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SKEW DIFFERENTIAL FIELDS, DIFFERENTIAL AND DIFFERENCE EQUATIONS

by

Marius van der Put

Dedicated to Jean-Pierre Ramis on the occasion of his 60th birthday

Abstract. — The central question is: Let a differential or difference equation over a field K be isomorphic to all its Galois twists w.r.t. the group $\mathrm{Gal}(K/k)$. Does the equation descend to k? For a number of categories of equations an answer is given.

***Résumé* (Corps différentiels non commutatifs, équations différentielles et équations aux différences)**

On étudie la descente sur un corps k d'une équation différentielle ou aux différences donnée sur un corps K et qui est isomorphe à toutes ses conjuguées sous l'action du groupe de Galois $\mathrm{Gal}(K/k)$ de K sur k. On traite le cas de plusieurs classes d'équations.

Introduction

Rationality questions for differential modules and differential operators are strongly related to skew differential fields. This theme has been developed in [**H-P**]. An open question in [**H-P**] has found an answer, namely the existence and unicity of the differentiation on a skew field of finite dimension over its center, that is, a differential field in the usual sense. The present paper, *written in honour of Jean-Pierre Ramis*, reviews these descent problems but now in the context of meromorphic differential equations. A remarkable family of examples is the result. Equally surprising is that descent does hold for meromorphic q-difference equations. This is shown using recent work of J.-P. Ramis and J. Sauloy on moduli for these equations. Finally, it is shown that descent does not hold for meromorphic ordinary difference equations.

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1. The construction of skew differential fields

Let k denote a differential field having characteristic 0. The skew fields, or more generally the algebras F, that we consider here are central, simple, finite dimensional over their center k. A differentiation on F, extending the differentiation of k, is an additive map $\partial : F \to F$ such that $\partial(ab) = \partial(a)b + a\partial(b)$ holds for all $a, b \in F$. Moreover, we require that $\partial(a) = a'$ for every $a \in k$. For special cases, such differentiations are constructed in [**H-P**]. Here we prove a general result on differentiations.

Theorem 1.1. — *Let k be a differential field of characteristic 0. Let F be a central, simple algebra over k of finite dimension. Suppose that F contains a maximal commutative subfield K which is Galois over k. There exists a differentiation ∂_0 on F extending the differentiation of k and having the property $\partial_0 K \subset K$.*

Moreover, for any differentiation ∂ on F extending the differentiation of k, there is an element $c \in F$ (unique up to an element in k) such that $\partial(a) = \partial_0(a) + ac - ca$ for all $a \in F$.

Proof. — The asumptions on F imply that F is a crossed product algebra (see [**Bl**], Chapitre IV). The structure of F is the following:

The elements of F are uniquely given by expressions $\sum_{\sigma \in G} d_\sigma[\sigma]$, where G is the Galois group of K/k and all $d_\sigma \in K$. The multiplication is given by the rules $[\sigma]d = \sigma(d)[\sigma]$ (for $\sigma \in G$, $d \in K$) and $[\sigma][\tau] = c(\sigma,\tau)[\sigma\tau]$. Here $(\sigma,\tau) \mapsto c(\sigma,\tau)$ is a 2-cocycle representing an element of $H^2(G, K^*)$.

Let $'$ denote the unique differentiation on K, extending the one of k. Then $(\sigma,\tau) \mapsto c(\sigma,\tau)'/c(\sigma,\tau)$ is a 2-cocycle for $H^2(G, K)$. Since the latter group is trivial (see [**Se**]), there are elements $\{a(\sigma)\}$ in K such that $a(\sigma) + {}^{\sigma}a(\tau) - a(\sigma\tau) = c(\sigma,\tau)'/c(\sigma,\tau)$. Now, define ∂_0 by the formula

$$\partial_0\big(\sum d_\sigma[\sigma]\big) = \sum (d'_\sigma + d_\sigma a(\sigma))[\sigma].$$

The verification that ∂_0 has the required properties is straightforward.

Let ∂ be another derivation on F extending the one of k. Then $\partial - \partial_0$ is a k-linear derivation on F. It is known that these derivations are given by $a \mapsto [a, c] := ac - ca$ for $c \in F$. (See [**Ren**], Corollaire 3 on p. 111). □

We note that the differentiation ∂ on F, extending the differentiation of k, is *almost unique* if one prescribes that ∂ is the usual differentiation on the maximal commutative subfield K of F. Indeed, $\partial(a) = \partial_0(a) + [a, c]$ for some $c \in F$. For $a \in K$ one has $[a, c] = 0$. Further, K is a maximal commutative subfield of F and, thus, $c \in K$.

2. Skew differential fields over $\mathbf{R}(\{x\})$

Notations. — $k := \mathbf{R}(\{x\})$, $K := \mathbf{C}(\{x\})$ are the fields of convergent Laurent series over $\mathbf{R}$ and $\mathbf{C}$. The differentiation on these fields is given by $f \mapsto f' := xdf/dx$.

Hamilton's quaternion field is denoted by $\mathbf{H} := \mathbf{R}1 + \mathbf{R}i + \mathbf{R}j + \mathbf{R}k$. Then $F = \mathbf{H} \otimes_{\mathbf{R}} k$ is a quaternion field with center k. Let $\| \ \|$ denote the usual norm on $\mathbf{H}$. The differentiation on F is defined by $(h \otimes f)' = h \otimes f'$ for all $h \in \mathbf{H}$ and $f \in k$. The elements of F are represented by convergent Laurent series with coefficients in $\mathbf{H}$. Thus, an element of F has the form $\sum a_n x^n$ with all $a_n \in \mathbf{H}$ and such that only finitely many negative powers of x are present and, moreover, there are positive constants C, R with $\|a_n\| \leqslant CR^n$ for all $n \geqslant 0$. One observes that $(\sum a_n x^n)' = \sum n a_n x^n$.

Consider the 1-dimensional differential module $M = Fe$ over F, defined by the formula $\partial e = de$. After identifying F with M, via $v \mapsto ve$, one has $\partial(v) = v' + vd$. For d we make the choice $d = i + x^{-1}j$. One can consider M as a differential module over K of dimension 2, by the obvious inclusion $K \subset F$. Further, M is also a differential module over k of dimension 4.

Proposition 2.1. — $\mathrm{End}_{k[\partial]}(M)$, *the* $\mathbf{R}$*-algebra of the endomorphisms of the* k*-differential module* M, *is equal to* $\mathbf{H}$.

Proof. — Every k-linear map $L : M \to M$ has uniquely the form $L(v) = va_0 + iva_1 + jva_2 + kva_3$ with $a_0, \ldots, a_3 \in F$. A calculation shows that

$$(\partial L - L\partial)(v) = v(a_0' + [a_0, d]) + iv(a_1' + [a_1, d]) + jv(a_2' + [a_2, d]) + kv(a_3' + [a_3, d]).$$

Hence $L \in \mathrm{End}_{k[\partial]}(M)$ if and only if $a_i' + [a_i, d] = 0$ for $i = 0, \ldots, 3$. Therefore, the proposition follows from the statement:

The only solutions $a \in F$ *of* $a' + [a, d] = 0$ *are* $a \in \mathbf{R}$.

The proof of this statement is as follows. Write $a = \sum a_n x^n$ with all $a_n \in \mathbf{H}$. Then, $a' + [a, d] = 0$ translates into

$$\sum (na_n + [a_n, i] + [a_{n+1}, j])x^n = 0.$$

For $n > 0$ and $t = t_0 + t_1 i + t_2 j + t_3 k \in \mathbf{H}$ one has

$$nt + [t, i] = nt_0 + nt_1 i + (nt_2 + 2t_3)j + (nt_3 - 2t_2)k.$$

It follows that $\|nt + [t, i]\| \geqslant n\|t\|$.

For $s = s_0 + s_1 i + s_2 j + s_3 k \in \mathbf{H}$ one has $[s, j] = 2s_i k - 2s_3 i$ and thus $\|[s, j]\| \leqslant 2\|s\|$. One concludes that for $n > 0$ one has $\|a_{n+1}\| \geqslant \frac{n}{2}\|a_n\|$. If some $a_m \neq 0$ with $m > 0$, then $a_n \neq 0$ for all $n \geqslant m$. Moreover, for a suitable constant $C > 0$ one has $\|a_n\| \geqslant C2^{-n}n!$ for all $n \geqslant m$. This contradicts the assumption that the Laurent series a is convergent. The conclusion is that $a_n = 0$ for all $n \geqslant 1$.

$0 \cdot a_0 + [a_0, i] + [a_1, j] = 0$ implies that $a_0 \in \mathbf{R} + \mathbf{R}i$. After subtracting from a a real number, one has $a_0 \in \mathbf{R}i$. In the sequel we will write $*$ for a non-zero real number. Suppose that $a_0 = *i$. Then $-a_{-1} + [a_{-1}, i] + [a_0, j] = 0$ implies that $a_{-1} = *j + *k$. The equation $-2a_{-2} + [a_{-2}, i] + [a_{-1}, j] = 0$ implies $a_{-2} = *i$. By induction, one finds that $a_{-2m} = *i$ and $a_{-2m-1} = *j + *k$. This contradicts the fact that a is a Laurent series. One concludes that $a_0 = 0$. □

In the proof of the following corollary we will use some ideas and results of [**H-P**]. For any differential fields $k \subset K$, one says that a differential module M over K descends to k, if there exists a differential module N over k such that $M \cong K \otimes_k N$. Suppose that K/k is a Galois extension with group G. For a differential module M over K and for $\sigma \in G$, one defines the twisted differential module ${}^{\sigma}M$ by:

- ${}^{\sigma}M$ is equal to M as an additive group.
  For $f \in K$ and $m \in {}^{\sigma}M$ one puts $fm = \sigma^{-1}(f)m$.
- The ∂ on ${}^{\sigma}M$ coincides with ∂ on M.

If M descends to k, then clearly ${}^{\sigma}M \cong M$ for all $\sigma \in G$. The descent problem of [**H-P**] asks whether the converse is true. In general, there is an obstruction given by the class of a 2-cocycle.

Corollary 2.2. — *We keep the above notations.*

(a) *$M = Fe$ is an irreducible differential module over k.*

(b) *Let σ denote the non-trivial element of the Galois group of K/k. Then the twisted differential module ${}^{\sigma}M$ over K is isomorphic to M.*

(c) *The K-differential module M does not descend to k.*

(d) *Let $\widehat{k} = \mathbf{R}((x))$ and $\widehat{K} = \mathbf{C}((x))$. The $\widehat{K}$-differential module $\widehat{K} \otimes_K M$ descends to $\widehat{k}$.*

Proof

(a) Suppose that M is reducible as a K-differential module. Let $N \subset M$ be a 1-dimensional K-submodule. Then jN is also a 1-dimensional K-submodule and $M = N + jN$. In particular, M is semi-simple as K-differential module. If M is irreducible as K-differential module, then M is semi-simple, too. According to [**H-P**], proposition 2.7, M is also semi-simple as k-differential module. Since $\mathrm{End}_{k[\partial]}(M)$ is a skew field, one has that M is irreducible as k-differential module.

(b) The map $\Phi(\sigma) : M \to M$, given by $fe \mapsto jfe$, is a σ-linear bijection commuting with ∂. This proves the statement.

(c) Since $\Phi(\sigma)\Phi(\sigma) = -1$, the 2-cocycle class in $H^2(\{1, \sigma\}, \mathbf{C}^*)$, associated to M, is not trivial. It follows from [**H-P**], theorem 2.8, that the K-differential module M does not descend to k.

(d) Put $\widehat{M} = \widehat{K} \otimes_K M$. The twisted $\widehat{K}$-module ${}^{\sigma}\widehat{M}$ is isomorphic to $\widehat{M}$. According to [**H-P**], theorem 2.4, $\widehat{M}$ descends to $\widehat{k}$. □

Explicit calculations. — The element e of the K-differential module $M = Fe$ is a cyclic vector. The minimal monic operator $L_2 \in K[\partial]$ with $L_2 e = 0$ can be calculated to be

$$\delta^2 + \delta + (1 + x^{-2} - i).$$

Here, we prefer to write $\delta = xd/dx$ instead of ∂, since the latter may be confusing. Note that $\delta x = x\delta + x$. The corollary translates into the following:

L_2 is equivalent to its conjugate $\sigma(L_2) = \delta^2 + \delta + (1 + x^{-2} + i)$. In fact, one has

$$\sigma(L_2)x(\delta - i) = x(\delta - i - 2)L_2.$$

Furthermore, L_2 is not equivalent over K to an operator in $k[\delta]$.

Finally, L_2 is over $\widehat{K}$, equivalent to the operator $\delta^2 + \delta + x^{-2}$ in $k[\delta]$.

Now, we apply the methods of [**M-R**] (see also [**P-S2**]) in order to obtain the formal classification, the formal differential Galois group G_{formal}, the monodromy group and the differential Galois group G of the 2-dimensional differential module M over $K = \mathbf{C}(\{x\})$.

(i) $\widehat{M} := \widehat{K} \otimes_K M$ has a basis e_1, e_2 such that $\delta e_j = q_j e_j$ for $j = 1, 2$ and with $q_1 = ix^{-1}$, $q_2 = -ix^{-1}$. This is the formal classification. One observes that $\sigma(q_1) = q_2$. This implies what we know already, namely that $\widehat{M}$ descends to $\widehat{k} = \mathbf{R}((x))$.

(ii) The formal differential Galois group, *i.e.*, the differential Galois group of $\widehat{M}$ over $\widehat{K}$, is the group

$$G_{\text{formal}} = \left\{ \left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right) \mid c \in \mathbf{C}^* \right\} \cong \mathbf{G}_m.$$

The formal monodromy is the identity.

(iii) The topological monodromy group of $\delta^2 + \delta + (1 + x^{-2} - i)$ can be calculated at the point ∞. The local exponents are $-i$, $i - 1$ and the eigenvalues of the topological monodromy are $e^{2\pi}, e^{-2\pi}$.

(iv) The differential Galois group G is contained in SL_2, since the second exterior power of M corresponds to the equation $\delta - 1$. Since M is irreducible, the group G is an irreducible subgroup of SL_2. According to [**M-R**], the group G is generated, as an algebraic group, by G_{formal} and the Stokes matrices. There are two singular directions $\pm\pi/2$ and two Stokes matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$. Since G is irreducible both Stokes matrices are different from 1. One concludes that $G = \mathrm{SL}_2$.

(v) The two Stokes matrices correspond to the singular directions for $q_1 - q_2$ and $q_2 - q_1$. From these one sees that the formal solutions involve Gevrey-1 functions. This is explicitly seen in the calculation of proposition 2.1, where the factor $n!$ occurs as measure for the divergence.

(vi) According to [**M-R**] and [**P-S2**], the product of the two Stokes matrices is conjugated to the topological monodromy. In particular, $2 + ab = e^{2\pi} + e^{-2\pi}$. This confirms again that $a \neq 0 \neq b$.

Moreover, one can convince oneself that the Stokes matrices of the twisted differential module ${}^{\sigma}M$ are $\left(\begin{smallmatrix} 1 & -\overline{b} \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ -\overline{a} & 1 \end{smallmatrix}\right)$ (where $\overline{a}, \overline{b}$ are the complex conjugates of a, b). The statement ${}^{\sigma}M \cong M$ is equivalent to $b = -\overline{a}$.

(vii) We have shown that the above condition $b = -\overline{a}$ on the Stokes matrices does not imply that M descends to $\mathbf{R}(\{x\})$. The 'explanation' is rather deep. We will use differential modules. Let the differential module N over K be given by the matrix equation

$$\delta + \begin{pmatrix} 0 & -x^{-1} \\ x^{-1} & 0 \end{pmatrix} \quad \text{with } \delta = x\frac{d}{dx}.$$

Babbitt and Varadarajan have considered the moduli set BV consisting of equivalence classes of pairs (M, ϕ) consisting of a differential module M over K and an isomorphism $\phi : \widehat{K} \otimes_K M \to \widehat{K} \otimes_K N$. This moduli set has the structure of an algebraic variety over $\mathbf{C}$. In this special case $BV \cong \mathbf{A}^2_{\mathbf{C}}$. This is a coarse moduli space. An element $(a, b) \in BV$ corresponds to the two Stokes matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right)$ that multisummation associates with (M, ϕ).

There is also a fine moduli space $\mathbf{M}$, again $\cong \mathbf{A}^2_{\mathbf{C}}$, with a universal family on it, namely

$$\delta + \begin{pmatrix} \alpha & -x^{-1} + \beta \\ x^{-1} + \beta & -\alpha \end{pmatrix} \quad \text{with } (\alpha, \beta) \in \mathbf{M}(\mathbf{C}).$$

$\mathbf{M}$ has an obvious natural structure over $\mathbf{R}$. In this special case, the analytic morphism $\mathrm{St} : \mathbf{M} \to BV$ is explicitly known, namely

$$\mathrm{St}(\alpha, \beta) = i f(\alpha^2 + \beta^2) \cdot (\alpha + i\beta, \alpha - i\beta), \quad \text{where } f(t) = 2\,\frac{\sin\sqrt{t}}{\sqrt{t}}.$$

Take $\xi \in \mathbf{M}(\mathbf{C})$. Suppose that ξ is isomorphic to $\overline{\xi}$. Is there a $\eta \in \mathbf{M}(\mathbf{R})$ with ξ equivalent to η? This question translates into: Suppose that $\mathrm{St}(\xi) = \mathrm{St}(\overline{\xi})$. Is there an $\eta \in \mathrm{St}^{-1}(\mathrm{St}(\xi))$ with $\eta \in \mathbf{M}(\mathbf{R})$. One sees that the answer is negative in general.

(viii) The element e is also a cyclic element for M considered as differential module over k. The minimal monic operator $L_4 \in k[\delta]$ with $L_4 e = 0$ is calculated to be:

$$L_4 = \delta^4 + 2\delta^3 + (3 + 2x^{-2})\delta^2 + (2 - 2x^{-2})\delta + (2 + 4x^{-2} + x^{-4}).$$

Clearly, L_2 is an irreducible right hand factor of L_4 in the operator ring $K[\delta]$. Also $\sigma(L_2)$ is an irreducible right hand factor of L_4. Since L_2 and $\sigma(L_2)$ commute, one has in fact $L_4 = \sigma(L_2)L_2$. By the above, L_4 is irreducible in $k[\delta]$. However L_4 has a right hand factor in $\widehat{k}[\delta]$.

Remarks. — The above example can be largely extended as follows. Let F be the skew differential field considered above. On the left vector space $M := F^a$ over F one considers a structure ∂ of F-differential module given by $\partial(v) = v' + vD$, where $v' = (v_1, \ldots, v_a)' := (v_1', \ldots, v_a')$ and where $D \in \mathrm{Matr}(a, F)$. As in **[H-P]**, proposition 2.14, for a suitable choice of D, one has:

(a) $\mathrm{End}_{K[\partial]}(M) = \mathbf{C}$.

(b) *The twist* ${}^{\sigma}M$ *is isomorphic to the* K*-differential module* M.

(c) $\mathrm{End}_{k[\partial]}(M) = \mathbf{H}$ *and* M *is an irreducible* k*-differential module.*

(d) *The* K*-differential module* M *does not descend to* k.

(e) *The* $\widehat{K}$*-differential module* $\widehat{K} \otimes_K M$ *descends to* $\widehat{k}$.

Other skew fields over $k = \mathbf{R}(\{x\})$. — It is known that $K = \mathbf{C}(\{x\})$ is a C_1-field. It follows that the Brauer group of K is trivial. As a consequence, K is a splitting field for any skew field F, of finite dimension over its center k. Thus the Brauer group of k is equal to the cohomology group $H^2(\{1,\sigma\}, K^*)$. The map $\mathbf{C}^* \times \mathbf{Z} \times x\mathbf{C}\{x\} \to K^*$, given by $(c, n, f) \mapsto cx^n \exp(f)$ is an isomorphism of $\{1,\sigma\}$-modules. One calculates that $H^2(\{1,\sigma\}, \mathbf{C}^*)$ and $H^2(\{1,\sigma\}, \mathbf{Z})$ are both isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The $\{1,\sigma\}$-module $x\mathbf{C}\{x\}$ is an induced module and its H^2 is 0. Therefore the Brauer group of k is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. The three non-zero elements of this Brauer group correspond to the three distinct skew fields of finite dimension over their center k. One can give them explicitly as follows.

A quaternion algebra with center k is given by a basis b_0, b_1, b_2, b_3 over k and rules of multiplication: $b_0 = 1$; $b_i^2 = A_i \in k$ for $i = 1, 2, 3$; $b_1 b_2 = -b_2 b_1 = b_3$ and thus $A_1 A_2 = -A_3$.

For suitable choices of A_1, A_2 one obtains the three skew fields mentioned above, namely:

(1) $A_1 = -1$, $A_2 = -1$. This is the skew field $\mathbf{H} \otimes_{\mathbf{R}} k$ that we have used.
(2) $A_1 = -1$, $A_2 = x$.
(3) $A_1 = -1$, $A_2 = -x$.

The cases (2) and (3) lead to other examples of differential equations over k having interesting descent properties.

Another family of examples. — F denotes the quaternion field over $k = \mathbf{R}(\{x\})$ with basis $b_0, \ldots, b_3$ and multiplication given by $b_0 = 1$, $b_1 b_2 = -b_2 b_1 = b_3$ and $b_1^2 = -1$, $b_2^2 = x$. The differentiation $'$ on F, extending xd/dx on k, is defined by $b_0' = b_1' = 0$ and $b_2' = 1/2 \cdot b_2$, $b_3' = 1/2 \cdot b_3$. The field $K = \mathbf{C}(\{x\})$ is seen as a subfield of F by the identification of i with b_1.

One considers the differential module $M = Fe$ over F, defined by $\partial e = de$. Later on we will choose $d = b_1 + x^{-1} b_2$. Now M is a differential module over K of dimension 2, with basis $e, b_2 e$. The twisted module ${}^{\sigma}M$ is isomorphic to $M \otimes_K L$, where $L = Kr$ with $\partial r = 1/2 \cdot r$. Indeed, the σ-linear bijection $A(\sigma) : M \to M$, defined by $A(\sigma)m = b_2 m$, has the property $\partial A(\sigma) = A(\sigma)(\partial + 1/2)$. The second symmetric power $N := \mathrm{Sym}_K^2 M$ is a differential module of dimension 3 with basis $e \otimes e, b_2 e \otimes e, b_2 e \otimes b_2 e$. The σ-linear bijection $B : N \to N$, with formula $B(\sigma) m_1 \otimes m_2 = x^{-1} b_2 m_1 \otimes b_2 m_2$ commutes with ∂. Further $B(\sigma)B(\sigma)$ is the identity on N. Put $N^o = \{n \in N \mid B(\sigma)n = n\}$. Then one calculates that N^o is a vector space over k with basis

$$n_1^o = e \otimes e + x^{-1} b_2 e \otimes b_2 e,\ \ n_2^0 = b_1 e \otimes e - b_1 x^{-1} b_2 e \otimes b_2 e,\ \ n_3^o = b_2 e \otimes e.$$

Clearly N^o is a differential module over k and $K \otimes_k N^o \cong N$. In other words, N descends to k. For the choice $d = b_1 + x^{-1} b_2$, the minimal monic operator L_3 with

$L_3 n_3^o = 0$ can be calculated:

$$L_3 = \delta^3 - 1/2 \cdot \delta^2 + (4 - 4x^{-1})\delta + 4x^{-1} - 2.$$

Here one has written δ instead of ∂ in order to emphasize that δ is the operator xd/dx.

The main interesting properties of the 3-dimensional differential module N^o over k are:

***Proposition 2.3* (Properties of M and N^o)**

(i) *N^o is not the second symmetric power of a differential module over k.*

(ii) *For a finite field extension $L \supset k$, which is a splitting field of F, the module $L \otimes_k N^o$ is a second symmetric power. This holds in particular for $L = K$ and $L = k(\sqrt{x})$.*

(iii) *$\widehat{k} \otimes_k N^o$ is a second symmetric power.*

(iv) *$\widehat{M} := \widehat{K} \otimes_K M$ descends to $\widehat{k}$.*

Again with $d = b_1 + x^{-1}b_2$, one computes that the monic minimal operator L_2 for the cyclic element e of M (over the differential field K):

$$L_2 = \delta^2 + 1/2 \cdot \delta - x^{-1} + 1 - i/2.$$

In this formula we have again written $\delta := xd/dx$, in order to emphasize that the differentiation on K is given by $f \mapsto xdf/dx$. By construction L_3 is equivalent, but not equal (!), to the second symmetric power of L_2.

(i) states that L_3 is not equivalent over k to the second symmetric power of an operator in $k[\delta]$.

(ii) states that L_3 is equivalent to a second symmetric power in $L[\delta]$ if $L \supset k$ is a splitting field for F.

(iii) states that L_3 is equivalent over $\widehat{k}$ to the second symmetric power of an element in $k[\delta]$.

(iv) states that L_2 is equivalent over $\widehat{K}$ to an operator in $k[\delta]$.

Proof. — For the proof of (i) and (ii), we use methods of [**M-R**] and [**P-S2**]. In particular we determine the formal differential Galois group G_{formal}, the topological monodromy group, the differential Galois G, et cetera of L_2.

(a) G and G_{formal} are contained in $\{A \in \mathrm{GL}_2 \mid \det(A)^2 = 1\}$. (because of the term $1/2 \cdot \delta$ in L_2).

(b) The eigenvalues of the operator L_2 are $\pm x^{-1/2}$. (follows from a computation with the Riccati equation $u' + u^2 + 1/2 \cdot u + (-x^{-1} + 1 - i/2) = 0$).

(c) G_{formal} is the dihedral group generated by $\{\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right) \mid c \in \mathbf{C}^*\}$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (follows from the explicit form $\widehat{K}(x^{1/2}, \exp(2x^{-1/2}))$ of the Picard-Vessiot field).

(d) $G = \{A \in \mathrm{GL}_2 \mid \det(A)^2 = 1\}$.

(The topological monodromy can be calculated at ∞. The formal monodromy is easy to calculate. The product of the formal monodromy and the Stokes matrices is conjugated to the topological monodromy. Hence there are non-trivial Stokes matrices. Finally G, as algebraic group, is generated by G_{formal} and the Stokes matrices. Thus G is the above group).

The test for a differential module Z of dimension 3 to be a second symmetric power, is the existence of a 1-dimensional submodule of $\mathrm{Sym}^2 Z$, which corresponds to non-singular quadratic form having a rational point over the differential field. We refer to [**H-P**] for a proof of this and for the statement that the only candidate in case N^o is the quadratic form corresponding to F. Here one uses that $sym_K^4 M$ has no 1-dimensional submodules, which is a consequence of the above calculation of G. Now (i) and (ii) follow.

Let $\widehat{F}$ denote the quaternion field over $\widehat{k}$, given by the same formulas as F. Then $\widehat{M} := \widehat{K} \otimes_K M = \widehat{F}e$ is a differential module over $\widehat{F}$ with the same formula $\partial e = (b_1 + x^{-1}b_2)e$. We will show that there exists an element $f = ue$, with $u \in \widehat{F}$, $u \neq 0$, such that $\partial f = x^{-1}b_2 f$. Assuming this, one considers $\widehat{M}$ again as a $\widehat{K}$-differential module with cyclic vector f. The minimal monic operator $\widetilde{L}_2 \in \widehat{K}[\delta]$ with $\widetilde{L}_2 f = 0$ is easily calculated, namely

$$\widetilde{L}_2 = \delta^2 + 1/2 \cdot \delta - x^{-1}.$$

This proves (iv). Part (iii) follows at once from (iv), since L_3 is equivalent over $\widehat{k}$ to the second symmetric power of $\widetilde{L}_2$.

The element $u \in \widehat{F}$ that we are looking for must be a non-zero solution of the equation $\partial(ue) = x^{-1}b_2 ue$. This translates into

$$u' + [u, x^{-1}b_2] + ub_1 = 0.$$

Consider the operator $L : \widehat{F} \to \widehat{F}$, given by $L(u) := u' + [u, x^{-1}b_2] + ub_1$. One observes that:

$$L(1) = b_1 \qquad L(b_1 x) = 2b_3 + (b_1 - 1)x$$
$$L(b_2) = 1/2 \cdot b_2 + b_3 \qquad L(b_3) = 1/2 \cdot b_3 + 2b_1 - b_2$$

and for $n \geqslant 1$ one has the formulas

$$L(x^n) = (n + b_1)x^n \qquad L(b_1 x^{n+1}) = 2b_3 x^n + ((n+1)b_1 - 1)x^{n+1}$$
$$L(b_2 x^n) = ((n + 1/2)b_2 + b_3)x^n \qquad L(b_3 x^n) = ((n + 1/2)b_3 + 2b_1 - b_2)x^n.$$

Let u_0 be a non-trivial $\mathbf{R}$-linear combination of $1, b_1 x, b_2, b_3$ such that $L(u_0)$ lies in $\mathbf{R}[[x]]$-module W spanned by $x1, xb_1, xb_2, xb_3$. Let V be the $\mathbf{R}[[x]]$-module spanned by $x, x^2 b_1, xb_2, xb_3$. The formulas show that $L : V \to W$ is bijective. Let $u_1 \in V$ satisfy $L(u_1) = -L(u_0)$. Then $u = u_0 + u_1$ is a non-zero element of $\widehat{F}$ and $L(u) = 0$. □

The method of proposition 2.3 extends and produces a large class of differential equations over k, having interesting descent properties.

3. Descent for q-difference equations

In this section we compare q-difference equations over the fields $\mathbf{R}(\{x\})$ and $\mathbf{C}(\{x\})$. For this purpose one supposes that $q \in \mathbf{R}$ and $0 < |q| < 1$. Let σ denote the complex conjugation of $\mathbf{C}$ and its natural extension to an automorphism of $\mathbf{C}(\{x\})$ over $\mathbf{R}(\{x\})$. The automorphism ϕ on both fields $\mathbf{C}(\{x\})$ and $\mathbf{R}(\{x\})$ is defined by $\phi x = qx$. A q-difference module $M = (M, \Phi)$ over $\mathbf{R}(\{x\})$ or $\mathbf{C}(\{x\})$ is a finite dimensional vector space over that field together with a bijective additive map $\Phi : M \to M$ satisfying $\Phi(fm) = \phi(f)\Phi(m)$ for $m \in M$ and f in the field. For a q-difference module M over $\mathbf{C}(\{x\})$ one defines the twist ${}^{\sigma}M = ({}^{\sigma}M, {}^{\sigma}\Phi)$ as follows:

(i) ${}^{\sigma}M$ is equal to M as additive group, ${}^{\sigma}\Phi$ is equal to Φ and

(ii) ${}^{\sigma}M$ has a new structure as $\mathbf{C}(\{x\})$-vector space given by the formula $f * m := \sigma(f)m$.

***Theorem 3.1* (Descent for q-difference modules).** — *Let q-difference module M over $\mathbf{C}(\{x\})$ satisfy ${}^{\sigma}M \cong M$. There exists a q-difference module N over $\mathbf{R}(\{x\})$ such that $M \cong \mathbf{C} \otimes_{\mathbf{R}} N$.*

Proof. — We sketch here the proof which is based on the work of J.-P. Ramis and J. Sauloy ([**D-R-S-Z**] and a preprint in preparation) concerning moduli spaces for convergent q-difference equations (*i.e.*, q-difference modules over $\mathbf{C}(\{x\})$).

We consider first the case of a regular singular q-difference module M over $\mathbf{C}(\{x\})$. The classification of regular singular modules can be formulated as follows. There is a unique $\mathbf{C}$-vector space $W \subset M$ with the following properties:

(a) the natural map $\mathbf{C}(\{x\}) \otimes_{\mathbf{C}} W \to M$ is a bijection,

(b) $\Phi(W) = W$ and all eigenvalues λ of the restriction ψ of Φ to W satisfy $|q| < |\lambda| \leqslant 1$.

We write $W(M)$ and $\psi(M)$ for the W and ψ above. A morphism $f : M \to N$ between regular singular modules induces a linear map $\widetilde{f} : W(M) \to W(N)$ satisfying $\widetilde{f} \circ \psi(M) = \psi(N) \circ \widetilde{f}$. One obtains in this way a $\mathbf{C}$-linear bijection (with obvious notations):

$$\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}\big((W(M), \psi(M)), (W(N), \psi(N))\big).$$

The pair (W, ψ) associated to ${}^{\sigma}M$ is equal to $({}^{\sigma}W(M), \psi(M))$, where ${}^{\sigma}W(M)$ is the twist of $W(M)$. More precisely, ${}^{\sigma}W(M)$ coincides with $W(M)$ as an additive group. The new scalar multiplication on ${}^{\sigma}W(M)$ is given by $a * w = \overline{a}w$ (with $a \in \mathbf{C}$, $w \in W(M)$ and $\overline{a}$ the complex conjugate of a). If ${}^{\sigma}M \cong M$, then there exists an additive bijective $A : W(M) \to W(M)$, commuting with $\psi(M)$ and such that $A(aw) = \overline{a}A(w)$ for all $a \in \mathbf{C}$ and $w \in W(M)$. It is not dificult to verify that there exists a real vector space V together with an $\mathbf{R}$-linear automorphism τ such that $\mathbf{C} \otimes_{\mathbf{R}} (V, \tau) \cong (W(M), \psi(M))$. Moreover $N := \mathbf{R}(\{x\}) \otimes_{\mathbf{R}} (V, \tau)$, with its obvious structure of a q-difference module over $\mathbf{R}(\{x\})$, satisfies $\mathbf{C} \otimes_{\mathbf{R}} N \cong M$.

One associates, in a canonical way, to a general q-difference module M a Newton polygon and a decreasing slope filtration $\{M^{\geqslant\mu}\}$ of M by submodules. Each non zero quotient $M^{\geqslant\mu}/M^{>\mu}$ is a pure q-difference module. The latter means that the Newton polygon of this quotient has only one slope, namely μ.

A pure q-difference module such that its slope is an integer has the form $N\otimes E(n)$ with N a regular singular module and $E(n)=\mathbf{C}(\{x\})e$ with $\Phi e=x^ne$ (for a suitable integer n).

In the case of non-integer slopes μ one has to make a few modifications to describe the pure difference modules with slope μ. Let $d\geqslant 1$ denote the denominator of μ. First of all, one extends ϕ to an automorphism of $\mathbf{C}(\{x^{1/d}\})$ by $\phi(x^{1/d})=q^{1/d}x^{1/d}$ for a suitable choice of $q^{1/d}$. A coherent choice can be made as follows. Choose an element τ in the upper half plane with $e^{2\pi i\tau}=q$. Then $q^{1/d}:=e^{2\pi i\tau/d}$. One defines the module $E(\mu)^+:=\mathbf{C}(\{x^{1/d}\})e$ over $\mathbf{C}(\{x^{1/d}\})$ by $\Phi e=x^{\mu}e$. Let $E(\mu)$ denote $E(\mu)^+$, considered as a q-difference module of dimension d over the field $\mathbf{C}(\{x\})$. One can show that $E(\mu)$ does not depend on the choice of $q^{1/d}$. Moreover, one can show that any pure q-difference module with slope μ has uniquely the form $E(\mu)\otimes N$ where N is a regular singular module.

One associates to M the graded module $\mathrm{gr}(M):=\oplus_\mu M^{\geqslant\mu}/M^{>\mu}$. This is a direct sum of pure modules. If one works over the field of formal Laurent series, *i.e.*, the field $\mathbf{C}((x))$, then M is isomorphic to $\mathrm{gr}(M)$. In the convergent situation one can only say that M is a multiple extension of the pure modules ocurring in $\mathrm{gr}(M)$ (and taken in the correct order). The equivalence classes of the multiple extension are the $\mathbf{C}$-valued points of a finite dimensional moduli variety over $\mathbf{C}$, which we will call $\mathrm{Ext}(\mathrm{gr}(M))$. In fact, $\mathrm{Ext}(\mathrm{gr}(M))$ is an affine space over $\mathbf{C}$. In the simplest situation, one considers extensions of two pure modules M_1,M_2. Then $\mathrm{Ext}(\mathrm{gr}(M))$ is just $\mathrm{Ext}^1(M_1,M_2)$ (where M_1,M_2 are interpreted as left modules over $\mathbf{C}(\{x\})[\Phi,\Phi^{-1}]$). This is a vector space of finite dimension over $\mathbf{C}$. In short, the q-difference module M corresponds to a pair $(\mathrm{gr}(M),\xi)$ with ξ a (closed) point of $\mathrm{Ext}(\mathrm{gr}(M))$.

Now we return to the proof of the theorem. Let M be given and suppose that ${}^{\sigma}M\cong M$. This isomorphism is provided by an additive bijective map $A:M\to M$, which commutes with Φ and is semi-linear, *i.e.*, $A(fm)=\sigma(f)A(m)$ for all $m\in M$ and $f\in\mathbf{C}(\{x\})$. The map A preserves the canonical filtration and induces a semi-linear bijection $\mathrm{gr}(A):\mathrm{gr}(M)\to\mathrm{gr}(M)$, commuting with Φ. Each $E(\mu)$, as defined above, is obviously of the form $\mathbf{C}\otimes_{\mathbf{R}}E_{\mathbf{R}}(\mu)$, where $E_{\mathbf{R}}(\mu)$ is a q-difference module over $\mathbf{R}(\{x\})$. Using the above results for regular singular q-difference modules one concludes that $\mathrm{gr}(M)$ is equal to $\mathbf{C}\otimes_{\mathbf{R}}\mathrm{gr}(M)_{\mathbf{R}}$, where $\mathrm{gr}(M)_{\mathbf{R}}$ is a direct sum of pure modules over $\mathbf{R}(\{x\})$. The moduli variety $\mathrm{Ext}(\mathrm{gr}(M))$ is the complexification of a real affine variety $\mathrm{Ext}(\mathrm{gr}(M)_{\mathbf{R}})$. The closed point ξ of $\mathrm{Ext}(\mathrm{gr}(M))$ is invariant under complex conjugation and comes therefore from a real point η of $\mathrm{Ext}(\mathrm{gr}(M)_{\mathbf{R}})$. The q-difference module N over $\mathbf{R}(\{x\})$, given by the pair $(\mathrm{gr}(M)_{\mathbf{R}},\eta)$, has by construction the property $\mathbf{C}\otimes_{\mathbf{R}}N\cong M$. □

Remarks 3.2

(1) Descent of q-difference modules over the field $\mathbf{C}(x)$ (again with $q \in \mathbf{R}$ and $0 < |q| < 1$) does not hold. Indeed, one introduces the skew difference field $F := \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}(x)$ provided with the operator ϕ given by $\phi(x) = qx$ and ϕ is the identity on $\mathbf{H}$. One considers the one dimensional skew q-difference module Fe given by $\Phi(e) = (i + jx)e$. Let $N = Fe$, viewed as a 4-dimensional q-difference module over $\mathbf{R}(x)$. The essential step is to verify that $\mathrm{End}_{\mathbf{R}(x)[\Phi,\Phi^{-1}]}(N) = \mathbf{H}$. Assuming this, one defines $M = Fe$, viewed as a q-difference module over $\mathbf{C}(x)$ of dimension 2. A variation on the proof of corollary 2.2 will show that M is irreducible, ${}^{\sigma}M \cong M$ and M does not descend to $\mathbf{R}(x)$. The element e is a cyclic vector for M. The scalar q-difference equation corresponding to e can be calculated to be

$$y(q^2x) + i(q-1)y(qx) + (1 + q^2x^2 + q - 1)y(x) = 0 \text{ with } q \in \mathbf{R}, 0 < |q| < 1.$$

This equation is irreducible and equivalent to its complex conjugate. However, the equation is not equivalent to an equation over $\mathbf{R}(x)$.

For the verification of $\mathrm{End}_{\mathbf{R}(x)[\Phi,\Phi^{-1}]}(N) = \mathbf{H}$, one takes an $\mathbf{R}(x)$-linear map $L : N \to N$, which has the form $L(v) = va_0 + iva_1 + jva_2 + kva_3$ (where $v \in Fe = F$ and $a_0, \ldots, a_3 \in F$). The equation $\Phi L = L\Phi$ implies that each a_i satisfies the equation $\phi(a)(i + jx) = (i + jx)a$. The real vector space $V := \{a \in F | \phi(a)(i + jx) = (i + jx)a\}$ has dimension $\leqslant 4$, since it is the set of solutions of a 4-dimensional q-difference equation. Moreover, V is a subalgebra of F. Take a non zero $a \in V$ and let $P \in \mathbf{R}[x]$ be the monic polynomial of minimal degree such that $P \cdot a \in \mathbf{H} \otimes \mathbf{R}[x]$. The equality $\phi(a)d = da$ implies that $(1 + x^2)\phi(P) \cdot a \in \mathbf{H} \otimes \mathbf{R}[x]$. A small computation shows that P must be a power of x. Therefore $a \in \mathbf{H} \otimes \mathbf{R}[x, x^{-1}]$. Write $a = \sum_{i=s}^{t} a_i x^i$ with $s, t \in \mathbf{Z}$, $s \leqslant t$ and all $a_i \in \mathbf{H}$. Observing that all powers of a lie in V and that V is finite dimensional, one concludes that $a \in \mathbf{H}$. Now a commutes with $(i + jx)$ and this leads to $a \in \mathbf{R}$. The conclusion is that $L \in \mathbf{H}$, as required.

(2) The reason for the rather striking difference between theorem 3.1 and the examples in section 2 lies in the structure of the moduli space. In the first case, this moduli space has (in the situation that we are interested in) a natural real structure. In the second case, the moduli space is a coarse one.

4. Descent for ordinary difference equations

We consider the difference fields $F(\{x^{-1}\})$ and $F((x^{-1}))$, where F is either $\mathbf{R}$ or $\mathbf{C}$ and with the automorphism ϕ given by $\phi(x) = x + 1$. Let M be a difference module over $\mathbf{C}(\{x^{-1}\})$ such that its twist ${}^{\sigma}M$, where σ is the complex conjugation extended to $\mathbf{C}(\{x^{-1}\})$ in the obvious way, is isomorphic to M. The descent problem is to determine whether M is isomorphic to $\mathbf{C} \otimes_{\mathbf{R}} N$ for some difference module over $\mathbf{R}(\{x^{-1}\})$.

The meromorphic classification of ordinary difference modules is rather complicated. We will restrict ourselves to the class of the *regular difference modules.* We recall from [**P-S1**] some definitions and results. A difference module $M = (M, \Phi)$ is called regular singular if there is a $\mathbf{C}\{x^{-1}\}$-lattice $\Lambda \subset M$ that is invariant under Φ and moreover Φ is the identity on $\Lambda/x^{-1}\Lambda$. Equivalently, M is represented by a matrix difference equation $y(x+1) = Ay(x)$ with A a convergent matrix of the form $1 + A_2x^{-2} + \cdots$. Furthermore, M is regular if and only if $\mathbf{C}((x^{-1})) \otimes M$ is a trivial difference module over $\mathbf{C}((x^{-1}))$. The class of all regular difference modules forms a Tannakian category.

Theorem 4.1. *Descent does not hold for the category of the regular difference modules over* $\mathbf{C}(\{x^{-1}\})$.

For the construction of an example showing that descent does not hold, we replace the category of the regular difference modules by an equivalent Tannakian category. An object of this Tannakian category is a triple $(V, T_{\text{upper}}, T_{\text{lower}})$ where:

(a) V is a complex vector space of finite dimension,

(b) $T_{\text{upper}} = T_{\text{upper}}(u)$ is a $\mathbf{C}\{u\}$-linear automorphism of $\mathbf{C}\{u\} \otimes_{\mathbf{C}} V$ such that $T_{\text{upper}}(0) = 1$ and,

(c) $T_{\text{lower}} = T_{\text{lower}}(u^{-1})$ is a $\mathbf{C}\{u^{-1}\}$-linear automorphism of $\mathbf{C}\{u^{-1}\} \otimes_{\mathbf{C}} V$ such that $T_{\text{lower}}(0) = 1$.

We note that the symbol u stands for $e^{2\pi i x}$.

A morphism $f : (V, T_{\text{upper}}, T_{\text{lower}}) \to (V', T'_{\text{upper}}, T'_{\text{lower}})$ is a linear map $f : V \to V'$ satisfying $f \circ T_{\text{upper}} = T'_{\text{upper}} \circ f$ and $f \circ T_{\text{lower}} = T'_{\text{lower}} \circ f$.

There is a summation method for regular difference modules. The right summation F_{right} and the left summation F_{left} of a formal fundamental matrix F for M are compared by considering $F_{\text{right}}^{-1} F_{\text{left}}$. This matrix exists on an upper half plane and on a lower half plane and yields convergent matrices in the variables u and u^{-1}. These two matrices are the T_{upper} and T_{lower} above. The term V of the triple is the formal solution space of M (or of the matrix equation $y(x+1) = Ay(x)$). This describes the functor $M \mapsto (V, T_{\text{upper}}, T_{\text{lower}})$ from the category of the regular difference equations to the category of the triples. In subsections 8.5 and 10.1 of [**P-S1**] details and explicit calculations are given.

For a vector space V over $\mathbf{C}$, we write (as before) ${}^{\sigma}V$ for the twisted vector space. This means that ${}^{\sigma}V = V$ as additive group and the new scalar multiplication $*$ on ${}^{\sigma}V$ is given by $\lambda * v = \sigma(\lambda)v$ (with $\lambda \in \mathbf{C}$, $\sigma(\lambda) = \overline{\lambda} =$ the complex conjugate of λ and $v \in V$). A linear map $F : V \to W$ can also be considered as a map ${}^{\sigma}V \to {}^{\sigma}W$. This map is again linear and will be denoted by ${}^{\sigma}F$. A linear map $A : {}^{\sigma}V_1 \to V_2$ can also be considered as a semi-linear map $A : V_1 \to V_2$, *i.e.*, A is additive and $A(\lambda v) = \overline{\lambda} v$ for $\lambda \in \mathbf{C}$ and $v \in V_1$. A *real structure* for a complex vector space V is a real subspace $W \subset V$ such that the natural map $\mathbf{C} \otimes_{\mathbf{R}} W \to V$ is an isomorphism.

Suppose that a real structure for V is given. Then one defines complex conjugation on V by $\overline{w_1 + iw_2} = w_1 - iw_2$ for all $w_1, w_2 \in W$. Let $F : V \to V$ be a linear map and suppose that a real structure for V is given. Then one defines a linear map $\overline{F}$, called the complex conjugate of F, by the formula $\overline{F}(w) = \overline{F(w)}$ for all $w \in W$.

Let M correspond to the triple $(V, T_{\rm upper}(u), T_{\rm lower}(u^{-1}))$. Using the explicit formulation of subsection 10.1 of [**P-S1**], one can see that the triple corresponding to ${}^{\sigma}M$ is

$$({}^{\sigma}V, {}^{\sigma}T_{\rm lower}(u), {}^{\sigma}T_{\rm upper}(u^{-1})).$$

We will write $T_{\rm upper} = 1 + \sum_{n\geqslant 1} T_{{\rm upper},n}u^n$ and $T_{\rm lower} = 1 + \sum_{n\geqslant 1} T_{{\rm lower},n}u^{-n}$.

Suppose that M descends to $\mathbf{R}(\{x^{-1}\})$ and so $M = \mathbf{C}\otimes_{\mathbf{R}} N$ for some regular difference module N over $\mathbf{R}(\{x^{-1}\})$. Then the solution space V has a natural real structure, namely $V = \mathbf{C} \otimes_{\mathbf{R}} W$, where W is the real vector space $\ker(\Phi - 1, \mathbf{R}((x^{-1})) \otimes N)$. For this real structure one has that $\overline{T_{{\rm upper},n}} = T_{{\rm lower},n}$ for all $n \geqslant 1$. Equivalently, for each $n \geqslant 1$, the maps $T_{{\rm upper},n} + T_{{\rm lower},n}$ and $iT_{{\rm upper},n} - iT_{{\rm lower},n}$ are real (*i.e.*, they are invariant under conjugation).

On the other hand let be given a triple $(V, T_{\rm upper}, T_{\rm lower})$ together with a real structure for V such that $T_{{\rm upper},n} + T_{{\rm lower},n}$ and $iT_{{\rm upper},n} - iT_{{\rm lower},n}$ are real for all $n \geqslant 1$, then the corresponding regular difference module M descends to $\mathbf{R}(\{x^{-1}\})$.

By the above considerations, theorem 4.1 will be proven if we can produce an example of a triple $(V, T_{\rm upper}, T_{\rm lower})$ and an isomorphism

$$A : ({}^{\sigma}V, {}^{\sigma}T_{\rm lower}(u), {}^{\sigma}T_{\rm upper}(u^{-1})) \longrightarrow (V, T_{\rm upper}(u), T_{\rm lower}(u^{-1}))$$

such that V does not have a real structure for which all maps $T_{{\rm upper},n} + T_{{\rm lower},n}$ and $iT_{{\rm upper},n} - iT_{{\rm lower},n}$ are real.

We prefer to see the map A as a semi-linear isomorphism satisfying $AT_{{\rm upper},n} = T_{{\rm lower},n}A$ and $AT_{{\rm lower},n} = T_{{\rm upper},n}A$ for all $n \geqslant 1$. Since $Ai = -iA$, the latter conditions can also be formulated as A commutes with $T_{{\rm upper},n} + T_{{\rm lower},n}$ and $iT_{{\rm upper},n} - iT_{{\rm lower},n}$ for all $n \geqslant 1$. The example is now constructed as follows:

$\mathbf{H}$ denotes the skew field of the Hamilton quaternions over $\mathbf{R}$. Let $V = \mathbf{H}e$ be a 1-dimensional left vector space over $\mathbf{H}$. In particular, V is a 2-dimensional vector space over $\mathbf{C}$. The semi-linear map $A : V \to V$ is given by $A(he) = jhe$. Consider two sequences $\{A_n\}_{n\geqslant 1}$ and $\{B_n\}_{n\geqslant 1}$ of $\mathbf{H}$-linear maps from V to itself, with bounded norms. Define the $\mathbf{C}$-linear maps $T_{{\rm upper},n}, T_{{\rm lower},n}$ by the formulas $A_n = T_{{\rm upper},n} + T_{{\rm lower},n}$ and $B_n = iT_{{\rm upper},n} - iT_{{\rm lower},n}$. Put $T_{\rm upper} = 1 + \sum_{n\geqslant 1} T_{{\rm upper},n}u^n$ and $T_{\rm lower} = 1 + \sum_{n\geqslant 1} T_{{\rm lower},n}u^{-n}$. Then A is an isomorphism

$$({}^{\sigma}V, {}^{\sigma}T_{\rm lower}(u), {}^{\sigma}T_{\rm upper}(u^{-1})) \longrightarrow (V, T_{\rm upper}(u), T_{\rm lower}(u^{-1})).$$

Suppose that $W \subset V$ is a real structure such that all A_n, B_n are real. We may suppose that W contains e. Then W contains $\mathbf{R}[S]e$ for $S = A_n$ and $S = B_n$. We note that $\mathbf{R}[S]$ is a commutative subfield of $\mathbf{H}$. We may have chosen the A_n, B_n such that the fields $\mathbf{R}[S]$ with $S = A_n$ or $S = B_n$ are distinct maximal commutative subfields of $\mathbf{H}$. We conclude that W does not exist.

From the construction of this example for theorem 4.1 one can guess the explicit form of the difference module. Let $F := \mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}(\{x^{-1}\})$ denote the skew difference field with ϕ-action defined by $\phi(x) = x + 1$. The left vector space Fe is made into a skew difference module by $\Phi e = de$ for a suitable $d \in F$. One takes a d of the form $d = 1 + d_2 x^{-2} + d_3 x^{-3} + \cdots$. Then $M := Fe$, viewed as a difference module over $\mathbf{C}(\{x^{-1}\})$ of dimension 2, is regular and moreover ${}^{\sigma}M \cong M$. We follow the method explained in Remarks 3.2 part (1). In order to show that M does not descend to $\mathbf{R}(\{x^{-1}\})$, one has to consider $N := Fe$ as a difference module over $\mathbf{R}(\{x^{-1}\})$ of dimension 4. The essential step is to show that the $\mathbf{R}$-algebra of endomorphism of the difference module N is equal to $\mathbf{H}$. This amounts to showing that the only solutions $a \in F$ of the equation $\phi(a)d = da$ are the $a \in \mathbf{R}$. We note that there are more solutions in $\mathbf{H} \otimes_{\mathbf{R}} \mathbf{R}((x^{-1}))$. For a specific example, say $d = 1 + ix^{-2} + jx^{-3}$, one has to verify that the only *convergent* solutions a of $\phi(a)d = da$ are the $a \in \mathbf{R}$. It seems rather difficult to prove this by direct computation.

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