\int\limits_{D_F \backslash D_{\mathbb{A}_F}} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, r)\psi_D^{-1}(r)dr \not\equiv 0, \tag{4.13}$$

where

$$D = \left\{ r = \begin{pmatrix} 1 & \alpha & * & \gamma & \beta & * \\ & 1 & \delta & 0 & 0 & -\bar{\beta} \\ & & 1 & 0 & 0 & -\bar{\gamma} \\ & & & * & 1 & -\bar{\delta} & * \\ & & & & & 1 & -\bar{\alpha} \\ & & & & & & 1 \end{pmatrix} \in U_6 \right\}, \qquad \psi_D(r) = \psi_E(\alpha + \delta). \tag{4.14}$$

Note that D is obtained from V' by exchanging c and $-\bar{c}$ with the zeroes in coordinates (1,4),(3,6) in (4.10). This is done as follows. Let

$$Z = \left(\begin{smallmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{smallmatrix}\right) \in \mathrm{Res}_{E/F}\,\mathrm{GL}_3, \quad m(Z) = \left\{ \left(\begin{smallmatrix} z & \\ & z^* \end{smallmatrix}\right) \in U_6 \;\middle|\; z \in Z \right\}$$

$$X_0 = \left\{ \left(\begin{smallmatrix} 0 & t & e \\ & 0 & -\bar{t} \\ & & 0 \end{smallmatrix}\right) \;\middle|\; e + \bar{e} = 0 \right\}, \qquad X = \left\{ x \in \mathrm{Res}_{E/F} M_{3\times 3} \mid w_3 x + {}^t\overline{(w_3 x)} = 0 \right\}$$

$$\ell(X) = \left\{ \left(\begin{smallmatrix} I_3 & x \\ & I_3 \end{smallmatrix}\right) \;\middle|\; x \in X \right\}, \qquad \bar{\ell}(X) = \left\{ \bar{\ell}(x) = \left(\begin{smallmatrix} I_3 & \\ x & I_3 \end{smallmatrix}\right) \;\middle|\; x \in X \right\}.$$

Denote

$$\overline{Y}^{12} = \left\{ \bar{\ell}\left(\begin{smallmatrix} 0 & c & 0 \\ & 0 & -\bar{c} \\ & & 0 \end{smallmatrix}\right) \right\}, \quad \overline{Y}^{13} = \left\{ \bar{\ell}\left(\begin{smallmatrix} 0 & 0 & e \\ & 0 & 0 \\ & & 0 \end{smallmatrix}\right) \;\middle|\; e + \bar{e} = 0 \right\}, \quad X^{11} = \left\{ \ell\left(\begin{smallmatrix} t & & \\ & 0 & \\ & & -\bar{t} \end{smallmatrix}\right) \right\}$$

$$C = m(Z)\ell(X_0)\overline{Y}^{13}.$$

Then it is easy to check that C is a group, (it is a subgroup of V') and that the following properties are satisfied.

(i) Let $\psi_C = \psi_{V'}|_{C_{\mathbb{A}_F}}$. Then $\overline{Y}^{12}$ and X^{11} normalize C and (their adele points) preserve ψ_C.

(ii) $[X^{11}, \overline{Y}^{12}] \subset C$

(iii) The characters $\psi_C(xyx^{-1}y^{-1})$ on $X_F^{11}\backslash X_{\mathbb{A}_F}^{11}$ (resp. on $\overline{Y}_F^{12}\backslash\overline{Y}_{\mathbb{A}_F}^{12}$) as y (resp. x) varies in $\overline{Y}_F^{12}$ (resp. X_F^{11}) are all characters of $X_F^{11}\backslash X_{\mathbb{A}_F}^{11}$ (resp. $\overline{Y}_F^{12}\backslash\overline{Y}_{\mathbb{A}_F}^{12}$).

$$\begin{tikzcd}[column sep=small]
 & V'X^{11} = D\overline{Y}^{12} \arrow[dr, dash, "\overline{Y}^{12}"] & \\
V' = C\overline{Y}^{12} \arrow[ur, dash, "X^{11}"] & & CX^{11} = D \\
 & C \arrow[ul, dash, "\overline{Y}^{12}"] \arrow[ur, dash, "X^{11}"'] &
\end{tikzcd} \tag{4.15}$$

Let us check (iii), for example. We have

$$\begin{pmatrix} I_3 & \\ y & I_3 \end{pmatrix}\begin{pmatrix} I_3 & x \\ & I_3 \end{pmatrix}\begin{pmatrix} I_3 & \\ -y & I_3 \end{pmatrix}\begin{pmatrix} I_3 & -x \\ & I_3 \end{pmatrix} = \begin{pmatrix} I_3-xy & xyx \\ -yxy & I_3+yx+yxyx \end{pmatrix} \tag{4.16}$$

Now, for $y = \begin{pmatrix} 0 & c & 0 \\ & 0 & -\bar{c} \\ & & 0 \end{pmatrix}$, $x = \begin{pmatrix} t & & \\ & 0 & \\ & & \bar{t} \end{pmatrix}$, $yxy = 0$, and hence (4.16) equals (note that $\overline{\ell}(y) \in \overline{Y}^{12}$, $\ell(x) \in X^{11}$) $\begin{pmatrix} z & \\ & z^* \end{pmatrix}$, where $z = \begin{pmatrix} 1 & -ct & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$. Hence ψ_C applied to the l.h.s. of (4.16) equals $\psi_E^{-1}(ct)$, which represents a general character of t (resp. c), as c (resp. t) varies.

Let us explain now the equivalence of (4.12) and (4.13). Put $e_\xi(h) = \mathrm{Res}_{s=1} E(\xi_{\tau',s}, h)$. We have

$$\begin{aligned}
\int\limits_{V'_F\backslash V'_{\mathbb{A}_F}} e_\xi(v')\psi_{V'}^{-1}(v')dv' &= \int\limits_{\overline{Y}_F^{12}\backslash\overline{Y}_{\mathbb{A}_F}^{12}} \int\limits_{C_F\backslash C_{\mathbb{A}_F}} e_\xi(cy)\psi_C^{-1}(c)dcdy \\
&= \int\limits_{\overline{Y}_F^{12}\backslash\overline{Y}_{\mathbb{A}_F}^{12}} \sum_{\lambda\in E} \int\limits_{E\backslash\mathbb{A}_E} \int\limits_{C_F\backslash C_{\mathbb{A}_F}} e_\xi\Big(c\ell\begin{pmatrix} t & & \\ & 0 & \\ & & \bar{t} \end{pmatrix} y\Big)\psi_E^{-1}(\lambda t)\psi_C^{-1}(c)dtdcdy \\
&\underset{D=CX^{11}}{\overset{\uparrow}{=}} \int\limits_{\overline{Y}_F^{12}\backslash\overline{Y}_{\mathbb{A}_F}^{12}} \sum_{\lambda\in E} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_{C,\lambda}^{-1}(r)drdy.
\end{aligned}$$

Here, for $r = c\ell\begin{pmatrix} t & & \\ & 0 & \\ & & \bar{t} \end{pmatrix} \in CX^{11} = D$, $\psi_{C,\lambda}(r) = \psi_E(\lambda t)\psi_C(c)$. Let $y_0 \in \overline{Y}_F^{12}$. Then $e_\xi(ry) = e_\xi(y_0 ry) = e_\xi(y_0 r y_0^{-1} y_0 y)$. Recall that y_0 normalizes $D_{\mathbb{A}_F}$, and it preserves D_F. Also, for $r = c\cdot x$, $x \in X_{\mathbb{A}_F}^{11}$, $c \in C_{\mathbb{A}_F}$, $y_0^{-1}xy_0 = [y_0^{-1}, x]x \in C_{\mathbb{A}_F}X_{\mathbb{A}_F}^{11}$, and

$y_0^{-1}cy_0 \in C$, with $\psi_C(c) = \psi_C(y_0^{-1}cy_0)$. Thus, for each $y_0 \in \overline{Y}_F^{12}$, we have

$$\int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(y_0ry)\psi_{C,\lambda}^{-1}(r)dr \underset{\substack{\uparrow \\ \text{change variable} \\ r\mapsto y_0^{-1}ry_0}}{=} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry_0y)\psi_{C,\lambda}^{-1}(y_0^{-1}ry_0)dr$$

$$= \int\limits_{X_F^{11}\backslash X_{\mathbb{A}_F}^{11}} \int\limits_{C_F\backslash C_{\mathbb{A}_F}} e_\xi(cxy_0y)\psi_{C,\lambda}^{-1}([y_0^{-1},x]x)\psi_C^{-1}(c)dcdx.$$

We could take even a variable $y_\lambda \in \overline{Y}_F^{12}$, $\lambda \in E$, and get the same results. Take $y_\lambda = \overline{\ell}\begin{pmatrix} 0 & -\lambda & 0 \\ & 0 & \bar{\lambda} \\ & & 0 \end{pmatrix}$. Then for $x = \ell\begin{pmatrix} t & & \\ & 0 & \\ & & \bar{t} \end{pmatrix}$, we have seen in (4.16) that $\psi_{C,\lambda}([y_\lambda^{-1},x]x) = \psi_E(-\lambda t)\psi_E(\lambda t) = 1$. Put $\psi_D(xc) = \psi_C(c)$. We get that the l.h.s. of (4.12) equals

$$\int\limits_{\overline{Y}_F^{12}\backslash \overline{Y}_{\mathbb{A}_F}^{12}} \sum_{y_\lambda\in\overline{Y}_F^{12}} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry_\lambda y)\psi_D^{-1}(r)drdy = \int\limits_{\overline{Y}_{\mathbb{A}_F}^{12}} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_D^{-1}(r)drdy.$$

Thus, we have shown that

$$\int\limits_{V'_F\backslash V'_{\mathbb{A}_F}} e_\xi(v')\psi_{V'}^{-1}(v')dv' = \int\limits_{\overline{Y}_{\mathbb{A}_F}^{12}} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_D^{-1}(r)drdy. \tag{4.17}$$

We claim that the r.h.s. of (4.17) is not identically zero, if and only if

$$\int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(r)\psi_D^{-1}(r)dr \not\equiv 0,$$

which is (4.13). Indeed, assume that the r.h.s. of (4.17) is identically zero. Apply the convolution operator $\int_{\mathbb{A}_E}\phi(t)E_\tau(\ell\begin{pmatrix} t & & \\ & 0 & \\ & & -\bar{t} \end{pmatrix})dt$, for $\phi \in S(\mathbb{A}_E)$. We get (denoting $\phi(\ell\begin{pmatrix} t & & \\ & 0 & \\ & & -\bar{t} \end{pmatrix}) = \phi(t)$)

$$0 \equiv \int\limits_{X_{\mathbb{A}_F}^{11}} \int\limits_{\overline{Y}_{\mathbb{A}_F}^{12}} \int\limits_{D_F\backslash D_{\mathbb{A}_F}} \phi(x)e_\xi(r[y,x]xy)\psi_D^{-1}(r)drdydx$$

$$= \int\limits_{\overline{Y}_{\mathbb{A}_F}^{12}} \int\limits_{X_{\mathbb{A}_F}^{11}} \phi(x)\psi_D([y,x])dx \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_D^{-1}(r)drdy$$

$$= \int\limits_{\overline{Y}_{\mathbb{A}_F}^{12}} \widehat{\phi}(y) \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_D^{-1}(r)drdy.$$

In the one before last integral, we changed variable $r \mapsto r[y,x]^{-1}x^{-1}$. Recall that $\psi_D|_{X_{\mathbb{A}_F}^{11}} = 1$. In the last integral, $\widehat{\phi}(y) = \int_{X_{\mathbb{A}_F}^{11}}\phi(x)\psi_D([y,x])dx$. This is a Fourier transform of ϕ, since $x \mapsto \psi_D([y,x])$ is a general character of x, as y varies. Thus, (for all ξ)

$$\int\limits_{\overline{Y}^{12}_{\mathbb{A}_F}} \widehat{\phi}(y) \int\limits_{D_F\backslash D_{\mathbb{A}_F}} e_\xi(ry)\psi_D^{-1}(r)drdy \equiv 0,$$

for all $\phi \in S(X^{11}_{\mathbb{A}_F})$. This is equivalent to (4.13). In the passage from (4.12) to (4.13) we "exchanged" $\overline{Y}^{12}$ and X^{11}. (see (4.15)).

We have to prove (4.13). Let

$$X^{22} = \left\{\ell\begin{pmatrix}0&&\\&t&0\\&&0\end{pmatrix} \;\middle|\; t+\bar{t}=0\right\}.$$

Then X^{22} normalizes D and preserves ψ_D. Put $\widetilde{D} = D\cdot X^{22}$, and extend ψ_D to $\widetilde{D}$, by making it trivial on X^{22}. Denote this extension by $\psi_{\widetilde{D}}$. Let

$$X^{21}_+ = \left\{\ell\begin{pmatrix}0&0&0\\t&0&0\\0&-\bar{t}&0\end{pmatrix} \;\middle|\; \bar{t}=t\right\}$$

Then one can check that X^{21}_+ normalizes $\widetilde{D}$ and preserves $\psi_{\widetilde{D}}$. Let $D^+ = \widetilde{D}\cdot X^{21}_+$, and extend $\psi_{\widetilde{D}}$ to a character ψ_{D^+} of D^+, by making it trivial on X^{21}_+. In order to prove (4.13), it is enough to prove

$$\int\limits_{D^+_F\backslash D^+_{\mathbb{A}_F}} \mathrm{Res}_{s=1}\, E(\xi_{\tau',s},r)\psi^{-1}_{D^+}(r)dr \not\equiv 0 \tag{4.18}$$

Let

$$X^{21}_- = \left\{\ell\begin{pmatrix}0&0&0\\t&0&0\\0&-\bar{t}&0\end{pmatrix} \;\middle|\; \bar{t}=-t\right\}$$

We can "exchange" in (4.18) $\overline{Y}^{13}$ by X^{21}_-. More precisely, this is done as follows. Let $C^+ = m(Z)\ell(X)X^{21}_+$. This is a subgroup of D^+. Put $\psi_{C^+} = \psi_{D^+}|_{C^+}$. Then

(i) $\overline{Y}^{13}$ and X^{21}_- normalize C^+ and preserve ψ_{C^+}.

(ii) $[X^{21}_-,\overline{Y}^{12}] \subset C^+$

(iii) The characters $\psi_{C^+}(xyx^{-1}y^{-1})$ on $X^{21}_{-F}\backslash X^{21}_{-\mathbb{A}_F}$ (resp. on $\overline{Y}^{13}_F\backslash\overline{Y}^{13}_{\mathbb{A}_F}$) as y (resp. x) varies in $\overline{Y}^{13}_F$ (resp. X^{21}_{-F}) are all characters of $X^{21}_{-F}\backslash X^{21}_{-\mathbb{A}_F}$ (resp. $\overline{Y}^{13}_F\backslash\overline{Y}^{13}_{\mathbb{A}_F}$).

$$\begin{tikzcd}[column sep=small]
 & D^+X^{21}_- = U\overline{Y}^{13} \arrow[dl, dash, "X^{21}_-"'] \arrow[dr, dash, "\overline{Y}^{13}"] & \\
D^+ = C^+\overline{Y}^{13} \arrow[dr, dash, "\overline{Y}^{13}"'] & & U = C^+X^{21}_- \arrow[dl, dash, "X^{21}_-"] \\
 & C^+ &
\end{tikzcd}$$

Extend ψ_{C^+} to a character ψ_U of U by making it trivial on X_-^{21}. As before, (4.18) is equivalent to

$$\int\limits_{U_F\backslash U_{\mathbb{A}_F}} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, r)\psi_U^{-1}(r)dr \not\equiv 0. \tag{4.19}$$

Note that $r \in U_{\mathbb{A}_F}$ has the form

$$r = \begin{pmatrix} 1 & a & * & * & * & * \\ & 1 & b & * & * & * \\ & & 1 & 0 & * & * \\ & & & 1 & -\bar{b} & * \\ & & & & 1 & -\bar{a} \\ & & & & & 1 \end{pmatrix} \in U_6(\mathbb{A}_F)$$

and

$$\psi_U(r) = \psi_E(a+b).$$

U is a subgroup of the standard maximal unipotent subgroup N of U_6. Extend ψ_U to to ψ_N on $N_{\mathbb{A}_F}$ by making it trivial on the Siegel radical S. Clearly (4.19) will follow from the nonvanishing of the Fourier coefficient of $\mathrm{Res}_{s=1} E(\xi_{\tau',s}, \cdot)$ with respect to ψ_N along $N_F\backslash N_{\mathbb{A}_F}$. This last Fourier coefficient is just the constant term of $\mathrm{Res}_{s=1} E(\xi_{\tau',s}, \cdot)$ along S, followed by the Whittaker coefficient for the Levi part of the Siegel parabolic subgroup. Writing the constant term of $\mathrm{Res}_{s=1} E(\xi_{\tau',s}, \cdot)$ in terms of the intertwining operator, we see that the last Fourier coefficient is just a Whittaker coefficient applied to τ' with respect to the standard nondegenerate character defined by ψ_E, which is, of course, not identically zero. This completes the proof of Proposition 16. □

We now conclude that $\sigma_\psi(\tau) \neq 0$. For this, let

$$\gamma = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & I_2 & & \\ & & & 1 & \\ & & & -1 & 1 \end{pmatrix}.$$

Then, by Proposition 16,

$$\int\limits_{V_F\backslash V_{\mathbb{A}_F}} \mathrm{Res}_{s=1} E(\xi_{\tau',s}, \gamma^{-1}v\gamma)\psi_V^{-1}(v)dv \not\equiv 0. \tag{4.20}$$

Note that for $v \in V_{\mathbb{A}_F}$ of the form

$$v = \begin{pmatrix} 1 & 0 & a & b & * & * \\ & 1 & c & d & * & * \\ & & 1 & 0 & -\bar{d} & -\bar{b} \\ & & & 1 & -\bar{c} & -\bar{a} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}, \tag{4.21}$$

$$\gamma^{-1}v\gamma = \begin{pmatrix} 1 & 0 & a & b & * & * \\ & 1 & c-a & d-b & * & * \\ & & 1 & 0 & -\bar{d}+\bar{b} & -\bar{b} \\ & & & 1 & -\bar{c}+\bar{a} & -\bar{a} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

Change variables in (4.20), $c \mapsto c+a$, $d \mapsto d+b$. Let $\widetilde{\psi}$ be the character, which takes v in $V_{\mathbb{A}_F}$ of the form (4.21) to $\psi(a-b-d)$. Thus

$$\int\limits_{V_F\backslash V_{\mathbb{A}_F}} \mathrm{Res}_{s=1}\ E(\xi_{\tau',s}, v)\widetilde{\psi}^{-1}(v)dv \not\equiv 0. \tag{4.22}$$

Change variable in (4.22), $c \mapsto c+d$ (v of the form (4.21)). Consider the following subgroups.

$$J = \left\{ \begin{pmatrix} I_2 & x & y \\ & I_2 & x' \\ & & I_2 \end{pmatrix} \in V \,\middle|\, x = \begin{pmatrix} a & b \\ d & d \end{pmatrix} \right\}$$

$$K = \left\{ \begin{pmatrix} 1 & & & & & \\ & 1 & c & & & \\ & & 1 & & & \\ & & & 1 & -\bar{c} & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right\}$$

$$L = \left\{ \begin{pmatrix} 1 & t & & & \\ & 1 & & & \\ & & I_2 & & \\ & & & 1 & -\bar{t} \\ & & & & 1 \end{pmatrix} \right\}.$$

Put $\psi_J = \widetilde{\psi}\big|_J$. Then

(i) The subgroups K, L normalize J and preserve ψ_J.

(ii) $[K, L] \subset J$

(iii) The characters $\psi_J(xyx^{-1}y^{-1})$ describe general characters of x in $K_F\backslash K_{\mathbb{A}_F}$ (resp. $y \in L_F\backslash L_{\mathbb{A}_F}$) as y varies in L_F (resp. as x varies in K_F).

Note that $V = J \cdot K$. Denote $U' = JL$, and extend ψ_J to a character of U', by making it trivial on L. Now "exchange" K and L in (4.22). We get that

$$\int\limits_{U'_F\backslash U'_{\mathbb{A}_F}} \mathrm{Res}_{s=1}\, E(\xi_{\tau',s}, r)\psi_{U'}^{-1}(r)dr \not\equiv 0. \tag{4.23}$$

Note that $r \in U'_{\mathbb{A}_F}$ has the form

$$r = \begin{pmatrix} 1 & t & a & b & * & * \\ & 1 & d & d & * & * \\ & & 1 & 0 & -\bar{d} & -\bar{b} \\ & & & 1 & \bar{d} & -\bar{a} \\ & & & & 1 & -\bar{t} \\ & & & & & 1 \end{pmatrix}$$

and

$$\psi_{U'}(r) = \psi_E(a-b) \cdot \psi_E(d).$$

This means that the l.h.s. of (4.23) is the integration (4.7), which defines $\sigma_\psi(\tau)$, followed by the Whittaker coefficient with respect to ψ_E along $i(N)$, where N is the standard maximal unipotent subgroup of $G = U_3$. In particular $\sigma_\psi(\tau) \neq 0$, and we also showed that the ψ_E-Whittaker coefficient of $\sigma_\psi(\tau)$, as a representation of $U_3(\mathbb{A}_F)$ is nontrivial.

4.3. The tower property: Case $H = \mathrm{Sp}_8$, $\tau-$ on $\mathrm{GL}_4(\mathbb{A}_F)$, $G = \widetilde{\mathrm{Sp}}_4$

Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$, and $L(\tau, \frac{1}{2}) \neq 0$. (This implies in particular that $\widehat{\tau} = \tau$ and $\omega_\tau = 1$). Let $H = \mathrm{Sp}_8$, and let P be the Siegel parabolic subgroup of H. Let $\rho_{\tau,s} = \mathrm{Ind}_{P_{\mathbb{A}_F}}^{H_{\mathbb{A}_F}} \tau|\det\cdot|^{s-1/2}$, and consider the corresponding Eisenstein series $E(\xi_{\tau,s}, h)$ on $\mathrm{Sp}_8(\mathbb{A}_F)$, for a holomorphic, K-finite section $\xi_{\tau,s}$. $E(\xi_{\tau,s}, h)$ has a simple pole at $s = 1$, as data vary. Recall that the space of $\sigma_\psi(\tau)$ is spanned by the Fourier-Jacobi coefficients of type $(\psi_1, 1, \phi)$ of $\mathrm{Res}_{s=1} E(\xi_{\tau,s}, \cdot)$ along N_2. We repeat the definitions in this case

$$N_2 = \left\{ v = \begin{pmatrix} 1 & x & * & * & * \\ & 1 & y & t & * \\ & & I_4 & y' & * \\ & & & 1 & -x \\ & & & & 1 \end{pmatrix} \in \mathrm{Sp}_8 \right\}. \tag{4.24}$$

For $v \in N_2(\mathbb{A}_F)$ as in (4.24),

$$\psi_1(v) = \psi(x).$$

The group N_2 surjects onto the Heisenberg group $\mathcal{H}$ in five variables by

$$j(v) = (y; t),$$

for $v \in N_2$, as in (4.24). Let $\omega_{\psi^{-1}}$ be the Weil representation of $\widetilde{\mathrm{Sp}}_4(\mathbb{A}_F) \ltimes \mathcal{H}_{\mathbb{A}_F}$, acting on $S(\mathbb{A}_F^2)$, corresponding to the character ψ^{-1}. Denote, for $\phi \in S(\mathbb{A}_F^2)$, the corresponding theta series by $\theta_{\psi^{-1}}^\phi(\cdot)$. The representation $\sigma_\psi(\tau)$ of $\widetilde{\mathrm{Sp}}_4(\mathbb{A}_F)$ acts in the space of automorphic functions spanned by

$$\widetilde{g} \longmapsto \int_{N_2(F)\backslash N_2(\mathbb{A}_F)} \mathrm{Res}_{s=1} E(\xi_{\tau,s}, vj(g))\theta_{\psi^{-1}}^\phi(j(v)\widetilde{g})\psi_1^{-1}(v)dv. \tag{4.25}$$

Here g is the projection of $\widetilde{g}$ in $\widetilde{\mathrm{Sp}}_4(\mathbb{A}_F)$ onto $\mathrm{Sp}_4(\mathbb{A}_F)$, and we extend j to an embedding of $\mathrm{Sp}_4(\mathbb{A}_F) \cdot \mathcal{H}_{\mathbb{A}_F}$ inside $\mathrm{Sp}_8(\mathbb{A}_F)$ by $j(g) = \begin{pmatrix} I_2 & & \\ & g & \\ & & I_2 \end{pmatrix}$.

In order to prove that $\sigma_\psi(\tau)$ is cuspidal, we have to show that the constant terms along unipotent radicals (of parabolic subgroups of Sp_4) vanish on $\sigma_\psi(\tau)$. The tower property that we reveal when we compute these constant terms is that they are expressed in terms of "deeper descents" $\sigma_\psi^{(k)}(\tau)$ $(k < n = 2)$, which in our case means $k = 0, 1$. Here $\sigma_\psi^{(0)}(\tau)$ is simply the "space" of ψ-Whittaker coefficients on the group "$\widetilde{\mathrm{Sp}}_0(\mathbb{A}_F)$" which by definition is $\{1\}$, of the residue representation E_τ (acting on $\mathrm{Span}\{\mathrm{Res}_{s=1} E(\xi_{\tau,s}, \cdot)\}$). Since the ψ-Whittaker coefficient of $E(\xi_{\tau,s}, \cdot)$ is holomorphic at $s = 1$, the last space is zero dimensional, *i.e.* $\sigma_\psi^{(0)}(\tau) = 0$. The space $\sigma_\psi^{(1)}(\tau)$ is the space of automorphic functions on $\widetilde{\mathrm{Sp}}_2(\mathbb{A}_F) = \widetilde{\mathrm{SL}}_2(\mathbb{A}_F)$ spanned by

$$\widetilde{g} \longmapsto \int_{N_3(F)\backslash N_3(\mathbb{A}_F)} \mathrm{Res}_{s=1}\ E(\xi_{\tau,s}, uj'(g))\theta_{\psi^{-1}}^{\prime\varphi}(j'(u))\psi_2^{-1}(u)du.$$

Here $\varphi \in S(\mathbb{A}_F)$, and $\theta'^{\varphi}_{\psi^{-1}}(\cdot)$ is the theta series corresponding to the Weil representation $\omega'_{\psi^{-1}}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F) \ltimes \mathcal{H}'(\mathbb{A}_F)$, where $\mathcal{H}'$ is the Heisenberg group in three variables. The group N_3 is

$$N_3 = \left\{ u = \begin{pmatrix} z & x & y \\ & I_2 & x' \\ & & z^* \end{pmatrix} \in \mathrm{Sp}_8 \;\middle|\; z \in Z_3 = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}. \tag{4.26}$$

For $u \in N_3$, as in (4.26), $\psi_2(u) = \psi(z_{12} + z_{23})$, and $j'(u) = (x_{31}, x_{32}; y_{31})$ (the surjection $N_3 \to \mathcal{H}'$). Finally, for $g \in \mathrm{SL}_2(\mathbb{A}_F)$, $j'(g) = \begin{pmatrix} I_3 & & \\ & g & \\ & & I_3 \end{pmatrix}$.

There are two standard unipotent radicals of maximal parabolic subgroups of Sp_4:

$$R = \left\{ \begin{pmatrix} 1 & x & y \\ & I_2 & x' \\ & & 1 \end{pmatrix} \in \mathrm{Sp}_4 \right\}, \quad S = \left\{ \begin{pmatrix} I_2 & x \\ & I_2 \end{pmatrix} \in \mathrm{Sp}_4 \right\}.$$

Proposition 17

(a) *The constant term of elements of $\sigma_\psi(\tau)$ along R is a sum of certain integrals of elements of $\sigma_\psi^{(1)}(\tau)$.*

(b) *The constant term of elements of $\sigma_\psi(\tau)$ along S is a sum of certain integrals of elements of $\sigma_\psi^{(0)}(\tau)$.*

We conclude that if $\sigma_\psi^{(1)}(\tau) = 0$, then the elements of $\sigma_\psi(\tau)$ are cuspidal, in the sense that their constant terms along unipotent radical are all zero. Note, as we explained before that $\sigma_\psi^{(0)}(\tau)$ is zero. In general, we may consider $\sigma_\psi^{(k)}(\tau)$ for $k \leqslant 2n$. This is a representation of $\widetilde{\mathrm{Sp}}_{2k}(\mathbb{A}_F)$. The constant terms of the elements of $\sigma_\psi^{(k)}(\tau)$ along unipotent radicals turn out to be sums of elements of $\sigma_\psi^{(j)}(\tau)$, for $j < k$. The tower principle says that there is a first index k_0, such that $\sigma_\psi^{(k_0)}(\tau) \neq 0$, and then $\sigma_\psi^{(k_0)}(\tau)$ is cuspidal. We actually prove that $k_0 = n$.

Proof of Proposition 17(a). — Put, for short $e_\tau(h) = \mathrm{Res}_{s=1} E(\xi_{\tau,s}, h)$. We consider

$$c(e_\tau, \phi) = \int\limits_{R_F \backslash R_{\mathbb{A}_F}} \int\limits_{N_2(F) \backslash N_2(\mathbb{A}_F)} e_\tau(vj(r)) \theta^{\phi}_{\psi^{-1}}(j(v)r) \psi_1^{-1}(v) dv dr.$$

Since R splits in $\widetilde{\mathrm{Sp}}_4$, we identify R as a subgroup of $\widetilde{\mathrm{Sp}}_4$. Let $\gamma = \begin{pmatrix} I_2 & 1 & & \\ & & I_2 & \\ & & & I_2 \\ & 1 & & \end{pmatrix}$.

Denote the right γ-translate of e_τ by $\gamma \cdot e_\tau$. We have

$$c(e_\tau, \phi) = \int\limits_{R_F \backslash R_{\mathbb{A}_F}} \int\limits_{N_2(F) \backslash N_2(\mathbb{A}_F)} \gamma \cdot e_\tau(\gamma v j(r) \gamma^{-1}) \theta^{\phi}_{\psi^{-1}}(j(v)r) \psi_1^{-1}(v) dv dr. \tag{4.27}$$

Consider the group $\gamma N_2 j(R) \gamma^{-1}$. We have

$$\gamma N_2 j(R) \gamma^{-1} = T \cdot L \cdot Z \cdot X,$$

where

$$T=\left\{\begin{pmatrix}1&0&0&*&0\\ &I_2&*&*&*\\ &&I_2&*&0\\ &&&I_2&0\\ &&&&1\end{pmatrix}\in \mathrm{Sp}_8\right\},\qquad Z=\left\{\begin{pmatrix}1&&&&\\ &z&&&\\ &&I_2&&\\ &&&z^*&\\ &&&&1\end{pmatrix}\in \mathrm{Sp}_8\;\middle|\; z=\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\right\}$$

$$L=\left\{\begin{pmatrix}1&&&&\\ *&I_2&&&\\ &&I_2&&\\ &&&I_2&\\ &&&*&1\end{pmatrix}\in \mathrm{Sp}_8\right\},\qquad X=\left\{\begin{pmatrix}1&0&*&0&*\\ &I_2&0&0&0\\ &&I_2&0&*\\ &&&I_2&0\\ &&&&1\end{pmatrix}\in \mathrm{Sp}_8\right\}.$$

The integral (4.27) becomes

$$\int\limits_{X_F\backslash X_{\mathbb{A}_F}}\int\limits_{Z_F\backslash Z_{\mathbb{A}_F}}\int\limits_{L_F\backslash L_{\mathbb{A}_F}}\int\limits_{T_F\backslash T_{\mathbb{A}_F}}\gamma\cdot e_\tau(t\cdot\ell\cdot z\cdot x)\theta^{\phi}_{\psi^{-1}}((0,t_{34},t_{35},t_{38};t_{36})(\ell_{31},0,0,0;0)\widetilde{j}(x))$$

$$\cdot\psi^{-1}(z_{23})dtd\ell dzdx. \tag{4.28}$$

Here $\widetilde{j}$ is an isomorphism of X with R. It is the inverse to the conjugation by γ composed with j. The theta series in (4.28) equals

$$\sum_{\eta_1\in F}\sum_{\eta_2\in F}\omega_{\psi^{-1}}((\eta_1,0,0,0;0)(0,t_{34},t_{35},t_{38};t_{36})(\ell_{31},0,0,0;0)\widetilde{j}(x))\phi(0,\eta_2). \tag{4.29}$$

The inner sum in (4.29), as a function $(t_{34},t_{35},t_{38},t_{36})$, is left T_F invariant, for fixed η_1,ℓ_{31},x. In (4.28), we may interchange the $T_F\backslash T_{\mathbb{A}_F}$ integration and the summation over $\eta_1\in F$. Now change variable $t\mapsto \ell_{\eta_1}^{-1}t\ell_{\eta_1}$, where

$$\ell_{\eta_1}=\begin{pmatrix}1&&&&&&\\ 0&1&&&&&\\ \eta_1&&1&&&&\\ &&&I_2&&&\\ &&&&1&&\\ &&&&&1&\\ &&&&-\eta_1&0&1\end{pmatrix}\in L_F.$$

In (4.28), $\gamma e_\tau(t\cdot\ell\cdot z\cdot x)$ becomes $\gamma e_\tau(t\cdot(\ell_{\eta_1}\ell)\cdot z\cdot x)$, and in (4.29), the inner sum becomes $\sum_{\eta_2\in F}\omega_{\psi^{-1}}((0,t_{34},t_{35},t_{38};t_{36})(\eta_1+\ell_{31},0,0,0;0)\widetilde{j}(x)\phi(0,\eta_2)$. Now collapse $\int_{L_F\backslash L_{\mathbb{A}_F}}\sum_{\eta_1\in F}$ into $\int_{L^1_F\backslash L_{\mathbb{A}_F}}$, where

$$L^1=\left\{\begin{pmatrix}1&&&&\\ *&1&&&\\ &&I_4&&\\ &&&1&\\ &&&*&1\end{pmatrix}\in \mathrm{Sp}_8\right\}.$$

We get

$$\int\limits_{X_F\backslash X_{\mathbb{A}_F}}\int\limits_{Z_F\backslash Z_{\mathbb{A}_F}}\int\limits_{L^1_F\backslash L_{\mathbb{A}_F}}\int\limits_{T_F\backslash T_{\mathbb{A}_F}}\gamma e_\tau(t\cdot\ell\cdot z\cdot x)\cdot\psi^{-1}(z_{23})$$
$$\cdot\sum_{\eta\in F}\omega_{\psi^{-1}}((0,t_{34},t_{35},t_{38};t_{36})(\ell_{31},0,0,0;0)\widetilde{j}(x))\phi(0,\eta)dtd\ell dzdx. \tag{4.30}$$

Note that

$$\omega_{\psi^{-1}}(j'(x))\varphi(0,\eta)=\varphi(0,\eta).$$

We can conjugate x "back to the left" in (4.30) to get

$$\int\limits_{Z_F\backslash Z_{\mathbb{A}_F}} \int\limits_{L^1_F\backslash L_{\mathbb{A}_F}} \int\limits_{U_F\backslash U_{\mathbb{A}_F}} \gamma e_\tau(u\cdot\ell\cdot z)\sum_{\eta\in F}\omega_{\psi^{-1}}((0,u_{34},u_{35},u_{38};u_{36})\,(\ell_{31},0,0,0;0))$$

$$\phi(0,\eta)\psi^{-1}(z_{23})dud\ell dz, \tag{4.31}$$

where $U=T\cdot X$. Now take $\phi=\phi_1\otimes\phi_2$, $\phi_i\in S(\mathbb{A}_F)$. Denote by $\omega'_{\psi^{-1}}$ the Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A}_F)\cdot\mathcal{H}'(\mathbb{A}_F)$. Then

$$\omega_{\psi^{-1}}((0,u_{34},u_{35},u_{38};u_{36})(\ell_{31},0,0,0;0))\phi(0,\eta)=\phi_1(\ell_{31})\omega'_{\psi^{-1}}((u_{34},u_{35};u_{36}))\phi_2(\eta).$$

For such ϕ, (4.31) equals

$$\int\limits_{\mathbb{A}_F}\phi_1(y)\int\limits_{Z_F\backslash Z_{\mathbb{A}_F}} \int\limits_{L^1_F\backslash L^1_{\mathbb{A}_F}} \int\limits_{U_F\backslash U_{\mathbb{A}_F}} \gamma e_\tau(u\ell^1 z\ell_y)\theta'^{\phi_2}_{\psi^{-1}}((u_{34},u_{35};u_{36}))\psi^{-1}(z_{23})dud\ell^1dzdy.$$

Denote

$$\phi_1*(\gamma e_\tau)(h)=\int\limits_{\mathbb{A}_F}\phi_1(y)\cdot(\gamma e_\tau)(h\ell_y)dy.$$

Then

$$\begin{aligned}(4.32)\quad & c(e_\tau,\phi_1\otimes\phi_2)\\ &=\int\limits_{Z_F\backslash Z_{\mathbb{A}_F}} \int\limits_{L^1_F\backslash L^1_{\mathbb{A}_F}} \int\limits_{U_F\backslash U_{\mathbb{A}_F}} (\phi_1*(\gamma e_\tau))(u\ell^1z)\theta'^{\phi_2}_{\psi^{-1}}(i(u))\psi^{-1}(z_{23})dud\ell^1dz\end{aligned}$$

Here $i(u)=(u_{34},u_{35};u_{36})$. As we did in the previous section, we can exchange in (4.32) the subgroups L^1 and

$$V=\left\{\begin{pmatrix} \begin{smallmatrix}1&0&*\\&1&0\\&&1\end{smallmatrix} &&\\ & I_2 & \\ && \begin{smallmatrix}1&0&*\\&1&0\\&&1\end{smallmatrix}\end{pmatrix}\in\mathrm{Sp}_8\right\}.$$

Denote $Z'=VZ$ and let $\psi_{Z'}$ denote the character of $Z'_{\mathbb{A}_F}$, which is trivial on $V_{\mathbb{A}_F}$ and takes z in $Z_{\mathbb{A}_F}$ to $\psi(z_{23})$. As in (4.17), we get that

$$\begin{aligned}(4.33)\quad & c(e_\tau,\phi_1\otimes\phi_2)\\ &=\int\limits_{L^1_{\mathbb{A}_F}} \int\limits_{Z'_F\backslash Z'_{\mathbb{A}_F}} \int\limits_{U_F\backslash U_{\mathbb{A}_F}} \phi_1*(\gamma e_\tau)(uz'\ell^1)\theta'^{\phi_2}_{\psi^{-1}}(i(u))\psi^{-1}_{Z'}(z')dudz'd\ell^1.\end{aligned}$$

Consider the function on $F\backslash\mathbb{A}_F$

$$t\longmapsto\int\limits_{Z'_F\backslash Z'_{\mathbb{A}_F}} \int\limits_{U_F\backslash U_{\mathbb{A}_F}} \phi_1*(\gamma e_\tau)(uz'x_t\ell^1)\theta'^{\phi_2}_{\psi^{-1}}(i(u))\psi^{-1}_{Z'}(z')dudz', \tag{4.34}$$

where

$$x_t = \begin{pmatrix} 1 & t & & & \\ & 1 & & & \\ & & I_4 & & \\ & & & 1 & \\ & & & -t & 1 \end{pmatrix}.$$

Write the Fourier expansion of (4.34) (evaluated at zero)

$$(4.35) \qquad \sum_{\lambda \in F^*} \int_{N_3(F)\backslash N_3(\mathbb{A}_F)} \phi_1 * (\gamma e_\tau)(u\widehat{\lambda}\ell^1)\theta'^{\phi_2}_{\psi^{-1}}(j'(u))\psi_2^{-1}(u)du,$$

where $\widehat{\lambda} = \begin{pmatrix} \lambda & & \\ & I_6 & \\ & & \lambda^{-1} \end{pmatrix}$. See the paragraph before the statement of Proposition 17 for notation. Note that in (4.35) we did not include the constant coefficient, since it will contain as an inner integration the constant term of $\phi_1 * (\gamma e_\tau)$ along the radical of the standard parabolic subgroup of Sp_8, which preserves a line. This constant term is clearly zero. Note that the summand in (4.35), corresponding to λ, is an element of $\sigma_\psi^{(1)}(\tau)$ evaluated at $\widehat{\lambda}$. We proved

$$(4.36) \quad c(e_\tau, \phi \otimes \phi_2) \\ = \sum_{\lambda \in F^*} \int_{L^1_{\mathbb{A}_F}} \int_{N_3(F)\backslash N_3(\mathbb{A}_F)} \phi_1 * (\gamma e_\tau)(u\widehat{\lambda}\ell^1)\theta'^{\phi_2}_{\psi^{-1}}(j'(u))\psi_2^{-1}(u)du d\ell^1.$$

This completes the proof of Proposition 17a. □

4.4. Vanishing of $\sigma_\psi^{(k)}(\tau)$, for $k < n$: Case $H = \mathrm{SO}_8$, τ – on $\mathrm{GL}_4(\mathbb{A}_F)$

Let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$, such that $L^S(\tau, \Lambda^2, s)$ has a pole at $s = 1$. Let $H = \mathrm{SO}_8$, and let P be the Siegel parabolic subgroup of H. Let $\rho_{\tau,s} = \mathrm{Ind}_{P_{\mathbb{A}_F}}^{H_{\mathbb{A}_F}} \tau|\det\cdot|^{s-\frac{1}{2}}$, and consider, as before, the corresponding Eisenstein series $E(\xi_{\tau,s}, h)$. It has a simple pole at $s = 1$, as data vary. Recall that the representation $\sigma_\psi(\tau)$ of $\mathrm{SO}_5(\mathbb{A}_F)$ acts in the space spanned by the functions

$$(4.37) \qquad g \longmapsto \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \mathrm{Res}_{s=1}\ E(\xi_{\tau,s}, ui(g))\psi_{1,-1}^{-1}(u)du,$$

where

$$(4.38) \qquad N_1 = \left\{ u = \begin{pmatrix} 1 & v & * \\ & I_6 & v' \\ & & 1 \end{pmatrix} \in \mathrm{SO}_8 \right\}$$

$\psi_{1,-1}(u) = \psi(v_3 - v_4)$ (for $u \in N_1$, as in (4.38)). The isomorphism i sends SO_5 onto

$$\left\{ \begin{pmatrix} 1 & & \\ & h & \\ & & 1 \end{pmatrix} \in \mathrm{SO}_8 \;\middle|\; h \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

As explained in Section 1.2 and in the previous section, the constant term on $\sigma_\psi(\tau)$ with respect to the radical (in SO_5) $R = \left\{ \begin{pmatrix} 1 & x & * \\ & I_3 & x' \\ & & 1 \end{pmatrix} \in \mathrm{SO}_5 \right\}$ is expressed in terms of $\sigma_\psi^{(1)}(\tau)$, and the constant term on $\sigma_\psi(\tau)$ with respect to the Siegel radical (in SO_5) is expressed in terms of $\sigma_\psi^{(0)}(\tau)$, which is just the Whittaker coefficient on $\langle \mathrm{Res}_{s=1} E(\xi_{\tau,s}, \cdot)\rangle$, and is known to be trivial. See the guidelines to the proof of Theorem 4. We will show

Proposition 18. — *For τ as above, $H = \mathrm{SO}_8$, we have*

$$\sigma_\psi^{(1)}(\tau) = 0.$$

Proof. — The proof is using just the fact that at one unramified place ν, τ_ν is self-dual, and has a trivial central character. Fix such a place ν. By Proposition 15, the unramified constituent π_ν of $\rho_{\tau_1,1} = \mathrm{Ind}_{P_{F_\nu}}^{H_{F_\nu}} \tau_\nu |\det\cdot|^{1/2}$ is the unramified constituent of a representation of the form $\rho_{\mu_1,\mu_2} = \mathrm{Ind}_{Q_{F_\nu}}^{H_{F_\nu}} \mu_1 \circ \det \otimes \mu_2 \circ \det$. Here μ_1, μ_2 are unramified characters of F_ν^*, such that τ_ν is the unramified constituent of the representation of $\mathrm{GL}_4(F_\nu)$ induced from the standard Borel subgroup and its character defined by

$$\mathrm{diag}(t_1, \ldots, t_u) \longmapsto \mu_1\Big(\frac{t_1}{t_4}\Big)\mu_2\Big(\frac{t_2}{t_3}\Big).$$

Q is the standard parabolic subgroup of H, whose Levi part is isomorphic to $\mathrm{GL}(2) \times \mathrm{GL}(2)$. If $\sigma_\psi^{(1)}(\tau)$ is nontrivial, then the Jacquet module with respect to $(N_2(F_\nu), (\psi_\nu)_{2,-1})$, $J_{N_2(F_\nu),(\psi_\nu)_{2,-1}}(\rho_{\mu_1,\mu_2})$ is nontrivial. Thus, the proposition will be proved if we show that

$$J_{N_2(F_\nu),(\psi_\nu)_{2,-1}}(\rho_{\mu_1,\mu_2}) = 0. \tag{4.39}$$

We use Bruhat theory. Let Q_2 be the standard parabolic subgroup of H, whose Levi part is isomorphic to $\mathrm{GL}_2 \times \mathrm{SO}_4$. We first analyze $J_{N_2(F_\nu),(\psi_\nu)_{2,-1}} \Big(\mathrm{Res}_{Q_2(F_\nu)}(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}} \eta \otimes \pi)\Big)$, where $\eta = \mu_1 \circ \det$ and π is an irreducible representation (later to be specified as $\mathrm{Ind}\,\mu_2 \circ \det$). We apply Bruhat theory to study $\mathrm{Res}_{Q_2(F_\nu)}\Big(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}} \eta \otimes \pi\Big)$. This restriction has a filtration of $Q_2(F_\nu)$ – modules, with subquotients parametrized by $Q_2\backslash H/Q_2$. The quotient $Q_2\backslash H$ is isomorphic to the variety Y_2 of two dimensional isotropic subspaces of the (column) space F^8 (equipped with the quadratic form preserved by H). Let $\{e_1, \ldots, e_4, e_{-4}, \ldots, e_{-1}\}$ be the standard basis of F^8. Let $X^{(2)} = \mathrm{Span}\{e_1, e_2\}$ be the standard two dimensional isotropic subspace. The isomorphism $Q_2\backslash H \cong Y_2$ is given by $Q_2 h \mapsto h^{-1} \cdot X^{(2)}$. The orbits of Q_2 in Y_2 are parametrized by $r = \dim(X \cap X^{(2)})$, and $s = \dim(X \cap (X^{(2)})^\perp)$, $X \in Y_2$. Note that $0 \leqslant r \leqslant s \leqslant 2$. A representative is

$$X_{r,s} = \mathrm{Span}\{e_1, \ldots, e_r; e_3, \ldots, e_{2+s-r}; e_{-(r+1)}, \ldots, e_{-(2+r-s)}\}.$$

Choose (a Weyl element, for example) $w_{r,s} \in H$, such that $w_{r,s}^{-1} X^{(2)} = X_{r,s}$. The corresponding subquotients for $\mathrm{Res}_{Q_2(F_\nu)}\Big(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}} \eta \otimes \pi\Big)$ are

$$\Gamma_{r,s} = \mathrm{Ind}^{c\,Q_2(F_\nu)}_{w_{r,s}^{-1}Q_2(F_\nu)w_{r,s} \cap Q_2(F_\nu)} (\eta \otimes \pi \cdot \delta_{Q_2}^{\frac{1}{2}})^{w_{r,s}} \cdot \delta^{-1/2}.$$

(The factor $\delta^{-1/2}$ appears in order to make the induction normalized.) Consider, for example, the case $r=1$, $s=2$. Here, we have

(4.40)
$$w_{1,2}^{-1}Q_2(F_\nu)w_{1,2}\cap Q_2(F_\nu)=\left\{\begin{pmatrix} a_{11} & a_{12} & x_{11} & x_{12} & x_{13} & x_{14} & y_{11} & y_{12}\\ & a_{22} & 0 & x_{22} & x_{23} & x_{24} & y_{21} & y'_{11}\\ & & b_{11} & b_{12} & b_{13} & b_{14} & x'_{24} & x'_{14}\\ & & & c_{11} & c_{12} & b'_{13} & x'_{23} & x'_{13}\\ & & & c_{21} & c_{22} & b'_{12} & x'_{22} & x'_{12}\\ & & & & & b_{11}^{-1} & 0 & x'_{11}\\ & & & & & & a_{22}^{-1} & a'_{12}\\ & & & & & & & a_{11}^{-1}\end{pmatrix}\in H_{F_\nu}\right\}:=L_{12}$$

The representation $\xi_{1,2}=(\eta\otimes\pi\cdot\delta_{Q_2}^{1/2})^{w_{1,2}}$ takes elements of the form (4.40) to

$$|a_{11}b_{11}|^{5/2}\mu_1(a_{11}b_{11})\pi\begin{pmatrix} c_{11} & 0 & x'_{23} & c_{12}\\ x_{22} & a_{22} & y_{21} & x_{23}\\ 0 & 0 & a_{22}^{-1} & 0\\ c_{21} & 0 & x'_{22} & c_{22}\end{pmatrix}. \tag{4.41}$$

Let us prove that $J_{N_2(F_\nu),(\psi_\nu)_{2,-1}}(\Gamma_{1,2})=0$. Fit $\Gamma_{1,2}$ into an exact sequence $0\to S_2\to\Gamma_{1,2}\to S_1\to 0$, where S_2 is the subspace of functions in $\Gamma_{1,2}$ supported inside Ω, which consists of all matrices $\begin{pmatrix} a & * & *\\ & b & *\\ & & a^*\end{pmatrix}$ in $Q_2(F_\nu)$, such that a lies in the open Bruhat cell of $\mathrm{GL}_2(F_\nu)$. The support of these functions (in $\Gamma_{1,2}$) is compact modulo $L_{12}(F_\nu)$. S_1 is the space of smooth functions on the complement of Ω inside $Q_2(F_\nu)$, where left $L_{1,2}(F_\nu)$ – translations act by (4.41), and the support is compact modulo $L_{1,2}(F_\nu)$. Thus, we have to show that $J_{N_2(F_\nu),(\psi_\nu)_{2,-1}}\,(S_i)=0$; $i=1,2$. Let $f\in S_1$. We show that

$$\int\limits_{N_2(\mathcal{P}_\nu^{-M})}(\psi_\nu)_{2,-1}^{-1}(n)f\Big(x_2(t)\begin{pmatrix} I_2 & & \\ & k & \\ & & I_2\end{pmatrix}n\Big)dn=0, \tag{4.42}$$

for all $k\in\mathrm{SO}_4(\mathcal{O}_{F_\nu}), t\in F_\nu$;

$$x_2(t)=\begin{pmatrix} 1 & & & & & \\ & 1 & t & & & & \\ & & 1 & & & & \\ & & & I_2 & & & \\ & & & & 1 & -t & \\ & & & & & 1 & \\ & & & & & & 1\end{pmatrix}.$$

The support of the integrand in t depends on f, so we may take M large enough so that, in the support of f,

$$\begin{pmatrix} I_2 & & \\ & k & \\ & & I_2\end{pmatrix}x_2(t)\begin{pmatrix} I_2 & & \\ & k^{-1} & \\ & & I_2\end{pmatrix}\in N_2(\mathcal{P}_\nu^{-M}),$$

for all $k\in\mathrm{SO}_4(\mathcal{O}_{F_\nu})$. Making a change of variable in n, we may assume that $t=0$ in (4.42). Consider now the subintegration in (4.42) on $x_1(z)$, $|z|\leqslant q_\nu^M$, where

$$x_1(z)=\begin{pmatrix} 1 & z & & & \\ & 1 & & & \\ & & I_4 & & \\ & & & 1 & -z\\ & & & & 1\end{pmatrix}.$$

It gives

$$\int\limits_{|z|\leqslant q_\nu^M}\psi_\nu^{-1}(z)f\Big(x_1(z)\begin{pmatrix} I_2 & & \\ & k & \\ & & I_2\end{pmatrix}n\Big)dz=\Big(\int\limits_{|z|\leqslant q_\nu^M}\psi_\nu^{-1}(z)dz\Big)f\Big(\begin{pmatrix} I_2 & & \\ & k & \\ & & I_2\end{pmatrix}n\Big)=0.$$

Here we used that $\xi_{1,2}(x_1(z)) = \mathrm{id}$. This shows that $J_{N_2(F_\nu),(\psi_\nu)_{1,1}}(\Gamma_{1,2}) = 0$.

Let $f \in S_2$. We have to show that

$$(4.43) \qquad \int\limits_{N_2(\mathcal{P}_\nu^{-M})} (\psi_\nu)^{-1}_{2,-1}(n) f\Big(x_2(t)\begin{pmatrix} w & & \\ & k & \\ & & w^* \end{pmatrix} x_1(b)n\Big)dn = 0,$$

where $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. As before, we may assume that $t = b = 0$. Now consider the subintegration on

$$y(u) = \begin{pmatrix} 1 & & & & \\ & 1 & u & * & \\ & & I_4 & u' & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

The corresponding du-integration (with $b = 0$) is

$$\int\limits_{u\in(\mathcal{P}_\nu^{-M})^4} \psi^{-1}(u_2 - u_3) f\Bigg(\begin{pmatrix} 1 & 0 & uk^{-1} & 0 & * \\ & 1 & 0 & 0 & 0 \\ & & I_4 & 0 & ku' \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}\begin{pmatrix} w & & \\ & k & \\ & & w^* \end{pmatrix} n\Bigg)du$$

$$= \int\limits_{u\in(\mathcal{P}_\nu^{-M})^4} \psi^{-1}(u_2 - u_3)du \cdot f\Big(\begin{pmatrix} w & & \\ & k & \\ & & w^* \end{pmatrix} n\Big) = 0.$$

This proves that $J_{N_2(F_\nu),(\psi_\nu)_{2-1}}(S_2) = 0$. Similar arguments imply that r cannot be 1 or 2. Thus, $r = 0$. Similar arguments imply also that for $r = 0$, s cannot be 0 or 2. Put $w_{0,1} = w_1$. Then

$$(4.44) \qquad w_1^{-1}Q_2w_1 \cap Q_2 = \left\{\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & x_{14} & 0 & 0 \\ a_{21} & a_{22} & 0 & x_{22} & x_{23} & x_{24} & y & 0 \\ & & b_{11} & b_{12} & b_{13} & b_{14} & x'_{24} & x'_{14} \\ & & & c_{11} & c_{12} & b'_{13} & x'_{23} & 0 \\ & & & c_{21} & c_{22} & b'_{12} & x'_{22} & 0 \\ & & & & & b_{11}^{-1} & 0 & 0 \\ & & & & & & a_{22}^{-1} & 0 \\ & & & & & & a'_{21} & a_{11}^{-1} \end{pmatrix} \in H\right\} := L_1$$

The representation $\xi_1 = ((\eta \otimes \pi)\cdot \delta_{Q_2}^{1/2})^{w_1}$ takes an element of the form (4.44) to

$$(4.45) \qquad \Big|\frac{b_{11}}{a_{11}}\Big|^{5/2}\mu_1\Big(\frac{b_{11}}{a_{11}}\Big)\pi^\omega\begin{pmatrix} a_{22} & x_{23} & x_{22} & y \\ & c_{22} & c_{21} & x'_{22} \\ & c_{12} & c_{11} & x'_{23} \\ & & & a_{22}^{-1} \end{pmatrix}.$$

Here $\omega = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$.

As before, we consider appropriate analogs S'_1, S'_2 of the spaces S_1, S_2, and it remains to show that

$$(4.46) \qquad \int\limits_{N_2(\mathcal{P}_\nu^{-M})} (\psi_\nu)^{-1}_{2,-1}(n) f\Big(\begin{pmatrix} \alpha_i & & \\ & k & \\ & & \alpha_i^* \end{pmatrix} n\Big)dn = 0,$$

for $k \in \mathrm{SO}_4(\mathcal{O}_{F_\nu})$, and $\alpha_1 = I_2, \alpha_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$; f is in S'_1, S'_2 (respectively). In case $i = 2$, we consider the subintegration on $x_1(z)$, $|z| \leqslant q_\nu^M$, and we get $\big(\int_{|z|\leqslant q_\nu^M} \psi_\nu^{-1}(z)dz\big)\cdot$

$f\Big(\Big(\begin{smallmatrix}\alpha_2 & k & \\ & & \alpha_2^*\end{smallmatrix}\Big)n\Big)=0$. In case $i=1$, again consider $y(u)$, and the subintegration

$$(4.47)\qquad \int\limits_{u\in(\mathcal{P}_\nu^{-M})^4}\psi_\nu^{-1}(u_2-u_3)f\left(\left(\begin{smallmatrix}I_2 & k\\ & I_2\end{smallmatrix}\right)y(u)n\right)du$$

$$=\int\limits_{u\in(\mathcal{P}_\nu^{-M})^4}\psi_\nu^{-1}(u_2-u_3)f\Big(y(uk^{-1})\left(\begin{smallmatrix}I_2 & k\\ & I_2\end{smallmatrix}\right)n\Big)du.$$

Now take in (4.47) the subintegration on $u=(0,u_2,u_3,u_4)k$, $|u_i|\leqslant q_\nu^M$. We get

$$(4.48)\qquad \int\limits_{|u_i|\leqslant q_\nu^M}\psi_\nu^{-1}\Big(u\cdot k\Big(\begin{smallmatrix}0\\1\\-1\\0\end{smallmatrix}\Big)\Big)\pi^\omega\begin{pmatrix}1 & u_3 & u_2 & *\\ & 1 & 0 & -u_2\\ & & 1 & -u_3\\ & & & 1\end{pmatrix}f\Big(\Big(\begin{smallmatrix}I_2 & k\\ & I_2\end{smallmatrix}\Big)n\Big)du.$$

We must have $k\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}=\begin{pmatrix}*\\ *\\ *\\0\end{pmatrix}$, otherwise the du_4-integration results in zero. For such k, the vanishing of (4.48) follows from the fact that, by induction, $\pi=\mathrm{Ind}_{Q_2'(F_\nu)}^{\mathrm{SO}_4(F_\nu)}\mu_2\circ\det$ has zero Jacquet modules with respect to $N_2'=\left\{\left(\begin{smallmatrix}1 & v & *\\ & I & v'\\ & & 1\end{smallmatrix}\right)\in\mathrm{SO}_4\right\}$, and characters $\left(\begin{smallmatrix}1 & v & *\\ & I_2 & v'\\ & & 1\end{smallmatrix}\right)\mapsto\psi(av_1-a^{-1}v_2)$ (which in this case is easy to see, since these are Whittaker characters). This completes the proof of Proposition 18. $\square$

4.5. Unramified parameters of $\sigma_\psi(\tau)$: Case $H=\mathrm{SO}_8, G=\mathrm{SO}_5$ and τ on $\mathrm{GL}_4(\mathbb{A}_F)$. — We keep the notation of Section 4.4. From the explanations at the beginning of Section 4.4, it is clear that the next proposition determines the unramified parameters of (any summand of) $\sigma_\psi(\tau)$ at the place ν.

Proposition 19. — *We have an isomorphism of* $\mathrm{SO}_5(F_\nu)$*-modules*

$$J_{N_1(F_\nu),(\psi_\nu)_{1,-1}}\Big(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}}(\mu_1\circ\det\otimes\mu_2\circ\det)\Big)\cong\mathrm{Ind}_{B_\nu}^{\mathrm{SO}_5(F_\nu)}\mu_1\otimes\mu_2$$

Here B is the standard Borel subgroup of SO_5.

Proof. — The method is the same as in Section 4.4. Again consider $\eta=\mu_1\circ\det$ on $\mathrm{GL}_2(F_\nu)$ and $\pi=\mathrm{Ind}_{Q_2'(F_\nu)}^{\mathrm{SO}_4(F_\nu)}\mu_2\circ\det$. Let Q_1 be the standard parabolic subgroup of H which preserves an (isotropic) line. We analyze $\mathrm{Res}_{Q_1(F_\nu)}\Big(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}}\eta\otimes\pi\Big)$ using Bruhat theory. So consider $Q_2\backslash H/Q_1$. Identify, as in Sec.4.3, $Q_2\backslash H\cong Y_2$. The orbits of Q_1 in Y_2 are determined by $f=\dim(X\cap X^{(1)})$, and $s=\dim(X\cap(X^{(1)})^\perp)$, $X\in Y_2$. Here $X^{(1)}=Fe_1$. Note that $0\leqslant r\leqslant 1\leqslant s\leqslant 2$.

If $r=1$, then $e_1\in X$, and since X is isotropic, we get that $X\subset(X^{(1)})^\perp$, and so $s=2$. Thus, we may take as a representative $X=X^{(2)}$. The corresponding subquotient of $\mathrm{Res}_{Q_1(F_\nu)}\Big(\mathrm{Ind}_{Q_2(F_\nu)}^{H_{F_\nu}}\eta\otimes\pi\Big)$ is

$$T_{1,2}=\mathrm{Ind}^{c\,Q_1(F_\nu)}_{(Q_1\cap Q_2)(F_\nu)}((\eta\otimes\pi)\cdot\delta_{Q_2}^{1/2}\delta^{-1/2}.$$

We have

$$(4.49) \qquad Q_1 \cap Q_2 = \left\{ \begin{pmatrix} a_1 & * & * & * & * \\ & a_2 & * & * & * \\ & & b & * & * \\ & & & a_2^{-1} & * \\ & & & & a_1^{-1} \end{pmatrix} \in H \;\middle|\; b \in \mathrm{SO}_4 \right\},$$

and $(\eta \otimes \pi) \cdot \delta_{Q_2}^{1/2}$ takes an element of the form (4.49) to

$$|a_1 a_2|^{5/2} \mu_1(a_1 a_2) \pi(b).$$

Clearly, for f in the space of $T_{1,2}$, and $M \gg 0$,

$$(4.50) \qquad \int_{N_1(\mathcal{P}_\nu^{-M})} (\psi_\nu)_{1,-1}^{-1}(n) f\Big(\begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix} n \Big) dn = 0,$$

for any $k \in \mathrm{SO}_6(F_\nu)$. Indeed, $f\Big(\begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix} n \Big) = f \begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix}$, for any $n \in N_1(F_\nu)$. This shows that $J_{N_1(F_\nu),(\psi_\nu)_{1,-1}}(T_{1,2}) = 0$. Thus, we may assume that $r = 0$. If $s = 2$, we may take the representative $X = \mathrm{Span}\{e_2, e_3\}$. The corresponding representative in $Q_2 \backslash H / Q_1$ can be taken to be $w_2 = \begin{pmatrix} & I_3 & \\ 1 & & \\ & & 1 \\ & I_3 & \end{pmatrix}$ (so that $w_2^{-1} X^{(2)} = X$).

Let $T_2 = \mathrm{Ind}^{c\, Q_1(F_\nu)}_{w_2^{-1} Q_2(F_\nu) w_2 \cap Q_1(F_\nu)} ((\eta \otimes \pi) \delta_{Q_2}^{1/2})^{w_2} \delta^{-1/2}$. We have

$$(4.51) \qquad w_2^{-1} Q_2 w_2 \cap Q_1 = \left\{ \begin{pmatrix} a & 0 & x & z & e \\ & b & y & v & z' \\ & & c & y' & x' \\ & & & b^* & 0 \\ & & & & a^{-1} \end{pmatrix} \in H \;\middle|\; \begin{matrix} c \in \mathrm{SO}_2 \\ b \in \mathrm{GL}_2 \end{matrix} \right\}.$$

The representation $\xi_2 = ((\eta \otimes \pi) \delta_{Q_2}^{1/2})^{w_2}$ takes an element of the form (4.51) to

$$(4.52) \qquad |\det b|^{5/2} \mu_1(\det b) \pi^\omega \begin{pmatrix} a & x & e \\ & c & x' \\ & & a^{-1} \end{pmatrix},$$

where $\omega = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$. Consider, for f in the space of T_2, $M \gg 0$, and $k \in \mathrm{SO}_6(\mathcal{O}_\nu)$,

$$(4.53) \qquad \int_{N_1(\mathcal{P}_\nu^{-M})} (\psi_\nu)_{1,-1}^{-1}(n) f\Big(\begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix} n \Big) dn.$$

Consider the subintegration of (4.53) on $n(v) = \begin{pmatrix} 1 & v & * \\ & I_6 & v' \\ & & 1 \end{pmatrix}$, where $v = (0, 0, u_3, \ldots, u_6)k$, $|u_i| \leqslant q_\nu^M$. By (4.52), we get

$$(4.54) \qquad \int_{|u_i| \leqslant q_\nu^M} \psi_\nu^{-1}\left((0,0,u_3,\ldots,u_6) k \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \right) \pi^\omega \begin{pmatrix} 1 & u_3 & u_4 & -u_3 u_4 \\ & 1 & 0 & -u_4 \\ & & 1 & -u_3 \\ & & & 1 \end{pmatrix} f\Big(\begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix} n \Big) du.$$

We must have

$$k \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ * \\ 0 \\ 0 \end{pmatrix},$$

otherwise the $d(u_5, u_6)$- integration results in zero. For such k,

$$k \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ a \\ -a^{-1} \\ 0 \\ 0 \end{pmatrix}, \quad |a| = 1.$$

Thus (4.54) becomes (up to q_ν^{2M})

$$\int_{|u_i| \leqslant q_\nu^M} \psi_\nu^{-1}(au_3 - a^{-1}u_4)\pi^\omega \begin{pmatrix} 1 & u_3 & u_4 & -u_3u_4 \\ & 1 & 0 & -u_4 \\ & & 1 & -u_3 \\ & & & 1 \end{pmatrix} f\left(\begin{pmatrix} 1 & & \\ & k & \\ & & 1 \end{pmatrix} n\right) d(u_3, u_4), \tag{4.55}$$

which is zero for M large enough, exactly as in the end of Sec. 4.3. (This is a place to apply induction. Recall that $\pi = \mathrm{Ind}_{Q_2'(F_\nu)}^{\mathrm{SO}_4(F_\nu)} \mu_2 \circ \det$.) Note that k, n, a may be taken in compact sets, which depend on f only. Finally, let $r = 0$, $s = 1$. Here, a corresponding representative is $w_1 = \begin{pmatrix} & & 1 \\ & I_6 & \\ 1 & & \end{pmatrix} \begin{pmatrix} I_3 & & & \\ & & 1 & \\ & 1 & & \\ & & & I_3 \end{pmatrix}$. Let

$$T_1 = \mathrm{Ind}^{c\,Q_1(F_\nu)}_{w_1^{-1}Q_2(F_\nu)w_1 \cap Q_1(F_\nu)} ((\eta \otimes \pi) \cdot \delta_{Q_2}^{1/2})^{w_1} \delta^{-1/2}.$$

We have

$$w_1^{-1}Q_2 w_1 \cap Q_1 = \left\{ \begin{pmatrix} a & 0 & 0 & x & 0 \\ & b & y & z & x' \\ & & c & y' & 0 \\ & & & b^{-1} & 0 \\ & & & & a^{-1} \end{pmatrix} \in H \;\middle|\; c \in \mathrm{SO}_4 \right\}. \tag{4.56}$$

The representation $\xi_1 = ((\eta \otimes \pi) \cdot \delta_{Q_2}^{1/2})^{w_1}$ takes an element of the form (4.56) to

$$\left|\frac{b}{a}\right|^{5/2} \mu_1\left(\frac{b}{a}\right) \pi^\varepsilon(c), \tag{4.57}$$

where $\pi^\varepsilon(c) = \pi(\varepsilon c \varepsilon^{-1})$, $\varepsilon = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$. Using the same methods as before, we prove

$$J_{N_1(F_\nu),(\psi_\nu)_{1,-1}}(T_1) \cong \mathrm{Ind}_{Q_1'(F_\nu)}^{\mathrm{SO}_5(F_\nu)} \mu_1 \otimes \pi^\varepsilon|_{\mathrm{SO}_3(F_\nu)},$$

where Q_1' is the standard parabolic subgroup of SO_5, which preserves an isotropic line. Finally, it is easy to see that $\pi^\varepsilon|_{\mathrm{SO}_3(F_\nu)} \cong \mathrm{Ind}_{B_\nu'}^{\mathrm{SO}_3(F_\nu)} \mu_2$, for $\pi = \mathrm{Ind}_{Q_2'(F_\nu)}^{\mathrm{SO}_4(F_\nu)} \mu_2 \circ \det$. Here B' is the standard Borel subgroup of SO_3. This completes the proof of Proposition 19. □

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