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CURVATURE OF PENCILS OF FOLIATIONS

by

Alcides Lins Neto

Dedicated to J.-P. Ramis in his 60^{th} birthday

Abstract. — Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. In this case, we can define the *pencil* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ *of foliations generated by* $\mathcal{F}$ *and* $\mathcal{G}$. We can associate to a pencil $\mathcal{P}$ a meromorphic 2-form $\Theta = \Theta(\mathcal{P})$, the form of curvature of the pencil, which is in fact the Chern curvature (*cf.* [**Ch**]). When $\Theta(\mathcal{P}) \equiv 0$ we will say that the pencil is *flat*. In this paper we give some sufficient conditions for a pencil to be flat. (Theorem 2). We will see also how the flatness reflects in the pseudo-group of holonomy of the foliations of $\mathcal{P}$. In particular, we will study the set $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H} \text{ has a first integral}\}$ in some cases (Theorem 1).

***Résumé* (Courbure de pinceaux de feuilletages).** — Nous nous intéressons au *pinceau de feuilletages* $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ engendré par deux feuilletages $\mathcal{F}$ et $\mathcal{G}$ holomorphes singuliers distincts sur une surface complexe compacte M et appartenant à la même classe, *i.e.*, $N_{\mathcal{F}} = N_{\mathcal{G}}$. La forme de courbure du pinceau $\mathcal{P}$ est une 2-forme $\Theta = \Theta(\mathcal{P})$ qui coïncide avec la courbure de Chern (*cf.* [**Ch**]) ; lorsque $\Theta(\mathcal{P}) \equiv 0$ on dit que le pinceau est *plat*. Dans cet article, nous donnons des conditions suffisantes de platitude d'un pinceau (Théorème 2). Nous regardons comment se traduit la platitude dans le pseudo-groupe d'holonomie des feuilletages de $\mathcal{P}$ et, en particulier, nous étudions dans certains cas l'ensemble $\{\mathcal{H} \in \mathcal{P} \mid \mathcal{H} \text{ admet une intégrale première}\}$ (Théorème 1).

1. Introduction

Let $\mathcal{F}$ and $\mathcal{G}$ be two distinct singular holomorphic foliations on a compact complex surface M, with isolated singularities, in the same class, that is $N_{\mathcal{F}} = N_{\mathcal{G}}$. This means that there exists a Leray covering $(U_\alpha)_{\alpha \in A}$ of M by open sets, and collections $(\omega_\alpha)_{\alpha \in A}$, $(\eta_\alpha)_{\alpha \in A}$ and $(g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, such that

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(I) ω_α and η_α are holomorphic 1-forms on U_α which represent the foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. This means that $\mathcal{F}|_{U_\alpha}$ and $\mathcal{G}|_{U_\alpha}$ are defined by the differential equations $\omega_\alpha = 0$ and $\eta_\alpha = 0$, respectively. Since the singularities of $\mathcal{F}$ and $\mathcal{G}$ are isolated, we have $\mathrm{cod}_{\mathbb{C}}(\omega_\alpha = 0) \geqslant 2$ and $\mathrm{cod}_{\mathbb{C}}(\eta_\alpha = 0) \geqslant 2$ for every $\alpha \in A$.

(II) If $U_{\alpha\beta} \neq \varnothing$ then $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, $\omega_\alpha = g_{\alpha\beta} \cdot \omega_\beta$ and $\eta_\alpha = g_{\alpha\beta} \cdot \eta_\beta$ on $U_{\alpha\beta}$.

The class of the multiplicative cocycle $(g_{\alpha\beta})_{U_{\alpha\beta}\neq\varnothing}$ in $\mathrm{Pic}(M)$ defines $N_{\mathcal{F}}$ and $N_{\mathcal{G}}$, so that $N_{\mathcal{F}} = N_{\mathcal{G}}$. The *pencil generated by $\mathcal{F}$ and $\mathcal{G}$* is the family $\mathcal{P} = (\mathcal{F}_T)_{T\in\overline{\mathbb{C}}}$, where

(III) $\mathcal{F}_\infty = \mathcal{G}$ and if $T \in \mathbb{C}$, then $\mathcal{F}_T$ is represented on U_α by the form $\omega_\alpha^T := \omega_\alpha + T \cdot \eta_\alpha$.

The singular set of $\mathcal{F}_T$ is defined by $\mathrm{sing}(\mathcal{F}_T) \cap U_\alpha = \{\omega_\alpha^T = 0\}$. The tangency divisor of $\mathcal{F}$ and $\mathcal{G}$ is defined by $\mathrm{Tang}(\mathcal{F},\mathcal{G}) \cap U_\alpha = \{\omega_\alpha \wedge \eta_\alpha = 0\}$. Note that $\mathrm{sing}(\mathcal{F}_T)$ and $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ are analytic subsets of M and that $\mathrm{sing}(\mathcal{F}_T) \subset |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ for all $T \in \overline{\mathbb{C}}$. Since $\mathcal{F} \neq \mathcal{G}$, $|\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ is a proper analytic subset of pure dimension one. Let $W = M \setminus |\mathrm{Tang}(\mathcal{F},\mathcal{G})|$ and $W_\alpha = W \cap U_\alpha$. Since $\omega_\alpha \wedge \eta_\alpha(p) \neq 0$ for all $p \in W_\alpha$, there exists an unique holomorphic 1-form θ_α on W_α such that

$$(*) \qquad d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha \quad \text{and} \quad d\eta_\alpha = \theta_\alpha \wedge \eta_\alpha$$

for all $\alpha \in A$. It follows from $(*)$, (II) and $\omega_\alpha \wedge \eta_\alpha \not\equiv 0$ that, if $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \varnothing$ then, $\theta_\alpha = \theta_\beta + dg_{\alpha\beta}/g_{\alpha\beta}$ on $W_{\alpha\beta}$. Hence $d\theta_\alpha = d\theta_\beta$ on $W_{\alpha\beta}$ and we can define a holomorphic 2-form Θ on W by

$$(**) \qquad \Theta|_{U_\alpha} := d\theta_\alpha$$

It can be proved that the form Θ can be extended meromorphically to $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ (see §2). This extension will be called the *curvature of the pencil* $\mathcal{P}(\mathcal{F},\mathcal{G})$. We will say that the pencil is *flat* if $\Theta = 0$. Let us see some examples of flat pencils.

Example 1. — Let ω and η be two meromorphic closed 1-forms on some compact complex surface M, such that $\omega \wedge \eta \not\equiv 0$ and the divisors of poles and zeroes of ω and η coincide. Let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by ω and η, respectively. It is known that $N_{\mathcal{F}} = N_{\mathcal{G}}$ in this case (*cf.* [**Br**]). Moreover, the pencil generated by $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{P}(\mathcal{F},\mathcal{G})$, is defined by the pencil of forms $\omega_T = \eta + T \cdot \omega$. Therefore, it is flat. We will call a pencil like this *a pencil of closed forms*.

A particular case is given by some families of logarithmic forms in $\mathbb{C}P(2)$. Let $f_1, \ldots, f_k$, $k \geqslant 3$, be irreducible homogeneous polynomials of three variables such that $df_i \wedge df_j \neq 0$ if $i \neq j$. Given $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$, such that $\sum_{j=1}^k \lambda_j \cdot dg(f_j) = 0$, set $\omega_\lambda = \sum_{j=1}^k \lambda_j \cdot df_j/f_j$. The closed form ω_λ can be considered as meromorphic form on $\mathbb{C}P(2)$, so that the family $(\omega_\lambda)_\lambda$ generates a family of foliations $(\mathcal{F}_\lambda)_\lambda$ on $\mathbb{C}P(2)$. It can be checked that any pencil contained in this family is flat.

Another particular case, is the following: let M be the complex two torus $\mathbb{C}^2/\Gamma$, where $\Gamma = \mathbb{Z} \cdot v_1 \oplus \mathbb{Z} \cdot v_2 \oplus \mathbb{Z} \cdot v_3 \oplus \mathbb{Z} \cdot v_4$ is some lattice in $\mathbb{C}^2$, and $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2/\Gamma$ be the canonical projection. Consider an affine coordinate system (z, w) on $\mathbb{C}^2$ and let $\mathcal{F}$ and $\mathcal{G}$ be the foliations generated by the closed forms ω and η such that $\pi^*(\omega) = dz$ and $\pi^*(\eta) = dw$, respectively.

Example 2. — The pull-back of a flat pencil is a flat pencil. More precisely, let M and N be complex surfaces and $f\colon M - \rightarrow N$ be a meromorphic map. If $\mathcal{P} := \mathcal{P}(\mathcal{F}, \mathcal{G})$ is a pencil of foliations on N, then we can define the pencil $f^*(\mathcal{P}) = \mathcal{P}(f^*(\mathcal{F}), f^*(\mathcal{G}))$ on M. It is not difficult to prove that, if $\mathcal{P}$ is flat then $f^*(\mathcal{P})$ is also flat.

Example 3. — Suppose that the pencil $\mathcal{P}(\mathcal{F}, \mathcal{G})$ is defined by $\omega + T \cdot \eta$, where ω and η are meromorphic 1-forms, and there exists a closed meromorphic 1-form θ such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$. Then the pencil $\mathcal{P}(\mathcal{F}, \mathcal{G})$ is flat. Of course, the pencils of Example 1 are of this kind, because the forms ω and η are closed. However, the reader can find some examples in [**LN**] or [**LN-1**] which are not generated by closed forms. One example of this kind is the pencil $\mathcal{P}_1$ of foliations of degree two on $\mathbb{C}P(2)$ defined in some affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ by the the forms (see § 2.4 of [**LN**]):

$$(1) \qquad \begin{cases} \omega_1 = (4x - 9x^2 + y^2)dy - 6y(1 - 2x)dx \\ \eta_1 = 2y(1 - 2x)dy - 3(x^2 - y^2)dx. \end{cases}$$

A straightforward computation gives $d\omega_1 = \frac{5}{6}\frac{dP}{P} \wedge \omega_1$ and $d\eta_1 = \frac{5}{6}\frac{dP}{P} \wedge \eta_1$, where $P(x, y) = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4$. The other examples of [**LN**] can be obtained from the above one by pulling-back $\mathcal{P}_1$ by a meromorphic map $f\colon \mathbb{C}P(2) - \rightarrow \mathbb{C}P(2)$.

Another example is the pencil $\mathcal{P}_2$ of degree three generated by

$$(2) \qquad \begin{cases} \omega_2 = y(x^2 - y^2)dy - 2x(y^2 - 1)dx \\ \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x + y)(y^2 - 1)dx. \end{cases}$$

In this case, we have $d\omega_2 = \frac{3}{4}\frac{dQ}{Q} \wedge \omega_2$ and $d\eta_2 = \frac{3}{4}\frac{dQ}{Q} \wedge \eta_2$, where $Q(x, y) = (y^2 - 1)(x^2 + y^2 - 2x)(x^2 + y^2 + 2x)$.

We would like to observe that both pencils $\mathcal{P}_1$ and $\mathcal{P}_2$ are exceptional families of foliations in the sense of [**LN-1**]. This means the folowing: Let $\mathcal{F}_T^j$, $T \in \overline{\mathbb{C}}$, be the foliation defined in $\mathbb{C}^2 \subset \mathbb{C}P(2)$ by the form $\omega_j + T \cdot \eta_j$ ($\mathcal{F}_\infty^j$ defined by η_j), where ω_j and η_j are as in (j), $j = 1, 2$, of example 3. Then, for $j = 1, 2$, we have:

(a) The singularities of $\mathcal{F}_T^j$ are of constant analytic type. In other words, there is a finite subset $F_j \subset \overline{\mathbb{C}}$ such that if $T_1, T_2 \in \overline{\mathbb{C}} \setminus F_j$ then every singularity of $\mathcal{F}_{T_1}^j$ is locally analytically equivalent to some singularity of $\mathcal{F}_{T_2}^j$.

(b) If we set

$$E_j = \{T \in \overline{\mathbb{C}} \mid \mathcal{F}_T^j \text{ has a meromorphic first integral}\},$$

then E_j is countable and dense in $\overline{\mathbb{C}}$.

(c) Given $T \in E_j$ denote by $d_j(T)$ the degree of the generic level of the first integral of $\mathcal{F}^j_T$. Then, for any $m \in \mathbb{N}$ the set $\{T \in E_j \mid d_j(T) \leqslant m\}$ is finite. In particular, in both families, there are foliations with first integrals of arbitrarily large degrees.

Concerning the exceptional pencils above, we have the following result:

Theorem 1. — *Let* E_j, $j = 1, 2$, *be as in* (b). *Then*

$$\begin{cases} E_1 = \mathbb{Q} \cdot \langle 1, e^{2\pi i/3} \rangle \cup \{\infty\} \\ E_2 = \mathbb{Q} \cdot \langle 1, i \rangle \cup \{\infty\}. \end{cases}$$

where $\mathbb{Q} \cdot \langle a, b \rangle = \{q_1 \cdot a + q_2 \cdot b \mid q_1, q_2 \in \mathbb{Q}\}$.

In our last result we will give some sufficient condictions for the flatness of a pencil $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$ in terms of the singularities of the foliations in $\mathcal{P}$ and the components of the divisor of tangencies. In order to state it, let us consider the singularities of $\mathcal{F}_T$, $T \in \overline{\mathbb{C}}$. Without lost of generality, we will suppose that $\mathcal{F}$ and $\mathcal{G}$ have isolated singularities. This implies that the set $NI := \{T \in \overline{\mathbb{C}} \mid \mathcal{F}_T$ has non-isolated singularities$\}$ is finite. Set $IS := \overline{\mathbb{C}} \setminus NI$ and for each $T \in IS$, set $n(T) := \#(\mathrm{sing}(\mathcal{F}_T))$. Note that, if $T \in IS$ then $N_{\mathcal{F}_T} = N_{\mathcal{F}}$. It is well known that the number of singularities of $\mathcal{F}_T$, counted with multiplicities, is given by (*cf.* [**Br**]):

$$m(\mathcal{F}) = m(\mathcal{F}_T) = N_{\mathcal{F}}^2 + N_{\mathcal{F}}.K_M + c_2(M)$$

where K_M is the canonical bundle of M. Hence $n(T) \leqslant m(\mathcal{F})$ for all $T \in IS$. Let $n_0 = \max\{n(T) \mid T \in IS\}$ and $GP = \{T \in IS \mid n(T) = n_0\}$. We need a fact.

Lemma 1. — $\overline{\mathbb{C}} \setminus GP$ *is finite. Moreover, there exist holomorphic maps* $p_j \colon GP \to M$, $j = 1, \ldots, n_0$, *such that* $\mathrm{sing}(\mathcal{F}_T) = \{p_1(T), \ldots, p_{n_0}(T)\}$ *for all* $T \in GP$.

The proof of Lemma 1 is left for the reader.

Definition 1. — We say that the singularity p_j is *fixed* if the map $p_j \colon GP \to M$ is constant. Otherwise, we say that p_j is *movable*. For instance, if p is a singularity of the curve $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ then p is a singularity of all foliations of the pencil and it is a fixed singularity of the pencil.

Note that, for any movable singularity p_j of the pencil, the image $p_j(GP)$ is contained in some irreducible component C of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$. In this case we will say that *p_j is contained in C*.

Let $C \subset \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ be an irreducible component. We have two possibilities:

(A) C is invariant for both foliations $\mathcal{F}$ and $\mathcal{G}$. In this case, C is invariant for all foliations $\mathcal{F}_T$ in the pencil and we will say that C is *invariant for the pencil*.

(B) C is not invariant for the pencil. In this case, the set $IN(C) = \{T \in \overline{\mathbb{C}} \mid C$ is invariant for $\mathcal{F}_T\}$ is finite.

Remark 1. — Given an irreducible component C of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$, we have two possibilities: either C contains a movable singularity, or C does not contain movable singularities. In the second case, we will call C a *NI-component*. The reason is the following: let $(U_\alpha)_{\alpha \in A}$ be a covering of M by open sets and $(\omega_\alpha)_{\alpha \in A}$, $(\eta_\alpha)_{\alpha \in A}$ be collections of holomorphic 1-forms such that the foliations in the pencil are defined on U_α by $\omega_\alpha^T := \omega_\alpha + T \cdot \eta_\alpha$, $T \in \overline{\mathbb{C}}$. Given $p \in U_\alpha \cap C \setminus (\mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(\mathcal{G}))$, there exists an unique T_o such that $\omega_\alpha(p) + T_o \cdot \eta_\alpha(p) = 0$, because $\omega_\alpha(p)$ and $\eta_\alpha(p)$ are linearly dependent. However, since C does not contain movable singularities and $p \notin \mathrm{sing}(\mathcal{F}) \cup \mathrm{sing}(\mathcal{G})$, the unique possibility is that $\omega_\alpha(q) + T_o \cdot \eta_\alpha(q) = 0$ for all $q \in C \cap U_\alpha$. Hence, $T_o \in NI$ and the component C is contained in $\mathrm{sing}(\mathcal{F}_{T_o})$. Note that T_o depends only on C. We will use the notation $T_o = T(C)$. This happens for instance in the case of the Logarithmic forms (see Example 1).

The *divided foliation* associated to $T(C)$ is defined as follows: for each $\alpha \in A$, let $(f_\alpha = 0)$ be a reduced equation of $C \cap U_\alpha$. Since $\omega_\alpha^{T(C)}|_{C \cap U_\alpha} \equiv 0$, we can write $\omega_\alpha^{T(C)} = f_\alpha^\ell \cdot \widetilde{\omega}_\alpha$, where $\widetilde{\omega}_\alpha$ has isolated singularities and $\ell \in \mathbb{N}$, does not depend on α. The divided foliation, denoted by $\widetilde{\mathcal{F}}_{T(C)}$, is defined by the collection $(\widetilde{\omega}_\alpha)_{\alpha \in A}$. Note that $N_{\widetilde{\mathcal{F}}_{T(C)}} = N_{\mathcal{F}_{T(C)}} \otimes C^{-\ell}$.

Definition 2. We say that an irreducible component C of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is *nice*, if one of the following condictions hold:

(a) C is invariant for the pencil and contains a movable singularity $p_j(T)$ such that the function $T \in GP \mapsto BB(p_j(T), \mathcal{F}_T)$ is constant, where $BB(p_j(T), \mathcal{F}_T)$ denotes the Baum-Bott index of the singularity (*cf.* [**Br**]).

(b) C is an NI-component, invariant for the pencil.

(c) C is non-invariant for the pencil and C contains a movable singularity, say $p_j(T)$, such that $BB(p_j(T), \mathcal{F}_T) = 0$ for all $T \in GP$.

(d) C is an NI-component, non-invariant for the pencil. In this case, we ask that C is invariant for the divided foliation associated to $T(C)$.

The last result, characterizes when the pencil is flat, if we assume that the components of the divisor of tangencies have multiplicity one.

Theorem 2. — *Let $\mathcal{F}$ and $\mathcal{G}$ by two holomorphic foliations on a compact complex surface, such that $N_{\mathcal{F}} = N_{\mathcal{G}}$, and let Θ be the curvature of the pencil generated by them. Suppose that all components of* $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ *have multiplicity one. Then the following condictions are equivalent:*

(a) *The pencil is flat.*

(b) *All components of* $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ *are nice.*

(c) Θ *is holomorphic.*

Let us state one consequence.

Corollary. — *Let $\mathcal{F}$ and $\mathcal{G}$ by two holomorphic foliations on a compact complex surface M. Suppose that $N_{\mathcal{F}} = N_{\mathcal{G}}$ and* $\mathrm{Tang}(\mathcal{F}, \mathcal{G}) = \varnothing$. *Then the pencil generated by them is flat. Moreover, M is a complex 2-torus and $\mathcal{F}$, $\mathcal{G}$ are linear foliations.*

We observe that this corollary is a consequence of Theorem 2 and the classification of complex compact surfaces (see [**BPV**]). We would like to pose the following problems:

Problem 1. — Given a flat pencil $\mathcal{P} = \mathcal{P}(\mathcal{F}, \mathcal{G})$, describe the set

$$E(\mathcal{P}) = \{\alpha \in \overline{\mathbb{C}} \mid \mathcal{F}_\alpha \text{ has a first integral}\}.$$

Problem 2. — Give necessary and sufficient conditions for a pencil to be flat, like in Theorem 2. Recall that Theorem 2 is true only in the case that all components of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ have multiplicity one.

Problem 3. — Give necessary and sufficient conditions for a flat pencil to be a pencil of closed 1-forms. We observe that the pencils defined by logarithmic forms satisfy the following properties, when all components of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ have multiplicity one:

(a) All invariant components of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ are NI-components.

(b) All non-invariant components of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ are nice.

We note that the above conditions are necessary in the case that all components of $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ have multiplicity one. It seems that they are also sufficient in some cases.

2. Proofs

2.1. Proof of Theorem 1. — We will use the notation $\mathcal{F}^j_T$ (resp. $\mathcal{F}^j_\infty$) to denote the foliation defined by $\omega_j + T \cdot \eta_j$, $T \in \mathbb{C}$ (resp. η_j), where ω_j and η_j are as in (j) of example 3, $j = 1, 2$. First of all, we observe that, in both cases, it is easy to see that some foliations in the foliations in the pencils have first integrals. Given $\alpha \in E_j$ we will call g^j_α the first integral of $\mathcal{F}^j_\alpha$. For the pencil $\mathcal{P}_1$ we have:

$$\text{(3)} \qquad \begin{cases} g^1_\infty(x, y) = P(x, y)/(2x - 1)^3 \\ g^1_1(x, y) = P(x, y)/(y - x)^3 \\ g^1_{-1}(x, y) = P(x, y)/(y + x)^3 \end{cases}$$

where, $P(x, y) = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4$.

In particular, $1, -1, \infty \in E_1$. On the other hand, for the pencil $\mathcal{P}_2$ we have

$$\text{(4)} \qquad \begin{cases} g^2_0(x, y) = C_1(x, y) \cdot C_{-1}(x, y)/4L_1(y) \cdot L_{-1}(y) \\ g^2_\infty(x, y) = L_{-1}(y) \cdot C_1(x, y)/L_1(y) \cdot C_{-1}(x, y) \\ g^2_{1/2}(x, y) = L_1(y) \cdot C_1(x, y)/L_{-1}(y) \cdot C_{-1}(x, y) \end{cases}$$

where

$$\begin{cases} C_1(x,y) = x^2 + y^2 - 2x \\ C_{-1}(x,y) = x^2 + y^2 + 2x \\ L_1(y) = y - 1 \\ L_{-1}(y) = y + 1 \end{cases}$$

In particular, $0, \infty, 1/2 \in E_2$.

Note that, in all above cases, the generic level curves of g^j_α are elliptic curves. There is a difference between the two cases: for $j = 1$ the level curves, after normalization, are of the form $\mathbb{C}/\langle 1, e^{2\pi i/3}\rangle$, whereas for $j = 2$ they are of the form $\mathbb{C}/\langle 1, i\rangle$. In the case $j = 1$, the proof can be found in § 2.4 of [**LN**]. In the case $j = 2$, the fact that the level curves are elliptic can be proved by using the genus formula. For instance, in the case of g^2_∞, the level curve $L_c := (g^2_\infty = c)$, for generic $c \in \mathbb{C}$, has degree three and no singularities. Hence, $g(L_c) = (3-1)(3-2)/2 = 1$. The proof that the normalization L_c is $\mathbb{C}/\langle 1, i\rangle$ will be sketched next.

Let us give an idea of the proof that the pencil $\mathcal{P}_2$ is exceptional. This proof was done in § 2.2 of [**LN**] for another pencil (of degree four), but the idea is the same. First of all, the divisor of tangency of $\mathcal{F}^2_0$ and $\mathcal{F}^2_\infty$ is

$$Tg := \mathrm{Tang}(\mathcal{F}^2_0, \mathcal{F}^2_\infty) = C_1 + C_{-1} + L_1 + L_{-1} + L_\infty,$$

where L_∞ is the line at infinity of $\mathbb{C}^2 \subset \mathbb{C}P(2)$. The singular set of Tg, which are the fixed singularities of the pencil, is (in homogeneous coordinates)

(I) Fix:=$\{O := (0:0:1), A := (-1:1:1), B := (1:1:1), C := (1:-1:1), D := (-1:-1:1), E := (1:i:0), F := (1:-i:0), G := (1:0:0)\}$. For $T \notin \{1, -1, i, -i, \infty\}$ the points E, F, G are radial singularities for the foliation $\mathcal{F}^2_T$ (of type $1:1$), whereas the points A, B, C, D and O are singularities of type $2:1$. We say that a singularity is of type $p:q$ if the foliation has a local first integral of the form u^p/v^q, in some local coordinate system (u, v).

On the other hand, each component of Tg contains exactly one movable singularity of $\mathcal{F}^2_\alpha$, $\alpha \in \overline{\mathbb{C}}$:

(II) The points $P_{-1}(\alpha) := (\alpha, -1) \in L_{-1}$, $P_1(\alpha) := (-\alpha, 1) \in L_1$, $Q_{-1}(\alpha) := (-2/(1+\alpha^2), 2\alpha/(1+\alpha^2)) \in C_{-1}$ and $Q_1(\alpha) := (2/(1+\alpha^2), -2\alpha/(1+\alpha^2)) \in C_1$. These singularities are of the type $1:-4$ (with local first integral of the type $u \cdot v^4$).

(III) The point $P_\infty(\alpha) := [\alpha : 1 : 0] \in L_\infty$. This singularity is of the type $1:-2$.

The next step is to reduce the fixed singularities (which are dicritical) by blowing-ups. This can be done for all foliations in the pencil simultaneously by doing one blowing-up at each radial singularity and two at each singularity of the type $2:1$. After this procedure, we find a rational surface M and a bimeromorphism $\pi\colon M \to \mathbb{C}P(2)$. We will use the notation $\mathcal{F}_\alpha = \pi^*(\mathcal{F}^2_\alpha)$, $\alpha \in \overline{\mathbb{C}}$, and $\mathcal{P}$ for the pencil in M so obtained. The pencil $\mathcal{P}$ has ten invariant curves (rational): five of them are the strict transforms of the components of Tg and the other five are the divisors introduced in the first blowing-up at the singularities of the type $2:1$ (A, B, C, D, O). For each $\alpha \in E_2$, the foliation $\mathcal{F}_\alpha$, which corresponds to the first integral g_α^2, has also a first integral $g_\alpha := g_\alpha^2 \circ \pi$. We observe that g_α is holomorphic, because the foliation $\mathcal{F}_\alpha$ has no dicritical singularities. In fact, for any $\alpha \in \overline{\mathbb{C}}$, $\mathcal{F}_\alpha$ has ten singularities, one in each invariant curve, which are the folowing: four of the type $1:-4$, which come from the singularities $P_1(\alpha)$, $P_{-1}(\alpha)$, $Q_1(\alpha)$ and $Q_{-1}(\alpha)$, and six of the type $1:-2$. One of these six singularities come from $P_\infty(\alpha)$ and the other five are contained in the five invariant divisors introduced in the blowing-up procedure. We leave the details of the proof of these facts for the reader.

Let us describe briefly the (singular) fibration g_∞. We will denote by T_c the level curve $g_\infty^{-1}(c) \subset M$. It has three critical levels: T_0, T_1 and T_∞. If we call $U = M \setminus (T_0 \cup T_1 \cup T_\infty)$, then $f := g_\infty|_U : U \to \overline{\mathbb{C}} \setminus \{0, 1, \infty\} := W$ is a (regular) elliptic fibration. The main fact is the following

Lemma 2.1.1. — *If $\alpha \neq \infty$ then $\mathcal{F}_\alpha$ is tranverse to the fibers of f in all points of the set U.*

Proof. — Since the divisors introduced by π are contained in $T_0 \cup T_1 \cup T_\infty$, it is sufficient to prove that the foliations $\mathcal{F}_\infty^2$ and $\mathcal{F}_\alpha^2$ are transverse outside Tg, because $\pi|_U : U \to \pi(U) = \mathbb{C}P(2) \setminus Tg$ is a biholomorphism. On the other hand, we have:

$$\begin{aligned}(\omega_2 + \alpha \cdot \eta_2) \wedge \eta_2 &= 2(x^2 + y^2 - 2x)(x^2 + y^2 + 2x)(y-1)(y+1)dx \wedge dy \\ &= 2\, C_1 \cdot C_{-1} \cdot L_1 \cdot L_{-1}\, dx \wedge dy.\end{aligned}$$

Hence $\mathcal{F}_\alpha^2$ and $\mathcal{F}_\infty^2$ are transverse outside Tg, which implies the lemma. □

Now, we use Ehresmann's theory of foliations tranverse to a fibration (*cf.* [E-R]). According to this theory, if L is a leaf of $\mathcal{F}_\alpha|_U$ then $f|_L : L \to W$ is a covering map. Moreover, if we fix a (regular) fiber T_c and a closed curve $\gamma : [0,1] \to W = \overline{\mathbb{C}} \setminus \{0, 1, \infty\}$ with $\gamma(0) = \gamma(1) = c$, then we can define an automorphism $H_{\gamma,\alpha} : T_c \to T_c$, as follows: given $p \in T_c$, let $L_\alpha(p)$ be the leaf of $\mathcal{F}_\alpha$ through p. Since $f|_{L_\alpha(p)} : L_\alpha(p) \to W$ is a covering map, there exists an unique curve $\widehat{\gamma}$ on $L_\alpha(p)$ such that $f \circ \widehat{\gamma} = \gamma$ and $\widehat{\gamma}(0) = p$. The automorphism is defined by $H_{\gamma,\alpha}(p) = \widehat{\gamma}(1)$. It is called the *global holonomy transformation* associated to γ. We will use the following facts:

(i) For every $\alpha \in \mathbb{C}$ the automorphism $H_{\gamma,\alpha}$ is holomorphic and depends only of the the class of γ in $\Pi_1(W, c)$. This follows from Ehresmann's theory and the fact that the foliations are holomorphic.

(ii) If $\gamma_1, \gamma_2 \in \Pi_1(W, c)$ and $\alpha \in \mathbb{C}$ then $H_{\gamma_1 * \gamma_2, \alpha} = H_{\gamma_1,\alpha} \circ H_{\gamma_2,\alpha}$. In particular, for each $\alpha \in \mathbb{C}$, we can define an action $H_\alpha : \Pi_1(W, c) \to \mathrm{Aut}(T_c)$ by $H_\alpha(\gamma) = H_{\gamma,\alpha}$, called the *holonomy representation*. The image $H_\alpha(\Pi_1(W, c)) := G(\alpha, c)$ is called the *global holonomy group* of $\mathcal{F}_\alpha$.

(iii) For each fixed $\gamma \in \Pi_1(W, c)$, the map $H_\gamma\colon \mathbb{C} \times T_c \to T_c$ defined by $H_\gamma(\alpha, p) = H_{\gamma,\alpha}(p)$ is holomorphic. This follows from the theorem of holomorphic dependency of the solutions with respect to initial conditions and parameters and the fact that $H_{\gamma,\alpha}$ can be found by integrating the equation $\omega_2 + \alpha \cdot \eta_2 = 0$.

(iv) For any $p \in T_c$, the orbit of p by H_α coincides with the intersection of the leaf $L_\alpha(p)$ with the fiber T_c.

(v) If c_1 is another point of W and γ_1 is a curve in W connecting c_1 to c, then, for each $\alpha \in \mathbb{C}$ it can be defined a biholomorphism $F_\alpha\colon T_{c_1} \to T_c$ (by lifting γ_1 to leaves of $\mathcal{F}_\alpha$) such that

$$H_\alpha(\gamma_1^{-1} * \gamma * \gamma_1) = F^{-1} \circ H_\alpha(\gamma) \circ F.$$

In particular, the holonomy representations are conjugated and the fibration f is isotrivial, that is, all regular fibers are biholomorphic.

Now, consider the two closed curves $\gamma_0, \gamma_1 : [0,1] \to W$, where $\gamma_k(0) = \gamma_k(1) = c$, $k = 0, 1$, γ_0 goes around 0 once and γ_1 goes around ∞ once. It is known that γ_0, γ_1 generate $\Pi_1(W, c)$. We will call $f_{1,\alpha} = H_\alpha(\gamma_0)$ and $g_{1,\alpha} = H_\alpha(\gamma_1)$. Fix a holomorphic universal covering $P\colon \mathbb{C} \to T_c$ and let $f_\alpha, g_\alpha \in \mathrm{Aut}(\mathbb{C})$ be coverings of $f_{1,\alpha}$ and $g_{1,\alpha}$, respectively ($P \circ f_\alpha = f_{1,\alpha} \circ P$ and $P \circ g_\alpha = g_{1,\alpha} \circ P$).

Lemma 2.1.2. — *If we choose well the orientation of the curves γ_0 and γ_1, then for any $\alpha \in \mathbb{C}$ we have $f_\alpha(z) = i \cdot z + A(\alpha)$ and $g_\alpha(z) = i \cdot z + B(\alpha)$, where $A, B\colon \mathbb{C} \to \mathbb{C}$ are holomorphic.*

Idea of the proof. — The proof is analogous to the proof of Proposition 4 of § 2.2 of [**LN**], and so we will give only an idea. Let us consider the case of f_α. The critical fiber $T_0 := f^{-1}(0)$ of the fibration f contains the strict transforms, by $\pi\colon M \to \mathbb{C}P(2)$, of the curves C_1 and L_{-1}, which we call C and L, respectively. On the other hand, C_1 and L_{-1} contain the movable singularities $Q_1(\alpha)$ and $P_{-1}(\alpha)$ of $\mathcal{F}_\alpha^2$, which are of the type $1 : -4$. These singularities give origin to movable singularities of the pencil $\mathcal{P}$, $Q(\alpha) := \pi^{-1}(Q_1(\alpha)) \in C$ and $P(\alpha) = \pi^{-1}(P_{-1}(\alpha)) \in L$, which are also of the type $1 : -4$. Since $Q(\alpha)$ is the unique singularity of $\mathcal{F}_\alpha$ on C and C is a rational curve, $Q(\alpha)$ is linearizable for the foliation $\mathcal{F}_\alpha$ (because the holonomy of C is trivial, and so linearizable). The same argument applies to $P(\alpha)$, which is the unique singularity of $\mathcal{F}_\alpha$ on L. On the other hand, the foliation $\mathcal{F}_\alpha$ has an unique local smooth separatrix, say $S(\alpha)$, which is transversal to C. Since the quotient of the eigenvalues is $-1/4$, the holonomy of $S(\alpha)$, in a suitable coordinate system u of a transversal Σ, is linear of the form $u \mapsto e^{-2\pi i/4} \cdot u = -i \cdot u$. If we choose c near 0 then the separatrix $S(\alpha)$ cuts the fiber T_c in an unique point, say $p(\alpha)$. It can be checked that $f|_{S(\alpha)}\colon S(\alpha) \to D := f(S(\alpha))$ is a bijection. If we choose the curve γ_0 as a small circle sorrounding 0 contained in D, then when we go around γ_0 in order to evaluate $f_{1,\alpha}$ we see that $p(\alpha)$ is a fixed point of $f_{1,\alpha}$. Moreover, the section Σ can be choosed to be contained in T_c. This implies that $f_{1,\alpha}$ is locally conjugated to $u \mapsto \pm i \cdot u$. The sign $\pm$ depends on the orientation of γ_0. We choose this orientation in such a way that $f_{1,\alpha}$ is locally conjugated to $u \mapsto i \cdot u$. This implies that $f_{1,\alpha}$ has period four and that $f_\alpha(z) = i \cdot z + A(\alpha)$. Analogously, we can choose the orientation of γ_1 in such a way that $g_\alpha(z) = i \cdot z + B(\alpha)$. The maps $\alpha \in \mathbb{C} \mapsto A(\alpha), B(\alpha)$ are holomorphic by (iii). □

As a consequence of Lemma 2.1.2, we obtain that T_c is biholomorphic to $\mathbb{C}/\langle 1, i\rangle$. This implies that all regular fibers of f are biholomorphic to $\mathbb{C}/\langle 1, i\rangle$, because the fibration is isotrivial. We will fix an universal covering $P \colon \mathbb{C} \to T_c$ such that the associated lattice is $\langle 1, i\rangle$. The crucial result is the following:

Lemma 2.1.3. — *$A(\alpha)$ and $B(\alpha)$ are affine, that is, $A(\alpha) = a_1 \cdot \alpha + a_0$ and $B(\alpha) = b_1 \cdot \alpha + b_0$, where $a_0, a_1, b_0, b_1 \in \mathbb{C}$.*

Proof. — We need another lemma.

Lemma 2.1.4. — *Let $\mathcal{P}(\mathcal{F}, \mathcal{G})$ be a flat pencil on a surface M. Given $p \in M \setminus \mathrm{Tang}(\mathcal{F}, \mathcal{G})$, there exists a local coordinate system $(U, (x, y))$, $p \in U$, $(x, y) \colon U \to \mathbb{C}^2$, such that the foliation $\mathcal{F}_\alpha$ of the pencil, $\alpha \in \overline{\mathbb{C}}$, is defined on U by $dy + \alpha \cdot dx = 0$. Moreover, if $(V, (u, v))$ is another coordinate system such that $U \cap V \neq \varnothing$ is connected and $\mathcal{F}_\alpha|_V$ is defined by $dv + \alpha \cdot du = 0$, $\alpha \in \overline{\mathbb{C}}$, then $du = \lambda \cdot dx$ and $dv = \lambda \cdot dy$ on $U \cap V$, where $\lambda \in \mathbb{C}^*$.*

Proof. — Let $W \subset M \setminus \mathrm{Tang}(\mathcal{F}, \mathcal{G})$ be a small simply connected open neighborhood of p and ω, η be holomorphic 1-forms such that the foliation $\mathcal{F}_\alpha|_W$ is defined by $\omega + \alpha \cdot \eta = 0$. Note that $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_\infty = \mathcal{G}$ are defined on W by $\omega = 0$ and $\eta = 0$, respectively. Since $W \cap \mathrm{Tang}(\mathcal{F}, \mathcal{G}) = \varnothing$, we have $\omega \wedge \eta \neq 0$ on W. Hence, we can write $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, where θ is holomorphic on W. Since the pencil is flat, θ is closed. Therefore, there exists $h \in \mathcal{O}(W)$ such that $\theta = dh$. If we set $f = \exp(h)$ then we get

$$d\omega = \frac{df}{f} \wedge \omega \quad \text{and} \quad d\eta = \frac{df}{f} \wedge \eta \qquad \Longrightarrow \qquad d\Big(\frac{\omega}{f}\Big) = d\Big(\frac{\eta}{f}\Big) = 0.$$

Again, since W is simply connected, there exist $x, y \in \mathcal{O}(W)$ such that $dy = \omega/f$ and $dx = \eta/f$. The foliation $\mathcal{F}_\alpha$ is defined on W by $dy + \alpha \cdot dx = \frac{1}{f}(\omega + \alpha \cdot \eta) = 0$. Note that $dx \wedge dy \neq 0$ on W. It follows that $(x, y) \colon W \to \mathbb{C}^2$ is an immersion. This implies that we can take a smaller neighborhood $U \subset W$ of p such that $(x, y)|_U$ is a biholomorphism from U to an open set of $\mathbb{C}^2$.

Let $(V, (u, v))$ be another coordinate system such that $U \cap V \neq \varnothing$ is connected and $\mathcal{F}_\alpha|_V$ is defined by $dv + \alpha \cdot du = 0$. Note that $\mathcal{F}|_V$ and $\mathcal{G}|_V$ are defined by $dv = 0$ and $du = 0$, respectively. Since $\mathcal{F}_\alpha|_{U\cap V}$ is defined by $dy + \alpha \cdot dx$ and $du + \alpha \cdot dv = 0$, we get

$$(*) \qquad dv + \alpha \cdot du = h(x, y, \alpha)(dy + \alpha \cdot dx)$$

where h is holomorphic. Differenciating both members of $(*)$ with respect to α, we get

$$du = \frac{\partial h}{\partial \alpha}(dy + \alpha \cdot dx) + h \cdot dx \implies \frac{\partial h}{\partial \alpha} \equiv 0,$$

because du is a multiple of dx on $U \cap V$. Hence, $h(x, y, \alpha) = h(x, y)$, does not depend on α. Therefore, $du = h \cdot dx$ and $dv = h \cdot dy$ on $U \cap V$. This implies that $dh \wedge dy = dh \wedge dx = 0$ and $h \in \mathbb{C}^*$, is a constant. This finishes the proof of lemma 2.1.4. □

Let us finish the proof of Lemma 2.1.3. Fix $\alpha_0 \in \mathbb{C}$ and $p \in T_c$. Set $q = f_{1,\alpha_0}(p) \in T_c$. Denote by $L_\alpha(p)$ the leaf of $\mathcal{F}_\alpha$ through p. Let $\gamma_p \colon [0,1] \to L_{\alpha_0}(p)$ be the lifting of γ_0 on the leaf $L_{\alpha_0}(p)$ through the fibration f. Note that $\gamma_p(0) = p$ and $\gamma_p(1) = q$. Let $(U_n)_{1 \leqslant n \leqslant m}$ be a covering of $\gamma_p[0,1]$ by open sets as in Lemma 2.1.4. For each $n = 1, \ldots, m$ there exists a coordinate system (x_n, y_n) on U_n such $\mathcal{F}_\alpha|_{U_n}$ is defined by $dy_n + \alpha \cdot dx_n = 0$. We can choose the enumeration in such a way that there is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ of $[0,1]$ such that $\gamma_p[t_{n-1}, t_n] \subset U_n$, for all $n = 1, \ldots, m$. We can suppose that $U_n \cap U_{n+1}$ is connected for every $n = 1, \ldots, m-1$. It follows from Lema 2.1.4 that there exist constants $\lambda_n \in \mathbb{C}^*$ such that $dx_{n+1} = \lambda_n \cdot dx_n$ and $dy_{n+1} = \lambda_n \cdot dy_n$, $n = 1, \ldots, m-1$. Hence,

(i) $y_{n+1} = \lambda_n \cdot y_n + a_n$, where $a_n \in \mathbb{C}$, $n = 1, \ldots, m-1$.

Fix transversal sections to the foliation $\mathcal{F}_0$, $\Sigma_0, \ldots, \Sigma_m$, such that:

(ii) $\gamma_p(t_n) \in \Sigma_n$, $n = 0, 1, \ldots, m$.

(iii) $\Sigma_n \subset (x_n = ct)$, that is Σ_n is contained in a leaf of $\mathcal{F}_\infty$. Note that $\Sigma_0, \Sigma_m \subset T_c$.

Since $\mathcal{F}_\alpha$ is defined by $dy_n + \alpha \cdot dx_n = 0$ on U_n, the holonomy transformation of $\mathcal{F}_\alpha$, α near α_0, from the section $\Sigma_{n-1} \subset (x_n = c_1)$ to the section $\Sigma_n \subset (x_n = c_2)$, in terms of the parameter y_n is of the form $y_n \mapsto H_n(y_n, \alpha) = y_n - \alpha \cdot b_n$, $b_n = c_2 - c_1$. It follows from (i) that, in the section Σ_n, we have $y_{n+1} = \lambda_n \cdot y_n + a_n$, and so the holonomy transformation H_n, can be written in terms of the parameter y_{n+1} (in the immage) as $y_{n+1}(y_n, \alpha) = \lambda_n . H_n(y_n) + a_n = \lambda_n \cdot y_n - \alpha . \lambda_n \cdot b_n + a_n$. As the reader can check, this implies that the holonomy transformation from the section $\Sigma_0 \subset U_1 \cap T_c$ to the section $\Sigma_m \subset U_M \cap T_c$, which is the composition of the intermediate holonomies, is of the form

$$y_m = H(y_1, \alpha) = \mu \cdot y_1 + \alpha \cdot b + c, \quad \text{where } \mu \in \mathbb{C}^*,\ b, c \in \mathbb{C}.$$

Now, let us relate the parameters $y_1 \in \Sigma_1$ and $y_m \in \Sigma_m$ with the parametrization which comes from the universal covering $P \colon \mathbb{C} \to T_c$. Since $\mathcal{F}_0$ is transverse to T_c, there exists a neighborhood V of T_c such that

(iv) $f|_V \colon V \to D := f(V)$ is a trivial fibration. In particular, $V \simeq D \times T_c$, where $f|_V = \pi_1$, the first projection, and the fibers of the second projection $\pi_2 \colon V \to T_c$ are the leaves of $\mathcal{F}_0|_V$.

Let τ be a non-vanishing 1-form on T_c such that $P^*(\tau) = dz$.

Claim. — *There exist constants $k_1, k_m \in \mathbb{C}^*$ such that $dy_1|_{\Sigma_1} = k_1 \cdot \tau|_{\Sigma_1}$ and $dy_m|\Sigma_m = k_m \cdot \tau|_{\Sigma_m}$.*

Proof. — Set $\omega = \pi_2^*(\tau)$. Note that $\omega(p) \neq 0$, for all $p \in V$, and that the foliation $\mathcal{F}_0|_V$ is defined by $\omega = 0$. We can suppose that $D \subset \mathbb{C}$ and consider $x := f|_V : V \to \mathbb{C}$. This implies that $\mathcal{F}_\infty|_V$ is defined by $dx = 0$. We assert that there exists $g \in \mathcal{O}^*(D)$ such that the foliation $\mathcal{F}_\alpha|_V$ is defined by $\omega + \alpha \cdot g(x) \cdot dx = 0$.

In fact, since ω and dx are linearly independent on V, the foliation $\mathcal{F}_\alpha|_V$ is defined by a 1-form of the type $\omega_\alpha = \omega + g_\alpha \cdot dx$, where $g_\alpha \in \mathcal{O}^*(V)$. Since the fiber $T_x = f^{-1}(x)$ is compact, the function $g_\alpha|_{T_x}$ is constant. Hence, we can write $g_\alpha = g_\alpha(x)$ and $\omega_\alpha = \omega + g_\alpha(x) \cdot dx$. Fix a point $q \in V$ and a coordinate system $(U_q, (x_q, y_q))$ such that $U_q \subset V$ and $\mathcal{F}_\alpha|_{U_q}$ is defined by $dy_q + \alpha \cdot dx_q = 0$. It follows that $dy_q + \alpha \cdot dx_q = h_\alpha(\omega + g_\alpha(x) \cdot dx)$ on U_q, where $h_\alpha \in \mathcal{O}^*(U_q)$. Differentiating twice both members with respect to α and by an argument similar to the proof of Lemma 2.1.4, we get $\partial h_\alpha / \partial \alpha = 0$ and $\partial^2 g_\alpha / \partial \alpha^2 = 0$. This implies that $g_\alpha(x) = \alpha \cdot g(x)$, where $g \in \mathcal{O}^*(V)$.

Since ω and $g(x)dx$ are closed, they are locally exact and we can apply Lemma 2.1.4 to them and the forms dy_1 and dx_1. It follows that $dy_1 = k_1 \cdot \omega|_{U_1}$, $k_1 \in \mathbb{C}^*$. Similarly, $dy_m = k_m \cdot \omega|_{U_m}$, $k_m \in \mathbb{C}^*$. Hence, $dy_j|_{\Sigma_j} = k_j \cdot \tau|_{\Sigma_j}$, $j = 1, m$. □

Now, fix a disk $D_1 \subset \mathbb{C}$ such that $\phi_1 := P|_{D_1} : D_1 \to \Sigma_1$ is a biholomorphism. The claim implies that $\phi_1^*(dy_1) = k_1 \cdot dz$. Therefore, $y_1 \circ \phi_1(z) = k_1 \cdot z + d_1$, $d_1 \in \mathbb{C}$. Similarly, $y_m \circ \phi_m(z) = k_m \cdot z + d_m$, $d_m \in \mathbb{C}$ ($\phi_m = P|_{D_m}$). It follows that the holonomy transformation f_α can be written, in terms of the parameter $z \in \mathbb{C}$, as

$$f_\alpha(z) = k_m^{-1}.H(y_1 \circ \phi_1(z), \alpha) - k_m^{-1} \cdot d_m = i \cdot z + a_1 \cdot \alpha + a_0,$$

where $a_1 = k_m^{-1} \cdot b$ and $a_0 = k_m^{-1}(c - d_m) + \mu \cdot d_1$. Hence, $A(\alpha) = a_1 \cdot \alpha + a_0$, where $a_1, a_0 \in \mathbb{C}$. Similarly, $B(\alpha) = b_1 \cdot \alpha + b_0$. □

Now, the point $z_0 = A(\alpha)/(1 - i)$ is a fixed point of f_α. Let $Q_\alpha(z) = z - z_0$. The global holonomy group $G(\alpha, c)$ (viewed in the universal covering) is conjugated to the group generated by $F_\alpha(z) = Q_\alpha \circ f_\alpha \circ Q_\alpha^{-1}(z) = i \cdot z$ and $G_\alpha(z) = Q_\alpha \circ g_\alpha \circ Q_\alpha^{-1}(z) = i \cdot z + C(\alpha)$, where $C(\alpha) = B(\alpha) - A(\alpha) = a \cdot \alpha + b$, $a = b_1 - a_1$ and $b = b_0 - a_0$. Let us finish the proof of Theorem 1. We need two more results. We will give only an idea of the proof of these results (see Proposition 5 and its corollary in [**LN**]).

Lemma 2.1.5. — *The following assertions are equivalent:*

(a) *The group $G(\alpha, c)$ is finite.*
(b) *$G(\alpha, c)$ has a finite orbit in T_c.*
(c) *There exists $m \in \mathbb{N}$ such that $m \cdot C(\alpha) \in \langle 1, i \rangle$.*
(d) *$\mathcal{F}_\alpha$ has a first integral. In particular, $\alpha \in E_2$.*

Idea of the proof. — The proof of the equivalences (a) $\iff$ (b) $\iff$ (c) is based in the fact that the group generated by F_α and G_α is

$$G = \{z \mapsto c \cdot z + d \cdot C(\alpha) \mid c \in \{1, -1, i, -i\} \text{ and } d \in \langle 1, i \rangle\}.$$

This is done in Proposition 5 of [**LN**] in another case, but the proof is similar for the above case. On the other hand, if $\mathcal{F}_\alpha$ has a first integral, then all leaves of $\mathcal{F}_a$ are algebraic and cut T_c in a finite number of points. Hence, (d) $\Longrightarrow$ (b). Finally, if the group $G(\alpha, c)$ is finite, say $\# G(\alpha, c) = m$, then each leaf of $\mathcal{F}_\alpha$ cut each fiber $T_x = f^{-1}(x)$ in m points. This implies that all leaves $\mathcal{F}_\alpha$ are algebraic. There is a delicate point here, which involves the fact that the leaves of $\mathcal{F}_\alpha$ cut transversely the components of the critical fibers of f which are not invariant for $\mathcal{F}_\alpha$. We have not proved this fact here, but the proof can be done by studing carefully the blowing-up process π. We leave the details for the reader. Now, we can use Darboux's theorem which asserts that if all leaves of a foliation are algebraic then the foliation has a first integral. Therefore, (a) $\Longrightarrow$ (d). □

Lemma 2.1.6. *The map $\alpha \mapsto C(\alpha)$ is non-constant. In particular, $a \neq 0$.*

Idea of the proof. — If $\alpha \mapsto C(\alpha)$ were constant then all holonomy groups $G(\alpha, c)$ would be isomorphic. Therefore, it is sufficient to prove that there are $\alpha_0, \alpha_1 \in E_2$ such that $\#(G(\alpha_0, c)) \neq \#(G(\alpha_1, c))$. In the case of this pencil, we have $0, 1/2 \in E_2$ and the first integrals g_0^2 and $g_{1/2}^2$ given in (4). It can be checked by using Bézout's theorem and the explicit expressions for g_∞^2, g_0^2 and $g_{1/2}^2$ that the generic leaf of $\mathcal{F}_0$ cuts T_c in eight points, whereas the generic leaf of $\mathcal{F}_{1/2}$ cuts T_c in four points. This implies that $\#(G(0, c)) = 8$ and $\#(G(1/2, c)) = 4$. Therefore, $\alpha \mapsto C(\alpha)$ is not constant. □

End of the proof of Theorem 1. — We have seen that $C(\alpha) = a \cdot \alpha + b$, where $a \neq 0$. On the other hand, $0, 1/2 \in E_2$, which implies that there exist $m, n \in \mathbb{N}$ and $m_0, n_0, m_1, n_1 \in \mathbb{N}$ such that

$$m \cdot b = m_0 + n_0 \cdot i \quad \text{and} \quad n\Big(\frac{a}{2} + b\Big) = m_1 + n_1 \cdot i \quad \Longrightarrow \quad a, b \in \mathbb{Q} \cdot \langle 1, i \rangle.$$

Since $\mathbb{Q} \cdot \langle 1, i \rangle$ is a field, we get

$$m(\alpha \cdot a + b) \in \mathbb{Q} \cdot \langle 1, i \rangle,\ m \in \mathbb{N} \quad \iff \quad \alpha \in \mathbb{Q} \cdot \langle 1, i \rangle.$$

This finishes the proof in the case of the pencil $\mathcal{P}_2$.

In the case of the pencil $\mathcal{P}_1$ the proof is similar. In this case, the non-singular fibers of f are biholomorphic to $\mathbb{C}/\langle 1, k \rangle$ ($k = e^{\pi i/3}$) and the holonomy group of $\mathcal{F}_\alpha$ is isomorphic to the group generated by the transformations $F_\alpha(z) = k \cdot z$ and $G_\alpha(z) = k^2 \cdot z + C(\alpha)$ (in the universal covering), where and $C(\alpha) = a \cdot \alpha + b$, $a \neq 0$. This group is

$$G = \{z \mapsto c \cdot z + d \cdot C(\alpha) \mid c \in \{1, k, k^2, k^3, k^4, k^5\} \text{ and } d \in \langle 1, k \rangle\}.$$

By the analogous of Lemma 2.1.5 we have that $\alpha \in E_1$ if, and only if, there exists $m \in \mathbb{N}$ such that $m \cdot C(\alpha) \in \langle 1, k\rangle$. On the other hand, we know that $1, -1 \in E_1$, because we have the explicit first integrals g_1^1 and g_{-1}^1 (see (3)). Therefore, there exist $m, n \in \mathbb{N}$ and $m_0, n_0, m_1, n_1 \in \mathbb{Z}$ such that

$$m(a+b) = m_0 + n_0 \cdot k \text{ and } n(-a+b) = m_1 + n_1 \cdot k \quad \Longrightarrow \quad a, b \in \mathbb{Q} \cdot \langle 1, k\rangle.$$

Since $\mathbb{Q} \cdot \langle 1, k\rangle$ is a field, we get

$$m(a \cdot \alpha + b) \in \langle 1, k\rangle,\ m \in \mathbb{N} \quad \Longleftrightarrow \quad \alpha \in \mathbb{Q} \cdot \langle 1, k\rangle.$$

This finishes the proof of the theorem. □

2.2. Proof of Theorem 2. — Let $\mathcal{P}(\mathcal{F}, \mathcal{G})$ be a pencil of foliations on the compact complex surface M.

Definition 3. — Suppose that $\mathcal{F}$ and $\mathcal{G}$ are defined on an open set $U \subset M$ by $\omega = 0$ and $\eta = 0$, where ω and η are holomorphic 1-forms on U. We will say that (U, ω, η) are *compatible with the pencil* if the foliation $\mathcal{F}_\alpha$ is defined on U by $\omega + \alpha \cdot \eta = 0$, $\alpha \in \mathbb{C}$.

We need a Lemma.

Lemma 2.2.1. — *Let C be an irreducible component of* $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ *of multiplicity $k \geqslant 1$. There exists a finite set $F \subset |C|$ such that if $p \in |C| \setminus F$ then there is a holomorphic coordinate system $(U, (x, y))$ with $p \in U$, $x(p) = y(p) = 0$, $|C| \cap U = (y = 0)$, and holomorphic 1-forms ω and η, representing $\mathcal{F}|_U$ and $\mathcal{G}|_U$ respectively, such that (U, ω, η) is compatible with the pencil and*

(a) *If C is invariant for the pencil then*

$$\begin{cases} \omega = dy \\ \eta = P(x, y)\, dy - y^k\, dx \end{cases}$$

where $P \in \mathcal{O}(U)$. If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, then

$$\theta = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big) dy$$

In particular, $\Theta|_U = y^{-k}\, P_{xx}(x, y)\, dx \wedge dy$ in these coordinates.

(b) *If C is non-invariant for $\mathcal{F}$ (and so for the pencil) then*

$$\begin{cases} \omega = dx \\ \eta = y^k\, dy - Q(x, y)\, dx \end{cases}$$

where $Q \in \mathcal{O}(U)$. If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$, then

$$\theta = \frac{Q_y}{y^k}\, dx$$

In particular $\Theta|_U = -\frac{\partial}{\partial y}(y^{-k}\, Q_y)\, dx \wedge dy$ in these coordinates.

Proof. — Consider a covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of M by open sets and collections $\Omega = (\omega_\alpha)_{\alpha\in A}$, $\Xi = (\eta_\alpha)_{\alpha\in A}$ and $\Lambda = (g_{\alpha\beta})_{U_{\alpha\beta}\neq\varnothing}$, such that $(U_\alpha, \omega_\alpha, \eta_\alpha)$ is compatible with the pencil for every $\alpha \in A$ and, if $U_{\alpha\beta} \neq \varnothing$ then $\omega_\alpha = g_{\alpha\beta}\cdot\omega_\beta$ and $\eta_\alpha = g_{\alpha\beta}\cdot\eta_\beta$ on $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Let $F_1 = |C| \cap \operatorname{sing}(\mathcal{F})$. Given $p \in |C| \setminus F_1$, let $(V, (u, v))$ be a holomorphic coordinate system around p such that $u(p) = v(p) = 0$ and $V \cap |C| = (v = 0)$. We can suppose that $V \subset U_\alpha$, for some $\alpha \in A$. Suppose first that C is invariant for the pencil. Since $p \notin \operatorname{sing}(\mathcal{F})$ and C is invariant for $\mathcal{F}$, by taking a smaller V if necessary, we can suppose that the leaves of $\mathcal{F}|_C$ are the level curves of v, so that $\omega_\alpha|_V = f\cdot dv$, where $f \in \mathcal{O}^*(V)$. Set $\omega = f^{-1}\cdot\omega_\alpha = dv$ and $\eta = f^{-1}\cdot\eta_\alpha$. Note that (V, ω, η) is compatible with the pencil. Let $\eta = A(u, v)dv - B(u, v)du$. Since $\omega\wedge\eta = B(u, v)du\wedge dv$ and the multiplicity of C in $\operatorname{Tang}(\mathcal{F}, \mathcal{G})$ is k, then $B(u, v) = v^k\cdot b(u, v)$, where $b \in \mathcal{O}(V)$ and $b(u, 0) \not\equiv 0$. Let $F_V = \{(u, 0) \in |C|\cap V\,;\ b(u, 0) = 0\}$ and $F = \cup_V F_V \cup F_1$. We leave for the reader the proof that F is finite. If $p \in |C| \setminus F$ then, in the above coordinate system we have $b(0, 0) \neq 0$. Therefore, there exists a neighborhood U of p, with $U \subset\subset V$, and a function $x \in \mathcal{O}(U)$ such that $x(p) = 0$, $\partial x/\partial u = b$ and $\Phi(u, v) = (x(u, v), v)$ is biholomorphism onto $\Phi(U) \subset \mathbb{C}^2$. In the coordinate system $(x, y) := (x, v)$, we have $\omega = dy$ and

$$\eta = A\,dv - v^k\,b\,du = A\,dy - y^k\Big(dx - \frac{\partial x}{\partial v}dy\Big) = \Big(A + y^k\,\frac{\partial x}{\partial v}\Big)dy - y^k\,dx := P\,dy - y^k\,dx$$

Let us compute $\Theta|_U$. If θ is such that $d\omega = \theta\wedge\omega$ and $d\eta = \theta\wedge\eta$ then $\theta = \phi\cdot dy$, because $\omega = dy$ and $d\omega = 0$. Since

$$d\eta = (P_x + k\,y^{k-1})dx\wedge dy = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big)dy\wedge\eta$$

we get that

$$\theta = \Big(\frac{P_x}{y^k} + \frac{k}{y}\Big)dy \quad\Longrightarrow\quad \Theta|_U = d\theta = \frac{P_{xx}}{y^k}dx\wedge dy$$

Now, suppose that C is non-invariant for $\mathcal{F}$. Let

$$F_1 = \{p \in |C|:\ \mathcal{F}\text{ is tangent to } |C| \text{ at } p\}.$$

Clearly F_1 is finite and if $p \in |C| \setminus F_1$ then there exists a holomorphic coordinate system $(V, (u, v))$ around p such that $V \subset U_\alpha$ for some $\alpha \in A$, $u(p) = v(p) = 0$, $|C| \cap V = (v = 0)$ and the leaves of $\mathcal{F}|_V$ are the level curves of u. In this case, $\omega_\alpha|_V = f\cdot du$ where $f \in \mathcal{O}^*(V)$. Set $\omega := du = f^{-1}\omega_\alpha|_V$ and $\eta = f^{-1}\cdot\eta_\alpha|_V$. Note that (V, ω, η) is compatible with the pencil. Let $\eta = A\,dv - B\,du$, where $A, B \in \mathcal{O}(U)$. Since $\omega\wedge\eta = A\,du\wedge dv$ and C is a component of multiplicity k, we can write $A = v^k\cdot a$, where $a(u, 0) \not\equiv 0$. Let $F_V = \{(u, 0) \in |C|\cap V\,;\ a(u, 0) = 0\}$ and set $F = \cup_V F_V \cup F_1$. We leave for the reader the proof that F is finite. If $p \in |C| \setminus F$ then in the above coordinate system we have $a(0, 0) \neq 0$. We assert that there exists a coordinate system $(U, (x, y))$ around p such that $U \subset V$, $u = x$, $y = v\cdot\phi(u, v)$ and

$$(*)\qquad \frac{\partial\,y^{k+1}}{\partial v} = (k+1)\,v^k\,a(u, v)$$

In fact, in a neighborhood of $p = (0,0) \in V$, we can write $(k+1)\, v^k\, a(u,v) = \sum_{j=k}^{\infty} a_j(u)\, v^j$, where $a_k(0) = (k+1)\, a(0,0) \neq 0$. Let

$$\phi(u,v) = \sum_{j=k+1}^{\infty} \frac{1}{j}\, a_{j-1}(u)\, v^j := v^{k+1} \cdot b(u,v).$$

Note that $b(0,0) = a(0,0) \neq 0$ and $\partial\phi/\partial v = (k+1)\, v^k\, a(u,v)$. Let $U_1 \subset V$ be a simply connected open neighborhood of $(0,0)$ such that $b \in \mathcal{O}^*(U_1)$. Let $c \in \mathcal{O}^*(U_1)$ be such that $c^{k+1} = b$ and $y \in \mathcal{O}(U_1)$ be defined by $y(u,v) = v \cdot c(u,v)$. Note that $y^{k+1} = \phi$ and the map $\Phi(u,v) = (u, y(u,v)) = (x,y)$ is a biholomorphism from some open neighborhood $U \subset U_1$ onto an open subset of $\mathbb{C}^2$. Clearly, the coordinate system $(U,(x,y))$ satisfies $(*)$. In these coordinates, we have $\omega = dx$ and

$$\begin{aligned}\eta = v^k\, a(u,v)\, dv - B(u,v)\, du &= \frac{1}{k+1}\frac{\partial y^{k+1}}{\partial v}\, dv - B(u,v)\, du\\ &= y^k\, dy - \Big(\frac{1}{k+1}\frac{\partial y^{k+1}}{\partial u} + B(u,v)\Big) du := y^k\, dy - Q(x,y)\, dx\end{aligned}$$

If θ is such that $d\omega = \theta \wedge \omega$ and $d\eta = \theta \wedge \eta$ then $\theta = \psi \cdot dx$, because $\omega = dx$ and $d\omega = 0$. Since

$$d\eta = Q_y\, dx \wedge dy = \frac{Q_y}{y^k}\, dx \wedge \eta$$

we get that

$$\theta = \frac{Q_y}{y^k}\, dx \quad\Longrightarrow\quad \Theta = d\theta = -\frac{\partial}{\partial y}\Big(\frac{Q_y}{y^k}\Big)\, dx \wedge dy$$

This finishes the proof of Lemma 2.2.1. □

From now on, in this section, we will suppose that all irreducible components of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ have multiplicity one.

2.2.2. (b) $\Longrightarrow$ (c). Denote by D_∞ the divisor of poles of Θ. Let C be a component of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$. Suppose first that C is invariant for the pencil. Since the multiplicity of C in $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ is one, by Lemma 2.2.1, we can choose a coordinate system $(U,(x,y))$ such that $U \cap C = (y=0)$, $p = (0,0) \in U$ and

$$\begin{cases}\omega = dy\\ \eta = P(x,y)\, dy - y\, dx\end{cases} \tag{5}$$

Let $P(x,y) = p_0(x) + y\, p(x,y)$, where $p \in \mathcal{O}(U)$ and $p_0(x) = \sum_{j=0}^{\infty} a_j\, x^j$. Since $\Theta = h \cdot dx \wedge dy$, where $h = y^{-1}\, P_{xx} = y^{-1}\, p_0''(x) + p_{xx}(x,y)$, then $C \not\subset D_\infty$ if, and only if, $p_0(x) = a_0 + a_1\, x$. Note that the foliation $\mathcal{F}_T$ associated to $\eta_T = \eta + T \cdot \omega = (T + P(x,y))dy - y\, dx$, is defined on U by the vector field

$$X_T(x,y) = (T + p_0(x) + y\, p(x,y))\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Hence, the singularities of $\mathcal{F}_T$ on U are given by $y = T + p_0(x) = 0$.

We have two possibilities: either p_0 is a constant ($p_0(x) = a_0$), or p_0 is not a constant. In the first case, we get that $\eta_{-a_0} = y(p\,dy - dx)$. In this case, $-a_0 \in NI$ and there is no movable singularity on C. Moreover $\Theta = p_{xx}\,dx \wedge dy$, which implies that $C \not\subset D_\infty$. In the second case, there is a movable singularity on C: if $x(T)$ is such that $T + p_0(x(T)) = 0$ and $-T$ is a regular value of p_0 then $x(T)$ is a movable singularity of $\mathcal{P}$ and $T \in GP = \{T \in IS \mid n(T) = n_0\}$. Without lost of generality, we can suppose that this singularity satisfies (a) of Definition 2. This singularity is non-degenerate, in the sense that zero is not an eigenvalue of $DX_T(q(T))$, where $q(T) = (x(T), 0)$. In this case, the Baum-Bott index of $\mathcal{F}_T$ at $p(T)$ is given by (*cf.* [**Br**]):

$$
\begin{aligned}
B(T) := BB(q(T), \mathcal{F}_T) &= \frac{\operatorname{tr}^2(DX_T(q(T))}{\det(DX_T(q(T))} = \frac{(p_0'(x(T)) + 1)^2}{p_0'(x(T))} \\
&= p_0'(x(T)) + \frac{1}{p_0'(x(T))} + 2
\end{aligned} \tag{6}
$$

Since C is nice, we have $B'(T) \equiv 0$. As the reader can check, this condiction is equivalent to

$$
p_0''(x(T))\big(1 - \frac{1}{(p_0'(x(T)))^2}\big)x'(T) \equiv 0
$$

Since $q(T)$ is a movable singularity, we have $x'(T) \not\equiv 0$. Therefore, $p_0''(x(T)) \equiv 0$, which implies that $p_0'' \equiv 0$ and $p_0(x) = a_0 + a_1 x$ (note that $p_0'(x(T)) = \pm 1$ implies also that $p_0'' = 0$). Therefore, $C \not\subset D_\infty$.

Suppose now that C is non-invariant for $\mathcal{P}$. Consider a coordinate system $(U, (x, y))$ such that $U \cap C = (y = 0)$, $p = (0, 0) \in U$ and

$$
\begin{cases} \omega = dx \\ \eta = y\,dy - Q(x, y)\,dx \end{cases} \tag{7}
$$

where $Q(x, y) = q_0(x) + q_1(x)\,y + y^2\,q(x, y)$, where q_0, q_1 and q are holomorphic. Since $\Theta = -(y^{-1}\,Q_y)_y\,dx \wedge dy$, then $C \not\subset D_\infty$ if, and only if, $q_1(x) \equiv 0$. Note that the foliation $\mathcal{F}_T$ associated to $\eta_T = \eta + T \cdot \omega = y\,dy + (T - Q(x, y)dx$, is defined on U by the vector field

$$
X_T(x, y) = y\frac{\partial}{\partial x} + (q_0(x) + q_1(x)\,y + y^2\,q(x, y) - T)\frac{\partial}{\partial y}
$$

Hence, the singularities of $\mathcal{F}_T$ on U are given by $y = q_0(x) - T = 0$.

We have two possilities: either q_0 is a constant, or q_0 is not a constant. In the first case, we get $\eta_{q_0} = y[dy - (q_1(x) + y\,q(x, y))dx]$, and so $q_0 \in NI$ and there is no movable singularity on C. Since C is nice, the curve C is invariant for the divided foliation associated to q_0, which is defined by $\widetilde{\omega} = dy - (q_1(x) + y\,q(x, y))dx$ on U. But, $C \cap U = (y = 0)$ and this curve is invariant for $\widetilde{\omega} = 0$ if, and only if, $q_1 \equiv 0$. Therefore, $C \not\subset D_\infty$. In the second case, there is a movable singularity: $p(T) = (x(T), 0) \in U \cap C$, where $x(T)$ is such that $q_0(x(T)) - T = 0$. Set $q_0(0) = T_0$. If T is a regular value of q_0 near T_0, then $T \in GP$. Without lost of generality, we can suppose that this singularity satisfies (c) of Definition 2. This singularity is non-degenerate, and so:

$$B(T) := BB(p(T), \mathcal{F}_T) = \frac{\operatorname{tr}^2(DX_T(p(T))}{\det(DX_T(p(T))} = \frac{q_1^2(x(T))}{q_0'(x(T))} \tag{8}$$

Since C is nice, we get $B \equiv 0$, and so $q_1 \equiv 0$, which implies that $C \not\subset D_\infty$.

2.2.3. (a) $\Longrightarrow$ (b). — Suppose first that C is invariant for $\mathcal{P}(\mathcal{F}, \mathcal{G})$. Let $(U, (x, y))$ be a coordinate system like in (5), around a point $p = (0, 0) \in U \cap C$. Since $\Theta \equiv 0$, by Lemma 2.2.1, we have $P_{xx} = 0$. This implies that $P(x, y) = p_0(y) + p_1(y)x$, where p_0, p_1 are holomorphic. Hence, the singularities of $\mathcal{F}_T$ on $C \cap U$ are the solutions of $y = T + p_0(0) + p_1(0)x = 0$. We have two possibilities: either $p_1(0) \neq 0$, or $p_1(0) = 0$. If $p_1(0) = 0$, then $T = -p_0(0) \in NI$ and we are in the situation of (b) of Definition 2. Therefore, C is nice. If $p_1(0) \neq 0$, then C contains an unique movable singularity: $q(T) = (x(T), 0)$, where $x(T) = -(T + p_0(0))/p_1(0)$ (clearly $q(T) \in U$ for $|T + p_0(0)|$ small enough). This singularity is non-degenerate, and so by (6) we get:

$$BB(p(T), \mathcal{F}_T) = \frac{\operatorname{tr}^2(DX_T(p(T))}{\det(DX_T(p(T))} = \frac{(p_1(0)+1)^2}{p_1(0)}$$

Hence, C is nice in this case.

Suppose now that C is non-invariant for the pencil. Consider a coordinate system $(U, (x, y))$ around $p = (0, 0) \in U$ as in (7). Since $\Theta \equiv 0$, Lemma 2.2.1 implies that

$$\frac{\partial}{\partial y}(y^{-1} Q_y) = 0 \quad \Longrightarrow \quad Q(x, y) = q_0(x) + q_2(x)y^2$$

This implies that C is nice, as the reader can check by using (8).

2.2.4. (c) $\Longrightarrow$ (a). — Suppose that Θ is holomorphic. The idea is to use the well-known fact that

$$\Theta \equiv 0 \quad \Longleftrightarrow \quad \int_M \Theta \wedge \overline{\Theta} = 0 \quad \Longleftrightarrow \quad [\Theta] = 0 \text{ in } H^2_{DR}(M)$$

The proof will be based in the following:

Claim 1. — *$\int_M \Theta \wedge \overline{\Theta} = -2\pi i \int_M c_1(N_\mathcal{F}) \wedge \overline{\Theta}$, where $c_1(N_\mathcal{F})$ is any representative of the first Chern class of $N_\mathcal{F}$ in $H^2_{DR}(M)$.*

Proof. — Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a covering of M by open sets, $\Omega = (\omega_\alpha)_{\alpha \in A}$, $\Xi = (\eta_\alpha)_{\alpha \in A}$ and $\Lambda = (g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$ be as in (I), (II) and (III) of §1. Let $(\theta_\alpha)_{\alpha \in A}$ be a collection of 1-forms, where θ_α is meromorphic on U_α, $d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$ and $d\eta_\alpha = \theta_\alpha \wedge \eta_\alpha$. Recall that, if $U_{\alpha\beta} \neq \varnothing$ then $\theta_\alpha - \theta_\beta = dg_{\alpha\beta}/g_{\alpha\beta}$. On the other hand, by taking a C^∞ resolution of the additive cocycle $(dg_{\alpha\beta}/g_{\alpha\beta})_{U_{\alpha\beta} \neq \varnothing}$, we can write $dg_{\alpha\beta}/g_{\alpha\beta} = \mu_\alpha - \mu_\beta$, where the closed 2-form Λ defined by $\Lambda|_{U_\alpha} = \frac{i}{2\pi} d\mu_\alpha$, represents $c_1(N_\mathcal{F})$ on $H^2_{DR}(M)$ (*cf.* [**G-H**], p. 141). If $U_{\alpha\beta} \neq \varnothing$, then $dg_{\alpha\beta}/g_{\alpha\beta} = \theta_\alpha - \theta_\beta = \mu_\alpha - \mu_\beta$. Hence, we can define a C^∞ 1-form φ on $W := M \setminus \mathrm{Tang}(\mathcal{F},\mathcal{G})$ by $\varphi|_{U_\alpha\cap W} = \frac{i}{2\pi}(\theta_\alpha - \mu_\alpha)$. Note that $d\varphi = \frac{i}{2\pi}\Theta - \Lambda$. This implies that $d\varphi$ extends to a C^∞ form in M. Moreover,

$$\int_M (\tfrac{i}{2\pi}\Theta - \Lambda)\wedge\overline{\Theta} = \int_M d\varphi\wedge\overline{\Theta}. \tag{9}$$

The idea is to prove that $\int_M d\varphi\wedge\overline{\Theta} = 0$. Let us study the behavior of φ near an irreducible component of $\mathrm{Tang}(\mathcal{F},\mathcal{G})$. Set $\mathrm{Tang}(\mathcal{F},\mathcal{G}) = \sum_{j=1}^k C_j + \sum_{i=1}^\ell D_i$, where C_j is invariant for the pencil, $j = 1,\dots,k$, and D_i is non-invariant, $i = 1,\dots,\ell$. Consider first the non-invariant case. Let $p \in |D_i| \cap U_\alpha$ be a point such that we have a normal form like in (b) of Lemma 2.2.1, in a coordinate system $(U,(x,y))$, where $U \subset U_\alpha$. As we have seen, $\omega_\alpha|_U = f\,\omega$ and $\eta_\alpha|_U = f\,\eta$, where $f \in \mathcal{O}^*(U)$, $\omega = dx$ and $\eta = ydy - Q(x,y)dx$, $Q(x,y) = q_0(x) + q_1(x)\,y + y^2\,q(x,y)$. This implies that $\theta_\alpha = \theta + df/f$, where $\theta = \frac{Q_y}{y}dx$. Note that Θ is holomorphic in U if, and only if, Q_y/y is holomorphic, which implies that θ_a is holomorphic in U and φ is C^∞ in U. This implies that φ is C^∞ on $M \setminus |C|$, where $C = \sum_j C_j$ and $|C| = \cup_j |C_j|$.

Consider now a point $p \in |C|$. Let $(f_1\cdots f_k = 0)$ be a (reduced) equation of C in a small Stein neighborhood U of p. We assert that there exist $\lambda_1,\dots,\lambda_k \in \mathbb{C}$ and a C^∞ 1-form ν such that

$$\varphi|_U = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} + \nu. \tag{10}$$

In fact, suppose first that p belongs to an invariant component C_j and we have a normal form like in (a) of Lemma 2.2.1 on a coordinate system $(U,(x,y))$, where $U \subset U_\alpha$, for some $\alpha \in A$. As before, we have $\omega_\alpha|_U = f\cdot\omega = f\cdot dy$ and $\eta_\alpha|_U = f\cdot\eta$, where $f \in \mathcal{O}^*(U)$ and $\eta = P(x,y)\,dy - y\,dx$. From the first part of the proof and the fact that Θ is holomorphic, we get

$$\theta_\alpha = \theta + \frac{df}{f} = \frac{1+P_x}{y}dy + \frac{df}{f} = \frac{1}{2\pi i}\lambda_U\frac{dy}{y} + \varsigma_U, \tag{*}$$

where $\lambda_U \in \mathbb{C}$ and ς_U is a holomorphic 1-form. This implies that $\varphi|_U = \lambda_U\dfrac{dy}{y} + \nu_U$, where ν_U is a C^∞ 1-form.

Let us prove that λ_U depends only of C_j. It follows from (*) that

$$\frac{1}{2\pi i}\lambda_U = \mathrm{Res}(\theta_\alpha, C_j) = \frac{1}{2\pi i}\int_\gamma \theta_\alpha,$$

where γ is a small cicle surrounding C_j. If $\beta \in A$ is such that $U_\alpha \cap U_\beta \cap C_j \neq \varnothing$ then $\theta_\alpha - \theta_\beta = dg_{\alpha\beta}/g_{\alpha\beta}$. Hence,

$$\frac{1}{2\pi i}\int_\gamma \theta_\alpha = \frac{1}{2\pi i}\int_\gamma \theta_\beta,$$

if $\gamma \subset U_\alpha \cap U_\beta$. This proves that λ_U depends only of C_j. Set $\lambda_U = \lambda_j$.

Note that λ_j satisfies the following property

(A) Let $(f_{j\alpha}=0)$ be a reduced equation of $C_j\cap U_\alpha$. Then $\theta_\alpha-\frac{1}{2\pi i}\lambda_j df_{j\alpha}/f_{j\alpha}$ has no poles along $C_j\cap U_\alpha$.

We leave the proof of (A) for the reader. Let $p\in|C|\cap U_\alpha$ and $(f_{j\alpha}=0)$ be a reduced equation of C_j on U_α. It follows from (A) that $\theta_\alpha-\sum_{j=1}^k\frac{1}{2\pi i}\lambda_j\, df_{j\alpha}/f_{j\alpha}$ is holomorphic on U_α. Hence, $\nu=\varphi|_{U_\alpha}-\sum_{j=1}^k\lambda_j\, df_{j\alpha}/f_{j\alpha}$ is C^∞. This proves (10).

Let us prove that $\int_M d\varphi\wedge\overline{\Theta}=0$. We will consider two cases:

First case. — All the singularities of C are nodes. In this case, we can find a finite open covering $\mathcal{V}=(V_\alpha)_{\alpha\in A}$ of M with the following properties:

(i) For every $\alpha\in A$, V_α is a domain of a coordinate system $\psi_\alpha=(x_\alpha,y_\alpha)\colon U_\alpha\to\mathbb{C}^2$ such that $\psi_\alpha(U_\alpha)=D_2\times D_2$, where $D_r=\{z\in\mathbb{C}|\,|z|<r\}$.

(ii) If $U_\alpha=\psi_\alpha^{-1}(D_1\times D_1)$ then $\cup_\alpha U_\alpha=M$.

(iii) If $|C|\cap V_\alpha\neq\varnothing$ is smooth then $(y_\alpha=0)$ is an equation of $C\cap V_\alpha$. In particular, $\varphi|_{V_\alpha}=\lambda\,\frac{dy_\alpha}{y_\alpha}+\nu$, where $\lambda\in\mathbb{C}$ and ν is C^∞.

(iv) If $|C|\cap V_\alpha$ has a singularity in V_α then $(x_\alpha\cdot y_\alpha=0)$ is an equation of $C\cap V_\alpha$. In particular, $\varphi|_{V_\alpha}=\lambda_a\,\frac{dx_\alpha}{x_\alpha}+\lambda_b\,\frac{dy_\alpha}{y_\alpha}+\nu$, where $\lambda_a,\lambda_b\in\mathbb{C}$ and ν is C^∞.

In general, let $(f_\alpha=0)$ be an equation of $C\cap V_\alpha$. Let $(\varphi_\alpha)_{\alpha\in A}$ be a C^∞ partition of the unity such that $\operatorname{supp}(\varphi_\alpha)\subset V_\alpha$ for all $\alpha\in A$ and set $f=\exp(\sum_\alpha\varphi_\alpha\cdot\ln|f_\alpha|)$. If $\beta\in A$ is fixed, then

$$\begin{aligned}f|_{V_\beta}&=\exp\Big(\sum_{\alpha,V_{\alpha,\beta}\neq\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)\cdot\exp\Big(\sum_{\alpha,V_{\alpha,\beta}=\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)\\&=\exp\Big(\sum_{\alpha,V_{\alpha,\beta}\neq\varnothing}\varphi_\alpha\cdot\ln|g_{\alpha\beta}f_\beta|\Big)\cdot\exp\Big(\sum_{\alpha,V_{\alpha,\beta}=\varnothing}\varphi_\alpha\cdot\ln|f_\alpha|\Big)=|f_\beta|\cdot g_\beta\end{aligned}$$

where $g_\beta\colon V_\beta\to(0,+\infty)$ is C^∞.

(v) $f|_{V_\alpha}=|f_\alpha|\cdot g_\alpha$, where $g_\alpha\in C^\infty(V_\alpha)$. In particular, f can be extended continually to M as $f|_{|C|}\equiv0$.

(vi) $f>0$ on $M\smallsetminus|C|$ and $f^{-1}(0)=|C|$.

Set $M_\varepsilon=\{p\in M|\,f(p)\geqslant\varepsilon\}$ and $C_\varepsilon=\{p\in M|\,f(p)\leqslant\varepsilon\}$. For all $\varepsilon>0$ we have

$$\begin{aligned}\int_M d\varphi\wedge\overline{\Theta}&=\int_{M_\varepsilon}d\varphi\wedge\overline{\Theta}+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}=\int_{M_\varepsilon}d(\varphi\wedge\overline{\Theta})+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}\\&=\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}+\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}.\end{aligned}$$

Since $\lim_{\varepsilon\to0}\big(\int_{C_\varepsilon}d\varphi\wedge\overline{\Theta}\big)$, we get

(vii) $\int_M d\varphi\wedge\overline{\Theta}=\lim_{\varepsilon\to0}\big(\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}\big)$.

It is enough to prove that $\lim_{\varepsilon\to0}\big(\int_{\partial M_\varepsilon}\varphi\wedge\overline{\Theta}\big)=0$. In order to prove this fact, consider a covering $\{V_1:=V_{\alpha_1},\dots,V_n:=V_{\alpha_n}\}$ of $|C|$ by sets of $\mathcal{V}$, such that $\{U_j:=U_{\alpha_j}\mid1\leqslant j\leqslant n\}$ is still a covering of $|C|$. If $U=\cup_{j=1}^nU_j$ then there exists ε_0 such that, if $\varepsilon < \varepsilon_0$ then $\partial M_\varepsilon \subset V$. Hence, if $S_j(\varepsilon) = \partial M_\varepsilon \cap \overline{U}_\varepsilon$ and $I_j(\varepsilon) = \int_{S_j(\varepsilon)} |\varphi \wedge \overline{\Theta}|$, we get that

$$\Big|\int_{\partial M_\varepsilon} \varphi \wedge \overline{\Theta}\Big| \leqslant \sum_{j=1}^{n} I_j(\varepsilon), \quad \text{if } \varepsilon < \varepsilon_0$$

It follows that, it is sufficient to prove that $\lim_{\varepsilon\to 0} I_j(\varepsilon) = 0$ for all $j = 1, \ldots, n$. We will prove this fact in the case where V_j is like in (iv) and leave the other case for the reader.

Consider a coordinate system (x, y) on V_j as in (iv), that is $|C| \cap V_j = (x \cdot y = 0)$. As we have seen before, $\Theta|_{V_j} = g(x, y)\, dx \wedge dy$ and $\varphi|_{V_j} = \lambda_a \frac{dx}{x} + \lambda_b \frac{dy}{y} + \nu$, where $g \in \mathcal{O}(V_j)$, $\lambda_a, \lambda_b \in \mathbb{C}$ and ν is C^∞. Therefore, there exists a constant $c > 0$ such that on $\overline{U}_j$ we have

$$|\varphi \wedge \overline{\Theta}| \leqslant c\left(\Big|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}\Big| + \Big|\frac{dy}{y} \wedge d\overline{x} \wedge d\overline{y}\Big| + \Big|\nu \wedge d\overline{x} \wedge d\overline{y}\Big|\right)$$

If we set

$$A_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}\Big|,$$

$$B_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big|\frac{dy}{y} \wedge d\overline{x} \wedge d\overline{y}\Big|$$

$$C_j(\varepsilon) = \int_{S_j(\varepsilon)} \Big|\nu \wedge d\overline{x} \wedge d\overline{y}\Big|,$$

then $I_j(\varepsilon) \leqslant c.(A_j(\varepsilon) + B_j(\varepsilon) + C_j(\varepsilon))$. Hence, it is sufficient to prove that $\lim_{\varepsilon\to 0} A_j(\varepsilon) = \lim_{\varepsilon\to 0} B_j(\varepsilon) = \lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$. We will prove that $\lim_{\varepsilon\to 0} A_j(\varepsilon) = 0$ and leave the proof that $\lim_{\varepsilon\to 0} B_j(\varepsilon) = \lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$ for the reader (note that $\lim_{\varepsilon\to 0} C_j(\varepsilon) = 0$ because ν is C^∞). Given $0 < a < 1$, define

$$J(a, \varepsilon) = \int_{S_j(\varepsilon)\cap(|x|\geqslant a)} \Big|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}\Big| \quad \text{and} \quad K(a, \varepsilon) = \int_{S_j(\varepsilon)\cap(|x|\leqslant a)} \Big|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}\Big|$$

so that $A_j(\varepsilon) = J(a, \varepsilon) + K(a, \varepsilon)$. Since $|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}|$ is C^∞ on $(|x| \geqslant a)$, we get that $\lim_{\varepsilon\to 0} J(a, \varepsilon) = 0$ for all $a > 0$. Therefore, it is sufficient to prove that there exists $0 < a < 1$ such that $\lim_{\varepsilon\to 0} K(a, \varepsilon) = 0$.

Set $x = r e^{i\alpha}$ and $y = s e^{i\beta}$, so that $|\frac{dx}{x} \wedge d\overline{x} \wedge d\overline{y}| = 2|dr \wedge d\alpha \wedge d\overline{y}|$. In the coordinate system (r, α, y) we have $f(r, \alpha, y) = r \cdot s \cdot g(r, \alpha, y)$ (by (iv)), where $g \in C^\infty$ and $g > 0$. Since $\partial r \cdot g / \partial r(0, \alpha, y) = g(0, \alpha, y) > 0$, there exists $0 < a < 1$ such that the map $\psi(r, \alpha, y) = (r \cdot g(r, \alpha, y), \alpha, y) = (R, \alpha, y)$ is diffeomorphism from a neighborhood W of $(r = 0) \cap (|y| \leqslant 1)$ onto $W_1 = (R < \delta) \cap (|y| < 1 + \delta)$, where $W \sup(r \leqslant a) \cap (|y| \leqslant 1)$. Note that $\psi^{-1}(R, \alpha, y) = (R \cdot h(R, \alpha, y), \alpha, y)$, where h is C^∞. In the coordinate system (R, α, y) we have

$$S_j(\varepsilon) \cap W_1 = (R \cdot |y| = R \cdot s = \varepsilon) \cap (s \leqslant 1) := T(\varepsilon) \implies K(a, \varepsilon) = \int_{T(\varepsilon)} 2|d(R \cdot h) \wedge d\alpha \wedge d\overline{y}|$$

if $\varepsilon > 0$ is small. We assert that there exists a constant $c > 0$ such that $2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| \leqslant c\cdot R|ds\wedge d\alpha\wedge d\beta|$ on $T(\varepsilon)$, if ε is small (the restriction to $T(\varepsilon)$). In fact,

$$\begin{aligned}2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| &\leqslant 2R\,|dh\wedge d\alpha\wedge d\overline{y}| + 2|h|\,|dR\wedge d\alpha\wedge d\overline{y}|\\ &\leqslant 2R\,|h_R|\,|dR\wedge d\alpha\wedge d\overline{y}| + 2R\,|h_y|\,|d\alpha\wedge dy\wedge d\overline{y}|\\ &\quad + 2|h|\,|dR\wedge d\alpha\wedge d\overline{y}|\end{aligned}$$

Since $K := \psi((r\leqslant a)\cap(|y|\leqslant 1))$ is compact, $2|h|, 2|h_R|, 2|h_y|, R$ are bounded in K, so that there exists a constants $c_1 > 0$ such that

$$\begin{aligned}2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| &\leqslant c_1\big(R\,|d\alpha\wedge dy\wedge d\overline{y}| + |dR\wedge d\alpha\wedge d\overline{y}|\big)\\ &\leqslant c_1\big(2\,R\,|ds\wedge d\alpha\wedge d\beta| + |dR\wedge d\alpha\wedge d\overline{y}|\big)\end{aligned}$$

on K, because $|d\alpha\wedge dy\wedge d\overline{y}| = 2|ds\wedge d\alpha\wedge d\beta|$. On the other hand, $\overline{y} = s\cdot e^{-i\beta}$ and $R\cdot s = \varepsilon$ on $T(\varepsilon)$. Hence, if $\varepsilon > 0$ is small, we get

$$\begin{aligned}|dR\wedge d\alpha\wedge d\overline{y}| &= |d(R\,d\overline{y})\wedge d\alpha| = |d(-R\,s\,i\,e^{-i\beta}d\beta)\wedge d\alpha + d(R\,e^{-i\beta}ds)\wedge d\alpha|\\ &= |d(-\varepsilon\,i\,e^{-i\beta}d\beta)\wedge d\alpha + d(R\,e^{-i\beta}ds)\wedge d\alpha|\\ &= |d(R\,e^{-i\beta}ds)\wedge d\alpha| \leqslant R\,|ds\wedge d\alpha\wedge d\beta| + |dR\wedge ds\wedge d\alpha|\\ &= R\,|ds\wedge d\alpha\wedge d\beta|\end{aligned}$$

because $dR\wedge ds = 0$ on $T(\varepsilon)$. Therefore, on $T(\varepsilon)$ we have $2|d(R\cdot h)\wedge d\alpha\wedge d\overline{y}| \leqslant c\cdot R|ds\wedge d\alpha\wedge d\beta|$, where $c = 3\,c_1$. From this, we get that

$$K(a,\varepsilon) \leqslant c\int_{T(\varepsilon)} R|ds\wedge d\alpha\wedge d\beta| = c\,\varepsilon\int_{T(\varepsilon)}\Big|\frac{ds}{s}\wedge d\alpha\wedge d\beta\Big|.$$

On the other hand, the region $T(\varepsilon)$ in the real hypersurface $R\cdot s = \varepsilon$, is contained in a region of the form

$$T_1(\varepsilon) := \{(R,s,\alpha,\beta)\mid R\cdot s = \varepsilon,\ \alpha,\beta\in[0,2\pi],\ 1\geqslant s\geqslant \varepsilon/R_0\},$$

where $R_0 = \sup\{R(r,\alpha,y)\mid (r,\alpha,y)\in S_j(\varepsilon)\}$. This implies that

$$K(a,\varepsilon) \leqslant c\,\varepsilon\int_{T_1(\varepsilon)}\Big|\frac{ds}{s}\wedge d\alpha\wedge d\beta\Big| = 4\pi^2\,c\,\varepsilon\cdot|\log(\varepsilon/R_0)| \quad\Longrightarrow\quad \lim_{\varepsilon\to 0}K(a,\varepsilon) = 0$$

This finishes the proof of Claim 1 in the first case.

Second case: general case. — Consider a resolution of the curve C by blowing-ups $\pi\colon \widehat{M}\to M$ and let $C^* = \pi^{-1}(C)$, $\Theta^* = \pi^*(\Theta)$ and $\varphi^* = \pi^*(\varphi)$. Then $\int_M d\varphi\wedge\overline{\Theta} = 0$ if, and only if, $\int_{\widehat{M}} d\varphi^*\wedge\overline{\Theta}^* = 0$. Note that the singularities of C^* are of nodal type. It is sufficient to prove that $|C^*|$ admits an open covering satisfying (i), (ii), (iii) and (iv). Let $p\in\operatorname{sing}(C)$ (which is not a node) and $q\in\pi^{-1}(p)$. Since the singularities of C^* are nodes, we have two possibilities: either q is a smooth point of C^*, or q is in the normal crossing of two local components, say D_1 and D_2 of C^*. Let us consider, for instance, the second case. Let $(W,(x,y))$ be a coordinate system around p, where $C \cap U$ has a reduced equation $(f_1 \cdots f_k = 0)$. As we have seen, we can write $\varphi|_W = \sum_{j=1}^k \lambda_j \frac{df_j}{f_j} + \nu$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and ν is C^∞. Consider a coordinate system $(V, \phi = (u, v))$ around $q = (0, 0)$ such that $\pi(V) \subset W$, $\phi(V) = \{(u, v) \in \mathbb{C}^2 \mid |u|, |v| \leqslant 2\}$, $D_1 \cap V = (u = 0)$ and $D_2 \cap V = (v = 0)$. We have still two possibilities: either $\pi(D_1) = \pi(D_2) = \{p\}$, or $\pi(D_j) = \{p\}$ for just one $j \in \{1, 2\}$. Let us consider, for instance, the first case. In this case, if $\widehat{f_j}$ is the strict transform of f_j, then $F_j := \widehat{f_j}|_V \in \mathcal{O}^*(V)$. On the other hand, $f_j \circ \pi(u, v) = u^{m_j} \cdot v^{n_j} \cdot F_j$. Hence, in the coordinates (u, v) we have, $\pi^*(\varphi) = \lambda_a \frac{du}{u} + \lambda_b \frac{dv}{v} + \nu^*$, where $\lambda_a = \sum_j m_j \cdot \lambda_j$, $\lambda_b = \sum_j n_j \cdot \lambda_j$ and $\nu^* = \pi^*(\nu) + \sum_j \lambda_j dF_j / F_j$. Since $F_j \in \mathcal{O}^*(V)$ for all j, we get that ν^* is C^∞. We leave the proof of the other cases for the reader. This finishes the proof of Claim 1. □

Let us finish the proof of (c) $\Longrightarrow$ (a). Suppose by contradiction that Θ is holomorphic and $\Theta \not\equiv 0$. Let $Z := (\Theta)_0$ be the divisor of zeroes of Θ. Given a divisor D on M we will denote by $[D]$ its class in $\mathrm{Pic}(M)$. Since Θ is a non-vanishing section of $\Omega^2(M)$, we have $K_M = [Z]$. On the other hand, it is known that $\mathrm{Tang}(\mathcal{F}, \mathcal{G}) = K_M + N_{\mathcal{F}} + N_{\mathcal{G}}$ (*cf.* [**Br**]). Since $N_{\mathcal{F}} = N_{\mathcal{G}}$ we get that $2N_{\mathcal{F}} = \mathrm{Tang}(\mathcal{F}, \mathcal{G}) - [Z] = \sum_{j=1}^m n_j [D_j]$, where $n_j \in \mathbb{Z}$ and D_j is an irreducible component of $\mathrm{Tang}(\mathcal{F}, \mathcal{G}) \cup Z$, $1 \leqslant j \leqslant m$. It follows from Claim 1 that

$$\int_M \Theta \wedge \overline{\Theta} = \sum_{j=1}^m -i\pi\, m_j \int_M c_1(D_j) \wedge \overline{\Theta}$$

On the other hand, it is known that (*cf.* [**G-H**])

$$\int_M c_1(D_j) \wedge \overline{\Theta} = \int_{D_j} \overline{\Theta} = 0$$

because $\overline{\Theta}$ is a $(0, 2)$-form. This finishes the proof of Theorem 2. □

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