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CHAOS VERSUS RENORMALIZATION AT QUADRATIC S-UNIMODAL MISIUREWICZ BIFURCATIONS

by

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Abstract. — We study C^3 families of unimodal maps of the interval with negative Schwarzian derivative and quadratic critical point, transversally unfolding Misiurewicz bifurcations, and for these families we prove that existence of an absolutely continuous invariant probability measure ("chaos") and existence of a renormalization are prevalent in measure along the parameter. Moreover, the method also shows that existence of a renormalization is dense and chaos occurs with positive measure.

1. Introduction

The *quadratic family*

$$f_a : \begin{array}{ccc} [0,1] & \longrightarrow & [0,1] \\ x & \longmapsto & 4ax(1-x) \end{array}, \qquad a \in [0,1],$$

is the simplest model that shows the complexity arising in nonlinear dynamical systems. For a fixed value of the parameter a, supposed to vary along the interval $[0,1]$, one is interested to follow the behavior of iterates x_0, $x_1 = f_a(x_0)$, $x_2 = f_a(x_1)$, $\ldots$, in other words of orbits

$$\mathcal{O}(x_0) = \{f_a^n(x_0)\}_{n\geqslant 0}$$

starting at a point x_0. The set $\omega(x_0)$ of accumulation points of $\mathcal{O}(x_0)$ gives a clue of the asymptotic behavior of the orbit of x_0, and is called the *ω-limit set of* x_0. It turns out ([**7**]) that "typical" starting points $x_0 \in [0,1]$ have equal ω-limit sets. This could be stated as follows: *for each $a \in [0,1]$, there is a set $A = A_a$ such that $\omega(x_0) = A$ for Lebesgue almost every $x_0 \in [0,1]$*. Moreover, there are only three types of sets which A_a could be: (i) a *periodic orbit*, i.e. a set $\{p_0, p_1, \ldots, p_{k-1}\}$ such that $f_a(p_0) = p_1$, $f_a(p_1) = p_2$, $\ldots$, $f_a(p_{k-1}) = p_0$; (ii) a *periodic collection of pairwise disjoint intervals*

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$\{I_0, I_1, \ldots, I_{k-1}\}$ where $f_a(I_0) = I_1$, $f_a(I_1) = I_2$, $\ldots$, $f_a(I_k) = I_0$; or (iii) a *Cantor set* (i.e. a perfect and totally disconnected compact set) of zero Lebesgue measure.

The striking alternation of behavior of f_a has been revealed and proved along the last three decades. Among others, we know that: parameters for which the typical ω-limit set is a periodic orbit are dense (and contain intervals, implying also positive Lebesgue measure) ([**3**], [**8**]); parameters for which the typical ω-limit set is a collection of intervals have positive measure (following [**4**]); and parameters for which the typical ω-limit set is a Cantor set have zero Lebesgue measure ([**10**]).

Among parameters with a cycle of intervals as its typical ω-limit set, with total Lebesgue measure ([**9**], [**12**]) we find those for which there is an absolutely continuous (with respect to Lebesgue) f_a-invariant probability measure. In this case f_a is said to be *chaotic*, although more intuitive and not exactly equivalent definitions of "chaos" are available. This definition supplies at least some statistical properties for the mean growth of derivatives along orbits and imply some dynamical structure on the configuration space.

On the other hand, parameters where the typical ω-limit set is a non-hyperbolic periodic orbit are rare in measure. In other words, hyperbolicity is prevalent in measure for these parameters. Putting things altogether, we conclude that for almost all $a \in [0,1]$, the dynamics of f_a is *either hyperbolic or chaotic.*

A largely used concept in one-dimensional dynamics is the idea of renormalization. We say that f_a is *renormalizable* if there is a collection of pairwise disjoint intervals $\{I_0, I_1, \ldots, I_{k-1}\}$ properly contained in $[0,1]$ such that (i) the critical point $\frac{1}{2}$ of f_a belongs to, say, I_{k-1}; (ii) $f_a(I_{k-1}) \subset I_0$ and $f_a(\partial I_{k-1}) \subset \partial I_0$; (iii) $f_a : I_i \to I_{i+1}$ is a diffeomorphism for all $i = 0, \ldots, k-2$. In particular, if we call $I = I_{k-1}$, then the function $f_a^k|I$ resembles in many ways the general aspect of a quadratic function in $[0,1]$, since $f_a^k(I) \subset I$, $f_a^k(\partial I) \subset \partial I$ and $f_a^k|I$ has a single (quadratic) critical point (equal to $\frac{1}{2}$). By an affine rescaling a new function $g : [0,1] \to [0,1]$ could be defined, but in general we may not expect g to be quadratic.

Renormalization is a kind of reduction tool. For example, the behavior of typical orbits is completely determined by the restriction $f_a^k|I$, since we know (see [**13**] and references therein) that for Lebesgue almost every $x \in [0,1]$ there is $n = n(x)$ such that $f_a^n(x) \in I$. All subsequent iterates must remain inside the cycle from this iterate on, because of the invariance properties stated above. This suggests that no complete knowledge of the quadratic family could be achieved without the understanding of a larger class of functions which contains in particular the ones generated *via* renormalization. For this class, it would be desirable some qualitative dynamical similarity with quadratic functions, not only for technical reasons (proves with recursive arguments) but also for the sake of some universality in the conclusions.

In [**3**] and [**8**] (denseness of hyperbolicity), [**9**] joint with [**12**] (measure prevalence of chaos) and [**10**] (rareness of Cantor ω-limit sets), this larger class of functions to which the quadratic functions belong (and which is invariant under renormalization) is composed by all analytic functions f which are holomorphically extendible to a neighborhood U of $[0,1]$ in the complex plane, such that $f(U)$ contains the closure of U and f is a double branched covering between U and $f(U)$. Recently ([**1**]) there have been considered the case of real analytic functions, but even so some main arguments are based on constructions developed in the complex plane.

Among the results mentioned for the quadratic family, the positive measure of chaotic parameters, proved for the first time in [**4**], is the only one which has been stated for C^2 families (see for example [**16**] or [**13**] and references therein). The present work is an attempt to provide techniques restricted to the real setting, weakening smoothness considerably, in order to state results that go in the same direction as the ones of the previous paragraph. Unfortunately the extent of the conclusions cannot be as complete as the ones already proved for the quadratic family. The main reason is that our statements are of a local nature, that is, they are valid only for parameters in small intervals around some bifurcation values. This does not allow us to go beyond the first renormalization, where full families appear.

Here we deal with C^3 *unimodal* interval maps f, that is those with a single turning point c, with the (classical) additional hypothesis that the Schwarzian derivative

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

defined for all $x \neq c$, is non-positive. These functions will be called *S-unimodal.* From this hypothesis some *a priori* conclusions can be derived. For example, there is at most one periodic attractor and if it does exist then it must attract the critical orbit $\mathcal{O}(c)$ ([**15**]). Moreover, distortion of derivatives for powers of f can be uniformly controlled (see statements in [**13**]). This comes from two facts: first, if a diffeomorphism defined in an interval I has non-positive Schwarzian derivative, the ratio between its derivatives evaluated at two points can be bounded by a constant which depends only on the proportion between their mutual distance and their distance to the boundary of I, *but not on the diffeomorphism.* Second, powers of f have also non-positive Schwarzian derivatives, hence distortion bounds may be obtained whenever $f^n|I$ is a diffeomorphism for some I, independently of n.

To make clear the results we want to state below, it is convenient to relate renormalization with the classification of functions into three types we have made above, which are still valid for the larger class we are considering now (see [**7**]). First, we observe that if f is renormalizable then there is an interval $I^{(1)}$ containing the critical point and a number k_1 such that $f^{k_1}|I^{(1)}$ is a unimodal function. It may be that this function is also renormalizable, and in this case we say that f is (at least) twice renormalizable. We can take the maximum chain of renormalization intervals ordered by (proper) inclusion

$$[0,1] = I^{(0)} \supset I^{(1)} \supset I^{(2)} \dots .$$

If this chain has size $N+1$ then we say that f is N times renormalizable, and if its size is not finite ($N = \infty$) then we say that f is *infinitely renormalizable.* The case where the size is equal to 1 is called *non-renormalizable.*

It turns out that f is infinitely renormalizable if and only if typical points have a Cantor set as its ω-limit set ([**13**]). If f is N times renormalizable, its typical ω-limit set is determined by the N-th renormalization $g = f^{k_N}|I^{(N)}$. If g has an attracting fixed point, the ω-limit set is a periodic orbit, otherwise a collection of intervals. Here we are using the fact that if g had an attracting point of period greater or equal than two then g would be renormalizable, characterizing a contradiction.

We say that f is *Misiurewicz* if the critical point c is not recurrent, i.e. $c \notin \omega(c)$. It may happen that $\omega(c)$ is an attracting periodic orbit. If not, then $f(c)$ belongs to a hyperbolic invariant compact set $\Lambda = \Lambda_f$. From hyperbolic theory, we know that for g sufficiently near f (in the C^1 topology), there is a g-hyperbolic invariant compact set Λ_g such that $f|\Lambda_f$ and $g|\Lambda_g$ are conjugated by $h_g : \Lambda \to \Lambda_g$. The function $g \mapsto \Lambda_g$ is in fact C^1 and is called the *hyperbolic continuation* of Λ. Now we embed f in a C^3 family $(f_a)_a$, where $f_0 = f$, and call w the point belonging to Λ such that $w = f(c)$. As a varies, w has its continuation $w_a = h_{f_a}(w)$ and the critical point c has its continuation c_a, which is well defined by the Implicit Function Theorem, using that c is quadratic. We will say that $(f_a)_a$ is *transversal at* $a = 0$ if

$$\frac{d}{da}\,(f_a(c_a) - w_a) \neq 0.$$

Without loss of generality, we assume $c_a \equiv c$ and $\dfrac{d}{da}(f_a(c) - w_a) > 0$.

Theorem 1.1. — *Let $f : [0,1] \to [0,1]$ be a C^3 S-unimodal non-renormalizable Misiurewicz function, without periodic attractors. Let $(f_a)_a$ be a C^3 family with $f_0 = f$, transversal at $a = 0$. Then there is $\varepsilon > 0$ such that*

(1) *for almost all $a \in [-\varepsilon, \varepsilon]$, f_a is chaotic or renormalizable;*

(2) *parameters for which f_a is renormalizable constitute a countable union of closed intervals which is dense in $[-\varepsilon, \varepsilon]$;*

(3) *parameters for which f_a is at the same time non-renormalizable and Misiurewicz have zero Lebesgue measure in $[-\varepsilon, \varepsilon]$.*

All items of Theorem 1.1 are new for non-analytic families (the third item is analogous to the statements in [**14**])

As a corollary of the method, we are also able to show that parameters for which f_a is chaotic have positive Lebesgue measure in $[-\varepsilon, \varepsilon]$, assertion which has already been proved, even in more generality, for C^2 families (see [**16**] and [**13**], Chap.V, Section 6; in fact, they prove that the relative measure goes to one at the bifurcation value). The techniques, however, go in a totally different direction, since they work with exclusion of "bad" parameters (which in general include everyone for which there is a renormalization), showing then that the remaining ones have positive measure and reasonably good expansion properties (an absolutely continuous invariant probability measure, for instance). These methods however may exclude also some positive measure set of "good" parameters, for which one could also prove the existence of stochastic dynamics. Here, on the other hand, we show that chaos is prevalent in non-renormalizable dynamics *and* non-renormalizable dynamics occurs with positive measure in the parameter.

Our methods could also be useful to obtain precise estimates of the measure of chaotic parameters and even an upper bound for the Hausdorff dimension of non-renormalizable non-chaotic parameters, provided enough control was achieved in configuration space (see [**5**], for attempts in this direction for C^2 families).

After suitable changes in the conclusion, we could drop the assumption that the bifurcating map is "non-renormalizable" in Theorem 1.1 by writing, instead, that f is finitely renormalizable. In this case, f would be N times renormalizable ($N \geqslant 1$), Misiurewicz and without periodic attractors. Then for the transversal family $(f_a)_a$ we would have two possibilities: (i) f_a is at least N times renormalizable for all $a \in [-\varepsilon, \varepsilon]$, for $\varepsilon > 0$ small; (ii) f_a is at least N times renormalizable for $a \in [-\varepsilon, 0]$ and at least $N-1$ times renormalizable for $a \in (0, \varepsilon]$. The first statement might be rephrased, respectively, into: (i) almost every $a \in [-\varepsilon, \varepsilon]$ is chaotic or $N+1$ times renormalizable; (ii) almost all $a \in [-\varepsilon, 0]$ is chaotic and $N+1$ times renormalizable and almost all $a \in [0, \varepsilon]$ is chaotic or N times renormalizable. The proof would run on in the same way, with minor adaptations.

The proof of Theorem 1.1 uses a result proved in [**2**]. Some "starting conditions" must be satisfied for the functions f_a, $a \in [-\varepsilon, \varepsilon]$, allowing an inductive argument to work. This will be better explained in the next section.

2. Mounting the proof

Let $f : [0,1] \to [0,1]$ be an S-unimodal C^3 function and c its critical point. Assume that f is Misiurewicz, i.e. the critical point c is not recurrent, and f does not have a periodic attractor. The following definitions and Proposition 2.2 can be found in [**11**] (in fact without the Misiurewicz hypothesis).

Let $x \in [0,1]$ and $\tau(x) \neq x$ be such that

$$f(x) = f(\tau(x)),$$

and let $V_x = (x, \tau(x))$.

Definition 2.1. — A point $x \in [0,1]$ is *nice* if $f^n(x) \notin V_x$ for all $n \geqslant 1$. In this case V_x is a *nice interval*.

For example, every periodic orbit contains a nice point, for instance the one maximizing the value of f. Moreover, as f does not have a periodic attractor then there are periodic points arbitrarily near c, assuring arbitrarily small nice intervals.

Let $\mathcal{U}_x \subset [0,1]$ be the set of points that visit V_x at least once (including the points of V_x), and

$$\Lambda_x = [0,1] \smallsetminus \mathcal{U}_x.$$

The following Proposition is proved in [**11**].

Proposition 2.2

(1) *If I is a connected component of $\mathcal{U}_x$ then there is n such that $f^n : I \to V_x$ is monotone and onto. This function is called the* transfer map *from I to V_x.*

(2) *In this case, the intervals of the collection*

$$\{I, f(I), \dots, f^n(I) = V_x\}$$

are pairwise disjoint.

(3) *The set Λ_x is invariant and hyperbolic (hence with zero measure), and if $w \in \Lambda_x$ is such that $f^n(w) \notin \overline{V}_x$, $\forall n \geqslant 1$, then Λ_x accumulates from both sides on w (for short, $w \in \Lambda_x \smallsetminus \partial\Lambda_x$, where $\partial\Lambda_x$ denotes the set of points of Λ_x which belong to the boundary of a connected component of $(0,1) \smallsetminus \Lambda_x$).*

Proof. — See [**11**]. □

As f is Misiurewicz, there is a neighbourhood V of c such that $f^n(c) \notin V$, $\forall n \geqslant 1$. Take a hyperbolic periodic nice point y in V (all periodic points must be hyperbolic under the hypotheses, since $Sf \leqslant 0$ implies that nonhyperbolic periodic points must be attractors). Then $\overline{V}_y \subset V$ and, as $f^n(c) \notin \overline{V}_y$, $\forall n \geqslant 1$, it follows from Proposition 2.2 that $f(c) \in \Lambda_y \smallsetminus \partial\Lambda_y$. In other words, $f(c)$ is accumulated from both sides by arbitrarily small connected components of $\mathcal{U}_y$.

Now we define a new nice point as follows. Take $z \in V_y \cap [0,c)$ such that $f(z) \in \partial I$, for some connected component I of $\mathcal{U}_y$. As $f(c) \in \Lambda_y \smallsetminus \partial\Lambda_y$, z can be chosen arbitrarily near c, so that

$$\frac{|V_z|}{|V_y|}$$

can be as small as desired. With a minor modification in context, the following Proposition is also stated in [**11**].

Proposition 2.3. — *Let I be a connected component of $\mathcal{U}_z$ and, by Proposition 2.2, let n be such that $f^n : I \to V_z$ is monotone and onto. Then there is $\widehat{I} \supset I$ such that $f^n : \widehat{I} \to V_y$ is monotone and onto.*

Proof. — Let T be the maximal interval containing I such that $f^n|T$ is monotone and $f^n(T) \subset V_y$. It is easy to see by Proposition 2.2, item 2, that $I \subset \operatorname{int}(T)$. Supposing by contradiction that $f^n(T) \neq V_y$, there is at least one connected component L of $T \smallsetminus I$ such that $\overline{f^n(L)} \subset V_y$. By the maximality of T, there is $j < n$ such that $c \in \partial f^j(L)$. Again by Proposition 2.2, $f^j(I) \cap V_z = \varnothing$, hence $z \in f^j(L)$ (or $\tau(z) \in f^j(L)$). But $\overline{f^n(L)} \subset V_y$ implies $f^{n-j}(z) \in V_y$, contradiction, since $f(z) \in \Lambda_y$. □

Let $(f_a)_a$ be a C^3 family of S-unimodal functions with $f_0 = f$, transversal at $a = 0$, where a varies in the range $[-\varepsilon, \varepsilon]$, for some $\varepsilon > 0$. As y is a hyperbolic periodic point, it has a continuation y_a defined for small values of a. Also z has a continuation z_a, since it is a preimage of y. Moreover the hyperbolic sets Λ_y and Λ_z have continuations $\Lambda_{y,a}$ and $\Lambda_{z,a}$ and the whole "hyperbolic structure" is preserved. This could be stated as follows: for each sufficiently small a there is a homeomorphism

$$h_a : [0,1] \setminus V_z \longrightarrow [0,1] \setminus V_{z_a}$$

such that

$$f_a^n \circ h_a(x) = h_a \circ f_0^n(x),$$

whenever $\{x, f_0(x), \ldots, f_0^n(x)\} \subset [0,1] \setminus V_z$. In particular, Proposition 2.3 remains valid (if adapted to the continuations) for $a \in [-\varepsilon, \varepsilon]$.

Lemma 2.4. — *Let $f_a^n : I \to V_{z_a}$ be the transfer map of some preimage I of V_{z_a}, and let $f_a^n : \widehat{I} \to V_{y_a}$ be its extension. If $I \cap [f_a(z_a), 1] \neq \varnothing$ then $\widehat{I} \subset [f_a(z_a), 1]$.*

Proof. — Otherwise $f_a(z_a) \in \operatorname{int}(\widehat{I})$ and $f_a^n(f_a(z_a)) \in V_{y_a}$, contradiction, since by the choice of z the orbit of z_a never intersects V_{y_a}. □

Now we fix some notation, which the reader can follow with the help of Figure 1 (depicted for $a > 0$). Let $\widetilde{w} > w$ be a point of Λ_y, for $a = 0$, and $\widetilde{w_a}$ its continuation. Since

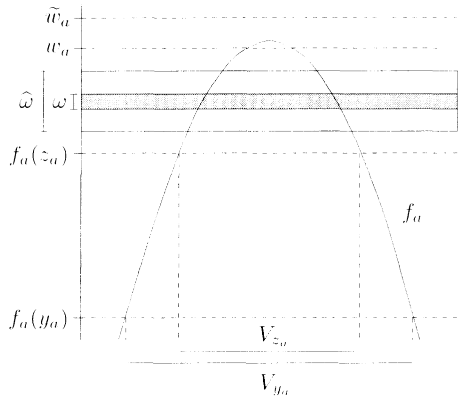

FIGURE 1. Mounting the proof

Λ_y accumulates from both sides in w, we may suppose that $|\widetilde{w}_a - w_a| \ll |w_a - f_a(z_a)|$. By requiring $\varepsilon > 0$ small enough we also beg that $f_a(c) < \widetilde{w}_a$, for all $a \in [-\varepsilon, \varepsilon]$. Let $\mathcal{W} = \mathcal{W}_a$ be the collection of preimages of V_{z_a} intersecting $[f_a(z_a), \widetilde{w}_a]$. For each $\omega = \omega_a \in \mathcal{W}$ let $W : \omega \to V_{z_a}$ be its transfer map, and let $\widehat{\omega} = \widehat{\omega}_a$ be its extension domain relative to V_{y_a}. Although hidden in the notation, we look at W as a function of both parameter and space, defined in the domain

$$\{(a,x)\ ;\ x \in \omega_a, a \in [-\varepsilon, \varepsilon]\}.$$

We will adopt capital letters to indicate two-variable dependence in other situations. For example, we write $F(a,x) = f_a(x)$, so that partial derivatives are denoted by F_a, F_x, F_{xx}, F_{xa}, etc. In this notation, compositions are denoted with respect to the second variable (configuration space), for example $W \circ F$ means the function $(W \circ F)(a,x) = W(a, F(a,x))$. The powers F^k are inductively defined as $F^k(a,x) = F(a, F^{k-1}(a,x))$ and we write F^k_x, F^k_a, etc. for their derivatives. The notation $(F_x)^k$, in turn, means the k-th power of the x-derivative of F. We sometimes treat these functions as functions of one-variable (the x-variable), writing expressions as $F(x)$, meaning $F(a,x)$, or $F|I$, meaning $f_a|I$, where I is an interval, whenever it is clear that the parameter is fixed.

In Section 5 we will show that the transversality of the family $(f_a)_a$ at $a = 0$ implies that the critical value $f_a(c)$ transversally crosses the hyperbolic set Λ_{z_a} not only at w_a, for $a = 0$, but also at nearby points for small parameter values. In particular, if we fix some preimage ω_0 of V_{z_a}, whose continuation is $\omega_{0,a}$, the set of parameters

$$J_0 = \{a \in [-\varepsilon, \varepsilon]\ ;\ f_a(c) \in \omega_{0,a}\}$$

is an interval, for ε small (see Figure 2, where J_0 occurs for $a < 0$). Moreover, we will

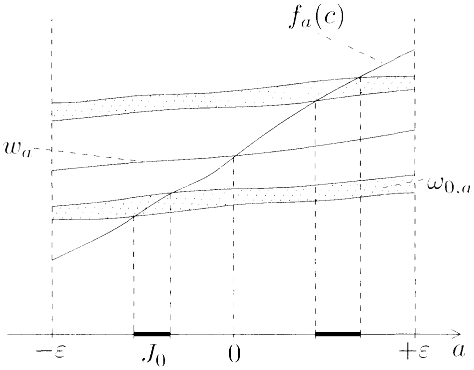

FIGURE 2. Evolution of the critical value

show that the set

$$\Gamma = \{a \in [-\varepsilon, \varepsilon]\ :\ f_a(c) \in \Lambda_{z_a}\}$$

has zero Lebesgue measure. Hence all of our assertions will be made for a fixed preimage ω_0 to which the critical value belongs, for parameters in the corresponding interval J_0.

We now focus our attention on the first return map $\Phi = \Phi_a$ of V_{z_a}, for parameters $a \in J_0$, for a fixed $\omega_0 = \omega_{0,a}$ (see Figure 3). The connected component of $\mathrm{dom}(\Phi)$ containing the critical point c is called the *central interval* and will be denoted by $\gamma_0 = \gamma_{0,a}$ (note that $\Phi(\partial\gamma_0) \subset \partial V_{z_a}$). The restriction

$$H \equiv \Phi|\gamma_0 = W_0 \circ F|\gamma_0$$

is called the *central branch*, where $W_0 : \omega_0 \to V_{z_a}$ is the transfer map associated to ω_0. The remaining connected components of $\mathrm{dom}(\Phi)$, together with the central interval, cover V_{z_a} up to measure zero. They form a collection which will be denoted by $\mathcal{P}$, where for each $\pi \in \mathcal{P}$ we have $F(\pi) = \omega$, for some $\omega \in \mathcal{W}$. In other words,

$$P \equiv \Phi|\pi = W \circ F|\pi : \pi \longrightarrow V_{z_a}$$

is a diffeomorphism, where $W : \omega \to V_{z_a}$ is the transfer map associated to ω. Each $\pi \in \mathcal{P}$ is called a *regular interval*.

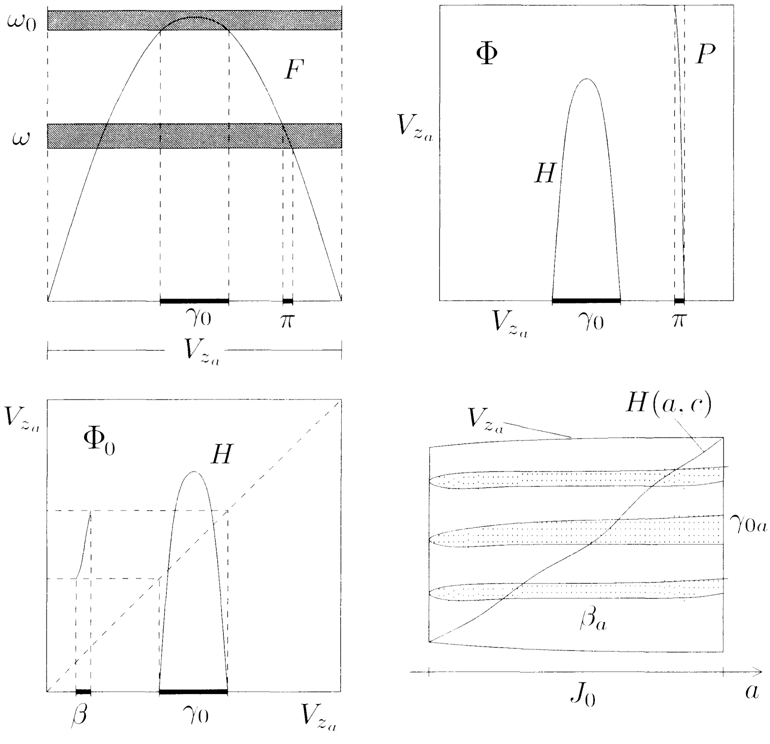

FIGURE 3. Return functions

A further refinement is made, obtaining from Φ a new map Φ_0, defined in V_{z_a} (up to measure zero). This map coincides with Φ in the central interval, and outside it corresponds to the first entry map into γ_0. The domain of Φ_0 is composed by the central image together with a collection $\mathcal{B}$ of intervals called the *preimages of the central interval.* To each preimage $\beta \in \mathcal{B}$ we define the diffeomorphism $B \equiv \Phi_0|\beta : \beta \to \gamma_0$, assigning $\pi_1, \pi_2, \ldots, \pi_n$ such that $\beta \subset \pi_1$, $P_1(\beta) \subset \pi_2, \ldots, (P_m \circ \cdots \circ P_1)(\beta) \subset \pi_{m+1}, \ldots, (P_n \circ \cdots \circ P_1)(\beta) = \gamma_0$, where $P_m : \pi_m \to V_{z_a}$ is the restriction of Φ to π_m, $m = 1, \ldots, n$. We also define

$$\mathcal{U}(\beta) = (P_n \circ \cdots \circ P_1)^{-1}(V_{z_a}),$$

which in particular coincides with π_1 in the case $n = 1$.

Of course all definitions above depend on the parameter a, which is allowed to vary in the interval J_0. Capital letters again are used to denote two-variable functions. The interval $\gamma_0 = \gamma_{0,a}$ is continuously defined for all $a \in J_0$. The same is true for each $\pi \in \mathcal{P}$ and $\beta \in \mathcal{B}$. Figure 3 shows what should be the evolution of the connected components of $\mathrm{dom}\Phi_0$ with respect to the parameter, along the interval J_0. Among others, we will show that $H(a, c)$ transversally crosses these components.

A number of requirements for the map Φ_0, which we call *starting conditions*, must be satisfied, in order to start an induction procedure, developed in [**2**], that proves Theorem 1.1. We separate these requirements into three parts, listed below. We are implicitly assuming non-positive Schwarzian derivative.

Geometry. — There is $\eta > 0$ small such that

$$\frac{|\gamma_{0,a}|}{|V_{z_a}|} < \eta, \quad \frac{|\beta_a|}{\mathrm{dist}(\beta_a, \gamma_{0,a})} < \eta, \quad \frac{|\beta_a|}{\mathrm{dist}(\beta_a, \partial V_{z_a})} < \eta,$$

for all $a \in J_0$ and $\beta \in \mathcal{B}$. Moreover, for each $\beta \in \mathcal{B}$, the diffeomorphism $B : \beta \to \gamma_0$ is extendible to a $\eta^{-1}|\beta|$-neighborhood of β, for all $a \in J_0$.

These conditions are uniform in the parameter and have been considered in previous works (see [**6**] and [**7**], for example).

Central branch. — $H_{xx} \neq 0$, $H_a \neq 0$ and there is $\delta_0 > 0$ small such that the quotients

$$|\gamma_0| \cdot \left|\frac{H_{xxx}}{H_{xx}}\right|, \quad |\gamma_0| \cdot \left|\frac{H_{ax}}{H_a}\right|, \quad |J_0| \cdot \left|\frac{H_{aa}}{H_a}\right|, \quad |J_0| \cdot \left|\frac{H_{xxa}}{H_{xx}}\right|,$$

are smaller than δ_0, for all $x \in \gamma_{0,a}$ and $a \in J_0$.

In particular, these conditions imply small distortion of H_{xx} and H_a along $x \in \gamma_{0,a}$ and $a \in J_0$.

Preimages of the central branch. — $|B_x| \geqslant 2$, for all $\beta \in \mathcal{B}$, $x \in \beta_a$, $a \in J_0$. For each $\beta \in \mathcal{B}$, let

$$J(\beta) = \{a \in J_0 \ ; \ \mathrm{Im}\, H \cap \mathcal{U}(\beta) \neq \varnothing \text{ or } |\,\mathrm{Im}\, H| \geqslant \tfrac{1}{7}|V_{z_a}|\}.$$

Let V be the mean value of $H_a(a,c)$ along $a \in J_0$. Then there is $\delta_1 > 0$ small such that the quotients

$$\left|\frac{B_a}{B_x V}\right|, \quad |\gamma_0| \cdot \left|\frac{B_{xx}}{(B_x)^2}\right|, \quad |\gamma_0| \cdot \left|\frac{B_{xa}}{(B_x)^2 V}\right|,$$
$$|\gamma_0| \cdot \left|\frac{B_{aa}}{(B_x)^2 V^2}\right|, \quad |\gamma_0|^2 \cdot \left|\frac{B_{xxx}}{(B_x)^3}\right|, \quad |\gamma_0|^2 \cdot \left|\frac{B_{xxa}}{(B_x)^3 V}\right|,$$

are smaller than δ_1, for all $x \in \beta_a$, $a \in J(\beta)$ and $\beta \in \mathcal{B}$. The first quotient implies that preimages are transversally crossed by the critical value of H, and the second implies small distortion of derivatives of the functions $B : \beta \to \gamma_0$.

The following Theorem is proved in [**2**], when Φ_0 is C^∞. In Appendix A we show that in fact C^3 is enough.

***Theorem 2.5* (Colli).** — *If Φ_0 satisfies the starting conditions* Geometry, Central Branch and Preimages of the Central Branch, *for sufficiently small $\eta > 0$, $\delta_0 > 0$ and $\delta_1 > 0$ then*

(1) *for almost all $a \in J_0$, f_a is chaotic or renormalizable;*

(2) *parameters for which f_a is renormalizable constitute a countable union of closed intervals which is dense in J_0;*

(3) *parameters for which f_a is chaotic have positive Lebesgue measure in J_0;*

(4) *parameters for which f_a is non-renormalizable and Misiurewicz have zero Lebesgue measure in J_0.*

Therefore we are left to prove that, given $\eta > 0$, $\delta_0 > 0$ and δ_1, there is a choice of V_z and $\varepsilon > 0$ such that for every map Φ_0 as above, constructed for $a \in J_0$, $J_0 \subset [-\varepsilon, \varepsilon]$, the starting conditions are satisfied with the constants η, δ_0 and δ_1.

In the proof we rely mostly on expansion estimates which comes from the Misiurewicz hypothesis. It is known that distortion of derivatives can be obtained using expansion along iterates, and the same will be true for the quotients mentioned above, related to distortion involving both the parameter and the configuration space. The estimates are, however, more delicate, and recovering of bad derivatives must be achieved in unusual manners, mainly when parameter is involved. We call *circular recovering* the ensemble of these techniques, which are developed in Section 4, and their first applications appear already in Section 5, where the first derivative with respect to the parameter appears.

In addition to expansion obtained from the proximity of a Misiurewicz bifurcation, the techniques exposed in Section 4 use also the geometry generated by the dynamics and *a priori* distortion coming from the hypothesis on the Schwarzian derivative.

We believe that this result could be stated without the Misiurewicz hypothesis, but some obstacles should be bypassed. First, a transversality condition should be formulated for germs of families unfolding a general non-renormalizable map. Second, some features of the geometry should be adapted. And third, some expansion would be desirable, unless a completely different approach could control the quotients *without expansion* (more or less like the Schwarzian derivative controls distortion even if little of the dynamics is known).

The Sections are organized as follows. In Section 3 we briefly discuss constants and their hierarchy, and state immediate consequences of non-positiveness of the Schwarzian derivative. The main one is Corollary 3.4, proving the Starting Conditions called "Geometry". We are left to obtain the remaining Starting Conditions, a task which is achieved step by step. In Section 4 we develop the techniques mentioned above which we call "circular recovering". There we deal with the expansion rates of the transfer maps $W : \omega \to V_{z_a}$, for $\omega \in \mathcal{W}$. In fact, more than simply estimating W_x, we also look at derivatives of intermediate iterates, like $F^i|\omega$ if $i < k$ and $W = F^k|\omega$. Moreover, we are able to recover not only "bad derivatives" but also "the square of bad derivatives", which is essential to Section 6. In Section 5 we explore the transversality assumption on the bifurcation and control the quotient W_a/W_x (and also intermediate iterations). This quotient is related with the way pre-images ω of V_{z_a} are crossed by the critical value. We also prove that the set of parameters Γ where the critical value does not belong to any of these pre-images has zero Lebesgue measure. In Section 6 the remaining quotients for the transfer maps W are controlled.

In Section 7 we obtain the Starting Conditions called "Central Branch". Estimates of Sections 5 and 6 are used, since the central branch $H : \gamma_0 \to V_{z_a}$ is the composition $W_0 \circ F|\gamma_0$ (recalling that W_0 is the transfer map of the pre-image ω_0 of V_{z_a} to which the critical value belongs).

In Section 8 we work with regular branches $P : \pi \to V_{z_a}$ and their compositions, which form the maps $B : \beta \to \gamma_0$. Recall that P is the composition $W \circ F|\pi$, for some $W : \omega \to V_{z_a}$, $\omega \in \mathcal{W}$. The goal is to control expansion of compositions, since there are also bad derivatives for some of the P's. But bad derivatives may be recovered as in Section 4, with ideas resembling "circular recovering".

In Section 9 we study the first derivative with respect to the parameter for compositions of regular branches and we achieve control on the first quotient B_a/B_x of the Starting Conditions "Pre-images of the central branch". The remaining quotients are obtained in Section 10.

Everywhere we have to work with mixed derivatives of compositions, using the formula stated in Appendix B. In Appendix A, as we said above, a key lemma in [**2**] is stated for C^3 families, instead of C^∞. The same approach could be useful whenever one has to deal with saddle-nodes and parameter distortion at the same time.

3. Conventions, distortion and geometry

We adopt the following convention on constants. We denote by C_0 a constant greater than 0 which is bigger than any constant used from now on which depends only on functions belonging to a C^3 small neighborhood of f_0. This includes universal constants which do not depend even on these functions. Next, we adopt C_y as the constant which depends also on the choice of y, and C_z as the constant depending on the choice of z. There will be some abuse of notation when we calculate things as "$3C_0^4$" and after all say that it is smaller than C_0. This means that if in some previous Lemma we have estimated something with $\widetilde{C_0}$ and now we are obtaining another estimate $\widehat{C_0} = 3\widetilde{C_0}^4$ then C_0 is greater than both $\widetilde{C_0}$ and $\widehat{C_0}$.

The Greek letter δ will be used as an auxiliary quantifier, appearing always as "given $\delta > 0$ there is...". We will choose δ sufficiently small such that the Starting Conditions are satisfied for given η, δ_0 and δ_1.

Remark that we have the freedom to choose V_z (independently of V_y) in such a way that the ratio $|V_z|/|V_y|$ is small. After the choice of V_z we can also choose ε small. For example, we define

$$r = r(z) = 2\frac{|V_z|}{|V_y|}$$

and choose ε small so that

$$\frac{|V_{z_a}|}{|V_{y_a}|} \leqslant r(z)$$

for all $a \in [-\varepsilon, \varepsilon]$. Moreover, the constant ε has to be chosen small to validate the constants C_0, C_y and C_z.

To be more precise, we will be interested not only on the ratio $|V_{z_a}|/|V_{y_a}|$, but on the size of V_{z_a} compared with both connected components of $V_{y_a} \setminus V_{z_a}$. But the involution function $\tau = \tau_a$ is Lipschitz with constant C_0, for $a \in [-\varepsilon, \varepsilon]$, so that $r(z)$ small also implies that V_{z_a} is uniformly small compared with its adjacent components of $V_{y_a} \setminus V_{z_a}$.

Below we introduce the small constant $\theta > 0$, which will be related to the extendibility of iterations of the map. It will directly depend on $r = r(z)$.

Other constants, $\sigma = \sigma(y) > 1$ and $\lambda = \sqrt{\sigma}$ will depend only on the choice of V_y (with ε small, of course), and will be related to the rate of expansion outside V_{y_a}.

Finally, we use the symbols "$\simeq$", "$\lesssim$" and "$\gtrsim$", in the following sense. For some fixed small constant $\xi > 0$, say $\xi = 10^{-3}$, $C \lesssim D$ whenever $D > 0$ and $C \leqslant (1+\xi)D$. Then $C \gtrsim D$ if and only if $D \lesssim C$ and $C \simeq D$ if and only if $C \lesssim D$ and $D \lesssim C$.

Non-positive Schwarzian derivative has its main consequence in the *Koebe principle*, which is restated in the following form.

Lemma 3.1. — *Given $\theta > 0$, there is $q > 0$ such that if $f : \widehat{I} \to f(\widehat{I})$ is a diffeomorphism, $Sf(x) \leqslant 0$ for all $x \in \widehat{I}$, $I \subset \widehat{I}$ is another interval and $f(I)$ is smaller than q times the size of each connected component of $f(\widehat{I}) \setminus f(I)$ then there is a $\theta^{-1}|I|$-neighborhood $\widetilde{I}$ of I in $\widehat{I}$ such that the derivative of f has small distortion in $\widetilde{I}$, that is*

$$\frac{f'(x)}{f'(y)} \simeq 1.$$

Proof. — See [**13**] for a detailed account. □

Lemma 3.1 has the following important Corollaries, which prove the Geometry of the Starting Conditions. They will be used in many points of this work.

Corollary 3.2. — *Given $\theta > 0$, if $r = r(z)$ is sufficiently small then $|\omega|$ is θ times smaller than the two connected components of $\widehat{\omega} \setminus \omega$.*

Proof. — The transfer map $W : \omega \to V_{z_a}$ is extendible to $W : \widehat{\omega} \to V_{y_a}$. But W has non-positive Schwarzian derivative, since it is a power of f_a and the sign of the Schwarzian derivative is preserved by compositions. Then the Koebe principle can be applied to W. □

Corollary 3.3. — *Given $\theta > 0$, if $r = r(z)$ is sufficiently small then*

$$\frac{|\pi|}{\operatorname{dist}(\pi, \partial V_{z_a})} < \theta,$$

for all $\pi \in \mathcal{P}$.

Proof. — By Lemma 2.4, $\widehat{\omega} \subset [F(z_a), 1]$, $\forall\, \omega \in \mathcal{W}$. Combining with Corollary 3.2, ω is as small as we want compared with $\operatorname{dist}(\omega, F(z_a))$, provided $r(z)$ is small. But for every $\pi \in \mathcal{P}$, $F(\pi) = \omega$, for some $\omega \in \mathcal{W}$. The Lemma follows, since F is approximately quadratic on V_{z_a}. □

Corollary 3.4. — *Given $\eta > 0$, if $r = r(z)$ and ε are sufficiently small then*

$$\frac{|\gamma_0|}{|V_{z_a}|} < \eta, \quad \frac{|\beta|}{\operatorname{dist}(\beta, \gamma_0)} < \eta, \quad \frac{|\beta|}{\operatorname{dist}(\beta, \partial V_{z_a})} < \eta,$$

for all $\beta \in \mathcal{B}$. Moreover, for each $\beta \in \mathcal{B}$, the diffeomorphism $B : \beta \to \gamma_0$ is extendible to a $\eta^{-1}|\beta|$-neighborhood of β.

Proof. — The first inequality can be obtained with ε small. The intervals $\omega = \omega_a$ accumulate (uniformly on a) in $w = w_a$. If ε is small then ω_0 must be small for every $J_0 \subset [-\varepsilon, \varepsilon]$ and γ_0 will be small as well, compared with V_{z_a}, whose size is approximately constant. Moreover γ_0 is small compared with each one of the connected components of $V_{z_a} \setminus \gamma_0$.

To prove the remaining assertions, observe that $\mathcal{U}(\beta)$ is into the connected component of $V_{z_a} \setminus \gamma_0$ to which β belongs, and $B : \beta \to \gamma_0$ is extendible to $B : \mathcal{U}(\beta) \to V_{z_a}$. By Lemma 3.1, if ε is small then there is an $\eta^{-1}|\beta|$-neighborhood of β in $\mathcal{U}(\beta)$. In particular the other inequalities are valid and B is extendible to this neighborhood. □

4. Circular recovering

In this Section we deal with expansion of derivatives along the iterates which send an interval $\omega \in \mathcal{W}$ onto V_{z_a}. We use Proposition 4.1 below, proved for example in [**13**], which assures some expansion of derivatives provided some simple information is given about the orbit. In the proof of this Proposition, a loss in expansion at a given iterate is compensated by the iterates following it, which is a kind of *forward recovering* of the derivative. Lemma 4.5 below says that the last loss of expansion in the derivative could also be recovered by the first iterates. This could be called a *backward recovering*. We call *circular recovering* the combined use of these techniques. The same ideas appear in Sections 8 and 9, in a slightly different context. They are in the core of this work and deserve a careful attention.

Proposition 4.1. — *There is $C_y > 0$, $\sigma = \sigma(y) > 1$ and $\varepsilon > 0$ such that if $a \in [-\varepsilon, \varepsilon]$ then $F = F(a, \cdot)$ has the following properties.*

1. *If $x, \ldots, F^{k-1}(x) \notin V_{y_a}$ then $|F_x^k(a, x)| \geqslant C_y^{-1}\sigma^k$.*
2. *If $x, \ldots, F^{k-1}(x) \notin V_{z_a}$ and $F^k(x) \in V_{y_a}$ then $|F_x^k(a, x)| \geqslant C_y^{-1}\sigma^k$.*
3. *If $x, \ldots, F^{k-1}(x) \notin V_{z_a}$ then $|F_x^k(a, x)| \geqslant C_y^{-1}\sigma^k \inf_{i=0,\ldots,k-1} |F_x(a, F^i(a, x))|$.*

This constant $\sigma = \sigma(y) > 1$ will be fixed from now on. The first consequence is bounded distortion for iterates outside V_{y_a}.

Lemma 4.2. — *Suppose $\varepsilon > 0$ small and $a \in [-\varepsilon, \varepsilon]$. There is $C_y > 0$ such that if T is an interval satisfying $F^i(T) \cap V_{y_a} = \varnothing$ for all $i = 0, \ldots, j-1$ then*

$$\frac{F_x^j(u)}{F_x^j(v)} \leqslant C_y$$

for all $u, v \in T$.

Proof. — Write

$$\big|\log|F_x^j(u)| - \log|F_x^j(v)|\big| = \left|\sum_{i=0}^{j-1} \log|F_x(F^i u)| - \log|F_x(F^i v)|\right|,$$

which is smaller than

$$\widetilde{C}_y \sum_{i=0}^{j-1} |F^i u - F^i v|,$$

where $\widetilde{C}_y = \max\{|\frac{\partial}{\partial x}\log|F_x(a, x)|| \ ; \ x \notin V_{y_a}\}$, remarking that $F_x(F^i u)$ and $F_x(F^i v)$ have the same sign for $i = 0, \ldots, j-1$. But Proposition 4.1 implies

$$1 \geqslant |F^j u - F^j v| \geqslant C_y^{-1}\sigma^{j-i}|F^i u - F^i v|,$$

proving the Lemma. □

Now we fix some $\omega \in \mathcal{W}$ and $x \in \omega$, and suppose $a \in [-\varepsilon, \varepsilon]$, for ε small. We write $W : \omega \to V_{z_a}$ as $W = F^k|\omega$. The next Lemma says that when the orbit visits the interval V_{y_a} the square of the derivative can be recovered by the next iterates until the next visit of the orbit to V_{y_a}.

Lemma 4.3. — *There is $C_y > 0$ such that if $u = F^l x \in V_{y_a} \smallsetminus V_{z_a}$ and $j \geqslant 2$ is the first integer such that $F^j u \in V_{y_a}$ then*

$$|F_x^{j-1}(Fu)| \cdot |F_x(u)|^2 \geqslant C_y^{-1}.$$

Proof. — Let $T = [Fu, \widetilde{w}_a]$. As F is approximately quadratic and ε is small then

$$|T| \leqslant C_0 |F_x(u)|^2.$$

Hence the Lemma will be proved if we show that

$$|F_x^{j-1}(Fu)| \cdot |T| \geqslant C_y^{-1}.$$

This in turn follows from $|F_x^i(Fu)| \cdot |T| \geqslant C_y^{-1}$, where i is the first integer such that $F^i(T) \cap V_{y_a} \neq \varnothing$, since $i \leqslant j-1$ and $|F_x^{j-1-i}(F^{i+1}u)| \geqslant C_y^{-1}\sigma^{j-1-i}$. Now $F^i(T)$ is an interval intersecting V_{y_a}, but with a point, say $F^i(\widetilde{w}_a)$, outside a neighborhood V containing the closure of V_{y_a} (see definitions of V and V_{y_a} in Section 2). This implies that there is $d > 0$ such that $|F^i(T)| \geqslant d$.

By Lemma 4.2,

$$d \leqslant |F^i(T)| \leqslant C_y |F_x^i(Fu)| \cdot |T|,$$

proving the Lemma. □

The following Lemma is a corollary of the proof of Lemma 4.3. It says that the square of a bad derivative $F_x(u)$, $u = F^l x$, may also be recovered by the *first iterates* of the orbit of x.

Lemma 4.4. — *Let $i_1 \geqslant 1$ be the first integer such that $F^{i_1} x \in V_{y_a}$. There is $C_y > 0$ such that if $u = F^l x$, $l < k$, is such that $u \in V_{y_a} \smallsetminus V_{z_a}$ then*

$$|F_x^{i_1}(x)| \cdot |F_x(u)|^2 \geqslant C_y^{-1}.$$

Proof. — As in the proof of Lemma 4.3, let $T = [Fu, \widetilde{w}_a]$. We want to show that $|F_x^{i_1}(x)| \cdot |T| \geqslant C_y^{-1}$. If i is the first integer such that $F^i(T) \cap V_{y_a} \neq \varnothing$ then $i \leqslant i_1$ (since $T \supset \omega$). Hence it suffices to show that $|F_x^i(x)| \cdot |T| \geqslant C_y^{-1}$. But by the bounded distortion of the derivative of $F^i|T$ and since $x \in T$ we have $d \leqslant |F^i(T)| \leqslant C_y |F_x^i(x)| \cdot |T|$, for some fixed $d > 0$, and the Lemma follows. □

If the square of a bad derivative is recovered by the first iterates then the same happens with the derivative itself. This is the content of the following Corollary. Let $\lambda = \lambda(y) = \sqrt{\sigma(y)}$, where σ is given by Proposition 4.1.

Corollary 4.5. — *Let $i_1 \geqslant 1$ be the first integer such that $F^{i_1}x \in V_{y_a}$. There is $C_y > 0$ such that if $u = F^l x$, $l \leqslant k$, is such that $u \in V_{y_a} \setminus V_{z_a}$ then*

$$|F_x^{i_1}(x)| \cdot |F_x(u)| \geqslant C_y^{-1}\lambda^{i_1}.$$

Proof. — By Lemma 4.4,

$$|F_x^{i_1}(x)|^{1/2} \cdot |F_x(u)| \geqslant C_y^{-1/2}.$$

On the other hand, $x, Fx, \dots, F^{i_1-1}x \notin V_{y_a}$, hence by Proposition 4.1

$$|F_x^{i_1}(x)|^{1/2} \geqslant C_y^{-1/2}(\sigma^{1/2})^{i_1}.$$

The Corollary is proved if we multiply both sides of the first inequality by $|F_x^{i_1}(x)|^{1/2}$ and then use the second inequality. □

The following two Corollaries will be directly applied in the following Sections.

Corollary 4.6. — *There are $C_y > 0$ and $\lambda = \lambda(y) > 1$ such that the following holds. For all $x \in \omega$, $\omega \in \mathcal{W}$ with transfer map $W = F^k|\omega : \omega \to V_{z_a}$ and $u = F^l x$, for $0 \leqslant l \leqslant k-1$, we have:*

(1) $|F_x^{k-l}(u)| \geqslant C_y^{-1}\lambda^{k-l}$.
(2) *If* $u \notin V_{y_a}$ *then* $|F_x^{k-l-1}(Fu)| \cdot |F_x(u)|^2 \geqslant C_y^{-1}\lambda^{k-l}$.
(3) *If* $u \in V_{y_a}$ *then* $|F_x^{k-l-1}(Fu)| \cdot |F_x(u)|^2 \geqslant C_y^{-1}\lambda^{s}$, *where*

$$s = \#\{l+1 \leqslant i < k\ ;\ F^i x \in V_{y_a}\}.$$

Proof. — The first inequality comes directly from Proposition 4.1. It is valid also for λ since $\lambda < \sigma$. The second inequality follows if we use the first and observe that if $u \notin V_{y_a}$ then $|F_x(u)| \geqslant C_0^{-1}|V_{y_a}| \geqslant C_y^{-1}$. For the last inequality we use Lemma 4.3 to assure that the square of the bad derivative is recuperated until the next visit of the orbit to V_{y_a}. From this moment on we use the expansion given by the first inequality, with unknown number of iterates surely greater or equal than s. □

Corollary 4.7. — *There are $C_y > 0$ and $\lambda = \lambda(y) > 1$ such that*

$$|F_x^j(x)| \geqslant C_y^{-1}\lambda^j$$

for all $x \in \omega$, $\omega \in \mathcal{W}$ with transfer map $W = F^k : \omega \to V_{z_a}$ and $1 \leqslant j \leqslant k$.

Proof. — Let $l \leqslant j$ be the last iterate such that $F^l x \in V_{y_a}$ and $1 \leqslant i_1 \leqslant l$ be the first iterate such that $F^{i_1}x \in V_{y_a}$. If $j = l$ then Proposition 4.1 implies the Corollary. Otherwise we write

$$|F_x^j(x)| = |F_x^{j-l-1}(F^{l+1}x)| \cdot |F_x(F^l x)| \cdot |F_x^{l-i_1}(F^{i_1}x)| \cdot |F_x^{i_1}(x)|.$$

As $F^{l+1}x, F^{l+2}x, \dots, F^j x \notin V_{y_a}$, by Proposition 4.1 we have

$$|F_x^{j-l-1}(F^{l+1}x)| \geqslant C_y^{-1}\lambda^{j-l-1}.$$

In addition, $F_x^{i_1}, \ldots, F^{l-1}x \notin V_{z_a}$ and $F^l x \in V_{y_a}$, hence again by Proposition 4.1 we have $|F_x^{l-i_1}(F^{i_1}x)| \geqslant C_y^{-1}\lambda^{l-i_1}$. Finally, by Corollary 4.5, $|F_x(F^l x)| \cdot |F_x^{i_1}(x)| \geqslant C_y^{-1}\lambda^{i_1}$. □

5. Exploring transversality

In this Section we combine the estimates of Section 4 with the transversality assumption. For $\omega \in \mathcal{W}$ with transfer map $W = F^k|\omega : \omega \to V_{z_a}$ we may define $x_\omega = x_{\omega,a} = W^{-1}(c)$ as the "center" of $\omega = \omega_a$.

Using the Glossary (at the end of this work), we obtain

$$\frac{d}{da}x_{\omega,a} = -\frac{W_a}{W_x}(a, x_{\omega,a}) = -\sum_{i=1}^{k} \frac{F_a \circ F^{i-1}}{F_x^i}(a, x_{\omega,a}).$$

We want in fact to give estimates on W_a/W_x for every $x \in \omega$ and even estimates on F_a^j/F_x^j, for every $x \in \omega$ and $j = 1, \ldots, k$, as in the following Lemma.

Lemma 5.1. — *There is $C_y > 0$ such that*

$$\left|\frac{F_a^j}{F_x^j}\right| \leqslant C_y,$$

for every $x \in \omega$, $\omega \in \mathcal{W}$ with transfer map $W = F^k|\omega : \omega \to V_{z_a}$ and $j = 1, \ldots, k$.

Proof. — By the Glossary,

$$\frac{F_a^j}{F_x^j} = \sum_{i=1}^{j} \frac{F_a \circ F^{i-1}}{F_x^i}.$$

But F_a is bounded by C_0 and $|F_x^i| \geqslant C_y^{-1}\lambda^i$, by Corollary 4.7. □

Lemma 5.2. — *Given $\delta > 0$, there are an integer $k = k(\delta, y) \geqslant 1$ and $\mu = \mu(\delta) > 0$ such that if $\omega_1, \omega_2 \in \mathcal{W}$ have transfer maps $W_s = F^{k_s}|\omega_s$, $s = 1, 2$ with $k_1, k_2 \geqslant k$ and moreover $x_s \in \omega_s$, $s = 1, 2$, satisfy $|x_1 - x_2| < \mu$ then*

$$\left|\frac{W_{1,a}}{W_{1,x}}(x_1) - \frac{W_{2,a}}{W_{2,x}}(x_2)\right| < \delta.$$

Proof. — Let $C_y > 0$ and $\lambda = \lambda(y)$ be as in Corollary 4.7 and let

$$C_0 > \max\{|F_a| \; ; \; a \in [-\varepsilon, \varepsilon], \; x \in [0, 1]\}.$$

Let $k = k(\delta, y)$ be such that

$$C_y C_0 \frac{\lambda^{-k}}{1 - \lambda^{-1}} < \frac{\delta}{4}.$$

Write

$$\frac{W_{s,a}}{W_{s,x}}(x_s) = \sum_{i=1}^{k_s} \frac{F_a \circ F^{i-1}}{F_x^i}(a, x_s),$$

for $s = 1, 2$. If $k_s > k$ then

$$\left|\sum_{i=k+1}^{k_s} \frac{F_a \circ F^{i-1}}{F_x^i}(a, x_s)\right| \leqslant C_y C_0 \frac{\lambda^{-k}}{1-\lambda^{-1}} < \frac{\delta}{4},$$

for $s = 1, 2$, using Corollary 4.7 and the choice of k. Then we are left to proving that

$$\left|\sum_{i=1}^{k} \frac{F_a \circ F^{i-1}}{F_x^i}(a, x_1) - \sum_{i=1}^{k} \frac{F_a \circ F^{i-1}}{F_x^i}(a, x_2)\right| < \frac{\delta}{2}.$$

But this is true if $|x_1 - x_2| < \mu$, for sufficiently small $\mu > 0$. $\square$

Let us see what are the consequences of Lemma 5.2. Let $\{\omega_N\}_N$ be a sequence converging to w at $a = 0$. In particular the centers $x_N = x_{\omega_N}$ converge to w and their continuations $a \mapsto x_{N,a}$ converge in the C^0 topology to $a \mapsto w_a$, for $a \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$ small. This is easy to be proved since the rates of expansion outside V_{z_a} are uniform. By Lemma 5.2, $\{a \mapsto x_{N,a}\}_N$ also converges in the C^1 topology. This leads to a formulae on $\dfrac{d}{da} w_a$:

$$\frac{d}{da} w_a = -\sum_{i=1}^{\infty} \frac{F_a \circ F^{i-1}}{F_x^i}(a, w_a).$$

Now let $\nu > 0$ be such that

$$F_a(0, c) - \frac{d}{da} w_a \bigg|_{a=0} \geqslant 2\nu,$$

by the transversality condition. This implies that if V_z is chosen sufficiently small, in order that every $\omega \in \mathcal{W}$ is forced to be near w, and $a \in [-\varepsilon, \varepsilon]$, for $\varepsilon > 0$ small, then

$$\frac{W_a}{W_x}(a, x) \geqslant \nu - F_a(0, c),$$

for every $x \in \omega = \omega_a$, $\omega \in \mathcal{W}$ and $a \in [-\varepsilon, \varepsilon]$.

Moreover, if $\varepsilon > 0$ is small then for every point $x \in \Lambda_{z_0}$ in $[f_0(z), \widetilde{w}]$, its continuation $x_a = h_a(x)$ has velocity smaller than $\frac{d}{da} f_a(c) - \nu$. This implies two things: (i) to each $x \in \Lambda_z$ corresponds (at most) a single point $a = a(x) \in [-\varepsilon, \varepsilon]$ in the parameter space such that $f_a(c) = x_a$ and (ii) for every $\omega \in \mathcal{W}$ the set $\{a \in [-\varepsilon, \varepsilon] \ ; \ f_a(c) \in \omega = \omega_a\}$ is an interval.

Define

$$\Gamma = \{a \in [-\varepsilon, \varepsilon] \ ; \ f_a(c) \in \Lambda_{z_a}\},$$

which is totally disconnected. We will prove below that $\mathrm{Leb}(\Gamma) = 0$. Each gap of Γ corresponds to the parameters for which $f_a(c)$ belongs to some $\omega = \omega_a \in \mathcal{W}$. The collection of gaps in the complement of Γ will be called $\mathcal{J}_0$, and from the next Section on we shall restrict our attention to a particular element J_0 of this collection, as already described in Section 2.

Lemma 5.3. — $\mathrm{Leb}(\Gamma) = 0$.

Proof. — Without loss of generality and for simplicity we will consider in this proof only the negative range $[-\varepsilon, 0]$ and will assume the following: for $a = -\varepsilon$, $f_a(c)$ belongs to the leftmost boundary point of some $\omega_1 = \omega_{1,a} \in \mathcal{W}$, with transfer map $W_1 = F^{n_1}|\omega_1 : \omega_1 \to V_{z_a}$, and any other $\omega = \omega_a \in \mathcal{W}$ between $\omega_{1,a}$ and w_a has transfer map $W = F^n|\omega : \omega \to V_{z_a}$ with $n > n_1$.

For each interval family $I = (I_a)_a$ let

$$J(I) = \{a \in [-\varepsilon, \varepsilon] \; ; \; f_a(c) \in I_a\}.$$

Let $T^1 = (T^1_a)_a$ be the family of intervals with boundary points $\partial_+ T^1_a = w_a$ and $\partial_- T^1_a$ the rightmost point of $\omega_{1,a}$. It is not difficult to see that the following reasoning is independent of a, so we omit the subindex. Let $n_2 \geqslant 1$ be the first integer such that $f^{n_2}(T^1)$ intersects V_z. Then $n_2 > n_1$ and $f^{n_2}(T^1)$ must contain V_z (in fact V_y), since $f^i(\partial_+ T^1) \notin V_y$, $\forall i \geqslant 0$, and $f^i(\partial_- T^1) \notin V_y$, $\forall i \geqslant n_1 + 1$ (by the definition of V_z and V_y). Therefore there is $\omega_2 \in \mathcal{W}$, $\omega_2 \subset T^1$, with transfer map $W_2 = F^{n_2}|\omega_2 : \omega_2 \to V_z$. Moreover, any other $\omega \in \mathcal{W}$ between ω_1 and ω_2 or else between ω_2 and w has transfer map $W = F^k|\omega : \omega \to V_z$ with $n > n_2$.

By Proposition 4.1, the expansion outside V_z is uniform, up to a constant which depends only on the choice of z. Therefore, analogously to Lemma 4.2, we have bounded distortion for iterates outside V_z, this time with a constant C_z. In this particular case, this means that

$$\frac{F^{n_2}_x(x)}{F^{n_2}_x(y)} \leqslant C_z$$

for every $x, y \in T^1$. Hence

$$\frac{|\omega_2|}{|T^1|} \geqslant C_z^{-1} |V_z|.$$

It is easy to see, because of the bounds on velocities, that

$$\frac{|J(\omega_2)|}{|J(T^1)|} \geqslant C_0^{-1} C_z^{-1} |V_z| \equiv \widetilde{C}_z^{-1}.$$

The interval $J(\omega_2)$ is in the complement of Γ. Hence at this stage $\mathrm{Leb}(\Gamma) \leqslant (1 - \widetilde{C}_z^{-1}) |J(T^1)|$. The argument continues by induction in the remaining connected components of $J(T^1) \smallsetminus J(\omega_2)$, and so on, in order that at every stage a definite z-dependent fraction of parameters not belonging to Γ is suppressed from the remaining ones. This proves the Lemma. □

6. Transfer maps

Let $\omega \in \mathcal{W}$ and $W = F^k|\omega : \omega \to V_{z_a}$ its transfer map. We have already established bounds on W_a / W_x in Section 5. In this Section we control the quotients

$$\frac{W_{xx}}{(W_x)^2}, \quad \frac{W_{xa}}{(W_x)^2}, \quad \frac{W_{xxx}}{(W_x)^3}, \quad \frac{W_{aa}}{(W_x)^2}, \quad \frac{W_{xxa}}{(W_x)^3}.$$

Once more we suppose that V_y is already chosen, and then take V_z sufficiently small. We always assume $\omega \in \mathcal{W}$ as above and $a \in [-\varepsilon, \varepsilon]$, for $\varepsilon > 0$ sufficiently small, but constants are independent of these choices. In the Lemmas we omit the argument of functions. It is implicit that they are calculated for $a \in [-\varepsilon, \varepsilon]$ and $x \in \omega$. If we write $F_x \circ F^{i-1}$, for example, it means $F_x(a, F^{i-1}(a, x))$. In this notation, $|F^i - c|$ is the distance from the critical point to the i-th iterate of F.

We start by proving a technical Lemma which is a direct consequence of Corollary 4.6. The goal is to bound the sum

$$S_j = \sum_{i=1}^{j} |(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2|^{-1},$$

where $j \leqslant k$, which appears in all Lemmas of this Section.

Lemma 6.1. — *There is $C_y > 0$ such that $S_j \leqslant C_y$, for all $j \leqslant k$.*

Proof. — This follows from Corollary 4.6. Separate the sum S_j into two sums: the first, containing only those $2 \leqslant i \leqslant j$ such that $F^{i-1} \in V_{y_a}$, is bounded by a y-dependent geometric series, following the third item of the Corollary, and the second, containing only those $1 \leqslant i \leqslant j$ such that $F^{i-1} \notin V_{y_a}$, is also bounded by a y-dependent geometric series, following the second item of the Corollary. □

Lemma 6.2. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|V_{z_a}| \cdot \left| \frac{W_{xx}}{(W_x)^2} \right| < \delta$$

for all $x \in \omega$, $\omega \in \mathcal{W}$. Moreover, if as above $W = F^k|\omega$ then

$$\frac{|V_{z_a}|}{|F_x^{k-j} \circ F^j|} \cdot \left| \frac{F_{xx}^j}{(F_x^j)^2} \right| < \delta,$$

for all $x \in \omega$ and $j = 1, \ldots, k-1$.

Proof. — Write

$$\frac{1}{F_x^{k-j} \circ F^j} \cdot \frac{F_{xx}^j}{(F_x^j)^2} = \sum_{i=1}^{j} \frac{F_{xx} \circ F^{i-1}}{(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2},$$

for $1 \leqslant j \leqslant k$. As $|F_{xx}|$ is bounded by C_0, the sum is bounded by $C_0 S_j$, where S_j was given above, hence by $C_0 C_y$, by Lemma 6.1. The Lemma is proved if we multiply by $|V_{z_a}|$ and take V_z sufficiently small. □

Lemma 6.3. — *Given $\delta > 0$, if V_z is small enough then*

$$|V_{z_a}|^2 \cdot \left| \frac{W_{xxx}}{(W_x)^3} \right| < \delta,$$

for all $x \in \omega$ and $\omega \in \mathcal{W}$.

Proof. — Write

$$\frac{W_{xxx}}{(W_x)^3} = S_1 + 3S_2,$$

where

$$S_1 = \sum_{i=1}^{k} \frac{F_{xxx} \circ F^{i-1}}{(F_x^{k-i} \circ F^i)^2 (F_x \circ F^{i-1})^3}$$

and

$$S_2 = \sum_{i=2}^{k} \frac{F_{xx} \circ F^{i-1}}{(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2} \cdot \frac{1}{F_x^{k-i+1} \circ F^{i-1}} \frac{F_{xx}^{i-1}}{(F_x^{i-1})^2}.$$

We start by estimating $|V_{z_a}|^2 S_2$. By Lemma 6.2,

$$\frac{|V_{z_a}|}{F_x^{k-i+1} \circ F^{i-1}} \frac{F_{xx}^{i-1}}{(F_x^{i-1})^2}$$

is smaller than δ, for every $i = 2, \ldots, k$, provided V_z is small. Using Lemma 6.1 as in Lemma 6.2 we have

$$|V_{z_a}|^2 |S_2| \leqslant |V_{z_a}| C_0 C_y \delta,$$

which is smaller than $\delta/6$ if V_z is sufficiently small.

Similarly, using the first item of Corollary 4.6, we bound $|V_{z_a}|^2 S_1$ by

$$C_0 C_y |V_{z_a}|^2 \sum_{i=1}^{k} |(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2|^{-1}$$

which is smaller than $\delta/2$, if V_z is sufficiently small, by Lemma 6.1. □

Let $k = k(\omega)$ be the transfer time from ω to V_{z_a}, for $\omega \in \mathcal{W}$. Let

$$N = \min\{k(\omega)\ ;\ \omega \in \mathcal{W}\}.$$

By the definition of $\mathcal{W}$, if V_z is small then any $\omega \in \mathcal{W}$ must be near w_a, hence N is big. In the following Lemma we use the fact that N/λ^N is as small as we wish, provided V_z is sufficiently small.

Lemma 6.4. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|V_{z_a}| \cdot \left| \frac{W_{xa}}{(W_x)^2} \right| < \delta$$

for all $x \in \omega$, $\omega \in \mathcal{W}$. Moreover

$$\frac{|V_{z_a}|}{|F_x^{k-j} \circ F^j|} \cdot \left| \frac{F_{xa}^j}{(F_x^j)^2} \right| < \delta,$$

for all $j < k$, where $k = k(\omega)$.

Proof. — Write

$$\frac{1}{F_x^{k-j}\circ F^j}\cdot\frac{F_{xa}^j}{(F_x^j)^2}=\sum_{i=1}^{j}\frac{F_{xa}\circ F^{i-1}}{W_x(F_x\circ F^{i-1})}+\sum_{i=2}^{j}\frac{F_{xx}\circ F^{i-1}}{(F_x^{k-i}\circ F^i)(F_x\circ F^{i-1})^2}\cdot\frac{F_a^{i-1}}{F_x^{i-1}}.$$

We have $|F_{xa}|<C_0$, $|W_x|\geqslant C_y^{-1}\lambda^k$ and $|F_x\circ F^{i-1}|\geqslant C_0^{-1}|V_{z_a}|$, therefore

$$|V_{z_a}|\cdot\left|\sum_{i=1}^{j}\frac{F_{xa}\circ F^{i-1}}{W_x(F_x\circ F^{i-1})}\right|<C_0^2C_y\frac{k}{\lambda^k},$$

which is smaller than $\delta/2$ if the choice of V_z implies N sufficiently big.

Moreover, by Lemma 5.1

$$\left|\frac{F_a^{i-1}}{F_x^{i-1}}\right|\leqslant C_y$$

But Lemma 6.1 implies that

$$C_y|V_{z_a}|\cdot\sum_{i=2}^{j}\left|\frac{F_{xx}\circ F^{i-1}}{(F_x^{k-i}\circ F^i)(F_x\circ F^{i-1})^2}\right|<\frac{\delta}{2}$$

if V_z is sufficiently small. □

Lemma 6.5. — *Given $\delta>0$, if V_z is sufficiently small then*

$$|V_{z_a}|\cdot\left|\frac{W_{aa}}{(W_x)^2}\right|<\delta,$$

for every $x\in\omega$, $\omega\in\mathcal{W}$.

Proof. — Write

$$\frac{W_{aa}}{(W_x)^2}=S_1+2S_2+S_3,$$

where

$$S_1=\sum_{i=1}^{k}\frac{F_{aa}\circ F^{i-1}}{W_xF_x^i},$$

$$S_2=\sum_{i=2}^{k}\frac{F_{xa}\circ F^{i-1}}{W_xF_x^i}\cdot\frac{F_a^{i-1}}{F_x^{i-1}}$$

and

$$S_3=\sum_{i=2}^{k}\frac{F_{xx}\circ F^{i-1}}{(F_x^{k-i}\circ F^i)(F_x\circ F^{i-1})^2}\cdot\left(\frac{F_a^{i-1}}{F_x^{i-1}}\right)^2.$$

The proof follows as in Lemma 6.4. Note that here denominators are slightly better, and $|F_x^i|$ can be estimated using Corollary 4.7. □

Lemma 6.6. — *Given $\delta>0$, if V_z is sufficiently small then*

$$|V_{z_a}|^2\cdot\left|\frac{W_{xxa}}{(W_x)^3}\right|<\delta,$$

for all $x\in\omega$, $\omega\in\mathcal{W}$.

Proof. — Write

$$\frac{W_{xxa}}{(W_x)^3} = S_1 + S_2 + S_3 + 2S_4 + 2S_5,$$

where

$$S_1 = \sum_{i=1}^{k} \frac{F_{xxa} \circ F^{i-1}}{W_x(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2},$$

$$S_2 = \sum_{i=2}^{k} \frac{F_{xxx} \circ F^{i-1})(F_x \circ F^{i-1})}{(F_x^{k-i} \circ F^i)^2(F_x \circ F^{i-1})^4} \cdot \frac{F_a^{i-1}}{F_x^{i-1}},$$

$$S_3 = \sum_{i=2}^{k} \frac{F_{xa} \circ F^{i-1}}{W_x F_x^i} \cdot \frac{1}{F_x^{k-i+1} \circ F^{i-1}} \cdot \frac{F_{xx}^{i-1}}{(F_x^{i-1})^2}$$

$$S_4 = \sum_{i=2}^{k} \frac{F_{xx} \circ F^{i-1}}{(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2} \cdot \frac{2}{F_x^{k-i+1} \circ F^{i-1}} \cdot \frac{F_{xa}^{i-1}}{(F_x^{i-1})^2}$$

and

$$S_5 = \sum_{i=2}^{k} \frac{F_{xx} \circ F^{i-1}}{(F_x^{k-i} \circ F^i)(F_x \circ F^{i-1})^2} \cdot \frac{F_a^{i-1}}{F_x^{i-1}} \cdot \frac{1}{F_x^{k-i+1} \circ F^{i-1}} \cdot \frac{F_{xx}^{i-1}}{(F_x^{i-1})^2}.$$

The only “new” term to pay attention is

$$\frac{|V_{z_a}|}{F_x^{k-i+1}} \cdot \frac{F_{xa}^{i-1}}{(F_x^{i-1})^2},$$

but it can be bounded using Lemma 6.4. □

7. Central branch

Now we fix J_0, the parameter interval such that the critical value belongs to $\omega_0 \in \mathcal{W}$. Therefore the central branch $H : \gamma_0 \to V_{z_a}$ of the first return map to V_{z_a} may be written as $H = W_0 \circ F$, where $W_0 : \omega_0 \to V_{z_a}$ is the transfer map associated to ω_0.

All the Lemmas below depend on the fact that V_z and ε are sufficiently small, so we omit it in the statements.

The following Lemma shows in particular that $H_a(a, x)$ is nonzero for every $x \in \gamma_0$, $a \in J_0$ and its sign is determined by the sign of $W_{0,x}$.

Lemma 7.1

$$\frac{H_a(a, x)}{W_{0,x}(a, F(a, x))} \geqslant \frac{\nu}{2},$$

for every $x \in \gamma_0$ and $a \in J_0$.

Proof. — Write

$$H_a(a, x) = W_{0,a}(a, F(a, x)) + W_{0,x}(a, F(a, x))F_a(a, x).$$

If V_z and ε are small then $F_a(a, x)$ is very near $F(0, c)$. But in Section 5 we have shown that

$$\frac{W_{0,a}}{W_{0,x}} \geqslant \nu - F_a(0, c),$$

and the estimate follows. □

An analogous statement is valid for H_{xx}.

Lemma 7.2

$$\frac{H_{xx}(a, x)}{W_{0,x}(a, F(a, x))} \simeq F_{xx}(0, c) < 0,$$

for all $x \in \gamma_0$ *and* $a \in J_0$.

Proof. — As the critical point is quadratic, $(F_x(a, x))^2 < C_0|\omega_0|$ for every $x \in \gamma_0$. Moreover, the function $W_0 : \omega_0 \to V_{z_a}$ has small distortion, by Lemma 3.1 and Proposition 2.3, implying that

$$|W_{0,x}| \simeq \frac{|V_{z_a}|}{|\omega_0|}.$$

Also, by continuity, $F_{xx}(a, x) \simeq F_{xx}(0, c)$. We have

$$\frac{H_{xx}}{W_{0,x}(a, F)} = (F_x)^2 W_{0,x}(a, F) \frac{W_{0,xx} \circ F}{(W_{0,x} \circ F)^2} + F_{xx},$$

hence the Lemma is proved if we show that

$$C_0 |V_{z_a}| \frac{W_{0,xx}}{(W_{0,x})^2}$$

is small for points in ω_0. But this is true by Lemma 6.2, choosing V_z small with respect to V_y. □

At this point we are ready to prove the four starting conditions relative to the central branch. For simplicity, we write from now on $W = W_0$ to designate the transfer map of ω_0.

Lemma 7.3. — *Given* $\delta > 0$, V_z *is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{H_{xxx}}{H_{xx}} \right| < \delta,$$

for all $x \in \gamma_0$ *and* $a \in J_0$.

Proof. — As $H = W \circ F$ we have

$$H_{xxx} = (W_x \circ F) F_{xxx} + (W_{xxx} \circ F)(F_x)^3 + 3(W_{xx} \circ F) F_x F_{xx}.$$

We analyze these three terms, each one divided by H_{xx} and multiplied by $|\gamma_0|$. By Lemma 7.2,

$$H_{xx} \simeq s_0 (W_x \circ F),$$

where $s_0 = F_{xx}(0,c)$. Hence

$$|\gamma_0| \cdot \left| \frac{(W_x \circ F) F_{xxx}}{H_{xx}} \right| < \frac{2C_0}{|s_0|} |\gamma_0|,$$

which is smaller than $\delta/3$ if V_z is small (since $|\gamma_0| \ll |V_{z_a}|$). The second term can be written as

$$|\gamma_0| (F_x)^3 \, \frac{(W_x \circ F)^3}{H_{xx}} \, \frac{W_{xxx} \circ F}{(W_x \circ F)^3}.$$

But $|F_x|^3 \leqslant C_0 |\gamma_0|^3$,

$$|W_x \circ F| \simeq \frac{|V_{z_a}|}{|\omega_0|}$$

and $|\gamma_0|^2 \leqslant C_0 |\omega_0|$, hence the second term is bounded by

$$\frac{2C_0^3}{|s_0|} |V_{z_a}|^2 \left| \frac{W_{xxx} \circ F}{(W_x \circ F)^3} \right|$$

which, according to Lemma 6.3, can be smaller than $\delta/3$ if V_z is sufficiently small. Similarly, the last term is bounded by

$$\frac{6C_0^4}{|s_0|} |V_{z_a}| \cdot \left| \frac{W_{xx} \circ F}{(W_x \circ F)^2} \right|,$$

which can be made smaller than $\delta/3$ by the choice of V_z, according to Lemma 6.2. □

Lemma 7.4. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{H_{ax}}{H_a} \right| < \delta$$

for all $x \in \gamma_0$ and $a \in J_0$.

Proof. — Writing $H = W \circ F$ we obtain

$$H_{xa} = (W_x \circ F) F_{xa} + (W_{xa} \circ F) F_x + (W_{xx} \circ F) F_x F_a$$

and then analyze each term when multiplied by $|\gamma_0| \cdot |H_a|^{-1}$. By Lemma 7.1 we have

$$|H_a(a,x)| \geqslant \frac{\nu}{2} |W_x \circ F|,$$

so that

$$|\gamma_0| \cdot \left| \frac{W_x \circ F}{H_a} \right| \cdot |F_{xa}| < \frac{2C_0}{\nu} |\gamma_0|,$$

which is smaller than $\delta/3$ if V_z (and hence γ_0) is sufficiently small. Also,

$$|\gamma_0| \cdot |F_x| \left| \frac{W_{xa} \circ F}{H_a} \right| < \frac{2}{\nu} |\gamma_0| \cdot |F_x| \cdot |W_x \circ F| \cdot \left| \frac{W_{xa} \circ F}{(W_x \circ F)^2} \right|.$$

As in the proof of the previous Lemma, $|F_x| < C_0 |\gamma_0|$, $|W_x \circ F| \simeq |V_{z_a}|/|\omega_0|$ and $|\gamma_0|^2/|\omega_0| \leqslant C_0$, so that $|\gamma_0| \cdot |F_x| \cdot |W_x \circ F| \leqslant 2C_0^2 |V_{z_a}|$. Then Lemma 6.4 implies that

$$\frac{4C_0^2}{\nu} |V_{z_a}| \cdot \left| \frac{W_{xa} \circ F}{(W_x \circ F)^2} \right| < \frac{\delta}{3},$$

provided V_z is small enough.

The same argument combined with Lemma 6.2 is applied to the last term, proving the Lemma. □

Lemma 7.5. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|J_0| \cdot \left|\frac{H_{aa}}{H_a}\right| < \delta$$

for all $x \in \gamma_0$ and $a \in J_0$.

Proof. — First observe that $|J_0| \leqslant C_0|\omega_0|$ and

$$|H_a| \geqslant \frac{\nu}{2}\,|W_x \circ F| \simeq \frac{\nu}{2} \cdot \frac{|V_{z_a}|}{|\omega_0|}.$$

Then write

$$H_{aa} = (W_x \circ F)F_{aa} + W_{aa} \circ F + (W_{xa} \circ F)F_a + (W_{xx} \circ F)(F_a)^2$$

and proceed as in the previous Lemmas, using also Lemma 6.5. □

Lemma 7.6. — *Given $\delta > 0$, if V_z is sufficiently small, then*

$$|J_0| \cdot \left|\frac{H_{xxa}}{H_{xx}}\right| < \delta,$$

for all $x \in \gamma_0$ and $a \in J_0$.

Proof. — The proof is similar to the previous Lemmas, after writing

$$\begin{aligned} H_{xxa} &= (W_x \circ F)F_{xxa} + (W_{xxa} \circ F)(F_x)^2 \\ &\quad + (W_{xxx} \circ F)F_a(F_x)^2 + (W_{xx} \circ F)(2F_{xa}F_x + F_aF_{xx}) + (W_{xa} \circ F)F_{xx}. \end{aligned}$$ □

8. Expansion of regular branch compositions

We aim at proving the starting conditions for preimages of the central branch. To any preimage $\beta \in \mathcal{B}$ is assigned a sequence of regular branches

$$\{P_m : \pi_m \longrightarrow V_{z_a}\}_{m=1,\dots,n}$$

such that $B : \beta \to \gamma_0$ is written as $B = P_n \circ \cdots \circ P_1|\beta$. Each P_m in turn is written as $P_m = W_m \circ F|\pi_m$, where $W_m : \omega_m \to V_{z_a}$ is the transfer map of $\omega_m \in \mathcal{W}$, $\omega_m \neq \omega_0$, $m = 1, \dots, n$.

This Section is devoted to estimate the expansion of regular branches and their compositions. The ideas involved here are very similar to the concept of forward recovering, mentioned in Section 4. A kind of backward recovering appears in Section 9, when dealing with the first parameter derivative.

The first estimates give absolute lower bounds for derivatives of regular branches. We will see that expansion may be not sure in some cases. Next we show that every time there is a loss of derivative for some P_m there is an immediate recuperation for P_{m+1}.

From now on we choose a constant $\theta > 0$, and take V_z small so that Corollaries 3.2 and 3.3 are satisfied. This constant will be chosen sufficiently small, according to the needing of various Lemmas until the end of the work. It is implicitly assumed that assertions are valid for every $\beta \in \mathcal{B}$ and constants do not depend on β.

Accordingly to Section 2, the intervals $\omega_0, \omega_1, \ldots, \omega_n$ have extension domains $\widehat{\omega}_0$, $\widehat{\omega}_1, \ldots, \widehat{\omega}_n$ which are mapped onto V_{y_a}. Take ω_m, for some $m = 1, \ldots, n$. We will say that ω_m is subordinated to ω_0 if $\omega_m \subset \widehat{\omega}_0$, that ω_0 is subordinated to ω_m if $\omega_0 \subset \widehat{\omega}_m$ and that ω_0 and ω_m are independent otherwise. By construction it turns out that one and only one of these situations occurs. In particular, this implies by Corollary 3.2 that in the first and third cases

$$\operatorname{dist}(\omega_m, \omega_0) \geqslant \theta^{-1}|\omega_m|,$$

and in the second case

$$\operatorname{dist}(\omega_m, \omega_0) \geqslant \theta^{-1}|\omega_0|.$$

Lemma 8.1. — *If ω_m is subordinated to ω_0, or else ω_m and ω_0 are independent, then*

$$|P_{m,x}(x)| \geqslant C_0^{-1}\theta^{-1},$$

for all $x \in \pi_m$.

Proof. — Write $P_m = W_m \circ F|\pi_m$, hence $|P_{m,x}| = |W_{m,x} \circ F| \cdot |F_x|$. As $W_m : \omega_m \to V_{z_a}$ is extendible to $W_m : \widehat{\omega_m} \to V_{y_a}$, and V_z is chosen small, then by Lemma 3.1

$$|W_{m,x} \circ F| \simeq \frac{|V_{z_a}|}{|\omega_m|}.$$

But the hypotheses imply that

$$|F_x| \geqslant C_0^{-1}\sqrt{\operatorname{dist}(\omega_m, \omega_0)} \geqslant C_0^{-1}\theta^{-1/2}|\omega_m|^{1/2}.$$

Therefore

$$|P_{m,x}| \geqslant C_0^{-1}\theta^{-1/2}\frac{|V_{z_a}|}{|\omega_m|^{1/2}}.$$

On the other hand, if ε is small then $|\widehat{\omega}_m| < 2|\operatorname{Im} F|V_{z_a}|$, hence

$$|\omega_m|^{1/2} \leqslant \sqrt{2}\theta^{1/2}|\operatorname{Im} F|V_{z_a}|^{1/2} \leqslant C_0\theta^{1/2}|V_{z_a}|,$$

and the Lemma follows. □

Lemma 8.2. — *If ω_0 is subordinated to ω_m then*

$$|P_{m,x}(x)| \geqslant C_0^{-1}\theta^{-3/2}\frac{|\omega_0|^{1/2}}{|V_{z_a}|},$$

for all $x \in \pi_m$.

Proof. — As in the proof of the previous Lemma,

$$|P_{m,x}| \geqslant C_0^{-1} \frac{|V_{z_a}|}{|\omega_m|} \theta^{-1/2} |\omega_0|^{1/2}.$$

On the other hand, also as in the previous Lemma,

$$|\omega_m| < C_0 \theta |V_{z_a}|^2,$$

and the Lemma follows. □

These two Lemmas suggest that a bad derivative may occur if ω_0 is subordinated to ω_m, as depicted in Figure 4. The problem is overcome with forward recuperation,

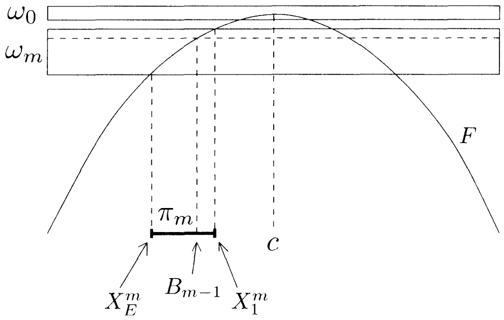

FIGURE 4. Possible bad derivative in the m-th iterate

which we describe now. Let

$$B_m \equiv P_m \circ \cdots \circ P_1 | \beta,$$

hence $B = B_n$. Let B_m denote the point $B_m(x)$, as we did before in other situations, when it is clear that there is no possibility of confusion. In this notation, $B_{m-1} \in \pi_m$, and $F(B_{m-1}) \in \omega_m$, see Figure 4. We call x_I^m and x_E^m, respectively, the innermost and the outermost boundary points of π_m, with respect to the critical point c.

Define

$$\rho_m = \rho_m(x) = \frac{|F(x_I^m) - F(B_{m-1})|}{|\omega_m|}.$$

By the small distortion property of $W_m : \omega_m \to V_{z_a}$, if $\rho_m < 1/3$ then

$$\frac{\operatorname{dist}(B_m, \partial V_{z_a})}{|V_{z_a}|} \simeq \rho_m.$$

We state two technical Lemmas to be used in the sequel.

Lemma 8.3. — $|P_{m,x}(x_E^m)| \gtrsim |P_{m,x}(y)|$, $\forall\, y \in \pi_m$.

Proof. — As W_m has small distortion, the distortion of P_m is mostly due to $F|\pi_m$. But $|F_x|$ increases as the distance from the critical point increases, since $F_x x \neq 0$. □

Lemma 8.4. — $\dfrac{|B_{m-1}-c|}{|x_E^m-c|} > C_0^{-1}\rho_m^{1/2}$.

Proof. — Since V_{z_a} is small and next to c the function F is nearly quadratic. □

As
$$\frac{1}{|\pi_m|}\int_{\pi_m}|P_{m,x}(y)|dy = \frac{|V_{z_a}|}{|\pi_m|}$$
and V_{z_a} is at least θ^{-1} times greater than π_m, according to Corollary 3.3, it is expected that $|P_{m,x}|$ is big for some points in π_m. In particular, by Lemma 8.3, $|P_{m,x}(x_E^m)| > \frac{3}{4}\theta^{-1}$. The following Lemma gives a lower bound for $|P_{m,x}(B_{m-1})|$ as a function of ρ_m.

Lemma 8.5. — $|P_{m,x}(B_{m-1})| > C_0^{-1}\theta^{-1}\rho_m^{1/2}$.

Proof. — By the remark above $|P_{m,x}(x_E^m)| > \frac{3}{4}\theta^{-1}$, hence
$$|P_{m,x}(B_{m-1})| = |P_{m,x}(x_E^m)|\cdot\frac{|P_{m,x}(B_{m-1})|}{|P_{m,x}(x_E^m)|} > \frac{3}{4}\theta^{-1}\frac{|W_{m,x}(F(B_{m-1}))|}{|W_{m,x}(F(x_E^m))|}\cdot\frac{|F_x(B_{m-1})|}{|F_x(x_E^m)|}.$$
But $W_{m,x}$ is almost constant in ω_m and
$$\frac{|F_x(B_{m-1})|}{|F_x(x_E^m)|} \simeq \frac{|B_{m-1}-c|}{|x_E^m-c|},$$
and the Lemma follows using Lemma 8.4. □

Corollary 8.6. — $|P_{n,x}(B_{n-1})| > C_0^{-1}\theta^{-1}$.

Proof. — As $F(B_{n-1}) \in W_n^{-1}(\gamma_0)$ then $\rho_n \simeq 1/2$. □

Corollary 8.7. — *If* $\rho_m \geqslant \frac{1}{3}$ *then* $|P_{m,x}(B_{m-1})| > C_0^{-1}\theta^{-1}$

Proof. — Directly from Lemma 8.5. □

The following Lemma is the central assertion for forward recovering.

Lemma 8.8. — *If* $\rho_m < \frac{1}{3}$ *then*
$$|(P_{m+1}\circ P_m)_x(B_{m-1})| \geqslant C_0^{-1}\theta^{-2}.$$

Proof. — First notice that $B_m = P_m(B_{m-1}) \in \pi_{m+1}$, and $|\pi_{m+1}| < \theta\mathrm{dist}(\pi_{m+1}, \partial V_{z_a})$ by Corollary 3.3. Moreover $\mathrm{dist}(B_m, \partial V_{z_a}) \simeq \rho_m|V_{z_a}|$, hence
$$\frac{\mathrm{dist}(y,\partial V_{z_a})}{|V_{z_a}|} \simeq \rho_m,$$
for all $y \in \pi_{m+1}$. This also implies $|\pi_{m+1}| \lesssim \theta\rho_m|V_{z_a}|$. On the other hand, since $\rho_m < 1/3$, $|\pi_{m+1}| \ll \mathrm{dist}(\pi_{m+1}, \gamma_0)$, therefore the distortion of P_{m+1} must be small. Then
$$|P_{m+1,x}| \simeq \frac{|V_{z_a}|}{|\pi_{m+1}|} \gtrsim \theta^{-1}\rho_m^{-1}.$$

Combining with Lemma 8.5 we get

$$|(P_{m+1} \circ P_m)_x(B_{m-1})| > C_0^{-1} \rho_m^{-1/2} \theta^{-2},$$

proving the Lemma. □

We obtain some useful Corollaries, but first we introduce the following notation. For $m_0 \leqslant m_1$, let

$$\Delta_{m_0,m_1} = \Delta_{m_0,m_1}(x) = \left| \prod_{m=m_0}^{m_1} P_{m,x}(B_{m-1}) \right|,$$

which is equal to $|(P_{m_1} \circ \cdot \circ P_{m_0})_x(B_{m_0-1})|$, by the Chain Rule. For example, with this notation,

$$|B_x| = \Delta_{1,n}.$$

Let also $\Delta_{m_0,m_1} \equiv 1$ if $m_0 > m_1$.

Corollary 8.9. — *$|B_x| > (C_0^{-1}\theta^{-1})^n \gg 1$. In fact, $\Delta_{m_0,n} > (C_0^{-1}\theta^{-1})^{n-m_0+1}$, for all $m_0 = 1, \ldots, n$.*

Proof. — We prove by (decreasing) induction on m_0, starting from $m_0 = n$. For $m_0 = n$ we use Corollary 8.6. Suppose now that $\Delta_{m,n} > (C_0^{-1}\theta^{-1})^{n-m+1}$, for all $m = m_0 + 1, \ldots, n$. We want to prove that $\Delta_{m_0,n} > (C_0^{-1}\theta^{-1})^{n-m_0+1}$. But if $\rho_{m_0} \geqslant \frac{1}{3}$ then

$$\Delta_{m_0,n} = |P_{m_0,x}(B_{m_0-1})| \cdot \Delta_{m_0+1,n} > (C_0^{-1}\theta^{-1})^{n-m_0+1},$$

by induction and Corollary 8.7. Otherwise $\rho_{m_0} < \frac{1}{3}$, then

$$\Delta_{m_0,n} = |(P_{m_0+1} \circ P_{m_0})_x(B_{m_0-1})| \cdot \Delta_{m_0+2,j} > (C_0^{-1}\theta^{-1})^{n-m_0+1},$$

by induction and Lemma 8.8. □

Corollary 8.10. — *If $|P_{m_1,x}(B_{m_1-1})| > C_0^{-1}\theta^{-1}$ then $\Delta_{m_0,m_1} > (C_0^{-1}\theta^{-1})^{m_1-m_0+1}$, for all $m_0 = 1, \ldots, m_1$.*

Proof. — As in the proof of Corollary 8.9, but now induction starts at m_1. □

Corollary 8.11. — *In general,*

$$\Delta_{m_0,m_1} > (C_0^{-1}\theta^{-1})^{m_1-m_0+1} \frac{|\omega_0|^{1/2}}{|V_{z_a}|},$$

for every $1 \leqslant m_0 \leqslant m_1 \leqslant n$.

Proof. — First we notice that

$$\frac{|\omega_0|^{1/2}}{|V_{z_a}|} < C_0 \theta^{1/2} \ll 1,$$

by Corollary 3.2 and Lemma 2.4, supposing also θ sufficiently small.

Suppose $m_0 < m_1$. If $\rho_{m_1-1} < 1/3$, then

$$|(P_{m_1} \circ P_{m_1-1})_x(B_{m_1-2})| > (C_0^{-1}\theta^{-1})^2,$$

by Lemma 8.8, and by induction, as in the preceding Corollaries,

$$\Delta_{m_0,m_1} > (C_0^{-1}\theta^{-1})^{m_1-m_0+1}.$$

Otherwise $\rho_{m_1-1} \geqslant 1/3$, implying, by Corollaries 8.7 and 8.10 that

$$\Delta_{m_0,m_1-1} > (C_0^{-1}\theta^{-1})^{m_1-m_0}.$$

Then we write

$$\Delta_{m_0,m_1} = |P_{m_1,x}(B_{m_1-1})| \cdot \Delta_{m_0,m_1-1}$$

and use Lemmas 8.1 and 8.2. These Lemmas are directly applied in the case $m_0 = m_1$. □

9. Parameter dependence of regular branches

As remarked in Section 2, for each $\beta \in \mathcal{B}$ the function $B : \beta \to \gamma_0$ is extendible to $B : \mathcal{U}(\beta) \to V_{z_a}$. By Corollary 3.4, as $|\gamma_0|/|V_{z_a}| < \eta$, provided V_z and ε are sufficiently small, we choose $\eta > 0$ so that, by Lemma 3.1, $\mathcal{U}(\beta)$ contains a $\theta^{-1}|\beta|$-neighborhood of β, for some small $\theta > 0$, where the derivative of B has small distortion. Moreover, $\mathcal{U}(\beta)$ is completely inside one of the connected components of $V_{z_a} \setminus \gamma_0$.

We have also defined the parameter interval

$$J(\beta) = \{a \in J_0 \ ; \ \operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing \text{ or } |\operatorname{Im} H| \geqslant \frac{1}{7}|V_{z_a}|\}.$$

Observe that, according to the notation of the previous Section, $\beta \subset \mathcal{U}(\beta) \subset \pi_1$, so that $\operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing$ implies $\operatorname{Im} H \cap \pi_1 \neq \varnothing$. On the other hand, $|\operatorname{Im} H| \geqslant \frac{1}{7}|V_{z_a}|$ implies $|\operatorname{Im} F \cap \omega_0| \geqslant \frac{1}{8}|\omega_0|$, by the small distortion of $W_0 : \omega_0 \to V_{z_a}$.

We now define $Y = Y(a, \beta)$, for $a \in J_0$, as the distance between $F(\beta)$ and $F(c)$ (see Figure 5). As $\beta \in \pi_1$ then $F(\beta) \in \omega_1$. We also define $Z = Z(a) = |\operatorname{Im} F|V_{z_a}|$, $X = X(a, \beta) = Z - Y$ and $\tau = X/Z$.

The underlining idea in this Section is to better control the derivative of $P_1 : \pi_1 \to V_{z_a}$. As $P_{1,x} = (W_{1,x} \circ F)F_x$, expansion depends on the relative position of $F(\beta)$ with respect to the critical value, controlled by τ.

Roughly speaking, we deal with the following situations. Fixing β and taking $a \in J(\beta)$ we may have $|\operatorname{Im} H \cap V_{z_a}| \geqslant \frac{1}{7}|V_{z_a}|$. In this case, since $|\operatorname{Im} F \cap \omega_0|$ is relatively large with respect to $|\omega_0|$ the derivative of F outside γ_0 is always bounded by something of the order of $|\omega_0|^{1/2}$, which will be enough to our purposes. Otherwise $a \in J(\beta)$ implies $\operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing$. In this case we may have τ small or not. If τ is small it means that β is near ∂V_{z_a}, and the derivative of F is not so small. But if τ is near 1 this means that β (and also π_1), is near the critical point. The consequence is that $\operatorname{Im} H$ occupies approximately one half of V_{z_a}, which in turn implies that $\operatorname{Im} F$

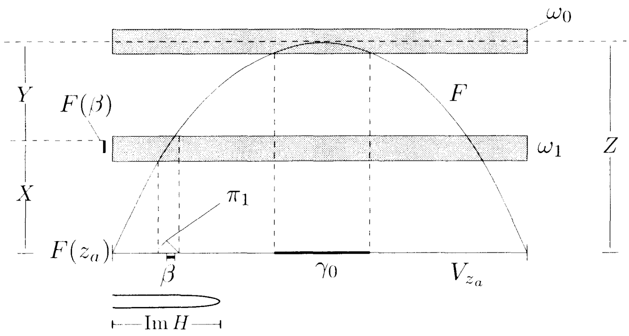

FIGURE 5. Placement of $F(\beta)$ with respect to the critical value

occupies approximately one half of ω_0. Once again derivatives outside γ_0 must be at least of the order of $|\omega_0|^{1/2}$.

The next two Lemmas quantify these arguments.

Lemma 9.1. — *If $a \in J(\beta)$ then $|\gamma_0| > C_0^{-1}\tau^{1/2}|\omega_0|^{1/2}$.*

Proof. — If $|\operatorname{Im} H| \geqslant \frac{1}{7}|V_{z_a}|$ then $|\operatorname{Im} F \cap \omega_0| \geqslant \frac{1}{8}|\omega_0|$, hence $|\gamma_0| > C_0^{-1}|\omega_0|^{1/2}$. Now it is enough to verify the inequality when $\operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing$ but $|\operatorname{Im} H| < \frac{1}{7}|V_{z_a}|$. By Lemma 2.4 and Corollary 3.2 $|\omega_1| < \theta X$, and also $\operatorname{dist}(\omega_1, F(z_a)) \simeq \tau|\operatorname{Im} F|V_{z_a}|$. As $|F_x| < C_0|V_{z_a}|$ in V_{z_a} then

$$\operatorname{dist}(\pi_1, \partial V_{z_a}) \geqslant C_0^{-1}|V_{z_a}|^{-1}\tau|\operatorname{Im} F|V_{z_a}|. \tag{1}$$

The assumption $\operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing$ implies $|\operatorname{Im} H| \gtrsim \operatorname{dist}(\pi_1, \partial V_{z_a})$, hence by the small distortion of $W_0 : \omega_0 \to V_{z_a}$

$$|\gamma_0| \geqslant C_0^{-1}\left[\operatorname{dist}(\pi_1, \partial V_{z_a}) \cdot \frac{|\omega_0|}{|V_{z_a}|}\right]^{1/2}.$$

Using Equation (1) we obtain the Lemma, taking into account that

$$\frac{|\operatorname{Im} F|V_{z_a}|^{1/2}}{|V_{z_a}|} \geqslant C_0^{-1}.$$

□

Lemma 9.2. — *If $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ then $|P_{1,x}| \geqslant C_0^{-1}\tau^{-1}\theta^{-1}$.*

Proof. — We write $|P_{1,x}| = |W_{1,x} \circ F| \cdot |F_x|$. By small distortion properties, $|W_{1,x}| \simeq |V_{z_a}|/|\omega_1|$. Moreover

$$|\omega_1| < C_0\tau\theta|V_{z_a}|^2,$$

since $|\omega_1| \leqslant \theta X = \tau\theta Z$ and $Z = |\operatorname{Im} F|V_{z_a}| \leqslant C_0|V_{z_a}|^2$. On the other hand, $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ implies $|F_x| \geqslant C_0^{-1}|V_{z_a}|$, and the Lemma follows. □

The goal of this Section is to show that

$$\frac{B_{m,a}}{B_{m,x}H_a}$$

is as small as desired, for all $a \in J(\beta)$ and $x \in \beta$, provided V_z and ε are sufficiently small. Here $B_m = P_m \circ P_{m-1} \circ \cdots \circ P_1$ and H_a is the mean value of $H(a,x)$, for $x \in \gamma_0$ and $a \in J_0$, based on the statements of Section 7. For $m = n$ this gives the first quotient of the starting conditions Preimages of the central branch. The cases $m < n$ will be used for the other quotients. We write

$$\frac{B_{m,a}}{B_{m,x}} = \sum_{t=1}^{m} \frac{P_{t,a} \circ B_{t-1}}{B_{t,x}},$$

therefore

$$\frac{B_{m,a}}{B_{m,x}H_a} = \frac{P_{1,a}}{P_{1,x}H_a} + \sum_{t=2}^{m} \frac{1}{B_{t-1,x}} \cdot \frac{P_{t,a} \circ B_{t-1}}{(P_{t,x} \circ B_{t-1})H_a}. \tag{2}$$

This last equation motivates the following Lemmas.

Lemma 9.3. — *For $m = 1, \ldots, n$,*

$$\left| \frac{P_{m,a} \circ B_{m-1}}{(P_{m,x} \circ B_{m-1})H_a} \right| < C_y \frac{|\omega_0|}{|V_{z_a}|} \cdot \frac{1}{|B_{m-1} - c|}.$$

Proof. — Write

$$\frac{P_{m,a} \circ B_{m-1}}{P_{m,x} \circ B_{m-1}} = \frac{F_a \circ B_{m-1}}{F_x \circ B_{m-1}} + \frac{1}{F_x \circ B_{m-1}} \cdot \frac{W_{m,a} \circ F \circ B_{m-1}}{W_{m,x} \circ F \circ B_{m-1}}.$$

We know that $|F_a| \leqslant C_0$, $|F_x \circ B_{m-1}| \geqslant C_0^{-1}|B_{m-1} - c|$ and $|W_{m,a}|/|W_{m,x}| \leqslant C_y$ (by Lemma 5.1). The Lemma follows using Lemma 7.1. $\square$

Lemma 9.4. — *If* $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ *then*

$$\left| \frac{P_{1,a}}{P_{1,x}H_a} \right| < C_y \theta.$$

Proof. — The quotient is evaluated at $B_0 = B_0(x) = x$. By the hypothesis, $|B_0 - c| \gtrsim \frac{1}{4}|V_{z_a}|$. By Lemma 9.3,

$$\left| \frac{P_{1,a}}{P_{1,x}H_a} \right| < C_y \frac{|\omega_0|}{|V_{z_a}|^2} < C_y \theta. \qquad \square$$

Lemma 9.5. — *If* $\operatorname{dist}(\pi_1, \gamma_0) < \frac{1}{4}|V_{z_a}|$ *and $a \in J(\beta)$ then*

$$\left| \frac{P_{m,a} \circ B_{m-1}}{(P_{m,x} \circ B_{m-1})H_a} \right| < C_y \frac{|\gamma_0|}{|V_{z_a}|},$$

for all $m = 1, \ldots, n$.

Proof. — If $a \in J(\beta)$ then $|\operatorname{Im} H| \geqslant \frac{1}{7}|V_{z_a}|$ or $\operatorname{Im} H \cap \mathcal{U}(\beta) \neq \varnothing$. In the latter case, $\operatorname{Im} H \cap \pi_1 \neq \varnothing$ and, by the hypotheses, $|\operatorname{Im} H| \gtrsim \frac{3}{8}|V_{z_a}|$. In any case, $|\operatorname{Im} H| \geqslant \frac{1}{7}|V_{z_a}|$, therefore $|\operatorname{Im} F \cap \omega_0| > \frac{1}{8}|\omega_0|$. This implies $|B_{m-1} - c| > C_0^{-1}|\omega_0|^{1/2}$, for all $m = 1, \dots, n$. By Lemma 9.3 the quotient of the statement is bounded by $C_y|\omega_0|^{1/2}/|V_{z_a}|$. By Lemma 9.1 and the hypothesis, implying τ bounded away from zero by C_0^{-1}, it follows

$$|\omega_0|^{1/2} < C_0\,|\gamma_0|$$

and the Lemma. □

The next Lemma is somehow analogous to the idea of backward recovering of Section 4.

Lemma 9.6. — *If $a \in J(\beta)$ and $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ then*

$$|P_{1,x}|^{-1} \cdot \left|\frac{P_{m,a} \circ B_{m-1}}{(P_{m,x} \circ B_{m-1})H_a}\right| < C_y \theta \frac{|\omega_0|^{1/2}}{|V_{z_a}|},$$

for all $m = 1, \dots, n$.

Proof. — By Lemma 9.1, $|B_{m-1} - c| > C_0^{-1}\tau^{1/2}|\omega_0|^{1/2}$. Putting into Lemma 9.3 and using Lemma 9.2 the Lemma follows. Observe also that $\tau \leqslant 1$. □

Lemma 9.7. — *Given $\delta > 0$, if V_z and ε are sufficiently small then*

$$\left|\frac{B_{m,a}}{B_{m,x} H_a}\right| < \delta,$$

for all $\beta \in \mathcal{B}$, $x \in \beta$, $a \in J(\beta)$ and $m = 1, \dots, n$, where $B = P_n \circ \cdots \circ P_1|\beta : \beta \to \gamma_0$ and $B_m = P_m \circ \cdots \circ P_1|\beta$. The value H_a indicates the mean value of $H(a, x)$ for $x \in \gamma_0$ and $a \in J_0$.

Proof. — We evaluate term by term the R.H.S of Equation (2) supposing always that $a \in J(\beta)$. We have to consider two separate cases: A) $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ and B) $\operatorname{dist}(\pi_1, \gamma_0) < \frac{1}{4}|V_{z_a}|$. The first term,

$$\frac{P_{1,a}}{P_{1,x} H_a},$$

is bounded by $C_y\theta$ if $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$, by Lemma 9.4, and bounded by $C_y|\gamma_0|/|V_{z_a}|$ if $\operatorname{dist}(\pi_1, \gamma_0) < \frac{1}{4}|V_{z_a}|$, by Lemma 9.5. In both cases the first term is bounded by $\delta/2$, provided θ is small, and this is guaranteed if V_z is small enough. We are left with the remaining terms, from $t = 2$ to $t = m$.

In Case B, where $\operatorname{dist}(\pi_1, \gamma_0) < \frac{1}{4}|V_{z_a}|$, Lemma 9.5 implies

$$\left|\frac{P_{t,a}}{P_{t,x} H_a}\right| < C_y \frac{|\omega_0|^{1/2}}{|V_{z_a}|}.$$

On the other hand,

$$|B_{t-1,x}|^{-1} = \Delta_{1,t-1}^{-1} \leqslant (C_0\theta)^{t-1} \frac{|V_{z_a}|}{|\omega_0|^{1/2}},$$

by Corollary 8.11. Hence we are left with

$$C_y \sum_{t=2}^{m} (C_0\theta)^{t-1},$$

which is smaller than $\delta/2$ if θ is sufficiently small.

In Case A, where $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$, we bound the t-th term by

$$\Delta_{2,t-1}^{-1} |P_{1,x}|^{-1} \cdot \left| \frac{P_{t,a} \circ B_{t-1}}{(P_{t,x} \circ B_{t-1}) H_a} \right|,$$

which is smaller than $C_y \theta (C_0\theta)^{t-2}$, by Corollary 8.11 and Lemma 9.6. Therefore the sum is smaller than $\delta/2$ if θ is small enough. □

10. Other derivatives

We keep the same notation introduced in the preceding Sections. The goal is to bound derivatives of $P : \pi \to V_{z_a}$, for all $\pi \in \mathcal{P}$, and take their compositions to bound derivatives of $B : \beta \to \gamma_0$, for all $\beta \in \mathcal{B}$. This will complete the proof of the starting conditions Preimages of the central branch.

Lemma 10.1. — *If V_z is small then*

$$|\gamma_0| \cdot \left| \frac{P_{xx}}{(P_x)^2} \right| < \frac{|\gamma_0|}{|V_{z_a}|} + C_0 |P_x|^{-1}.$$

Proof. — Write $P = W \circ F$ and

$$\frac{P_{xx}}{(P_x)^2} = \frac{F_{xx}}{P_x F_x} + \frac{W_{xx}}{(W_x)^2}.$$

We have

$$|\gamma_0| \cdot \left| \frac{W_{xx}}{(W_x)^2} \right| \ll \frac{|\gamma_0|}{|V_{z_a}|},$$

for V_z sufficiently small, by Lemma 6.2. This controls the second term. For the first, we have $|F_{xx}| < C_0$ and

$$\frac{|\gamma_0|}{|F_x|} < C_0. \qquad \square$$

Lemma 10.2. — *If V_z is small then*

$$|\gamma_0| \cdot \left| \frac{B_{m,xx}}{(B_{m,x})^2} \right| < \frac{|\gamma_0|}{|V_{z_a}|} + C_0 \sum_{t=1}^{m} \Delta_{t,m}^{-1},$$

for all $m = 1, \ldots, n$.

Proof. — Write

$$\frac{B_{m,xx}}{(B_{m,x})^2} = \sum_{t=1}^{m} \frac{1}{(P_m \circ \cdots \circ P_{t+1})_x \circ B_t} \cdot \frac{P_{t,xx} \circ B_{t-1}}{(P_{t,x} \circ B_{t-1})^2}.$$

Then

$$|\gamma_0| \cdot \left| \frac{B_{m,xx}}{(B_{m,x})^2} \right| \leqslant \sum_{t=1}^{m} \Delta_{t+1,m}^{-1} \left(\frac{|\gamma_0|}{|V_{z_a}|} + C_0 |P_{t,x} \circ B_{t-1}|^{-1} \right),$$

by Lemma 10.1. In the statement we separate the term $\Delta_{m+1,m}^{-1} \equiv 1$. □

Lemma 10.3. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{B_{xx}}{(B_x)^2} \right| < \delta,$$

for all $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J_0$.

Proof. — Put $m = n$ in Lemma 10.2. Then

$$\Delta_{t+1,n}^{-1} \leqslant (C_0 \theta)^{n-t}, \quad \Delta_{t,n}^{-1} \leqslant (C_0 \theta)^{n-t+1},$$

by Corollary 8.9. As V_z small implies θ small and $|\gamma_0|/|V_{z_a}|$ small, then the Lemma follows. □

Lemma 10.4. — *If V_z is sufficiently small then*

$$|\gamma_0|^2 \cdot \left| \frac{P_{xxx}}{(P_x)^3} \right| < \frac{|\gamma_0|^2}{|V_{z_a}|^2} + |P_x|^{-1},$$

for all $x \in \pi$, $\pi \in \mathcal{P}$ and $a \in J_0$.

Proof. — Write $P = W \circ F$ and

$$\frac{P_{xxx}}{(P_x)^3} = \frac{F_{xxx}}{(F_x)^2 W_x P_x} + \frac{W_{xxx}}{(W_x)^3} + 3 \frac{W_{xx}}{(W_x)^2} \cdot \frac{F_{xx}}{F_x P_x},$$

where W_x, W_{xx}, etc, mean $W_x \circ F$, $W_{xx} \circ F$, etc. The second term, multiplied by $|\gamma_0|^2$, is smaller than $|\gamma_0|^2/|V_{z_a}|^2$, by Lemma 6.3, if V_z is small. For the first term, $|\gamma_0|^2/(F_x)^2 < C_0$ and $|W_x| \gg C_0$ (by the choice of V_z) implies that it is bounded by $\frac{1}{2}|P_x|^{-1}$. For the third term, we have $|\gamma_0|/|F_x| < C_0$ and

$$|\gamma_0| \cdot \left| \frac{W_{xx}}{(W_x)^2} \right| \ll \frac{|\gamma_0|}{|V_{z_a}|} \ll C_0,$$

by Lemma 6.2, hence it is bounded by $\frac{1}{2}|P_x|^{-1}$, and the Lemma follows. □

Lemma 10.5. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0|^2 \cdot \left| \frac{B_{xxx}}{(B_x)^3} \right| < \delta,$$

for all $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J_0$.

Proof. — We write

$$\frac{B_{xxx}}{(B_x)^3} = S_1 + 3S_2,$$

where

$$S_1 = \sum_{t=1}^{n} \frac{P_{t,xxx}}{(P_{n,x}\dots P_{t+1,x})^2 (P_{t,x})^3}$$

and

$$S_2 = \sum_{t=2}^{n} \frac{P_{t,xx}}{P_{n,x}\dots P_{t+1,x}(P_{t,x})^2} \cdot \frac{1}{P_{n,x}\dots P_{t,x}} \frac{B_{t-1,xx}}{(B_{t-1,x})^2}.$$

For simplicity, we are omitting arguments, writing $P_{t,x}$ instead of $P_{t,x} \circ B_{t-1}$, etc. Using Lemma 10.4 and $\Delta_{n+1,n} \equiv 1$ we obtain

$$|\gamma_0|^2 |S_1| \leqslant \frac{|\gamma_0|^2}{|V_{z_a}|^2} + |P_{n,x}|^{-1} + \sum_{t=1}^{n-1} \Delta_{t+1,n}^{-2} + \sum_{t=1}^{n-1} \Delta_{t+1,n}^{-1} \Delta_{t,n}^{-1},$$

which is smaller than $\delta/2$ if θ is sufficiently small, by Corollary 8.9. On the other hand $|\gamma_0|^2|S_2|$ is bounded by

$$\sum_{t=2}^{n} \Delta_{t+1,n}^{-1} \Delta_{t,n}^{-1} \left(1 + C_0 |P_{t,x}|^{-1}\right) \cdot \left(1 + C_0 \sum_{s=1}^{t-1} \Delta_{s,t-1}^{-1}\right),$$

using Lemmas 10.1 and 10.2 and $|\gamma_0| \ll |V_{z_a}|$. It is straightforward to see that this sum is smaller than $\delta/6$, if θ is small, using Corollary 8.9. □

Lemma 10.6. — *If V_z is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{P_{xa}}{(P_x)^2 H_a} \right| < |P_x|^{-1} + \frac{|\omega_0|}{|V_{z_a}|^2},$$

for all $x \in \pi$, $\pi \in \mathcal{P}$ and $a \in J_0$.

Proof. — Write $P = W \circ F$ and

$$\frac{P_{xa}}{(P_x)^2} = \frac{F_{xa}}{F_x P_x} + \frac{1}{F_x} \frac{W_{xa}}{(W_x)^2} + \frac{W_{xx}}{(W_x)^2} \cdot \frac{F_a}{F_x}.$$

We analyze each term multiplied by $|\gamma_0|/|H_a|$, which is smaller than $C_0|\gamma_0|/|W_{0,x}|$. As $|\gamma_0|/|F_x| < C_0$, the first term can be bounded by

$$\frac{C_0}{|W_{0,x}|} \cdot |P_x|^{-1} < |P_x|^{-1},$$

if V_z is small. For the second term we still use that

$$|V_{z_a}| \frac{W_{xa}}{(W_x)^2}$$

is much smaller than C_0, by Lemma 6.4, and $|W_{0,x}| \simeq |V_{z_a}|/|\omega_0|$. The same in the third term, but now using Lemma 6.2. □

Lemma 10.7. — *If V_z is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{B_{m,xa}}{(B_{m,x})^2 H_a} \right| < m\Delta_{1,m}^{-1} + C_0 \left(\frac{|\omega_0|^{1/2}}{|V_{z_a}|} + \sum_{t=2}^{m} \Delta_{t,m}^{-1} \right),$$

for all $m = 1, \dots, n$, $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J(\beta)$.

Proof. — Write

$$\frac{B_{m,xa}}{(B_{m,x})^2} = S_1 + S_2,$$

where

$$S_1 = \sum_{t=1}^{m} \frac{P_{t,xa}}{P_{t,x} B_{m,x}}$$

and

$$S_2 = \sum_{t=2}^{m} \frac{1}{P_{m,x} \dots P_{t+1,x}} \cdot \frac{P_{t,xx}}{(P_{t,x})^2} \cdot \frac{B_{t-1,a}}{B_{t-1,x}}.$$

Then, by Lemma 10.6,

$$\frac{|\gamma_0|}{|H_a|} |S_1| \leqslant \sum_{t=1}^{m} \frac{P_{t,x}}{B_{m,x}} \cdot \left(|P_{t,x}|^{-1} + \frac{|\omega_0|}{|V_{z_a}|^2}, \right)$$

which is smaller than

$$m\Delta_{1,m}^{-1} + \frac{|\omega_0|}{|V_{z_a}|^2} \sum_{t=1}^{m} \Delta_{t+1,m}^{-1} \Delta_{1,t-1}^{-1}.$$

Using Corollary 8.11, we simplify

$$\frac{|\omega_0|}{|V_{z_a}|^2} \sum_{t=1}^{m} \Delta_{t+1,m}^{-1} \Delta_{1,t-1}^{-1} \leqslant \frac{|\omega_0|^{1/2}}{|V_{z_a}|} + \sum_{t=2}^{m} \Delta_{t,m}^{-1}.$$

On the other hand, as in Lemma 10.2,

$$\frac{|\gamma_0|}{|H_a|} |S_2| \leqslant \frac{|\gamma_0|}{|V_{z_a}|} + C_0 \sum_{t=2}^{m} \Delta_{t,m}^{-1},$$

using Lemmas 10.1 and 9.7. □

Lemma 10.8. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0| \cdot \left| \frac{B_{xa}}{(B_x)^2 H_a} \right| < \delta,$$

for all $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J(\beta)$.

Proof. — It is enough to apply Corollary 8.9 in Lemma 10.7, with θ small. □

Lemma 10.9. — *If V_z is sufficiently small then*

$$|\gamma_0|^2 \cdot \left| \frac{P_{xxa}}{(P_x)^3 H_a} \right| < \frac{|\omega_0|}{|V_{z_a}|^2} + |P_x|^{-1},$$

for all $x \in \pi$, $\pi \in \mathcal{P}$ and $a \in J_0$.

Proof. — We have

$$\frac{P_{xxa}}{(P_x)^3} = Q_1 + Q_2 + Q_3 + Q_4 + 4Q_5 + 2Q_6,$$

where

$$Q_1 = \frac{F_{xxa}}{P_x W_x (F_x)^2}, \quad Q_2 = \frac{W_{xxa}}{F_x (W_x)^3}, \quad Q_3 = \frac{W_{xxx}}{(W_x)^3} \cdot \frac{F_a}{F_x},$$

$$Q_4 = \frac{1}{P_x} \frac{W_{xa}}{(W_x)^2} \frac{F_{xx}}{(F_x)^2}, \quad Q_5 = \frac{1}{W_x} \frac{W_{xx}}{(W_x)^2} \frac{F_{xa}}{(F_x)^2}, \quad Q_6 = \frac{1}{W_x} \frac{W_{xx}}{(W_x)^2} \frac{F_{xx}}{(F_x)^2} \frac{F_a}{F_x}.$$

We multiply each of these terms by $|\gamma_0|^2/|H_a|$, which is smaller than $C_0|\gamma_0|^2|\omega_0|/|V_{z_a}|$, by Lemma 7.1, using then the following estimates, which are valid for V_z sufficiently small: $|F_{xxa}|$, $|F_{xx}|$, $|F_a|$, $|F_{xa}| < C_0$, $|\gamma_0| < C_0|F_x|$, $C_0|W_x|^{-1} < 1$, $C_0|\omega_0| < |V_{z_a}|$, $C_0|\gamma_0| < |V_{z_a}|$ and Lemmas 6.2, 6.4, 6.6, 6.3, with $\delta = C_0^{-1}$ or $\delta = 1$. □

Lemma 10.10. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0|^2 \cdot \left| \frac{B_{xxa}}{(B_x)^3 H_a} \right| < \delta,$$

for all $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J(\beta)$.

Proof. — Write

$$\frac{|\gamma_0|^2}{H_a} \frac{B_{xxa}}{(B_x)^3} \leqslant |S_1| + |S_2| + |S_3| + 4|S_4| + 2|S_5|,$$

where

$$S_1 = \sum_{t=1}^{n} \Delta_{t+1,n}^{-2} \Delta_{1,t-1}^{-1} \cdot |\gamma_0|^2 \frac{P_{t,xxa}}{(P_{t,x})^3 H_a},$$

$$S_2 = \sum_{t=2}^{n} \Delta_{t+1,n}^{-2} \cdot |\gamma_0|^2 \frac{P_{t,xxx}}{(P_{t,x})^3} \cdot \frac{B_{t-1,a}}{B_{t-1,x} H_a},$$

$$S_3 = \sum_{t=2}^{n} \Delta_{1,n}^{-1} \Delta_{t+1,n}^{-1} \cdot |\gamma_0| \frac{P_{t,xa}}{(P_{t,x})^2 H_a} \cdot |\gamma_0| \frac{B_{t-1,xx}}{(B_{t-1,x})^2},$$

$$S_4 = \sum_{t=2}^{n} \Delta_{t+1,n}^{-1} \Delta_{t,n}^{-1} \cdot |\gamma_0| \frac{P_{t,xx}}{(P_{t,x})^2} \cdot |\gamma_0| \frac{B_{t-1,xa}}{(B_{t-1,x})^2 H_a},$$

$$S_5 = \sum_{t=2}^{n} \Delta_{t+1,n}^{-1} \Delta_{t,n}^{-1} \cdot |\gamma_0| \frac{P_{t,xx}}{(P_{t,x})^2} \cdot |\gamma_0| \frac{B_{t-1,xx}}{(B_{t-1,x})^2} \cdot \frac{B_{t-1,a}}{B_{t-1,x} H_a}.$$

By Lemma 10.9,

$$|S_1| \leqslant \sum_{t=1}^{n} \Delta_{t+1,n}^{-2} \Delta_{1,t-1}^{-1} \left(\frac{|\omega_0|}{|V_{z_a}|^2} + |P_{t,x}|^{-1} \right),$$

which is smaller than $\delta/5$ if θ is small, by Corollaries 8.9 and 8.11. With the other sums we proceed in the same way, using the same Corollaries and also Lemmas 9.7, 10.2, 10.7, 10.4, 10.6 and 10.1. □

Lemma 10.11. — *If V_z is sufficiently small then*

$$|\gamma_0| \cdot \left|\frac{P_{aa}}{(P_x)^2(H_a)^2}\right| < C_0 \frac{|\omega_0|^2}{|V_{z_a}|^2}|P_x|^{-1} + \frac{|\omega_0|^2}{|V_{z_a}|^3}|x-c|^{-1},$$

for all $x \in \pi$, $\pi \in \mathcal{P}$ and $a \in J_0$.

Proof. — Write

$$\frac{P_{aa}}{(P_x)^2} = \frac{F_{aa}}{P_x F_x} + \frac{W_{aa}}{(F_x)^2(W_x)^2} + \frac{2F_a W_{xa}}{(F_x)^2(W_x)^2} + \left(\frac{F_a}{F_x}\right)^2 \frac{W_{xx}}{(W_x)^2}.$$

Then we use $|\gamma_0| < C_0|F_x|$, $|F_x|^{-1} < C_0|x-c|^{-1}$, $|H_a|^{-1} < C_0|\omega_0|/|V_{z_a}|$ and Lemmas 6.5, 6.4 and 6.2, with $\delta = C_0^{-1}$. □

Lemma 10.12. — *Given $\delta > 0$, if V_z is sufficiently small then*

$$|\gamma_0| \cdot \left|\frac{B_{aa}}{(B_x)^2(H_a)^2}\right| < \delta,$$

for all $x \in \beta$, $\beta \in \mathcal{B}$ and $a \in J(\beta)$.

Proof. — Write

$$\frac{|\gamma_0|^2}{H_a^2}\frac{B_{aa}}{(B_x)^2} \leqslant |S_1| + 2|S_2| + |S_3|,$$

where

$$S_1 = \sum_{t=1}^{n} \Delta_{1,t-1}^{-2}\Delta_{t+1,n}^{-1} \cdot |\gamma_0| \frac{P_{t,aa}}{(P_{t,x})^2(H_a)^2},$$

$$S_2 = \sum_{t=2}^{n} \Delta_{t+1,n}^{-1}\Delta_{1,t-1}^{-1} \cdot |\gamma_0| \frac{P_{t,xa}}{(P_{t,x})^2 H_a} \cdot \frac{B_{t-1,a}}{B_{t-1,x}H_a},$$

$$S_3 = \sum_{t=2}^{n} \Delta_{t+1,n}^{-1} \cdot |\gamma_0| \frac{P_{t,xx}}{(P_{t,x})^2} \cdot \left(\frac{B_{t-1,a}}{B_{t-1,x}H_a}\right)^2.$$

Using Lemma 10.6, Lemma 9.7 (with $\delta = 1$) and Corollaries 8.9 and 8.11 we get

$$|S_2| \leqslant (n-1)(C_0\theta)^n + \frac{|\omega_0|^{1/2}}{|V_{z_a}|}(n-1)(C_0\theta)^{n-1},$$

which is smaller than $\delta/6$ if θ is small. It is also easy to see that $|S_3|$ is smaller than $\delta/3$ if θ is small, using Lemma 10.1 and Corollary 8.9. The difficult part is contained in S_1. By Lemma 10.11 we have

$$|S_1| \leqslant \sum_{t=1}^{n} \Delta_{1,t-1}^{-2}\Delta_{t+1,n}^{-1}\left(C_0\frac{|\omega_0|^2}{|V_{z_a}|^2}|P_{t,x}|^{-1} + \frac{|\omega_0|^2}{|V_{z_a}|^3}|B_{t-1}-c|^{-1}\right).$$

If we separate into two sums, the first one is bounded by

$$\frac{|\omega_0|^2}{|V_{z_a}|^2}\Delta_{1,n}^{-1}\sum_{t=1}^{n}\Delta_{1,t-1}^{-1},$$

where it is implicit that $\Delta_{1,0} \equiv 1$. By Corollary 8.11, $\sum_{t=1}^{n} \Delta_{1,t-1}^{-1} \leqslant 2|V_{z_a}|/|\omega_0|^{1/2}$, so that the first sum is bounded by

$$|\omega_0| \cdot \frac{|\omega_0|^{1/2}}{|V_{z_a}|}(C_0\theta)^n,$$

which is smaller than $\delta/6$ if θ is small. For the second sum,

$$S = \frac{|\omega_0|^2}{|V_{z_a}|^3} \sum_{t=1}^{n} \Delta_{1,t-1}^{-2} \Delta_{t+1,n}^{-1} \cdot |B_{t-1} - c|^{-1},$$

we use the ideas of Section 9.

We have two cases: A) $\operatorname{dist}(\pi_1, \gamma_0) \geqslant \frac{1}{4}|V_{z_a}|$ and B) $\operatorname{dist}(\pi_1, \gamma_0) < \frac{1}{4}|V_{z_a}|$. In Case B, $|\gamma_0| > C_0^{-1}|\omega_0|^{1/2}$ (see Lemma 9.5, for example), which implies, by Corollaries 8.9 and 8.11,

$$S \leqslant C_0 \frac{|\omega_0|}{|V_{z_a}|^2} < \frac{\delta}{6},$$

if V_z is sufficiently small. In Case A we have

$$|B_{t-1} - c|^{-1} \leqslant 2|\gamma_0|^{-1} < C_0 \tau^{-1/2} |\omega_0|^{-1/2},$$

for all $t = 1, \ldots, n$, by Lemma 9.1, $|B_0 - c|^{-1} < 5|V_{z_a}|^{-1}$ by hypothesis and $|P_{1,x}|^{-1}$, by Lemma 9.2, hence

$$S \leqslant \frac{|\omega_0|^2}{|V_{z_a}|^3} \left\{ C_0 \tau^{1/2} \theta |\omega_0|^{-1/2} \sum_{t=2}^{n} \Delta_{1,t-1}^{-1} \Delta_{2,t-1}^{-1} \Delta_{t+1,n}^{-1} + \Delta_{2,n}^{-1} \cdot 5|V_{z_a}|^{-1} \right\},$$

which is smaller than

$$5 \frac{|\omega_0|^2}{|V_{z_a}|^4} \Delta_{2,n}^{-1} + C_0 \frac{|\omega_0|^{1/2}}{|V_{z_a}|} \theta \sum_{t=2}^{n} (C_0\theta)^{n-t},$$

by Corollaries 8.9 and 8.11. If V_z is small the Lemma is proved. □

A. Appendix

As remarked at the end of Section 2, Theorem 2.5 is proved in [**2**] assuming C^∞ differentiability. This hypothesis is used only for estimates of derivatives near saddle-node bifurcations, where a map is considered as a time-one map of a flow. Here we are able to reduce the needed differentiability to 3, obtaining the same bounds (Lemmas S.7 and S.8 of [**2**]) without any embedding into a flow. In addition, as the arguments are direct, they allow much more control on constants.

The proof of Theorem 2 is made in [**2**] by induction, starting from the map Φ_0 defined in Section 2. The central interval γ_0 together with the preimages of the central branch β belonging to the collection $\mathcal{B}_0$ form the set of connected components of the domain of Φ_0, contained in $\gamma_{-1} \equiv V_{z_a}$.

The central branch $H_0 = \Phi_0|\gamma_0$ is unimodal and $H_0(\partial\gamma_0) \subset \partial\gamma_{-1}$. We also have the diffeomorphic branches $B = \Phi_0|\beta : \beta \to \gamma_0$. The map Φ_{n+1}, for $n \geqslant 0$, is defined by induction with domain in γ_n, with a central branch $H_{n+1} = \Phi_{n+1}|\gamma_{n+1} : \gamma_{n+1} \to \gamma_n$, $H_{n+1}(\partial\gamma_{n+1}) \subset \partial\gamma_n$ and with diffeomorphic branches $B = \Phi_{n+1}|\beta : \beta \to \gamma_{n+1}$, for β in the collection $\mathcal{B}_{n+1}$. The map H_{n+1} is the critical component of the Φ_n-first entry map into γ_n *after escaping* from this same γ_n (and not the Φ_n-first return map to γ_n, as usual). The maps $B : \beta \to \gamma_{n+1}$ are the branches of the Φ_n-first entry map into γ_{n+1}.

At all stages of the induction, the maps Φ_n are shown to satisfy the same three sets of conditions Geometry, Central Branch and Preimages of the Central Branch of Section 2, with small and uniform constants $\eta > 0$, $\delta_0 > 0$ and $\delta_1 > 0$. One of the main steps in the proof is the analysis of H_n-iterates near the creation of a saddle-node fixed point for H_n. In [**2**], this analysis is resumed in Lemmas S.7 and S.8.

The function H_n is a two-variable function $H_n = H_n(a, x)$, defined for $x \in \gamma_n = \gamma_{n,a}$ and $a \in J$, where J is some interval. As a varies along J, $H_n(a, c)$ crosses $\gamma_{n-1} = \gamma_{n-1,a}$. For simplicity we assume $c = 0$ and $H_n(0, 0) = 0$. The starting conditions named Central Branch imply that there are non-zero constants S_n and V_n such that

$$1 - 2\delta_0 \leqslant \frac{H_{n,xx}(a, x)}{2S_n} \leqslant 1 + 2\delta_0, \quad 1 - 2\delta_0 \leqslant \frac{H_{n,a}(a, x)}{V_n} \leqslant 1 + 2\delta_0,$$

for all $x \in \gamma_{n,a}$ and $a \in J$, if δ_0 is sufficiently small.

The sign of $S_n \cdot V_n$ is always the same as the sign of $S_0 \cdot V_0$, which we suppose to be negative, without loss of generality. If we do a linear coordinate change $x \mapsto -S_n x$, $a \mapsto -S_n V_n a$ we normalize H_n so that

$$|H_{n,xx}(a, x) + 2| \leqslant 4\delta_0, \quad |H_{n,a}(a, x) - 1| \leqslant 2\delta_0.$$

The starting conditions Central Branch are kept unaffected by linear changes of coordinates. Integrating these two last inequalities we have

$$|H_{n,x}(a, x) + 2x| \leqslant 4\delta_0|x|,$$

$$|H_n(a, x) - (H_n(a, 0) - x^2)| \leqslant 2\delta_0 x^2$$

and

$$|H_n(a, 0) - a| \leqslant 2\delta_0|a|.$$

In fact, we are only concerned here with negative values of the parameter, where the saddle-node appears. Let a_s be the least (and unique) value for which there is a (unique) solution for the equation

$$H_n(a, x) = x$$

and let x_s be such that $H_n(a_s, x_s)$.

By solving the equation $H_{n,x}(x_s) = 1$ we get

$$-\frac{1}{2}\frac{1}{1 - 2\delta_0} \leqslant x_s \leqslant -\frac{1}{2}\frac{1}{1 + 2\delta_0}.$$

On the other hand, $a_s \in [a_s^1, a_s^2]$, where a_s^1 and a_s^2 are, respectively, the first parameter values for which $(1-2\delta_0)(a-x^2) = x$ and $(1+2\delta_0)(a-x^2) = x$ have a solution. Hence

$$-\frac{1}{4}\frac{1}{(1-2\delta_0)^2} \leqslant a_s \leqslant -\frac{1}{4}\frac{1}{(1+2\delta_0)^2}.$$

These values are very near $a = -\frac{1}{4}$ and $x = -\frac{1}{2}$, which are the bifurcation values for $(a,x) \mapsto a - x^2$. Now we normalize H_n again by linear changes of coordinates both in a and x so that $a_s = -\frac{1}{4}$ and $x_s = -\frac{1}{2}$. The values of $H_{n,a}$ and $H_{n,xx}$ do change, but are still very near 1 and -2, if δ_0 is sufficiently small.

For the sake of simplicity, we write $H = H_n$, in these coordinates. For every such H the starting conditions Central Branch are satisfied, H_{xx} is near -2, H_a is near 1, and the values of the saddle-node bifurcation are given by $(a_s, x_s) = (-\frac{1}{4}, -\frac{1}{2})$. The constant δ_0 regulates the proximity to the function $(a,x) \mapsto a - x^2$. We call $\mathcal{H} = \mathcal{H}_{\delta_0}$ the set of functions satisfying these conditions. Since here we are only interested in a bounded region of the plane and the parameter space near the saddle-node bifurcation, we can fix the domain of each $H \in \mathcal{H}$ as

$$\{(a,x)\ ;\ (a,x) \in [-10,10] \times [-10,10]\}.$$

Every constant appearing in the estimates will be uniform among the functions $H \in \mathcal{H}$, provided δ_0 is sufficiently small.

Let $a_0 < a_s$ be such that $|H(a,0)| \geqslant 2$ for every $a \leqslant a_0$. This is the lowest parameter value we are interested in, since all iterates outside the critical region $H^{-1}([H^2(0), H(0)])$ have some expansion (approximately greater or equal than 4) and can be treated by other methods. For $H(a,x) = a - x^2$, we have $a_0 = -2$ and in the remaining cases there is an error of the order of δ_0 about -2.

For $a > a_0$ we are concerned with iterates $x, Hx, \ldots, H^j x$, where $|x| < 2$, $x \notin H^{-1}([H^2(0), H(0)])$ and $|H^j(x)| \geqslant 2$. For each a we associate the number $l = l(a)$ which gives the maximal j. In other words,

$$l = l(a) = \min\{j \geqslant 1\ ;\ |H^j(H(0))| \geqslant 2\}.$$

Now for $x \in (-2,2) \setminus H^{-1}([H^2(0), H(0)])$ and j as above we denote

$$F_S(a,x) = H^j(a,x).$$

We aim at proving the following Lemma (corresponding to Lemma S.8 in [**2**]).

Lemma A.1

There is $C > 0$ such that for all $H \in \mathcal{H}$, $x \in (-2,2) \setminus H^{-1}([H^2(0), H(0)])$ and $a_0 < a < a_s = -\frac{1}{4}$, we have

$$|F_{S,x}|^{-1},\ \left|\frac{F_{S,xx}}{(F_{S,x})^2}\right|,\ \left|\frac{F_{S,xxx}}{(F_{S,x})^3}\right| \leqslant C,$$

$$|F_{S,x}|,\ \left|\frac{F_{S,xx}}{F_{S,x}}\right| \leqslant Cl^2,$$

$$|F_{S,a}|, \left|\frac{F_{S,xa}}{F_{S,x}}\right|, \left|\frac{F_{S,xxa}}{(F_{S,x})^2}\right| \leqslant Cl^3,$$
$$|F_{S,aa}| \leqslant Cl^6,$$

where $l = l(a)$ as above. Moreover, if $x \in [H^2(0), H(0)]$ then

$$|F_{S,a}| \geqslant C^{-1}l^3$$

and

$$|F_{S,x}|,\ |F_{S,xx}| \leqslant C.$$

This Lemma will be proved in the following way. We will fix $a_1 < a_s$ and define

$$l_0 = \max_{H\in\mathcal{H}} \max_{a_0\leqslant a\leqslant a_1} l(a).$$

Also we let $x_l < -\frac{1}{2} < x_r$ be such that $x_l > -2$, $x_r < H(0)$ and some conditions stated below are satisfied. The order of choice is this one: first x_l and x_r, then a_1 and finally δ_0. If δ_0 is sufficiently small then the constant C will be uniform for all $H \in \mathcal{H}$.

Iterates done for $a_0 \leqslant a \leqslant a_1$ and outside $[x_l, x_r]$ for $a > a_1$ are in (uniformly) finite number, so that they contribute only with constants to the Lemma. The main problem lies on the "unbounded" part $[a_1, a_s] \times [x_l, x_r]$, which is solved if we prove the following Lemma (Lemma S.7 in [**2**]).

Lemma A.2. — *There is $C > 0$ such that for $a > a_1$*

(1) $C^{-1} \leqslant |H^j_x| \leqslant Cl^2$, *for all* $x \in [x_l, x_r]$*;*
(2) $C^{-1} \leqslant |H^j_x| \leqslant C$, *for all* $x \in [Hx_r, x_r]$*;*
(3) $|H^j_a| \leqslant Cl^3$, *for all* $x \in [x_l, x_r]$*;*
(4) $|H^j_a| \geqslant C^{-1}l^3$, *for all* $x \in [Hx_r, x_r]$*;*
(5) $|H^j_{aa}| \leqslant Cl^6$, *for all* $x \in [x_l, x_r]$*;*
(6) $|H^j_{xx}| \leqslant C|H^j_x|^2$, *for all* $x \in [x_l, x_r]$*;*
(7) $|H^j_{xxx}| \leqslant C|H^j_x|^3$, *for all* $x \in [x_l, x_r]$*;*
(8) $|H^j_{xa}| \leqslant C|H^j_x|l^3$, *for all* $x \in [x_l, x_r]$*;*
(9) $|H^j_{xxa}| \leqslant C|H^j_x|^2l^3$, *for all* $x \in [x_l, x_r]$*.*

Establishing the relation between a and l is one of the main steps in the proof of Lemma A.2. We change coordinates again by $x \mapsto x + \frac{1}{2}$ and $a \mapsto a + \frac{1}{4}$, so that now the saddle-node occurs for $(a, x) = (0, 0)$. We also regard a_0, a_1, x_l, x_r in the new coordinates. Then we suppose that x_l and x_r are chosen so that

$$\frac{2}{3} \leqslant H_x \leqslant \frac{3}{2},$$

for all $x \in [H^3x_l, x_r]$. Moreover we take a_1 such that if $a \geqslant a_1$ and $H^j(a, x_r) < x_l$ then $j \geqslant 10$ (actually these choices are somewhat arbitrary).

The following Lemma compares H with purely quadratic functions. It is a direct consequence of the assumed proximity to $(a, x) \mapsto a - x^2$.

Lemma A.3. — *If $H \in \mathcal{H}$, δ_0 is sufficiently small, $a_1 \leqslant a < 0$ and $x \in [H^2 x_l, x_r]$ then*

$$\frac{5}{4}(a - x^2) \leqslant H(a,x) - x \leqslant \frac{3}{4}(a - x^2).$$

A *fundamental domain* for H is any interval of the form $[Hx, x]$. The smaller fundamental domain in $[x_l, x_r]$ has size equal to $\min |H(a,x) - x|$, which is greater or equal than $\frac{3}{4}a$, by Lemma A.3.

Consider now a fundamental domain $[H^{i+1}x_r, H^i x_r]$, $i \geqslant 0$. Let m be the first integer such that $H^m x_r < x_l$ (m differs from l by a finite amount). The power H^{m-i} maps $[H^{i+1}x_r, H^i x_r]$ diffeomorphically onto $[H^{m+1}x_r, H^m x_r]$. Note that H^{m-i} is extendible to the adjacent fundamental domains, so that the image extends to $[H^{m+2}x_r, H^{m-1}x_r]$. As the Schwarzian derivative of H is non-positive, the distortion of the power map derivative is bounded. In other words, there is $C_1 > 0$ such that

$$\frac{H_x^{m-i}(x_1)}{H_x^{m-i}(x_2)} \leqslant C_1,$$

for every $x_1, x_2 \in [H^{i+1}x_r, H^i x_r]$ and $0 \leqslant i \leqslant m-1$. In particular, by the Mean Value Theorem and the estimate on the least size of a fundamental domain, there is $C > 0$ such that

$$C^{-1} < |H_x^j| < Ca^{-1},$$

for all $x \in [x_l, x_r]$, where j here is the first integer such that $H^j x < x_l$.

Another consequence is that if $x \in [Hx_r, x_r]$ then $|H_x^j| < C$. This proves the first two items of Lemma A.2, provided we have Lemma A.7 below, relating a and l. To prove this Lemma, however, we need three others.

Lemma A.4. — *If i_0 is such that $|Hx - x|$ is not monotone in $[H^{i_0+1}x_r, H^{i_0}x_r]$ then*

$$|H^{i_0+1}x_r - H^{i_0}x_r| \leqslant \frac{25}{8}a.$$

Proof. — Let x_c be the unique point where $\min |Hx - x|$ is attained. Then x_c belongs to $[H^{i_0+1}x_r, H^{i_0}x_r]$ and, by Lemma A.3, $|Hx_c - x_c| \leqslant \frac{5}{4}a$. But $[H^{i_0+1}x_r, H^{i_0}x_r] \subset [Hx_c, H^{-1}x_c]$, hence

$$|H^{i_0+1}x_r - H^{i_0}x_r| \leqslant |Hx_c - x_c| + |H^{-1}x_c - x_c|.$$

By the choice of x_l and x_r,

$$|H^{-1}x_c - x_c| \leqslant \frac{3}{2}|Hx_c - x_c|,$$

and the Lemma is proved. □

Lemma A.5. — *If $|Hx - x|$ is monotone in $[H^{i+1}x_r, H^i x_r]$ then*

$$\frac{2}{3} \leqslant \int_{H^{i+1}x_r}^{H^i x_r} \frac{1}{|Hx - x|}dx \leqslant \frac{3}{2}.$$

Proof. — By the Mean Value Theorem, there is x_i in $[H^{i+1}x_r, H^i x_r]$ such that

$$\int_{H^{i+1}x_r}^{H^i x_r} \frac{1}{|Hx - x|} dx = \frac{|H^{i+1}x_r - H^i x_r|}{|Hx_i - x_i|}.$$

As the maximum and the minimum values of $|Hx - x|$ are attained at the boundary of $[H^{i+1}x_r, H^i x_r]$, we compare them with $|H^{i+1}x_r - H^i x_r|$ using the supposition $\frac{2}{3} \leqslant |H_x| \leqslant \frac{3}{2}$, and the Lemma is proved. □

Lemma A.6. — *If i_0 is such that $|Hx - x|$ is not monotone in $[H^{i_0+1}x_r, H^{i_0}x_r]$ then*

$$\int_{H^{i_0+1}x_r}^{H^{i_0}x_r} \frac{1}{|Hx - x|} dx \leqslant 5.$$

Proof. — This is a consequence of Lemma A.4, since

$$\int_{H^{i_0+1}x_r}^{H^{i_0}x_r} \frac{1}{|Hx - x|} dx \leqslant |H^{i_0+1}x_r - H^{i_0}x_r| \max |Hx - x|^{-1} \leqslant \frac{25}{8} a \cdot \frac{4}{3} a^{-1}. \qquad \square$$

Lemma A.7. — *There is $C > 0$ such that*

$$C^{-1}a^{-1/2} \leqslant l(a) \leqslant Ca^{-1/2},$$

for all $a_0 \leqslant a < 0$.

Proof. — It is enough to prove the same statement for m instead of l and for $a \geqslant a_1$. By Lemmas A.5 and A.6, we have

$$\frac{2}{3}\left(\int_{H^m x_r}^{x_r} \frac{1}{|Hx - x|} dx - 5\right) \leqslant m - 1 \leqslant \frac{3}{2} \int_{H^m x_r}^{x_r} \frac{1}{|Hx - x|} dx.$$

Applying Lemma A.3 to the left inequality, we get

$$\begin{aligned}
m &\geqslant -4 + \frac{2}{3} \cdot \frac{4}{5} \int_{x_l}^{x_r} \frac{1}{|a| + x^2} dx \\
&\geqslant -4 + \frac{1}{2} |a|^{-1/2} \left(\arctan \frac{x_r}{\sqrt{|a_1|}} - \arctan \frac{x_l}{\sqrt{|a_1|}} \right) \\
&\geqslant C^{-1}|a|^{-1/2},
\end{aligned}$$

where C is fixed after the choice of x_l, x_r and a_1. On the other hand,

$$\int_{H^m x_r}^{x_l} \frac{1}{|Hx - x|} dx \leqslant \int_{Hx_l}^{x_l} \frac{1}{|Hx - x|} dx \leqslant 1,$$

since the maximum of $|Hx - x|^{-1}$ in $[Hx_l, x_l]$ is attained in x_l. Then

$$\begin{aligned} m &\leqslant \frac{5}{2} + \frac{3}{2}\int_{x_l}^{x_r} \frac{1}{|Hx - x|} dx \\ &\leqslant \frac{3}{2}\cdot\frac{4}{3}|a|^{-1/2}\left(\arctan\frac{x_r}{\sqrt{|a|}} - \arctan\frac{x_l}{\sqrt{|a|}}\right) \\ &\leqslant \frac{5}{2} + 2\pi|a|^{-1/2} \leqslant C|a|^{-1/2}. \end{aligned}$$

□

Now we are able to prove the remaining assertions of Lemma A.2.

Lemma A.8. — $|H_a^j| \leqslant Cl^3$, *for all* $x \in [x_l, x_r]$.

Proof. — For $x \in [x_l, x_r]$, write

$$H_a^j = H_x^j \sum_{i=1}^{j} \frac{H_a \circ H^{i-1}}{H_x^i} \leqslant 2\sum_{i=1}^{j} |H_x^{j-i} \circ H^i|.$$

But this last sum is bounded by $Cjl^2 \leqslant Cl^3$. □

Lemma A.9. — *If* $x \in [Hx_r, x_r]$ *then* $|H_a^j| \geqslant C^{-1}l^3$.

Proof. — As in the previous Lemma,

$$H_a^j \geqslant \frac{3}{4}\sum_{i=1}^{j} H_x^{j-i},$$

since $H_a \simeq 1$. By bounded distortion,

$$H_x^{j-i} \geqslant C^{-1}|H^{i+1}x_r - H^i x_r|^{-1},$$

hence, similarly to the proof of Lemma A.7,

$$\begin{aligned} |H_a^j| &\geqslant C^{-1}\sum_{i=1}^{m} \frac{|H^{i+1}x_r - H^i x_r|}{|H^{i+1}x_r - H^i x_r|^2} \\ &\geqslant C^{-1}\int_{x_l}^{x_r} \frac{1}{(|a| + x^2)^2} dx \geqslant C^{-1}|a|^{-3/2} \geqslant C^{-1}l^3. \end{aligned}$$

□

Lemma A.10. — $|H_{xx}^j| \leqslant C|H_x^j|^2$, *for all* $x \in [x_l, x_r]$.

Proof. — Writing

$$\frac{H_{xx}^j}{(H_x^j)^2} = \sum_{i=1}^{j} \frac{H_{xx} \circ H^{i-1}}{(H_x^{j-i+1} \circ H^{i-1})(H_x \circ H^{i-1})}$$

we get

$$|H_{xx}^j| \leqslant C(H_x^j)^2 \sum_{i=1}^{j} \frac{1}{|H_x^{j-i+1} \circ H^{i-1}|}.$$

Using bounded distortion,

$$|H_x^{j-i+1}|^{-1} \leqslant C|H^i x - H^{i-1}x|,$$

for $i = 1, \ldots, j$. As the sum of the sizes of the fundamental domains is bounded, the Lemma follows. □

Lemma A.11. — *$|H_{aa}^j| \leqslant Cl^6$, for all $x \in [x_l, x_r]$.*

Proof. — Write

$$\frac{H_{aa}^j}{(H_x^j)^2} = S_1 + 2S_2 + S_3,$$

where

$$S_1 = \sum_{i=1}^{j} \frac{H_{aa} \circ H^{i-1}}{H_x^i H_x^j}$$

$$S_2 = \sum_{i=2}^{j} \frac{H_{xa} \circ H^{i-1}}{H_x^j (H_x \circ H^{i-1})} \cdot \frac{H_a^{i-1}}{H_x^{i-1}}$$

$$S_3 = \sum_{i=2}^{j} \frac{H_{xx} \circ H^{i-1}}{(H_x^{j-i+1} \circ H^{i-1})(H_x \circ H^{i-1})} \cdot \left(\frac{H_a^{i-1}}{H_x^{i-1}}\right)^2.$$

Then, as in Lemma A.8,

$$(H_x^j)^2 |S_1| \leqslant C \sum_{i=1}^{j} |H_x^{j-i}| \leqslant Cl^3.$$

In addition,

$$(H_x^j)^2 |S_2| \leqslant C \sum_{i=2}^{j} \sum_{t=1}^{i-1} |H_x^{j-t} \circ H^t| \leqslant Cl^4,$$

as in the proof of Lemma A.8. Finally,

$$\begin{aligned}(H_x^j)^2 |S_3| &\leqslant C \sum_{i=2}^{j} |H_x^{j-i+1} \circ H^{i-1}|^{-1} \left(\sum_{t=1}^{i-1} |H_x^{j-t} \circ H^t|\right)^2 \\ &\leqslant Cl^6 \sum_{i=2}^{j} |H_x^{j-i+1} \circ H^{i-1}|^{-1} \leqslant Cl^6,\end{aligned}$$

where the last inequality is similar to the proof of Lemma A.10. □

Lemma A.12. — *$|H_{xxx}^j| \leqslant C|H_x^j|^3$, for all $x \in [x_l, x_r]$.*

Proof. — Similar to Lemma A.10. It is enough to bound $H_{xxx}/(H_x)^3$ by

$$C\left(\sum_{i=1}^{j} |H^i x_r - H^{i-1} x_r|\right)^2. \qquad \square$$

Lemma A.13. — *$|H_{xa}^j| \leqslant C|H_x^j| l^3$, for all $x \in [x_l, x_r]$.*

Proof. — Write $H^j_{xa}/H^j_x = S_1 + S_2$, where

$$S_1 = \sum_{i=1}^{j} \frac{H_{xa} \circ H^{i-1}}{(H_x \circ H^{i-1})}$$

and

$$S_2 = H^j_x \sum_{i=2}^{j} \frac{H_{xx} \circ H^{i-1}}{(H^{j-i+1}_x \circ H^{i-1})(H_x \circ H^{i-1})} \cdot \frac{H^{i-1}_a}{H^{i-1}_x}.$$

The techniques employed in the previous Lemmas lead to $|S_1| \leqslant Cl$ and $|S_2| \leqslant Cl^3$. □

Lemma A.14. — *$|H^j_{xxa}| \leqslant C|H^j_x|^2 l^3$, for all $x \in [x_l, x_r]$.*

Proof. — Write

$$\frac{H^j_{xxa}}{(H^j_x)^2} = H^j_x(S_1 + S_2 + S_3 + 4S_4 + 2S_5),$$

where

$$S_1 = \frac{1}{H^j_x} \sum_{i=1}^{j} \frac{H_{xxa} \circ H^{i-1}}{(H^{j-i+1}_x \circ H^{i-1})(H_x \circ H^{i-1})},$$

$$S_2 = \sum_{i=2}^{j} \frac{H_{xxx} \circ H^{i-1}}{(H^{j-i+1}_x \circ H^{i-1})^2 (H_x \circ H^{i-1})} \cdot \frac{H^{i-1}_a}{H^{i-1}_x},$$

$$S_3 = \sum_{i=2}^{j} \frac{H_{xa} \circ H^{i-1}}{H^j_x H^i_x} \cdot \frac{1}{H^{j-i+1}_x \circ H^{i-1}} \frac{H^{i-1}_{xx}}{(H^{i-1}_x)^2},$$

$$S_4 = \sum_{i=2}^{j} \frac{H_{xx} \circ H^{i-1}}{(H^{j-i+1}_x \circ H^{i-1})^2 (H_x \circ H^{i-1})} \cdot \frac{H^{i-1}_{xa}}{(H^{i-1}_x)^2},$$

$$S_5 = \sum_{i=2}^{j} \frac{H_{xx} \circ H^{i-1}}{(H^{j-i+1}_x \circ H^{i-1})^2 (H_x \circ H^{i-1})} \cdot \frac{H^{i-1}_a}{H^{i-1}_x} \cdot \frac{H^{i-1}_{xx}}{(H^{i-1}_x)^2},$$

then proceed as in the previous Lemmas. □

B. Glossary

The formula below appear in many places of this work. They give mixed derivatives of compositions of parameter dependent diffeomorphisms. Let $\{F_i\}_{i=1,\dots,j}$, $F_i = F_i(a,x)$, be a sequence of diffeomorphisms and $G = G_j$ its composition $G_j = F_j \circ \cdots \circ F_1$. Consider also the partial compositions $G_i = F_i \circ \cdots \circ F_1$ and $Q_i = F_j \circ \cdots \circ F_i$. To simplify the notation, we omit the points where the functions are evaluated.

$$\frac{G_a}{G_x} = \sum_{i=1}^{j} \frac{F_{i,a}}{G_{i,x}} \tag{3}$$

$$\frac{G_{xx}}{(G_x)^2} = \sum_{i=1}^{j} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \tag{4}$$

$$\frac{G_{xa}}{(G_x)^2} = \sum_{i=1}^{j} \frac{F_{i,xa}}{F_{i,x}G_{j,x}} + \sum_{i=2}^{j} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} \tag{5}$$

$$\frac{G_{aa}}{(G_x)^2} = \sum_{i=1}^{j} \frac{F_{i,aa}}{G_{i,x}G_{j,x}} + 2\sum_{i=2}^{j} \frac{F_{i,xa}}{F_{i,x}G_{j,x}} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} + \sum_{i=2}^{j} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \left(\frac{G_{i-1,a}}{G_{i-1,x}}\right)^2 \tag{6}$$

$$\frac{G_{xxx}}{(G_x)^3} = \sum_{i=1}^{j} \frac{F_{i,xxx}F_{i,x}}{(Q_{i+1,x})^2(F_{i,x})^4} + 3\sum_{i=2}^{j} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2} \tag{7}$$

$$\begin{aligned} \frac{G_{xxa}}{(G_x)^3} &= \sum_{i=1}^{j} \frac{F_{i,xxa}}{G_{j,x}Q_{i+1,x}(F_{i,x})^2} \\ &\quad + \sum_{i=2}^{j} \frac{F_{i,xxx}F_{i,x}}{(Q_{i+1,x})^2(F_{i,x})^4} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} + \sum_{i=2}^{j} \frac{F_{i,xa}}{F_{i,x}G_{j,x}} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2} \\ &\quad + 2\sum_{i=2}^{j} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \left[\frac{2}{Q_{i,x}} \frac{G_{i-1,xa}}{(G_{i-1,x})^2} + \frac{G_{i-1,a}}{G_{i-1,x}} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2}\right] \end{aligned} \tag{8}$$

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