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# $b$-FUNCTIONS AND INTEGRABLE SOLUTIONS OF HOLONOMIC $\mathcal{D}$-MODULE

*by*

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*À Jean-Pierre Ramis, à l'occasion de son 60$^e$ anniversaire.*

***Abstract.*** — A famous theorem of Harish-Chandra shows that all invariant eigendistributions on a semi-simple Lie group are locally integrable functions. We give here an algebraic version of this theorem in terms of polynomials associated with a holonomic $\mathcal{D}$-module.

***Résumé*** **($b$-fonctions et solutions intégrables des modules holonomes).** — Un célèbre théorème de Harish-Chandra montre que les distributions invariantes propres sur un groupe de Lie semi-simple sont des fonctions localement intégrables. Nous donnons ici une version algébrique de ce théorème en termes de polynômes associés à un $\mathcal{D}$-module holonome.

## Introduction

Let $G_{\mathbb{R}}$ be a real semisimple Lie group and $\mathfrak{g}_{\mathbb{R}}$ be its Lie algebra. An *invariant eigendistribution* $T$ on $G_{\mathbb{R}}$ is a distribution which is invariant under conjugation by elements of $G_{\mathbb{R}}$ and is an eigenvector of every bi-invariant differential operator on $G_{\mathbb{R}}$. The main examples of such distributions are the characters of irreducible representations of $G_{\mathbb{R}}$. A famous theorem of Harish-Chandra sets that all invariant eigendistributions are $L^1_{\mathrm{loc}}$-functions on $G_{\mathbb{R}}$ [**4**]. After transfer to the Lie algebra by the exponential map, such a distribution satisfies a system of partial differential equations.

In the language of $\mathcal{D}$-modules, these equations define a holonomic $\mathcal{D}$-module on the complexified Lie algebra $\mathfrak{g}$. We call this module the Hotta-Kashiwara module as it has been defined and studied first in [**6**]. In [**20**], J. Sekiguchi extended these results to symmetric pairs. He proved in particular that a condition on the symmetric pair is needed to extend Harish-Chandra theorem. In several papers, Levasseur and Stafford [**15, 16, 17**] gave an algebraic proof of the main part of Harish-Chandra theorem.

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In [**3**], we defined a class of holonomic $\mathcal{D}$-modules, which we called *tame $\mathcal{D}$-modules*. These $\mathcal{D}$-modules have no quotients supported by a hypersurface and their distribution solution are locally integrable. We proved in particular that the Hotta-Kashiwara module is tame, recovering Harish-Chandra theorem. The definition of tame is a condition on the roots of the $b$-functions which are polynomials attached to the $\mathcal{D}$-module and a stratification of the base space. However, the proof of the fact that the Hotta-Kashiwara module is tame involved some non algebraic vector fields.

The first aim of this paper is to give a completely algebraic version of Harish-Chandra theorem. We give a slightly different definition of tame and an algebraic proof of the fact that the Hotta-Kashiwara module is tame. This proof is different from the proof of [**3**] and gives more precise results on the roots of the $b$-functions. However our first proof was still valid in the case of symmetric pairs while the present proof uses a morphism of Harish-Chandra which does not exist in that case.

Our second aim is to answer to a remark made by Varadarajan during the Ramis congress. He pointed the fact that an invariant eigendistribution, considered as a distribution on the Lie algebra by the exponential map, is not a solution of the Hotta-Kashiwara module. A key point in the original proof of Harish-Chandra is precisely the proof that after multiplication by a function, the eigendistribution is solution of the Hotta-Kashiwara module (see [**23**]). The study of the Hotta-Kashiwara module did not bypass this difficult step. Here we consider a family of holonomic $\mathcal{D}$-module, which we call (H-C)-modules; this family includes the Hotta-Kashiwara modules but also the module satisfied directly by an eigendistribution. We prove that these modules are tame and get a direct proof of Harish-Chandra theorem.

## 1. $V$-filtration and $b$-functions

We first recall the definition and a few properties of the classical $V$-filtration, then we give a new definition of quasi-homogeneous $b$-functions and of tame $\mathcal{D}$-modules. We end this section with a result on the inverse image of $\mathcal{D}$-modules which will be a key point of the proof in the next section.

**1.1. Standard $V$-filtrations.** — In this paper, $(X, \mathcal{O}_X)$ is a smooth algebraic variety defined over $k$, an algebraically closed field of characteristic 0. The sheaf of differential operators with coefficients in $\mathcal{O}_X$ is denoted by $\mathcal{D}_X$. Results and proofs are still valid if $k = \mathbb{C}$, $X$ is a complex analytic manifold and $\mathcal{D}_X$ is the sheaf of differential operators with holomorphic coefficients.

Let $Y$ be a smooth subvariety of $X$ and $\mathcal{I}_Y$ the ideal of definition of $Y$. The $V$-filtration along $Y$ is given by [**10**]:

$$V_k\mathcal{D}_X = \{\, P \in \mathcal{D}_X|_Y \mid \forall\, l \in \mathbb{Z}, P\mathcal{I}_Y^l \subset \mathcal{I}_Y^{l+k} \,\}$$

(with $\mathcal{I}_Y^l = \mathcal{O}_X$ if $l \leqslant 0$).

This filtration has been widely used in the theory of $\mathcal{D}$-modules, let us recall some of its properties (for the details, we refer to [**19**], [**12**], [**18**], [**14**]). The associated graded ring $\mathrm{gr}_V\mathcal{D}_X$ is the direct image by $p : T_YX \to X$ of the sheaf $\mathcal{D}_{T_YX}$ of differential operators on the normal bundle $T_YX$. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, a $V\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite, *i.e.* if, locally, there are sections $(u_1, \dots, u_N)$ of $\mathcal{M}$ and integers $(k_1, \dots, k_N)$ such that $V_k\mathcal{M} = \sum V_{k-k_i}\mathcal{D}_X u_i$.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module provided with a good $V$-filtration, the associated graded module is a coherent $\mathrm{gr}_V\mathcal{D}_X$-module and if $\mathcal{N}$ is a coherent submodule of $\mathcal{M}$ the induced filtration is a good filtration (see [**19**, Chapter III, Proposition 1.4.3] or [**18**]).

Let $\theta_Y$ be the Euler vector field of the fiber bundle $T_YX$, that is the vector field verifying $\theta_Y(f) = kf$ when $f$ is a function on $T_YX$ homogeneous of degree $k$ in the fibers of $p$. A *b-function* along $Y$ for a coherent $\mathcal{D}_X$-module with a good $V$-filtration is a polynomial $b$ such that

$$\forall k \in \mathbb{Z}, \qquad b(\theta_Y + k)\mathrm{gr}_V^k\mathcal{M} = 0$$

If the good $V$-filtration is replaced by another, the roots of $b$ are translated by integers. Here, we always fix the filtration, in particular, if the $\mathcal{D}_X$-module is of the type $\mathcal{D}_X/\mathcal{I}$, the good filtration will be induced by the canonical filtration of $\mathcal{D}_X$.

**1.2. Quasi-homogeneous $V$-filtrations and quasi-$b$-functions.** — Let $\varphi = (\varphi_1, \dots, \varphi_d)$ be a polynomial map from $X$ to the vector space $W = k^d$ and $m_1, \dots, m_d$ be strictly positive and relatively prime integers. We define a filtration on $\mathcal{O}_X$ by:

$$V_k^\varphi\mathcal{O}_X = \sum_{\langle m,\alpha\rangle = -k} \mathcal{O}_X\varphi^\alpha$$

with $\alpha \in \mathbb{N}^d$, $\langle m, \alpha\rangle = \sum m_i\alpha_i$ and $\varphi^\alpha = \varphi_1^{\alpha_1}\cdots\varphi_d^{\alpha_d}$. If $k \geqslant 0$ we set $V_k^\varphi\mathcal{O}_X = \mathcal{O}_X$.

This filtration extends to $\mathcal{D}_X$ by:

(1) $$V_k^\varphi\mathcal{D}_X = \{\, P \in \mathcal{D}_X \mid \forall l \in \mathbb{Z}, PV_l^\varphi\mathcal{O}_X \subset V_{l+k}^\varphi\mathcal{O}_X \,\}$$

***Definition 1.2.1.*** A $(\varphi, m)$-weighted Euler vector field is a vector field $\eta$ in $\sum_i \varphi_i\mathcal{V}_X$ such that $\eta(\varphi_i) = m_i\varphi_i$ for $i = 1, \dots, d$. ($\mathcal{V}_X$ is the sheaf of vector fields on $X$.)

***Lemma 1.2.2.*** *Any $(\varphi, m)$-weighted Euler vector field is in $V_0^\varphi\mathcal{D}_X$ and if $\eta_1$ and $\eta_2$ are two $(\varphi, m)$-weighted Euler vector fields, $\eta_1 - \eta_2$ is in $V_{-1}^\varphi\mathcal{D}_X$.*

The map $\varphi$ may be not defined on $X$ but on an étale covering of $X$. More precisely, let us consider an étale morphism $\nu : X' \to X$ and a morphism $\varphi : X' \to W = k^d$. If $m_1, \dots, m_d$ are strictly positive and relatively prime integers, we define $V_k^\varphi\mathcal{O}_X$ as the sheaf of functions on $X$ such that $f_\circ\nu$ is in $V_k^\varphi\mathcal{O}_{X'}$. This defines a $V$-filtration on $\mathcal{O}_X$ and on $\mathcal{D}_X$ by the formula (1). The map $TX' \to TX \times_X X'$ is an isomorphism and a vector field $\eta$ on $X$ defines a unique vector field $\nu^*(\eta)$ on $X'$. By definition, a vector field $\eta$ on $X$ is a $(\varphi, m)$-weighted Euler vector field if $\nu^*(\eta)$ is a $(\varphi, m)$-weighted Euler vector field on $X'$.

***Definition 1.2.3***. — Let $u$ be a section of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. A polynomial $b$ is a quasi-$b$-function of type $(\varphi, m)$ for $u$ if there exist a $(\varphi, m)$-weighted Euler vector field $\eta$ and a differential operator $Q$ in $V_{-1}^{\varphi}\mathcal{D}_X$ such that $(b(\eta) + Q)u = 0$.

The quasi-$b$-function is said *regular* if the order of $Q$ as a differential operator is less or equal to the order of the polynomial $b$ and *monodromic* if $Q = 0$.

The quasi-$b$-function is said *tame* if the roots of $b$ are strictly greater than $-\sum m_i$.

These definitions are valid for any map $\varphi$ but here we always assume that $\varphi$ is smooth. Then if $Y = \varphi^{-1}(0)$, we say for short that $b$ is a quasi-$b$-function of total weight $|m| = \sum m_i$ along $Y$. Remark that lemma 1.2.2 shows that the definition is independent of the $(\varphi, m)$-weighted Euler vector field $\eta$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A $V^{\varphi}\mathcal{D}_X$-filtration on $\mathcal{M}$ is a good filtration if it is locally finite.

***Definition 1.2.4***. — Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $V^{\varphi}\mathcal{M}$ a good $V^{\varphi}\mathcal{D}_X$-filtration. A polynomial $b$ is a quasi-$b$-function of type $(\varphi, m)$ for $V^{\varphi}\mathcal{M}$ if, for any $k \in \mathbb{Z}$, $b(\eta + k)V_k^{\varphi}\mathcal{M} \subset V_{k-1}^{\varphi}\mathcal{M}$ where $\eta$ is a $(\varphi, m)$-weighted Euler vector field.

The quasi-$b$-function is *monodromic* if $b(\eta + k)V_k^{\varphi}\mathcal{M} = 0$.

Definition 1.2.3 is a special case of definition 1.2.4 if $\mathcal{D}_X u$ is provided with the filtration induced by the canonical filtration of $\mathcal{D}_X$.

Recall that if $\mathcal{M}$ is a $\mathcal{D}_X$-module its inverse image by $\nu$ is its inverse image as an $\mathcal{O}_X$-module, that is:

$$\nu^+\mathcal{M} = \mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{M} = \mathcal{D}_{X' \to X} \otimes_{\nu^{-1}\mathcal{D}_X} \nu^{-1}\mathcal{M}$$

where $\mathcal{D}_{X' \to X}$ is the $(\mathcal{D}_{X'}, \nu^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_{X'} \otimes_{\nu^{-1}\mathcal{O}_X} \nu^{-1}\mathcal{D}_X$.

***Lemma 1.2.5***. — *Let $\nu : X' \to X$ be an étale morphism and let $\varphi$ be a morphism $X' \to W = k^d$. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

*The polynomial $b$ is a quasi-b-function of type $(\varphi, m)$ for a section $u$ of $\mathcal{M}$ if and only if it is a quasi-b-function of type $(\varphi, m)$ for the section $1 \otimes u$ of $\nu^+\mathcal{M}$.*

*Proof*. — If $\nu : X' \to X$ is étale, the canonical morphism $\mathcal{D}_{X'} \to \mathcal{D}_{X' \to X}$ given by $P \mapsto P(1 \otimes 1)$ is an isomorphism and defines an injective morphism $\nu^* : \nu^{-1}\mathcal{D}_X \to \mathcal{D}_{X'}$.

Conversely, the morphism $\widetilde{\nu} : \nu_*\mathcal{O}_{X'} \to \mathcal{O}_X$ given by $\widetilde{\nu}(f)(x) = \sum_{y \in \nu^{-1}(x)} f(y)$ extends to a morphism $\nu_*\mathcal{D}_{X'} \to \mathcal{D}_X$.

These two morphism are compatible with the $V$-filtration defined by $\varphi$ and, by definition, a vector field $\eta$ on $X$ is a $(\varphi, m)$-weighted Euler vector field if and only if $\nu^*(\eta)$ is a $(\varphi, m)$-weighted Euler vector field on $X'$. If $(b(\eta) + R)u = 0$ we have $(b(\nu^*\eta) + \nu^* R)(1 \otimes u) = 0$ and conversely, if $(b(\nu^*\eta) + R_1)(1 \otimes u) = 0$ then $(b(\eta) + \nu_* R_1)u = 0$. □

***Remark 1.2.6***. — In [**3**] we gave an other definition of the $V^*$-filtration and quasi-$b$-function. The two definitions are essentially equivalent in the analytic framework but may differ in the algebraic case. More precisely, the filtration in [**3**] is given by a vector field $\eta$ which we called positive definite. For a given $V^\varphi$-filtration, we may find a defining vector field with coefficients in formal power series (or in convergent series if $k = \mathbb{C}$) but in general not in rational functions. The definition of [**3**] is more intrinsic in the analytic case but not suitable here.

**1.3. Tame $\mathcal{D}$-modules.** — Let us recall that a stratification of the manifold $X$ is a union $X = \bigcup_\alpha X_\alpha$ such that

- For each $\alpha$, $\overline{X}_\alpha$ is an algebraic subset of $X$ and $X_\alpha$ is its regular part.
- $\{X_\alpha\}_\alpha$ is locally finite.
- $X_\alpha \cap X_\beta = \varnothing$ for $\alpha \neq \beta$.
- If $\overline{X}_\alpha \cap X_\beta \neq \varnothing$ then $\overline{X}_\alpha \supset X_\beta$.

If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, its characteristic variety $Ch(\mathcal{M})$ is a homogeneous lagrangian subvariety of $T^*X$ hence there exists a stratification $X = \bigcup X_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup_\alpha \overline{T^*_{X_\alpha}X}$ [**9**, Ch. 5]. The set of points of $X$ where $Ch(\mathcal{M})$ is contained in the zero section of $T^*X$ is a non empty Zarisky open subset of $X$, its complementary is the *singular support* of $\mathcal{M}$.

For the next definition, we consider a cyclic $\mathcal{D}_X$-module with a canonical generator $\mathcal{M} = \mathcal{D}_X u = \mathcal{D}_X/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_X$.

***Definition 1.3.1***. — The cyclic holonomic $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X u$ is *tame* if there is a stratification $X = \bigcup X_\alpha$ of $X$ such that $Ch(\mathcal{M}) \subset \bigcup_\alpha \overline{T^*_{X_\alpha}X}$ and, for each $\alpha$, a tame quasi-$b$-function associated with $X_\alpha$.

With definition 1.2.3, this means that for each $\alpha$, there is a smooth map $\varphi_\alpha$ from a Zarisky open set of $X$ to a vector space $V$ such that $X_\alpha = \varphi_\alpha^{-1}(0)$, positive integers $m_1, \ldots, m_d$, a $(\varphi, m)$-weighted Euler vector field $\eta$ and a quasi-$b$-function $b_\alpha$ for $u$ with roots $> -\sum m_i$. A subvariety of $X$ is conic for $\eta_\alpha$ if it is invariant under the flow of $\eta_\alpha$. The module $\mathcal{M}$ is *conic tame* if it satisfy definition 1.3.1 and if moreover the singular support of $\mathcal{M}$ is conic for each $\eta_\alpha$.

The following property of a tame $\mathcal{D}_X$-module has been proved in [**3**]:

***Theorem 1.3.2***. — *If the $\mathcal{D}_X$-module $\mathcal{M}$ is tame then it has no quotient with support in a hypersurface of $X$.*

If $M$ is a real analytic manifold and $X$ its complexification, we also proved:

***Theorem 1.3.3***. — *Let $\mathcal{M}$ be a holonomic and tame $\mathcal{D}_X$-module, assume that its singular support is the complexification of a real subvariety of $M$, then $\mathcal{M}$ has no distribution solution on $M$ with support in a hypersurface. If $\mathcal{M}$ is conic-tame, its distribution solutions are in $L^1_{\mathrm{loc}}$.*

***Remark 1.3.4***. — It is important to note that the definition of *tame* and the conclusions of theorem 1.3.3 depend of the choice of a generator for $\mathcal{M}$.

**1.4. Inverse image.** — Let $\varphi : X \to W$ and $\varphi' : X' \to W'$ be two morphisms from smooth algebraic varieties $X$ and $X'$ to the vector spaces $W = k^d$ and $W' = k^{d'}$, let $m_1, \dots, m_d$ and $m'_1, \dots, m'_{d'}$ be strictly positive integers. Let $f : X' \to X$ and $F : W' \to W$ be two morphisms such that $\varphi \circ f = F \circ \varphi'$. We assume that $F$ is *quasi-homogeneous*, that is $F = (F_1, \dots, F_d)$ with $F_i(\lambda^{m'_1}x_1, \dots, \lambda^{m'_{d'}}x_{d'}) = \lambda^{m_i}F(x_1, \dots, x_{d'})$.

If $\mathcal{N}$ is a $\mathcal{D}_X$-module its inverse image by $f$ is:

$$f^+\mathcal{N} = \mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{N} = \mathcal{D}_{X'\to X} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{N}$$

where $\mathcal{D}_{X'\to X}$ is the $(\mathcal{D}_{X'}, f^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$.

We define a filtration on $\mathcal{D}_{X'\to X}$ by

$$V_k^{\varphi'}\mathcal{D}_{X'\to X} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}V_j^{\varphi}\mathcal{D}_X$$

By the hypothesis, $g \circ f$ is a section of $V_k^{\varphi'}\mathcal{O}_{X'}$ for any $g$ section of $V_k^{\varphi}\mathcal{O}_X$, hence the filtration on $\mathcal{D}_{X'\to X}$ is compatible with the corresponding filtrations on $\mathcal{D}_{X'}$ and $\mathcal{D}_X$.

If a $\mathcal{D}_X$-module $\mathcal{N}$ is provided with a $V^{\varphi}$-filtration, this defines a $V^{\varphi'}\mathcal{D}_{X'}$-filtration on $f^+\mathcal{N}$ by

$$V_k^{\varphi'}f^+\mathcal{N} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}V_j^{\varphi}\mathcal{N} = \sum_{i+j=k} V_i^{\varphi'}\mathcal{D}_{X'\to X} \otimes f^{-1}V_j^{\varphi}\mathcal{N} \tag{2}$$

The $V$-filtration has not all the good properties of the usual filtration, in particular non invertible elements may have an invertible principal symbol. In the proof of theorem 1.4.1 we introduce its formal completion given by:

$$\widehat{\mathcal{D}}_{X|Y} = \varinjlim_k V_k\widehat{\mathcal{D}}_{X|Y} \qquad \text{with} \qquad V_k\widehat{\mathcal{D}}_{X|Y} = \varprojlim_l V_k\mathcal{D}_X/V_{k-l}\mathcal{D}_X$$

By definition the graded ring of $\widehat{\mathcal{D}}_{X|Y}$ is the same than the graded ring of $\mathcal{D}_X$. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module provided with a good $V$-filtration, its completion $\widehat{V}\mathcal{M}$ is defined in the same way and has the same associated graded module than $V\mathcal{M}$. The following properties may be found in [**19**] and [**14**].

The sheaf $\widehat{\mathcal{D}}_{X|Y}$ is a coherent and noetherian, flat over $\mathcal{D}_X$. We remind that a coherent sheaf of rings $\mathcal{A}$ is noetherian if any increasing sequence of coherent $\mathcal{A}$-submodules of a coherent $\mathcal{A}$-module is stationary. The sheaf of rings $V_0\widehat{\mathcal{D}}_{X|Y}$ is also coherent and noetherian.

If $\mathcal{M}$ is a $\mathcal{D}_X$-module provided with a good $V$-filtration, the associated graded module is a coherent $\mathrm{gr}_V\mathcal{D}_X$-module and if $\mathcal{N}$ is a coherent submodule of $\mathcal{M}$ the induced filtration is a good filtration. If $\kappa : (\widehat{\mathcal{D}}_{X|Y})^N \to \mathcal{M}$ is a filtered morphism which defines a surjective graded morphism $\mathrm{gr}_V(\widehat{\mathcal{D}}_{X|Y})^N \to \mathrm{gr}_V\mathcal{M} \to 0$ then $\kappa$ is surjective.

As $\widehat{\mathcal{D}}_{X|Y}$ is flat over $\mathcal{D}_X$, if $\mathcal{M}$ is coherent we have $\widehat{V}\mathcal{M} = \widehat{\mathcal{D}}_{X|Y} \otimes_{\mathcal{D}_X} \mathcal{M}$. Remark also that $\widehat{V}\mathcal{O}_X$, the completion of $\mathcal{O}_X$ for the $V$-filtration, is the formal completion of $\mathcal{O}_X$ along $Y$ usually denoted by $\mathcal{O}_{\widehat{X|Y}}$ and $\widehat{\mathcal{D}}_{X|Y}$ is a $\mathcal{O}_{\widehat{X|Y}}$-module.

After completion by the $V$-filtration, we get a similar formula:

$$\widehat{V}_k^{\varphi'} f^+\mathcal{N} = \sum_{i+j=k} \widehat{V}_i^{\varphi'}\mathcal{O}_{X'} \otimes f^{-1}\widehat{V}_j^{\varphi}\mathcal{N} \tag{3}$$

Let $Y = \varphi^{-1}(0)$ and $Y' = \varphi'^{-1}(0)$, let $p : T_YX \to X$ and $p' : T_{Y'}X' \to X'$ be the normal bundles, $\widetilde{f} : T_{Y'}X' \to T_YX$ be the map induced by $f$,

***Theorem 1.4.1***. — *We assume that $\varphi'$ is smooth on $X'$. If $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module provided with a good $V^{\varphi}\mathcal{D}_X$-filtration, then $f^+\mathcal{N}$ is holonomic, ${p'}^{-1}\mathrm{gr}_{V^{\varphi'}}f^+\mathcal{N}$ is equal to $\widetilde{f}^+p^{-1}\mathrm{gr}_{V^{\varphi}}\mathcal{N}$ and isomorphic to the graded module associated with a good $V^{\varphi'}\mathcal{D}_{X'}$-filtration of $f^+\mathcal{N}$.*

*Proof*. — We recall that if $\mathcal{N}$ is coherent, then $f^+\mathcal{N}$ is not coherent in general but if $\mathcal{N}$ is holonomic, then $f^+\mathcal{N}$ is holonomic [**8**].

The filtration on $\mathcal{N}$ is a good $V^{\varphi}\mathcal{D}_X$-filtration hence we may assume that there are sections $(u_1, \ldots, u_q)$ of $\mathcal{N}$ and integers $(k_1, \ldots, k_q)$ such that $V_k^{\varphi}\mathcal{N} = \sum V_{k-k_i}^{\varphi}\mathcal{D}_X u_i$. Let $\mathcal{D}_{X'\to X}[N]$ be the sub-$\mathcal{D}_{X'}$-module of $\mathcal{D}_{X'\to X}$ generated by the sections of $\mathcal{D}_X$ of order less or equal to $N$. This submodule is finitely generated hence coherent. For each $N$, $(u_1, \ldots, u_q)$ defines a canonical morphism $(\mathcal{D}_{X'\to X}[N])^q \to f^+\mathcal{N}$ and the family of the images of these morphisms is an increasing sequence of coherent submodules of the coherent $\mathcal{D}_{X'}$-module $f^+\mathcal{N}$. As $\mathcal{D}_{X'}$ is a noetherian sheaf of rings, this sequence is stationary, hence there is some $N_0$ such that for each $N > N_0$, the morphism $(\mathcal{D}_{X'\to X}[N])^q \to f^+\mathcal{N}$ is onto. The filtration $V^{\varphi'}\mathcal{D}_{X'\to X}$ induces a good filtration on $\mathcal{D}_{X'\to X}[N]$ hence, for $N > N_0$ a good filtration on $f^+\mathcal{N}$ which is denoted by $V_k^{\varphi'}[N]f^+\mathcal{N}$. To prove the theorem, we will prove that if $N$ is large enough, $\mathrm{gr}_V f^+\mathcal{N}$ is equal to the graded module $\mathrm{gr}_{V[N]}f^+\mathcal{N}$ associated with the good filtration $V_k^{\varphi'}[N]f^+\mathcal{N}$.

We assume first that the integers $m'_i$ are equal to 1, that is that the $V^{\varphi'}$-filtration is the usual $V$-filtration on the non singular variety $Y' = \varphi'^{-1}(0)$. For $N > N_0$,

$p'^{-1}\mathrm{gr}_{V[N]}f^+\mathcal{N}$ is a coherent $\mathcal{D}_{T_{Y'}X'}$-module. A direct calculation shows that $p'^{-1}\mathrm{gr}_V f^+\mathcal{N} = \widetilde{f}^+ p^{-1}\mathrm{gr}_V^{\varphi}\mathcal{N}$. If $\mathcal{N}$ is holonomic then $\mathrm{gr}_V^{\varphi}\mathcal{N}$ is also holonomic [**12**, Cor 4.1.2.] hence $p'^{-1}\mathrm{gr}_V f^+\mathcal{N}$ is holonomic hence coherent.

Consider the completion $\widehat{V}f^+\mathcal{N}$ of $f^+\mathcal{N}$ for the $V$-filtration and $\widehat{V}[N]f^+\mathcal{N}$ of $f^+\mathcal{N}$ for the $V[N]$-filtration. The graded module of $\widehat{V}f^+\mathcal{N}$ is equal to the graded module of $Vf^+\mathcal{N}$ which is coherent. Let $u_1,\dots,u_M$ be local sections of $\widehat{V}f^+\mathcal{N}$ whose classes generate the graded module, then $u_1,\dots,u_M$ generate $\widehat{V}f^+\mathcal{N}$ as a filtered $V\widehat{\mathcal{D}}_{X'|Y'}$-module and applying the same result to the kernel of $(V\widehat{\mathcal{D}}_{X'|Y'})^M \to \widehat{V}f^+\mathcal{N}$ we get that $\widehat{V}f^+\mathcal{N}$ admits a filtered presentation

$$(V\widehat{\mathcal{D}}_{X'|Y'})^L \longrightarrow (V\widehat{\mathcal{D}}_{X'|Y'})^M \longrightarrow \widehat{V}f^+\mathcal{N} \longrightarrow 0.$$

This shows in particular that each $\widehat{V}_k f^+\mathcal{N}$ is a coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module. We know that, for any $N$, $\mathrm{gr}_{V[N]}f^+\mathcal{N}$ is coherent hence for the same reason, each $\widehat{V}_k[N]f^+\mathcal{N}$ is a coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module.

Consider the family of the images of $\widehat{V}_k[N]f^+\mathcal{N}$ in $\widehat{V}_k f^+\mathcal{N}$, it is an increasing sequence of coherent sub-modules of the coherent $V_0\widehat{\mathcal{D}}_{X'|Y'}$-module $\widehat{V}_k f^+\mathcal{N}$ hence it is stationary because the sheaf of rings $V_0\widehat{\mathcal{D}}_{X'|Y'}$ is noetherian. Moreover, the filtration $\widehat{V}f^+\mathcal{N}$ is separated hence the maps $\widehat{V}_k[N]f^+\mathcal{N} \to \widehat{V}_k f^+\mathcal{N}$ are injective and the union of the images is all $\widehat{V}_k f^+\mathcal{N}$, so there is some $N_0$ such that for any $N > N_0$, $\widehat{V}_k[N]f^+\mathcal{N} = \widehat{V}_k f^+\mathcal{N}$. This implies that $\mathrm{gr}_V f^+\mathcal{N} = \mathrm{gr}_{V[N]}f^+\mathcal{N}$ is the graded module associated with a good $V$-filtration of $f^+\mathcal{N}$.

Assume now that the numbers $m'_i$ are positive integers. Let $W'' = W'$, we define the ramification map $F_m : W'' \to W'$ by $F(s_1,\dots,s_d) = (s_1^{m'_1},\dots,s_d^{m'_d})$ and the corresponding map $f_m : X'' = X' \times_{W'} W'' \to X$. Applying the first part of the proof, we get $\widehat{V}[N]f_m^+ f^+\mathcal{N} = \widehat{V}f_m^+ f^+\mathcal{N}$ if $N$ is large. The formula (3) shows that

$$\widehat{V}f_m^+ f^+\mathcal{N} = \widehat{V}\mathcal{O}_{X''} \otimes_{f^{-1}\widehat{V}\mathcal{O}_{X'}} f^{-1}\widehat{V}_j^{\varphi}\mathcal{N} = \mathcal{O}_{\widehat{W}} \otimes_{\mathcal{O}_{\widehat{V'}}} f^{-1}\widehat{V}_j^{\varphi}\mathcal{N}.$$

Here $\mathcal{O}_{\widehat{W}}$ is the set of formal power series in $(s_1,\dots,s_d)$ while $\mathcal{O}_{\widehat{V'}}$ is the set of formal power series in $(s_1^{m'_1},\dots,s_d^{m'_d})$ hence $\mathcal{O}_{\widehat{W}}$ is a finite free $\mathcal{O}_{\widehat{V'}}$-module. So, if $M' = \sum m'_i$, $\widehat{V}f_m^+ f^+\mathcal{N}$ is isomorphic to $(\widehat{V}f^+\mathcal{N})^{M'}$ as a $\widehat{V}\mathcal{O}_{X'}$-module.

In the same way, $\widehat{V}[N]f_m^+ f^+\mathcal{N}$ is isomorphic to $(\widehat{V}[N]f^+\mathcal{N})^{M'}$, hence $\widehat{V}[N]f^+\mathcal{N} = \widehat{V}[N]f^+\mathcal{N}$. This shows that $\mathrm{gr}_{V^{\varphi'}}f^+\mathcal{N}$ is the graded module associated with $V^{\varphi'}[N]f^+\mathcal{N}$ which is a good filtration of $f^+\mathcal{N}$. □

***Remark 1.4.2***. — The result was known when $f$ is a submersion, $Y' = f^{-1}(Y)$ and the $V$-filtrations being the usual $V$-filtrations along $Y$ and $Y'$ [**14**]. The introduction of the weights $m_i$ and $m'_i$ allows $f$ to be non submersive and $Y'$ to be a proper subvariety of $f^{-1}(Y)$; the relation between the weights is given by the quasi-homogeneity of $F$.

***Corollary 1.4.3.*** — *Under the hypothesis of theorem 1.4.1, if $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module provided with a good $V^{\varphi}\mathcal{D}_X$-filtration, $f^+\mathcal{N}$ is provided with a good $V^{\varphi'}\mathcal{D}_{X'}$-filtration such that a polynomial $b$ is a quasi-b-function of type $(\varphi, m)$ for the filtration of $\mathcal{N}$ if and only if $b$ is a quasi-b-function of type $(\varphi', m')$ for the filtration of $f^+\mathcal{N}$.*

*Proof.* — Let $\eta'$ be a $(\varphi', m')$-weighted Euler vector field, then $\eta = f_*\eta'$ is a $(\varphi, m)$-weighted Euler vector field. As definition 1.2.4 is independent of the $(\varphi, m)$-weighted Euler vector field, we may assume that the quasi-$b$-function for $\mathcal{N}$ is relative to $\eta$.

By definition, for any $Q$ in $\mathcal{D}_{X'\to X}$, we have $\eta' Q = Q\eta$ hence for any polynomial $b(\eta')Q = Qb(\eta)$ which shows the corollary. □

***Corollary 1.4.4.*** — *Under the hypothesis of theorem 1.4.1, if $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module and $u$ a section of $\mathcal{N}$ with a quasi-b-function of type $(\varphi, m)$, then the section $1_{X'\to X} \otimes u$ of $f^+\mathcal{N}$ has the same polynomial $b$ as a quasi-b-function of type $(\varphi', m')$.*

*Proof.* — Recall that $1_{X'\to X}$ is the canonical section $1 \otimes 1$ in $\mathcal{D}_{X'\to X} = \mathcal{O}_{X'} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$. If $u$ is a section of $\mathcal{N}$, we set on $\mathcal{D}_X u$ the filtration image of the filtration of $\mathcal{D}_X$. Then, by definition of the filtration $V^{\varphi'}[N]f^+\mathcal{N}$ used in the proof of theorem 1.4.1, $1_{X'\to X} \otimes u$ is of order 0 for this filtration. Then corollary 1.4.4 is a special case of corollary 1.4.3. □

## 2. Reductive Lie algebras

**2.1. Statement of the main theorem.** — Let $G$ be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. The group $G$ acts on $\mathfrak{g}$ by the adjoint action hence on the symmetric algebra $S(\mathfrak{g})$ identified to the space $\mathcal{O}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$. By Chevalley's theorem, the space $\mathcal{O}(\mathfrak{g}^*)^G \simeq S(\mathfrak{g})^G$ of invariant polynomials on $\mathfrak{g}^*$ is equal to a polynomial algebra $k[Q_1, \ldots, Q_l]$ where $Q_1, \ldots, Q_l$ are algebraically independent. These spaces are graded and we denote $S_+(\mathfrak{g})^G = \oplus_{k>0} S_k(\mathfrak{g})^G$. It is also the set $\mathcal{O}_+(\mathfrak{g}^*)^G$ of invariant polynomials vanishing at $\{0\}$, their common roots are the nilpotent elements of $\mathfrak{g}^*$.

The differential of the adjoint action induces a Lie algebra morphism $\tau : \mathfrak{g} \to \mathrm{Der}S(\mathfrak{g}^*)$ by:

$$(\tau(A)f)(x) = \frac{d}{dt} f\left(\exp(-tA)\cdot x\right)|_{t=0} \quad \text{for } A \in \mathfrak{g},\ f \in S(\mathfrak{g}^*) = \mathcal{O}(\mathfrak{g}),\ x \in \mathfrak{g}$$

*i.e.* $\tau(A)$ is the vector field on $\mathfrak{g}$ whose value at $x \in \mathfrak{g}$ is $[x, A]$. We denote by $\tau(\mathfrak{g})$ the set of all vector fields $\tau(A)$ for $A \in \mathfrak{g}$. It generates the set of vector fields on $\mathfrak{g}$ tangent to the orbits of $G$.

Let $\mathcal{D}_{\mathfrak{g}}^G$ be the sheaf of differential operators on $\mathfrak{g}$ invariant under the adjoint action of $G$. The principal symbol $\sigma(P)$ of such an operator $P$ is a function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ invariant under the action of $G$. Examples of such invariant functions are the elements of $S(\mathfrak{g})^G$ identified to functions on $\mathfrak{g} \times \mathfrak{g}^*$ constant in the variables of $\mathfrak{g}$. If $F$ is a subsheaf of $\mathcal{D}_{\mathfrak{g}}^G$, we denote by $\sigma(F)$ the sheaf of the principal symbols of all elements of $F$.

***Definition 2.1.1***. — A subsheaf $F$ of $\mathcal{D}_{\mathfrak{g}}^G$ is of (H-C)-type if $\sigma(F)$ contains a power of $S_+(\mathfrak{g})^G$. An (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module is the quotient $\mathcal{M}_F$ of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and by $F$.

The main result of this paper is

***Theorem 2.1.2***. — *Any $\mathcal{D}_{\mathfrak{g}}$-module of (H-C)-type is holonomic and conic-tame.*

Here (H-C) stands for Harish-Chandra. There are two main examples of such $\mathcal{D}_{\mathfrak{g}}$-modules which we describe now.

***Example 2.1.3***. — An element $A$ of $\mathfrak{g}$ defines a vector field with constant coefficients on $\mathfrak{g}$ by:

$$(A(D_x)f)(x) = \frac{d}{dt} f(x+tA)|_{t=0} \quad \text{for } f \in S(\mathfrak{g}^*),\ x \in \mathfrak{g}$$

By multiplication, this extends to an injective morphism from $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$; we identify $S(\mathfrak{g})$ with its image and denote by $P(D_x)$ the image of $P \in S(\mathfrak{g})$. If $F$ is a finite codimensional ideal of $S(\mathfrak{g})^G$, its graded ideal contains a power of $S_+(\mathfrak{g})^G$ hence when it is identified to a set of differential operators with constant coefficients, $F$ is a subsheaf of $\mathcal{D}_{\mathfrak{g}}$ of (H-C)-type and $\mathcal{M}_F$ is a $\mathcal{D}_{\mathfrak{g}}$-module of (H-C)-type. If $\lambda \in \mathfrak{g}^*$, the module $\mathcal{M}_\lambda^F$ defined by Hotta and Kashiwara [**6**] is the special case where $F$ is the set of polynomials $Q - Q(\lambda)$ for $Q \in S(\mathfrak{g})^G$.

***Example 2.1.4***. — The enveloping algebra $U(\mathfrak{g})$ is the algebra of left invariant differential operators on $G$. It is filtered by the order of operators and the associated graded algebra is isomorphic by the symbol map to $S(\mathfrak{g})$. This map is a $G$-map and defines a morphism from the space of bi-invariant operators on $G$ to the space $S(\mathfrak{g})^G$. This map is a linear isomorphism, its inverse is given by a symmetrization morphism [**22**, Theorem 3.3.4.]. We assume that $k = \mathbb{C}$. Then, through the exponentional map a bi-invariant operator $P$ defines a differential operator $\widetilde{P}$ on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of $G$ (because the exponential intertwines the adjoint action on the group and on the algebra) and the principal symbol $\sigma(\widetilde{P})$ is equal to $\sigma(P)$.

Let $U$ be an open subset of $\mathfrak{g}$ where the exponential is injective and $U_G = \exp(U)$. Let $T$ be an invariant eigendistribution on $U_G$ and $\widetilde{T}$ the distribution on $U$ given by $\langle T, \varphi\rangle = \langle \widetilde{T}, \varphi_o \exp\rangle$. As $T$ is invariant and eigenvalue of all bi-invariant operators, $\widetilde{T}$ is solution of an (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module. As this module is conic-tame by theorem 2.1.2, the results of theorems 1.3.2 and 1.3.3 are true for it, hence $\widetilde{T}$ and $T$ are a $L^1_{\mathrm{loc}}$-function.

As $\mathfrak{g}$ is reductive, it is the direct sum $\mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ of its center and of the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. We choose a non-degenerate $G$-invariant symmetric bilinear form $\kappa$ on $\mathfrak{g}$ which extend the Killing form of $[\mathfrak{g}, \mathfrak{g}]$. This defines an isomorphism from $\mathfrak{g}^*$ to $\mathfrak{g}$ and the cotangent bundle $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ is identified with $\mathfrak{g} \times \mathfrak{g}$. Then if $\mathcal{N}(\mathfrak{g})$ is the nilpotent cone of $\mathfrak{g}$, the characteristic variety of an (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module is a subset of:

$$\{ (x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0, y \in \mathcal{N}(\mathfrak{g}) \}$$

so it is a holonomic $\mathcal{D}_{\mathfrak{g}}$-module [**6**].

**2.2. Stratification of a reductive Lie algebra.** — In this section, we define the stratification which will be used to prove that an (H-C)-type module is tame. This stratification is classical (see [**1**] for example).

The stratification of a reductive Lie algebra is the direct sum of the center by the stratification of the semi-simple part, so we may assume that $\mathfrak{g}$ is semi-simple. An element $X$ of $\mathfrak{g}$ is said to be *semisimple* (resp. *nilpotent*) if $\mathrm{ad}(X)$ is semisimple (resp $\mathrm{ad}(X)$ is nilpotent). Any $X \in \mathfrak{g}$ may be decomposed in a unique way as $X = S + N$ where $S$ is semisimple, $N$ is nilpotent and $[S, N] = 0$ (Jordan decomposition). An element $X$ is said to be *regular* if the dimension of its centralizer $\mathfrak{g}^X = \{ Z \in \mathfrak{g} \mid [X, Z] = 0 \}$ is minimal, that is equal to the rank of $\mathfrak{g}$. The set $\mathfrak{g}_{rs}$ of semisimple regular elements of $\mathfrak{g}$ is Zarisky dense and its complementary $\mathfrak{g}'$ is defined by a $G$-invariant polynomial equation $\Delta(X) = 0$. The function $\Delta$ may be defined from the characteristic polynomial of $\mathrm{ad}(X)$:

$$\det(T \cdot \mathrm{Id} - \mathrm{ad}(X)) = T^n + \sum \lambda_i(X) T^i$$

Here $n$ is the dimension of $\mathfrak{g}$. Then $\lambda_0 \equiv 0$, the rank $l$ of $\mathfrak{g}$ is the lowest $i$ such that $\lambda_i \not\equiv 0$ and $\Delta(X) = \lambda_l(X)$. This function is homogeneous of degree $n - l$.

The set $\mathfrak{N}(\mathfrak{g})$ of nilpotent elements of $\mathfrak{g}$ is a cone equal to:

$$\mathfrak{N}(\mathfrak{g}) = \{ X \in \mathfrak{g} \mid \forall P \in \mathcal{O}(\mathfrak{g})^G \; P(X) = P(0) \}$$

and the set of nilpotent orbits is finite and define a stratification of $\mathfrak{N}$ [**11**, Cor 3.7.].

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\mathfrak{W}$ the Weyl group $\mathfrak{W}(\mathfrak{g}, \mathfrak{h})$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the root system associated with $\mathfrak{h}$. For each $\alpha \in \Phi$ we denote by $\mathfrak{g}_\alpha$ the root subspace corresponding to $\alpha$ and by $\mathfrak{h}_\alpha$ the subset $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$ (they are all 1-dimensional). Let $\mathcal{F}$ be the set of the subsets $P$ of $\Phi$ which are closed and symmetric that is such that $(P + P) \cap \Phi \subset P$ and $P = -P$. For each $P \in \mathcal{F}$ we define $\mathfrak{h}_P = \sum_{\alpha \in P} \mathfrak{h}_\alpha$, $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}_\alpha$, $\mathfrak{h}_P^\perp = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P \}$ and $(\mathfrak{h}_P^\perp)' = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in P, \alpha(H) \neq 0 \text{ if } \alpha \notin P \}$.

The following results are well-known (see [**2**, Ch. VIII, § 3]):

a) $\mathfrak{q}_P = \mathfrak{h}_P + \mathfrak{g}_P$ is a semisimple Lie subalgebra of $\mathfrak{g}$ stable under $\mathrm{ad}\,\mathfrak{h}$ and $\mathfrak{h}_P^\perp$ is an orthocomplement of $\mathfrak{h}_P$ for the Killing form, $\mathfrak{h}_P$ is a Cartan subalgebra of $\mathfrak{q}_P$. The

Weyl group $\mathfrak{W}_P$ of $(\mathfrak{q}_P, \mathfrak{h}_P)$ is identified to the subgroup $\mathfrak{W}'$ of $\mathfrak{W}$ of elements whose restriction to $\mathfrak{h}_P^\perp$ is the identity.

b) $\mathfrak{h} + \mathfrak{g}_P$ is a reductive Lie subalgebra of $\mathfrak{g}$ stable under $\operatorname{ad}\mathfrak{h}$. For any $S \in \mathfrak{h}_P^\perp$, $\mathfrak{h} + \mathfrak{g}_P \subset \mathfrak{g}^S$ and $(\mathfrak{h}_P^\perp)' = \{ S \in \mathfrak{h}_P^\perp \mid \mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P \}$.

c) Conversely, if $S \in \mathfrak{h}$, there exists a subset $P$ of $\Phi$ which is closed and symmetric such that $\mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P$. $P$ is unique up to a conjugation by $\mathfrak{W}$.

To each $P$ of $\mathcal{F}$ and each nilpotent orbit $\mathfrak{O}$ of $\mathfrak{q}_P$ we associate a conic subset of $\mathfrak{g}$

$$S_{(P,\mathfrak{O})} = \bigcup_{X \in (\mathfrak{h}_P^\perp)'} G \cdot (X + \mathfrak{O}) \tag{4}$$

where $G \cdot (X + \mathfrak{O})$ is the union of orbits of points $X + \mathfrak{O}$.

If $X = S + N$ is the Jordan decomposition of $X \in \mathfrak{g}$, the semisimple part $S$ belongs to a Cartan subalgebra which we may assume to be $\mathfrak{h}$ because they are all conjugate. Hence there is some $P$ in $\mathcal{F}$ such that $\mathfrak{g}^S = \mathfrak{h} + \mathfrak{g}_P$. Then, if the orbit of $N$ in $\mathfrak{q}_P$ is $\mathfrak{O}$, $X \in S_{(P,\mathfrak{O})}$. For a detailed proof of the fact that it is a stratification, see [**3**].

**2.3. Polynomials and differentials.** — Let us begin with some elementary calculations. If $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index of $\mathbb{N}^n$ we denote $|\beta| = \sum \beta_i$ and $\beta! = \beta_1! \cdots \beta_n!$, if $\alpha$ is another element of $\mathbb{N}^n$, we denote by $\alpha \leqslant \beta$ the relation $\alpha_1 \leqslant \beta_1, \dots, \alpha_n \leqslant \beta_n$.

***Lemma 2.3.1.*** — *Let $\beta \in \mathbb{N}^n$ and $M = |\beta|$, let $N \in \mathbb{N}$ such that $N \leqslant M$, then*

$$\sum_{\substack{|\alpha|=N \\ \alpha\leqslant\beta}} \frac{\beta!}{\alpha!(\beta-\alpha)!} = \frac{M!}{N!(M-N)!}$$

*Proof*

$$\sum_{\alpha\leqslant\beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} x^\alpha = \prod_{i=1}^{n} \sum_{\alpha_i=0}^{\beta_i} \frac{\beta_i!}{\alpha_i!(\beta_i-\alpha_i)!} x_i^{\alpha_i} = (1+x_1)^{\beta_1} \cdots (1+x_n)^{\beta_n}$$

hence if $t = x_1 = \cdots = x_n$ we get:

$$\sum_{\alpha\leqslant\beta} \frac{\beta!}{\alpha!(\beta-\alpha)!} t^{|\alpha|} = (1+t)^M$$

and the coefficient of $t^N$ in both side of the equality gives the lemma. □

***Lemma 2.3.2.*** — *Let us denote $x = (x_1, \dots, x_n)$, $D_x = (D_{x_1}, \dots, D_{x_n})$, $x^\alpha = (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$ and $D_x^\alpha = (D_{x_1}^{\alpha_1}, \dots, D_{x_n}^{\alpha_n})$, let $\theta = \sum x_i D_{x_i}$, then:*

$$\sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha D_x^\alpha = \theta(\theta-1)\cdots(\theta-N+1)$$

*Proof.* — To prove the equality of the two differential operators we have to show that they give the same result when acting on a monomial $x^\beta$, so lemma 2.3.1 gives:

$$\sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha D_x^\alpha x^\beta = \sum_{\substack{|\alpha|=N \\ \alpha\leqslant\beta}} \frac{N!}{\alpha!} \frac{\beta!}{(\beta-\alpha)!} x^\beta = \frac{|\beta|!}{(|\beta|-N)!} x^\beta = \theta(\theta-1)\cdots(\theta-N+1)x^\beta$$

□

**Proposition 2.3.3.** — *Let $p_1,\ldots,p_n(\xi)$ be homogeneous polynomial on $X=\mathbb{C}^n$ and assume that:*

$$\bigcap_{i=1}^{n}\{p_i(\xi)=0\}=\{0\}$$

*Let $\mathcal{I}$ be the ideal of $\mathcal{D}_X$ generated by $p_1(D_x),\ldots,p_n(D_x)$ and $\mathcal{M}=\mathcal{D}_X/\mathcal{I}$. The $\mathcal{D}_X$ module $\mathcal{M}$ is holonomic and the b-function of $\mathcal{M}$ relative to $\{0\}$ is equal to*

$$b(\theta)=\theta(\theta-1)\cdots(\theta+n-M)$$

*where $M$ is the sum of the degrees of the polynomials $p_1,\ldots,p_n$ and $\theta$ the Euler vector field of $X$. This b-function is monodromic in the canonical coordinates of $\mathbb{C}^n$.*

*Proof.* — The Nullstellensatz shows that there is some integer $M_1$ such that the monomial $\xi^\alpha$ are in the ideal generated by $p_1,\ldots,p_n$ if $|\alpha|>M_1$. In fact it is known that the lowest $M_1$ is $M-n$ (the proof uses the Hilbert polynomial). Then lemma 2.3.2 shows that the $b$-function of $\mathcal{M}$ divides $\theta\cdots(\theta+n-M)$. It has been proved by T. Torrelli [**21**] that all integers $0,\ldots,M-n$ appear effectively as roots of $b$. □

**Proposition 2.3.4.** — *Let $p_1,\ldots,p_n$ be the same polynomials as in the previous proposition and let $P_1,\ldots,P_n$ be differential operators such that $\sigma(P_i)=p_i$. Let $\mathcal{I}$ be the ideal of $\mathcal{D}_X$ generated by the operators $P_1,\ldots,P_n$ and $\mathcal{M}=\mathcal{D}_X/\mathcal{I}$. The $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic and the b-function of $\mathcal{M}$ relative to $\{0\}$ is equal to*

$$b(\theta)=\theta(\theta-1)\cdots(\theta+n-M)$$

*The b-function of $\mathcal{M}$ along a vector subspace $L$ of $\mathbb{C}^n$ divides the same polynomial b.*

*Proof.* — Each function $\xi^\alpha$ for $|\alpha|=N=M-n+1$ is written as $\xi^\alpha=\sum q_i^\alpha(\xi)p_i(\xi)$ and

$$b(\theta)=\sum_{|\alpha|=N}\frac{N!}{\alpha!}x^\alpha D_x^\alpha=\sum_{\substack{|\alpha|=N\\ i=1,\ldots,n}}\frac{N!}{\alpha!}x^\alpha q_i^\alpha(D_x)p_i(D_x)$$

$$=\sum_{\substack{|\alpha|=N\\ i=1,\ldots,n}}\frac{N!}{\alpha!}x^\alpha q_i^\alpha(D_x)P_i(x,D_x)+\sum_{\substack{|\alpha|=N\\ i=1,\ldots,n}}\frac{N!}{\alpha!}x^\alpha q_i^\alpha(D_x)\left(p_i(D_x)-P_i(x,D_x)\right)$$

By definition, the order for the $V$-filtration along $\{0\}$ is always less then the usual order with equality if the operator has constant coefficients. So the order for the $V$-filtration of $\sum\frac{N!}{\alpha!}x^\alpha q_i^\alpha(D_x)(p_i(D_x)-P_i(x,D_x))$ is strictly less than the order of $\sum \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) p_i(D_x)$ which is the order of $b(\theta)$, that is 0. On the other hand $\sum \frac{N!}{\alpha!} x^\alpha q_i^\alpha(D_x) P_i(x, D_x)$ is in the ideal $\mathcal{I}$ hence $b(\theta)$ is a $b$-function.

For the second part, we choose linear coordinates of $\mathbb{C}^n$ such that $L = \{ (x, t_1, \dots, t_d) \in \mathbb{C}^n \mid t = 0 \}$ and we write

$$b(\langle t, D_t \rangle) = \sum_{|\beta| = N} \frac{N!}{\beta!} t^\beta D_t^\beta$$

As all $D_t^\beta$ for $|\beta| = N$ are in the ideal generated by $p_1(D_x, Dt), \dots, p_n(D_x, Dt)$ the proof is the same then before. □

**2.4. Proof of the main theorem.** — Let $\varphi : Y \to X$ be an algebraic map. A vector field $u$ on $Y$ is said to be *tangent to the fibers of* $\varphi$ if $u(f \circ \varphi) = 0$ for all $f$ in $\mathcal{O}_X$. A differential operator $P$ is said to be *invariant under* $\varphi$ if there exists a $k$-endomorphism $A$ of $\mathcal{O}_X$ such that $P(f \circ \varphi) = A(f) \circ \varphi$ for all $f$ in $\mathcal{O}_X$. If we assume from now that $\varphi$ is dominant, $A$ is uniquely determined by $P$ and is a differential operator on $X$. We denote by $A = \varphi_*(P)$ the image of $P$ in $\mathcal{D}_X$ under this ring homomorphism.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\mathfrak{W}$ the Weyl group $\mathfrak{W}(\mathfrak{g}, \mathfrak{h})$. The Chevalley theorem shows that $\mathcal{O}(\mathfrak{g})^G$ is equal to $k[P_1, \dots, P_l]$ where $(P_1, \dots, P_l)$ are algebraically independent invariant polynomials and $l$ is the rank of $\mathfrak{g}$, that the set of polynomials on $\mathfrak{h}$ invariant under $\mathfrak{W}$ is $\mathcal{O}(\mathfrak{h})^{\mathfrak{W}} = k[p_1, \dots, p_l]$ where $p_j$ is the restriction to $\mathfrak{h}$ of $P_j$ and that the restriction map $P \mapsto P|_{\mathfrak{h}}$ defines an isomorphism of $\mathcal{O}(\mathfrak{g})^G$ onto $\mathcal{O}(\mathfrak{h})^{\mathfrak{W}}$ [**22**, §4.9.]. The space $W = \mathfrak{h}/\mathfrak{W}$ is thus isomorphic to $k^l$ and the functions $P_1, \dots, P_l$ define two morphisms $\psi : \mathfrak{g} \to W$ and $\varphi : \mathfrak{h} \to W$ by $\psi(x) = (P_1(x), \dots, P_l(x))$ and $\varphi(z) = (p_1(z), \dots, p_l(z))$.

An operator $Q$ of $\mathcal{D}_{\mathfrak{g}}^G$ transforms invariant functions into invariant functions hence is invariant under $\psi$ and $\psi_*(Q)$ is a differential operator on $W$. A vector field of $\tau(\mathfrak{g})$ annihilates the functions $P_1, \dots, P_l$ hence is tangent to the fibers of $\psi$. In the same way, let $\mathcal{D}_{\mathfrak{h}}^{\mathfrak{W}}$ be the space of differential operators on $\mathfrak{h}$ which are invariant under the action of the Weyl group $\mathfrak{W}$, they are invariant under $\varphi$ and define operators on $W$ through $\varphi_*$.

Let $\mathcal{D}(\mathfrak{g})^G$ (resp. $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$) be the set of global sections of $\mathcal{D}_{\mathfrak{g}}^G$ (resp. of $\mathcal{D}_{\mathfrak{h}}^{\mathfrak{W}}$). The morphism of Harish-Chandra [**5**] is a morphism of sheaves of rings $\delta : \mathcal{D}(\mathfrak{g})^G \to \mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$ which satisfies the following properties:

(1) If $f \in \mathcal{O}(\mathfrak{g})^G \simeq \mathcal{O}(\mathfrak{h})^{\mathfrak{W}}$ then $\delta(P)(f|_{\mathfrak{h}}) = \Delta^{1/2} P(f) \Delta^{-1/2}|_{\mathfrak{h}}$.

(2) If $f \in \mathcal{O}(\mathfrak{g})^G$, $\delta(f)$ is the restriction of $f$ to $\mathfrak{h}$

(3) If $f \in S(\mathfrak{g})^G$ and $f$ is considered as a constant coefficients operator, then $\delta(f)$ is the restriction of $f$ to $\mathfrak{h}^*$.

(4) The morphism $\delta$ is surjective onto $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$.

(5) The kernel of $\delta$ is $\mathcal{D}(\mathfrak{g})^G \cap \mathcal{D}(\mathfrak{g}) \tau(\mathfrak{g})$.

The last two results have been proved algebraically by Levasseur and Stafford in [**15**] and [**16**]. Let $E$ be the Euler vector field of $\mathfrak{g}$ and $\vartheta$ the Euler vector field of $\mathfrak{h}$. The function $\Delta$ is homogeneous of degree $n-l$ (2.2) hence $\delta(E)$ is equal to $\vartheta-(n-l)/2$.

Let $\mathcal{D}_W[d^{-1}]$ be the sheaf of differential operators on $W$ with poles on $\{d=0\}$ and $\mathcal{D}(W)[d^{-1}]$ be the ring of its global sections. The function $\Delta$ is invariant hence of the form $d(P_1,\dots,P_l)$ and the formula $Q\mapsto d^{1/2}Qd^{-1/2}$ defines an isomorphism $\gamma$ of $\mathcal{D}_W[d^{-1}]$. We get a diagram:

$$\begin{array}{ccc} \mathcal{D}(\mathfrak{g})^G & \xrightarrow{\ \delta\ } & \mathcal{D}(\mathfrak{h})^{\mathfrak{W}} \\ {\scriptstyle\psi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*} \\ \mathcal{D}(W)[d^{-1}] & \xrightarrow{\ \gamma\ } & \mathcal{D}(W)[d^{-1}] \end{array} \tag{5}$$

If $f$ is a polynomial on $W$ and $Q$ an operator of $\mathcal{D}_{\mathfrak{g}}^G$ we have $\varphi_*(\delta(P))(f)=\gamma(\psi_*(P))(f)$ from the definitions hence the diagram is commutative. We can avoid the denominators $[d^{-1}]$ in the diagram because of the following lemma:

***Lemma 2.4.1**.* *The morphism $\gamma$ sends the image of $\psi_*$ into $\mathcal{D}(W)$ while its inverse $\gamma^{-1}$ sends the image of $\varphi_*$ into $\mathcal{D}(W)$.*

*Proof.* This commutativity of the diagram shows that if an operator of $\mathcal{D}(W)$ is in the range of $\psi_*$ then its image under $\gamma$ is in $\mathcal{D}(W)$.

Conversely let us choose a positive system of roots for $(\mathfrak{g},\mathfrak{h})$ and define a function by $\pi=\Pi_{\alpha>0}\alpha$. Then $\pi$ is a product of distinct linear forms, its square $\pi^2$ is equal to the restriction of $\Delta$ to $\mathfrak{h}$ and it is changed to $-\pi$ under a reflection of the Weyl group.

Let $P\in\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}$ and $f\in\mathcal{O}_{\mathfrak{h}}^{\mathfrak{W}}$, by definition the function $Pf$ is invariant under $\mathfrak{W}$ while the function $\pi^{-1}P(\pi f)$ is in $\mathcal{O}_{\mathfrak{h}}[\pi^{-1}]$ and is invariant under $\mathfrak{W}$. Hence the function $\tau=P(\pi f)$ is in $\mathcal{O}_{\mathfrak{h}}$ and changes its sign under the action of reflections.

Let $z$ be a point of $\{\pi=0\}$, there exists a root $\alpha$ such that $\alpha(z)=0$. Let $s$ be the reflection which let the hyperplane $\{\alpha=0\}$ invariant. We have $\tau(z)=\tau(z^s)=-\tau(z)$ hence $\tau(z)=0$. As $\tau$ vanishes on $\{\pi=0\}$ and $\pi$ has multiplicity 1, $\tau$ is divisible by $\pi$ and $\pi^{-1}P(\pi f)$ has no denominator.

So the operator $\pi^{-1}P\pi$ is in $\mathcal{D}(\mathfrak{h})^{\mathfrak{W}}[\pi^{-1}]$ but applied to an invariant polynomial it gives a polynomial. Its image under $\varphi_*$ is thus a differential operator of $\mathcal{D}(W)[d^{-1}]$ which sends any polynomial to a polynomial hence an operator of $\mathcal{D}(W)$. □

Let $F$ be an (H-C)-type subsheaf of $\mathcal{D}_{\mathfrak{g}}^G$, we define four $\mathcal{D}$-modules:

$\mathcal{M}_F$ is the (H-C)-type $\mathcal{D}_{\mathfrak{g}}$-module. It is equal to the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and $F$.

- $\mathcal{N}_F$ is the quotient of $\mathcal{D}_W$ by the ideal generated by $\psi_*(F)$.
- $\mathcal{M}_F^{\mathfrak{h}}$ is the quotient of $\mathcal{D}_{\mathfrak{h}}$ by the ideal generated by $\delta(F)$.
- $\mathcal{N}_F^{\mathfrak{h}}$ is the quotient of $\mathcal{D}_W$ by the ideal generated by $\varphi_*(\delta(F))$.

Let $1_{\mathfrak{g}\to W}$ be the canonical generator of $\mathcal{D}_{\mathfrak{g}\to V}$ as defined in the proof of corollary 1.4.4 and $u_{\mathfrak{g}\to W}$ its class in $\psi^+\mathcal{N}_F$. We denote by $\mathcal{M}_F^0$ the $\mathcal{D}_{\mathfrak{g}}$-submodule of $\psi^+\mathcal{N}_F$ generated by $u_{\mathfrak{g}\to W}$.

***Theorem 2.4.2***. — *The module $\mathcal{M}_F^0$ is conic-tame.*

In this section we prove this theorem and in the next section we prove that $\mathcal{M}_F$ is isomorphic to $\mathcal{M}_F^0$.

***Proposition 2.4.3***. — *Let $n$ be the dimension of $\mathfrak{g}$, $l$ its rank. Then there exit some positive integer $N$ such that*

$$b(T) = (T-N)\cdots T(T+1)\cdots\Big(T+\frac{n-l}{2}\Big)$$

*is a quasi-b-function of total weight $(n+l)/2$ for $\mathcal{N}_F$ along $\{0\}$. Moreover, $N=0$ if $\sigma(F)=S_+(\mathfrak{g})$.*

*Proof*. — We recall that the rank $l$ of the algebra $\mathfrak{g}$ is the dimension of a Cartan subalgebra and that the degrees $n_1,\dots,n_l$ of the generators $P_1,\dots,P_l$ of $\mathcal{O}(\mathfrak{g})^G$ are called the primitive degrees of $\mathfrak{g}$ and that their sum is $(n+l)/2$ [**22**]. The map $\psi:\mathfrak{g}\to W$ is defined by $(P_1,\dots,P_l)$, hence if $E=\sum x_iD_{x_i}$ is the Euler vector field of $\mathfrak{g}$, $\eta=\psi_*(E)$ is equal to $\sum n_it_iD_{t_i}$.

The morphism $\delta$ is graded and its restriction to $S(\mathfrak{g})^G$ is the map $Q\mapsto q=Q|_{\mathfrak{h}}$ hence $\sigma(\delta(F))$ the set of principal symbols contains a power of $S_+(\mathfrak{h})^{\mathfrak{W}}$ (and is equal to $S_+(\mathfrak{h})^{\mathfrak{W}}$ if $\sigma(F)=S_+(\mathfrak{g})$). We may then apply proposition 2.3.4 to the module $\mathcal{M}_F^{\mathfrak{h}}$ and we find that its $b$-function is equal to $b_0(\vartheta)=\vartheta(\vartheta-1)\cdots(\vartheta-M)$ where $\vartheta$ is the Euler vector field of $\mathfrak{h}$ and $M$ is a positive integer equal to $(n-l)/2$ if $\sigma(\delta(F))=S_+(\mathfrak{h})^{\mathfrak{W}}$. This means that there exist differential operators $R,A_1,\dots,A_l$ on $\mathfrak{h}$ such that $R$ is of order $-1$ for the $V$-filtration in $\{0\}$ and

$$b_0(\vartheta)+R(z,D_z)=A_1(z,D_z)q_1(z,D_z)+\cdots+A_l(z,D_z)q_l(z,D_z)$$

The action of $\mathfrak{W}$ on $\mathcal{D}_{\mathfrak{h}}$ does not affect the $V$-filtration and $b_0(\vartheta)$ and all $q_i(z,D_z)$ are invariant under the Weyl group, so if we take the mean value (that is $\frac{1}{\#\mathfrak{W}}\sum_{w\in\mathfrak{W}}P^w$) we find the same relation with $R$ and all $A_i$ invariant under $\mathfrak{W}$.

Applying $\varphi_*$ and $\gamma^{-1}$ we find

$$b_0(\gamma^{-1}(\varphi_*(\vartheta)))+\gamma^{-1}(\varphi_*(R))=B_1\psi_*(Q_1)+\cdots+B_l\psi_*(Q_l) \tag{6}$$

with $B_1,\dots,B_l$ in $\mathcal{D}(W)$ (lemma 2.4.1) and $\gamma^{-1}(\varphi_*(q_i))=\psi_*(Q_i)$ (the diagram 5 is commutative).

As $\varphi=(p_1,\dots,p_l)$ and $p_i$ has degree $n_i$, $\varphi_*(\vartheta)$ is equal to $\eta=\sum n_it_iD_{t_i}$. We have $\gamma^{-1}(\eta)=d^{-1/2}\varphi_*(\vartheta)d^{1/2}=\varphi_*(\Delta^{-1/2}\vartheta\Delta^{1/2})$ and the function $\Delta$ is homogeneous of degree $n-l$ hence $\Delta^{-1/2}\vartheta\Delta^{1/2}=\vartheta+(n-l)/2$ and $\gamma^{-1}(\eta)=\eta+(n-l)/2$. □

***Proposition 2.4.4.*** — *For each nilpotent orbit $S$ of codimension $r$, $\mathcal{M}_F^0$ has a $b$-function of total weight $(n+r)/2$ along $S$ equal to*

$$b(T) = (T-N)\cdots T(T+1)\cdots\Big(T+\frac{n-l}{2}\Big)$$

*with $N=0$ if $\sigma(F) = S_+(\mathfrak{g})$. Here $n$ is the dimension of $\mathfrak{g}$ and $l$ its rank.*

*All roots of $b$ are strictly greater than $-(n+r)/2$ hence this $b$-function is tame.*

*Proof.* — Let us consider first the null orbit $S=\{0\}$. We apply corollary 1.4.3 to $X' = \mathfrak{g}$, $X = W$, $f = \psi$, $W' = \mathfrak{g}$, $\varphi'$ is the identity map of $\mathfrak{g}$, $\varphi$ the identity map of $W$ and $F : \mathfrak{g}^Y \to W$ is the map $\psi$. The weights $(m'_1,\dots,m'_n)$ on $\mathfrak{g}$ are $(1,\dots,1)$, that is the $V$-filtration on $\mathfrak{g}$ is the usual $v$-filtration relative to $\{0\}$, and the weights $(m_1,\dots,m_l)$ on $W$ are the primitive degrees considered in the proof of Proposition 2.4.3. Then $F$ is quasi-homogeneous and we get directly the result for $S=\{0\}$.

Consider now the nilpotent orbit $S$ of maximal dimension, then $S$ is the smooth part of the nilpotent cone and $\psi : \mathfrak{g}\to W$ is smooth on $S$. We apply corollary 1.4.3 to $X' = \mathfrak{g}$, $X = W$, $f=\psi$, $W' = W$, $\varphi' = \psi$, $\varphi$ and $F$ are both the identity map of $W$. The weights on $\mathfrak{g}$ and on $W$ are the weights $(m_1,\dots,m_l)$ considered on $W$ in the case of the null orbit.

We consider now a non null nilpotent orbit $S$. Let $X\in S$, by the Jacobson-Morozov theorem, we can find $H$ and $Y$ in $\mathfrak{g}$ such that $(H,X,Y)$ is a $\mathfrak{sl}_2$-triple. They generate a Lie algebra isomorphic to $\mathfrak{sl}_2$ which acts on $\mathfrak{g}$ by the adjoint representation. The theory of $\mathfrak{sl}_2$-representations shows $\mathfrak{g}$ splits into a direct sum $\bigoplus_{i=1}^r E(\lambda_i)$ of irreducible submodules. The dimension of $E(\lambda_i)$ is $\lambda_i+1$ hence $n = \sum(\lambda_i+1)$. Moreover $\mathfrak{g} = [X,\mathfrak{g}]\oplus\mathfrak{g}^Y$, $\dim\mathfrak{g}^Y = r$ and we can select a basis $(Y_1,\dots,Y_r)$ of $\mathfrak{g}^Y$ such that $[H,Y_i] = -\lambda_i Y_i$. The tangent space to $S$ at $X$ is $[X,\mathfrak{g}]$ hence $r$ is the codimension of $S$.

The map $\nu : G\times\mathfrak{g}^Y\to\mathfrak{g}$ given by $\nu(g,Z) = g\cdot(X+Z)$ is a submersion because its tangent map is the map $\mathfrak{g}\times\mathfrak{g}^Y\to\mathfrak{g}$ given by $(Z',Z)\mapsto[Z',X]+Z$. Let $\mathfrak{g}_1$ be a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}^X\oplus\mathfrak{g}_1$, we have $[\mathfrak{g},X] = [\mathfrak{g}_1,X]$. We choose functions $(\alpha_1,\dots,\alpha_r)$ on $G$ whose differentials at the unit $e$ of $G$ are the equations of $\mathfrak{g}_1$ in $\mathfrak{g}$ and define $A = \{\, g\in G \mid \alpha_1(g) = \cdots = \alpha_r(g)\,\}$. Then there is a Zarisky open subset $U$ of $A\times\mathfrak{g}^Y$ containing $(e,0)$ which is smooth and such that the map $\nu : U\to\mathfrak{g}$ is étale.

Let $(s_1,\dots,s_r)$ be the coordinates of $\mathfrak{g}^Y$ associated with the basis $(Y_1,\dots,Y_r)$, they define functions $(s_1,\dots,s_r)$ on $U$ and $\nu(s^{-1}(0))$ is equal to $S$. Let $\eta_0 = \nu^*E$ on $U$ ($E$ be the Euler vector field of $\mathfrak{g}$). A standard calculation [**23**, Part I, §5.6], shows that $\eta_0(s_i) = (\lambda_i/2+1)s_i$ hence the map $F_0 : \mathfrak{g}^Y\to\mathfrak{g}$ defined by $F_0(Z) = X+Z$ is quasi-homogeneous if the weights on $\mathfrak{g}^Y$ are $m'_i = (\lambda_i/2+1)$ for $i=1,\dots,r$ and the weights on $\mathfrak{g}$ are $(1,\dots,1)$. The map $F : \mathfrak{g}^Y\to W$ defined by $F(Z) = \psi(X+Z)$ is thus quasi-homogeneous with the weights $(m'_1,\dots,m'_r)$ on $\mathfrak{g}^Y$ and $(m_1,\dots,m_l)$ on $W$.

Now, we apply corollary 1.4.3 to $X' = U$, $X = W$, $f = \psi_\circ\nu$, $W' = \mathfrak{g}^Y$, $\varphi'$ the projection $U \to \mathfrak{g}^Y$ and $F : \mathfrak{g}^Y \to W = k^l$ given by $F(Z) = \psi(X + Z)$. This gives the $b$-function for $\nu^+\mathcal{M}_F^0$ and thus for $\mathcal{M}_F^0$ by lemma 1.2.5. □

Let us now consider the non-nilpotent strata of the stratification of $\mathfrak{g}$ (§ 2.2):

***Proposition 2.4.5***. — *The module $\mathcal{M}_F^0$ admits a tame quasi-b-function $b_S$ along each stratum $S$.*

*More precisely, if the stratum is $S_{(P,\mathfrak{O})}$ according to definition (4) and $\mathfrak{q}_P$ the associated semi-simple Lie subalgebra of $\mathfrak{g}$, then*

a) *$b_S$ depends only on $P$ and its roots are integers greater or equal to $-(m - k)/2$ where $m$ is the dimension of $\mathfrak{q}_P$ and $k$ its rank.*

b) *The total weight of $b_S$ is equal to $(m + r)/2$ where $r$ is the codimension of $\mathfrak{O}$ in $\mathfrak{q}_P$.*

*In particular, on the stratum of codimension 1 in $\mathfrak{g}$, the roots of the usual b-function of $\mathcal{M}_F^0$ are half integers greater or equal to $-1/2$.*

*Proof*. — We fix a Cartan subalgebra of $\mathfrak{g}$ and a subset $P$ of roots with the notations of § 2.2. This define a semi-simple algebra $\mathfrak{q}_P$ to which are associated the maps $\psi_P : \mathfrak{q}_P \to W_P$ and $\varphi_P : \mathfrak{h}_P \to W_P$. Here $W_P$ is a vector space of dimension the rank of $\mathfrak{q}_P$. The Cartan subalgebra $\mathfrak{h}$ splits into the direct sum $\mathfrak{h} = \mathfrak{h}_P \oplus \mathfrak{h}_P^\perp$ and this define a map $\varphi'_P = \varphi_P \otimes 1 : \mathfrak{h} \to W = k^l \oplus \mathfrak{h}_P^\perp$.

Let $S \in \mathfrak{h}_P^\perp$, we know from section 2.2 that $\mathfrak{g}^S = \mathfrak{h}_P^\perp \oplus \mathfrak{q}_P$ and as $S$ is semisimple we have $\mathfrak{g} = [\mathfrak{g}, S] \oplus \mathfrak{g}^S$. The map $\nu : G \times \mathfrak{g}^S \to \mathfrak{g}$ defined by $\nu(g, Z) = g \cdot (Z + S)$ is thus a submersion. Let $(\alpha_1, \dots, \alpha_r)$ be functions on $G$ whose differentials at $e$ are the equations of $[\mathfrak{g}, S]$ in $\mathfrak{g}$ and define $A = \{\, g \in G \mid \alpha_1(g) = \cdots = \alpha_r(g) \,\}$. Then there is a Zarisky open subset $U$ of $A \times \mathfrak{g}^S$ containing $(e, 0)$ which is smooth and such that the map $\nu : U \to \mathfrak{g}$ is étale. Let $\psi' : U \to W$ be defined as the composition of the canonical projection $A \times \mathfrak{g}^S \to \mathfrak{g}^S$ and of $\psi_P$.

Now we follow the proof of proposition 2.4.3 with the same notations. Applying the second part of proposition 2.3.4 to $L = \{0\} \times (\mathfrak{h}_P^\perp)'$, we find that $\mathcal{N}_F$ admits a monodromic $b$-function along $L$ which is equal to $b_0(\vartheta_P) = \vartheta_P(\vartheta_P - 1) \cdots (\vartheta_P - N' + 1)$ $(\vartheta_P - N')$ where $\vartheta_P$ is the Euler vector field of $\mathfrak{h}_P$ and $N'$ is less or equal to $N = (n - l)/2$ with $n$ the dimension of $\mathfrak{g}$. This means that there exists $l$ differential operators $R, A_1, \dots, A_l$ on $\mathfrak{h}$ that we may assume invariant under $\mathfrak{W}_P$, with $R$ of order $-1$ for the $V$filtration associated with $L$ such that $b_0(\vartheta_P) + R = A_1(z, D_z)q_1(D_z) + \cdots + A_l(z, D_z)q_l(D_z)$. If $\lambda = 0$ we have $R = 0$.

As these operators are invariant under $\mathfrak{W}_P$ hence under $\varphi'$ we may apply $\varphi'_*$ and $\gamma^{-1}$ and find an equation $b_0(\gamma^{-1}(\eta)) + \gamma^{-1}\varphi_*(R) = B_1\psi_*(Q_1) + \cdots + B_l\psi_*(Q_l)$ with $B_1, \dots, B_l$ in $\mathcal{D}(W)$ and $\eta = \varphi_*(\vartheta)$. In the coordinates of $W$ defined by the isomorphism $\varphi' : k^l \oplus (\mathfrak{h}_P^\perp)' \to W$, the vector $\eta$ is equal to $\sum n_i t_i D_{t_i}$ where the $n_i$ are the primitive degrees of $\mathfrak{h}_P$, it is associated with the manifold $L' = \varphi'(\{0\} \oplus (\mathfrak{h}_P^\perp)')$.

As $\Delta$ is the product of $\Delta_P$ by a function which does not vanish on a neighborhood of $L$, the function $d$ which defines the morphism $\gamma$ is the product of the corresponding function $d_P$ associated with $\mathfrak{h}_P$ by a function $\varrho$ which does not vanish in a neighborhood of $L'$. So we have

$$\gamma^{-1}(\eta) = \varrho^{-1/2} d_P^{-1/2} \eta d_P^{1/2} \varrho^{1/2} = \varrho^{-1/2} (\eta + N_P) \varrho^{1/2} = (\eta + N_P) + a$$

where $N_P$ is $(m-k)/2$ ($m$ is the dimension of $\mathfrak{q}_P$, $k$ its rank) and $a$ is a function which vanishes on $L'$ hence of order at most $-1$ for the $V^\eta$-filtration. The operator $\gamma^{-1}\varphi_*(R)$ is also of order $-1$ for the $V^\eta$-filtration.

We have proved that $\mathcal{N}_F$ admits a $b(\eta)$-function along $L'$ which is equal to $b(T) = (T - N_P)\cdots(T - N_P + N)$. The end of the proof is the same than to the proof of proposition 2.4.4. □

Proposition 2.4.5 shows that $\mathcal{M}_F^0$ is tame. To prove theorem 2.4.2 we have still to prove that it is conic. This come from the fact that the singular support of $\mathcal{M}_F^0$ is conic for the Euler vector field of $\mathfrak{g}$ and the vector fields associated with the strata are equal to this Euler vector field modulo vector fields tangent to the orbits.

**2.5. Isomorphism with the inverse image.** — We recall that $\mathcal{M}_F^0$ is the submodule of $\psi^+\mathcal{N}_F$ generated by $u_{\mathfrak{g}\to W}$, it is the image of the morphism $\mathcal{M}_F \to \psi^+\mathcal{N}_F$.

***Theorem 2.5.1***. *The canonical morphism $\mathcal{M}_F \to \mathcal{M}_F^0$ is an isomorphism.*

*Proof*

*1st step: From semi-simple Lie algebras to reductive algebras.* — Assume that the result has been proved for semi-simple Lie algebras and let $\mathfrak{g}$ be a reductive algebra, direct sum of its center and a semisimple Lie algebra. By induction, we may assume that $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$ with $\mathfrak{c}$ subspace of the center of dimension 1 and $\mathfrak{g}'$ reductive Lie algebra for which the result has been proved.

Let $t$ a coordinate of $\mathfrak{c}$ and $\tau$ the corresponding coordinate of the dual space $\mathfrak{c}^*$. By the hypothesis, there is a differential operator in $F$ whose principal symbol is equal to some power $\tau^q$. This means that $\mathfrak{g}' = \{t = 0\}$ is non characteristic for $\mathcal{M}_F$. Let $\mathcal{K}$ be the kernel of $\mathcal{M}_F \to \mathcal{M}_F^0$. We have an exact sequence $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M}_F \longrightarrow \mathcal{M}_F^0 \longrightarrow 0$ of non characteristic $\mathcal{D}_\mathfrak{g}$-modules. As the inverse image is an exact functor in the non characteristic case, this gives an exact sequence $0 \longrightarrow \mathcal{K}/t\mathcal{K} \longrightarrow \mathcal{M}_F/t\mathcal{M}_F \longrightarrow \mathcal{M}_F^0/t\mathcal{M}_F^0 \longrightarrow 0$. If we prove that $\mathcal{K}/t\mathcal{K} = 0$, we will have $\mathcal{K} = 0$ (as $\mathcal{K}$ is non characteristic).

So, we have to prove that $\mathcal{M}_F/t\mathcal{M}_F \to \mathcal{M}_F^0/t\mathcal{M}_F^0$ is injective. Here we use the same proof than in [**13**, Lemma 2.2.3.]. In fact, as $\mathcal{D}_{\mathfrak{g}'}$-module $\mathcal{M}_F/t\mathcal{M}_F$ is generated by the classes of $1, D_t, \ldots, D_t^{q-1}$ and the submodule generated by $D_t^{q-1}$ is a module on $\mathfrak{g}'$ of the same type than $\mathcal{M}_F$ for which the theorem is true. Then we consider the quotient of $\mathcal{M}_F/t\mathcal{M}_F$ by the module generated by $D_t^{q-1}$ and argue by induction.

*2nd step: The result is true at points $X \in \mathfrak{g}$ whose semi-simple part is non zero*

By the first step, we may assume that $\mathfrak{g}$ is semisimple. Let $S$ be a non zero semisimple element of $\mathfrak{g}$, $\mathfrak{g}^S$ its centralizer and $G^S$ the corresponding group. The spaces $\mathcal{O}(\mathfrak{g})^G$ and $\mathcal{O}(\mathfrak{g}^S)^{G^S}$ are isomorphic hence the space $W_S$ associated with $\mathfrak{g}^S$ is equal to $W$ and the map $\psi_S : \mathfrak{g}^S \to W$ is the restriction of $\psi : \mathfrak{g} \to W$. Thus the sheaf of differential operators on $\mathfrak{g}^S$ invariant under the action of $G^S$ is isomorphic to $\mathcal{D}_{\mathfrak{g}}^G$.

By induction on the dimension of $\mathfrak{g}$, we may assume that the theorem is true for $\mathfrak{g}^S$ hence that the morphism $\mathcal{M}_F^S \to \psi_S^+ \mathcal{N}_F$ is injective. Here $\mathcal{M}_F^S$ is the $\mathcal{D}_{\mathfrak{g}^S}$ module associated with $F$ and $\mathcal{N}_F$ the quotient of $\mathcal{D}_W$ by the ideal generated by $F$. By definition, the germ at $S$ of $\psi^+\mathcal{N}_F$ is $(\psi^+\mathcal{N}_F)_S = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} (\psi_S^+\mathcal{N}_F)_S$. On the other hand, we have $(\mathcal{D}_{\mathfrak{g}}/\mathcal{D}_{\mathfrak{g}}\tau(\mathfrak{g}))_S = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} (\mathcal{D}_{\mathfrak{g}^S}/\mathcal{D}_{\mathfrak{g}^S}\tau(\mathfrak{g}^S))_S$ hence $\mathcal{M}_{F,S} = \mathcal{O}_{\mathfrak{g},S} \otimes_{\mathcal{O}_{\mathfrak{g}^S,S}} \mathcal{M}_{F,S}^S$. The morphism $\mathcal{M}_F \to \psi^+\mathcal{N}_F$ is thus injective at the point $S$ hence at all the orbits whose closure contains $S$ that is in particular at all points $X$ whose semisimple part in the Jordan decomposition is $S$.

*3nd step: The case of nilpotent orbits.* Let $\mathcal{K}$ be the kernel of $\mathcal{M}_F \to \mathcal{M}_F^0$. By the second step, we may assume that the theorem is true at all non nilpotent points of $\mathfrak{g}$ that is that $\mathcal{K}$ is supported by the nilpotent cone. Let $\mathcal{K}(\mathfrak{g})^G$ be the set of global sections of $\mathcal{K}$ invariant under $G$, we get an exact sequence

$$0 \longrightarrow \mathcal{K}(\mathfrak{g})^G \longrightarrow \mathcal{M}(\mathfrak{g})^G \longrightarrow \mathcal{M}_0(\mathfrak{g})^G \longrightarrow 0$$

and by [**7**, lemma 3.2.] we have $\mathcal{M}(\mathfrak{g})^G = \mathcal{M}_0(\mathfrak{g})^G = \mathcal{N}_F$ hence $\mathcal{K}(\mathfrak{g})^G = 0$. Then $\mathcal{K} = 0$ by [**17**, lemma 3.2.]. □

Remark that the third step is also a consequence of the property (5) of the Harish-Chandra morphism which has been proved by Levasseur-Stafford [**16**].

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