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SUB-ACTIONS FOR ANOSOV DIFFEOMORPHISMS

by

Artur O. Lopes & Philippe Thieullen

Dedicated to Jacob Palis

Abstract. — We show a positive Livsic type theorem for $\mathcal{C}^2$ Anosov diffeomorphisms f on a compact boundaryless manifold M and Hölder observables A. Given $A : M \to \mathbb{R}$, α-Hölder, we show there exist $V : M \to \mathbb{R}$, β-Hölder, $\beta < \alpha$, and a probability measure μ, f-invariant such that

$$A \leqslant V \circ f - V + \int A \, d\mu.$$

We apply this inequality to prove the existence of an open set $\mathcal{G}_\beta$ of β-Hölder functions, β small, which admit a unique maximizing measure supported on a periodic orbit. Moreover the closure of $\mathcal{G}_\beta$, in the β-Hölder topology, contains all α-Hölder functions, α close to one.

1. Introduction

We consider a compact riemannian manifold M of dimension $d \geqslant 2$ without boundary and a $\mathcal{C}^2$ transitive Anosov diffeomorphism $f : M \to M$. The tangent bundle TM admits a continuous Tf-invariant splitting $TM = E^u \bigoplus E^s$ of expanding and contracting tangent vectors. We assume M is equiped with a riemannian metric and there exists a constant $C(M)$, depending only on M and the metric and constants depending on f

$$0 < \Lambda_s < \lambda_s < 1 < \lambda_u < \Lambda_u$$

such that for all $n \in \mathbb{Z}$

$$\begin{cases} C(M)^{-1}\lambda_u^n \leqslant \|T_x f^n \cdot v\| \leqslant C(M)\Lambda_u^n & \text{for all } v \text{ in } E_x^u, \\ C(M)^{-1}\Lambda_s^n \leqslant \|T_x f^n \cdot v\| \leqslant C(M)\lambda_s^n & \text{for all } v \text{ in } E_x^s. \end{cases}$$

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Livsic theorem [**5**] asserts that, if $A : M \to M$ is a given Hölder function and satisfies $\int A\,d\mu = 0$ for all f-invariant probability measure μ, then A is equal to a coboundary V (which is Hölder too), that is:

$$A = V \circ f - V.$$

What hapens if we only assume $\int A\,d\mu \geqslant 0$ for all f-invariant probability measure μ? We denote by $\mathcal{M}(f)$, the set of f-invariant probability measures and $m(A,f) = \sup\big\{\int A\,d\mu \mid \mu \in \mathcal{M}(f)\big\}$.

For a β-Holder function V

$$\text{Höld}_\beta(V) = \sup_{0<d(x,y)} \Big\{ \frac{|V(x) - V(y)|}{d(x,y)^\beta} \Big\}.$$

We prove the following:

Theorem 1. — *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism on a compact manifold M without boundary. For any given α-Hölder function $A : M \to \mathbb{R}$, there exists a β-Hölder function $V : M \to \mathbb{R}$, that we call sub-action, such that:*

$$A \leqslant V \circ f - V + m(A,f),$$

and

$$\beta = \alpha \frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}, \quad \text{Höld}_\beta(V) \leqslant \frac{C(M)}{\min(1-\lambda_u^{-\alpha}, 1-\lambda_s^{\alpha})^2}\,\text{Höld}_\alpha(A)$$

where $C(M)$ is some constant depending only on M and the metric.

By analogy with Hamiltonian mechanics and the way we define V from A, we may interpret A as a lagrangian and V as a sub-action. This result extends a similar one we obtained in [**4**] for expanding maps of the circle (see [**2**] [**6**] for related results). The same techniques of [**4**] also apply for the one-directional shift as it is mentioned in [**4**].

The proof we give here is for bijective smooth systems, and we obtain V continuous in all M. Our result can not be derived (via Markov partition) directly from an analogous result for the bi-directional shift.

Corollary 2. — *The hypothesis are the same as in theorem 1. The following statements are equivalent:*

(i) $A \geqslant V \circ f - V$ *for some bounded measurable function* V,
(ii) $\int A\,d\mu \geqslant 0$ *for all f-invariant probability measure* μ,
(iii) $\sum_{k=0}^{p-1} A \circ f^k(x) \geqslant 0$ *for all $p \geqslant 1$ and point x periodic of period* p,
(iv) $A \geqslant V \circ f - V$ *for some Hölder function* V.

The proof of that corollary is straightforward and uses (for (iii) $\Rightarrow$ (ii)) the fact that the convex hull of periodic measures is dense in the set of all f-invariant probability measures for topological dynamical systems satisfying the shadowing lemma (see Lemma 5). F. Labourie suggested to us the following corollary:

Corollary 3. — *The hypothesis are the same as in theorem 1. If A satisfies $\int A\,d\mu \geqslant 0$ for all $\mu \in \mathcal{M}(f)$ and $\sum_{k=0}^{p-1} A \circ f^k(x) > 0$ for at least one periodic orbit x of period p then $\int A\,d\lambda > 0$ for all probability measure λ giving positive mass to any open set.*

Again the proof is straitforward: $R = A - V \circ f + V \geqslant 0$ for some continuous V and $\int R\,d\lambda = 0$ for such a measure λ implies $R = 0$ everywhere and in particular $\sum_{k=0}^{p-1} A \circ f(x) = 0$ for all periodic orbit x.

Any measure μ satisfying $\int A\,d\mu = m(A, f)$ is called a maximizing measure and since A is continuous, such a measure always exists. It is then natural to ask the following two questions: For which A, the set of maximizing measures is reduced to a single measure ? In the case there exists a unique maximizing measure, to what kind of compact set, the support of this measure looks like ?

The following theorem gives a partial answer for "generic" functions A.

Theorem 4. — *Let $f : M \to M$ be a $\mathcal{C}^2$ transitive Anosov diffeomorphism and $\beta < \ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$. Then there exists an open set $\mathcal{G}_\beta$ of β-Hölder functions (open in the $\mathcal{C}^\beta$-topology) such that:*

(i) *any A in $\mathcal{G}_\beta$ admits a unique maximizing measure μ_A;*

(ii) *the support of μ_A is equal to a periodic orbit and is locally constant with respect to $A \in \mathcal{G}_\beta$;*

(iii) *any α-Hölder function with $\alpha > \beta \ln(\Lambda_u/\lambda_s)/\ln(1/\lambda_s)$ is contained in the closure of $\mathcal{G}_\beta$ (the closure is taken with respect to the $\mathcal{C}^\beta$-topology).*

The proof of Theorem 4 is a simplification of what we gave in [**4**] in the one-dimensional setting. The existence of sub-actions is in both cases the main ingredient of the proof.

Now we will concentrate in one of our main results, namely, Theorem 1; the basic idea is the following: given a finite covering of M by open sets $\{U_1, \dots, U_l\}$ with sufficiently small diameter, we construct a Markov covering (and not a Markov partition) $\{R_1, \dots, R_l\}$ of rectangles: each R_i contains U_i and satisfies

$$x \in U_i \cap f^{-1}(U_j) \Longrightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j),$$

where $W^s(x, R_i)$ denotes the local stable leaf through x restricted to R_i. We then associate to each R_i a local sub-action V_i, defined on R_i by:

$$V_i(x) = \sup\big\{S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geqslant 0,\ y \in W^s(x, R_i)\big\}$$

where $\Delta^s(y, x)$ is a kind of cocycle along the stable leaf $W^s(x)$:

$$\Delta^s(y, x) = \sum_{n \geqslant 0} (A \circ f^n(y) - A \circ f^n(x)),$$

and where $S_n(A - m) = \sum_{k=0}^{n-1} (A - m) \circ f^k$.

This family $\{V_1, \dots, V_l\}$ of local sub-actions satisfies the inequality:

$$x \in U_i \cap f^{-1}(U_j) \Longrightarrow V_i(x) + A(x) - m \leqslant V_j \circ f(x)$$

and enable us to construct a global sub-action V:

$$V(x) = \sum_{i=1}^{l} \theta_i(x) V_i(x)$$

where $\{\theta_1, \dots, \theta_l\}$ is a smooth partition of unity associated to the covering $\{U_1, \dots, U_l\}$. The main difficulty is to prove that each V_i is Hölder on R_i.

2. Existence of sub-actions

We continue our description of the dynamics of transitive Anosov diffeomorphisms (for details information, see Bowen's monography [**3**]). All the results we are going to use depend on a small constant of expansiveness $\varepsilon^* > 0$ (by definition this constant says that any pseudo-orbit can be followed by true orbits) depending on f and M in the following way:

$$\varepsilon^* = C(M)^{-1} \min\Big(\frac{\lambda_u - 1}{\|D^2 f\|_\infty}, \frac{1-\lambda_s}{\|D^2 f\|_\infty}\Big)$$

where $C(M) \geqslant 1$ is a constant depending only on M and the riemannian metric. At each point x, one can define its local stable manifold $W^s_\varepsilon(x)$ for every $\varepsilon < \varepsilon^*$:

$$W^s_\varepsilon(x) = \big\{y \in M \mid d(f^n(x), f^n(y)) \leqslant \varepsilon \;\forall n \geqslant 0\big\}$$

which are $\mathcal{C}^2$ embeded closed disks of dimension $d^s = \dim E^s_x$ and tangent to E^s_x. In the same manner, $W^u_\varepsilon(x)$ is defined replacing f by f^{-1}. If two points x, y are close enough, $d(x,y) < \delta$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ have a unique point in common, called $[x,y]$:

$$[x,y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y) = W^s_{\varepsilon^*}(x) \cap W^u_{\varepsilon^*}(y),$$

where $\varepsilon = K^*\delta$ and K^* is again a large constant depending on M and f:

$$K^* = \frac{C(M)}{\min(1-\lambda_u^{-1}, 1-\lambda_s)}.$$

This estimate is in fact a particular case of Bowen's shadowing lemma:

***Lemma 5* (Bowen)**. — *If δ is small enough, $\delta < \varepsilon^*/K^*$, if $(x_n)_{n\in\mathbb{Z}}$ is a bi-infinite δ-pseudo-orbit, that is, $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$, then there exists a unique true orbit $\{f^n(x)\}_{n\in\mathbb{Z}}$ which ε-shadow $(x_n)_{n\in\mathbb{Z}}$, that is $d(f^n(x), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$ with $\varepsilon = K^*\delta$.*

This lemma (see [**3**]) for proof) is the main ingredient for constructing (dynamical) rectangles. A rectangle R is a closed set of diameter less than ε^*/K^* satisfying:

$$x, y \in R \Longrightarrow [x,y] \in R.$$

We will not use the notion of proper rectangles but will use instead the notion of Markov covering.

Definition 6. — Let $\mathcal{U} = \{U_1, \dots, U_l\}$ be a covering of M by open sets of diameter less than $\varepsilon^*/(K^*)^2$. We call a Markov covering associated to $\mathcal{U}$, a finite set $\mathcal{R} = \{R_1, \dots, R_l\}$ of rectangles of diameter less than ε^*/K^* satisfying:

$$\begin{aligned}
&U_i \subset R_i \\
&x \in U_i \cap f^{-1}(U_j) \Longrightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j) \\
&y \in f(U_i) \cap U_j \Longrightarrow f^{-1}(W^u(y, R_j)) \subset W^u(f^{-1}(y), R_i) \\
&\forall\, j,\ \exists\, i,\ f(U_i) \cap U_j \neq \varnothing
\end{aligned}$$

where $W^s(x, R_i) = W^s_{\varepsilon^*}(x) \cap R_i$ and $W^u(y, R_j) = W^u_{\varepsilon^*}(y) \cap R_j$.

An easy consequence of the shadowing lemma shows there always exist such Markov coverings:

Proposition 7. — *For every covering $\mathcal{U}$ of M by open sets such that the diameter of each U_i is less than $\varepsilon^*/(K^*)^2$, there exists a Markov covering $\mathcal{R}$ by rectangles of diameter less than ε^*/K^*.*

Proof. — Given $\mathcal{U} = \{U_1, \dots, U_l\}$ such a covering, we define the following compact space of $\varepsilon^*/(K^*)^2$ pseudo-orbits:

$$\Sigma = \{\omega = (\dots, \omega_{-2}, \omega_{-1} \mid \omega_0, \omega_1, \dots) \quad \text{s.t.} \quad U_{\omega_n} \cap f^{-1}(U_{\omega_{n+1}}) \neq \varnothing\}.$$

Here ω is a sequence of indices in $\{1, \dots, l\}$ and Σ is a subshift of finite type where $i \to j$ is a possible transition iff $U_i \cap f^{-1}(U_j)$ is not empty. Given such $\omega \in \Sigma$, we choose for all $n \in \mathbb{Z}$, $x_n \in U_{\omega_n}$ so that $f(x_n) \in U_{\omega_{n+1}}$. Then $(x_n)_{n\in\mathbb{Z}}$ is a $\varepsilon^*/(K^*)^2$ pseudo-orbit which corresponds to a unique true orbit $(f^n(x))_{n\in\mathbb{Z}}$ satisfying:

$$d(f^n(x), U_{\omega_n}) < \varepsilon^*/K^* \quad \forall\, n \in \mathbb{Z}.$$

Since ε^* is a constant of expansiveness, there can exists at most one point x satisfying the previous inequality for all n. We call that point $\pi(\omega)$ and notice that the map

$$\pi : \Sigma \to M$$

is surjective (for $\mathcal{U}$ is a covering), commutes with the left shift σ, $f \circ \pi = \pi \circ \sigma$, is continuous by expansiveness (in fact Hölder if Σ is equiped with the standard metric). Also notice that π may not be finite-to-one. We first construct a Markov cover on Σ as usual by the braket

$$[\omega, \omega'] = (\dots, \omega'_{-2}, \omega'_{-1} \mid \omega_0, \omega_1, \dots)$$

where $\omega = (\omega_n)_{n\in\mathbb{Z}}$, $\omega' = (\omega'_n)_{n\in\mathbb{Z}}$ and $\omega'_0 = \omega_0$. By uniqueness in the construction of $\pi(\omega)$, we get

$$\begin{aligned}
&\pi([\omega, \omega']) = [\pi(\omega), \pi(\omega')] \\
&\pi([i]) = R_i \text{ is a rectangle of } M \text{ containing } U_i \\
&\pi(W^s(\omega, [i])) = W^s(\pi(\omega), R_i) \quad \text{whenever } \omega_0 = i
\end{aligned}$$

where $[i]$, $i = 1, \dots, l$, is the cylinder $\{\omega \in \Sigma \mid \omega_0 = i\}$ and $W^s(\omega, [i])$ is the symbolic stable set $\{\omega' \in \Sigma \mid \omega'_n = \omega_n \ \forall n \geqslant 0\}$. (For the proof of the last equality, we just notice: if $x = \pi(\omega)$, $y \in W^s(x, R_i)$ and $y = \pi(\omega')$ then $\pi([\omega, \omega']) = y$ and $[\omega, \omega'] \in W^s(\omega, [i])$.) To finish the proof we only show

$$x \in U_i \cap f^{-1}(U_j) \Longrightarrow f(W^s(x, R_i)) \subset W^s(f(x), R_j).$$

Indeed, $x = \pi(\omega)$ for some $\omega = (\cdots, \omega_{-1} \mid i, j, \omega_2, \cdots)$ and

$$\sigma(W^s(\omega, [i]) \subset W^s(\sigma(\omega), [j]).$$

To conclude, we apply π on both sides. □

Definition 8. — Let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a Markov covering of M associated to some open covering $\mathcal{U} = \{U_1, \dots, U_l\}$. We define a local sub-action by

$$V_i(x) = \sup\{\, S_n(A - m) \circ f^{-n}(y) + \Delta^s(y, x) \mid n \geqslant 0, \ y \in W^s(x, R_i), \,\}$$

for $x \in U_i$, and where $S_n B = \sum_{k=0}^{n-1} B \circ f^k$, $\Delta^s(y, x) = \sum_{k \geqslant 0} (A \circ f^k(y) - A \circ f^k(x))$ and the supremum is taken over all $n \geqslant 0$ and points $y \in W^s(x, R_i)$.

Before showing V_i is a (finite!) Hölder function on each R_i, let's conclude the proof of Theorem 1:

Proof of Theorem 1. — Let $\mathcal{U} = \{U_1, \dots, U_l\}$ be an open covering of M, $\{R_1, \dots, R_l\}$ a Markov covering associated to $\mathcal{U}$ and $\{\theta_1, \dots, \theta_l\}$ a partition of unity adapted to $\mathcal{U}$. Let $\{V_1, \dots, V_l\}$ constructed as above and

$$V = \sum_i \theta_i V_i.$$

Suppose we have proved that $x \in U_i \cap f^{-1}(U_j)$ implies

$$V_i(x) + (A - m)(x) \leqslant V_j \circ f(x).$$

Multiplying this inequality by $\theta_i(x)\theta_j \circ f(x)$ and summing over i and j (whether or not $i \to j$ is a possible transition), we get

$$V(x) + (A - m)(x) \leqslant V \circ f(x) \quad (\forall x \in M).$$

We now prove the local sub-cohomological equation: if $x \in U_i \cap f^{-1}(U_j)$ and $y \in W^s(x, R_i)$, then $f(y) \in W^s(f(x), R_j)$ and

$$\begin{aligned} S_n(A - m) \circ f^{-n}(y) &+ \Delta^s(y, x) + (A - m)(x) \\ &= S_{n+1}(A - m) \circ f^{-(n+1)} \circ f(y) + \Delta^s(f(y), f(x)) \leqslant V_j \circ f(x). \end{aligned}$$

Taking the supremum over all $n \geqslant 0$ and all $y \in W^s(x, R_i)$, we get indeed

$$V_i(x) + (A - m)(x) \leqslant V_j \circ f(x).$$

That finishes the proof of theorem 1. □

We now come to our main technical lemma. We notice that, even in the case where A is Lipschitz, we only obtain a Hölder sub-action.

Lemma 9. — *If A is α-Hölder on M, R is a rectangle and V is defined as in Definition 8, then V is β-Hölder on R with exponent*

$$\beta = \alpha \frac{|\ln \lambda_s|}{\ln \Lambda_u + |\ln \lambda_s|} < \alpha.$$

Proof. — We divide the proof into four steps:

Step one. — If $d(x,x') < \varepsilon^*$ and x, x' are on the same stable leaf, then

$$\Delta^s(x,x') \leqslant \sum_{n\geqslant 0} |A\circ f^n(x) - A\circ f^n(x')| \leqslant C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha} d(x,x')^\alpha,$$

for some constant $C(M)$ depending only on M and the metric.

Indeed, it follows from the contraction $d(f^k(x), f^k(x')) \leqslant C(M)\lambda_s^k d(x,x')$ for $k\geqslant 0$ and the fact that A is α-Hölder.

Step two. — For every $n\geqslant 1$, $x,x'\in M$ such that $d(f^k(x), f^k(x')) < \varepsilon^*/K^*$ for all $0\leqslant k\leqslant n$, then

$$\sum_{k=0}^{n-1} |A\circ f^k(x) - A\circ f^k(x')| \leqslant K(M,f)\max(d(x,x')^\alpha, d(f^n(x), f^n(x'))^\alpha),$$

$$\text{where}\quad K(M,f) = C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha}, 1-\lambda_s^\alpha)^2}.$$

Indeed, one can build $w=[x,x']$; then on the one hand, $d(x,w)\leqslant\varepsilon^*$ and x,w are on the same stable leaf; on the other hand, $d(f^n(w), f^n(x'))\leqslant\varepsilon^*$ and $f^n(w)$ and $f^n(x')$ are on the same unstable leaf. We conclude by applying step one and the estimates:

$$d(x,w)\leqslant K^* d(x,x'),\quad d(f^n(w), f^n(x'))\leqslant K^* d(f^n(x), f^n(x')).$$

Step three. — We show that $V(x)$ is finite for every $x\in R$. It is precisely here that the choice of the normalizing constant $m(A,f)$ is important.

Indeed, since a transitive Anosov diffeomorphism is mixing (see [**3**]), there exists an integer $\tau^*\geqslant 1$ such that, for every finite orbit $\{f^{-n}(y),\dots,f^{-1}(y),y\}$, n arbitrary, $f^{\tau^*}(B(y,\varepsilon^*/K^*))$ contains $f^{-n}(y)$. Thanks to the shadowing lemma, there exists a periodic orbit z, of period $n+\tau^*$, satisfying

$$d(f^{-k}(z), f^{-k}(y))\leqslant\varepsilon^*\qquad(\forall\, k=0,1,\dots,n).$$

Using step two, $\sum_{k=1}^n (A\circ f^{-k}(y) - A\circ f^{-k}(z))$ is uniformly bounded in n by some constant $C(M,f)$. As any periodic orbit is associated to an invariant probability, then, $\sum_{k=1}^{n+\tau^*}(A\circ f^{-k}(z) - m(A,f))\leqslant 0$.

Without lost of generality we can assume $m(A,f)=0$. Therefore, we get

$$\begin{aligned}\sum_{k=1}^{n} A\circ f^{-k}(y) &\leqslant C(M,f)+\sum_{k=1}^{n+\tau^*} A\circ f^{-k}(z)+\tau^*\|A\|_\infty\\ &\leqslant C(M,f)+\tau^*\|A\|_\infty.\end{aligned}$$

Step four. — We finally prove that V is Hölder on R. Let $n\geqslant 0$, $x,x'\in R$, $y\in W^s(x,R)$ and define $y'=[x',y]$ belonging to R since R is a rectangle and to the same local unstable manifold as y. Then for some N we are going to choose soon: let $B=A-m(A,f)$,

$$\begin{aligned}S_nB\circ f^{-n}(y)+\Delta^s(y,x)\leqslant S_nB\circ f^{-n}(y')&+\Delta^s(y',x')\\ &+\sum_{k=-n}^{N-1}|A\circ f^k(y)-A\circ f^k(y')| \qquad (=\Sigma_1)\\ &+\sum_{k=0}^{N-1}|A\circ f^k(x)-A\circ f^k(x')| \qquad (=\Sigma_2)\\ &+|\Delta^s(f^N(y),f^N(x))| \qquad (=\Sigma_3)\\ &+|\Delta^s(f^N(y'),f^N(x'))| \qquad (=\Sigma_4)\end{aligned}$$

We now bound from above each Σ_i with respect to $d(x,x')$:

$$\begin{aligned}\Sigma_1 &\leqslant C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_u^{-\alpha}}d(f^N(y),f^N(y'))^\alpha,\\ \Sigma_2 &\leqslant C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha},1-\lambda_s^\alpha)^2}\max(d(x,x')^\alpha,d(f^N(x),f^N(x'))^\alpha),\\ \Sigma_3 &\leqslant C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha}d(f^N(y),f^N(x)),\\ \Sigma_4 &\leqslant C(M)\frac{\text{Höld}_\alpha(A)}{1-\lambda_s^\alpha}d(f^N(y'),f^N(x'))^\alpha.\end{aligned}$$

We now choose $N=N(x,x')$ by $\lambda_s^t\varepsilon^*=\Lambda_u^t d(x,x')$, $N=[t]+1$ and then choose $\widetilde{\varepsilon}\geqslant\varepsilon^*$ so that $\lambda_s^N\widetilde{\varepsilon}=\Lambda_u^N d(x,x')$. Then

$$\begin{gathered}d(f^N(x),f^N(x'))\leqslant C(M)\Lambda_u^N d(x,x')\leqslant C(M)\lambda_s^N\widetilde{\varepsilon},\\ d(f^N(y),f^N(x)) \text{ or } (f^N(y'),f^N(x'))\leqslant C(M)\lambda_s^N\varepsilon^*\leqslant C(M)\lambda_s^N\widetilde{\varepsilon}.\end{gathered}$$

In particular, we get first $d(f^N(y),f^N(y'))\leqslant 3C(M)\lambda_s^N\widetilde{\varepsilon}$ and next:

$$\begin{gathered}\Sigma_1+\cdots+\Sigma_4\leqslant 6C(M)\frac{\text{Höld}_\alpha(A)}{\min(1-\lambda_u^{-\alpha},1-\lambda_s^\alpha)^2}(\lambda_s^N\widetilde{\varepsilon})^\alpha=K(M,f)(\lambda_s^N\widetilde{\varepsilon})^\alpha,\\ S_nB\circ f^{-n}(y)+\Delta^s(y,x)\leqslant S_nB\circ f^{-n}(y')+\Delta^s(y',x')+K(M,f)(\lambda_s^N\widetilde{\varepsilon})^\alpha,\\ V(x)\leqslant V(x')+K(M,f)(\lambda_s^N\widetilde{\varepsilon})^\alpha.\end{gathered}$$

But

$$\lambda_s^N \widetilde{\varepsilon} = d(x, x')^{\ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)}. \qquad \square$$

Remark 10. — We have not used explicitly the fact that the stable foliation W^s is Hölder but our proof (step four) is close to showing W^s is Hölder of exponent $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Proof. — We show that if $\varepsilon < \varepsilon^*/K^*$, $d(x,x') \leqslant \varepsilon$, $y \in W^s_\varepsilon(x)$, $y' \in W^s_\varepsilon(x')$ and $y \in W^u_{\varepsilon^*}(y')$ then

$$d(y,y') \leqslant 3C(M)^2 d(x,x')^\gamma$$

where $\gamma = \ln(\lambda_u/\lambda_s)/\ln(\Lambda_u/\lambda_s)$.

Indeed we choose $t > 0$ real such that $\lambda_s^t \varepsilon = \Lambda_u^t d(x,x')$, $N = [t]+1$, and $\widetilde{\varepsilon}$ close to ε so that $\lambda_s^N \widetilde{\varepsilon} = \Lambda_u^N d(x,x')$ where $\widetilde{\varepsilon}/\varepsilon$ varies between 1 and Λ_u/λ_s. Then

$$d(f^N(x), f^N(y)) \text{ or } d(f^N(x'), f^N(y')) \text{ or } d(f^N(x), f^N(x')) \leqslant C(M)\lambda_s^N \widetilde{\varepsilon},$$
$$d(f^N(y), f^N(y')) \leqslant 3C(M)\lambda_s^N \widetilde{\varepsilon},$$
$$d(y,y') \leqslant 3C(M)^2 (\lambda_s/\lambda_u)^N \widetilde{\varepsilon} = 3C(M)^2 d(x,x')^\gamma. \qquad \square$$

3. Maximizing periodic measures

The proof of Theorem 4 requires two ingredients: the first one is the notion of sub-actions we have already studied, the second is the notion of strongly non-wandering points we are going to explain.

Definition 11. — Given $A \in \mathcal{C}^\beta(M)$ and $m = m(A,f)$, a point $x \in M$ is said to be strongly non-wandering with respect to A, if for any $\varepsilon > 0$, there exist $n \geqslant 1$ and $y \in M$ such that

$$y \in B(x,\varepsilon), \quad f^n(y) \in B(x,\varepsilon) \quad \text{and} \quad \Big| \sum_{k=0}^{n-1} (A-m)\circ f^k(y) \Big| < \varepsilon$$

where $B(x,\varepsilon)$ denotes the ball centered at x and radius ε. We call $\Omega(A,f)$ the set of strongly non-wandering points.

The first non-trivial but easy observation is that $\Omega(A,f)$ is non-empty; more precisely:

Lemma 12. — *The set $\Omega(A,f)$ is compact forward and backward f-invariant and contains the support of any maximizing measure.*

Proof. — If μ is maximizing, by Atkinson's theorem [**1**], for almost μ-point x, the Birkhoff's sums $\sum_{k=0}^{n-1}(A-m)\circ f^k$ are recurrent (in the sense of random walk theory) to $\int (A - m)\, d\mu = 0$: that is, for any Borel set B of positive μ-measure and for any $\varepsilon > 0$, the set

$$\Big\{x \in B \mid \exists n \geqslant 1\ f^n(x) \in B \text{ and } \big|\textstyle\sum_{k=0}^{n-1}(A - m) \circ f^k(x)\big| < \varepsilon\Big\}$$

has positive μ-measure. Since by definition of the support of a measure, any ball $B(x, \varepsilon)$ has positive μ-measure, we have proved that $\mathrm{supp}(\mu)$ is included in $\Omega(A, f)$. □

The second observation is that any Hölder function A is cohomologuous to $m(A, f)$ on $\Omega(A, f)$, more precisely:

Lemma 13. — *Let A be a $\mathcal{C}^0$-function and assume A admits a $\mathcal{C}^0$ sub-action V, then*

$$\Omega(A, f) \subseteq \Sigma_V(A, f) = \big\{x \in M \mid A - m = V \circ f - V\big\}$$

and any f-invariant measure μ whose support in contained in $\Omega(A, f)$ is maximizing.

The set $\Sigma_V(A, f)$ will play an important role later and it is convenient to to give it a name:

Definition 14. — Let A be a $\mathcal{C}^0$-function and V be a sub-action of A.

(i) We call the set $\Sigma_V(A, f) = \big\{x \in M \mid A - m = V \circ f - V\big\}$, the V-action-set of A.

(ii) Two points x, y of the V-action-set are said to be V-connected and we shall write $x \stackrel{V}{\to} y$, if for every $\varepsilon > 0$, there exist $n \geqslant 1$ and $z \in M$ (not necessarily in $\Sigma_V(A, f)$) such that

$$x \in B(z, \varepsilon), \quad y \in B(f^n(z), \varepsilon), \quad |S_N(A - m)(z) - (V(y) - V(x))| < \varepsilon.$$

Notice that, if V is β-Hölder for some $\beta > 0$, using the shadowing lemma, one can prove that $x \stackrel{V}{\to} y$ and $y \stackrel{V}{\to} u$ imply $x \stackrel{V}{\to} u$. This is so, because if z_x and n_x are the ones for $x \stackrel{V}{\to} y$ in (ii) above, and if z_y and n_y are the ones for $y \stackrel{V}{\to} u$ in (ii) above, then considering the pseudo-orbit $z_x, \ldots, f^{n_x}(z_x), z_y, \ldots, f^{n_y}(z_y)$, we can find by shadowing the z for $x \stackrel{V}{\to} u$ in (ii) above.

Proof of Lemma 13. — Define $R = V \circ f - V - A + m$ and choose $x \in \Omega(A, f)$. Then $\sum_{k=0}^{n_i-1}(A - m) \circ f^k(y_i)$ converges to 0 for a sequence of points y_i and a sequence of integers n_i such that y_i converges to x, n_i converges to $+\infty$ and $f^{n_i}(y_i)$ converges to x. Since R is non-negative,

$$0 \leqslant R(y_i) \leqslant \sum_{k=0}^{n_i-1} R \circ f^k(y_i) = V \circ f^{n_i}(y_i) - V(y_i) - \sum_{k=0}^{n_i-1}(A - m) \circ f^k(y_i)$$

converges to 0 and by continuity of R: $R(x) = 0$. □

Definition 15. — For any $\beta > 0$, define

$$\mathcal{G}_\beta = \{A \in \mathcal{C}^\beta(M) \mid \Omega(A, f) \text{ is a periodic orbit}\}.$$

Our next goal is to show that $\mathcal{G}_\beta$ is open in the $\mathcal{C}^\beta$ topology. We could have choosen a bigger set: the set of A in $\mathcal{C}^\beta(M)$ such that $\Omega(A,f)$ is minimal and is dynamically isolated (i.e. there exists U, open, containing $\Omega(A,f)$ as the only f-invariant compact set inside U) and the proof below would again be the same.

Lemma 16. — *For any $\beta>0$, $\mathcal{G}_\beta$ is open in the $\mathcal{C}^\beta$ topology and $\Omega(A,f)$ is locally constant as a function of A in $\mathcal{G}_\beta$.*

Proof. — Let $A\in\mathcal{G}_\beta$. We want to show that $\Omega(A,f)=\Omega(B,f)$ whenever B is sufficiently close to A in the $\mathcal{C}^\beta$ topology. By contradiction: let U be an isolating open set of the periodic orbit $\Omega(A,f)=\mathrm{orb}(p)$ and $\{A_n\}$ be a sequence of β-Hölder observables converging to A in the $\mathcal{C}^\beta$ topology such that $\Omega(A,f)$ is not included in U for each n.

Each A_n admits (Theorem 1) a γ-Hölder subaction V_n with γ-Hölder norm uniformly bounded and $\gamma=\beta\ln(1/\lambda_s)/\ln(\Lambda_u/\lambda_s)$. By Ascoli, $\{V_n\}$ admits a subsequence converging in the $\mathcal{C}^0$ topology to some γ-Hölder function V. Since the set of non-empty compact sets is compact with respect to the Hausdorff topology, we may assume that $\{\Omega(A_n,f)\}$ has a sub-sequence converging to some compact invariant set K. Each A_n satisfies:

$$A_n-m(A_n,f)\leqslant V_n\circ f-V_n \qquad (\forall\, x\in M),$$
$$A_n-m(A_n,f)=V_n\circ f-V_n \qquad (\forall\, x\in\Omega(A_n,f)).$$

By continuity of $m(A,f)$ with respect to A (for the $\mathcal{C}^0$ topology),

$$A-m(A,f)\leqslant V\circ f-V \qquad (\forall\, x\in M)$$
$$A-m(A,f)=V\circ f-V \qquad (\forall\, x\in K).$$

We have assumed that each $\Omega(A_n,f)\setminus U$ is not empty, then $K\setminus U$ is not empty too. Let $x_0\in K\setminus U$, the ω-limit set $\omega(x_0)$ and the α-limit set $\alpha(x_0)$ of x_0 are compact invariant sets included in $\Omega(A,f)$, necessarily:

$$\omega(x_0)=\alpha(x_0)=\mathrm{orb}(p)\subset\overline{\mathrm{orb}(x_0)}\subset\Sigma_V(A,f).$$

Since p is V-connected to x_0 and x_0 is V-connected to p, x_0 is V-connected to itself which is equivalent to $x_0\in\Omega(A,f)$. We just have obtained a contradiction. □

Proof of Theorem 4. — Let β given and A, α-Hölder with:

$$\beta<\widetilde{\beta}=\alpha\frac{\ln(1/\lambda_s)}{\ln(\Lambda_u/\lambda_s)}.$$

According to Theorem 1, there exists V, $\widetilde{\beta}$-Hölder, satisfying:

$$A-m\leqslant V\circ f-V \qquad (\forall\, x\in M).$$

Define $R = V \circ f - V - A + m$, $\phi_n = \min(R, 1/n)$ and $B_n = A + \phi_n$. Then ϕ_n is $\widetilde{\beta}$-Hölder with $\text{Höld}_{\widetilde{\beta}}(\phi_n) \leqslant \text{Höld}_{\widetilde{\beta}}(R)$ and

$$A - m \leqslant B_n - m \leqslant V \circ f - V \qquad (\forall x \in M)$$
$$B_n - m = V \circ f - V \qquad (\forall x \in \{R < 1/n\}).$$

In particular $m(B_n, f) = m(A, f)$ and the V-action set of B_n contains a neighborhood $\{R < 1/n\}$ of $\Omega(A, f)$. Using the shadowing lemma, we construct a periodic orbit $\text{orb}(p)$ inside $\{R < 1/n\}$ and we just have proved a perturbation B_n of A satisfies

$$\text{orb}(p) \cup \Omega(A, f) \subset \Omega(B_n, f).$$

Let ψ_n be any $\widetilde{\beta}$-Hölder function with small $\widetilde{\beta}$-Hölder norm satisfying:

$$\psi_n(x) = 0 \qquad (\forall x \in \text{orb}(p))$$
$$\psi_n(x) > 0 \qquad (\forall x \in M \setminus \text{orb}(p)).$$

Then $A_n = B_n - \psi_n = A + \phi_n - \psi_n$ is such that the minimizing measure has support on $\text{orb}(p)$, and A_n has small $\mathcal{C}^0$ norm and (possibly large) uniform $\widetilde{\beta}$-Hölder norm. Therefore (A_n) converges to A in the $\mathcal{C}^\beta$-topology and each A_n has a unique maximizing measure which is supported on a periodic orbit. □

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