Astérisque

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Astérisque, tome 298 (2005), p. 271-318

<http://www.numdam.org/item?id=AST_2005__298__271_0>

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A GUIDE TO THE REDUCTION MODULO p OF SHIMURA VARIETIES

by

Michael Rapoport

Abstract. — This is a survey of recent work on the reduction of Shimura varieties with parahoric level structures.

***Résumé* (Un guide à la réduction modulo p des variétés de Shimura).** — Cet article est un survol de resultats sur la réduction des variétés de Shimura à structure de niveau parahorique.

This report is based on my lecture at the Langlands conference in Princeton in 1996 and the series of lectures I gave at the semestre Hecke in Paris in 2000. In putting the notes for these lectures in order, it was my original intention to give a survey of the activities in the study of the reduction of Shimura varieties. However, I realized very soon that this task was far beyond my capabilities. There are impressive results on the reduction of "classical" Shimura varieties, like the Siegel spaces or the Hilbert-Blumenthal spaces, there are deep results on the reduction of specific Shimura varieties and their application to automorphic representations and modular forms, and to even enumerate all these achievements of the last few years in one report would be very difficult. Instead, I decided to concentrate on the reduction modulo p of Shimura varieties *for a parahoric level structure* and more specifically on those aspects which have a *group-theoretic interpretation.* Even in this narrowed down focus it was not my aim to survey all results in this area but rather to serve as a guide to those problems with which I am familiar, by putting some of the existing literature in its context and by pointing out unsolved questions. These questions or conjectures are of two different kinds. The first kind are open even for those Shimura varieties which are moduli spaces of abelian varieties. Surely these conjectures are the most urgent

2000 *Mathematics Subject Classification.* — 11G18, 11G40, 14G35.
Key words and phrases. — Parahoric subgroups, μ-admissible and μ-permissible set, affine Deligne-Lusztig varieties, semi-simple zeta function.

and the most concrete and the most tractable. The second kind are known for these special Shimura varieties. Here the purpose of the conjectures resp. questions is to extend these results to more general cases, *e.g.* to Shimura varieties of Hodge type.

As a general rule, I wish to stress that I would not be surprised if some of the conjectures stated here turn out to be false, especially in cases of very bad ramification. But I believe that even in these cases I should not be too far off the mark, and that a suitable modification of these conjectures gives the correct answer. My motivation in running the risk of stating precise conjectures is that I wanted to point out directions of investigation which seem promising to me.

The guiding principle of the whole theory presented here is to give a group-theoretical interpretation of phenomena found in special cases in a formulation which makes sense for a general Shimura variety. This is illustrated in the first section which treats some aspects of the elliptic modular case from the point of view taken in this paper. The rest of the article consists of two parts, the local theory and the global theory. Their approximate contents may be inferred from the table of contents below.

I should point out that the development in these notes is very uneven and that sometimes I have gone into the nitty gritty detail, whereas at other times I only give a reference for further developments. My motivation for this is that I wanted to give a real taste of the whole subject — in the hope that it is attractive enough for a student, one motivated enough to read on and skip parts which he finds unappealing.

In conclusion, I would like to stress, as in the introduction of [**R2**], the influence of the ideas of V. Drinfeld, R. Kottwitz, R. Langlands and T. Zink on my way of thinking about the circle of problems discussed here. In more recent times I also learned enormously from G. Faltings, A. Genestier, U. Görtz, T. Haines, J. de Jong, E. Landvogt, G. Laumon, B.C. Ngô, G. Pappas, H. Reimann, H. Stamm, and T. Wedhorn, but the influence of R. Kottwitz continued to be all-important. I am happy to express my gratitude to all of them. I also thank T. Ito, R. Kottwitz and especially T. Haines for their remarks on a preliminary version of this paper. I am grateful to the referee for his careful reading of the paper and his helpful remarks.

During the semestre Hecke I benefited from the financial support of the Ministère de la Recherche and the Humboldt Foundation (Prix Gay-Lussac/Humboldt).

Contents

1. Motivation: The elliptic modular curve

In this section we illustrate the problem of describing the reduction modulo p of a Shimura variety in the simplest case. Let $\boldsymbol{G} = GL_2$ and let $(\boldsymbol{G}, \{h\})$ be the usual Shimura datum. Let $\mathbf{K} \subset \boldsymbol{G}(\mathbf{A}_f)$ be an open compact subgroup of the form $\mathbf{K} = K^p \cdot K_p$ where K^p is a sufficiently small open compact subgroup of $\boldsymbol{G}(\mathbf{A}_f^p)$. Let $G = \boldsymbol{G} \otimes_{\mathbf{Q}} \mathbf{Q}_p$. We consider the cases where K_p is one of the following two parahoric subgroups of $G(\mathbf{Q}_p)$,

(i) $K_p = K_p^{(i)} = GL_2(\mathbf{Z}_p)$ (*hyperspecial maximal parahoric*)
(ii) $K_p = K_p^{(ii)} = \{g \in GL_2(\mathbf{Z}_p);\ g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\}$ (*Iwahori*)

The corresponding Shimura variety $\mathrm{Sh}(\boldsymbol{G}, h)_{\mathbf{K}}$ is defined over $\mathbf{Q}$. It admits a model $\mathrm{Sh}(G, h)_{\mathbf{K}}$ over $\operatorname{Spec} \mathbf{Z}_{(p)}$ by posing the following moduli problem over $(Sch/\mathbf{Z}_{(p)})$:

(i) an elliptic curve E with a level-K^p-structure.
(ii) an isogeny of degree p of elliptic curves $E_1 \to E_2$, with a level-K^p-structure.

The description of the point set $\mathrm{Sh}(G, h)_{\mathbf{K}}(\overline{\mathbf{F}}_p)$ takes in both cases (i) and (ii) the following form,

$$\mathrm{Sh}(G, h)_{\mathbf{K}}(\overline{\mathbf{F}}_p) = \coprod_{\varphi} I_\varphi(\mathbf{Q}) \backslash X(\varphi)_{K_p} \times X^p / K^p. \tag{1.1}$$

Here the sum ranges over the isogeny classes of elliptic curves and $I_\varphi(\mathbf{Q}) = \mathrm{End}_{\mathbf{Q}}(E)^\times$ is the group of self-isogenies of any element of this isogeny class. Furthermore, X^p/K^p may be identified with $\boldsymbol{G}(\mathbf{A}_f^p)/K^p$, with the action of $I_\varphi(\mathbf{Q})$ defined by the ℓ-adic representation afforded by the rational Tate module. The set $X(\varphi)_{K_p}$ is the most interesting ingredient.

Let $\mathcal{O} = W(\overline{\mathbf{F}}_p)$ be the ring of Witt vectors over $\overline{\mathbf{F}}_p$ and $L = \operatorname{Fract} \mathcal{O}$ be its fraction field. We denote by σ the Frobenius automorphism of L. Let N denote the rational Dieudonné module of E. Then N is a 2-dimensional L-vector space, equipped with a σ-linear bijective endomorphism F (the crystalline Frobenius). Then in case (i)

(hyperspecial case), the set $X(\varphi)_{K_p^{(i)}}$ has the following description

$$\begin{aligned} X(\varphi)_{K_p^{(i)}} &= \{\Lambda;\ p\Lambda \subsetneqq F\Lambda \subsetneqq \Lambda\} \\ &= \{\Lambda; \mathrm{inv}(\Lambda, F\Lambda) = \mu\}. \end{aligned} \tag{1.2}$$

Here Λ denotes a $\mathcal{O}$-lattice in N. The set of $\mathcal{O}$-lattices in N may be identified with $G(L)/G(\mathcal{O})$. We have used the elementary divisor theorem to establish an identification

$$\mathrm{inv} : G(L)\backslash[G(L)/G(\mathcal{O}) \times G(L)/G(\mathcal{O})] = G(\mathcal{O})\backslash G(L)/G(\mathcal{O}) \simeq \mathbf{Z}^2/S_2. \tag{1.3}$$

Furthermore $\mu = (1,0) \in \mathbf{Z}^2/S_2$ is the conjugacy class of one-parameter subgroups associated to $\{h\}$.

In case (ii) (Iwahori case), the set $X(\varphi)_{K_p^{(ii)}}$ has the following description,

$$X(\varphi)_{K_p^{(ii)}} = \{p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2;\ p\Lambda_1 \subsetneqq F\Lambda_1 \subsetneqq \Lambda_1, p\Lambda_2 \subsetneqq F\Lambda_2 \subsetneqq \Lambda_2\}. \tag{1.4}$$

Here again Λ_1, Λ_2 denote $\mathcal{O}$-lattices in N.

In either case $X(\varphi)_{K_p}$ is equipped with an operator Φ which under the bijection (1.1) corresponds to the action of the Frobenius automorphism on the left hand side.

Let us describe the set $X(\varphi)_{K_p^{(ii)}}$ in the manner of the second line of (1.2). The analogue in this case of the relative position of two chains of inclusions of $\mathcal{O}$-lattices in N, $p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2$ and $p\Lambda_2' \subsetneqq \Lambda_1' \subsetneqq \Lambda_2'$ is given by the identification analogous to (1.3),

$$\mathrm{inv} : G(L)\backslash[G(L)/G_0(\mathcal{O}) \times G(L)/G_0(\mathcal{O})] = G_0(\mathcal{O})\backslash G(L)/G_0(\mathcal{O}) \xrightarrow{\sim} \mathbf{Z}^2 \rtimes S_2. \tag{1.5}$$

Here $G_0(\mathcal{O})$ denotes the standard Iwahori subgroup of $G(\mathcal{O})$ and on the right appears the extended affine Weyl group $\widetilde{W}$ of GL_2. It is now a pleasant exercise in the Bruhat-Tits building of PGL_2 to see that

$$\begin{aligned} &\{p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2,\ p\Lambda_2' \subsetneqq \Lambda_1' \subsetneqq \Lambda_2';\ p\Lambda_1 \subsetneqq \Lambda_1' \subsetneqq \Lambda_1,\ p\Lambda_2 \subsetneqq \Lambda_2' \subsetneqq \Lambda_2\} \\ &\qquad = \{(g,g') \in (G(L)/G_0(\mathcal{O}))^2;\ \mathrm{inv}(g,g') \in \mathrm{Adm}(\mu)\}. \end{aligned} \tag{1.6}$$

Here $\mathrm{Adm}(\mu)$ is the following subset of $\widetilde{W}$,

$$\mathrm{Adm}(\mu) = \{t_{(1,0)}, t_{(0,1)}, t_{(1,0)} \cdot s\}. \tag{1.7}$$

Here $t_{(1,0)}$ and $t_{(0,1)}$ denote the translation elements in $\widetilde{W} = \mathbf{Z}^2 \rtimes S_2$ corresponding to (1,0) resp. (0,1) in $\mathbf{Z}^2$, and s denotes the non-trivial element in S_2.

For $w \in \widetilde{W}$ and any σ-linear automorphism F of N, let us introduce the *affine Deligne-Lusztig variety*,

$$X_w(F) = \{g \in G(L)/G_0(\mathcal{O});\ \mathrm{inv}(g, Fg) = w\}. \tag{1.8}$$

Then we may rewrite (1.4) in the following form, where F_φ denotes the crystalline Frobenius associated to φ,

$$X(\varphi)_{K_p^{(ii)}} = \bigcup_{w \in \mathrm{Adm}(\mu)} X_w(F_\varphi). \tag{1.9}$$

This is analogous to the second line in (1.2) which may be viewed as a *generalized* affine Deligne-Lusztig variety corresponding to the hyperspecial parahoric $K_p^{(i)}$. It should be pointed out that in this special case the union (1.9) is spurious: only one of the summands is non-empty, for a fixed isogeny class φ. For more general Shimura varieties this is no longer true.

The model $\mathrm{Sh}(G,h)_{\mathbf{K}}$ is smooth over $\operatorname{Spec}\mathbf{Z}_{(p)}$ in the hyperspecial case, but it has bad reduction in the Iwahori case. In the latter case there is the famous picture of the special fiber where two hyperspecial models meet at the supersingular points.

Such a global picture is not known in more general cases. The nature of the singularities in the special fiber in the Iwahori case can be understood in terms of the associated *local model*.

We consider the lattice chain $p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2$ in $\mathbf{Q}_p^2$, where $\Lambda_2 = \mathbf{Z}_p^2$ and $\Lambda_1 = p\mathbf{Z}_p \oplus \mathbf{Z}_p$. Let $\mathcal{M}^{\mathrm{loc}}(G,\mu)_{K_p^{(ii)}}$ be the join of $\mathbf{P}(\Lambda_1)$ and $\mathbf{P}(\Lambda_2)$ over $\mathbf{Z}_p$ (= scheme-theoretic closure of the common generic fiber $\mathbf{P}^1_{\mathbf{Q}_p}$ in $\mathbf{P}(\Lambda_1) \times_{\operatorname{Spec}\mathbf{Z}_p} \mathbf{P}(\Lambda_2)$). Then we obtain a diagram of schemes over $\operatorname{Spec}\mathbf{Z}_p$

$$\begin{array}{ccc} \widetilde{\mathrm{Sh}}(G,h)_{\mathbf{K}} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\widetilde{\lambda}} & \\ \mathrm{Sh}(G,h)_{\mathbf{K}} & & \mathcal{M}^{\mathrm{loc}}(G,\mu)_{K_p^{(ii)}}. \end{array} \tag{1.10}$$

Here π is the principal homogeneous space under the group scheme $\mathcal{G}$ over $\operatorname{Spec}\mathbf{Z}_p$ attached to $K_p^{(ii)}$ (with $\mathcal{G}(\mathbf{Z}_p) = K_p^{(ii)}$), which adds to the isogeny of degree p, $E_1 \to E_2$ over a base scheme S, and its level-K^p-structure, a trivialization of the DeRham homology modules,

$$\begin{array}{ccccc} H_{\mathrm{DR}}(E_1) & \longrightarrow & H_{\mathrm{DR}}(E_2) & \longrightarrow & H_{\mathrm{DR}}(E_1) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \Lambda_1 \otimes_{\mathbf{Z}_p} \mathcal{O}_S & \longrightarrow & \Lambda_2 \otimes_{\mathbf{Z}_p} \mathcal{O}_S & \longrightarrow & \Lambda_1 \otimes_{\mathbf{Z}_p} \mathcal{O}_S. \end{array} \tag{1.11}$$

The morphism $\widetilde{\lambda}$ is given by the Hodge filtration of the DeRham homology.

The diagram (1.10) can be interpreted as a relatively representable morphism of algebraic stacks

$$\lambda : \mathrm{Sh}(G,h)_{\mathbf{K}} \longrightarrow [\mathcal{M}^{\mathrm{loc}}(G,\mu)_{K_p^{(ii)}}/\mathcal{G}]. \tag{1.12}$$

This morphism λ is smooth of relative dimension $\dim\mathcal{G} = 4$. The analogue of λ in the hyperspecial case is a smooth morphism of relative dimension 4,

$$\lambda : \mathrm{Sh}(G,h)_{\mathbf{K}} \longrightarrow [\mathbf{P}^1_{\mathbf{Z}_p}/GL_{2\mathbf{Z}_p}]. \tag{1.13}$$

At this point we have met in this special case all the main actors which will appear in the sequel: the admissible subset of the extended affine Weyl group, affine Deligne-Lusztig varieties, the sets $X(\varphi)_{K_p}$ (later denoted by $X(\mu,b)_K$), local models

etc. These definitions can be given purely in terms of the p-adic group GL_2 and its parahoric subgroup K_p. This will be the subject matter of the local part (sections 2–6). On the other hand, the enumeration of isogeny classes and the description (1.1) of the points in the reduction are global problems. These are addressed in the global part (sections 7–10).

We conclude this section with the definition of a *Shimura variety of PEL-type*. The guiding principle of the theory is to investigate the moduli problems related to them and then to express these findings in terms of the Shimura data associated to them.

Let B denote a finite-dimensional semi-simple $\mathbf{Q}$-algebra, let $*$ be a positive involution on B, let $V \neq (0)$ be a finitely generated left B-module and let $\langle\ ,\ \rangle$ be a non-degenerate alternating bilinear form $\langle\ ,\ \rangle : V \times V \to \mathbf{Q}$ of the underlying $\mathbf{Q}$-vector space such that $\langle bv, w\rangle = \langle v, b^* w\rangle$ for all $v, w \in V$, $b \in B$. We denote by G the group of B-linear symplectic similitudes of V. This is an algebraic group $\boldsymbol{G}$ over $\mathbf{Q}$. We assume that $\boldsymbol{G}$ is a *connected*, hence reductive, algebraic group (this essentially excludes the orthogonal case). We let $h : \mathbf{C}^\times \to \boldsymbol{G}(\mathbf{R})$ be an algebraic homomorphism which defines on $V_{\mathbf{R}}$ a Hodge structure of type $(-1,0) + (0,-1)$ and which satisfies the usual Riemann conditions with respect to $\langle\ ,\ \rangle$, compare [**W**], 1.3.1. These data define by Deligne a Shimura variety $\mathrm{Sh}(\boldsymbol{G}, h)$ over the Shimura field $\mathbf{E}$.

We now fix a prime number p. Let $G = \boldsymbol{G} \otimes_{\mathbf{Q}} \mathbf{Q}_p$. We consider an order O_B of B such that $O_B \otimes \mathbf{Z}_p$ is a maximal order of $B \otimes \mathbf{Q}_p$. We assume that $O_B \otimes \mathbf{Z}_p$ is invariant under the involution $*$. We also fix a multichain $\mathcal{L}$ of $O_B \otimes \mathbf{Z}_p$-lattices in $V \otimes \mathbf{Q}_p$ which is self-dual for $\langle\ ,\ \rangle$, [**RZ2**]. Let $K = K_p$ be the stabilizer of $\mathcal{L}$ in $G(\mathbf{Q}_p)$. Most of the time K is a parahoric subgroup.

Finally we fix an open compact subgroup $K^p \subset \boldsymbol{G}(\mathbf{A}_f^p)$, which will be assumed sufficiently small. Let $\mathbf{K} = K^p \cdot K_p$.

We fix embeddings $\overline{\mathbf{Q}} \to \mathbf{C}$ and $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. We denote by $\boldsymbol{p}$ the corresponding place of $\mathbf{E}$ over p and by $E = \mathbf{E}_{\boldsymbol{p}}$ the completion and by κ the residue field of $\mathcal{O}_E$. We then define a moduli problem $\mathrm{Sh}(G, h)_{\mathbf{K}}$ over $\operatorname{Spec} \mathcal{O}_E$, *i.e.*, a set-valued functor, as follows. It associates to a scheme S over $\mathcal{O}_E$ the following data up to isomorphism ([**RZ2**], 6.9).

(1) An $\mathcal{L}$-set of abelian varieties up to prime-to-p isogeny, $A = \{A_\Lambda;\ \Lambda \in \mathcal{L}\}$.
(2) A $\mathbf{Q}$-homogeneous principal polarization $\overline{\lambda}$ of the $\mathcal{L}$-set A.
(3) A K^p-level structure

$$\overline{\eta} : H_1(A, \mathbf{A}_f^p) \simeq V \otimes \mathbf{A}_f^p \mod K^p,$$

which respects the bilinear forms on both sides up to a constant in $(\mathbf{A}_f^p)^\times$.

We require an identity of characteristic polynomials for each $\Lambda \in \mathcal{L}$,

$$\mathrm{char}(b; \mathrm{Lie}\, A_\Lambda) = \mathrm{char}(b; V_h^{0,-1}), \quad b \in O_B.$$

This moduli problem is representable by a quasi-projective scheme whose generic fiber is the initial Shimura variety $\mathrm{Sh}(\boldsymbol{G}, h)_{\mathbf{K}}$ (or at least a finite union of isomorphic copies of $\mathrm{Sh}(\boldsymbol{G}, h)_{\mathbf{K}}$).

However, contrary to the optimistic conjecture in [**RZ2**], this does not always provide us with a good integral model of the Shimura variety, *e.g.* flatness may fail. However, if the center of B is a product of field extensions which are unramified at p and excluding the orthogonal case as above, then $\mathrm{Sh}(G, h)_{\mathbf{K}}$ is a good integral model of the Shimura variety [**G1**], [**G2**]. For most of the remaining cases there is a closed subscheme of the above moduli space which is a good model [**PR1**], [**PR2**]. However, these closed subschemes cannot be defined in terms of the moduli problem of abelian varieties. Still, they can be analyzed and can serve as an experimental basis for the predictions which are the subject of this report.

PART I
LOCAL THEORY

2. Parahoric subgroups

Let G be a connected reductive group over a complete discretely valued field L with algebraically closed residue field. Kottwitz [**K4**] defines a functorial surjective homomorphism

$$\widetilde{\kappa}_G : G(L) \longrightarrow X^*(\widehat{Z}(G)^I). \tag{2.1}$$

Here $I = \mathrm{Gal}(\overline{L}/L)$ denotes the absolute Galois group of L and $\widehat{Z}(G)$ denotes the center of the Langlands dual group. For instance, if $G = GL_n$, then the target group is $\mathbf{Z}$ and $\widetilde{\kappa}_G(g) = \operatorname{ord} \det g$; if $G = GSp_{2n}$, then again the target group is $\mathbf{Z}$ and $\widetilde{\kappa}_G(g) = \operatorname{ord} c(g)$, where $c(g) \in L^\times$ is the multiplier of the symplectic similitude g.

Let $\mathcal{B} = \mathcal{B}(G_{\mathrm{ad}}, L)$ denote the Bruhat-Tits building of the adjoint group over L. Then $G(L)$ acts on $\mathcal{B}$.

Definition 2.1. — The *parahoric subgroup associated to a facet* $\boldsymbol{a}$ of $\mathcal{B}$ is the following subgroup of $G(L)$,

$$K_{\boldsymbol{a}} = \mathrm{Fix}(\boldsymbol{a}) \cap \mathrm{Ker}\, \widetilde{\kappa}_G.$$

If $\boldsymbol{a}$ is a maximal facet, *i.e.* an alcove, then the parahoric subgroup is called an *Iwahori subgroup.*

Remarks 2.2

(i) We have

$$K_{g\boldsymbol{a}} = gK_{\boldsymbol{a}}g^{-1}, \quad g \in G(L).$$

In particular, since all alcoves are conjugate to each other, all Iwahori subgroups are conjugate.

(ii) This notion of a parahoric subgroup coincides with the one by Bruhat and Tits [**BT2**], 5.2.6., *cf.* [**HR**]. Let $\mathcal{O}_L$ be the ring of integers in L. There exists a smooth group scheme $\mathcal{G}_a$ over $\operatorname{Spec}\mathcal{O}_L$, with generic fiber equal to G and with connected special fiber such that

$$K_a = \mathcal{G}_a(\mathcal{O}_L).$$

(iii) Let $G = T$ be a torus. Then there is precisely one parahoric subgroup K of $T(L)$. Then

$$K = \mathcal{T}^0(\mathcal{O}_L).$$

Here $\mathcal{T}^0$ denotes the identity component of the Néron model of T. For the group of connected components of the special fiber of the Néron model one has[(1)]

$$\pi_0(\overline{T}) = X_*(T)_I.$$

2.3. Let S be a maximal split torus in G and T its centralizer. Since by Steinberg's theorem G is quasi-split, T is a maximal torus. Let $N = N(T)$ be the normalizer of T. Let

$$\tilde{\kappa}_T : T(L) \longrightarrow X^*(\hat{T}^I) = X_*(T)_I \tag{2.2}$$

be the Kottwitz homomorphism associated to T and let $T(L)_1$ be its kernel. The factor group

$$\widetilde{W} = N(L)/T(L)_1 \tag{2.3}$$

will be called the *Iwahori Weyl group associated to* S. It is an extension of the relative Weyl group

$$W_0 = N(L)/T(L). \tag{2.4}$$

Namely, we have an exact sequence induced by the inclusion of $T(L)_1$ in $T(L)$,

$$0 \longrightarrow X_*(T)_I \longrightarrow \widetilde{W} \longrightarrow W_0 \longrightarrow 1. \tag{2.5}$$

The reason for the name given to $\widetilde{W}$ comes from the following fact [**HR**].

Proposition 2.4. — *Let K_0 be the Iwahori subgroup associated to an alcove contained in the apartment associated to the maximal split torus S. Then*

$$G(L) = K_0 \cdot N(L) \cdot K_0,$$

and the map $K_0 n K_0 \mapsto n \in \widetilde{W}$ induces a bijection

$$K_0 \backslash G(L) / K_0 \simeq \widetilde{W}.$$

More generally, let K and K' be parahoric subgroups associated to facets contained in the apartment corresponding to S. Let

$$\widetilde{W}^K = (N(L) \cap K)/T(L)_1, \quad \textit{resp.} \ \widetilde{W}^{K'} = (N(L) \cap K')/T(L)_1.$$

(1) See Notes at the end, n° 1

Then

$$K\backslash G(L)/K' \simeq \widetilde{W}^K\backslash\widetilde{W}/\widetilde{W}^{K'}.$$

For the structure of $\widetilde{W}$ we have the following fact [**HR**].

Proposition 2.5. — *Let x be a special vertex in the apartment corresponding to S, and let K be the corresponding parahoric subgroup. The subgroup W^K of $\widetilde{W}$ projects isomorphically to the factor group W_0 and the exact sequence (2.5) presents $\widetilde{W}$ as a semidirect product,*

$$\widetilde{W} = W_0 \ltimes X_*(T)_I.$$

Sometimes for $\nu \in X_*(T)_I$ we write t_ν for the corresponding element of $\widetilde{W}$.

Let S_{sc} resp. T_{sc} resp. N_{sc} be the inverse images of $S\cap G_{\mathrm{der}}$ resp. $T\cap G_{\mathrm{der}}$ resp. $N\cap G_{\mathrm{der}}$ in the simply connected covering G_{sc} of the derived group G_{der}. Then S_{sc} is a maximal split torus of G_{sc}, and T_{sc} resp. N_{sc} is its centralizer resp. normalizer. Hence

$$W_a = N_{\mathrm{sc}}(L)/T_{\mathrm{sc}}(L)_1 \tag{2.6}$$

is the Iwahori Weyl group of G_{sc}. This group is called the *affine Weyl group associated to S*, for the following reason. Let us fix a special vertex x in the apartment corresponding to S. Then there exists a reduced root system ${}^x\Sigma$ such that Proposition 2.5 (applied to G_{sc} instead of G) presents W_a as the affine Weyl group associated (in the sense of Bourbaki) to ${}^x\Sigma$,

$$W_a = W({}^x\Sigma) \ltimes Q^\vee({}^x\Sigma), \tag{2.7}$$

cf. [**T**], 1.7, compare also [**HR**]. In other words, we have an identification $W_0 \simeq W({}^x\Sigma)$ and $X_*(T_{\mathrm{sc}})_I \simeq Q^\vee({}^x\Sigma)$ compatibly with the semidirect product decompositions (2.7) and Proposition 2.5. In particular, W_a is a Coxeter group.

There is a canonical injective homomorphism $W_a \to \widetilde{W}$ which induces an injection from $X_*(T_{\mathrm{sc}})_I$ into $X_*(T)_I$. In fact, W_a is a normal subgroup of $\widetilde{W}$ and $\widetilde{W}$ is an extension,

$$1 \longrightarrow W_a \longrightarrow \widetilde{W} \longrightarrow X_*(T)_I/X_*(T_{\mathrm{sc}})_I \longrightarrow 1. \tag{2.8}$$

The affine Weyl group W_a acts simply transitively on the set of alcoves in the apartment of S, *cf.* [**T**], 1.7. Since $\widetilde{W}$ acts transitively on the set of these alcoves and W_a acts simply transitively, $\widetilde{W}$ is the semidirect product of W_a with the normalizer Ω of a base alcove, *i.e.* the subgroup of $\widetilde{W}$ which preserves the alcove as a set,

$$\widetilde{W} = W_a \rtimes \Omega. \tag{2.9}$$

In the sequel we will often identify Ω with $X_*(T)_I/X_*(T_{\mathrm{sc}})_I$.

Remarks 2.6

(i) Let $K = K_{\boldsymbol{a}}$ be a parahoric subgroup and $\mathcal{G} = \mathcal{G}_{\boldsymbol{a}}$ the corresponding group scheme over $\operatorname{Spec} \mathcal{O}_L$, *cf.* Remarks 2.2, (ii). Then $\widetilde{W}^K$ can be identified with the Weyl group of the (maximal reductive quotient of the) special fiber $\overline{\mathcal{G}} = \mathcal{G} \otimes_{\mathcal{O}_L} k$ of the group scheme $\mathcal{G}$.

(ii) Assume in Proposition 2.5 above that x is a *hyperspecial* vertex. In this case $S = T$ and W_0 is the *absolute* Weyl group of G. In this case we have

$$\widetilde{W} = W_0 \ltimes X_*(S)$$

and, since $\widetilde{W}^K = W_0$,

$$\widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K = X_*(S)/W_0.$$

3. μ-admissible and μ-permissible set

We continue with the notation of the previous section. In particular, we let N resp. T be the normalizer resp. centralizer of a maximal split torus S over L. Let

$$W = N(\overline{L})/T(\overline{L}) \tag{3.1}$$

be the absolute Weyl group of G. Then $H^1(I, T) = (0)$ and $W_0 = W^I$ is the set of invariants.

Let $\{\mu\}$ be a conjugacy class of cocharacters of G over $\overline{L}$. We denote by the same symbol the corresponding W-orbit in $X_*(T)$. We associate to $\{\mu\}$ a W_0-orbit $\Lambda = \Lambda(\{\mu\})$ in $X_*(T)_I$, as follows. Let B be a Borel subgroup containing T, defined over L (which exists since G is automatically quasi-split). We denote the corresponding closed (absolute) Weyl chamber in $X_*(T)_{\mathbf{R}}$ by $\overline{C}_B$. Let $\mu_B \in \{\mu\}$ be the unique element in $\overline{C}_B$. Then the W_0-orbit Λ of the image λ of μ_B in $X_*(T)_I$ is well-determined since any two choices of B are conjugate under an element of W_0.

Lemma 3.1. — *All elements in Λ are congruent modulo $X_*(T_{\mathrm{sc}})_I$.*

Proof. — Let us fix a special vertex x and let us identify W_0 with $W({}^x\Sigma)$, *cf.* (2.7). We claim that for any $\lambda \in X_*(T)_I$ and any $w \in W_0$ we have

$$w(\lambda) - \lambda \in Q^\vee({}^x\Sigma). \tag{3.2}$$

By induction on the length of w (w.r.t. some ordering of the roots) we may assume that $w = s_\alpha$ is a reflection about a simple root $\alpha \in {}^x\Sigma$. But

$$s_\alpha(\lambda) - \lambda = -\langle \lambda, \alpha \rangle \cdot \alpha^\vee. \tag{3.3}$$

The assertion will follow, once we show that $\langle \lambda, \alpha \rangle$ is an integer. This follows, since the image of $X_*(T)_I$ in $X_*(T_{\mathrm{ad}})_I \otimes \mathbf{R} = X_*(S_{\mathrm{sc}}) \otimes \mathbf{R}$ lies in the lattice of coweights $P^\vee$ for ${}^x\Sigma$ (this holds since $P^\vee$ acts simply transitively on the set of special vertices in the apartment and these are preserved under the subgroup $X_*(T)_I$ of $\widetilde{W}$, compare [**HR**]). □

We shall denote by $\tau = \tau(\{\mu\}) \in \Omega$ the common image of all elements of Λ. Let us now fix an alcove $\boldsymbol{a}$ in the apartment corresponding to S. This defines a Bruhat order on the affine Weyl group W_a which we extend in the obvious way to the semidirect product $\widetilde{W} = W_a \rtimes \Omega$, *cf.* (2.9).

Using this Bruhat order we can now introduce the *μ-admissible subset of* $\widetilde{W}$ [**KR1**],

$$\mathrm{Adm}(\mu) = \{w \in \widetilde{W};\ w \leqslant \lambda \text{ for some } \lambda \in \Lambda\} \tag{3.4}$$

Here we consider the elements $\lambda \in X_*(T)_I$ as elements of $\widetilde{W}$ (*translation elements*). Note that by definition all elements in $\mathrm{Adm}(\mu)$ have image τ in Ω.

More generally, let $\boldsymbol{a}'$ be a facet of $\boldsymbol{a}$ and let K be the corresponding parahoric subgroup. Then the Bruhat order on $\widetilde{W}$ induces a Bruhat order on the double coset space $\widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K$ characterized by

$$\begin{aligned} &\widetilde{W}^K w_1 \widetilde{W}^K \leqslant \widetilde{W}^K w_2 \widetilde{W}^K \\ &\iff \exists\, w_1' \in \widetilde{W}^K w_1 \widetilde{W}^K \text{ and } \exists\, w_2' \in \widetilde{W}^K w_2 \widetilde{W}^K \text{ such that } w_1' \leqslant w_2' \\ &\iff \text{the same holds for } w_1' \text{ and } w_2' \text{ the unique elements of minimal length} \\ &\qquad \text{in their respective double cosets.} \end{aligned} \tag{3.5}$$

We then define the *μ-admissible subset of* $\widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K$,

$$\mathrm{Adm}_K(\mu) = \{w \in \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K;\ w \leqslant \widetilde{W}^K \lambda \widetilde{W}^K \text{ for some } \lambda \in \Lambda\}. \tag{3.6}$$

Since the element of minimal length in a double coset is smaller than any element in it, the natural map

$$\mathrm{Adm}(\mu) \longrightarrow \mathrm{Adm}_K(\mu) \tag{3.7}$$

is surjective. In other words,

$$\mathrm{Adm}_K(\mu) = \text{ image of } \mathrm{Adm}(\mu) \text{ under } \widetilde{W} \longrightarrow \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K. \tag{3.8}$$

We next introduce another subset of $\widetilde{W}$. We first note that the apartment in $\mathcal{B}(G_{\mathrm{ad}}, L)$ corresponding to S is a principal homogeneous space under $X_*(S_{\mathrm{ad}})_{\mathbf{R}}$. Let Λ_{ad} be the image of Λ under the natural map

$$X_*(T)_I \longrightarrow X_*(T_{\mathrm{ad}})_I \longrightarrow X_*(T_{\mathrm{ad}})_I \otimes \mathbf{R} = X_*(S_{\mathrm{ad}})_{\mathbf{R}}.$$

We denote by $\mathcal{P}_\mu = \mathrm{Conv}(\Lambda_{\mathrm{ad}})$ the convex hull of Λ_{ad}. Now we can define the *μ-permissible subset* of $\widetilde{W}$,

$$\mathrm{Perm}(\mu) = \{w \in \widetilde{W};\ w \equiv \tau \mod W_a \text{ and } w(a) - a \in \mathcal{P}_\mu, \text{ for all } a \in \boldsymbol{a}\}. \tag{3.9}$$

Note that by convexity it suffices to impose the second condition in (3.9) for the vertices a_i of $\boldsymbol{a}$. Again there is a variant for a parahoric subgroup K corresponding to a facet $\boldsymbol{a}'$ of $\boldsymbol{a}$. Since $\widetilde{W}^K \subset W_a$ [**HR**], the first condition in the next definition

makes sense,

$$\text{(3.10)}\quad \mathrm{Perm}_K(\mu) = \big\{w \in \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K;\ w \equiv \tau \mod W_a \text{ and } w(a) - a \in \mathcal{P}_\mu, \text{ for all } a \in \boldsymbol{a}'\big\}.$$

Let us check that the second condition also depends only on the double coset of w. Let us write $\widetilde{W} = W_0 \rtimes X_*(T)_I$ (corresponding to the choice of a special vertex which defines the inclusion of W_a in $\widetilde{W}$). If $x \in \widetilde{W}^K$, let $x = x_0 \cdot t_\nu$ with $x_0 \in W_0$ and $\nu \in X_*(T)_I$. Then for $a \in \boldsymbol{a}'$ we have $x(a) = a$ which implies $x_0(a) + x_0(\nu) = a$. Hence

$$\begin{aligned} xw(a) - a &= x_0 w(a) + x_0(\nu) - a = x_0 w(a) - x_0(a) \\ &= x_0(w(a) - a) \in x_0 \mathcal{P}_\mu = \mathcal{P}_\mu, \end{aligned}$$

which proves our claim.

There is a natural map

$$\text{(3.11)}\qquad \mathrm{Perm}(\mu) \longrightarrow \mathrm{Perm}_K(\mu).$$

However, in contrast to (3.8) it is not clear whether this map is surjective, compare Proposition 3.10 below.

An important question is to understand the relation between the sets $\mathrm{Adm}(\mu)$ and $\mathrm{Perm}(\mu)$. In any case, the elements t_λ for $\lambda \in \Lambda$ are contained in both of them. In fact, these elements are maximal in $\mathrm{Adm}(\mu)$. The following fact is proved in [**KR1**]. We repeat the proof.

Proposition 3.2. — *Let G be split over L. Then*

$$\mathrm{Adm}(\mu) \subset \mathrm{Perm}(\mu).$$

In fact, $\mathrm{Perm}(\mu)$ *is closed under the Bruhat order.*

Note that, by the preliminary remarks above, the second claim implies the first. The significance of the second claim becomes more transparent when we discuss local models in Section 6. Namely, in many cases $\mathrm{Perm}(\mu)$ is supposed to parametrize the set of Iwahori-orbits in the special fiber of the local model and, since the latter is in these cases a closed subvariety of an affine flag variety, it contains with an orbit also all orbits in its closure. The question of when $\mathrm{Adm}(\mu)$ coincides with $\mathrm{Perm}(\mu)$ is closely related to the flatness property of local models. In all cases known so far, this flatness property was established by proving that the special fiber is reduced and that the generic points of the irreducible components of the special fiber are in the closure of the generic fiber. The property $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ is supposed to mean that the only irreducible components of the special fiber are "the obvious ones" indexed by t_λ for $\lambda \in \Lambda$, for which the liftability problem should be visibly true (*cf.* Görtz [**G1**]–[**G3**] for various cases).

Returning to Proposition 3.2, we note that when G is split over L, we have $S = T$ and the action of I is trivial. Furthermore $W_0 = W$. The reflections in the affine

Weyl group will be written as $s_{\beta-m}$ where β is a root in the sense of the euclidean root system and $m \in \mathbf{Z}$. The proposition is a consequence of the following lemma.

Lemma 3.3. — *Let $\mathcal{P}$ be a W_0-stable convex polygon in $X_*(S_{\mathrm{ad}})_{\mathbf{R}}$. Let $x, y \in \widetilde{W}$ with $x \leqslant y$. Let $v \in \boldsymbol{a}$ and put $v_x = x(v)$, $v_y = y(v)$. Then*

$$\text{if } v_y \in v + \mathcal{P}, \quad \text{then } v_x \in v + \mathcal{P}.$$

Proof. — We may assume that $x = s_{\beta-m}y$, with $\ell(x) < \ell(y)$. Since $\beta - m$ separates $\boldsymbol{a}$ from $y(\boldsymbol{a})$, it weakly separates $v \in \boldsymbol{a}$ from v_y. Now

$$(\beta - m)(v) = \beta(v) - m \tag{3.12}$$

$$(\beta - m)(v_y) = \beta(v_y) - m. \tag{3.13}$$

Hence we have 2 cases:

- First case: $\beta(v) \leqslant m \leqslant \beta(v_y)$
- Second case: $\beta(v_y) \leqslant m \leqslant \beta(v)$.

Now

$$v_x = s_{\beta-m}(v_y) = v_y - [\beta(v_y) - m]\beta^\vee. \tag{3.14}$$

Hence in either case, v_x lies on the segment joining v_y with $v_y - [\beta(v_y) - \beta(v)] \cdot \beta^\vee$. Hence it suffices to show that $s_\beta(v_y) + \beta(v)\beta^\vee \in v + \mathcal{P}$.

But $v_y = p + v$, with $p \in \mathcal{P}$, hence

$$\begin{aligned} s_\beta(v_y) &= s_\beta(p) + s_\beta(v) \\ &= s_\beta(p) + v - \beta(v) \cdot \beta^\vee. \end{aligned} \tag{3.15}$$

Hence $s_\beta(v_y) + \beta(v)\beta^\vee = s_\beta(p) + v \in v + \mathcal{P}$. □

The converse inclusion is not true in general. We have the following result which generalizes [**KR1**] valid for minuscule μ.

Theorem 3.4 **(Haines, Ngo [HN1])**. — *Let G be either GL_n or GSp_{2n}. In the case of GSp_{2n} assume that the dominant representative of $\{\mu\}$ is a sum of minuscule dominant coweights. Then* $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$.

It may be conjectured that we have equality in general in Proposition 3.2, if μ is a sum of minuscule dominant coweights. Note that, in the case of GL_n, this condition on μ is automatically satisfied. On the other hand, Haines and Ngo ([**HN1**]) have shown by example that for any split group G of rank $\geqslant 4$ not of type A_n, there exists a dominant coweight μ such that $\mathrm{Adm}(\mu) \neq \mathrm{Perm}(\mu)$. In [**HN1**] the result for GSp_{2n} is obtained by relating the sets $\mathrm{Adm}(\mu)$ resp. $\mathrm{Perm}(\mu)$ with the corresponding sets for the "ambient" GL_{2n}. It would be interesting to clarify this relation in other cases as well.

In the sequel, until Proposition 3.10, we investigate the intersections of $\mathrm{Adm}(\mu)$ resp. $\mathrm{Perm}(\mu)$ with the translation subgroup of $\widetilde{W}$. These results are taken from unpublished notes of Kottwitz, as completed by Haines. They will not be used elsewhere.

***Proposition 3.5* (Kottwitz, Haines).** — *Let G be split over L. Then*

$$X_*(T) \cap \mathrm{Adm}(\mu) = X_*(T) \cap \mathrm{Perm}(\mu).$$

To prove this we need a few more lemmas which will also be useful for other purposes. For the time being we assume G split. We denote by R the set of roots and by R^+ resp. Δ the set of positive resp. simple roots for a fixed ordering.

Lemma 3.6. — *Let $\nu \in X_*(S)$. Then*

$$\ell(t_\nu) = \langle \nu, 2w(\varrho) \rangle,$$

where $\varrho = \frac{1}{2}\Sigma_{\alpha>0}\alpha$ and where $w \in W$ is such that $w^{-1}(\nu)$ is dominant. In other words, if ν is dominant then

$$\ell(t_\nu) = \langle \nu, 2\varrho \rangle \quad \textit{and} \quad \ell(t_{w(\nu)}) = \ell(t_\nu), \ \textit{all } w \in W.$$

Proof. — This is an immediate consequence of [**IM**], Prop. 1.23. □

Lemma 3.7. — *For any $\beta \in R^+$ we have*

$$\ell(s_\beta) < \langle \beta^\vee, 2\varrho \rangle.$$

Proof. — It suffices to prove the weak inequality since the left hand side is an odd integer and the right hand side an even integer (twice the height of the coroot $\beta^\vee$). We use induction on $\ell(s_\beta)$, the case $\ell(s_\beta) = 1$ being trivial. So assume $\ell(s_\beta) \geqslant 3$. We first claim that

$$\exists\ \alpha \in \Delta \ \text{ such that } \ \ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) - 2. \tag{3.16}$$

Indeed, let $\alpha \in \Delta$ such that $\ell(s_\alpha s_\beta) = \ell(s_\beta) - 1$. Then also $\ell(s_\beta s_\alpha) = \ell(s_\beta) - 1$. Hence there are 2 possible configurations of $s_\alpha s_\beta s_\alpha, s_\beta s_\alpha, s_\alpha s_\beta, s_\beta$ in the Bruhat order,

(1) $s_\alpha s_\beta s_\alpha < s_\alpha s_\beta < s_\beta$ and $s_\alpha s_\beta s_\alpha < s_\beta s_\alpha < s_\beta$

(2) $s_\alpha s_\beta < s_\beta$ and $s_\alpha s_\beta < s_\alpha s_\beta s_\alpha$ and $s_\beta s_\alpha < s_\beta$ and $s_\beta s_\alpha < s_\alpha s_\beta s_\alpha$.

It suffices to show that case (2) does not arise. In case 2, by Lemma 4.1 of [**H3**] we have $s_\alpha s_\beta s_\alpha = s_\beta$. Hence β and $s_\alpha(\beta)$ are proportional, hence $s_\alpha(\beta) = \pm\beta$. The minus sign cannot occur since α is the only root in R^+ sent by s_α into R^-. Hence $s_\alpha(\beta) = \beta$, i.e. $\langle \alpha, \beta^\vee \rangle = \langle \alpha^\vee, \beta \rangle = 0$. Hence $s_\beta(\alpha) = \alpha \in R^+$. But for any $\gamma \in R^+$ and $w \in W_0$ we have $w^{-1}(\gamma) > 0 \Leftrightarrow w \leqslant s_\gamma w$. Hence $s_\beta < s_\alpha s_\beta$, a contradiction.

Now start with α satisfying (3.16) and put $\beta' = s_\alpha(\beta)$. Then by induction hypothesis

$$\ell(s_\beta) - 2 = \ell(s_{\beta'}) \leqslant \langle \beta'^\vee, 2\varrho \rangle = \langle \beta^\vee, 2\varrho \rangle - 2\langle \beta^\vee, \alpha \rangle.$$

Hence it suffices to show that $\langle \beta^\vee, \alpha \rangle \geqslant 1$. But if $\langle \beta^\vee, \alpha \rangle \leqslant 0$, then $s_\beta(\alpha) = \alpha - \langle \beta^\vee, \alpha \rangle \beta \in R^+$. Hence, arguing as before, $s_\beta < s_\alpha s_\beta$, a contradiction. □

Lemma 3.8. — *Let $\nu \in X_*(S)$ be dominant. Let $\beta \in R^+$ such that $\nu - \beta^\vee$ is dominant. Then $t_{\nu-\beta^\vee} \leqslant t_\nu$ in the Bruhat order on $\widetilde{W}$.*

[Here the Bruhat order on $\widetilde{W}$ is defined by the alcove $\boldsymbol{a}$ in $X_*(S_{\mathrm{ad}})_{\mathbf{R}}$ with apex 0 and bounded by hyperplanes $\alpha = 0$ for $\alpha \in \Delta$.]

Proof. — We use the identity

$$s_{\beta-1} \cdot s_\beta = t_{\beta^\vee}, \quad \beta \in R. \tag{3.17}$$

Indeed, this follows from the expression

$$s_{\beta+k}(x) = x - \langle \beta, x \rangle \beta^\vee - k\beta^\vee, \quad x \in X_*(S_{\mathrm{ad}})_{\mathbf{R}}. \tag{3.18}$$

This last identity also shows

$$t_\nu \cdot s_\beta = s_{\beta-m} \cdot t_\nu, \quad \beta \in R, \quad \nu \in X_*(S). \tag{3.19}$$

Here $m = \langle \beta, \nu \rangle$.

The assertion of the lemma follows from the following two statements.

$$t_\nu \cdot s_\beta \leqslant t_\nu \tag{3.20}$$

$$t_{\nu-\beta^\vee} \leqslant t_\nu \cdot s_\beta. \tag{3.21}$$

Let us prove (3.20), i.e.

$$s_{\beta-m} \cdot t_\nu \leqslant t_\nu, \quad m = \langle \beta, \nu \rangle. \tag{3.22}$$

It is enough to show that $\beta - m$ separates $\boldsymbol{a}$ from $t_\nu(\boldsymbol{a})$. But

$$\begin{aligned} (\beta - m)(\boldsymbol{a}) &= \beta(\boldsymbol{a}) - m \\ (\beta - m)(t_\nu(\boldsymbol{a})) &= \beta(\boldsymbol{a}) \subset [0, 1]. \end{aligned} \tag{3.23}$$

Hence it suffices to know that $m \geqslant 1$. But since $\nu - \beta^\vee$ is dominant we have

$$\langle \beta, \nu - \beta^\vee \rangle \geq 0, \text{ i.e. } m = \langle \beta, \nu \rangle \geqslant \langle \beta, \beta^\vee \rangle = 2. \tag{3.24}$$

Now let us prove (3.21). It follows with (3.17) that both sides of (3.21) differ by a reflection, since

$$t_{\nu-\beta^\vee} = t_\nu \cdot t_{\beta^\vee}^{-1} = (t_\nu \cdot s_\beta) \cdot s_{\beta-1}. \tag{3.25}$$

Hence it suffices to prove that $\ell(t_{\nu-\beta^\vee}) < \ell(t_\nu \cdot s_\beta)$. But by Lemma 3.6 we have, since ν and $\nu - \beta^\vee$ are dominant,

$$\ell(t_{\nu-\beta^\vee}) = \ell(t_\nu) - \langle 2\varrho, \beta^\vee \rangle < \ell(t_\nu) - \ell(s_\beta) \leqslant \ell(t_\nu s_\beta). \tag{3.26}$$

For the first inequality we used Lemma 3.7. □

Remark 3.9 **(Haines).** — In the course of the proof of Lemma 3.8 we proved the chain of inequalities

$$t_{\nu-\beta^\vee} \leqslant t_\nu s_\beta \leqslant t_\nu$$

which is stronger than the assertion of the Lemma. Here is a simpler argument for the assertion of Lemma 3.8 which does not make use of Lemma 3.7. To prove $t_{\nu-\beta^\vee} \leqslant t_\nu$, it is enough to prove the inequalities

$$t_{\nu-\beta^\vee} \leqslant s_\beta t_{\nu-\beta^\vee} = s_{\beta-1} t_\nu \leqslant t_\nu.$$

The middle equality follows from $s_{\beta-1} = t_{\beta^\vee} s_\beta$, *cf.* (3.17). The left inequality holds because $\nu - \beta^\vee$ dominant implies $\ell(s_\beta t_{\nu-\beta^\vee}) = \ell(s_\beta) + \ell(t_{\nu-\beta^\vee})$. We used that for any dominant λ and any simple reflection s we have

$$\ell(st_\lambda) = \sum_{\substack{\alpha>0 \\ s(\alpha)<0}} |\langle \alpha, \lambda\rangle + 1| + \sum_{\substack{\alpha>0 \\ s(\alpha)>0}} |\langle \alpha, \lambda\rangle| \tag{3.27}$$

([**IM**], Prop. 1.23). For the right inequality, it is enough to show that $\boldsymbol{a}$ and $t_\nu(\boldsymbol{a})$ are on opposite sides of the hyperplane fixed by $s_{\beta-1}$. But using $m = \langle \beta, \nu\rangle = \langle \beta, (\nu - \beta^\vee) + \beta^\vee\rangle \geqslant 2$, we have

$$(\beta - 1)(\boldsymbol{a}) \subset [-1, 0]$$
$$(\beta - 1)(t_\nu \boldsymbol{a}) = m - 1 + \beta(\boldsymbol{a}) \subset [m-1, m] \subset [1, \infty).$$

Proof of Proposition 3.5. — Let $\nu \in X_*(T) \cap \mathrm{Perm}(\mu)$ and let us prove that t_ν is μ-admissible. Since ν is μ-permissible we have $\mu - \nu \in X_*(S_{\mathrm{sc}})$. Let us first assume that ν is dominant. Then $\nu \overset{!}{\leqslant} \mu$, i.e. $\mu - \nu$ is a non-negative sum with integer coefficients of simple coroots. Now if λ and λ' are dominant coweights, then

$$\ell(t_{\lambda+\lambda'}) = \ell(t_\lambda) + \ell(t_{\lambda'}).$$

Hence if ν, μ and λ are dominant coweights, then

$$t_\nu \leqslant t_\mu \Longleftrightarrow t_{\lambda+\nu} \leqslant t_{\lambda+\mu}. \tag{3.28}$$

Returning to our $\nu \overset{!}{\leqslant} \mu$, there exists a sequence of coweights $\nu = \nu_0, \nu_1, \ldots, \nu_r = \mu$, such that consecutive terms look like $\lambda - \beta^\vee, \lambda$, for some positive coroot $\beta^\vee$. Adding a sufficiently dominant coweight to this sequence, we can assume that all terms are dominant and even regular dominant.[(2)] This reduces us to the special case of Lemma 3.8 where both ν and $\nu - \beta^\vee$ are regular dominant. Applying Lemma 3.8 we conclude

$$t_\nu \leqslant t_{\nu_1} \leqslant \cdots \leqslant t_\mu, \tag{3.29}$$

hence t_ν is μ-admissible.

If ν is arbitrary there exists a conjugate under $w \in W$ which is dominant, and any such conjugate by Lemma 3.6 has the same length. By a general lemma of Haines [**H3**], Lemma 4.5, elements of $\widetilde{W}$ which are conjugate under a simple reflection and of

[(2)] This argument, due to the referee, allows us to avoid Stembridge's Lemma, *e.g.* [**R3**], used in the original proof.

the same length are simultaneously μ-admissible. The result follows by induction by writing w as a product of simple reflections and using Lemma 3.6 repeatedly.[3] □

In the preceding considerations we looked at the situation for an Iwahori subgroup. Let us now make some comments on the subsets $\mathrm{Adm}_K(\mu)$ and $\mathrm{Perm}_K(\mu)$ of ${}^{\widetilde{W}^K}\backslash\widetilde{W}/\widetilde{W}^K$ for an arbitrary parahoric subgroup K, *cf.* (3.6) and (3.10). First of all we note that as a consequence of Proposition 3.2 and the surjectivity of (3.8) we have the following statement.

Proposition 3.10. — *Let G be split over L. Then*

$$\mathrm{Adm}_K(\mu) \subset \mathrm{Perm}_K(\mu).$$

If $\mathrm{Adm}(\mu)=\mathrm{Perm}(\mu)$, *and* $\mathrm{Perm}(\mu)\to\mathrm{Perm}_K(\mu)$ *is surjective, then also* $\mathrm{Adm}_K(\mu) = \mathrm{Perm}_K(\mu)$.

We note that, as proved in [**KR1**], all these statements hold true if G is equal to GL_n or to GSp_{2n} and μ is minuscule.[4]

Let now G be split over L, and let K be a special maximal parahoric subgroup. We may take the vertex fixed by K to be the origin in the apartment. This identifies

$$ {}^{\widetilde{W}^K}\backslash\widetilde{W}/\widetilde{W}^K = X_*(S)/W = X_*(S)\cap\overline{C}, \tag{3.30}$$

where $X_*(S)\cap\overline{C}$ are the dominant elements for some ordering of the roots. Let us choose $\mu\in W(\mu)$ dominant and introduce the partial order as before (3.28) (*i.e.*, the difference is a sum of positive coroots).

Proposition 3.11. — *Let G be split over L and let K be a special maximal parahoric subgroup. With the notations introduced we have*

$$\mathrm{Adm}_K(\mu) = \mathrm{Perm}_K(\mu) = \{\nu\in X_*(S)\cap\overline{C};\ \nu\overset{!}{\leqslant}\mu\}.$$

Proof. — Let $\nu\in\mathrm{Perm}_K(\mu)$. Then ν and μ have the same image in $X_*(S)/X_*(S_{\mathrm{sc}})$. Hence the condition on ν to be μ-permissible, which says that $t_\nu(0)\in\mathcal{P}_\mu$, is equivalent to

$$\nu\overset{!}{\leqslant}\mu. \tag{3.31}$$

Hence it suffices to show that (3.31) implies $t_\nu\leqslant t_\mu$. But this is shown by the proof of Proposition 3.5 above. □

Corollary 3.12. — *In the situation of the previous proposition assume that μ is minuscule. Then* $\mathrm{Adm}_K(\mu)$ *consists of one element, namely $\mu\in X_*(T)/W$.*

Proof. — Indeed in this case the set appearing in the statement of the Proposition consists of μ only, *cf.* [**K1**]. □

[3] See Notes at the end, n° 2
[4] See Notes at the end, n° 3

4. Affine Deligne-Lusztig varieties

In this section we change notations. Let F be a finite extension of $\mathbf{Q}_p$ and let L be the completion of the maximal unramified extension of F in a fixed algebraic closure $\overline{F}$ of F. We denote by σ the relative Frobenius automorphism of L/F. Let G be a connected reductive group over F and let $\widetilde{G}$ be the group over L obtained by base change. Let $\mathcal{B} = \mathcal{B}(G_{\mathrm{ad}}, L)$ be the Bruhat-Tits building of $\widetilde{G}_{\mathrm{ad}}$. The Bruhat-Tits building of G_{ad} over F can be identified with the set of $\langle\sigma\rangle$-invariants in $\mathcal{B}$.

We fix a maximal split torus $\widetilde{S}$ of $\widetilde{G}$ which is defined over F (such tori exist by [**BT2**], 5.1.12.) We also fix a facet $\boldsymbol{a}'$ in the apartment corresponding to $\widetilde{S}$ which is invariant under $\langle\sigma\rangle$. Let $\widetilde{K} = \widetilde{K}(\boldsymbol{a}')$ be the corresponding parahoric subgroup of $\widetilde{G}(L)$. The subgroup $K = \widetilde{K} \cap G(F)$ is called *the parahoric subgroup of $G(F)$ corresponding to $\boldsymbol{a}'$*. [The subgroup K determines $\boldsymbol{a}'$ uniquely, and hence we obtain a bijection between the set of parahoric subgroups of $G(F)$, the set of σ-invariant parahoric subgroups of $G(L)$ and the set of σ-invariant facets of $\mathcal{B}$.] From Prop. 2.4 we have a map (with obvious notation),

$$\mathrm{inv}: \ \widetilde{G}(L)/\widetilde{K} \times \widetilde{G}(L)/\widetilde{K} \longrightarrow \widetilde{W}^{\widetilde{K}} \backslash \widetilde{W} / \widetilde{W}^{\widetilde{K}}, \tag{4.1}$$

in which the target space can be identified with the quotient of the source by the diagonal action of $\widetilde{G}(L)$.

Definition 4.1. — Let $w \in \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^K$ and $b \in G(L)$. The *generalized affine Deligne-Lusztig variety associated to w and b* is the set

$$X_w(b) = \{g \in \widetilde{G}(L)/\widetilde{K};\ \mathrm{inv}(b\sigma(g), g) = w\}.$$

When $\widetilde{K}$ is an Iwahori subgroup, in which case $\widetilde{W}^{\widetilde{K}}$ is trivial, *i.e.* $w \in \widetilde{W}$, this set is called *the affine Deligne-Lusztig variety associated to w and b*.

Let

$$J_b(F) = \{h \in G(L);\ h^{-1}b\sigma(h) = b\}.$$

Then $J_b(F)$ acts on $X_w(b)$ via $g \mapsto hg$.

Remarks 4.2

(i) If $b' \in G(L)$ is σ-conjugate to b, *i.e.* $b' = h^{-1}b\sigma(h)$, then the map $g \mapsto g' = hg$ induces a bijection

$$X_w(b) \xrightarrow{\sim} X_w(b').$$

Sometimes it is useful to indicate the parahoric subgroup K in the notation. If $K' \subset K$, then, denoting by w a representative of w in $\widetilde{W}^{\widetilde{K}'} \backslash \widetilde{W} / \widetilde{W}^{\widetilde{K}'}$, there is a natural map

$$X_w(b)_{K'} \longrightarrow X_w(b)_K.$$

(ii) One could hope to equip $X_w(b)$ with the structure of an algebraic variety locally of finite type over the residue field $\mathbf{F}$ of $\mathcal{O}_L$.

(iii) The name given to this set derives from the analogue where F is replaced by the finite field $\mathbf{F}_q, L$ by the algebraic closure $\mathbf{F}$ of $\mathbf{F}_q$ and where $\widetilde{K} = B(\mathbf{F})$ for a Borel subgroup of G defined over $\mathbf{F}_q$. Then $\widetilde{W}$ is the geometric Weyl group of G and with $b = 1$ we obtain the usual Deligne-Lusztig variety associated to w [**DL**]. In this analogy, $J_b(F)$ becomes the group of rational points $G(\mathbf{F}_q)$. If instead of a Borel subgroup we consider a conjugacy class of parabolic subgroups defined over $\mathbf{F}_q$, we obtain the *generalized* Deligne-Lusztig varieties, compare [**DM**].

For the classical (generalized) Deligne-Lusztig varieties, there is a simple formula for their dimensions. For affine Deligne-Lusztig varieties such a formula is unknown. In fact, it is an open problem to determine the pairs (w, b) for which $X_w(b) \neq \varnothing$.[(5)]

Example 4.3. — Let $G = GL_2$ and $K = K_0$ = standard Iwahori subgroup. We associate to $b\sigma$ its slope vector $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Q}^2$. Then $\lambda_1 \geqslant \lambda_2$ with $\lambda_1 + \lambda_2 \in \mathbf{Z}$ and $\lambda_i \in \mathbf{Z}$ if $\lambda_1 \neq \lambda_2$. If $X_w(b) \neq \varnothing$ then the image of w in

$$X_*(T)_I / X_*(S_{\mathrm{sc}}) = \mathbf{Z}$$

coincides with $\lambda_1 + \lambda_2$. Conversely, let us assume this and let us enumerate the $w \in \widetilde{W}$ for which $X_w(b) \neq \varnothing$. We distinguish cases.

(a) b basic, *i.e.* $\lambda_1 = \lambda_2$.

(a1) $\lambda_1 + \lambda_2$ odd.

In this case $X_w(b) \neq \varnothing \Leftrightarrow \ell(w)$ is even and then $\dim X_w(b) = \ell(w)/2$.

(a2) $\lambda_1 + \lambda_2$ even.

In this case $X_w(b) \neq \varnothing \Leftrightarrow$ either the projection of w to W_a is trivial, in which case $\dim X_w(b) = 0$, or $\ell(w)$ is odd, in which case $\dim X_w(b) = (\ell(w) + 1)/2$.

(b) b hyperbolic, *i.e.* $\lambda_1 \neq \lambda_2$.

In this case $X_w(b) \neq \varnothing \Leftrightarrow$ either $w = t_{(\lambda_1, \lambda_2)}$, in which case $\dim X_w(b) = 0$, or $\ell(w) > \ell(t_{(\lambda_1, \lambda_2)})$, and $\ell(w) \equiv \lambda_1 - \lambda_2 + 1 \bmod 2$, in which case $\dim X_w(b) = \frac{\ell(w) - (\lambda_1 - \lambda_2 - 1)}{2}$.

Before going on, we recall some definitions of Kottwitz [**K2**], [**K4**]. Let $B(G)$ be the set of σ-conjugacy classes of elements of $G(L)$. The homomorphism $\widetilde{\kappa}_{\widetilde{G}}$, *cf.* (2.1), induces a map

$$\kappa_G : B(G) \longrightarrow X^*(\widehat{Z}(G)^\Gamma). \tag{4.2}$$

Here $\Gamma = \mathrm{Gal}(\overline{F}/F)$ denotes the absolute Galois group of F. We also recall the Newton map,

$$\overline{\nu} : B(G) \longrightarrow \mathfrak{A}^+. \tag{4.3}$$

Here the notation is as follows. Let G^* be the quasisplit inner form of G. Let B^* be a Borel subgroup of G^* defined over F and let T^* be a maximal torus in B^*. Then

(5) See Notes at the end, n° 4

$\mathfrak{A} = X_*(T^*)^\Gamma_{\mathbf{R}}$ and $\mathfrak{A}^+$ denotes the intersection of $\mathfrak{A}$ with the positive Weyl chamber in $X_*(T^*)_{\mathbf{R}}$ corresponding to B^*. For instance, if $G = GL_n$, then the Newton map associates to $b \in G(L)$ the slopes of the isocrystal $(L^n, b\sigma)$ with multiplicity of each slope equal to the dimension of the corresponding isotypical component, in decreasing order. An element $b \in B(G)$ is called *basic* if $\overline{\nu}_b$ is central, *i.e.* if $\overline{\nu}_b \in X_*(Z)_{\mathbf{R}}$. This is the analogue for general G of an isoclinic isocrystal. At the opposite extreme of the basic elements of $B(G)$ are the unramified elements. Namely, let $G = G^*$ be the quasisplit, and let A be a maximal split torus contained in T^*. Let $b \in A(L)$. Then $\overline{\nu}_b$ is the unique dominant element in the conjugacy class of $\mathrm{ord}(b) \in X_*(A) \subset \mathfrak{A}$ (this follows from the functoriality of the Newton map).

Let now $\{\mu\}$ be a conjugacy class of one-parameter subgroups of G. Then $\{\mu\}$ determines a well-defined element μ^* in $X_*(T^*)_{\mathbf{R}}$ lying in the positive Weyl chamber (use an inner isomorphism of G with G^* over $\overline{F}$). Let

$$\overline{\mu}^* = [\Gamma : \Gamma_{\mu^*}]^{-1} \cdot \sum_{\tau \in \Gamma/\Gamma_{\mu^*}} \tau(\mu^*). \tag{4.4}$$

Then $\overline{\mu}^* \in \mathfrak{A}^+$. On the other hand, $\{\mu\}$ determines a well-defined element $\mu^\natural$ of $X^*(\widehat{Z}(G)^\Gamma)$.

We define a finite subset $B(G, \mu)$ of $B(G)$ as the set of $b \in B(G)$ satisfying the following two conditions,

$$\kappa_G(b) = \mu^\natural \tag{4.5}$$

$$\overline{\nu}_b \leqslant \overline{\mu}^*, \tag{4.6}$$

cf. [**K4**], section 6. Here in (4.6) there appears the usual partial order on $\mathfrak{A}^+$, for which $\nu \leqslant \nu'$ if $\nu' - \nu$ is a nonnegative linear combination of simple relative coroots.

The motivation for the definition of $B(G, \mu)$ comes from the following fact. We return to the notation of the beginning of this section. Let us assume that G is quasisplit over F and $\widetilde{G}$ split over L, *i.e.*, G is unramified. Let K be a hyperspecial maximal parahoric subgroup. Then $T = \widetilde{S}$ and $\widetilde{W}^{\widetilde{K}} \backslash \widetilde{W} / \widetilde{W}^{\widetilde{K}}$ can be identified with $X_*(\widetilde{S})/W_0$.

Proposition 4.4 **([RR])**. — *Let $\mu \in X_*(\widetilde{S})/W_0$. For $b \in G(L)$ let $[b] \in B(G)$ be its σ-conjugacy class. Then*

$$X_\mu(b) \neq \varnothing \Longrightarrow [b] \in B(G, \mu).$$

This is the group theoretic version of Mazur's inequality between the Hodge polygon of an F-crystal and the Newton polygon of its underlying F-isocrystal.[(6)]

Example 4.5. — Let $G = GL_n$ and let $T = S$ be the group of diagonal matrices, and K the stabilizer of the standard lattice O_F^n in F^n. Then, with the choice of the upper

[(6)] See Notes at the end, n° 5

triangular matrices for B^*,

$$\mathfrak{A}^+ = (\mathbf{R}^n)_+ = \{\nu = (\nu_1, \dots, \nu_n) \in \mathbf{R}^n;\ \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n\}.$$

For the usual partial order on $\mathfrak{A}^+$ we have $\nu \leqslant \nu'$ iff

$$\sum_{i=1}^{r} \nu_i \leqslant \sum_{i=1}^{r} \nu_i' \quad \text{for } r = 1, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \nu_i'.$$

Let $b \in G(L)$ and let $(N, \Phi) = (L^n, b \cdot \sigma)$ be the corresponding isocrystal of dimension n. If M is an $\mathcal{O}_L$-lattice in N we have

$$\mu(M) = \mathrm{inv}(M, \Phi(M)) \in (\mathbf{Z}^n)_+.$$

Here $(\mathbf{Z}^n)_+ = \mathbf{Z}^n \cap (\mathbf{R}^n)_+$ and $\mu(M) = (\mu_1, \dots, \mu_n)$ iff there exists a $\mathcal{O}_L$-basis $e_1, \dots, e_n$ of M such that $\pi^{\mu_1} e_1, \dots, \pi^{\mu_n} e_n$ is a $\mathcal{O}_L$-basis of $\Phi(M)$. By π we denoted a uniformizer of F. Denoting by $\overline{\nu}_b$ the Newton vector of the isocrystal (N, Φ), Mazur's inequality states that

$$\overline{\nu}_b \leqslant \mu(M),$$

i.e. $[b] \in B(G, \mu(M))$.

Conjecture 4.6. — *The converse in the implication of Proposition 4.4 holds.*

In this direction we have the following results.[7]

Theorem 4.7. — *The converse implication in Proposition 4.4 holds in either of the following cases.*

(i) [**KR2**] *$G = GL_n$ or $G = GSp_{2n}$.*

(ii) [**R3**] *The derived group of G is simply connected, $\mu \in X_*(\widetilde{S})/W_0$ is $\langle\sigma\rangle$-invariant and $b \in A(F)$, where A denotes a maximal F-split torus in G.*

Whereas the individual affine Deligne-Lusztig varieties are very difficult to understand, the situation seems to change radically when we form a suitable finite union of them. This is the subject of the next section.

5. The sets $X(\mu, b)_K$

We continue with the notation of the previous section. Let $\{\mu\}$ be a conjugacy class of one-parameter subgroups of G. Equivalently, $\{\mu\}$ is a W-orbit in $X_*(T)$ where T denotes the centralizer of $\widetilde{S}$. We again introduce the subsets $\mathrm{Adm}_{\widetilde{K}}(\mu)$ resp. $\mathrm{Perm}_{\widetilde{K}}(\mu)$ of $\widetilde{W}^{\widetilde{K}} \backslash \widetilde{W} / \widetilde{W}^{\widetilde{K}}$ and again set $K = \widetilde{K} \cap G(F)$.

Let $b \in G(L)$. Then we define the following set, a finite union of generalized affine Deligne-Lusztig varieties,

$$X(\mu, b)_K = \{g \in \widetilde{G}(L)/\widetilde{K};\ \mathrm{inv}(g, b\sigma(g)) \in \mathrm{Adm}_{\widetilde{K}}(\mu)\}. \tag{5.1}$$

[7] See Notes at the end, n° 6

Let $E \subset \overline{F}$ be the field of definition of $\{\mu\}$. Let $E_0 = E \cap L$ and $r = [E_0 : F]$. We note that σ acts in compatible way on $\widetilde{W}$ and its subgroup $X_*(T)_I$, and that the map (4.1) is compatible with this action.

Lemma 5.1. — *The subset $\Lambda(\{\mu\})$ of $X_*(T)_I$ (defined shortly after (3.1)) is invariant under σ^r. Hence also the subsets $\mathrm{Perm}_{\widetilde{K}}(\mu)$ and $\mathrm{Adm}_{\widetilde{K}}(\mu)$ of $\widetilde{W}^{\widetilde{K}} \backslash \widetilde{W} / \widetilde{W}^{\widetilde{K}}$ are invariant under σ^r.*

Proof. — Let $\nu \in X_*(T)$ with image $[\nu]_I$ in $X_*(T)_I$. Then

$$\sigma^r([\nu]_I) = [\tau(\nu)]_I,$$

for $\tau \in \mathrm{Gal}(\overline{L}/F)$ an arbitrary lifting of σ^r. We take for τ an extension of the automorphism $\mathrm{id} \otimes \sigma^r$ of $EL = E \otimes_{E_0} L$. Then $\tau \in \mathrm{Gal}(\overline{L}/E)$ and hence preserves the orbit $\{\mu\}$ in $X_*(T)$. Furthermore τ normalizes $\mathrm{Gal}(\overline{L}/L)$. Hence if $\mu \in \overline{C}_B \cap \{\mu\}$ for a Borel subgroup defined over L, then $\tau(\mu) \in \overline{C}_{B'} \cap \{\mu\}$ for another Borel subgroup defined over L. Hence if $\lambda \in X_*(T)_I$ denotes the image of μ, then $\sigma^r(\lambda) = w_0(\lambda)$ for some $w_0 \in W_0$ which implies the first assertion. The second assertion follows (for $\mathrm{Adm}_{\widetilde{K}}(\mu)$ use that $w_1 \leqslant w_2$ implies $\sigma(w_1) \leqslant \sigma(w_2)$). □

Using this lemma we can now define an operator Φ on $X(\mu, b)_K$ by

$$\Phi(g) = (b\sigma)^r \cdot g \cdot \sigma^{-r} = b \cdot \sigma(b) \cdots \sigma^{r-1}(b) \cdot \sigma^r(g). \tag{5.2}$$

Let us check that Φ indeed preserves the set $X(\mu, b)_K$. We have

$$\begin{aligned} \mathrm{inv}(\Phi(g), b\sigma(\Phi(g)) &= \mathrm{inv}(\sigma^r(g), \sigma^r(b\sigma(g))) \\ &= \sigma^r(\mathrm{inv}(g, b\sigma(g)). \end{aligned} \tag{5.3}$$

The claim follows from Lemma 5.1.

In the context of Remark 4.2., (ii), the set $X(\mu, b)_K$ may be expected to be the set of $\mathbf{F}$-points of an algebraic variety over $\mathbf{F}$, and Φ would define a Weil descent datum over the residue field κ_E of E in the sense of [**RZ2**].

As mentioned at the end of the last section, whereas it seems difficult to understand when the individual affine Deligne-Lusztig varieties which make up $X(\mu, b)_K$ are non-empty, their union seems to behave better, at least in the cases when $\{\mu\}$ is minuscule.

Conjecture 5.2. — *Let $\{\mu\}$ be minuscule.*

(a) *$X(\mu, b)_K \neq \varnothing$ if and only if the class $[b]$ of b in $B(G)$ lies in the subset $B(G, \mu)$.*
(b) *For $K \subset K'$, the induced map $X(\mu, b)_K \to X(\mu, b)_{K'}$ is surjective.*

Remark 5.3

Suppose that K is hyperspecial. Then, if $X(\mu, b)_K \neq \varnothing$, it follows that $[b] \in B(G, \mu)$, *cf.* Prop. 4.4. This holds even when $\{\mu\}$ is not minuscule. In general, it

is not clear whether the hypothesis that $\{\mu\}$ be minuscule is indeed necessary in Conjecture 5.2.[8]

In the direction of Conjecture 5.2 we first note the following easy observation.

Lemma 5.4. — *If* $X(\mu,b)_K \neq \varnothing$, *then*

$$\kappa(b) = \mu^\natural.$$

Proof. — We consider the composition $\widetilde{\kappa} = \widetilde{\kappa}_G$ of $\widetilde{\kappa}_{\widetilde{G}}$ and the natural surjection,

$$G(L) \longrightarrow X^*(\widehat{Z}(G)^I) \longrightarrow X^*(\widehat{Z}(G)^\Gamma). \tag{5.4}$$

The map $\widetilde{\kappa}_G$ induces κ_G on $B(G)$. If $g\widetilde{K} \in X(\mu,b)_K$, then $g^{-1}b\sigma(g) = k_1 w k_1$ with $k_1, k_2 \in \widetilde{K}$ and with $w \in \mathrm{Adm}_K(\mu)$. Since $k_1, k_2 \in \mathrm{Ker}\,\widetilde{\kappa}_{\widetilde{G}}$ we conclude that

$$\widetilde{\kappa}(b) = \widetilde{\kappa}(g^{-1}b\sigma(g)) = \widetilde{\kappa}(k_1 w k_2) = \widetilde{\kappa}(w).$$

But

$$\widetilde{W}^{\widetilde{K}} w \widetilde{W}^{\widetilde{K}} \leqslant \widetilde{W}^{\widetilde{K}} t_{\mu'} \widetilde{W}^{\widetilde{K}},$$

for a conjugate μ' of μ. Since $\widetilde{W}^{\widetilde{K}} \subset W_a$ and $W_a \subset \mathrm{Ker}\,\widetilde{\kappa}$ (since $\widetilde{\kappa}(G_{\mathrm{sc}}(L)) = 0$), we conclude that $\widetilde{\kappa}(w) = \widetilde{\kappa}(t_{\mu'})$. If $\mu' = w_0(\mu)$ for $w_0 \in W_0$ we have

$$\widetilde{\kappa}(t_{\mu'}) = \widetilde{\kappa}(w_0 t_\mu w_0^{-1}) = \widetilde{\kappa}(t_\mu),$$

hence $\widetilde{\kappa}(b) = \widetilde{\kappa}(t_\mu) = \mu^\natural$. □

Theorem 5.5 (**[KR2]**). — *Conjecture 5.2 holds for minuscule* μ, *in the cases* $G = R_{F'/F}(GL_n)$ *and* $G = R_{F'/F}(GSp_{2n})$, *where* F' *is an unramified extension of* F.

In fact, in loc. cit. also the case when G is an inner form of GL_n is treated.
We also mention the following case when Conjecture 5.2a) holds.

Proposition 5.6. — *Assume that* G *splits over* L *and that the center of* G *is connected. Let* $b \in G(L)$ *be such that* $[b] \in B(G)_{\mathrm{basic}}$. *Let* K_0 *be an Iwahori subgroup defined over* F. *Then* $X(\mu,b)_{K_0} \neq \varnothing \Leftrightarrow [b] \in B(G,\mu)$.

Proof. — It is obvious that if $[b] \in B(G)_{\mathrm{basic}}$, then $[b] \in B(G,\mu)$ iff $\kappa(b) = \mu^\natural$. Hence one implication ($\Rightarrow$) follows from Lemma 5.4. Now let $[b] \in B(G,\mu) \cap B(G)_{\mathrm{basic}}$.

Claim. — *Let* N *be the normalizer of a maximal torus* S *which splits over* L, *and let* $\widetilde{K} \subset G(L)$ *be a* σ*-stable Iwahori subgroup corresponding to an alcove in the apartment of* S. *Then there exists a representative* b' *of* $[b]$ *in* $N(L)$ *which normalizes* $\widetilde{K}$.

[8] See Notes at the end, n° 7.

Proof of Claim. — Since the center of G is connected, the map $G(L) \to G_{\mathrm{ad}}(L)$ is surjective. Hence we may replace G by G_{ad}, in which case $B(G)_{\mathrm{basic}} = H^1(F, G)$. Hence any representative of b defines an inner form of G which splits over L. Assume that there exists a representative b' of $[b]$ in $N(L)$ as in the claim. Then in the corresponding inner form of G there exists a maximal torus which splits over F^{un} and an F-rational Iwahori subgroup fixing an alcove in the apartment for this torus. Conversely, if the inner form of G corresponding to a representative of $[b]$ has this property, then this representative normalizes this maximal torus and the Iwahori subgroup. Now, since any inner form of G contains a maximal torus which splits over F^{un} and an F-rational Iwahori subgroup fixing an alcove in the apartment for this torus, such a representative must exist, which proves the claim. □

Let $g \in G(L)$ be such that $\widetilde{K} = g\widetilde{K}_0 g^{-1}$. Then $gb'\sigma(g)^{-1} \in \widetilde{K}_0 w \widetilde{K}_0$, where $w \in \widetilde{W}$ normalizes $\widetilde{K}_0$. It follows that the component of w in W_a is trivial, hence by Lemma 5.4, $w \leqslant t_\mu$. If $b' = h\widetilde{b}\sigma(h)^{-1}$ then $gh\widetilde{K}_0 \in X_w(b\sigma) \subset X(\mu, \widetilde{b})_{K_0}$. □

The following statement yields an inequality which goes in a sense in the opposite direction to that defining $B(G, \mu)$.

Proposition 5.7. — *Let G be split over F. Let S be a maximal split torus over F. Let $b \in S(L)$. Let $\widetilde{K}_0$ denote the Iwahori subgroup fixing an alcove in the apartment of $\mathcal{B}$ corresponding to S. Let $w \in \widetilde{W}$ such $X_w(b\sigma) \neq \varnothing$, i.e.,*

$$\exists\ g \in G(L) : g^{-1} b\sigma(g) \in \widetilde{K}_0 w \widetilde{K}_0.$$

Then for the translation by the Newton point $\nu_b \in X_*(S) \cap \mathfrak{A}^+$ of b we have

$$t_{\nu_b} \leqslant w.$$

Proof. — Let $\mathcal{A}$ be the apartment in $\mathcal{B}$ corresponding to $\widetilde{S} = S \otimes_F L$, and let $\boldsymbol{a}_0 \subset \mathcal{A}$ be the base alcove fixed by $\widetilde{K}_0$, and let $\boldsymbol{a} = g\boldsymbol{a}_0$, for $g \in X_w(b\sigma)$. Let α be the automorphism of $\mathcal{B}$ induced by $b\sigma$. Then we have for the component w_a of w in the affine Weyl group

$$w_a = \mathrm{inv}(\boldsymbol{a}, \alpha(\boldsymbol{a})). \tag{5.5}$$

Let $\mathcal{C}$ be any quartier corresponding to the positive vector Weyl chamber, after a choice of a special vertex of $\boldsymbol{a}_0$. Only the germ of $\mathcal{C}$ will be relevant to us, *i.e.* $\mathcal{C}$ up to translation by an element of $X_*(S_{\mathrm{sc}})_{\mathbf{R}}$. Let $\varrho_{\mathcal{A},\mathcal{C}}$ be the corresponding retraction, *i.e.* $\varrho_{\mathcal{A},\mathcal{C}} = \varrho_{\mathcal{A},\boldsymbol{a}'}$ for some alcove $\boldsymbol{a}'$ far into the quartier, compare [**BT1**], Prop. 2.9.1. Then we have the following two statements.

$$\alpha \circ \varrho_{\mathcal{A},\mathcal{C}}(\boldsymbol{a}) = \varrho_{\mathcal{A},\mathcal{C}} \circ \alpha(\boldsymbol{a}) \tag{5.6}$$

$$\text{For any two alcoves } \boldsymbol{a}, \boldsymbol{a}', \quad \mathrm{inv}(\varrho_{\mathcal{A},\mathcal{C}}(\boldsymbol{a}), \varrho_{\mathcal{A},\mathcal{C}}(\boldsymbol{a}')) \leqslant \mathrm{inv}(\boldsymbol{a}, \boldsymbol{a}'). \tag{5.7}$$

To see (5.6), note that α preserves $\mathcal{A}$ and the germ of $\mathcal{C}$, hence

$$\alpha \circ \varrho_{\mathcal{A},\mathcal{C}} \circ \alpha^{-1}(\boldsymbol{a}) = \varrho_{\alpha(\mathcal{A}),\alpha(\mathcal{C})}(\boldsymbol{a}) = \varrho_{\mathcal{A},\alpha(\mathcal{C})}(\boldsymbol{a}). \tag{5.8}$$

Since α also preserves the germ of $\mathcal{C}$, it follows that $\varrho_{\mathcal{A},\alpha(\mathcal{C})}(\boldsymbol{a}) = \varrho_{\mathcal{A},\mathcal{C}}(\boldsymbol{a})$.

To see (5.7), let $\boldsymbol{a} = \boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_\ell = \boldsymbol{a}'$ be a minimal gallery Γ between $\boldsymbol{a}$ and $\boldsymbol{a}'$. This corresponds to a minimal decomposition of $x = \mathrm{inv}(\boldsymbol{a}, \boldsymbol{a}')$,

$$x = s_1 \cdots s_\ell. \tag{5.9}$$

Here $s_1, \ldots, s_\ell$ are the reflections around the walls of type $\boldsymbol{a}_0 \cap \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{\ell-1} \cap \boldsymbol{a}_\ell$ of the base simplex ([**BT1**], 2.3.10). The image of Γ under $\varrho = \varrho_{\mathcal{A},\mathcal{C}}$ is a gallery $\overline{\Gamma}$ between $\overline{\boldsymbol{a}} = \varrho(\boldsymbol{a})$ and $\overline{\boldsymbol{a}}' = \varrho(\boldsymbol{a}')$. Furthermore, the type $\overline{s}_1, \ldots, \overline{s}_\ell$ of $\overline{\Gamma}$ is identical with that of Γ ([**BT1**], 2.3.4.) Let us replace $\overline{\Gamma}$ by a minimal gallery. Then we can write $\overline{x} = \mathrm{inv}(\overline{\boldsymbol{a}}, \overline{\boldsymbol{a}}')$ as

$$\overline{x} = s_{i_1} \cdots s_{i_k} \tag{5.10}$$

([**BT1**], 2.1.9 and 2.1.11). Hence $\overline{x} \leqslant x$, which proves (5.7).

We now apply this to $\boldsymbol{a}$ and $\alpha(\boldsymbol{a})$. But for any $\overline{\boldsymbol{a}}$ contained in $\mathcal{A}$ we have

$$\mathrm{inv}(\overline{\boldsymbol{a}}, \alpha(\overline{\boldsymbol{a}})) = (t_\nu)_a. \tag{5.11}$$

Here $\nu = \nu_b \in X_*(S)$. Hence using (5.6) and (5.7),

$$\begin{aligned}(t_\nu)_a = \mathrm{inv}(\varrho(\boldsymbol{a}), \alpha\varrho(\boldsymbol{a})) &= \mathrm{inv}(\varrho(\boldsymbol{a}), \varrho(\alpha(\boldsymbol{a})) \\ &\leqslant \mathrm{inv}(\boldsymbol{a}, \alpha(\boldsymbol{a})) = w_a.\end{aligned}$$

Taking into account the definition of the Bruhat order on $\widetilde{W}$, the assertion follows. □

Whereas Proposition 5.6 concerned the case of a basic element b, the following proposition treats the other extreme, namely, unramified elements b.

Proposition 5.8. — *Let $G, S, \widetilde{K}_0$ and $b \in S(L)$ be as in the previous proposition. Assume that G_{der} is simply connected. Then $X(\mu, b)_{K_0} \neq \varnothing \Leftrightarrow [b] \in B(G, \mu)$.*

Proof. — If $X(\mu, b)_{K_0} \neq \varnothing$, there exists $w \in \mathrm{Adm}(\mu)$ such that $X_w(b\sigma) \neq \varnothing$. This implies $\kappa(w) = \kappa(b)$ and $t_{\nu_b} \leqslant w$, by Lemma 5.4 and Proposition 5.7. Since $w \leqslant t_{\mu'}$ for some conjugate μ' of μ it follows that $t_{\nu_b} \leqslant t_{\mu'}$ which implies that $W_0 t_{\nu_b} W_0 \leqslant W_0 t_\mu W_0$ and hence $\nu_b \overset{!}{\leqslant} \mu$, *i.e.* $[b] \in B(G, \mu)$.

Conversely, let $[b] \in B(G, \mu)$. Hence ν_b and μ are both dominant elements in $X_*(S)$ with $\nu_b \leqslant \mu$. However, for any alcove $\boldsymbol{a}$ in the apartment $\mathcal{A}$ corresponding to S, we have $\mathrm{inv}(\boldsymbol{a}, b\sigma(\boldsymbol{a})) = (t_{\nu_b})_a$, hence $X_{t_{\nu_b}}(b\sigma) \neq \varnothing$. Hence it suffices to see that $t_{\nu_b} \in \mathrm{Adm}(\mu)$. But $\nu_b \leqslant \mu$, hence since G_{der} is simply connected, $\nu_b \overset{!}{\leqslant} \mu$. Therefore by Proposition 3.11 $t_{\nu_b} \leqslant t_\mu$, *i.e.* $t_{\nu_b} \in \mathrm{Adm}(\mu)$. □

As mentioned above, the sets $X(\mu, b)_K$ should have the structure of an algebraic variety over the residue field $\mathbf{F}$ of $\mathcal{O}_L$, at least when $\{\mu\}$ is minuscule. For their dimension there is a conjectural formula when b is basic. To state it we first mention the following result. The set $B(G, \mu)$ is partially ordered (a finite poset) by $[b] \leqslant [b']$

iff $\overline{\nu}_{[b]} \leqslant \overline{\nu}_{[b']}$ in $\mathfrak{A}^+$. That this is indeed a partial order follows from the fact that the map $(\overline{\nu}, \kappa) : B(G) \to \mathfrak{A}^+ \times X^*(\widehat{Z}(G)^\Gamma)$ is injective [**K4**], [**RR**].

Theorem 5.9 **(Chai [C2])**. — *Assume G quasisplit over F.*

(i) *Any subset of $B(G,\mu)$ has a join, i.e. a supremum.*

(ii) *The poset $B(G,\mu)$ is ranked, i.e. any two maximal chains between two comparable elements have the same length.*

(iii) *Let $[b],[b'] \in B(G,\mu)$ with $[b] \leqslant [b']$. Then the length of the maximal chain between $[b]$ and $[b']$ is given by*

$$\text{length}([b],[b']) = \sum_{i=1}^{\ell} ([\langle \omega_i, \overline{\nu}_{[b']} \rangle - \langle \omega_i, \overline{\mu}^* \rangle] - [\langle \omega_i, \overline{\nu}_{[b]} \rangle - \langle \omega_i, \overline{\mu}^* \rangle]).$$

Here $\omega_1, \ldots, \omega_\ell$ are the fundamental F-weights of the adjoint group G_{ad}, *i.e.* $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for any simple relative coroot $\alpha_j^\vee$. Also $[x]$ denotes the greatest integer $\leqslant x$.

We note that $B(G,\mu)$ has a unique minimal element, namely the unique basic element $[b_0]$ in $B(G,\mu)$, and a unique maximal element, namely the μ-ordinary element $[b_1] = [b_\mu]$ for which $\overline{\nu}_{[b_1]} = \overline{\mu}^*$. Given (i) and (ii) of Theorem 5.9, the formula in (iii) is equivalent to

$$\text{length}([b],[b_1]) = -\sum_{i=1}^{\ell} [\langle \omega_i, \overline{\nu}_{[b]} \rangle - \langle \omega_i, \overline{\mu}^* \rangle]. \tag{5.12}$$

The dimension formula for $X(\mu,b)_K$ may now be given as follows.

Conjecture 5.10. — *Let K be a hyperspecial maximal parahoric. Let $[b] = [b_0] \in B(G,\mu)$ be basic. Then $X(\mu,b)_K$ is equidimensional of dimension*

$$\begin{aligned} \dim X(\mu,b)_K &= \langle 2\varrho, \overline{\mu}^* \rangle - \text{length}([b],[b_\mu]) \\ &= \langle 2\varrho, \overline{\mu}^* \rangle + \sum_{i=1}^{\ell} [-\langle \omega_i, \overline{\mu}^* \rangle]. \end{aligned}$$

Here ϱ denotes the half-sum of all positive roots[9].

The motivation for this formula comes from global considerations connected with the Newton strata of Shimura varieties, compare Theorem 7.4 below. It would be interesting to extend this conjecture to the non-basic case [10] and also to the case when K is no longer hyperspecial.

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[9] See Notes at the end, n° 8

[10] *Added in October 2004*: In the light of recent results of Chai and Oort, and of Mierendorff, the following formula seems reasonable,

$$\dim X(\mu,b)_K = \langle 2\varrho, \overline{\mu}^* - \overline{\nu}_{[b]} \rangle + \sum_{i=1}^{\ell} [-\langle \omega_i, \overline{\mu}^* - \overline{\nu}_{[b]} \rangle]$$

6. Relations to local models

We continue with the notation of the last two sections. In particular, G denotes a connected reductive group over F and $\{\mu\}$ is a conjugacy class of one-parameter subgroups of G. Again E is the field of definition of $\{\mu\}$. Let K be a parahoric subgroup of $G(F)$ and $\widetilde{K}$ the corresponding parahoric subgroup of $G(L)$. We denote by $\mathcal{G} = \mathcal{G}_K$ the group scheme over $\mathcal{O}_F$ corresponding to K, *cf.* Remark 2.2, (ii).

To these data one would like to associate the *local model*, a projective scheme $\mathcal{M}^{\mathrm{loc}} = \mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ over $\operatorname{Spec}\mathcal{O}_E$, equipped with an action of $\mathcal{G}_{\mathcal{O}_E}$, at least if $\{\mu\}$ is minuscule. It is not clear at the moment how to characterize $\mathcal{M}^{\mathrm{loc}}$ or how to construct it in general. It should have at least the following properties.

(i) $\mathcal{M}^{\mathrm{loc}}$ is flat over $\operatorname{Spec}\mathcal{O}_E$ with generic fiber isomorphic to G/P_μ. Here P_μ denotes the variety of parabolic subgroups in the conjugacy class of parabolic subgroups corresponding to $\{\mu\}$.

(ii) There is an identification of the geometric points of the special fiber,

$$\mathcal{M}^{\mathrm{loc}}(\overline{\kappa}_E) = \{g \in G(L)/\widetilde{K};\ \widetilde{K}g\widetilde{K} \in \mathrm{Adm}_{\widetilde{K}}(\mu)\}.$$

(iii) $\mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ is functorial in K and in G.

Examples 6.1

(1) If K is hyperspecial, then we set $\mathcal{M}^{\mathrm{loc}}(G,\mu)_K = \mathcal{G}_{\mathcal{O}_E}/\mathcal{P}_\mu$, where $\mathcal{P}_\mu$ is in the conjugacy class of parabolic subgroups in $\mathcal{G}_{\mathcal{O}_E}$ corresponding to $\{\mu\}$. In this case $\mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ is smooth over $\operatorname{Spec}\mathcal{O}_E$. Property (ii) follows from Corollary 3.12.

(2) Let V be an F-vector space of dimension n. Let $G = GL(V)$ and let $\{\mu\}$ be minuscule of weight r for some $0 \leqslant r \leqslant n$, *i.e.* $\omega_r \in \{\mu\}$, where $\omega_r(t) = \mathrm{diag}(t,\dots,t,1,\dots,1)$ with r times t and $n-r$ times 1. Let $e_1,\dots,e_n$ be a basis of V and, for $0 \leqslant i \leqslant n-1$, let

$$\Lambda_i = \mathrm{span}_{\mathcal{O}_F}\{\pi^{-1}e_1,\dots,\pi^{-1}e_i,e_{i+1},\dots,e_n\}.$$

For a non-empty subset I of $\{0,1,\dots,n-1\}$, let $K = K_I$ be the parahoric subgroup of $G(F)$ which is the common stabilizer of the lattices Λ_i, for $i \in I$. The local model $\mathcal{M}^{\mathrm{loc}} = \mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ for this triple $(G,\{\mu\},K_I)$ represents the following moduli problem on $(Sch/\operatorname{Spec}\mathcal{O}_F)$ (in this case $E = F$). To S the functor associates the set of commutative diagrams of $\mathcal{O}_S$-modules,

$$\begin{array}{ccccccccc}
\Lambda_{i_0,S} & \longrightarrow & \Lambda_{i_1,S} & \longrightarrow & \cdots & \longrightarrow & \Lambda_{i_m,S} & \xrightarrow{\pi} & \Lambda_{i_0,S} \\
\cup & & \cup & & & & \cup & & \cup \\
\mathcal{F}_{i_0} & \longrightarrow & \mathcal{F}_{i_1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_{i_m} & \longrightarrow & \mathcal{F}_{i_0}\,.
\end{array}$$

Here $I = \{i_0 < i_1 < \cdots < i_m\}$ and we have set $\Lambda_{i,S} = \Lambda_i \otimes_{\mathcal{O}_F} \mathcal{O}_S$. It is required that $\mathcal{F}_{i_j}$ is a locally free $\mathcal{O}_S$-module of rank r which is locally a direct summand of $\Lambda_{i_j,S}$. The main result of the paper [**G1**] of Görtz is that $\mathcal{M}^{\mathrm{loc}}$ satisfies the conditions (i) and (ii) above.

(3) Let V be a F-vector space of dimension $2n$ with a symplectic form $\langle\ ,\ \rangle$. Let $G = GSp(V, \langle\ ,\ \rangle)$ and let $\{\mu\}$ be minuscule of weight n. Let $e_1, \dots, e_{2n}$ be a symplectic basis of V, i.e.

$$\langle e_i, e_j \rangle = 0,\ \langle e_{i+n}, e_{j+n} \rangle = 0,\ \langle e_i, e_{2n-j+1} \rangle = \delta_{ij}$$

for $i, j = 1, \dots, n$. Let I be a non-empty subset of $\{0, \dots, 2n-1\}$ which with $i \neq 0$ also contains $2n - i$. Let $K = K_I$ be the parahoric subgroup of $G(F)$ which is the common stabilizer of the lattices Λ_i for $i \in I$. The local model $\mathcal{M}^{\mathrm{loc}} = \mathcal{M}^{\mathrm{loc}}(G, \mu)_K$ for the triple $(G, \{\mu\}, K_I)$ represents the moduli problem on $(Sch/\mathcal{O}_F)$ which to S associates the objects $(\mathcal{F}_{i_0}, \dots, \mathcal{F}_{i_m})$ of the local model for $(GL(V), \{\mu\}, K_I)$ as in Example (2) above which satisfy the following additional conditions. If $0 \in I$, then $\mathcal{F}_0$ is isotropic for the symplectic form on $\Lambda_{0,S}$, and for each $i \in I$ with $i \neq 0$, the composition

$$\mathcal{F}_i \longrightarrow \Lambda_{i,S} \simeq \widehat{\Lambda}_{2n-i,S} \longrightarrow \widehat{\mathcal{F}}_{2n-i}$$

is the zero map. Here "hat" denotes the dual $\mathcal{O}_S$-module.

By the main result of [**G2**], $\mathcal{M}^{\mathrm{loc}}$ satisfies the conditions (i) and (ii) above.

(4) Let $G = R_{F'/F}(GL_n)$ or $G = R_{F'/F}(GSp_{2n})$, where F' is a totally ramified extension. Let $\{\mu\}$ be a minuscule conjugacy class of one-parameter subgroups and let K be a parahoric subgroup of $G(F)$. In [**PR1**] resp. [**PR2**] local models $\mathcal{M}^{\mathrm{loc}}(G, \mu)_K$ are constructed which satisfy conditions (i) and (ii) above. But in these cases it seems difficult to describe the functors that these local models represent.

In all these examples, property (ii) for local models can be considerably strengthened by identifying the special fiber of $\mathcal{M}^{\mathrm{loc}}(G, \mu)_K$ with a closed subscheme of the partial flag variety corresponding to $\widetilde{K}$ of the loop group over κ_E associated to G, [**G1**], [**PR2**]. Here, to be on the safe side, we are assuming G split. Via this identification there is a link between the theory of local models and the geometric Langlands program of Beilinson, Drinfeld et al. [**BD**].

The true significance of the local models becomes more transparent when they appear in the global context of Shimura varieties, *cf.* (7.1). It is in this context that they were first introduced, originally by P. Deligne and G. Pappas [**DP**] for the Hilbert-Blumenthal varieties, then by J. de Jong [**J**] for the Siegel moduli space and, still later, in [**RZ2**]. Here we explain their relation with the sets $X(\mu, b)_K$. Let

$$\widetilde{K}_1 = \ker(\mathcal{G}(\mathcal{O}_L) \longrightarrow \mathcal{G}(\overline{\kappa}_E)).$$

Let

$$X(\mu, b)_{K_1} = \{g \in G(L)/\widetilde{K}_1;\ \mathrm{inv}(g\widetilde{K}, b\sigma(g)\widetilde{K}) \in \mathrm{Adm}_K(\mu)\}.$$

In other words, $X(\mu, b)_{K_1}$ is the inverse image of $X(\mu, b)_K$ under $G(L)/\widetilde{K}_1 \to G(L)/\widetilde{K}$. We define a map

$$\widetilde{\gamma} : X(\mu, b)_{K_1} \longrightarrow \mathcal{M}^{\mathrm{loc}}(G, \mu)_K(\overline{\kappa}_E) \tag{6.1}$$

by

$$\widetilde{\gamma}(g\widetilde{K}_1) = g^{-1}b\sigma(g)\cdot\widetilde{K}.$$

This is well-defined since $\widetilde{K}_1$ acts trivially on $\mathcal{M}^{\mathrm{loc}}(\overline{\kappa}_E)$. Noting that $\widetilde{K}/\widetilde{K}_1$ is a principal homogeneous space under $\mathcal{G}(\overline{\kappa}_E)$, we may write $\widetilde{\gamma}$ more suggestively as a map on geometric points of algebraic stacks,

$$\gamma : X(\mu,b)_K \longrightarrow [\mathcal{M}^{\mathrm{loc}}/\mathcal{G}\otimes_{\mathcal{O}_F}\mathcal{O}_E](\overline{\kappa}_E). \tag{6.2}$$

It should be possible, at least if $\{\mu\}$ is minuscule, to equip $X(\mu,b)_K$ with the structure of an algebraic variety over $\overline{\kappa}_E$ and the map γ should be induced by a morphism of algebraic stacks over $\overline{\kappa}_E$,

$$X(\mu,b)_K \longrightarrow [\mathcal{M}^{\mathrm{loc}}\otimes_{\mathcal{O}_E}\overline{\kappa}_E/\mathcal{G}\otimes_{\mathcal{O}_F}\overline{\kappa}_E].$$

Furthermore, this morphism should be compatible with Weil descent data over κ_E on source and target.

After Görtz's theorems the most interesting question is the following conjecture. A variant was also proposed by G. Pappas [**P**], Conj. 2.12.

Conjecture 6.2. — *Assume that G is unramified over F. Let $\mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ be the local model over* $\operatorname{Spec}\mathcal{O}_E$, *corresponding to a parahoric subgroup K of $G(F)$ and a minuscule conjugacy class of cocharacters $\{\mu\}$, with its action of $\mathcal{G}\otimes_{\mathcal{O}_F}\mathcal{O}_E$. Then there exists a $\mathcal{G}\otimes_{\mathcal{O}_F}\mathcal{O}_E$-equivariant blowing up in the special fiber $\widetilde{\mathcal{M}}^{\mathrm{loc}}(G,\mu)_K \to \mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ which has semistable reduction.*

In Example 6.1, (2) the conjecture above is trivial for $r = 1$ (in this case $\mathcal{M}^{\mathrm{loc}}$ has semistable reduction). For $r = 2$, Faltings [**F2**] has constructed an equivariant blowing-up with semistable reduction, *i.e.*, the conjecture holds in this case, compare also [**L**]. In Example 6.1, (3) the existence of a semistable blowing-up is due to de Jong [**J**] for $n = 2$ and to Genestier [**Ge**] for $n = 3$.

PART II
GLOBAL THEORY

7. Geometry of the reduction of a Shimura variety

In the global part we use the following notation. Let $(\boldsymbol{G},\{h\})$ be a Shimura datum, *i.e.* $\boldsymbol{G}$ is a connected reductive group over $\mathbf{Q}$ and $\{h\}$ a $G(\mathbf{R})$-conjugacy class of homomorphisms from $R_{\mathbf{C}/\mathbf{R}}\boldsymbol{G}_m$ to $\boldsymbol{G}_{\mathbf{R}}$ satisfying the usual axioms. We fix a prime number p. We let $\mathbf{K}$ be an open compact subgroup of $\boldsymbol{G}(\mathbf{A}_f)$ which is of the form $\mathbf{K} = K^p\cdot K$, where $K^p \subset G(\mathbf{A}_f^p)$ and where $K = K_p$ is a parahoric subgroup of $G(\mathbf{Q}_p)$. We also assume that K^p is sufficiently small to exclude torsion phenomena.

The corresponding Shimura variety $\mathrm{Sh}(\boldsymbol{G},\{h\})_{\mathbf{K}}$ is a quasi-projective variety defined over the Shimura field $\mathbf{E}$, a finite number field contained in the field $\overline{\mathbf{Q}}$ of algebraic numbers. It is the field of definition of the conjugacy class $\{\mu_h\}$, where μ_h is the cocharacter corresponding to $h \in \{h\}$. Let $G = \boldsymbol{G} \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $F = \mathbf{Q}_p$. After fixing an embedding of $\overline{\mathbf{Q}}$ into an algebraic closure of $\mathbf{Q}_p$, we obtain a conjugacy class $\{\mu\}$ of one-parameter subgroups of G. It is defined over the completion E of $\mathbf{E}$ in the p-adic place $\boldsymbol{p}$ induced by this embedding. We have therefore obtained by localization a triple $(G,\{\mu\},K)$ as in the local part relative to $F = \mathbf{Q}_p$.

We make the basic assumption that the Shimura variety has a good integral model over $\mathrm{Spec}\,\mathcal{O}_{\mathbf{E},\boldsymbol{p}}$. Although we do not know how to characterize it, or how to construct it in general, we know a good number of examples all related to moduli spaces of abelian varieties. The "facts" stated below all refer to these moduli spaces, and the conjectures also concern these moduli spaces or are extrapolations to the general case. We denote by $\mathrm{Sh}(G,h)_{\mathbf{K}}$ the model over $\mathrm{Spec}\,\mathcal{O}_E$ which is obtained by base change $\mathcal{O}_{\mathbf{E},\boldsymbol{p}} \to \mathcal{O}_E$ from this good integral model.

The significance of the local model $\mathcal{M}^{\mathrm{loc}}(G,\mu)_K$ is given by the relatively representable morphism of algebraic stacks over $\mathrm{Spec}\,\mathcal{O}_E$,

$$\lambda : \mathrm{Sh}(G,h)_{\mathbf{K}} \longrightarrow [\mathcal{M}^{\mathrm{loc}}(G,\mu)_K/\mathcal{G}_{\mathcal{O}_E}]. \tag{7.1}$$

Here $\mathcal{G}$ is the group scheme over $\mathrm{Spec}\,\mathbf{Z}_p$ corresponding to K_p. The morphism λ is smooth of relative dimension $\dim G$. This statement is proved in those cases where an integral model of the Shimura variety $\mathrm{Sh}(\boldsymbol{G},h)_{\mathbf{K}}$ exists [**PR2**], [**RZ2**]. The crux is, just as in (1.11), to find a normal form for the DeRham homology modules of a polarized $\mathcal{L}$-set of abelian varieties up to prime-to-p isogeny. Once this is established, the assertion is a consequence of the theorem of Grothendieck-Messing.

We now turn to the special fiber. By associating to each point of $\mathrm{Sh}(G,h)_{\mathbf{K}}(\overline{\kappa}_E)$ the isomorphism class of its rational Dieudonné module (again this makes sense only for those Shimura varieties which are moduli spaces of abelian varieties), we obtain a map

$$\delta : \mathrm{Sh}(G,h)_{\mathbf{K}}(\overline{\kappa}_E) \longrightarrow B(G).$$

We note that by Proposition 4.4 the image of δ is contained in $B(G,\mu)$, provided that G is unramified and K is hyperspecial.

Conjecture 7.1. — $\mathrm{Im}(\delta) = B(G,\mu)$.

In particular, we expect that the basic locus is non-empty. Let $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}} = \mathrm{Sh}(G,h)_{\mathbf{K}} \otimes_{\mathcal{O}_E} \kappa_E$. The *basic locus* $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K},\mathrm{basic}}$ is the set of points whose image under δ is the unique basic element $[b_0]$ of $B(G,\mu)$. More generally, for $[b] \in B(G)$, let

$$\mathcal{S}_{[b]} = \delta^{-1}([b]). \tag{7.2}$$

***Proposition 7.2* ([RR]).** — *Each $\mathcal{S}_{[b]}$ is a locally closed subvariety of $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}}$. Furthermore, for $[b],[b'] \in B(G)$, we have*

$$\mathcal{S}_{[b]} \cap \mathrm{closure}(\mathcal{S}_{[b']}) \neq \varnothing \Longrightarrow [b] \leqslant [b'].$$

This is the group-theoretic version of Grothendieck's semicontinuity theorem, according to which the Newton vector of an isocrystal decreases under specialization (in the natural partial order on $(\mathbf{R}^n)_+$, *cf.* (4.5)). The subvarieties $\mathcal{S}_{[b]}$ are called the *Newton strata of* $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}}$. The Newton stratification of the special fiber is very mysterious.

***Questions 7.3*.** — Assume that K_p is hyperspecial. Let $[b],[b'] \in B(G,\mu)$ with $[b] \leqslant [b']$.

(i) Is $\mathcal{S}_{[b]} \cap \mathrm{closure}(\mathcal{S}_{[b']}) \neq \varnothing$?

(ii) Is $\mathcal{S}_{[b]} \cap \mathrm{closure}(\mathcal{C}) \neq \varnothing$, for every irreducible component $\mathcal{C}$ of $\mathcal{S}_{[b']}$?

(iii) Is $\mathcal{S}_{[b]} \subset \mathrm{closure}(\mathcal{S}_{[b']})$?

Obviously, (i) is implied by (iii). Here part (iii) has become known as the strong Grothendieck conjecture and (i) as the weak Grothendieck conjecture, although this denomination is somewhat abusive.

***Theorem 7.4* (Oort [O]).** — *For the Shimura variety associated to GSp_{2n} (*the Siegel moduli space*), question 7.3 (iii) has an affirmative answer. Also in this case, each Newton stratum $\mathcal{S}_{[b]}$ is equidimensional of codimension in $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}}$ equal to*

$$\mathrm{codim}\, \mathcal{S}_{[b]} = \mathrm{length}([b],[b_\mu]),$$

compare Theorem 5.9. Here $[b_\mu] = [b_1]$ denotes the μ-ordinary element of $B(G,\mu)$, cf. (5.12)[(11)].

***Conjecture 7.5* (Chai [C2]).** — *Assume K_p hyperspecial. Each Newton stratum $\mathcal{S}_{[b]}$ is equidimensional of codimension given by the above formula.*

Since the basic element in $B(G,\mu)$ is minimal, the basic locus $\overline{\mathrm{Sh}(G,h)}_{K,\mathrm{basic}}$ is a closed subvariety of the special fiber. This variety has been studied in many cases ([**LO**], [**Ka**], [**Ri**]); it is conceivable that one can give a group-theoretical "synthetic" description of it in general. At the other extreme is the μ-ordinary element $[b_\mu] \in B(G,\mu)$. It is the unique maximal element of $B(G,\mu)$, *cf.* (5.12).

***Conjecture 7.6* (Chai).** — *Let K_p be hyperspecial. The orbit of any point of $\mathcal{S}_{[b_\mu]}$ under $\boldsymbol{G}(\mathbf{A}_f^p)$ is dense in $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}}$.*

Here the action of $\boldsymbol{G}(\mathbf{A}_f^p)$ is via Hecke correspondences. In this direction we have the following results.

***Theorem 7.7* (Chai [C1]).** — *The conjecture 7.6 is true for the Siegel moduli space.*

(11) See Notes at the end n° 9

***Theorem 7.8* (Wedhorn [W]).** — *We assume that the Shimura variety* $\mathrm{Sh}(G,h)_{\mathbf{K}}$ *corresponds to a PEL-moduli problem of abelian varieties. Let* K_p *be hyperspecial. The* μ*-ordinary stratum* $\mathcal{S}_{[b_\mu]}$ *is dense in* $\overline{\mathrm{Sh}(G,h)}_{\mathbf{K}}$.

The hypothesis that K_p be hyperspecial in Wedhorn's theorem is indeed necessary, as the examples of Stamm [**S**] relative to the Hilbert-Blumenthal surfaces with Iwahori level structure at p and of Drinfeld [**D**] relative to a group which is ramified at p show.

We finally relate the maps γ and λ. Let $[b] \in B(G,\mu)$ and let $b \in G(L)$ be a representative of $[b]$. As in Definition 4.1, we let

$$J_b(\mathbf{Q}_p) = \{g \in G(L);\ g^{-1}b\sigma(g) = b\}.$$

The Newton stratum $\mathcal{S}_{[b]}$ has a covering $\widetilde{\mathcal{S}}_b$, for which we fix an isomorphism of the isocrystal in the variable point $x \in \mathcal{S}_{[b]}$ with the model isocrystal with G-structure determined by b. Then $\widetilde{\mathcal{S}}_b$ is a principal homogeneous space under $J_b(\mathbf{Q}_p)$ over $\mathcal{S}_{[b]}$. The relation between γ and λ is then given by a commutative diagram of morphisms of algebraic stacks (over $\overline{\kappa}_E$, or even compatible with Weil descent data over κ_E),

$$\begin{array}{ccc} \widetilde{\mathcal{S}}_b & \xrightarrow{\widetilde{\lambda}} & X(\mu,b)_{K_p} \\ \downarrow & & \downarrow \gamma \\ \mathcal{S}_{[b]} & \xrightarrow{\lambda} & \left[\overline{\mathcal{M}}^{\mathrm{loc}}(G,\mu)_{K_p}/\overline{\mathcal{G}}_{\kappa_E}\right]. \end{array}$$

Here $\overline{\mathcal{M}}^{\mathrm{loc}}$ resp. $\overline{\mathcal{G}}$ denote the special fibers of the local model resp. the group scheme corresponding to K_p, and by λ we denoted the restriction of (7.1) to $\mathcal{S}_{[b]}$. The commutativity of the diagram above reflects the well-known compatibility between DeRham homology and Dieudonné theory.

8. Pseudomotivic and quasi-pseudomotivic Galois gerbs

In this section and the next one we wish to give a conjectural description of the point set of $\mathrm{Sh}(G,h)_{\mathbf{K}}(\overline{\kappa}_E)$ with its action of the Frobenius automorphism. This description is modeled on the one given in [**LR**], but differs from it in an important detail, compare Remark 9.3. The idea is to partition the point set into "isogeny classes", as was done in the case of the elliptic modular curve in (1.1), and then to describe the point set of the individual isogeny classes in a manner reminiscent of (1.1) and (1.2) resp. (1.9) in the elliptic modular case. According to an idea of Grothendieck, the set of isogeny classes will be described in terms of representations of certain Galois gerbs. In this section we introduce these Galois gerbs, following Reimann's book [**Re1**].

We first explain our terminology concerning Galois gerbs. Let k be a field of characteristic zero and let $\Gamma = \mathrm{Gal}(\overline{k}/k)$ be the Galois group of a chosen algebraic

closure. A *Galois gerb over* k is an extension of topological groups

$$1 \longrightarrow G(\overline{k}) \longrightarrow \mathcal{G} \xrightarrow{q} \Gamma \longrightarrow 1. \tag{8.1}$$

Here G denotes a linear algebraic group over $\overline{k}$ and is called the *kernel* of $\mathcal{G}$. The topology on Γ is the Krull topology and the topology on $G(\overline{k})$ is the discrete topology. The extension $\mathcal{G}$ is required to satisfy the following two conditions:

(i) For any representative $g_\sigma \in \mathcal{G}$ of $\sigma \in \Gamma$, the automorphism $g \mapsto g_\sigma g g_\sigma^{-1}$ of $G(\overline{k})$ is a σ-linear algebraic automorphism.

(ii) Let K/k be a finite extension over which G is defined. Let $\Gamma_K = \mathrm{Gal}(\overline{k}/K)$ be the corresponding subgroup of Γ. We choose a section of $\mathcal{G} \to \Gamma$ over Γ_K such that the automorphism

$$g \longmapsto g_\sigma g g_\sigma^{-1}, \quad g \in G(\overline{k})$$

defines the K-structure on G. Then the resulting bijection

$$q^{-1}(\Gamma_K) = G(\overline{k}) \times \Gamma_K$$

is a homomorphism.

A *morphism between Galois gerbs* $\varphi : \mathcal{G} \to \mathcal{G}'$ is a continuous map of extensions which induces the identity map on Γ and an algebraic homomorphism on the kernel groups. Two homomorphisms φ_1 and φ_2 are called *equivalent* if there exists $g' \in G'(\overline{k})$ with $\varphi_2 = \mathrm{Int}(g) \circ \varphi_1$. A *neutral gerb* is one isomorphic to the semi-direct product

$$\mathcal{G}_G = G(\overline{k}) \rtimes \Gamma$$

associated to an algebraic group G over k. Sometimes we will have to consider a slightly more general notion. Let k' be a Galois extension of k contained in $\overline{k}$ with Galois group $\Gamma' = \mathrm{Gal}(k'/k)$. Then one defines in the obvious way the notion of a k'/k-Galois gerb, which is an extension

$$1 \longrightarrow G(k') \longrightarrow \mathcal{G} \longrightarrow \Gamma' \longrightarrow 1, \tag{8.2}$$

where G is an algebraic group defined over k'. A k'/k-Galois gerb defines in the obvious way a $\overline{k}/k$-Galois gerb (i.e. a Galois gerb): one first pulls back the extension by the surjection $\Gamma \to \Gamma'$ and then pushes out via $G(k') \to G(\overline{k}) = (G \otimes_{k'} \overline{k})(\overline{k})$.

In the sequel we will have to deal with projective limits of Galois gerbs of the previous kind. We transpose the above terminology to them. In particular two morphisms of pro-Galois gerbs will be called equivalent if they are projective limits of equivalent morphisms of Galois gerbs (in [**Re1**], B.1.1, this is called *algebraically equivalent*).

An important example is given by the Dieudonné gerb over $\mathbf{Q}_p$, *cf.* [**Re1**], B.1.2. For every $n \in \mathbf{Z}$, $n \geq 1$, there is an explicitly defined $\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p$-gerb $\mathcal{D}_n$ with kernel group $\boldsymbol{G}_m$. For n' divisible by n there is a natural homomorphism $\mathcal{D}_{n'} \to \mathcal{D}_n$ inducing the map $x \mapsto x^{n'/n}$ on the kernel groups. Let

$$\mathcal{D}_0 = \varprojlim \mathcal{D}_n \tag{8.3}$$

be the pro-$\mathbf{Q}_p^{\mathrm{un}}/\mathbf{Q}_p$-Galois gerb defined by this projective system. Then in $\mathcal{D}_0$ there is an explicit representative d_σ of the Frobenius element. The *Dieudonné gerb* $\mathcal{D}$ is the pro-$\overline{\mathbf{Q}}_p/\mathbf{Q}_p$-Galois gerb defined by $\mathcal{D}_0$.

Another Galois gerb of relevance to us is the *weight gerb* $\mathcal{W}$. This is the Galois gerb over $\mathbf{R}$ with kernel $\boldsymbol{G}_m$ which is defined by the fundamental cocycle of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ ($w_{\varrho,\sigma} = -1$ if $\varrho = \sigma =$ complex conjugation; otherwise $w_{\varrho,\sigma} = 1$).

We now recall some pertinent facts about the pro-Galois gerbs appearing in the title of this section. We fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ and for every place ℓ of $\mathbf{Q}$ an embedding $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_\ell$. Let $L/\mathbf{Q}$ be a finite Galois extension contained in $\overline{\mathbf{Q}}$.

There is an initial object $(Q^L, \nu(\infty)^L, \nu(p)^L)$ in the category of all triples (T, ν_∞, ν_p) where T is a $\mathbf{Q}$-torus which splits over L and such that $\nu_\infty, \nu_p \in X_*(T)$ satisfy

$$[L_\infty : \mathbf{R}]^{-1} \cdot \mathrm{Tr}_{L/\mathbf{Q}}(\nu_\infty) + [L_p : \mathbf{Q}_p]^{-1} \cdot \mathrm{Tr}_{L/\mathbf{Q}}(\nu_p) = 0, \tag{8.4}$$

cf. [**Re1**], B.2.2.

Similarly, assume that L is a CM-field and denote by L_0 its maximal totally real subfield. Then there is an initial object $(P^L, \nu(\infty)^L, \nu(p)^L)$ in the category of all triples (T, ν_∞, ν_p) where T is a $\mathbf{Q}$-torus which splits over L and such that $\nu_\infty, \nu_p \in X_*(T)$ are defined over $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively and such that

$$\nu_\infty + [L_p : \mathbf{Q}_p]^{-1} \cdot \mathrm{Tr}_{L/L_0}(\nu_p) = 0, \tag{8.5}$$

cf. [**Re1**], B.2.3. Since obviously condition (8.5) implies condition (8.4), there is a canonical morphism

$$(Q^L, \nu(\infty)^L, \nu(p)^L) \longrightarrow (P^L, \nu(\infty)^L, \nu(p)^L). \tag{8.6}$$

If $L \subset L'$ then we obtain morphisms of tori in the opposite direction,

$$Q^{L'} \longrightarrow Q^L, \quad P^{L'} \longrightarrow P^L. \tag{8.7}$$

Let Q resp. P denote the pro-torus defined by this projective system. Then there are homomorphisms of pro-tori over $\mathbf{Q}$,

$$\nu(\infty) : \boldsymbol{G}_{m,\mathbf{R}} \longrightarrow Q_{\mathbf{R}}, \quad \text{resp. } \nu(p) : \mathbf{D} \longrightarrow Q_{\mathbf{Q}_p} \tag{8.8}$$

whose composite with $Q \to Q^L$ is $\nu(\infty)^L$ if L is totally imaginary resp. is $[L_p : \mathbf{Q}_p] \cdot \nu(p)^L$. Here $\mathbf{D}$ denotes the pro-torus with character group equal to $\mathbf{Q}$.

Similarly, we obtain

$$\nu(\infty) : \boldsymbol{G}_m \longrightarrow P_{\mathbf{R}}, \quad \nu(p) : \mathbf{D} \longrightarrow P_{\mathbf{Q}_p}. \tag{8.9}$$

We can now introduce the pro-Galois gerbs which will be relevant for the theory of Shimura varieties.

A *quasi-pseudomotivic Galois gerb* is a pro-Galois gerb $\mathfrak{Q}$ over $\mathbf{Q}$ with kernel Q together with morphisms

$$\begin{aligned} \zeta_\infty &: \mathcal{W} \longrightarrow \mathfrak{Q}_{\mathbf{R}} \\ \zeta_p &: \mathcal{D} \longrightarrow \mathfrak{Q}_{\mathbf{Q}_p} \\ \zeta_\ell &: \Gamma_\ell \longrightarrow \mathfrak{Q}_{\mathbf{Q}_\ell}, \quad \ell \neq \infty, p \end{aligned} \tag{8.10}$$

such that $\nu(\infty)$ is induced by ζ_∞ resp. $\nu(p)$ is induced by ζ_p on the kernel group $\boldsymbol{G}_{m_{\mathbf{R}}}$ of the weight gerb $\mathcal{W}$ resp. the kernel group $\mathbf{D}$ of the Dieudonné gerb $\mathcal{D}$. In addition, a coherence condition on the family $\{\zeta_\ell;\ \ell \neq \infty, p\}$ is imposed, *cf.* [**Re1**], B.2.7. Similarly one defines a *pseudomotivic Galois gerb* $(\mathfrak{P}, \zeta_\ell^P)$. These pro-Galois gerbs are uniquely defined up to an isomorphism preserving the morphisms ζ_ℓ up to equivalence for $\ell = \infty, p$ and $\ell \neq p$. Furthermore, these isomorphisms are unique up to equivalence. There is a morphism

$$\mathfrak{Q} \longrightarrow \mathfrak{P} \tag{8.11}$$

compatible with the morphisms ζ_ℓ resp. ζ_ℓ^P and inducing the homomorphism (8.6) above on the kernel groups.

For a pair (T, μ) consisting of a $\mathbf{Q}$-torus T and an element $\mu \in X_*(T)$, there is associated a morphism of Galois gerbs

$$\psi_\mu : \mathfrak{Q} \longrightarrow \mathcal{G}_T, \tag{8.12}$$

cf. [**Re1**], B.2.10. This morphism factors through $\mathfrak{P}$ if and only if the following two conditions are satisfied,

(i) the image of $\nu(\infty)$ in $X_*(T)$ is defined over $\mathbf{Q}$

(ii) the image of $\nu(p)$ in $X_*(T) \otimes \mathbf{Q}$ satisfies the Serre condition, *i.e.*, it is defined over a CM-field and its weight is defined over $\mathbf{Q}$, *cf.* [**Re1**], B.2.11. This is the case if T itself satisfies the Serre condition, *i.e.*, $(\mathrm{id} + \iota)(\tau - \mathrm{id}) = (\tau - \mathrm{id})(\mathrm{id} + \iota)$ in $\mathrm{End}(X_*(T))$, where ι denotes the complex conjugation and $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is arbitrary.

Remark 8.1. — The pseudomotivic Galois gerb was introduced in [**LR**] with the aim of describing the points in the reduction of a Shimura variety when $(G, \{h\})$ satisfies the Serre condition. When this last condition is dropped, the pseudomotivic Galois gerb cannot suffice for this purpose. However, the quasi-pseudomotivic Galois gerb in [**LR**], introduced there to cover the cases when the Serre condition fails, does not exist (there is a fatal error in the construction of loc. cit.). Two replacements have been suggested, one by Pfau [**Pf**] and one by Reimann [**Re1**]. We follow here the latter.

9. Description of the point set in the reduction

In this section we return to the notation used in section 7. Therefore $\mathrm{Sh}(G, h)_{\mathbf{K}}$ is an integral model over $\mathrm{Spec}\, \mathcal{O}_E$ of the Shimura variety associated to $(\boldsymbol{G}, \{h\}, \mathbf{K} =$

$K^p \cdot K_p$), and $\{\mu_h\}$ is the associated conjugacy class of cocharacters of the reductive group $\boldsymbol{G}$ over Spec $\mathbf{Q}$. Our purpose is to describe the set $\mathrm{Sh}(G,h)_{\mathbf{K}}(\overline{\kappa}_E)$ of our model over Spec $\mathcal{O}_E$ of the Shimura variety $\mathrm{Sh}(\boldsymbol{G},h)_{\mathbf{K}}$. We make the blanket assumption that the derived group of $\boldsymbol{G}$ is simply connected.

The description of the points in the reduction will be in terms of admissible morphisms of pro-Galois gerbs

$$\varphi : \mathfrak{Q} \longrightarrow \mathcal{G}_G. \tag{9.1}$$

Definition 9.1. — A morphism (9.1) of pro-Galois gerbs over $\mathbf{Q}$ is called *admissible* if it satisfies the four conditions a)–d) below.

Let $D = G/G_{\mathrm{der}}$ and let μ_D be the image of $\{\mu_h\}$ in $X_*(D)$. The first condition is global:

(a) *The composition* $\mathfrak{Q} \to \mathfrak{G}_{\boldsymbol{G}} \to \mathfrak{G}_D$ *is equivalent to* ψ_{μ_D}, *cf. (8.12).*

The next three conditions will be local, one for each place of $\mathbf{Q}$. To formulate the next condition we remark that for $h \in \{h\}$ with corresponding weight homomorphism $w_h : \boldsymbol{G}_{m,\mathbf{R}} \to \boldsymbol{G}_{\mathbf{R}}$ the pair $(w_h, \mu_h(-1))$ corresponds to a morphism of Galois gerbs over $\mathbf{R}$,

$$\xi_\infty : \mathcal{W} \longrightarrow \mathcal{G}_{\boldsymbol{G}_{\mathbf{R}}}. \tag{9.2}$$

(b) *The composition* $\varphi \circ \zeta_\infty$ *is equivalent to* ξ_∞.

(c) *For any* $\ell \neq \infty, p$ *the composition* $\varphi \circ \zeta_\ell$ *is equivalent to the canonical section* ξ_ℓ *of* $\mathcal{G}_{\boldsymbol{G}_{\mathbf{Q}_\ell}}$.

For the final condition we remark that (the equivalence class of) the composition $\varphi \circ \zeta_p : \mathcal{D} \to \mathcal{G}_{\boldsymbol{G}_{\mathbf{Q}_p}}$ defines an element $[b] = [b(\varphi_p)] = [b(\varphi)]$ of $B(G)$. More precisely, let $\mathcal{D}_0$ be the explicit unramified version of the Dieudonné gerb as in [**Re1**], B.2, compare (8.3). Then there exists a morphism

$$\theta_0 : \mathcal{D}_0 \longrightarrow G(\mathbf{Q}_p^{\mathrm{un}}) \rtimes \widehat{\mathbf{Z}} \tag{9.3}$$

such that $\varphi \circ \zeta_p$ is equivalent to the pullback $\overline{\theta}_0$ of θ_0 to Γ. Then $[b]$ is the class of $b = \theta_0(d_\sigma)$, where $d_\sigma \in \mathcal{D}_0$ is the explicit representative of the Frobenius σ.

(d) *The element* $[b]$ *lies in* $B(G,\mu)$.

We note that, whereas the local components $\varphi \circ \zeta_\infty$ and $\varphi \circ \zeta_\ell$ ($\ell \neq p$) are uniquely determined up to equivalence by the Shimura data, the p-component $\varphi \circ \zeta_p$ is allowed to vary over a finite set of equivalence classes.

To every admissible morphism φ we shall associate a set $S(\varphi)$ with an action from the right of $Z(\mathbf{Q}_p) \times G(\mathbf{A}_f^p)$ and a commuting action of an automorphism Φ. Here Z denotes the center of G. For $\ell \neq \infty, p$ let

$$X_\ell = \{g \in G(\overline{\mathbf{Q}}_\ell); \quad \mathrm{Int}(g) \circ \xi_\ell = \varphi \circ \zeta_\ell\}. \tag{9.4}$$

By condition (c) this set is non-empty. We put

$$X^p = \prod\nolimits'_{\ell \neq \infty,\, p} X_\ell, \tag{9.5}$$

where the restricted product is explained in [**Re1**], B.3.6. The group $\boldsymbol{G}(\mathbf{A}_f^p)$ acts simply transitively on X^p. We also put $X_p = X(\mu, b)_{K_p}$ in the notation of (5.1), where $b \in G(L)$ is as above. It is equipped with commuting actions of $Z(\mathbf{Q}_p)$ and an operator Φ (*cf.* (5.2)). Finally we introduce the group of automorphisms $I_\varphi = \mathrm{Aut}(\varphi)$. The group $I_\varphi(\mathbf{Q})$ obviously operates on X^p. Let $g_p \in G(\overline{\mathbf{Q}}_p)$ be such that

$$\varphi_p \circ \zeta_p = \mathrm{Int}\, g_p \circ \overline{\theta}_0 \tag{9.6}$$

where the notation is as in the formulation of condition (d) above. Then we obtain an embedding

$$I_\varphi(\mathbf{Q}) \longrightarrow J_b(\mathbf{Q}_p), \quad h \longmapsto g_p\, h\, g_p^{-1} \tag{9.7}$$

where $J_b(\mathbf{Q}_p)$ is the group associated to the element b which acts on $X_p = X(\mu, b)_{K_p}$, *cf.* Definition 4.1. We now define

$$S(\varphi)_{K_p} = \varprojlim_{K^p} I_\varphi(\mathbf{Q}) \backslash X_p \times X^p / K^p, \quad \text{resp. } S(\varphi)_{\mathbf{K}} = I_\varphi(\mathbf{Q}) \backslash X_p \times X^p / K^p, \tag{9.8}$$

where the limit is over all open compact subgroups $K^p \subset \boldsymbol{G}(\mathbf{A}_f^p)$. On $S(\varphi)_{K_p}$ we have commuting actions of the automorphism Φ and of $Z(\mathbf{Q}_p) \times \boldsymbol{G}(\mathbf{A}_f^p)$ from the right.

Conjecture 9.2. — *Assume that the derived group of $\boldsymbol{G}$ is simply connected. Then for every sufficiently small K^p there is a model $\mathrm{Sh}(G, \{h\})_{\mathbf{K}}$ of $\mathrm{Sh}(\boldsymbol{G}, \{h\})_{\mathbf{K}}$ over $\mathrm{Spec}\, \mathcal{O}_{\mathbf{E}_{(p)}}$ such that the point set of its special fiber is a disjoint sum of subsets invariant under the actions of the Frobenius automorphism over κ_E and of $Z(\mathbf{Q}_p)$ and $\boldsymbol{G}(\mathbf{A}_f^p)$,*

$$\mathrm{Sh}(G, h)_{\mathbf{K}}(\overline{\kappa}_E) = \coprod_{\varphi} \mathrm{Sh}(G, h)_{\mathbf{K}, \varphi},$$

and for each φ a bijection

$$\mathrm{Sh}(G, h)_{\mathbf{K}, \varphi} = S(\varphi)_{\mathbf{K}},$$

which carries the action of the Frobenius automorphism over κ_E on the left into the action of Φ on the right and which commutes with the actions of $Z(\mathbf{Q}_p)$ and of $\boldsymbol{G}(\mathbf{A}_f^p)$ (for variable K^p) on both sides. Here the disjoint union is taken over a set of representatives of equivalence classes of admissible morphisms $\varphi : \mathfrak{Q} \to \mathcal{G}_{\boldsymbol{G}}$.

We remark that if D splits over a CM-field and the weight homomorphism w_h is defined over $\mathbf{Q}$, every admissible morphism $\varphi : \mathfrak{Q} \to \mathcal{G}_{\boldsymbol{G}}$ factors through $\mathfrak{P}$, *cf.* [**Re1**], B.3.9.

Note that we are not proposing a characterization of the model $\mathrm{Sh}(G, h)_{\mathbf{K}}$. In the case where K_p is hyperspecial, such a characterization was suggested by Milne [**M2**]. In this case we expect $\mathrm{Sh}(G, h)_{\mathbf{K}}$ to be smooth over $\mathrm{Spec}\, \mathcal{O}_{\mathbf{E}_{(p)}}$. In [**Re3**],

Reimann gives a wider class of parahoric subgroups K_p for which one should expect the smoothness of this model, and he conjectures that this class is exhaustive.

Conjecture 9.2 has been proved by Reimann [**Re1**], Prop. 6.10 (and Remark 4.9) and Prop. 7.7, in the following case: $\boldsymbol{G}$ is the multiplicative group of a quaternion algebra over a totally real field in which p is unramified, which is either totally indefinite or which is split at all primes above p. Furthermore, K_p is a maximal compact subgroup of $G(\mathbf{Q}_p)$. It has been proved for Shimura varieties of PEL-type by Milne [**M1**], when K_p is hyperspecial. In [**LR**] it is shown how the conjecture is related to a hypothetical good theory of motives, compare also [**M3**].

Remark 9.3. — We note that if Conjecture 5.2 holds, then each summand in Conjecture 9.2 is non-empty. In [**LR**] (apart from a very special case of bad reduction) it was assumed that K_p is hyperspecial, and the admissibility condition (d) was replaced by the condition that $X(\mu, b)_{K_p}$ be non-empty. From Remark 5.3 it follows that then $[b] \in B(G, \mu)$, *i.e.* condition d) above holds.

Remark 9.4.. — Assume Conjecture 9.2. In [**RZ2**] it was shown that in certain very rare cases the Shimura variety $\mathrm{Sh}(G, \{h\})_{\mathbf{K}}$ admits a p-adic uniformization by (products of) Drinfeld upper half spaces. The proof in loc. cit. is a generalization of Drinfeld's proof [**D**] of Cherednik's uniformization theorem in dimension one. From the proof in [**RZ2**] it is clear that this can occur only when all admissible morphisms are locally equivalent, provided that all summands in Conjecture 9.2 are non-empty. It comes to the same to ask that $\varphi_p \circ \zeta_p$ is basic for any admissible morphism φ. In [**K4**] it is shown that when G is adjoint simple such that $B(G, \mu)$ consists of a single element (which is then basic), then (G, μ) is the adjoint pair associated to $(D^{\times}_{1/n}, (1, 0, \ldots, 0))$ or $(D^{\times}_{-1/n}, (1, \ldots, 1, 0))$, where $D^{\times}_{1/n}$ resp. $D^{\times}_{-1/n}$ denotes the inner form of GL_n associated to the central division algebra of invariant $1/n$ resp. $-1/n$. In other words, this result of Kottwitz implies in conjunction with Conjecture 9.2 and Conjecture 5.2 that there is no hope of finding cases of p-adic uniformization essentially different from those in [**RZ2**]. In particular, in all cases of p-adic uniformization the uniformizing space will be a product of Drinfeld upper half spaces.

10. The semi-simple zeta function

We continue with the notation of the previous section. One ultimate goal of the considerations of the previous section is to determine the local factor of the zeta function of $\mathrm{Sh}(\boldsymbol{G}, h)_{\mathbf{K}}$ at p. Our present approach is through the determination of the local *semi-simple* zeta function [**R2**]. We refer to [**HN2**], § 3.1 for a systematic exposition of the concepts of the semi-simple zeta function and semi-simple trace of Frobenius. The decisive property of the semi-simple trace of Frobenius on representations of the local Galois group is that it factors through the Grothendieck group. In the case of good reduction, the semi-simple zeta function coincides with the usual zeta function.

To calculate the semi-simple zeta function we may use the Lefschetz fixed point formula. Let κ_E^n be the extension of degree n of κ_E contained in $\overline{\kappa}_E$. For $x \in \overline{\mathrm{Sh}(G,h)}_K(\kappa_E^n)$ we introduce the semisimple trace

$$\mathrm{Contr}_n(x) = \mathrm{tr}^{\mathrm{ss}}(Fr_n;\ R\Psi_x(\overline{\mathbf{Q}}_\ell)). \tag{10.1}$$

Here $R\Psi(\overline{\mathbf{Q}}_\ell)$ denotes the complex of nearby cycles. By Fr_n we denote the geometric Frobenius in $\mathrm{Gal}(\overline{\kappa}_E/\kappa_E^n)$. This is the contribution of x to the Lefschetz fixed point formula over κ_E^n. In the case of good reduction, or more generally if x is a smooth point of $\mathrm{Sh}(G,h)_{\mathbf{K}}$, then $\mathrm{Contr}_n(x) = 1$.

For an admissible homomorphism $\varphi : \mathfrak{Q} \to \mathcal{G}_{\boldsymbol{G}}$ as in Conjecture 9.2, we introduce the *contribution of* φ (or its equivalence class) *to the Lefschetz fixed point formula over* κ_E^n,

$$\mathrm{Contr}_n(\varphi) = \sum_{x \in \mathrm{Sh}(G,h)_{\mathbf{K},\varphi}(\kappa_E^n)} \mathrm{Contr}_n(x). \tag{10.2}$$

Definition 10.1. — A morphism $\varphi : \mathfrak{Q} \to \mathcal{G}_{\boldsymbol{G}}$ is called special if there exists a maximal torus $T \subset G$ and an element $\mu \in X_*(T)$ which defines a one-parameter subgroup of G in the conjugacy class $\{\mu_h\}$ such that φ is equivalent to $i \circ \psi_\mu$, *cf.* (8.12). Here $i : \mathcal{G}_T \to \mathcal{G}_{\boldsymbol{G}}$ denotes the canonical morphism defined by the inclusion of T in G.

If K_p is hyperspecial (and G_{der} is simply connected, as is assumed throughout this section), then every admissible morphism is special ([**LR**], Th. 5.3), at least if it factors through $\mathfrak{P}$.

Conjecture 10.2. — *We have* $\mathrm{Contr}_n(\varphi) = 0$ *unless* φ *is special.*

For some cases of this conjecture related to GL_2, compare [**R1**] and [**Re1**]. Note that this is really a conjecture about bad reduction. In the case of good reduction the cancellation phenomenon predicted by Conjecture 10.2 cannot occur since each point x in $\mathrm{Sh}(G,h)_{\mathbf{K}}(\kappa_E^n)$ contributes 1 in this case. This is compatible with the remark immediately preceding the statement of the conjecture, which says that the conjecture is empty if K_p is hyperspecial.

Let us explain how one would like to give a group-theoretic expression for $\mathrm{Contr}_n(x)$. Let x be represented by $(x_p, x^p) \in X_p \times X^p/K^p$ under the bijection (9.8). Let $n' = n \cdot r$, where as shortly after (5.1) $r = [\kappa_E : \mathbf{F}_p]$. Since x is fixed under the n-th power of the Frobenius over κ_E, it is fixed under the n'-th power of the absolute Frobenius and we obtain an equation of the form

$$(\Phi^{n'} x_p, x^p) = h \cdot (x_p, x^p), \tag{10.3}$$

for some $h \in I_\varphi(\mathbf{Q})$. By [**K1**], Lemma 1.4.9, it follows that there exists $c \in G(L)$ such that

$$c \cdot h^{-1} \cdot \Phi^{n'} \cdot c^{-1} = \sigma^{n'}. \tag{10.4}$$

This is an identity in the semi-direct product $G(L) \rtimes \langle\sigma\rangle$, where L is the completion of the maximal unramified extension $\mathbf{Q}_p^{\mathrm{un}}$ of $\mathbf{Q}_p$. Let $\mathbf{Q}_{p^{n'}}$ be the fixed field of $\sigma^{n'}$. Then the element $\delta \in G(L)$ which is defined by the equation

$$c \cdot (b\sigma) \cdot c^{-1} = \delta\sigma \tag{10.5}$$

lies in $G(\mathbf{Q}_{p^{n'}})$. Here $b \in G(L)$ is the element defined before (9.3). Also, always by [**K1**], we have that $x'_p = c \cdot x_p$ lies in $G(\mathbf{Q}_{p^{n'}})/\widetilde{K}^{\langle\sigma^{n'}\rangle}$. To simplify the notation put $K_{p^{n'}} = \widetilde{K}_p^{\langle\sigma^{n'}\rangle}$. Let

$$\mathcal{H} = \mathcal{H}(G(\mathbf{Q}_{p^{n'}})/\!/K_{p^{n'}}) \tag{10.6}$$

be the Hecke algebra corresponding to the parahoric subgroup $K_{p^{n'}}$. It may be conjectured that there exists an element $\phi_p^n \in \mathcal{H}$ with the following property. Let $g'_p \in G(\mathbf{Q}_{p^{n'}})$ be a representative of x'_p. Then

$$\mathrm{Contr}_n(x) = \phi_p^n(g_p'^{-1}\delta\sigma(g'_p)). \tag{10.7}$$

Appealing to [**K1**], 1.5, we therefore obtain that the contribution of the admissible homomorphism $\varphi : \mathfrak{Q} \to \mathcal{G}_{\boldsymbol{G}}$ to the Lefschetz fixed point formula over κ_E^n is a sum of terms of the form

$$v \cdot O_h(\phi^p) \cdot TO_\delta(\phi_p^n). \tag{10.8}$$

Here $O_h(\phi^p)$ is the orbital integral over $h \in I_\phi(\mathbf{Q})$ of the characteristic function of K^p and $TO_\delta(\phi_p^n)$ the twisted orbital integral of ϕ_p^n over the twisted conjugacy class of $\delta \in G(\mathbf{Q}_{p^{n'}})$. Furthermore, v is a certain volume factor.

For the function ϕ_p^n there is the following conjecture.

***Conjecture 10.3* (Kottwitz).** — *Assume that G splits over $\mathbf{Q}_{p^{n'}}$. Let $K^\circ_{p^{n'}}$ be an Iwahori subgroup of $G(\mathbf{Q}_{p^{n'}})$ contained in $K_{p^{n'}}$. Then ϕ_p^n is the image of $p^{n'\cdot\langle\varrho,\mu\rangle} \cdot z_\mu$ under the homomorphism of Hecke algebras*

$$\mathcal{H}(G(\mathbf{Q}_{p^{n'}})/\!/K^\circ_{p^{n'}}) \longrightarrow \mathcal{H}(G(\mathbf{Q}_{p^{n'}})/\!/K_{p^{n'}}).$$

Here z_μ denotes the Bernstein function in the center of the Iwahori Hecke algebra associated to μ, compare [**H2**], *2.3.*

Recall that the center of $\mathcal{H}(G(\mathbf{Q}_{p^{n'}})/\!/K^\circ_{p^{n'}})$ has a basis as a $\mathbf{C}$-vector space formed by the Bernstein functions z_λ, where λ runs through the conjugacy classes of one-parameter subgroups of G.

In this direction we have the following facts.

Theorem 10.4. — *Conjecture 10.3 holds in the following cases.*

(1) (Haines, Ngo [**HN2**]): *$G = GL_n$ or $G = GSp_{2n}$.*
(2) (Haines [**H2**]): *G is an inner form of GL_n and $\{\mu\} \ni \omega_1$ (the Drinfeld case).*

It would be interesting to extend the statement of Kottwitz' conjecture to the general case. Once this is done (and the corresponding conjecture proved!) it remains to calculate the sum over all equivalence classes of admissible homomorphisms φ of the expressions (10.8). In fact, one would like to replace the twisted orbital integrals in (10.8) by an ordinary orbital integral of a suitable function on $G(\mathbf{Q}_p)$ and compare the resulting expression with the trace of a suitable function on $G(\mathbf{A})$ in the automorphic spectrum. When the Shimura variety is not projective, one also has to deal with the contribution of the points on the boundary. Even when the Shimura variety is projective, the phenomenon of L-indistinguishability complicates the picture. But, at least these complications are of a different nature from the ones addressed in this report. They are of a group-theoretic nature, not of a geometric nature.

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Notes added June 2003

We give here some complements and mention some recent developments. A list of supplementary references can be found at the end.

(1) *The Iwahori subgroup of a torus, compare Remarks 2.2, (iii).* At the request of the referee we give more details. Let L be a complete discretely valued field with algebraically closed residue field. Let T be a torus over L. Then T has a lft Néron model over $\operatorname{Spec} \mathcal{O}_L$ (*cf.* [**BLR**] = S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Springer Verlag; 10.1, Prop. 6). Let $\mathcal{T}^1$ be the maximal subgroup scheme of finite type over $\operatorname{Spec} \mathcal{O}_L$ and let $\mathcal{T}^\circ$ be its identity component.

Proposition

(a) *$\mathcal{T}^1(\mathcal{O}_L)$ is the unique maximal bounded subgroup $\widetilde{K}$ of $T(L)$ and $K := \mathcal{T}^\circ(\mathcal{O}_L)$ is a subgroup of finite index in $\widetilde{K}$.*

(b) *K is the Iwahori subgroup of $T(L)$.*

(c) *For the group of connected components of the special fiber $\overline{\mathcal{T}}$ of $\mathcal{T}$, we have*

$$\pi_0(\overline{\mathcal{T}}) = X_*(T)_I.$$

Proof

(a) By loc. cit., $\mathcal{T}^1(\mathcal{O}_L)$ is a bounded subgroup, and hence is contained in $\widetilde{K}$. On the other hand, $\widetilde{K}$ has a Néron model in the rigid-analytic sense which is contained in $\mathcal{T}$ (*cf.* S. Bosch & K. Schlöter, Néron models in the setting of formal and rigid geometry, *Math. Ann.* **301** (1995), p. 339–362), and therefore contained in $\mathcal{T}^1$ by maximality. Hence $\widetilde{K} \subset \mathcal{T}^1(\mathcal{O}_L)$. This proves the first claim, and the second is immediate.

(b) If T is induced, then $X_*(T)_I = \operatorname{Hom}(X^*(T)^I, \mathbf{Z})$, and hence

$$\operatorname{Ker} \widetilde{\kappa}_T = \widetilde{K} = \{x \in T(L);\ |\chi(x)| = 1,\ \forall \chi \in X^*(T)^I\}.$$

On the other hand, T is a product of tori of the form $R_{F'/F}(\boldsymbol{G}_m)$ for a finite extension L' of L and then $\mathcal{T}^1$ is a product of group schemes of the form $R_{\mathcal{O}_{F'}/\mathcal{O}_F}(\boldsymbol{G}_m)$ and hence is connected, whence $\mathcal{T}^1(\mathcal{O}_L) = \mathcal{T}^\circ(\mathcal{O}_L) = K$.

In the general case we choose induced tori R, S over L and an exact sequence of I-modules,

$$X_*(S) \longrightarrow X_*(R) \longrightarrow X_*(T) \longrightarrow 0.$$

We obtain a commutative diagram with surjective vertical arrows,

$$\begin{array}{ccccccc} S(L) & \longrightarrow & R(L) & \longrightarrow & T(L) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \widetilde{\kappa}_S} & & \downarrow{\scriptstyle \widetilde{\kappa}_R} & & \downarrow{\scriptstyle \widetilde{\kappa}_T} & & \\ X_*(S)_I & \longrightarrow & X_*(R)_I & \longrightarrow & X_*(T)_I & \longrightarrow & 0. \end{array}$$

Hence the map $\operatorname{Ker} \widetilde{\kappa}_R \to \operatorname{Ker} \widetilde{\kappa}_T$ is surjective. On the other hand, the map on lft Néron models $\mathcal{R}(\mathcal{O}_L) \to \mathcal{T}(\mathcal{O}_L)$ is surjective and this implies that $\mathcal{R}^\circ(\mathcal{O}_L) \to \mathcal{T}^\circ(\mathcal{O}_L)$

is also surjective, *cf.* [BLR], 9.6, Lemma 2. From $K_R = \operatorname{Ker} \widetilde{\kappa}_R$ we conclude $K_T = \operatorname{Ker} \widetilde{\kappa}_T$.

(c) This now follows easily from the surjectivity of the Kottwitz map $\widetilde{\kappa}_T$. □

If we replace L by an arbitrary complete discretely valued field F *with perfect residue field* k, then (a) and (b) are still true (with L replaced by F). In (c) the LHS has to be interpreted as the group of connected components of $\overline{T} \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$.

(2) *The equality* $X_*(T) \cap \operatorname{Adm}(\mu) = X_*(T) \cap \operatorname{Perm}(\mu)$ *and the closure relations in affine Grassmannians, compare Prop. 3.5.* As T. Haines pointed out, Lemma 3.8 can be used to give a simple proof of the closure relations in the affine Grassmannian (for the case of SL_n, compare A. Beauville, Y. Laszlo: Conformal blocks and generalized theta functions, *Comm. Math. Phys.* **164** (1994), p. 385–419).

Let LG denote the loop group associated to the split group G over a field k. We denote by $I \subset K$ the loop group versions of the Iwahori and special maximal parahoric subgroup. We obtain the natural morphism between the affine flag variety and the affine Grassmannian,

$$\pi : LG/I \longrightarrow LG/K.$$

We choose the convention of viewing elements of the extended affine Weyl group $\widetilde{W}$ as elements in LG by sending $\lambda \in X_*(T)$ to $t^{-\lambda} := \lambda(t^{-1})$, and $w \in W_0$ to any lift in the normalizer $N_G(T) \subset G \subset LG$. With this convention, we have the BN-pair axioms

$$IwIsI = IwsI, \quad \text{if } w < ws$$
$$IwIsI = IwI \cup IwsI, \quad \text{if } ws < w,$$

where the Bruhat order $\leqslant$ is defined using the simple reflections s through the walls of the alcove fixed by I. (It is a subtle point, but it is important to use the element t^{-1} rather than t for embedding $X_*(T) \hookrightarrow LG$ here.)

Then we have the usual Demazure resolution

$$\overline{I\tau I/I} \times_I \overline{Is_1 I/I} \times_I \cdots \times_I \overline{Is_r I/I} \longrightarrow \overline{IwI/I},$$

where $w = \tau s_1 \cdots s_r$ is any reduced expression. This being a proper morphism, and an isomorphism over the inverse image of IwI/I, is all we need to conclude that the closure relations are given by the Bruhat order,

$$\overline{IwI/I} = \coprod_{w' \leqslant w} Iw'I/I.$$

Now we want to use this information to deduce the analogous fact for affine Grassmannians:

$$\overline{Kt_\mu K/K} = \coprod_{\lambda \leqslant \mu} Kt_\lambda K/K.$$

(Note that here $t_\lambda = \lambda(t^{-1})$ as above.)

Now Lemma 3.8 states that for dominant coweights λ, μ, we have

$$\lambda \leqslant \mu \Longleftrightarrow w_0 t_\lambda \leqslant w_0 t_\mu.$$

This follows since $w_0 t_\lambda \leqslant w_0 t_\mu \Leftrightarrow t_\lambda \leqslant t_\mu$. Now it follows easily that the set of elements $w' \leqslant w_0 t_\mu$ is the set of all elements $w_1 t_\lambda w_2$, where $w_1, w_2 \in W_0$, and λ is dominant with $\lambda \leqslant \mu$. So we have from the closure relations in the affine flag variety

$$\overline{Iw_0 t_\mu I/I} = \bigcup_{w' \leqslant w_0 t_\mu} Iw'I/I = \bigcup_{\lambda \leqslant \mu} \Big(\bigcup_{w_1, w_2 \in W_0} Iw_1 t_\lambda w_2 I/I \Big).$$

Applying π to both sides, we deduce the analogous fact for affine Grassmannians

$$\overline{Kt_\mu K/K} = \bigcup_{\lambda \leqslant \mu} Kt_\lambda K/K,$$

by using the following lemma.

Lemma

(1) $\pi(\cup_{w_1, w_2 \in W_0} Iw_1 t_\lambda w_2 I/I) = Kt_\lambda K/K$.

(2) $\pi(\overline{Iw_0 t_\mu I/I}) = \overline{Kt_\mu K/K}$.

Proof. — The first statement is easy. As for the second statement, the continuity of π gives

$$\pi(\overline{Iw_0 t_\mu I/I}) \subset \overline{\pi(Iw_0 t_\mu I/I)} \subset \overline{Iw_0 t_\mu K/K} \subset \overline{Kt_\mu K/K}.$$

On the other hand, since π is proper, $\pi(\overline{Iw_0 t_\mu I/I})$ is a closed set containing each $\pi(Iwt_\mu I/I) = Iwt_\mu K/K$ (for $w \in W_0$), hence also the closure of their union $Kt_\mu K/K$. □

(3) *The surjectivity of* $\mathrm{Perm}(\mu) \to \mathrm{Perm}_K(\mu)$ *and the equality* $\mathrm{Adm}_K(\mu) = \mathrm{Perm}_K(\mu)$, *compare Prop. 3.10.* This has now been proved by U. Görtz [**G3**] for $G = GL_n$ and arbitrary μ (loc. cit., Cor. 7.6) and for $G = GSp_{2n}$ and μ a sum of minuscule coweights (loc. cit., Cor. 7.10). Also, the referee points out that it might be useful in this context to generalize Deodhar's Lemma (Lemma 6.3 in [**HN1**]). This might help to prove in general the inclusion $\mathrm{Perm}_K^{\mathrm{st}}(\mu) \subset \mathrm{Adm}_K(\mu)$, compare [**HN1**]; here $\mathrm{Perm}_K^{\mathrm{st}}(\mu)$ is the obvious generalization to the parahoric case of the *strongly permissible set* $\mathrm{Perm}^{\mathrm{st}}(\mu)$ of loc. cit. As T. Haines points out, this together with the equality $\mathrm{Perm}_K^{\mathrm{st}}(\mu) = \mathrm{Perm}_K(\mu)$ for $G = GL_n$ could be used to show that $\mathrm{Adm}_K^{GSp}(\mu) = \mathrm{Perm}_K^{GL}(\mu) \cap \widetilde{W}(GSp)$, compare [**HN1**].

(4) *Structure of the affine Deligne-Lusztig varieties, compare Example 4.3.* The question for which pairs (w, b) the affine Deligne-Lusztig variety $X_w(b)$ is non-empty has been investigated by D. Reuman in his Chicago PhD thesis [**Reu1**], for $G = SL_3$ and $G = Sp_4$, and for various b (mostly $b = 1$, but also for others). For the question on the dimension of affine Deligne-Lusztig varieties compare [**Reu2**]; see also note (7) below.

(5) *Group-theoretic version of Mazur's inequality, compare Prop. 4.4.* The proof in [**RR**] is by reduction to the usual Mazur inequality between the Hodge polygon and the Newton polygon of an F-crystal. Recently R. Kottwitz [**K5**] has given

a purely group-theoretical proof by reduction to the inequality of Arthur-Harish-Chandra. That paper also contains a group-theoretic version of the Hodge-Newton decomposition of an F-crystal.

(6) *Converse to Mazur's inequality, compare Conjecture 4.6.* In [**KR2**] there is an abstract criterion (Prop. 4.6) for the non-emptyness of $X_\mu(b)$, valid for any split group. In [**K5**] this is generalized to the case of an unramified group. As to the actual converse to Mazur's inequality, this has been proved by C. Leigh [**Le**] in the case of an arbitrary classical split group. For GL_n, there is a proof, different from [**KR2**], by reduction to the converse to Fontaine's inequality, using Laffaille's theorem, *cf.* [**FR**]. For a general unramified group J.-P. Wintenberger [**Wi**] deduces the converse, in case μ is minuscule, from his theorem mentioned in (7) below.

(7) *On the sets $X(\mu,b)_K$, compare Conjecture 5.2 and Remark 5.3.* In the light of recent developments, it seems that the hypothesis that μ be minuscule is not necessary for part a) of Conjecture 5.2. Indeed, Wintenberger [**Wi**], Th. 2 proves that if $[b] \in B(G,\mu)$, then $X(b,\mu)_K \neq \varnothing$ for any parahoric subgroup, in case G is quasi-split and $\{\mu\}$ defined over L (in a slightly stronger form, replacing the usual Bruhat order in the definition of $\mathrm{Adm}(\mu)$ by the *weak* Bruhat order).

Now assume that G is unramified and that $X(b,\mu)_K \neq \varnothing$ for some Iwahori subgroup K. Then, as pointed out by the referee, it follows conversely that $[b] \in B(G,\mu)$. Indeed, let $K' = K_a$ be the hyperspecial parahoric subgroup over F fixing a vertex a of the alcove $\boldsymbol{a}$ corresponding to K. Let $w \in \mathrm{Adm}(\mu)$ with $X_w(b)_K \neq \varnothing$. Put $\nu = w(a) - a$. Then $X_\nu(b)_{K'} \neq \varnothing$. Since $w \in \mathrm{Adm}(\mu) \subset \mathrm{Perm}(\mu)$ (*cf.* Prop. 3.2), ν lies in the convex hull of $W_0\mu$. By Prop. 4.4 it follows that $[b] \in B(G,\nu) \subset B(G,\mu)$.

(8) *The dimension formula for $X(\mu,b)_K$, compare Conjecture 5.10.* Note that $\dim X(\mu,b)_K = \dim X_\mu(b)_K$ (recall that K is hyperspecial). The dimension formula is easy to verify for $G = GL_2$. It is compatible with the dimension formulae in Example 4.3 for $X_w(b)$ for $G = GL_2$ and the Iwahori subgroup K_0 in the following sense. Denoting by $\pi : G(L)/\widetilde{K}_0 \to G(L)/\widetilde{K}$ the natural projection, we have for any dominant λ

$$\pi^{-1}(X_\lambda(b)_K) = \bigcup_{w \in W_0 t_\lambda W_0} X_w(b)_{K_0},$$

and $\dim \pi^{-1}(X_\lambda(b)_K) = \dim X_\lambda(b)_K + \dim \widetilde{K}/\widetilde{K}_0$, compare [**Reu2**].

Let G be split, and assume that $\mu^\natural = 0$. Then $\overline{\mu}^* = \mu$ and $\langle \omega_i, \mu \rangle \in \mathbf{Z}\ \forall\, i = 1, \ldots, \ell$. The conjecture in this case then states

$$\dim X_\mu(b) = \langle \varrho, \mu \rangle.$$

This formula was rediscovered by D. Reuman [**Reu2**] and proved by him for $G = SL_2, SL_3, Sp_4$. He mentions work in progress with R. Kottwitz which seems to confirm this formula in general (under the assumptions spelled out above).

(9) *On the generalized Newton stratification, compare Questions 7.3 and Theorem 7.4.* The strong stratification property of the generalized Newton stratification ((iii)

of Question 7.3) has also been proved for Shimura varieties associated to certain unitary similitude groups, as well as the codimension formula analogous to Theorem 7.4, compare [**O**] and [**BW**]. These questions are also addressed in the survey [**R4**].

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