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ON THE STOKES GEOMETRY OF HIGHER ORDER PAINLEVÉ EQUATIONS

by

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Abstract. — We show several basic properties concerning the relation between the Stokes geometry (*i.e.*, configuration of Stokes curves and turning points) of a higher order Painlevé equation with a large parameter and the Stokes geometry of (one of) the underlying Lax pair. The higher-order Painlevé equation with a large parameter to be considered in this paper is one of the members of P_J-hierarchy with $J = \mathrm{I}$, II-1 or II-2, which are concretely given in Section 1. Since we deal with higher order equations, the Stokes curves may cross; some anomaly called the Nishikawa phenomenon may occur at the crossing point, and in this paper we analyze the mechanism why and how the Nishikawa phenomenon occurs. Several examples of Stokes geometry are given in Section 5 to visualize the core part of our results.

Résumé **(Sur la géométrie de Stokes des équations de Painlevé d'ordre supérieur)**

Nous exhibons plusieurs propriétés fondamentales liant, d'une part, la géométrie de Stokes (*i.e.*, la configuration des courbes de Stokes et des points tournants) d'une équation de Painlevé d'ordre supérieur à grand paramètre et, d'autre part, la géométrie de Stokes de l'une des paires de Lax sous-jacentes. L'équation de Painlevé d'ordre supérieur à grand paramètre considérée est l'une des équations de la hiérarchie P_J pour $J = \mathrm{I}$, II-1 ou II-2 que nous détaillons dans le paragraphe 1. Les équations étant d'ordre supérieur leurs lignes de Stokes peuvent se croiser et l'anomalie connue sous le nom de « phénomène de Nishikawa » peut se produire aux points de croisement. Nous analysons le mécanisme par lequel ce phénomène de Nishikawa apparaît. Plusieurs exemples de géométrie de Stokes sont donnés dans le paragraphe 5 en vue d'une visualisation de la partie centrale de nos résultats.

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0. Introduction

This paper is the first of a series of our papers on the exact WKB analysis of higher order Painlevé equations. For the sake of the clarity and the uniformity of the description we restrict our consideration in this paper to the $P_{\mathrm{I}}, P_{\mathrm{II}\text{-}1}$ and $P_{\mathrm{II}\text{-}2}$ hierarchies with a large parameter η, which are described explicitly in Section 1. Although these hierarchies are basically the same as those discussed by Shimomura ([**S2**]), Gordoa-Pickering ([**GP**]) and Gordoa-Joshi-Pickering ([**GJP**]), we need to appropriately introduce a large parameter η in their coefficients together with the underlying systems of linear differential equations (the so-called Lax pairs) so that we may develop the WKB analysis of the hierarchies in question. As is evident in the series of papers ([**KT1, AKT2, KT2, T1**]; see [**KT3**] for their résumé), the relations between the Stokes geometry for (one of) the Lax pair and the appropriately defined Stokes geometry for the Painlevé equation play the key role in the WKB analysis of the traditional Painlevé equations, *i.e.*, the second order differential equations first studied by Painlevé and Gambier. One of our main purposes of this paper is to show that the relations observed for the traditional Painlevé equations remain to hold for each member in the Painlevé hierarchies considered in this paper (Section 2). Another main purpose of this paper is to analyze why the novel and interesting phenomena numerically discovered by one of us (Y.N.) should occur in our context (Section 3). To analytically detect where the phenomena (the so-called Nishikawa phenomena) are observed, we introduce the notion of new Stokes curves in Section 4. In Section 5 we present several illuminating examples of Stokes geometry for higher order Painlevé equations and the Stokes geometry of their underlying Lax pair. Appendix A gives a proof of some properties of auxiliary functions $\mathcal{K}_j$ and K_j used in Sections 1 and 2 to write down the $P_{\mathrm{II}\text{-}1}$-hierarchy with a large parameter. In Appendix B we note that the P_{I}-hierarchy with a large parameter is equivalent to a hierarchy discussed by Gordoa and Pickering ([**GP**]) if a large parameter is appropriately introduced.

As the discussion of [**KT1**] etc. uses a Lax pair of single differential equations, the results there may look pretty different from the results in this paper, where a Lax pair of 2×2 systems is used, that is, the framework of Flaschka-Newell ([**FN**]) and Jimbo-Miwa ([**JM**]) is used instead of the framework of Okamoto ([**O**]); in particular, the apparent singularities which played an important role in [**KT1**] etc. do not appear in this paper. Hence we end this introduction with briefly recalling the geometric results in [**KT1**] which are reformulated for a Lax pair of matrix equations. For the sake of simplicity we consider only the first Painlevé equation. Thus, following [**JM**], we start with the following Lax pair:

$$(0.1) \qquad \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & \text{(0.1.a)} \\[2ex] \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0. & \text{(0.1.b)} \end{cases}$$

where

$$A = \begin{pmatrix} v(t,\eta) & 4(x-u(t,\eta)) \\ x^2 + u(t,\eta)x + u(t,\eta)^2 + t/2 & -v(t,\eta) \end{pmatrix} \tag{0.2}$$

and

$$B = \begin{pmatrix} 0 & 2 \\ x/2 + u(t,\eta) & 0 \end{pmatrix}. \tag{0.3}$$

That is, we consider an isomonodromic deformation (with respect to the variable t) of the first matrix equation (0.1.a); the second equation (0.1.b) explicitly describes this deformation. To obtain (0.1) we have introduced a large parameter η to the equation (C.2) of [**JM**, p. 437] so that the resulting compatibility condition may become the first Painlevé equation with a large parameter η in [**KT1**] etc. We have also interchanged the first component and the second component of the unknown vector ψ for the sake of uniformity of presentation in this paper. The compatibility condition of the equations (0.1.a) and (0.1.b), *i.e.*,

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB - BA) = 0 \tag{0.4}$$

can be readily seen to be equivalent to the following system (H_{I}):

$$(H_{\mathrm{I}}): \begin{cases} \dfrac{du}{dt} = \eta v \\ \dfrac{dv}{dt} = \eta(6u^2 + t) \end{cases}. \tag{0.5}$$

We next construct the so-called 0-parameter solution $(\widehat{u}, \widehat{v})$ of (H_{I}) which has the following form:

$$\widehat{u}(t,\eta) = \widehat{u}_0(t) + \eta^{-1}\widehat{u}_1(t) + \cdots, \tag{0.6}$$

$$\widehat{v}(t,\eta) = \widehat{v}_0(t) + \eta^{-1}\widehat{v}_1(t) + \cdots. \tag{0.7}$$

It is known that, although $(\widehat{u}, \widehat{v})$ is a divergent series, it is Borel summable. Note that

$$6\widehat{u}_0^2 + t = 0 \quad \text{and} \quad \widehat{v}_0 = 0 \tag{0.8}$$

hold and that $\widehat{u}_j$ and $\widehat{v}_j$ $(j \geq 1)$ are recursively determined. Substituting $(\widehat{u}, \widehat{v})$ into the coefficients of A and B, we let A_0 and B_0 denote their top degree part in η, that is,

$$A_0 = \begin{pmatrix} 0 & 4(x - \widehat{u}_0(t)) \\ x^2 + \widehat{u}_0(t)x + \widehat{u}_0(t)^2 + t/2 & 0 \end{pmatrix}, \tag{0.9}$$

$$B_0 = \begin{pmatrix} 0 & 2 \\ x/2 + \widehat{u}_0(t) & 0 \end{pmatrix}. \tag{0.10}$$

To consider the linearization of (H_{I}) at $(\widehat{u}, \widehat{v})$, we set $u = \widehat{u} + \Delta u$ and $v = \widehat{v} + \Delta v$ in (0.5) and consider the part linear in $(\Delta u, \Delta v)$. (Although the terminology "linearization" used here has a completely different meaning from that used in [**JM**], we hope there is no fear of confusions; in [**JM**] etc., the linearization of (H_{I}) means the system (0.1) of linear differential equations.) Then we obtain

$$\frac{d}{dt}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix} = \eta\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}. \tag{0.11}$$

Let C and C_0 respectively denote

$$\begin{pmatrix}0 & 1\\ 12\widehat{u} & 0\end{pmatrix} \tag{0.12}$$

and

$$\begin{pmatrix}0 & 1\\ 12\widehat{u}_0 & 0\end{pmatrix}. \tag{0.13}$$

Concerning the matrices A_0, B_0 and C_0 we find the following several relations. First of all, (0.8) immediately entails

$$A_0 = 2(x - \widehat{u}_0)B_0. \tag{0.14}$$

This relation leads to the following

Fact A

(i) *The equation* (0.1.a) *has one double turning point* $x = \widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$.

(ii) *It has one simple turning point* $x = -2\widehat{u}_0(t)$ *if* $\widehat{u}_0 \neq 0$, *and this point is a turning point of the equation* (0.1.b).

Here and in what follows we use the terminology "a turning point" for a matrix equation like (0.1.a) to mean, as usual, a point where eigenvalues of its highest degree part in η (*i.e.*, the matrix A_0 in the case of (0.1.a)) merge. In other words, a turning point is a zero of the discriminant of the characteristic equation of the highest degree part, and it is said to be simple (resp. double) if it is a simple (resp. double) zero of the discriminant. We next obtain

$$12\widehat{u}_0(t)\widehat{u}_0(t)' + 1 = 0 \tag{0.15}$$

by differentiating (0.8). Then this relation proves the following

Fact B. — *The eigenvalues* $\lambda_\pm$ *of* A_0 (*i.e.*, $\pm 2(x - \widehat{u}_0)\sqrt{x + 2\widehat{u}_0}$) *and the eigenvalues* $\mu_\pm$ *of* B_0 (*i.e.*, $\pm\sqrt{x + 2\widehat{u}_0}$) *satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{0.16}$$

The following Fact C might look too trivial to note, but for the sake of later references we note it here.

Fact C. — *We find*

$$\det(\nu - C_0) = 4\det(\mu - B_0)\Big|_{x=\widehat{u}_0, \mu=\nu/2}. \tag{0.17}$$

In what follows a point is called a turning point of a non-linear equation when it is a turning point of the linearization of the non-linear equation at a 0-parameter solution. (Hence, logically speaking, we have to specify the 0-parameter solution to define the notion of a turning point. However, the situation is usually obvious and we omit the explicit reference to the 0-parameter solution unless it is confusing.)

The following Fact D (actually together with Facts A, B and C) is observed for all traditional Painlevé equations with due modifications and it plays a crucially important role in reducing each Painlevé transcendent to Painlevé I near its simple turning point. (*Cf.* [**KT1, KT2**] and [**KT3**].)

Fact D

(i) *At the turning point $t = 0$ of the equation* (0.11), *the double turning point $x = \widehat{u}_0(t)$ merges with the simple turning point $x = -2\widehat{u}_0(t)$ in the Stokes geometry of* (0.1.a).

(ii) *We find*

$$\frac{1}{2}\int_0^t (\nu_+ - \nu_-)dt = \int_{-2\widehat{u}_0(t)}^{\widehat{u}_0(t)} (\lambda_+ - \lambda_-)dx, \tag{0.18}$$

where $\nu_\pm$ are the eigenvalues of the matrix C_0.

Since a Stokes curve of (0.1.a) that emanates from a turning point a is, by definition, a curve defined by

$$\operatorname{Im}\int_a^x (\lambda_+ - \lambda_-)dx = 0, \tag{0.19}$$

and since a Stokes curve of (0.11) that emanates from its turning point τ (actually $\tau = 0$) is given by

$$\operatorname{Im}\int_\tau^t (\nu_+ - \nu_-)dt = 0, \tag{0.20}$$

the relation (0.18) entails the following important

Fact E. — *If t ($\neq 0$) lies on a Stokes curve of* (0.11), *the Stokes geometry of* (0.1.a) *becomes degenerate in the sense that its two turning points are connected by a Stokes curve.*

In this manner the Stokes geometry of (0.11), *i.e.*, the Stokes geometry of (H_{I}) is closely related with that of (0.1.a), one of the underlying Lax pair whose monodromy data (including Stokes multipliers) are preserved.

Remark 0.1. — As is common in the literature (*e.g.*, [**V**]) in the exact WKB analysis (*i.e.*, WKB analysis based on the Borel resummation), we employ the above definition of a Stokes curve, that is, the definition making use of the imaginary part of the quantity in question; considering the imaginary part, not the real part, is most appropriate in view of the definition of the Borel resummation.

Remark 0.2. — Because of the simple character of the Stokes geometry of (0.1.a) its degeneracy occurs only when the parameter t lies on a Stokes curve of (0.11). As we will see in Section 3, this is not always the case for the higher order Painlevé equations. However, Fact E, together with Facts A, B, C and D, will be confirmed with due modifications in Section 2 for each member in the P_J-hierarchy with $J = \mathrm{I}$, II-1 or II-2.

1. P_J-hierarchy with a large parameter ($J = \mathrm{I}$, II-1 or II-2)

The purpose of this section is to explicitly write down the P_J-hierarchy with a large parameter ($J = \mathrm{I}$, II-1 or II-2) together with the underlying Lax pair.

1.1. P_{I}-hierarchy with a large parameter. — The P_{I}-hierarchy with a large parameter η is, by definition, the following family of systems of non-linear equations which are labeled by a positive integer m. As one can readily see, the first member of the family, *i.e.*, $(P_{\mathrm{I}})_1$ is reduced to (P_{I}), the Painlevé I equation with a large parameter η (in the notation of [**KT3**] etc.). This fact justifies the name "P_{I}-hierarchy". It was introduced (in a form somewhat different from the expression below) by Shimomura ([**S1, S2**]) in studying the most degenerate Garnier system. It is essentially the same as the P_{I}-hierarchy proposed earlier by Gordoa and Pickering ([**GP**]) through a particular reduction of KdV-hierarchy in a similar way as in the case of $P_{\text{II-1}}$-hierarchy discussed in the next subsection (*cf.* Appendix B). See also [**KS**].

Definition 1.1.1 **(P_{I}-hierarchy with a large parameter η)**

$$(1.1.1)\qquad (P_{\mathrm{I}})_m : \begin{cases} \dfrac{du_j}{dt} = 2\eta v_j \quad (j = 1, \dots, m), & (1.1.1.\mathrm{a}) \\ \dfrac{dv_j}{dt} = 2\eta(u_{j+1} + u_1 u_j + w_j) \quad (j = 1, \dots, m), & (1.1.1.\mathrm{b}) \\ u_{m+1} = 0, \end{cases}$$

where w_j is a polynomial of u_l and v_l ($1 \leqslant l \leqslant j$) that is determined by the following recursive relation:

$$(1.1.2)\quad w_j = \frac{1}{2}\left(\sum_{k=1}^{j} u_k u_{j+1-k}\right) + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\left(\sum_{k=1}^{j-1} v_k v_{j-k}\right) + c_j + \delta_{jm} t \quad (j = 1, \dots, m).$$

Here c_j is a constant and δ_{jm} stands for Kronecker's delta.

Remark 1.1.1

(i) $(P_{\mathrm{I}})_1$ is equivalent to

$$u_1'' = \eta^2(6u_1^2 + 4c_1 + 4t). \tag{1.1.3}$$

(ii) $(P_{\mathrm{I}})_2$ is equivalent to

$$u_1'''' = \eta^2(20u_1u_1'' + 10(u_1')^2) + \eta^4(-40u_1^3 - 16c_1u_1 + 16c_2 + 16t). \tag{1.1.4}$$

(iii) $(P_{\mathrm{I}})_3$ is equivalent to

$$\begin{aligned} u_1^{(6)} = \eta^2(28u_1u_1^{(4)} + 56u_1'u_1^{(3)} + 42(u_1'')^2) - \eta^4(280u_1^2u_1'' + 280u_1(u_1')^2 \\ + 16c_1u_1'') + \eta^6(280u_1^4 + 96c_1u_1^2 - 64c_2u_1 - 32c_1^2 + 64c_3 + 64t). \end{aligned} \tag{1.1.5}$$

To present the underlying Lax pair we first introduce the following polynomials in x with coefficients u_j, etc.

$$U(x) = x^m - \sum_{j=1}^{m} u_j x^{m-j}, \tag{1.1.6}$$

$$V(x) = \sum_{j=1}^{m} v_j x^{m-j}, \tag{1.1.7}$$

$$W(x) = \sum_{j=1}^{m} w_j x^{m-j}. \tag{1.1.8}$$

We then let A and B denote the following matrices:

$$A = \begin{pmatrix} V(x)/2 & U(x) \\ (2x^{m+1} - xU(x) + 2W(x))/4 & -V(x)/2 \end{pmatrix} \tag{1.1.9}$$

$$B = \begin{pmatrix} 0 & 2 \\ u_1 + x/2 & 0 \end{pmatrix}. \tag{1.1.10}$$

Now the required Lax pair is given by

$$(L_{\mathrm{I}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0. & (1.1.11.\mathrm{a}) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0. & (1.1.11.\mathrm{b}) \end{cases} \tag{1.1.11}$$

In order to prove that $(P_{\mathrm{I}})_m$ is the condition for the compatibility of (1.1.11.a) and (1.1.11.b), we first show the following

Lemma 1.1.1. *The system of equations $(P_{\mathrm{I}})_m$ together with the relation (1.1.2) entails*

$$\frac{dw_j}{dt} = 2\eta u_1 v_j + \delta_{jm} \quad (j = 1, \ldots, m). \tag{1.1.12}$$

Proof. — When $m = 1$ the conclusion is obvious. Hence, we suppose $m > 1$. It, then, follows from (1.1.2) that

$$w_1 = \frac{1}{2}u_1^2 + c_1. \tag{1.1.13}$$

Thus we find by (1.1.1.a)

$$w_1' = 2\eta u_1 v_1. \tag{1.1.14}$$

We, now, use the induction on j. Suppose that (1.1.12) holds for $j = 1, \dots, j_0 < m$. Then, by differentiating w_{j_0+1} determined by (1.1.2), we find

$$\begin{aligned} w'_{j_0+1} = \frac{1}{2}\Big(\sum_{k=1}^{j_0+1}(u'_k u_{j_0+2-k} + u_k u'_{j_0+2-k})\Big) \\ + \sum_{k=1}^{j_0}(u'_k w_{j_0+1-k} + u_k w'_{j_0+1-k}) \\ - \frac{1}{2}\Big(\sum_{k=1}^{j_0}(v'_k v_{j_0+1-k} + v_k v'_{j_0+1-k})\Big) + \delta_{j_0+1,m}. \end{aligned} \tag{1.1.15}$$

Then, the induction hypothesis together with $(P_1)_m$ entails

$$\begin{aligned} w'_{j_0+1} &= 2\eta\Big(\sum_{l=1}^{j_0+1} v_{j_0+2-l}u_l + \sum_{k=1}^{j_0} v_k w_{j_0+1-k} + \sum_{k=1}^{j_0} u_k u_1 v_{j_0+1-k} \\ &\qquad - \sum_{k=1}^{j_0}(u_{k+1} + u_1 u_k + w_k)v_{j_0+1-k}\Big) + \delta_{j_0+1,m} \\ &= 2\eta\Big(v_{j_0+1}u_1 + \sum_{p=1}^{j_0} v_{j_0+1-p}u_{p+1} + \sum_{k=1}^{j_0} v_k w_{j_0+1-k} \\ &\qquad + \sum_{k=1}^{j_0} u_k u_1 v_{j_0+1-k} - \sum_{k=1}^{j_0} u_{k+1}v_{j_0+1-k} - \sum_{l=1}^{j_0} w_{j_0+1-l}v_l \\ &\qquad - \sum_{k=1}^{j_0} u_1 u_k v_{j_0+1-k}\Big) + \delta_{j_0+1,m} \\ &= 2\eta v_{j_0+1}u_1 + \delta_{j_0+1,m}. \end{aligned} \tag{1.1.16}$$

Thus, the induction proceeds, completing the proof of (1.1.12). □

We, now, prove the following

Proposition 1.1.1. — *$(P_1)_m$ is the compatibility condition for* (1.1.11.a) *and* (1.1.11.b).

Proof. — The compatibility condition for (1.1.11.a) and (1.1.11.b) is given by

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta[A, B] = 0. \tag{1.1.17}$$

It follows from the definition of matrices A and B that

$$[A,B]=\begin{pmatrix} u_1U+xU-x^{m+1}-W & 2V \\ -u_1V-(xV)/2 & x^{m+1}-xU+W-u_1U \end{pmatrix}. \tag{1.1.18}$$

Writing down (1.1.17) componentwise, we find the following three relations.

$$\eta^{-1}\frac{\partial V}{\partial t}+2(u_1U+xU-x^{m+1}-W)=0, \tag{1.1.19}$$

$$\eta^{-1}\frac{\partial U}{\partial t}+2V=0, \tag{1.1.20}$$

$$\eta^{-1}(-x\frac{\partial U}{\partial t}+2\frac{\partial W}{\partial t}-2)-4u_1V-2xV=0. \tag{1.1.21}$$

Clearly, (1.1.20) is the same as (1.1.1.a). As the part of (1.1.19) with degree $m+1$ or m in x trivially vanishes, the relation (1.1.19) is reduced to

$$\eta^{-1}\frac{\partial v_j}{\partial t}+2(-u_1u_j-u_{j+1}-w_j)=0 \quad (j=1,\ldots,m). \tag{1.1.22}$$

This is nothing but (1.1.1.b). Note that $u_{m+1}=0$ by the definition. Let us, next, write down the coefficients of like powers in x in (1.1.21). The coefficient of x^m is

$$\eta^{-1}\frac{\partial u_1}{\partial t}-2v_1=0, \tag{1.1.23}$$

that of x^{m-j} $(1\leqslant j\leqslant m-1)$ is

$$\eta^{-1}\left(\frac{\partial u_{j+1}}{\partial t}+2\frac{\partial w_j}{\partial t}\right)-4u_1v_j-2v_{j+1}=0, \tag{1.1.24}$$

and that of x^0 is

$$\eta^{-1}\left(2\frac{\partial w_m}{\partial t}-2\right)-4u_1v_m=0. \tag{1.1.25}$$

Then, Lemma 1.1.1 proves that (1.1.24) is reduced to

$$\eta^{-1}\frac{\partial u_{j+1}}{\partial t}=2v_{j+1} \quad (j=1,\ldots,m-1). \tag{1.1.26}$$

The same lemma entails that (1.1.25) is a trivial relation. The combination of (1.1.23) and (1.1.26) is again the same as (1.1.1.a). Thus we have confirmed that $(P_{\mathrm{I}})_m$ is the compatibility condition of (1.1.11.a) and (1.1.11.b.). □

1.2. $P_{\mathrm{II}\text{-}1}$-hierarchy with a large parameter. — The $P_{\mathrm{II}\text{-}1}$-hierarchy (with a large parameter) is a hierarchy obtained by a similarity reduction of the KdV hierarchy. As is shown by Gordoa and Pickering in [**GP**], this hierarchy together with its underlying Lax pair can be reproduced also by their scheme called "nonisospectral scattering problems". Here, following the formulation of [**GP**], we define the $P_{\mathrm{II}\text{-}1}$-hierarchy with a large parameter in the following manner:

Definition 1.2.1 **($P_{\text{II-1}}$-hierarchy with a large parameter η)**

$$(P_{\text{II-1}})_m : \Big(\eta^{-1}\frac{\partial}{\partial t} + 2v\Big)K_m + g(2tv + \eta^{-1}) + c = 0. \tag{1.2.1}$$

Here, m is a positive integer that labels a member of the hierarchy, $v = v(t)$ is an unknown function, g ($\neq 0$) and c are constants, and K_j is a polynomial of v and its derivatives defined by the following recursive relation

$$\eta^{-1}\partial_t K_{j+1} = (\eta^{-3}\partial_t^3 - 4\eta^{-1}(v^2 - \eta^{-1}v')\partial_t - 2(2vv' - \eta^{-1}v''))K_j \tag{1.2.2}$$

for $j \geqslant 0$ with $K_0 = 1/2$ and $\partial_t = \partial/\partial t$.

Remark 1.2.1. — Although the differentiation ∂_t appears in the left-hand side of the recursive relation (1.2.2), we can define each K_j so that it becomes a polynomial only of v and its derivatives and independent of any integrated terms like $\partial_t^{-1}v$. For the proof see Appendix A. For example, first few members of K_j are given as follows:

$$K_0 = 1/2, \tag{1.2.3}$$

$$K_1 = -v^2 + \eta^{-1}v', \tag{1.2.4}$$

$$K_2 = 3v^4 - 6\eta^{-1}v^2v' + \eta^{-2}\big((v')^2 - 2vv''\big) + \eta^{-3}v^{(3)}, \tag{1.2.5}$$

$$\begin{aligned} K_3 = {} & -10v^6 + 30\eta^{-1}v^4v' + \eta^{-2}\big(10v^2(v')^2 + 20v^3v''\big) \\ & +\eta^{-3}\big(-10(v')^3 - 40vv'v'' - 10v^2v^{(3)}\big) \\ & +\eta^{-4}\big(-(v'')^2 + 2v'v^{(3)} - 2vv^{(4)}\big) + \eta^{-5}v^{(5)}. \end{aligned} \tag{1.2.6}$$

Remark 1.2.2. — By an induction we can also show that

$$K_j = \frac{(-1)^j 2^{j-1}(2j-1)!!}{j!}v^{2j} + O(\eta^{-1}), \tag{1.2.7}$$

where $(2j-1)!! = (2j-1)\cdot(2j-3)\cdot\cdots 3\cdot 1$.

Remark 1.2.3

(i) $(P_{\text{II-1}})_1$ is

$$\eta^{-2}v'' = v^3 - g(2tv + \eta^{-1}) - c. \tag{1.2.8}$$

This is equivalent to (P_{II}), the Painlevé II equation with a large parameter η.

(ii) $(P_{\text{II-1}})_2$ is

$$\eta^{-4}v^{(4)} = \eta^{-2}(10v^2v'' + 10v(v')^2) - 6v^5 - g(2tv + \eta^{-1}) - c. \tag{1.2.9}$$

The underlying Lax pair of (1.2.1) is

$$(L_{\text{II-1}})_m : \begin{cases} \Big(\dfrac{\partial}{\partial x} - \eta A\Big)\psi = 0, & (1.2.10.\text{a}) \\ \Big(\dfrac{\partial}{\partial t} - \eta B\Big)\psi = 0. & (1.2.10.\text{b}) \end{cases} \tag{1.2.10}$$

where

$$A=\frac{1}{4xg}\begin{pmatrix}-\eta^{-1}\partial_t T_m & 2T_m\\ 2qT_m-\eta^{-2}\partial_t^2 T_m & \eta^{-1}\partial_t T_m\end{pmatrix},\quad B=\begin{pmatrix}0 & 1\\ q & 0\end{pmatrix}. \tag{1.2.11}$$

Here, T_m and q respectively denote the following functions:

$$T_m = gt+\sum_{k=0}^{m}(4x)^k K_{m-k}, \tag{1.2.12}$$

$$q = x+v^2-\eta^{-1}v'. \tag{1.2.13}$$

Our $P_{\text{II-1}}$-hierarchy (1.2.1) is obtained from the hierarchy

$$(\partial_t+2v)\mathcal{K}_m+g(2tv+1)+c=0 \tag{1.2.14}$$

discussed by Gordoa and Pickering through the scaling

$$v\longmapsto \eta^{1/(2m+1)}v,\quad t\longmapsto \eta^{2m/(2m+1)}t,\quad g\longmapsto g,\quad c\longmapsto \eta c. \tag{1.2.15}$$

Here, $\mathcal{K}_j$ is a polynomial of v and its derivatives, satisfying the recursive relation

$$\partial_t\mathcal{K}_{j+1}=(\partial_t^3+4(v'-v^2)\partial_t+2(v'-v^2)')\mathcal{K}_j. \tag{1.2.16}$$

Note that by the scaling (1.2.15) $\mathcal{K}_j$ is transformed to $\eta^{2j/(2m+1)}K_j$ and each K_j can be written as

$$K_j=K_j[v,\eta]=K_{j,0}[v]+\eta^{-1}K_{j,1}[v]+\cdots+\eta^{-2j+1}K_{j,2j-1}[v] \tag{1.2.17}$$

with $K_{j,l}$ being a polynomial of v and its derivatives independent of η. As is explained also in [**GP**, III, pp. 5751–5755], (1.2.14) is the compatibility condition for the following system of linear ordinary differential equations:

$$\begin{cases} 4xg\dfrac{\partial}{\partial x}\psi=(-\partial_t T_m+2T_m\dfrac{\partial}{\partial x})\psi,\\ \left(\dfrac{\partial^2}{\partial t^2}+v'-v^2-x\right)\psi=0,\end{cases} \tag{1.2.18}$$

or for the system equivalent to it:

$$\frac{\partial}{\partial x}\psi=\widetilde{A}\psi,\quad \frac{\partial}{\partial t}\psi=\widetilde{B}\psi, \tag{1.2.19}$$

where

$$\widetilde{A}=\frac{1}{4xg}\begin{pmatrix}-\partial_t T_m & 2T_m\\ 2qT_m-\partial_t^2 T_m & \partial_t T_m\end{pmatrix},\quad \widetilde{B}=\begin{pmatrix}0 & 1\\ q & 0\end{pmatrix}. \tag{1.2.20}$$

Here,

$$T_m=gt+\sum_{k=0}^{m}(4x)^k\mathcal{K}_{m-k}\quad\text{and}\quad q=x+v^2-v'. \tag{1.2.21}$$

As a matter of fact, by straightforward computations we readily find that

$$\frac{\partial \widetilde{A}}{\partial t} - \frac{\partial \widetilde{B}}{\partial x} + [\widetilde{A}, \widetilde{B}] = \begin{pmatrix} 0 & 0 \\ \Delta & 0 \end{pmatrix} \tag{1.2.22}$$

with

$$\Delta = -\frac{1}{4xg}(\partial_t - 2v)\partial_t\big\{(\partial_t + 2v)\mathcal{K}_m + g(2tv+1)\big\}. \tag{1.2.23}$$

Thus (1.2.14) is the compatibility condition for the Lax pair (1.2.19) with (1.2.20). Our Lax pair (1.2.10) and (1.2.11) are obtained from (1.2.19) and (1.2.20) through the scaling (1.2.15) and $x \mapsto \eta^{2/(2m+1)}x$.

1.3. $P_{\text{II-2}}$-hierarchy with a large parameter. — The $P_{\text{II-2}}$-hierarchy with a large parameter is obtained from the hierarchy introduced by Gordoa-Joshi-Pickering in [**GJP**, p. 337] through an appropriate scaling of the variables and constants. Here, we content ourselves with explicitly listing up the final results and we refer the reader to [**N1**] and [**N2**] for the details of the discussion.

Definition 1.3.1 **($P_{\text{II-2}}$-hierarchy with a large parameter η)**

$$(P_{\text{II-2}})_m : \begin{cases} K_{m+1} + \displaystyle\sum_{j=1}^{m-1} c_j K_j + gt = 0, \\ L_{m+1} + \displaystyle\sum_{j=1}^{m-1} c_j L_j \qquad\quad = \delta. \end{cases} \tag{1.3.1}$$

Here, g ($\neq 0$), c_j and δ are constants, and K_j and L_j are polynomials of unknown functions u, v and their derivatives defined by the following recursive relation

$$\eta^{-1}\partial_t \begin{pmatrix} K_{j+1} \\ L_{j+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \eta^{-1}u' + u\eta^{-1}\partial_t - \eta^{-2}\partial_t^2 & 2\eta^{-1}\partial_t \\ 2\eta^{-1}v\partial_t + \eta^{-1}v_t & u\eta^{-1}\partial_t + \eta^{-2}\partial_t^2 \end{pmatrix}\begin{pmatrix} K_j \\ L_j \end{pmatrix} \tag{1.3.2}$$

($j \geqslant 0$) with $K_0 = 2$ and $L_0 = 0$.

Remark 1.3.1. — As in the case of $P_{\text{II-1}}$-hierarchy, we can show that K_j and L_j become polynomials of u, v and their derivatives. For the proof see [**N1**] and [**N2**]. First few members of K_j and L_j are given as follows:

$$\begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.3.3}$$

$$\begin{pmatrix} K_2 \\ L_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u^2 + 2v - \eta^{-1}u' \\ 2uv + \eta^{-1}v' \end{pmatrix}, \tag{1.3.4}$$

$$\begin{pmatrix} K_3 \\ L_3 \end{pmatrix} = \left(\frac{1}{2}\right)^2\begin{pmatrix} u^3 + 6uv - 3\eta^{-1}uu' + \eta^{-2}u'' \\ 3u^2v + 3v^2 + 3\eta^{-1}uv' + \eta^{-2}v'' \end{pmatrix}. \tag{1.3.5}$$

Remark 1.3.2

(i) $(P_{\mathrm{II}-2})_1$ is reduced to

$$\eta^{-2}u'' = 2u^3 + 2g\left(2tu + \eta^{-1}\right) + 4\delta. \tag{1.3.6}$$

(ii) $(P_{\mathrm{II}-2})_2$ is reduced to

$$\begin{aligned}\eta^{-4}u^{(4)} = \frac{1}{2u^2}\Big[&\eta^{-4}\left(-4(u')^2u'' + 3u(u'')^2 + 4uu'u^{(3)}\right) + 16\eta^{-3}guu' \\ &+ \eta^{-2}(-16gt(u')^2 + 5u^3(u')^2 + 16gtuu'' + 10u^4u'') - 24\eta^{-1}gu^3 \\ &+ \left(16g^2t^2u - 16{c_1}^2u^3 - 48\delta u^3 - 16gtu^4 - 24c_1u^5 - 5u^7\right)\Big].\end{aligned} \tag{1.3.7}$$

The underlying Lax pair of $(P_{\text{II-2}})_m$ is

$$(L_{\text{II-2}})_m : \begin{cases} \left(\dfrac{\partial}{\partial x} - \eta A\right)\psi = 0, & (1.3.8.\mathrm{a}) \\ \left(\dfrac{\partial}{\partial t} - \eta B\right)\psi = 0, & (1.3.8.\mathrm{b}) \end{cases} \tag{1.3.8}$$

where

$$A = A^{(m)} + c_{m-1}A^{(m-2)} + c_{m-2}A^{(m-3)} + \cdots + c_1A^{(0)} \tag{1.3.9}$$

$$B = \begin{pmatrix} -x + u/2 & 1 \\ -v & x - u/2 \end{pmatrix}. \tag{1.3.10}$$

Here, $A^{(j)}$ denotes

$$A^{(j)} = \frac{1}{g}\begin{pmatrix} -(2x-u)T_j - \eta^{-1}\partial_tT_j & 2T_j \\ \begin{array}{r}-2vT_j - \eta^{-1}\partial_t\left\{(2x-u)T_j\right. \\ \left.+\partial_tT_j + K_{j+1}\right\}\end{array} & (2x-u)T_j + \eta^{-1}\partial_tT_j \end{pmatrix}, \tag{1.3.11}$$

where

$$T_m = \frac{1}{2}\sum_{j=0}^{m} x^{m-j}K_j. \tag{1.3.12}$$

2. Relations between the Stokes geometry of the (P_J)-hierarchies and that of their underlying Lax pairs

In this section we prove that the relations, being similar to the Facts A $\sim$ E for the traditional Painlevé equations explained in Introduction, also hold between the Stokes geometry of a member in the (P_J)-hierarchies (J = I, II-1 and II-2) and that of its underlying Lax pair.

2.1. Case of the (P_{I})-hierarchy. — As in the case of the traditional Painlevé equations, we first construct what we call the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ of the following form:

$$\widehat{u}_j(t,\eta) = \widehat{u}_{j,0}(t) + \eta^{-1}\widehat{u}_{j,1}(t) + \cdots, \tag{2.1.1}$$

$$\widehat{v}_j(t,\eta) = \widehat{v}_{j,0}(t) + \eta^{-1}\widehat{v}_{j,1}(t) + \cdots. \tag{2.1.2}$$

Substituting these expansions into (1.1.1.a) and (1.1.1.b), we readily find that $\widehat{v}_{j,0}$ $(j = 1,\ldots,m)$ identically vanishes and $\widehat{u}_{j,0}$ should satisfy

$$\widehat{u}_{j+1,0} + \widehat{u}_{1,0}\widehat{u}_{j,0} + \widehat{w}_{j,0} = 0 \quad (j = 1,\ldots,m). \tag{2.1.3}$$

We can also observe that $\widehat{u}_{j,k}$ and $\widehat{v}_{j,k}$ $(k \geq 1)$ are recursively determined once $\widehat{v}_{j,0}$ is taken to be zero and $\widehat{u}_{j,0}$ is chosen so that it satisfies the algebraic equation (2.1.3). Note that the top order part $\widehat{w}_{j,0}$ of w_j satisfies a recursive relation

$$\widehat{w}_{j,0} = \frac{1}{2}\Big(\sum_{k=1}^{j}\widehat{u}_{k,0}\widehat{u}_{j+1-k,0}\Big) + \sum_{k=1}^{j-1}\widehat{u}_{k,0}\widehat{w}_{j-k,0} + c_j + \delta_{jm}t \quad (j = 1,\ldots,m) \tag{2.1.4}$$

corresponding to (1.1.2), and that (2.1.3) together with (2.1.4) recursively determines each $\widehat{u}_{j,0}$ $(j = 1,\ldots,m)$ as a polynomial of $\widehat{u}_{1,0}$. In particular, as $\widehat{u}_{m+1,0} = 0$ by the definition, (2.1.3) for $j = m$ provides an algebraic equation for $\widehat{u}_{1,0}$. Hence all $\widehat{u}_{j,0}$ and $\widehat{v}_{j,0}$ are determined algebraically and the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ is thus constructed.

Remark 2.1.1. — By using an induction on j we can verify that $\widehat{u}_{j,0}$ is a polynomial of $\widehat{u}_{1,0}$ with degree at most j. Furthermore, letting $(-1)^{j-1}\alpha_j\widehat{u}_{1,0}^{j}$ denote the top degree part of $\widehat{u}_{j,0}$, we obtain the following recursive relation for $\{\alpha_j\}$ as a consequence of (2.1.3) and (2.1.4):

$$\alpha_{j+1} = \alpha_j + \frac{1}{2}\Big(\sum_{k=1}^{j}\alpha_k\alpha_{j+1-k}\Big) - \sum_{k=1}^{j-1}\alpha_k(\alpha_{j+1-k} - \alpha_{j-k}) \quad (j = 1,\ldots,m) \tag{2.1.5}$$

and $\alpha_1 = 1$. Since

$$\widetilde{\alpha}_j = (-2)^j\frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-j+1)}{j!} = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2j-1)}{j!} \tag{2.1.6}$$

satisfies the same recursive relation (2.1.5), we can conclude that $\alpha_j = \widetilde{\alpha}_j \neq 0$. Thus, $\widehat{u}_{1,0}$ is a solution of an algebraic equation with degree exactly equal to $m+1$ and, roughly speaking, there exist $m+1$ 0-parameter solutions of $(P_{\mathrm{I}})_m$.

We, next, substitute the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$ of $(P_{\mathrm{I}})_m$ into the coefficients A and B respectively given by (1.1.9) and (1.1.10), *i.e.*, the coefficients of its underlying Lax pair. Then, their top order parts A_0 and B_0 become

$$A_0 = \begin{pmatrix} V_0(x)/2 & U_0(x) \\ (2x^{m+1} - xU_0(x) + 2W_0(x))/4 & -V_0(x)/2 \end{pmatrix}, \tag{2.1.7}$$

$$B_0 = \begin{pmatrix} 0 & 2 \\ \widehat{u}_{1,0} + x/2 & 0 \end{pmatrix}, \tag{2.1.8}$$

where $U_0(x)$, $V_0(x)$ and $W_0(x)$ respectively denote the top order parts (in η) of $U(x)$, $V(x)$ and $W(x)$, that is,

$$U_0(x) = x^m - \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m-j}, \tag{2.1.9}$$

$$V_0(x) = \sum_{j=1}^{m} \widehat{v}_{j,0} x^{m-j}, \tag{2.1.10}$$

$$W_0(x) = \sum_{j=1}^{m} \widehat{w}_{j,0} x^{m-j}. \tag{2.1.11}$$

Here, it follows from (2.1.3) that

$$\begin{aligned} &2x^{m+1} - xU_0(x) + 2W_0(x) \\ &\quad = x^{m+1} + \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} + 2\sum_{j=1}^{m} \widehat{w}_{j,0} x^{m-j} \\ &\quad = x^{m+1} + \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} - 2\sum_{j=1}^{m} (\widehat{u}_{j+1,0} + \widehat{u}_{1,0}\widehat{u}_{j,0}) x^{m-j} \\ &\quad = x^{m+1} + 2\widehat{u}_{1,0} x^m - \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m+1-j} - 2\widehat{u}_{1,0} \sum_{j=1}^{m} \widehat{u}_{j,0} x^{m-j} \\ &\quad = (x + 2\widehat{u}_{1,0}) U_0(x) \end{aligned} \tag{2.1.12}$$

holds. This immediately entails

$$A_0 = \frac{U_0(x)}{2} B_0, \tag{2.1.13}$$

and hence, as a generalization of Fact A for the traditional Painlevé equations, we obtain the following

Proposition 2.1.1

(i) *The equation* (1.1.11.a) *has* m (*generically*) *double turning points* (*which will be denoted by* $x = b_1(t), \ldots, x = b_m(t)$ *in what follows*), *and each double turning point is a root of* $U_0(x) = 0$.

(ii) *It has one (generically) simple turning point* $x = -2\widehat{u}_{1,0}(t)$, *(which will be denoted by* $x = a(t)$ *for short in what follows), and this point is simultaneously a turning point of the equation* (1.1.11.b).

We can also prove Fact B in a quite general context, that is, even for $(P_{\mathrm{I}})_m$ we have

Proposition 2.1.2. — *The eigenvalues* $\lambda_\pm$ *of* A_0 *and the eigenvalues* $\mu_\pm$ *of* B_0 *satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{2.1.14}$$

For the proof of Proposition 2.1.2 see [**T2**], where the method of diagonalization for the Lax pair $(L_{\mathrm{I}})_m$ is used to prove the proposition in question.

Now, to define the Stokes geometry of $(P_{\mathrm{I}})_m$, we consider the linearization of $(P_{\mathrm{I}})_m$ at the 0-parameter solution $(\widehat{u}_j, \widehat{v}_j)$, that is, we take the part linear in $(\Delta u_j, \Delta v_j)$ after the substitution $u_j = \widehat{u}_j + \Delta u_j$ and $v_j = \widehat{v}_j + \Delta v_j$ in $(P_{\mathrm{I}})_m$. We then obtain

$$\begin{cases} \dfrac{d}{dt}\Delta u_j = 2\eta \Delta v_j \quad (j = 1, \ldots, m), \\ \dfrac{d}{dt}\Delta v_j = 2\eta(\Delta u_{j+1} + \widehat{u}_1 \Delta u_j + \widehat{u}_j \Delta u_1 + \Delta w_j) \quad (j = 1, \ldots, m). \end{cases} \tag{2.1.15}$$

This defines a system of first order linear ordinary differential equations for $(\Delta u_j, \Delta v_j)$. We write this system as

$$\frac{d}{dt}\begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \vdots \\ \Delta v_m \end{pmatrix} = \eta C(t,\eta) \begin{pmatrix} \Delta u_1 \\ \Delta v_1 \\ \Delta u_2 \\ \Delta v_2 \\ \vdots \\ \Delta v_m \end{pmatrix}. \tag{2.1.16}$$

As in the case of the traditional Painlevé equations, we then call a turning point (resp. Stokes curve) of (2.1.16) a turning point (resp. Stokes curve) of our non-linear equation $(P_{\mathrm{I}})_m$. That is, if we let C_0 denote the top order part (*i.e.*, the part of order 0 in η) of the coefficient matrix $C(t,\eta)$ of the right-hand side of (2.1.16), a turning point τ of $(P_{\mathrm{I}})_m$ is a point where two eigenvalues $\nu_j(t)$ $(j = 1, 2)$ of C_0 merge and a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from τ is given by $\operatorname{Im}\int_\tau^t (\nu_1 - \nu_2)dt = 0$. To write down C_0 in an explicit manner, we note the following

Lemma 2.1.1

$$\Delta w_j = \widehat{u}_{1,0}\Delta u_j + O(\eta^{-1}) \qquad (j = 1, \ldots, m). \tag{2.1.17}$$

Proof. — In parallel with the proof of Lemma 1.1.1, we use the induction on j to prove (2.1.17). In the case of $j = 1$ (1.1.13) immediately entails

$$\Delta w_1 = \widehat{u}_1 \Delta u_1. \tag{2.1.18}$$

We now suppose that (2.1.17) holds for $j = 1, \ldots, j_0 (< m)$. It follows from (1.1.2) that

$$\begin{aligned}\Delta w_{j_0+1} = \sum_{k=1}^{j_0+1} \widehat{u}_{j_0+2-k} \Delta u_k \\ + \sum_{k=1}^{j_0} (\widehat{u}_{j_0+1-k} \Delta w_k + \widehat{w}_{j_0+1-k} \Delta u_k) - \sum_{k=1}^{j_0} \widehat{v}_{j_0+1-k} \Delta v_k.\end{aligned} \tag{2.1.19}$$

Then by the induction hypothesis together with the fact $\widehat{v}_{j,0} = 0$ we find

$$\Delta w_{j_0+1} = \sum_{k=1}^{j_0+1} \widehat{u}_{j_0+2-k} \Delta u_k + \sum_{k=1}^{j_0} (\widehat{u}_{j_0+1-k} \widehat{u}_{1,0} + \widehat{w}_{j_0+1-k}) \Delta u_k + O(\eta^{-1}). \tag{2.1.20}$$

Since we know by (2.1.3) that $\widehat{u}_{j+1,0} + \widehat{u}_{1,0} \widehat{u}_{j,0} + \widehat{w}_{j,0} = 0$ holds for $j = 1, \ldots, m$, we obtain from (2.1.20) the following:

$$\Delta w_{j_0+1} = \widehat{u}_{1,0} \Delta u_{j_0+1} + O(\eta^{-1}). \tag{2.1.21}$$

This completes the proof of (2.1.17). $\square$

In view of (2.1.15) and Lemma 2.1.1 we find that the explicit form of C_0 is given by

$$C_0 = \left(\begin{array}{cc|cc|cc|c} 0 & 2 & & & & & \\ 6\widehat{u}_{1,0} & 0 & 2 & & & & \\ \hline 0 & & 0 & 2 & & & \\ 2\widehat{u}_{2,0} & & 4\widehat{u}_{1,0} & 0 & 2 & & \\ 0 & & & & 0 & 2 & \\ 2\widehat{u}_{3,0} & & & & 4\widehat{u}_{1,0} & 0 & \\ \vdots & & & & & & \ddots \end{array} \right). \tag{2.1.22}$$

This leads to the following

Proposition 2.1.3. — *We have the relation*

$$\begin{aligned}\det(\nu - C_0) &= 4^m \prod_{j=1}^{m} \det(\mu - B_0) \Big|_{x=b_j(t), \mu=\nu/2} \\ &= \prod_{j=1}^{m} \big(\nu^2 - 4(2\widehat{u}_{1,0}(t) + b_j(t))\big),\end{aligned} \tag{2.1.23}$$

where $b_j(t)$ denotes a double turning point of (1.1.11.a)*, i.e., a root of* $U_0(x) = 0$ (*cf. Proposition 2.1.1*).

Proof. — Expanding

$$\det(2\mu - C_0) = 4^m \begin{vmatrix} \mu & -1 & & & & & \\ -3\widehat{u}_{1,0} & \mu & -1 & & & & \\ 0 & & \mu & -1 & & & \\ -\widehat{u}_{2,0} & & -2\widehat{u}_{1,0} & \mu & -1 & & \\ 0 & & & & \mu & -1 & \\ -\widehat{u}_{3,0} & & & & -2\widehat{u}_{1,0} & \mu & \\ \vdots & & & & & & \ddots \end{vmatrix} \tag{2.1.24}$$

with respect to the first column, we find

$$\begin{aligned} \det(2\mu - C_0) &= 4^m[\mu^2(\mu^2 - 2\widehat{u}_{1,0})^{m-1} - 3\widehat{u}_{1,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-1} \\ &\qquad - \widehat{u}_{2,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-2} - \cdots - \widehat{u}_{m,0}] \\ &= 4^m[(\mu^2 - 2\widehat{u}_{1,0})^m - \widehat{u}_{1,0}(\mu^2 - 2\widehat{u}_{1,0})^{m-1} - \cdots - \widehat{u}_{m,0}] \\ &= 4^m U_0(\mu^2 - 2\widehat{u}_{1,0}). \end{aligned} \tag{2.1.25}$$

This immediately entails (2.1.23). □

Proposition 2.1.3 claims that $\pm 2\sqrt{2\widehat{u}_{1,0}(t) + b_j(t)}$ is an eigenvalue of C_0 for $j = 1, \ldots, m$. We can thus label each eigenvalue of C_0 by a combination of the index j and the sign; we let $\nu_{j,\pm}$ denote $\pm 2\sqrt{2\widehat{u}_{1,0}(t) + b_j(t)}$ in what follows. Note that $\nu_{j,+} + \nu_{j,-} = 0$ holds for every j.

It also follows from Proposition 2.1.3 that $\det(\nu - C_0) = 0$ has the form $f(\nu^2, t)$ with some polynomial f of degree m. This implies that there are two kinds of turning points for $(P_{\mathrm{I}})_m$: (i) A turning point where the degree 0 part of f vanishes ("a turning point of the first kind"), and (ii) a turning point where the discriminant of f vanishes ("a turning point of the second kind"). Then, as in the case of the traditional Painlevé equations, we can obtain the following relations between the Stokes geometry of $(P_{\mathrm{I}})_m$ and that of its underlying Lax pair $(L_{\mathrm{I}})_m$.

Proposition 2.1.4

(i) *Let $t = \tau^{\mathrm{I}}$ be a turning point of the first kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{I}}$ a double turning point $x = b_j(t)$ merges with the simple turning point $x = a(t) = -2\widehat{u}_{1,0}(t)$ in the Stokes geometry of* (1.1.11.a). *Consequently the two eigenvalues $\nu_{j,\pm}$ of C_0 merge and vanish at $t = \tau^{\mathrm{I}}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t} (\nu_{j,+} - \nu_{j,-})dt = \int_{a(t)}^{b_j(t)} (\lambda_+ - \lambda_-)dx. \tag{2.1.26}$$

(ii) *Let $t = \tau^{\mathrm{II}}$ be a turning point of the second kind of $(P_{\mathrm{I}})_m$. Then at $t = \tau^{\mathrm{II}}$ a double turning point $x = b_j(t)$ merges with another double turning point $x = b_{j'}(t)$. Consequently two eigenvalues $\nu_{j,+}$ and $\nu_{j',+}$ of C_0 merge at $t = \tau^{\mathrm{II}}$, and so do $\nu_{j,-}$*

and $\nu_{j',-}$. Furthermore the following relation holds:

$$\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,+}-\nu_{j',+})dt=-\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,-}-\nu_{j',-})dt=\int_{b_{j'}(t)}^{b_j(t)}(\lambda_+-\lambda_-)dx. \tag{2.1.27}$$

Proof. — We first consider the case of a turning point $t=\tau^{\mathrm{I}}$ of the first kind. Proposition 2.1.3 implies that $2\widehat{u}_{1,0}(t)+b_j(t)$ vanishes at $t=\tau^{\mathrm{I}}$ for some j. This immediately entails that $x=b_j(t)$ merges with $x=-2\widehat{u}_{1,0}(t)$ at $t=\tau^{\mathrm{I}}$ and that $\nu_{j,\pm}$ merge and vanish there. Note that Proposition 2.1.3 also implies

$$\nu_{j,+}(t)-\nu_{j,-}(t)=2(\mu_+(x,t)-\mu_-(x,t))\Big|_{x=b_j(t)}. \tag{2.1.28}$$

Hence it follows from Proposition 2.1.2 that

$$\begin{aligned}\frac{d}{dt}\int_{a(t)}^{b_j(t)}(\lambda_+-\lambda_-)dx&=\int_{a(t)}^{b_j(t)}\frac{\partial}{\partial t}(\lambda_+-\lambda_-)dx\\&=\int_{a(t)}^{b_j(t)}\frac{\partial}{\partial x}(\mu_+-\mu_-)dx\\&=(\mu_+-\mu_-)\Big|_{x=b_j(t)}\\&=\frac{1}{2}(\nu_{j,+}-\nu_{j,-}).\end{aligned} \tag{2.1.29}$$

Integrating (2.1.29) from τ^{I} to t, we then obtain (2.1.26).

We next consider the case of a turning point $t=\tau^{\mathrm{II}}$ of the second kind. Proposition 2.1.3 again implies that $2\widehat{u}_{1,0}(t)+b_j(t)$ coincides with $2\widehat{u}_{1,0}(t)+b_{j'}(t)$ at $t=\tau^{\mathrm{II}}$ for some j and j'. This entails that $x=b_j(t)$ merges with $x=b_{j'}(t)$ at $t=\tau^{\mathrm{II}}$ and that $\nu_{j,+}$ and $\nu_{j',+}$ merge there. The proof of the relation (2.1.27) is similar to that of (2.1.26). □

As an immediate consequence of the relations (2.1.26) and (2.1.27) we also observe the following important

Proposition 2.1.5. — *If t lies on a Stokes curve of $(P_{\mathrm{I}})_m$ emanating from a turning point $t=\tau^{\mathrm{I}}$ (resp. $t=\tau^{\mathrm{II}}$) of the first (resp. second) kind, the Stokes geometry of* (1.1.11.a) *becomes degenerate in the sense that its two turning points $x=b_j(t)$ and $x=a(t)$ (resp. $x=b_j(t)$ and $x=b_{j'}(t)$) are connected by a Stokes curve.*

Propositions 2.1.4 and 2.1.5 are natural generalizations to $(P_{\mathrm{I}})_m$ of Facts D and E for the traditional Painlevé equations explained in Introduction.

2.2. Case of the $P_{\text{II-1}}$-hierarchy. — As in the case of the P_{I}-hierarchy, by substituting

$$v=\widehat{v}(t,\eta)=\widehat{v}_0(t)+\eta^{-1}\widehat{v}_1(t)+\cdots \tag{2.2.1}$$

into (1.2.1) and comparing like powers of η, we can construct the 0-parameter solution $\widehat{v}(t,\eta)$ of $(P_{\text{II-1}})_m$. In this case the top order part $\widehat{v}_0$ satisfies

$$2\widehat{v}_0 K_{m,0}(\widehat{v}_0) + 2gt\widehat{v}_0 + c = 0, \tag{2.2.2}$$

or more explicitly

$$\frac{(-1)^m 2^m (2m-1)!!}{m!}\widehat{v}_0^{2m+1} + 2gt\widehat{v}_0 + c = 0 \tag{2.2.3}$$

(*cf.* Remark 1.2.2).

We then substitute the 0-parameter solution $\widehat{v}(t,\eta)$ of $(P_{\text{II-1}})_m$ into the coefficients A and B of the underlying Lax pair (1.2.10). Their top order parts A_0 and B_0 are given by

$$A_0 = \frac{1}{2xg}\begin{pmatrix} 0 & T_{m,0} \\ q_0 T_{m,0} & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ q_0 & 0 \end{pmatrix}, \tag{2.2.4}$$

where

$$T_{m,0} = gt + \sum_{k=0}^{m}(4x)^k K_{m-k,0}\Big|_{v=\widehat{v}_0}, \tag{2.2.5}$$

$$q_0 = x + \widehat{v}_0^2. \tag{2.2.6}$$

Thus

$$A_0 = \frac{T_{m,0}}{2xg}B_0 \tag{2.2.7}$$

holds and hence we obtain

Proposition 2.2.1

(i) *The equation* (1.2.10.a) *has m* (*generically*) *double turning points* (*which will be denoted by $x = b_1(t)$, ..., $x = b_m(t)$ in what follows*), *and each double turning point $x = b_j(t)$ is a root of $T_{m,0} = 0$, that is,*

$$T_{m,0} = 2^{2m-1}\prod_{j=1}^{m}(x - b_j(t)). \tag{2.2.8}$$

(ii) *It has one* (*generically*) *simple turning point $x = -(\widehat{v}_0(t))^2$,* (*which will be denoted by $x = a(t)$ for short in what follows*), *and this point is simultaneously a turning point of the equation* (1.2.10.b).

The following proposition corresponding to Fact B also holds for $(P_{\text{II-1}})_m$.

Proposition 2.2.2. — *The eigenvalues $\lambda_\pm$ of A_0 and the eigenvalues $\mu_\pm$ of B_0 satisfy the following relation:*

$$\frac{\partial}{\partial t}\lambda_\pm = \frac{\partial}{\partial x}\mu_\pm. \tag{2.2.9}$$

(For the proof of Proposition 2.2.2 see [**T2**].)

Now we consider the linearization of $(P_{\text{II-1}})_m$ at the 0-parameter solution $\widehat{v}$. Let ΔK_j denote the linear (in Δv) part of K_j after the substitution $v = \widehat{v} + \Delta v$, then the linearization of $(P_{\text{II-1}})_m$ is

$$(\eta^{-1}\partial_t + 2\widehat{v})\Delta K_m + 2K_m\Big|_{v=\widehat{v}}\Delta v + 2gt\Delta v = 0. \tag{2.2.10}$$

Since K_j is a polynomial of v and its derivatives, there exists a (formal) differential operator

$$p_j(t, \eta^{-1}\partial_t; \eta^{-1}) = p_{j,0}(t, \eta^{-1}\partial_t) + \eta^{-1}p_{j,1}(t, \eta^{-1}\partial_t) + \cdots \tag{2.2.11}$$

for which the following relation holds:

$$\Delta K_j = p_j(t, \eta^{-1}\partial_t; \eta^{-1})\Delta v. \tag{2.2.12}$$

In terms of this operator $p_j(t, \eta^{-1}\partial_t; \eta^{-1})$, the characteristic equation (*i.e.*, the top order part (with respect to η) of the symbol obtained by replacing $\eta^{-1}\partial_t$ by ν) of (2.2.10) is expressed as

$$C(t, \nu) = (\nu + 2\widehat{v}_0)p_{m,0}(t, \nu) + 2K_{m,0}\Big|_{v=\widehat{v}_0} + 2gt. \tag{2.2.13}$$

This $C(t, \nu)$ corresponds to the characteristic equation of C_0 in the case of P_{I}-hierarchy.

Proposition 2.2.3. — *We find*

$$C(t, \nu) = 4^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{x=b_j(t), \mu=\nu/2}. \tag{2.2.14}$$

Proof. — We first note that the right-hand side of (2.2.14) becomes

$$\begin{aligned}4^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{x=b_j, \mu=\nu/2} &= 4^m \prod_{j=1}^{m} (\mu^2 - \widehat{v}_0^2 - x)\Big|_{x=b_j, \mu=\nu/2} \\ &= 4^m \prod_{j=1}^{m} \Big(\frac{\nu^2}{4} - \widehat{v}_0^2 - b_j\Big) \\ &= 2T_{m,0}\Big|_{x=(\nu^2 - 4v_0{}^2)/4}.\end{aligned} \tag{2.2.15}$$

To calculate the left-hand side of (2.2.14), we use the recursive relation (1.2.2). Considering the linear (in Δv) part of both sides of (1.2.2), we find that $\{p_{j,0}\}$ should satisfy the following recursive relation:

$$\nu p_{j+1,0}(t, \nu) = (\nu^3 - 4\widehat{v}_0^2\nu)p_{j,0}(t, \nu) + 2\nu(\nu - 2\widehat{v}_0)K_{j,0}\Big|_{v=\widehat{v}_0}, \tag{2.2.16}$$

that is,

$$p_{j+1,0}(t, \nu) = (\nu^2 - 4\widehat{v}_0^2)p_{j,0}(t, \nu) + 2(\nu - 2\widehat{v}_0)K_{j,0}\Big|_{v=\widehat{v}_0}. \tag{2.2.17}$$

By solving this recursive relation with the initial condition

$$p_{1,0}(t,\nu) = \nu - 2\widehat{v}_0, \tag{2.2.18}$$

we obtain

$$p_{m,0}(t,\nu) = 2(\nu - 2\widehat{v}_0)\sum_{k=0}^{m-1}(\nu^2 - 4\widehat{v}_0^{m-j-1})K_{k,0}\Big|_{v=\widehat{v}_0}. \tag{2.2.19}$$

It then follows from (2.2.13) that the left-hand side of (2.2.14) becomes

$$2\sum_{k=0}^{m}(\nu^2 - 4\widehat{v}_0^2)^{m-k}K_{k,0}\Big|_{v=\widehat{v}_0} + 2gt, \tag{2.2.20}$$

which coincides with $2T_{n,0}\big|_{x=(\nu^2-4\widehat{v}_0^2)/4}$. This completes the proof of Proposition 2.2.3. □

Thus the same propositions as Propositions 2.1.1 $\sim$ 2.1.3 in the case of the P_{I}-hierarchy hold for the $P_{\text{II-1}}$-hierarchy also. In particular, since it follows from Proposition 2.2.3 that $C(t,\nu)$ is of the form $f(\nu^2, t)$ with some polynomial f of degree m, we can define a turning point of the first kind and that of the second kind also for the $P_{\text{II-1}}$-hierarchy in a manner similar to the case of the P_{I}-hierarchy. For both kinds of the turning points we can verify the following relations, similar to those for the P_{I}-hierarchy, between the Stokes geometry of $(P_{\text{II-1}})_m$ and that of its underlying Lax pair $(L_{\text{II-1}})_m$.

Proposition 2.2.4

(i) *Let $t = \tau^{\mathrm{I}}$ be a turning point of the first kind of $(P_{\text{II-1}})_m$. Then at $t = \tau^{\mathrm{I}}$ a double turning point $x = b_j(t)$ merges with the simple turning point $x = a(t) = -(\widehat{v}_0(t))^2$ in the Stokes geometry of* (1.2.10.a). *Consequently the two roots $\nu_{j,\pm}$ of $C(t,\nu)$ merge and vanish at $t = \tau^{\mathrm{I}}$. Furthermore the following relation holds:*

$$\frac{1}{2}\int_{\tau^{\mathrm{I}}}^{t}(\nu_{j,+} - \nu_{j,-})dt = \int_{a(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{2.2.21}$$

(ii) *Let $t = \tau^{\mathrm{II}}$ be a turning point of the second kind of $(P_{\text{II-1}})_m$. Then at $t = \tau^{\mathrm{II}}$ a double turning point $x = b_j(t)$ merges with another double turning point $x = b_{j'}(t)$. Consequently two roots $\nu_{j,+}$ and $\nu_{j',+}$ of $C(t,\nu)$ merge at $t = \tau^{\mathrm{II}}$, and so do $\nu_{j,-}$ and $\nu_{j',-}$. Furthermore the following relation holds:*

$$\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,+} - \nu_{j',+})dt = -\int_{\tau^{\mathrm{II}}}^{t}(\nu_{j,-} - \nu_{j',-})dt = \int_{b_{j'}(t)}^{b_j(t)}(\lambda_+ - \lambda_-)dx. \tag{2.2.22}$$

Proposition 2.2.5. — *If t lies on a Stokes curve of $(P_{\text{II-1}})_m$ emanating from a turning point $t = \tau^{\mathrm{I}}$ (resp. $t = \tau^{\mathrm{II}}$) of the first (resp. second) kind, the Stokes geometry of* (1.2.10.a) *becomes degenerate in the sense that its two turning points $x = b_j(t)$ and $x = a(t)$ (resp. $x = b_j(t)$ and $x = b_{j'}(t)$) are connected by a Stokes curve.*

We omit the proof of Propositions 2.2.4 and 2.2.5 as it is the same as that of Propositions 2.1.4 and 2.1.5.

2.3. Case of the $P_{\text{II-2}}$-hierarchy. — As is discussed in [**N1**] and [**N2**], the relations between the Stokes geometry of each member of the hierarchy and that of its underlying Lax pair, similar to those for the P_{I}-hierarchy and $P_{\text{II-1}}$-hierarchy, can be confirmed also for the $P_{\text{II-2}}$-hierarchy. We refer the reader to [**N1**] and [**N2**] for their precise formulation and the details of the proofs. Here, we only explain the core part of the discussion. For the sake of simplicity of the notations, we restrict our consideration to the case where $c_0 = c_1 = \cdots = c_{m-1} = 0$.

Substituting the 0-parameter solution

$$\widehat{u}(t,\eta) = \widehat{u}_0(t) + \eta^{-1}\widehat{u}_1(t) + \cdots, \tag{2.3.1}$$

$$\widehat{v}(t,\eta) = \widehat{v}_0(t) + \eta^{-1}\widehat{v}_1(t) + \cdots \tag{2.3.2}$$

of $(P_{\text{II-2}})_m$ into the coefficients A and B of the underlying Lax pair (1.3.8), we find that their top order parts A_0 and B_0 become

$$A_0 = \frac{1}{g}\begin{pmatrix} -(2x-\widehat{u}_0)T_{m,0} & 2T_{m,0} \\ -2\widehat{v}_0 T_{m,0} & (2x-\widehat{u}_0)T_{m,0} \end{pmatrix}, \tag{2.3.3}$$

$$B_0 = \begin{pmatrix} -x+\widehat{u}_0/2 & 1 \\ -\widehat{v}_0 & x-\widehat{u}_0/2 \end{pmatrix}, \tag{2.3.4}$$

where

$$T_{m,0} = \frac{1}{2}\sum_{j=0}^{m} x^{m-j} K_{j,0}\Big|_{u=\widehat{u}_0, v=\widehat{v}_0}. \tag{2.3.5}$$

This immediately entails that

$$A_0 = \frac{2T_{m,0}}{g} B_0. \tag{2.3.6}$$

Hence, if we let $x = b_j(t)$ $(1 \leqslant j \leqslant m)$ denote a root of $T_{m,0} = 0$, each $b_j(t)$ becomes a (generically) double turning point of the equation (1.3.8.a). Note that in this case there exist two (generically) simple turning points, denoted by $x = a_1(t)$ and $x = a_2(t)$ in what follows, since the characteristic equation of B_0 is a quadratic polynomial in x.

We, next, consider the linearization of $(P_{\text{II-2}})_m$ at $(u,v) = (\widehat{u},\widehat{v})$. Letting ΔK_j and ΔL_j respectively denote the linear part of K_j and L_j in $(\Delta u, \Delta v)$ after the substitution $(u,v) = (\widehat{u},\widehat{v}) + (\Delta u, \Delta v)$, we find that the linearization of $(P_{\text{II-2}})_m$ is

$$\Delta K_{m+1} = \Delta L_{m+1} = 0. \tag{2.3.7}$$

Let $C(t,\nu)$ denote its characteristic equation, then we obtain

$$C(t,\nu) = (-1)^m \prod_{j=1}^{m} \det(\mu - B_0)\Big|_{\mu=\nu/2, x=b_j}. \tag{2.3.8}$$

As in the preceding two subsections, (2.3.8) enables us to define a turning point of the first kind and that of the second kind also for the $P_{\text{II-2}}$-hierarchy. The key relation (2.3.8) can be proved in a similar manner as in Section 2.2; That is, since K_j and L_j are polynomials of u, v and their derivatives, there exists a 2×2 matrix of differential operators

$$D_j(t,\eta^{-1}\partial_t;\eta^{-1}) = D_{j,0}(t,\eta^{-1}\partial_t) + \eta^{-1}D_{j,1}(t,\eta^{-1}\partial_t) + \cdots \tag{2.3.9}$$

satisfying

$$\begin{pmatrix}\Delta K_j\\ \Delta L_j\end{pmatrix} = D_j(t,\eta^{-1}\partial_t;\eta^{-1})\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}. \tag{2.3.10}$$

Then, in terms of $D_j(t,\eta^{-1}\partial_t;\eta^{-1})$, $C(t,\nu)$ is expressed as

$$C(t,\nu) = \det D_{m+1,0}(t,\nu). \tag{2.3.11}$$

On the other hand, considering the linear (in $(\Delta u, \Delta v)$) part of both sides of (1.3.2) and taking its top order term, we find

$$D_{j+1,0}(t,\nu) = \begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix} D_{j,0}(t,\nu) + \frac{1}{2}K_{j,0}I_2, \tag{2.3.12}$$

where I_2 stands for the 2×2 identity matrix. By solving this recursive relation under the condition $D_{1,0}(t,\nu) = I_2$, we obtain

$$D_{m+1,0} = \frac{1}{2}\sum_{j=0}^{m} K_{m-j,0}\begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix}^j. \tag{2.3.13}$$

Hence (2.3.5) and (2.3.13) entail that

$$\begin{aligned} D_{m+1,0}(t,\nu) &= \prod_{j=1}^{m}\left(\begin{pmatrix}(\widehat{u}_0-\nu)/2 & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2\end{pmatrix} - b_j I_2\right)\\ &= \prod_{j=1}^{m}\begin{pmatrix}(\widehat{u}_0-\nu)/2-b_j & 1\\ \widehat{v}_0 & (\widehat{u}_0+\nu)/2-b_j\end{pmatrix}. \end{aligned} \tag{2.3.14}$$

The relation (2.3.8) immediately follows from (2.3.4), (2.3.11) and (2.3.14).

3. The inevitability of the Nishikawa phenomenon

In a computer-assisted study of the Stokes geometry for $(P_{\text{II-2}})_2$ Nishikawa ([**N1**]) found the following intriguing phenomenon:

There exist points outside the union of all Stokes curves for $(P_{\text{II-2}})_2$ where the Stokes geometry of (1.3.8.a) degenerates. Furthermore the totality of such points forms a curved ray emanating from the intersection of two Stokes curves for $(P_{\text{II-2}})_2$.

The purpose of this section is to show why and how such a phenomenon, which is now known as the Nishikawa phenomenon, should be observed. To fix the notations we consider the case $(P_{\mathrm{I}})_2$, although the reasoning equally applies to $(P_J)_m$ with $m \geqslant 2$ and $J = \mathrm{I}$, II-1 or II-2. We note that the phenomena studied below are not observed when $m = 1$, *i.e.*, for the traditional Painlevé equations. One important reason for this is the fact that the number of the double turning points of the equation (1.1.11.a) is 1 when $m = 1$; at least two double turning points seem to be needed for the occurrence of a Nishikawa phenomenon.

Let T be a crossing point of two Stokes curves of $(P_{\mathrm{I}})_2$. Suppose that, when t lies on one of the Stokes curves of $(P_{\mathrm{I}})_2$, a (double) turning point A is connected with a (simple) turning point C by a Stokes curve in the Stokes geometry of the linear equation (1.1.11.a) and that another (double) turning point B is similarly connected with C by a Stokes curve of (1.1.11.a) when t lies on the other Stokes curve of $(P_{\mathrm{I}})_2$; the (topological) configuration of the Stokes curves of (1.1.11.a) when $t = T$ is seen in Figure 3.1. (As we study the configuration of Stokes curves both for the Painlevé equations (*i.e.*, in t-variable) and for one of the underlying Lax pair (*i.e.*, in x-variable), we put throughout this article a sign $\underline{t|}$ or $\underline{x|}$ to each figure for the convenience of the reader.) In what follows, having these geometrical situations in mind, we label the two Stokes curves of $(P_{\mathrm{I}})_2$ crossing at T as $[AC]$ and $[BC]$ respectively.

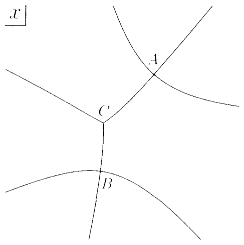


FIGURE 3.1

Let us move around the point T from t_1 to t_4 as designated by the arrows shown in Figure 3.2.

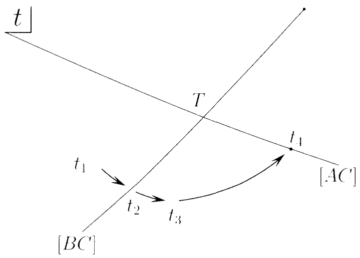


FIGURE 3.2

To fix the notation let us suppose that the configuration of Stokes curves of (1.1.11.a) at t_j ($j = 1, 2, 3$) is as in Figure 3.3.j. The letters $a \sim g$ label the directions into which Stokes curves asymptotically flow. Such configurations are really observed, for example, near the crossing point of Stokes curves of $(P_{\mathrm{I}})_2$ shown in Figure 5.1.2(i) in Section 5.

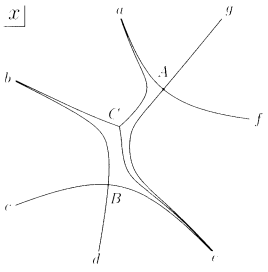


FIGURE 3.3.1

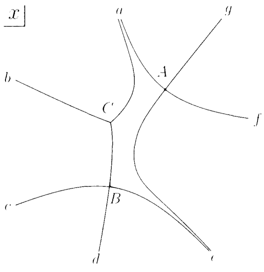


FIGURE 3.3.2

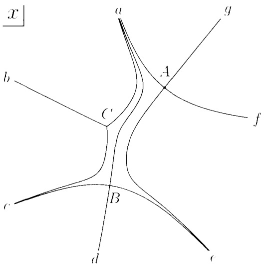


FIGURE 3.3.3

Note that we can detect the configuration in Fig. 3.3.3 by the relation (2.1.26) without resorting to the computer-assisted numerical computations; the Stokes curve emanating from B and flowing to the direction b in Fig. 3.3.1 should now go to some direction looking at C on the left side, but the number of directions to which Stokes curves of (1.1.11.a) flow is 7 and they are exhausted by $a \sim g$. Since (1.1.11.a) is a 2×2 system, its Stokes curves do not cross. Hence the only direction to which the Stokes curve in question flow is the direction a. The same reasoning applies to the Stokes curve emanating from C and flowing to the direction e in Fig. 3.3.1. Thus Fig. 3.3.3 is a logical consequence of Fig. 3.3.1 and Fig. 3.3.2.

Now, is it possible to reach a point t_4 in $[AC]$ with keeping the topological configuration designated in Fig. 3.3.3 ? For the convenience of the reader we give the configuration of Stokes curves of (1.1.11.a) when $t = t_4$ in Fig. 3.3.4.

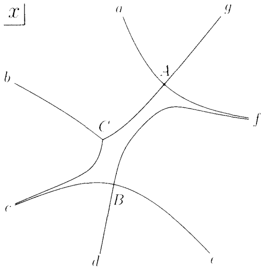


FIGURE 3.3.4

The answer to the above question is clearly "No", because no Stokes curve can connect A and C; if such a Stokes curve existed, it should cross the Stokes curve emanating from B and flowing to the direction a or d, and it should contradict the requirement that no Stokes curves should cross for 2×2 systems. Thus the Stokes curve emanating from B and flowing to the direction a should swing further and hit the turning point A as in Fig. 3.3.5 at some point $t = t_5$ during the journey of t from t_3 to t_4.

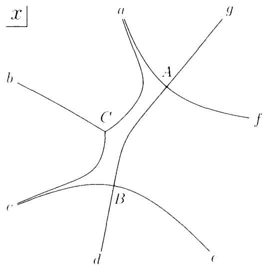


FIGURE 3.3.5

We can then smoothly continue our journey; we find the configuration shown in Fig. 3.3.6 after t passes through t_5, as is detected by (2.1.26). Then it is natural to find the configuration shown in Fig. 3.3.4 as we continue our journey to reach $t = t_4$.

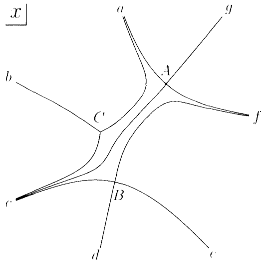


FIGURE 3.3.6

Summing up, during the journey from t_2 to t_4, unanticipated degeneracy of the Stokes geometry of (1.1.11.a) inevitably occurs at some point, and the totality of such points is a (curved) ray emanating from T. This explains why and how the Nishikawa phenomenon should occur.

We note that the above discussion makes essential use of the fact that, although $(P_{\mathrm{I}})_2$ is equivalent to the fourth order equation (and hence its Stokes curves may, and really do, cross), the Lax pair associated with it consists of 2×2 systems.

4. Introduction of a new Stokes curve to explain the Nishikawa phenomenon

The purpose of this section is to introduce a "new" Stokes curve so that the Nishikawa phenomenon may be naturally interpreted as the occurrence of degeneracy of the Stokes geometry of the underlying Lax pair when the parameter t lies on the new Stokes curve. Introduction of a new Stokes curve was first done by Berk-Nevins-Roberts ([**BNR**]) for a linear differential operator with holomorphic coefficients so that the connection formula for WKB solutions may be consistently written down near crossing points of Stokes curves. Because of the complexity of the equation in question, the reasoning of Berk et al. cannot be applied to our case. Instead, in introducing new Stokes curves for the linearization of $(P_J)_m$ such as (2.1.16) we use the graph-theoretical structure of the Stokes curves of the linear equation (1.1.11.a),(1.2.10.a) or (1.3.8.a).

Now, as Nishikawa ([**N1**]) has numerically observed, it is not always the case that we encounter Nishikawa phenomena near a crossing point of Stokes curves for the linearization of Painlevé equations, or for short, Fréchet derivatives. To characterize a crossing point of Stokes curves near which we observe Nishikawa phenomena we make some preparatory discussions.

Let us suppose that two Stokes curves for a Fréchet derivative cross transversally at a point T. By the Fact E for $(P_J)_m$ (*cf.* Proposition 2.1.5 and Proposition 2.2.5) each of the Stokes curves corresponds to a pair of turning points of (1.1.11.a) (or (1.2.10.a) or (1.3.8.a)) which are connected by a Stokes curve. Then either one of the following two situations is observed at $t = T$:

Case I: These two pairs share one turning point.

Case II: The four turning points are mutually distinct.

In what follows, we say in Case I that the two Stokes curves of (1.1.11.a) etc. (each of which connects a pair of turning points) are hinged by the shared turning point. We also call the shared turning point a hinging turning point (*cf.* Fig. 4.1). Using these terminologies, we further classify the situations in Case I.

Case Ia: The hinged two Stokes curves of (1.1.11.a) are adjacent at the hinging turning point.

Case Ib: The hinged two Stokes curves of (1.1.11.a) are not adjacent.

Note that, if the hinging turning point $x(T)$ in Fig. 4.1 is simple, then Case Ib is never realized; in fact, only 3 Stokes curves emanate from a simple turning point, and hence two Stokes curves are always adjacent there.

A crossing point T is said to be Lax-adjacent, or for short, LA if the configuration of Stokes curves of (1.1.11.a) etc. at $t = T$ is classified as in Case Ia. Otherwise, it is said to be non-Lax-adjacent or non-LA for short. An important property of two adjacent Stokes curves of (1.1.11.a) etc. is that the dominance relation of each of the Stokes curves is opposite (if the angle formed by the two Stokes curves does not

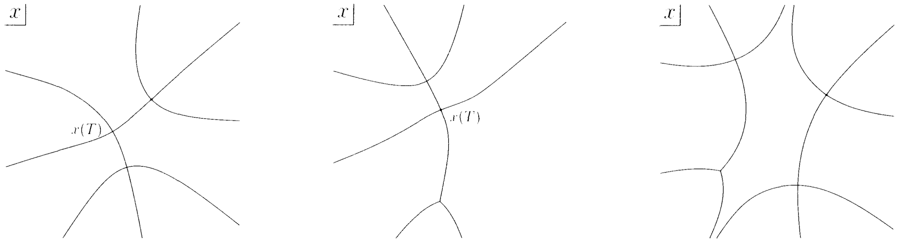


FIGURE 4.1. Example of configurations of relevant Stokes curves of (1.1.11.a) etc. in Cases Ia, Ib and II. In Cases Ia and Ib $x(T)$ designates the hinging turning point, while the pairs are not hinged in Case II.

contain the cut that fixes the branch of the characteristic values of (1.1.11.a) etc.) . In what follows we use this property in a substantial manner.

A new Stokes curve is, by definition, not introduced at a non-LA crossing point. At an LA crossing point T we introduce new Stokes curves that pass through T, following the rules given below. Here and in what follows, we attach the symbol "$(j,+) > (k,-)$" etc., to each (ordinary) Stokes curve to mean

$$\mathrm{Re}\int_{\tau}^{t}(\nu_{j,+}-\nu_{k,-})dt > 0 \tag{4.1}$$

holds on the Stokes curve in question. Here, $\nu_{j,+}$ (resp. $\nu_{k,-}$) designates the relevant characteristic root of the Fréchet derivative which is labeled by $(j,+)$ (resp. $(k,-)$), that is, $\nu_{j,+}$ and $\nu_{k,-}$ are solutions of the equation

$$\det(\nu - C_0) = 0. \tag{4.2}$$

(*Cf.* (2.1.23), (2.2.14) and (2.3.8)) We choose the lower end point of the integral in (4.1) to be the turning point from which the Stokes curve emanates. We also note that two symbols like $(j,+) > (k,-)$ and $(k,+) > (j,-)$ are attached to a Stokes curve which emanates from a turning point of the second kind; this means that two Stokes curves determined respectively by $\mathrm{Im}\int_{\tau}^{t}(\nu_{j,+}-\nu_{k,-})dt = 0$ and $\mathrm{Im}\int_{\tau}^{t}(\nu_{k,+}-\nu_{j,-})dt = 0$ sit on one and the same curve.

Rules for introducing new Stokes curves

Case A. — *At a Lax-adjacent crossing point T of two Stokes curves C_1 and C_2 respectively emanating from turning points $\tau_1 = \tau_1^{\mathrm{I}}$ and $\tau_2 = \tau_2^{\mathrm{I}}$ of the first kind.*

In this case, using the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4) and the assumption that T is an LA crossing point, we can find a simple turning point $a(t)$ and two double turning points $b_j(t)$ and $b_k(t)$ for which the configuration of relevant Stokes curves of (1.1.11.a) etc. contains the following portion at $t = T$ (Fig. 4.2).

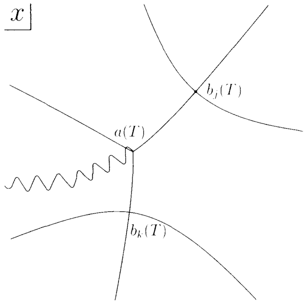


FIGURE 4.2

Here, the wiggly line designates a cut to fix the branch of $\sqrt{-\det A_0}$. Since τ_1 is a turning point of the first kind, we can find characteristic roots $\nu_{j,\pm}$ so that they satisfy

$$\nu_{j,-} = -\nu_{j,+} \tag{4.3}$$

and

$$\nu_{j,+}(\tau_1) = \nu_{j,-}(\tau_1) = 0. \tag{4.4}$$

(*Cf.* the remark after Proposition 2.1.3.) Letting Q_0 denote $-\det A_0$, we may assume

$$\frac{1}{2}\int_{\tau_1}^{t} (\nu_{j,+} - \nu_{j,-})dt = 2\int_{a(t)}^{b_j(t)} \sqrt{Q_0}\, dx \tag{4.5}$$

by replacing $\nu_{j,+}$ and $\nu_{j,-}$ if necessary. To fix the notation let us suppose that the Stokes curve C_1 is labeled by $(j,+) > (j,-)$. We then find

$$\operatorname{Re}\int_{a(T)}^{b_j(T)} \sqrt{Q_0}\, dx = \frac{1}{4}\int_{\tau_1}^{T} (\nu_{j,+} - \nu_{j,-})dt > 0. \tag{4.6}$$

With a similar reasoning we find characteristic roots $\nu_{k,\pm}$ satisfying

$$\nu_{k,-} = -\nu_{k,+}, \tag{4.7}$$

$$\nu_{k,+}(\tau_2) = \nu_{k,-}(\tau_2) = 0 \tag{4.8}$$

and

$$\int_{\tau_2}^{t} (\nu_{k,+} - \nu_{k,-})dt = 4\varepsilon\int_{a(t)}^{b_k(t)} \sqrt{Q_0}\, dx \tag{4.9}$$

with $\varepsilon = \pm 1$. In view of the location of the cut in Fig. 4.2, we find from (4.6)

$$\operatorname{Re}\int_{a(T)}^{b_k(T)} \sqrt{Q_0}\, dx < 0. \tag{4.10}$$

Hence the Stokes curve C_2 is labeled as $(k,+) > (k,-)$ (resp. $(k,-) > (k,+)$) if $\varepsilon = -1$ (resp. $\varepsilon = 1$). Then we introduce a new Stokes curve by the following:

$$\mathrm{Im} \int_T^t (\nu_{j,+} - \nu_{k,-})dt = \mathrm{Im} \int_T^t (\nu_{k,+} - \nu_{j,-})dt = 0 \tag{4.11}$$

if $\varepsilon = -1$, and

$$\mathrm{Im} \int_T^t (\nu_{j,+} - \nu_{k,+})dt = \mathrm{Im} \int_T^t (\nu_{k,-} - \nu_{j,-})dt = 0 \tag{4.12}$$

if $\varepsilon = 1$. At this moment we label a new Stokes curve by just the pair(s) of indices of the characteristic roots appearing in the definition of the curve, that is, we do not use the inequality symbol. To be concrete, the curve defined by (4.11) (resp. (4.12)) is labeled as $(j,+;k,-), (k,+;j,-)$ (resp. $(j,+;k,+), (k,-;j,-)$). Thus the resulting configuration of (ordinary and new) Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.3.

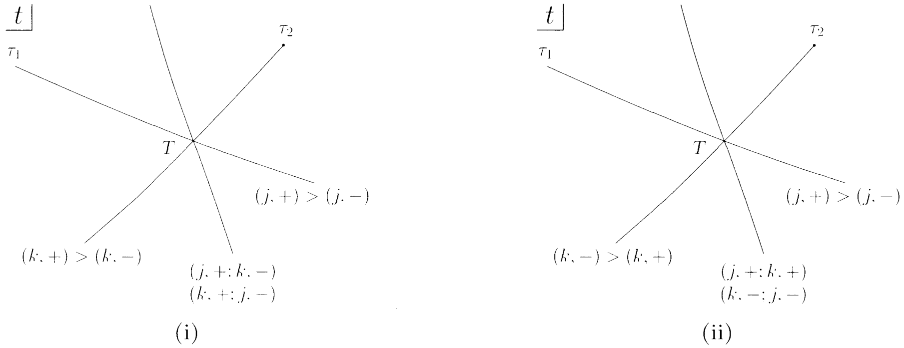


FIGURE 4.3

Case B. — *At a Lax-adjacent crossing point T of two Stokes curves C_1 and C_2 respectively emanating from a turning point $\tau_1 = \tau_1^{\mathrm{I}}$ of the first kind and from a turning point $\tau_2 = \tau_2^{\mathrm{II}}$ of the second kind.*

By the same reasoning as in Case A we find a simple turning point $a(t)$ and two double turning points $b_j(t)$ and $b_k(t)$ for which the configuration of Stokes curves of (1.1.11.a) etc. contains the portion designated in Fig. 4.4 (or its mirror image) at $t = T$.

Let us choose characteristic roots $\nu_{j,\pm}$ so that they satisfy (4.3) $\sim$ (4.6). To fix the situation we assume the Stokes curve C_1 is labeled as $(j,+) > (j,-)$. By the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4.), we find $\nu_{k,\pm}$ for which the

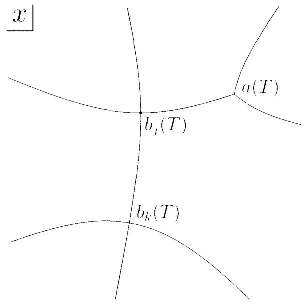


FIGURE 4.4

following relation holds with appropriate $\sigma = \pm$ and $\varepsilon = \pm 1$:

$$\int_{\tau_2}^{t} (\nu_{k,\sigma} - \nu_{j,+}) dt = 2\varepsilon \int_{b_j(t)}^{b_k(t)} \sqrt{Q_0}\, dx. \tag{4.13}$$

Hence we find

$$\nu_{k,\sigma} - \nu_{j,+} = 2\varepsilon \frac{d}{dt}\left(\int_{b_j(t)}^{a(t)} \sqrt{Q_0}\, dx + \int_{a(t)}^{b_k(t)} \sqrt{Q_0}\, dx \right). \tag{4.14}$$

On the other hand, (4.3) and (4.5) entail

$$\nu_{j,+} = 2\frac{d}{dt} \int_{a(t)}^{b_j(t)} \sqrt{Q_0}\, dx. \tag{4.15}$$

Thus we conclude $\varepsilon = +1$ in (4.14). Then, as σ is rather conventional in our current context, we consider both situations. (If we consider the problem globally, not localizing the problem near T, σ should be fixed in concrete problems. See [**NT**] for this point.) Since we have labeled C_1 as $(j,+) > (j,-)$, we find

$$\int_{a(T)}^{b_j(T)} \sqrt{Q_0}\, dx > 0. \tag{4.16}$$

Hence the Lax-adjacency assumption implies

$$\int_{b_j(T)}^{b_k(T)} \sqrt{Q_0}\, dx > 0. \tag{4.17}$$

This means that C_2 is labeled as

$$(k,+) > (j,+) \text{ and } (j,-) > (k,-) \text{ if } \sigma = + \tag{4.18}$$

$$(k,-) > (j,+) \text{ and } (j,-) > (k,+) \text{ if } \sigma = -. \tag{4.19}$$

The required new Stokes curve is then given by

$$\operatorname{Im} \int_{T}^{t} (\nu_{k,+} - \nu_{k,-}) dt = 0. \tag{4.20}$$

Thus the resulting configuration of (ordinary and new) Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.5.

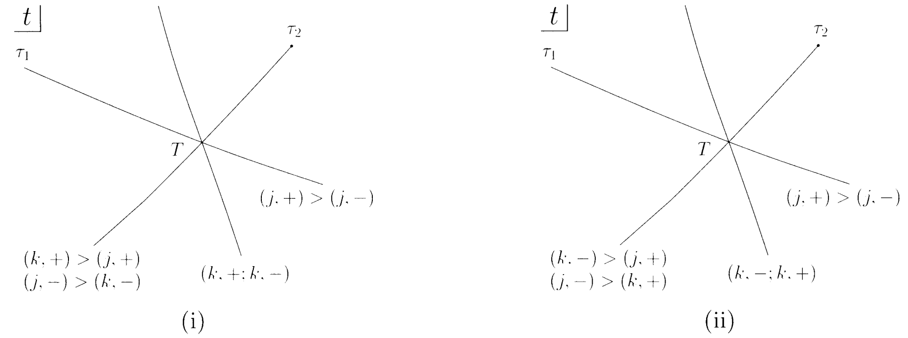


FIGURE 4.5

Case C. — *At a Lax-adjacent crossing point T of two Stokes curves C_1 and C_2 respectively emanating from turning points $\tau_1 = \tau_1^{\mathrm{II}}$ and $\tau_2 = \tau_2^{\mathrm{II}}$ of the second kind.*

In this case, using the Fact D for $(P_J)_m$ (*cf.* Proposition 2.1.4 and Proposition 2.2.4.) we find three double turning points $b_j(t), b_k(t)$ and $b_l(t)$ for which the configuration of Stokes curves of (1.1.11.a) etc. contains the following portion at $t = T$:

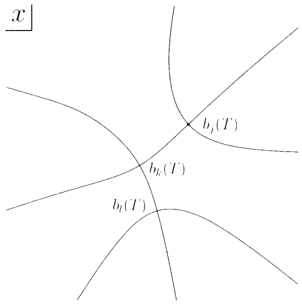


FIGURE 4.6

To fix the situation, let us choose characteristic roots $\nu_{j,\pm}$ and $\nu_{k,\pm}$ so that they satisfy the following:

$$\nu_{j,-} = -\nu_{j,+} \text{ and } \nu_{k,-} = -\nu_{k,+}, \tag{4.21}$$

$$\nu_{j,+}(\tau_1) = \nu_{k,+}(\tau_1) \neq 0, \tag{4.22}$$

$$\begin{aligned} \int_{\tau_1}^{t} (\nu_{j,+} - \nu_{k,+})dt &= -\int_{\tau_1}^{t} (\nu_{j,-} - \nu_{k,-})dt \\ &= 2\int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\, dx. \end{aligned} \tag{4.23}$$

We also assume

$$\operatorname{Re} \int_{b_k(T)}^{b_j(T)} \sqrt{Q_0}\, dx > 0. \tag{4.24}$$

Otherwise stated, the Stokes curve C_1 is labeled as $(j,+) > (k,+)$ and $(k,-) > (j,-)$. In parallel with the argument in Case B, we find characteristic roots $\nu_{l,\pm}$ for which the following relation holds with appropriate $\sigma = \pm$ and $\varepsilon = \pm 1$:

$$\int_{\tau_2}^{t} (\nu_{l,\sigma} - \nu_{k,+})dt = 2\varepsilon \int_{b_k(t)}^{b_l(t)} \sqrt{Q_0}\, dx. \tag{4.25}$$

Then we have

$$\begin{aligned} \nu_{l,\sigma} - \nu_{k,+} &= 2\varepsilon \frac{d}{dt}\Big(\int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\, dx + \int_{b_j(t)}^{b_l(t)} \sqrt{Q_0}\, dx \Big) \\ &= 2\varepsilon \frac{d}{dt}\Big(\int_{b_j(t)}^{b_l(t)} \sqrt{Q_0}\, dx) + \varepsilon(\nu_{j,+} - \nu_{k,+}) \Big). \end{aligned} \tag{4.26}$$

Hence we conclude $\varepsilon = +1$. Again in parallel with Case B, we do not fix σ. Since we have assumed (4.24), the Lax-adjacency assumption entails

$$\operatorname{Re} \int_{b_k(T)}^{b_l(T)} \sqrt{Q_0}\, dx < 0. \tag{4.27}$$

As $\varepsilon = +1$ in (4.25), we find that the Stokes curve C_2 is labeled as

$$(k,+) > (l,+) \text{ and } (l,-) > (k,-) \text{ if } \sigma = + \tag{4.28}$$

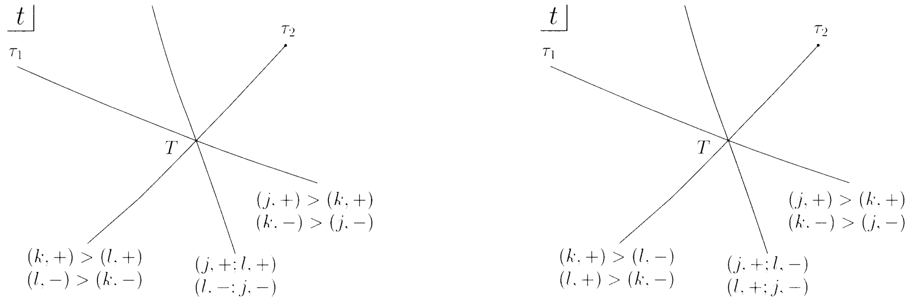
or

$$(k,+) > (l,-) \text{ and } (l,+) > (k,-) \text{ if } \sigma = -. \tag{4.29}$$

Then the required new Stokes curve is given by

$$\operatorname{Im} \int_{T}^{t} (\nu_{j,+} - \nu_{l,\sigma})dt = 0. \tag{4.30}$$

Thus the resulting configuration of Stokes curves near $t = T$ is either one of the following two graphs given in Fig. 4.7.



(i) (ii)

FIGURE 4.7

There exist crossing points of an ordinary Stokes curve and a new Stokes curve introduced above. However, no Nishikawa phenomena have been observed near them, at least in the examples so far studied. (*Cf.* [**N1**]; see also § 5.4). Hence we do not try to define the "secondary" new Stokes curves in this article. At the same time we surmise that we need such new Stokes curves in some more complicated examples.

Now, the importance and the naturality of the notion of new Stokes curves are shown by the following

Theorem 4.1. — *If t lies on a new Stokes curve introduced above, then the imaginary part of the integral $\int_{x_1(t)}^{x_2(t)} \sqrt{Q_0}\,dx$ vanishes for appropriately chosen turning points $x_1(t)$ and $x_2(t)$ of the equation* (1.1.11.a), (1.2.10.a) *or* (1.3.8.a). *To be more concrete, we find the following:*

(i) *In Case A, $x_1(t) = b_k(t)$ and $x_2(t) = b_j(t)$.*
(ii) *In Case B, $x_1(t) = a(t)$ and $x_2(t) = b_k(t)$.*
(iii) *In Case C, $x_1(t) = b_l(t)$ and $x_2(t) = b_j(t)$.*

Proof. — As the reasoning is the same for all cases, we prove the theorem only in the case (i). In what follows we use the notations in Rules above. Let us consider the case where $\varepsilon = -1$ in (4.9). Then, summing up (4.5) and (4.9), we find

$$\int_{\tau_1}^{t} (\nu_{j,+} - \nu_{j,-})dt + \int_{\tau_2}^{t} (\nu_{k,+} - \nu_{k,-})dt = 4 \int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\,dx. \tag{4.31}$$

Since T is a crossing point of Stokes curves C_1 and C_2,

$$\operatorname{Im} \int_{\tau_1}^{T} (\nu_{j,+} - \nu_{j,-})dt = \operatorname{Im} \int_{\tau_2}^{T} (\nu_{k,+} - \nu_{k,-})dt = 0 \tag{4.32}$$

holds. Therefore we obtain

$$\operatorname{Im} \int_{T}^{t} (\nu_{j,+} - \nu_{j,-} + \nu_{k,+} - \nu_{k,-})dt = 4 \operatorname{Im} \int_{b_k(t)}^{b_j(t)} \sqrt{Q_0}\,dx. \tag{4.33}$$

Since the left-hand side of (4.33) vanishes by the definition (4.11) of a new Stokes curve, we find the required fact. □

Remark 4.1. — If $x_1(t)$ and $x_2(t)$ are connected by a Stokes curve of (1.1.11.a) etc., then we find

$$\mathrm{Im}\int_{x_1(t)}^{x_2(t)}\sqrt{Q_0}\,dx = 0, \tag{4.34}$$

but not vice versa. The point is that a Stokes curve of (1.1.11.a) etc. is, by definition, an integral curve of the vector field $\mathrm{Im}\sqrt{Q_0}\,dx$ that emanates from a turning point. (*Cf.* [**AKT1**, p. 80])

As a matter of fact, Rules stated above are somewhat loose. A more precise description of a new Stokes curve should be as follows:

If the (real 1-dimensional) curve defined by (4.34) is non-singular,

$$\mathrm{Re}\int_{x_1(t)}^{x_2(t)}\sqrt{Q_0}\,dx \tag{4.35}$$

is monotonically decreasing or increasing along the curve. In particular, we can always find a point ω in the curve where the integral

$$\int_{x_1(t)}^{x_2(t)}\sqrt{Q_0}\,dx \tag{4.36}$$

vanishes at $t = \omega$. Then, in an analogy with the case of linear differential operators with holomorphic coefficients (*cf.* [**BNR**],[**AKT1**]), the part of the new Stokes curve which contains ω should be designated by a dotted line (near $t = T$) in the precise definition of a new Stokes curve. As a matter of fact the dotted part of a new Stokes curve is irrelevant to the degeneracy of the Stokes geometry of (1.1.11.a) etc.. This can be confirmed by a similar reasoning as is given in § 3 once concrete description of a new Stokes curve is given. As a typical example we analyze the example we studied in § 3. This time we consider the configuration of the Stokes curves of (1.1.11.a) etc. at $t = t_j$ ($j = 6, 7, 8$) designated in Fig. 4.8.

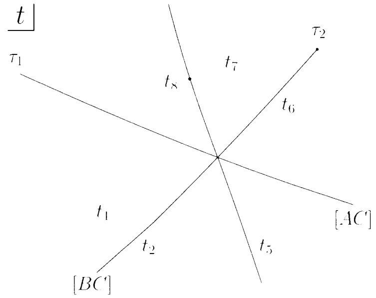


FIGURE 4.8. (*Cf.* Figure 3.2.)

If we move from t_2 to t_6 along $[BC]$, we cross $[AC]$ at T. Hence it follows from (2.1.26) that the Stokes curve emanating from A and flowing to the direction e and the Stokes curve emanating from C and flowing to the direction a in Fig. 3.3.2 should interchange the directions to which they flow when t reaches t_6, as shown in Fig. 4.9.6.

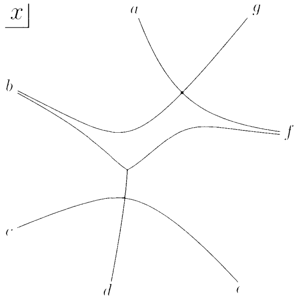


FIGURE 4.9.6

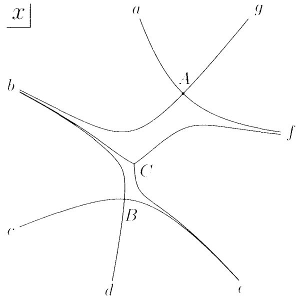


FIGURE 4.9.7

Again by (2.1.26) and the comparison of Fig. 3.3.1 and Fig. 3.3.2, we find from Fig. 4.9.6 that the configuration of Stokes curves at $t = t_7$ is given in Fig. 4.9.7.

Now we know that the unanticipated degeneracy of Stokes curves occurs at t_5 (*cf.* §3), and we can confirm that the point t_5 lies on the new Stokes curve described in Fig. 4.3. As the unanticipated degeneracy at t_5 means that A and B are connected by a Stokes curve (*cf.* Fig. 3.3.5), we label the curve as $[AB]$. Is it, then, possible to reach a point t_8 where A and B are connected by a Stokes curve with keeping the topological configuration designated in Fig. 4.9.7? The answer is clearly "No" by the same reasoning as in §3, *i.e.*, by the fact that no Stokes curves are allowed to cross each other for a 2×2 system, like (1.1.11.a). Otherwise stated, if A and B were really connected by a Stokes curve at $t = t_8$, either (A and C) or (B and C) should be connected by a Stokes curve before t reaches t_8. But, neither Stokes curve $[AC]$ nor $[BC]$ exists between t_7 and t_8. This means that A and B are not connected by a Stokes curve at t_8, although

$$\operatorname{Im} \int_A^B \sqrt{Q_0}\, dx = 0 \tag{4.37}$$

holds at $t = t_8$. As a matter of fact, some numerical computation shows that (4.36) vanishes at some point ω near t_8. Thus the precise description of the Stokes curves would be as in Fig. 4.10.

Finally we note that we can actually label a new Stokes curve not by just a pair like $(k, +; k, -)$ but by a more informative label like $(k, +) > (k, -)$; the sign of

$$\operatorname{Re} \int_{x_1(T)}^{x_2(T)} \sqrt{Q_0}\, dx \tag{4.38}$$

can be effectively used for this purpose. Concerning these subtle issues we will report in our forthcoming paper.

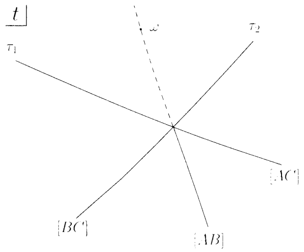


FIGURE 4.10

5. Examples of Stokes geometry

5.1. As the simplest example of the linearization of a higher order Painlevé equation, we study $(P_{\mathrm{I}})_2$ in this subsection. In this case the configuration of the Stokes curves are shown in Figure 5.1.1. However, if we want to understand the global structure of the configuration, we should take into account the Riemann sheet structure of the coefficients of $(P_{\mathrm{I}})_2$; the coefficients contain a multi-valued function $\widehat{u}_{1,0}$ defined by

$$5\widehat{u}_{1,0}^3 + 2c_1\widehat{u}_{1,0} - 2c_2 - 2t = 0. \tag{5.1}$$

Hence we first prepare three sheets which describes the Riemann sheet structure of $\widehat{u}_{1,0}$, and we then draw the Stokes curves of the linearization of $(P_{\mathrm{I}})_2$ on each sheet. The resulting configurations are described in Figure 5.1.2(j) ($j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}$) where we have chosen $c_1 = 1 - 1.7i$ and $c_2 = 0$. We note the singular points of $\widehat{u}_{1,0}$ are given by the zeros of discriminant of (5.1), which are coincident with the turning points τ_1^{I} and τ_2^{I} of $(P_{\mathrm{I}})_2$ of the first kind. The wiggly lines in Figure 5.1.2(j) designate the cuts to describe the global structure of $\widehat{u}_{1,0}$ with the additional information that the singularity of $\widehat{u}_{1,0}$ is of the square-root type. We note that, if we take into account the sheet structure of $\widehat{u}_{1,0}$, the points τ_1^{II} and τ_2^{II} on the first sheet (*i.e.*, in Figure 5.1.2(i)), for example, are not the turning points (of the second kind).

We next draw the new Stokes curves in Figure 5.1.2(j) to find the following Figure 5.1.3(j) ($j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}$). Here, we employ the precise definition of a new Stokes curve given in Remark 4.1; we will see below that the dotted part is irrelevant to the topological change of the configuration of the Stokes geometry of the linear equation (1.1.11.a). In Figure 5.1.5(i).j (resp. Figure 5.1.5(ii).k), we concretely describes the configuration of Stokes curves of (1.1.11.a) when t moves around the crossing point $t = T_{(\mathrm{i})}$ (resp. $T_{(\mathrm{ii})}$) of Stokes curves in Figure 5.1.3(i) (resp. Figure 5.1.3(ii)). The configuration for $t = T_{(\mathrm{i})}$ (resp. $t = T_{(\mathrm{ii})}$) is also given in Figure 5.1.4(i) (resp. Figure 5.1.4(ii)). The specific points to be considered are labeled by $t = t_j$ ($j = 1, \dots 12$) in Figure 5.1.3(i) and by $t = t_k$ ($k = 13, \dots 18$) in Figure 5.1.3(ii). The reader readily finds that the topological changes occur only at $t = t_j$ or $t = t_k$ that lies on an ordinary Stokes curve or on the solid line part of a new Stokes curve.

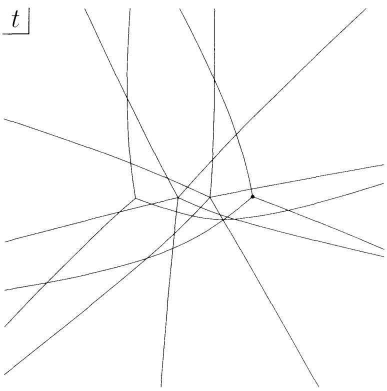


FIGURE 5.1.1

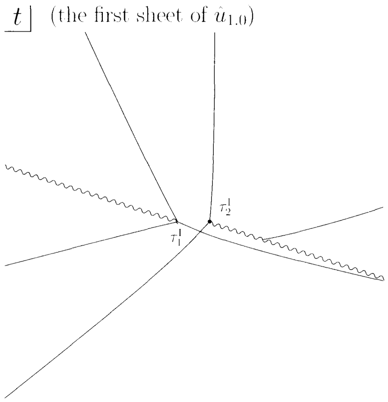


FIGURE 5.1.2(i)

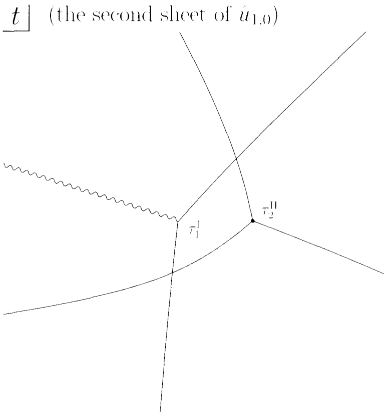


FIGURE 5.1.2(ii)

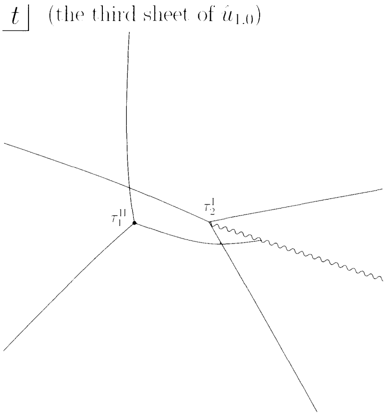


FIGURE 5.1.2(iii)

5.2. Since the number of double turning points of (1.1.11.a) is 2 for $(P_{\mathrm{I}})_2$, we need to try to study $(P_{\mathrm{I}})_3$, for example, to find a crossing point of two Stokes curves both emanating from a turning point of the second kind. (Case C in Section 4.) Fortunately we can really locate it in the Stokes geometry of $(P_{\mathrm{I}})_3$ (with $c_1 = 1.2+0.8i$, $c_2 = -1.7-1.5i$ and $c_3 = i$). The Stokes geometry (without the detailed consideration of the sheet structure) is given in Figure 5.2.1. We concentrate our attention to the turning points τ_1^{II} and τ_2^{II} specified in Figure 5.2.1 and we present in Figure 5.2.3 the configuration of Stokes curves of (1.1.11.a) at the crossing point T of the Stokes curve for $(P_{\mathrm{I}})_3$ emanating from τ_1^{II} and that from τ_2^{II}. The configuration of the Stokes curves for $t = t_j$ specified in Figure 5.2.2 is given respectively by Figure 5.2.4.j.

5.3. In studying $(P_{\mathrm{II}\text{-}1})_m$, one might wonder there would be any effect of the singularity at $x = 0$ in the equation (1.2.10.a). As some Stokes curves of (1.2.10.a) flow

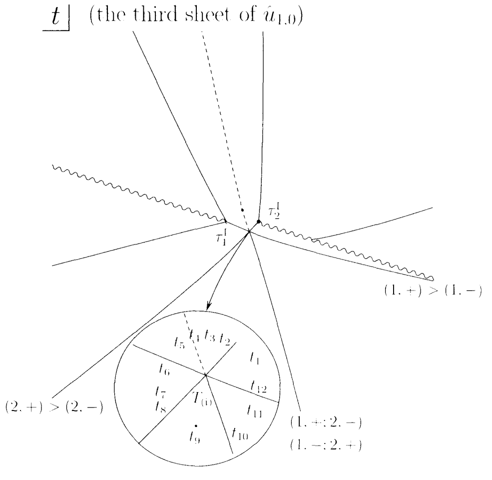


FIGURE 5.1.3(i)

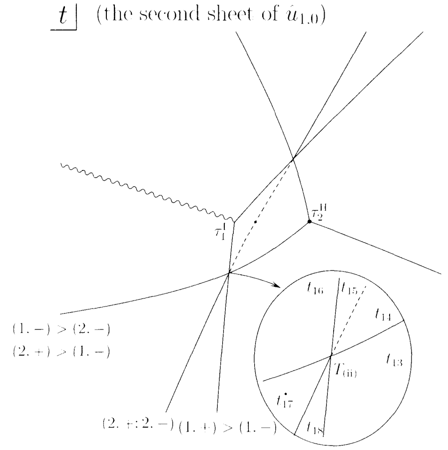


FIGURE 5.1.3(ii)

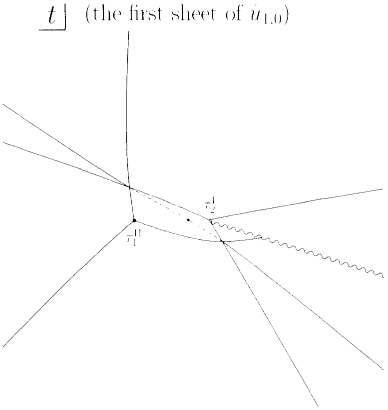


FIGURE 5.1.3(iii)

into the singular point $x = 0$ besides the points at infinity, the appearance of the Stokes geometry of (1.2.10.a) is somewhat different from that of the Stokes geometry of (1.1.11.a). But, nothing peculiar is observed concerning the relation between the Stokes geometry of the linearization of $(P_{\text{II-1}})_m$ and that of the linear equation (1.2.10.a). In order to show this we present the Stokes geometry of $(P_{\text{II-1}})_2$ with $g = -1/2$ and $c = 0.5 - 0.8i$, again ignoring the detailed sheet structure (*cf.* [**NT**]). We concentrate our attention to turning points τ^{I} and τ^{II} in Figure 5.3.1, and we present the enlarged figure of the Stokes curve emanating from τ^{I} and that from τ^{II},

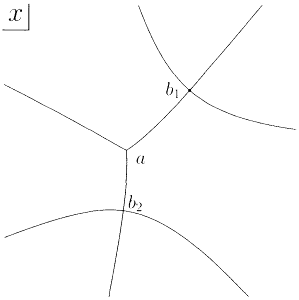


FIGURE 5.1.4(i)

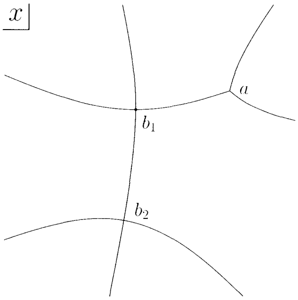


FIGURE 5.1.4(ii)

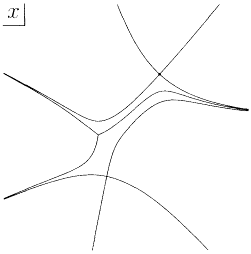


FIGURE 5.1.5(i).1

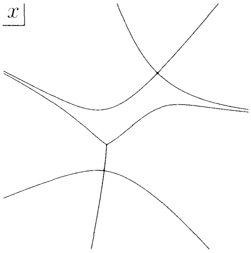


FIGURE 5.1.5(i).2

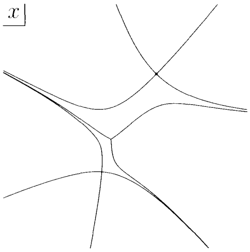


FIGURE 5.1.5(i).3

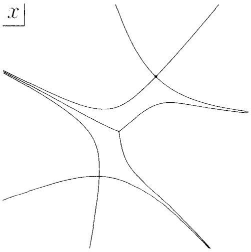


FIGURE 5.1.5(i).4

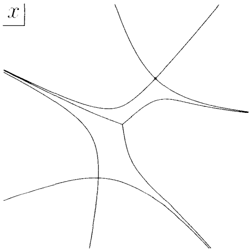


FIGURE 5.1.5(i).5

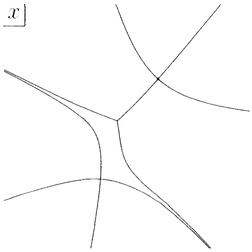


FIGURE 5.1.5(i).6

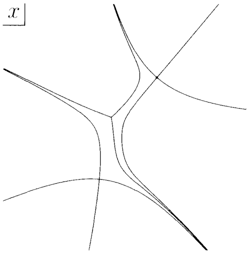


FIGURE 5.1.5(i).7

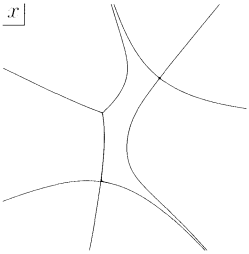


FIGURE 5.1.5(i).8

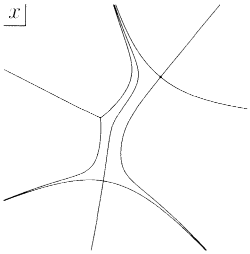


FIGURE 5.1.5(i).9

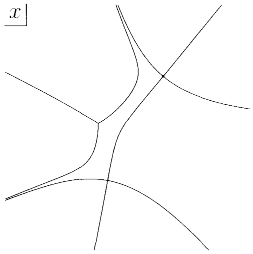


FIGURE 5.1.5(i).10

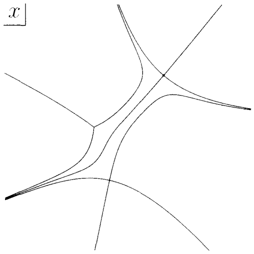


FIGURE 5.1.5(i).11

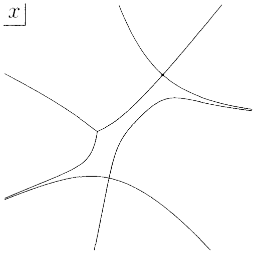


FIGURE 5.1.5(i).12

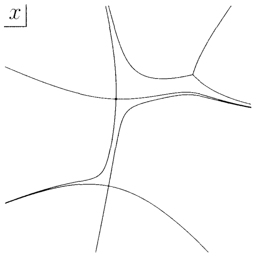


FIGURE 5.1.5(ii).13

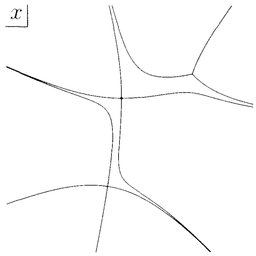


FIGURE 5.1.5(ii).14

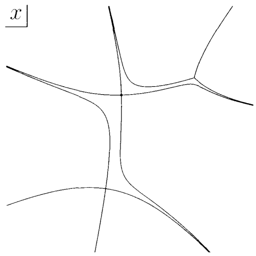


FIGURE 5.1.5(ii).15

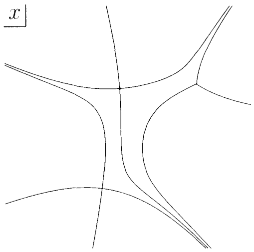


FIGURE 5.1.5(ii).16

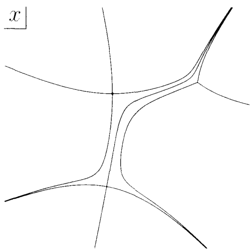


FIGURE 5.1.5(ii).17

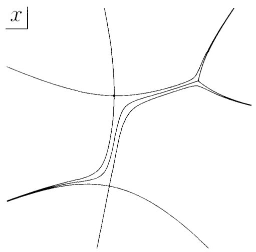


FIGURE 5.1.5(ii).18

together with the required new Stokes curve at the crossing point T. The configuration of the Stokes curves of (1.2.10.a) for $t = T$ is given by Figure 5.3.3 and that for $t = t_j$ ($j = 1, \ldots, 6$) is given respectively by Figure 5.3.4.j.

5.4. In connection with a remark before Theorem 4.1, we show an example of a crossing point of a new Stokes curve and an ordinary Stokes curve. The example is observed for $(P_{\text{II-2}})_2$ with $c = 9.8 - 0.1i$, $g = 7{,}6 + 6.6i$ and $\delta = -6.2 - 5.6i$, as we show below. The Stokes geometry of the linearization of $(P_{\text{II-2}})_2$ is given by Figure 5.4.1, and we concentrate our attention to the portion of Figure 5.4.1 that is enlarged in Figure 5.4.2; we focus our attention to the Stokes curve C_j ($j = 1, 2, 3$) respectively emanating from the turning point τ_j ($j = 1, 2, 3$), the new Stokes curve C_4 emanating from the crossing point T_0 of C_2 and C_3 and the crossing point T of

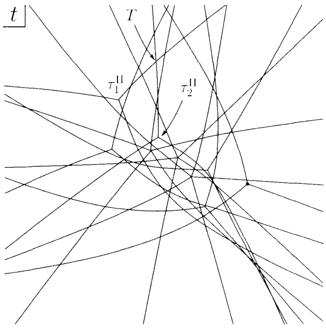


FIGURE 5.2.1

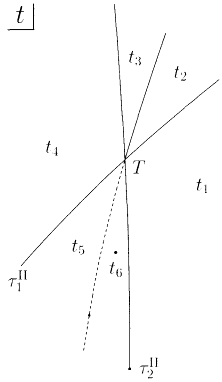


FIGURE 5.2.2

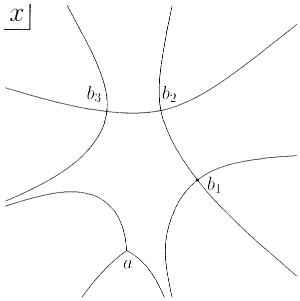


FIGURE 5.2.3

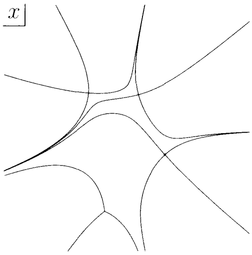


FIGURE 5.2.4.1

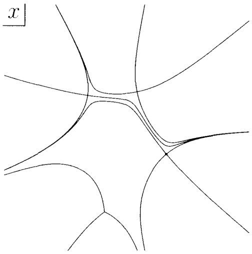


FIGURE 5.2.4.2

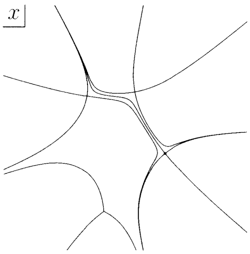


FIGURE 5.2.4.3

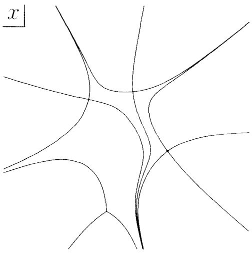


FIGURE 5.2.4.4

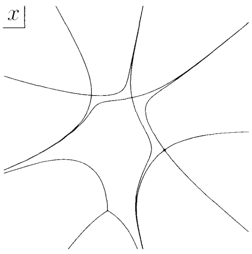


FIGURE 5.2.4.5

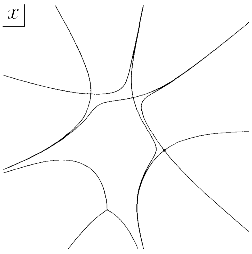


FIGURE 5.2.4.6

the Stokes curve C_1 and the new Stokes curve C_4; the configuration of Stokes curves of (1.3.8.a) at $t = T$ is given by Figure 5.4.3. Although we do not include the figures of the configuration of Stokes curves when the parameter t moves around T, we note that the topological change is observed only when t lies on C_1 or C_4.

Acknowledgements. — In making this research the authors, especially T. Koike and Y. Takei, benefited much from their visits to the University of Sydney in 2003 and 2004, supported by JSPS Japan-Australia Research Cooperative Program. They express their sincere gratitude to Prof. N. Joshi for her kind hospitality and to Prof.

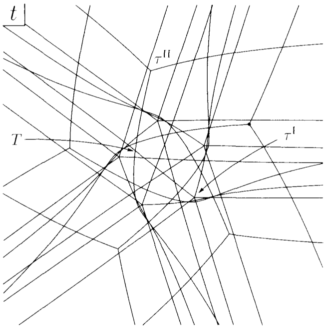


FIGURE 5.3.1

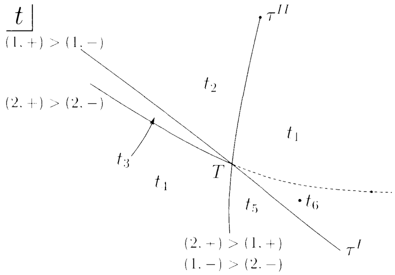


FIGURE 5.3.2

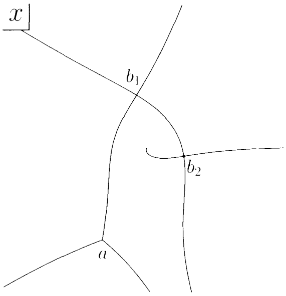


FIGURE 5.3.3

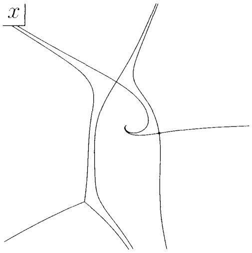


FIGURE 5.3.4.1

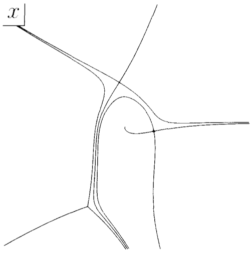


FIGURE 5.3.4.2

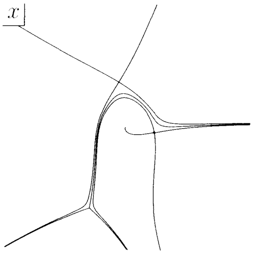


FIGURE 5.3.4.3

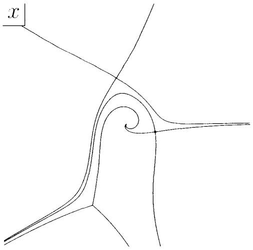


FIGURE 5.3.4.4

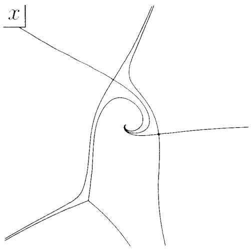


FIGURE 5.3.4.5

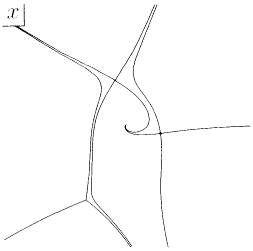


FIGURE 5.3.4.6

T. Aoki, Prof. P.R. Gordoa, Prof. N. Joshi, Prof. A. Pickering and Prof. S. Shimomura for many valuable discussions for hierarchies of higher-order Painlevé equations.

Appendix A

Some properties of $\mathcal{K}_j$ and K_j

Let us first consider $\{\mathcal{F}_j\}$ defined by the following recursive relation:

$$\partial_t \mathcal{F}_{j+1} = (\partial_t^3 + 4u\partial_t + 2u')\mathcal{F}_j \tag{A.1}$$

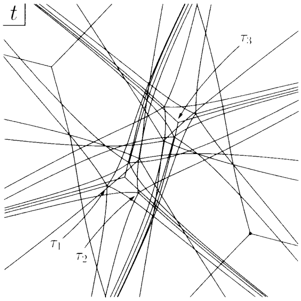


FIGURE 5.4.1

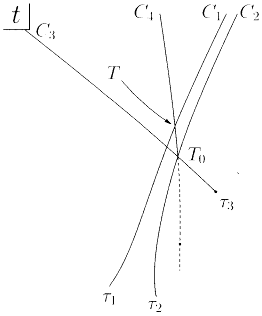


FIGURE 5.4.2

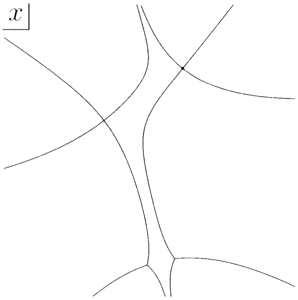


FIGURE 5.4.3

with $\mathcal{F}_0 = 1/2$. Here and in what follows, $'$ denotes the differentiation with respect to the variable t. Then, as is proved in [**DT**, Introduction], the following lemma holds for $\{\mathcal{F}_j\}$ thus defined.

Lemma A.1. — *If $\{\mathcal{F}_j\}$ satisfy* (A.1) *and each $\mathcal{F}_j$ does not contain a constant term, then the following relation holds:*

$$\mathcal{F}_{n+1} = -\sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} + 4u\sum_{j=0}^{n} \mathcal{F}_{n-j}\mathcal{F}_j + 2\sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t^2\mathcal{F}_j - \sum_{j=0}^{n} \partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j. \tag{A.2}$$

Once the relation (A.2) is confirmed, we can readily find that all $\mathcal{F}_j$ are polynomials of u and its derivatives by using an induction. Note that the recursive relation (A.1) itself does not fix integration constants in each step; here, we fix them to be 0.

In what follows we present a proof of Lemma A.1 along the line of [**DT**] for the reader's convenience. (See also [**L**] for another proof different from below.)

Proof. — Multiplying both sides of (A.1) by $\mathcal{F}_{n-j}$ and taking the sum from $j = 0$ to n, we obtain

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t\mathcal{F}_{j+1} = \sum_{j=0}^{n} \mathcal{F}_{n-j}\partial_t^3\mathcal{F}_j + \sum_{j=0}^{n} \mathcal{F}_{n-j}(4u\partial_t + 2u')\mathcal{F}_j. \tag{A.3}$$

The left-hand side of (A.3) can be written as

$$\mathcal{F}_0\partial_t\mathcal{F}_{n+1} + \frac{1}{2}\partial_t\sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1} = \frac{1}{2}\partial_t\left(\mathcal{F}_{n+1} + \sum_{j=0}^{n-1} \mathcal{F}_{n-j}\mathcal{F}_{j+1}\right). \tag{A.4}$$

On the other hand, since

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}{\partial_t}^3\mathcal{F}_j = \partial_t\left(\sum_{j=0}^{n} \mathcal{F}_{n-j}{\partial_t}^2\mathcal{F}_j - \frac{1}{2}\sum_{j=0}^{n} \partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j\right) \tag{A.5}$$

and

$$\sum_{j=0}^{n} \mathcal{F}_{n-j}(4u\partial_t + 2u')\mathcal{F}_j = 2\partial_t\left(u\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j\right), \tag{A.6}$$

the right-hand side of (A.3) becomes

$$\partial_t\left(\sum_{j=0}^{n}\mathcal{F}_{n-j}\partial_t^2\mathcal{F}_j - \frac{1}{2}\sum_{j=0}^{n}\partial_t\mathcal{F}_{n-j}\partial_t\mathcal{F}_j + 2u\sum_{j=0}^{n}\mathcal{F}_{n-j}\mathcal{F}_j\right). \tag{A.7}$$

This proves the lemma. □

Straightforward computations show that

$$\mathcal{F}_1 = u, \tag{A.8}$$

$$\mathcal{F}_2 = 3u^2 + u'', \tag{A.9}$$

$$\mathcal{F}_3 = 10u^3 + 5(u')^2 + 10uu'' + u^{(4)}, \tag{A.10}$$

$$\begin{aligned}\mathcal{F}_4 = 35u^4 + 70(u(u')^2 + u^2u'') + 21(u'')^2 \\ + 28u'u^{(3)} + 14uu^{(4)} + u^{(6)}.\end{aligned} \tag{A.11}$$

The polynomials $\{\mathcal{F}_j\}$ of u and its derivatives have the following scaling property:

Lemma A.2. — *Under the scaling $u \mapsto \lambda^2 u$, $t \mapsto \lambda^{-1}t$, $\{\mathcal{F}_j\}$ is transformed as*

$$\mathcal{F}_j \longmapsto \lambda^{2j}\mathcal{F}_j. \tag{A.12}$$

Employing what is called the Miura map $u = v' - v^2$, we now define a new family $\{\mathcal{K}_j\}$ of polynomials by

$$\mathcal{K}_j = \mathcal{F}_j\Big|_{u=v'-v^2}. \tag{A.13}$$

Then we can readily find that $\{\mathcal{K}_j\}$ satisfies the recursive relation (1.2.16). Hence these polynomials $\{\mathcal{K}_j\}$ coincide with those introduced in Section 1.2 to define the hierarchy (1.2.14) of Gordoa and Pickering. The following scaling property of $\{\mathcal{K}_j\}$ is also an immediate consequence of Lemma A.2:

Lemma A.3. — *Under the scaling $v \mapsto \lambda v$, $t \mapsto \lambda^{-1}t$, $\{\mathcal{K}_j\}$ is transformed as*

$$\mathcal{K}_j \longmapsto \lambda^{2j}\mathcal{K}_j. \tag{A.14}$$

Finally, as is explained in Section 1.2, $\{K_j\}$ defined by the recursive relation (1.2.2) is obtained from $\{\mathcal{K}_j\}$ through the scaling $v \mapsto \eta^{1/(2m+1)}v$, $t \mapsto \eta^{2m/(2m+1)}t$ and $\mathcal{K}_j \mapsto \eta^{2j/(2m+1)}K_j$. Hence K_j also becomes a polynomial of v and its derivatives.

Appendix B
Another formulation of the P_{I}-hierarchy

In [**GP**] Gordoa and Pickering discuss the following hierarchy of differential equations:

$$\mathcal{G}_{m+1} + gt = 0, \tag{B.1}$$

where g is a non-zero constant and $\{\mathcal{G}_j\}$ is defined by (B.2) below in terms of constants $\{\delta_j\}$ and $\{\mathcal{F}_j\}$ given in Appendix A.

$$\mathcal{G}_j = \mathcal{F}_j + \delta_1 \mathcal{F}_{j-1} + \cdots + \delta_j \mathcal{F}_0 = \sum_{k=0}^{j} \delta_k \mathcal{F}_{j-k} \quad (\delta_0 = 1). \tag{B.2}$$

Remark B.1. — We may assume $\delta_1 = 0$ without loss of generality. We also note that g may be changed to be an arbitrary non-zero constant by an appropriate scaling of u and t.

Remark B.2. — $\{\mathcal{G}_j\}$ satisfies

$$\partial_t \mathcal{G}_{j+1} = (\partial_t^3 + 4u\partial_t + 2u')\mathcal{G}_j. \tag{B.3}$$

Note that each $\mathcal{G}_j$ contains the constant term $\delta_j/2$. Hence an argument similar to that employed in the proof of Lemma A.1 entails that

$$\begin{aligned} \mathcal{G}_{n+1} = -\sum_{j=0}^{n-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1} + 4u \sum_{j=0}^{n} \mathcal{G}_{n-j}\mathcal{G}_j \\ + 2\sum_{j=0}^{n} \mathcal{G}_{n-j}\partial_t^2 \mathcal{G}_j - \sum_{j=0}^{n} \partial_t \mathcal{G}_{n-j}\partial_t \mathcal{G}_j \\ + \frac{1}{2}\delta_{n+1} + \frac{1}{4}\sum_{j=0}^{n-1} \delta_{n-j}\delta_{j+1}. \end{aligned} \tag{B.4}$$

We now introduce a large parameter η to (B.1) through a scaling

$$u \longmapsto \eta^{2\alpha} u, \quad t \longmapsto \eta^{\beta} t, \quad x \longmapsto \eta^{2\alpha} x, \quad g \longmapsto \eta^{2(m+1)\alpha-\beta} g, \quad \delta_j \longmapsto \eta^{2\alpha j}\delta. \tag{B.5}$$

Here, α and β are arbitrary constants satisfying $\alpha + \beta = 1$. Under this scaling $\{\mathcal{G}_j\}$ is transformed as

$$\mathcal{G}_j \longmapsto \eta^{2j\alpha} G_j, \tag{B.6}$$

where

$$G_j = \sum_{k=0}^{j} \delta_k F_{j-k}. \tag{B.7}$$

We thus obtain from (B.1) the following hierarchy of differential equations with a large parameter η:

$$G_{m+1} + gt = 0. \tag{B.8}$$

We now claim that the hierarchy (B.8) is equivalent to the P_{I}-hierarchy formulated in Section 1.1. That is, we can prove the following

Proposition B.1. — *Assume that $g = 2^{2m+1}$ and that $\delta_1 = 0$. Then, for a given solution u of* (B.8), *if we let u_j and v_j $(1 \leqslant j \leqslant m)$ be respectively given*

$$u_j = -2^{1-2j} G_j, \tag{B.9}$$

$$v_j = -2^{-2j} \eta^{-1} \partial_t G_j, \tag{B.10}$$

(u_j, v_j) satisfies $(P_{\mathrm{I}})_m$ with

$$c_j = 2^{-2j-2}\Big(\delta_{j+1} + \frac{1}{2}\sum_{k=0}^{j-1} \delta_{j-k}\delta_{k+1}\Big) \quad (1 \leqslant j \leqslant m). \tag{B.11}$$

Proof. — If we define w_j by

$$w_j = \begin{cases} 2^{-2j-1}\left(G_{j+1} - 2G_1 G_j - \eta^{-2}\partial_t^2 G_j\right) & (1 \leqslant j \leqslant m-1), \\ -2^{-2m-1}\left(2G_1 G_m + \eta^{-2}\partial_t^2 G_m\right) & (j = m), \end{cases} \tag{B.12}$$

we readily find that u_j, v_j and w_j satisfy the system (1.1.1). Thus what remains to be verified is that u_j, v_j and w_j thus defined should satisfy the recursive relation (1.1.2). Note that it follows from (B.4) and (B.6) that G_j satisfies the following relation:

$$\begin{aligned} G_{n+1} = &- \sum_{j=0}^{n-1} G_{n-j} G_{j+1} + 4u \sum_{j=0}^{n} G_{n-j} G_j \\ &+ 2\eta^{-2} \sum_{j=0}^{n} G_{n-j} \partial_t^2 G_j - \eta^{-2} \sum_{j=0}^{n} \partial_t G_{n-j} \partial_t G_j \\ &+ \frac{1}{2}\delta_{n+1} + \frac{1}{4}\sum_{j=0}^{n-1} \delta_{n-j}\delta_{j+1}. \end{aligned} \tag{B.13}$$

Using this relation (B.13), we obtain

$$\begin{aligned} &\text{L.H.S of (1.1.2)} - \text{R.H.S of (1.1.2)} \\ &= \begin{cases} 2^{-2j-2}\Big(\delta_{j+1} + \dfrac{1}{2}\displaystyle\sum_{k=0}^{j-1} \delta_{j-k}\delta_{k+1}\Big) - c_j & (j \neq m), \\ -2^{-2m-1}(G_{m+1} + 2^{2m+1} t) \\ \qquad + 2^{-2m-2}\Big(\delta_{m+1} + \dfrac{1}{2}\displaystyle\sum_{k=0}^{m-1} \delta_{m-k}\delta_{k+1}\Big) - c_m & (j = m). \end{cases} \end{aligned} \tag{B.14}$$

Hence (B.11) entails (1.1.2). This completes the proof of Proposition B.1. □

Each member of the hierarchy (B.8) has the following Lax pair:

$$\frac{\partial}{\partial x}\psi = \eta A\psi, \quad \frac{\partial}{\partial t}\psi = \eta B\psi, \tag{B.15}$$

where

$$A = \frac{1}{g}\begin{pmatrix} -\eta^{-1}\partial_t T_m & 2T_m \\ 2(x-u)T_m - \eta^{-2}\partial_t^2 T_m & \eta^{-1}\partial_t T_m \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ x-u & 0 \end{pmatrix} \tag{B.16}$$

and

$$T_m = \sum_{j=0}^{m} (4x)^j G_{m-j}. \tag{B.17}$$

As the form of this Lax pair is similar to that of $(L_{\text{II-1}})_m$ (*i.e.*, the underlying Lax pair of the $P_{\text{II-1}}$-hierarchy), we can develop a similar argument as in Section 2.2 also for the hierarchy (B.8). This gives us another proof of Propositions 2.1.1 $\sim$ 2.1.5 for the P_{I}-hierarchy.

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