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Astérisque, tome 286 (2003), p. 1-24

<http://www.numdam.org/item?id=AST_2003__286__1_0>

Astérisque
286, 2003, p. 1–24

ON THE MATHEMATICAL CONTRIBUTIONS OF JACOB PALIS

by

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Abstract. — A Conference on Dynamical Systems celebrating the 60th birthday of Jacob Palis was held at IMPA (Instituto de Matemática Pura e Aplicada) in Rio de Janeiro from July 19-28, 2000. This article is a revised and expanded version of a lecture I gave at the Conference. Many additions, including the list of references and the entire sections below on *Homoclinic Bifurcations*, *Cantor Sets and Fractal Invariants*, *Non-Hyperbolic Systems*, and *A Unifying View of Dynamics*, were made later by Marcelo Viana. It was decided to preserve the flavor of the lecture by keeping the narrative in the first person. I am grateful to Marcelo for his contributions to this paper. In my opinion, they greatly improved the presentation of the mathematical scope and influence of Jacob Palis.

Introduction

Let me begin just by saying that Jacob has made many, many contributions to Mathematics. I will not talk about all of them because, in fact, in one hour it's impossible to discuss in any detail all of them. I pick some of what I consider to be the main contributions, and there will be relatively little that is new for experts, but I hope you will be reminded of many experiences during the last thirty or some years of the development of Dynamical Systems.

First, to my mind his primary mathematical contributions fit into three categories:

– global stability related to the concepts of structural stability and Ω-stability;

– bifurcation theory, which is how systems depending on parameters change, how their structure changes.

– formulation of some general ideas and conjectures, that motivated several very interesting recent results in this field.

I will talk about these aspects of his work a little bit later. Besides these types of subjects there are many other ancillary results, many interesting kinds of things.

2000 ***Mathematics Subject Classification.*** — 37Dxx.
Key words and phrases. — Hyperbolic, Morse-Smale, structural stability, Axiom A, bifurcation theory, homoclinic tangency, tubular families, Ω-stability, stability conjecture.

But, together with the mathematical contributions that he has been making, one has to appreciate and understand the overview and direction of research that Jacob is responsible for. At the present time he is at

– 35 graduate students, and some 30 grand-students, originating from 10 different countries mainly in Latin America, as you can see in his academic tree (attached to this paper).

Some of these students have become main figures in the whole theory of Dynamical Systems, in fact in the world of Mathematics. You know who they are as well as I do, I don't need to mention names. It's a testimony to his vision, his generosity, and the freedom of ideas that he's encouraged, that he is such an inspiration to so many people.

In addition, I think it's really fair to say that in our time Jacob Palis has been one of the main figures responsible for the development of Mathematics and Science, primarily in Latin America[(1)] and, in fact, in many other places, through his

– organization of meetings, symposia, workshops, and the support of sciences and Mathematics in developing countries, most notably, that I'm familiar with, in Trieste.

He has facilitated the contacts between scientists who have had great difficulty in traveling to the west for political or other reasons. They were able to establish contacts with western mathematicians in the settings of meetings, workshops, and schools where one can get to meet many people. I myself met a number of people from mainland China in Trieste, at a time when it was extremely difficult for them to travel to Western Europe. Jacob has been one of the primary organizers and supporters of such occasions.

Moreover, he has been responsible, in great measure, for

– the tremendous growth of IMPA, this wonderful institute, as a researcher and, more recently, also as the Director.

I think it's fair to say that IMPA has become the principal center for Mathematics in Latin America and, certainly, one of the world centers for Dynamical Systems. In no small measure is this due to his efforts and, again, his vision.

I want to go now toward some of the mathematical developments Jacob has accompanied in his many years of activity.

Structural Stability

Let me go back to 1960. Let M be a compact connected smooth manifold without boundary, and let us consider the space $\mathcal{D}^r(M)$ of C^r diffeomorphisms on M, and the

(1) The impact of Jacob Palis's work throughout Latin America was the subject of another lecture at the Conference, by Alberto Verjovsky.

space $\mathcal{X}^r(M)$ of C^r vector fields on M, as well as certain distinguished well-known subsets of these

$\mathcal{D}^r_{ss}(M)$ = set of C^1 structurally stable diffeomorphisms on M,
$\mathcal{X}^r_{ss}(M)$ = set of C^1 structurally stable vector fields on M.

This notion of *structural stability* means that under any small C^1 perturbation, the entire orbit structure persists after a global continuous coordinate change. As far as I know, it was first presented by Andronov and Pontrjagin in 1937. They introduced these systems, that they called rough systems, or coarse systems, and the primary part of the paper [**2**] was to characterize them among vector fields in the two dimensional disk which were nowhere tangent to the boundary. And what they described in that paper was that a vector field X is structurally stable if and only if

(a) X has only finitely many critical elements (singular points and periodic orbits), all hyperbolic,

(b) and there are no saddle connections.

The next principal result connected to structural stability we will mention was due to Maurício Peixoto in a paper [**53**] that was published in 1959. There, he studied various general properties of structurally stable systems and proved that the Andronov-Pontrjagin systems formed an open and dense subset of the set of all vector fields on the two dimensional disk which were nowhere tangent to the boundary. Later, in [**54**], in a somewhat surprising way, he proved the following theorem: on a compact oriented surface M^2,

– the structurally stable vector fields $\mathcal{X}^r_{ss}(M^2)$ form a dense open set in the space $\mathcal{X}^r(M^2)$ and

– they are completely characterized by the Andronov-Pontrjagin conditions (a) and (b), and the additional condition that the α- and ω-limit sets of every point x are critical elements.

As far as I know, originally this paper was thought to prove that the result is true for all surfaces (not necessarily orientable), but that's still not known, except in the case of genus two, where Carlos Gutierrez [**18**] proved the general result, and in the C^1 topology, where it is a consequence of Pugh's closing-lemma [**56**].

This led to two main questions at the time:

– Is $\mathcal{X}^r_{ss}(M)$ non-empty, that is, do structurally stable systems exist on any manifold?

– Is $\mathcal{X}^r_{ss}(M)$ always dense in the space $\mathcal{X}^r(M)$ of all vector fields?

Also the analogous questions for C^r diffeomorphisms on compact manifolds.

Well, to some people's disappointment, the second question, the density, has a negative answer. That was discovered by Smale around 1964 or 65. He found out that on any manifold in dimension bigger than or equal to 4 there were open sets of vector fields which were not structurally stable. That dimension was then made optimal by Bob Williams in the end of the 60's [**68**]: he found more detailed versions of Smale's theorem, and a counter-example in dimension 3.

Around the same time, in the 60's, in the Soviet Union, Anosov studied other kinds of structurally stable systems. The systems that he called C-diffeomorphisms [**3**], where the entire space had a splitting into two continuous distributions invariant by the derivative, one of which was exponentially expanded and the other exponentially contracted under iterates. These systems, now well known, were coined the name Anosov diffeomorphisms by Smale in his 1967 paper [**65**] in the Bulletin of the AMS. What Anosov was able to to prove for these systems was that

- they formed an open subset of the set of all C^1 diffeomorphisms on a manifold
- and they were structurally stable systems.

The methods were related (I don't know, in fact, in which order) to his celebrated result that geodesic flows on manifolds with negative curvature were structurally stable and had the flow version of these Anosov conditions.

At this time, in the mid 60's, what was then the status of this kind of mathematics? We had high dimensional examples of structurally stable systems. They exhibited very complicated recurrence, and they were only known in special manifolds. In fact, for the Anosov systems the existence of the invariant bundles of course brings with it topological obstructions. So, for example on surfaces, Anosov diffeomorphisms only exist on the torus. And in higher dimensions, also only on very special manifolds. In fact, for a while it was felt that the only manifolds that admitted Anosov diffeomorphisms were the tori, of any dimension. Smale found examples using other kinds of Lie groups, non-Abelian Lie groups, but still they were very special in the kinds of manifolds that can exhibit them.

What about simple recurrence, that is, systems that don't have complicated recurrent orbits? Motivated by gradient systems, which Smale sort of used for going back and forward between dynamical systems and topology, a special class of dynamical systems, which we now call Morse-Smale systems, was defined. In the diffeomorphism case, these are systems where the non-wandering set consists of a finite number of hyperbolic periodic orbits, and if you have two such orbits their stable and unstable manifolds are transverse. Analogous definitions were given for vector fields, where the non-wandering set consists of finitely many critical points and periodic orbits all hyperbolic, and with the transversality conditions.

Smale was able to prove that there was a residual set of gradient systems (a residual set of functions) on any compact manifold that were Morse-Smale, and their time-one maps were Morse-Smale diffeomorphisms. The easy part of this is to realize that a Morse function has only hyperbolic critical points as its non-wandering set. But it's not so obvious to get the transversality condition: that is a consequence of a general approximation theorem, the Kupka-Smale theorem, which was done in those days. And Smale conjectured that,

– Morse-Smale systems form an open set in the space of all dynamical systems, both $\mathcal{D}^r(M)$ and $\mathcal{X}^r(M)$

– and every Morse-Smale system is structurally stable.

And then, in a remarkable result in 1967, in his thesis [**38**] Jacob Palis proved that the first statement, the openness statement, held in general. And he proved the second statement, that Morse-Smale systems were structurally stable, for any systems, diffeomorphisms and vector fields, in dimension less or equal to 3.

A Geometric Approach

To indicate some of the difficulties which Jacob had to overcome in proving this theorem, let's take a simple example of a Morse-Smale diffeomorphism on the 2-sphere as indicated in Figure 1, where we have six fixed points as the non-wondering set. The

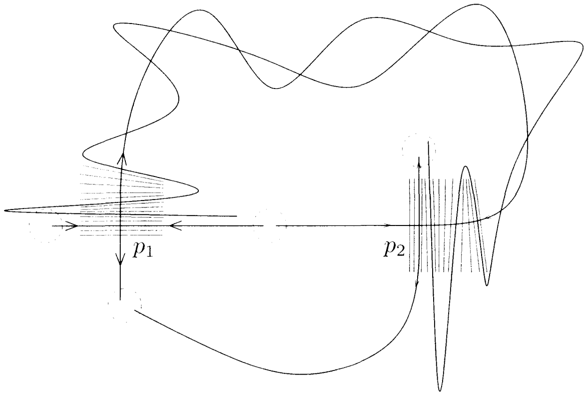

FIGURE 1. Tubular families

circles represent sources and sinks, and we have two saddle points, I denote p_1 and p_2, such that the unstable manifold of p_1 has some transverse intersection, a heteroclinic saddle connection, with the stable manifold of p_2.

Well, it was known earlier that there was a local stability phenomenon for hyperbolic fixed or periodic points, the Grobman-Hartman theorem. Locally, the system can be topologically linearized, that is, on a neighborhood of each periodic point the map is topologically conjugate to its derivative at the periodic point. But you need to do much more to get a global conjugacy, of course, you have to preserve stable and unstable manifolds globally. And orbits near the saddle points in the past go near the sources, and in the future go near the sinks. So, to have some conjugacy between a system like this and its perturbation it's not enough to look at local pictures, you have to glue them together in a special way. And the gluing is not obvious at all, because the local linearizations are very special, so how you glue this in some compatible way was a major problem.

And here there was the first major development that Palis came up with, which were the so-called tubular families, or invariant foliations, that I'll describe in some detail. They turned out to be very important for many later developments, as we'll see. These were invariant foliations defined in a neighborhood of each periodic point, one family for the stable direction and another for the unstable direction, and they were compatible: if two leaves from different periodic points intersect, then one contains the other. The construction of this is not at all obvious, it's still technically quite difficult — a very intricate geometric construction. The tubular families have different dimensions, in general. And the intricacies of this construction is what forced the restriction to dimension 3 in Jacob's thesis, the higher dimension analogue only came later.

In particular, initially it was thought that topological questions would arise in this connection, since one has to extend maps defined on certain subsets to bigger sets. It was thought that the annulus conjecture, a major unsolved problem at the time, was related to the higher dimension analogue of this tubular families method. Well, I'm not sure about the exact details of how these problems were overcome, but together with Smale in 1968 or 69, the general construction of tubular families was given, and the general structural stability of Morse-Smale systems in any dimension was proved [**42**].

It's important to notice that there is a lot of freedom in the construction of these tubular families. The conjugacies are not unique. The existence of invariant manifolds covering the whole manifold was crucial to Anosov in his treatment of structural stability. Those invariant manifolds are unique, and so the conjugacies, if they are near the identity, are unique for Anosov systems. Here they are highly non-unique, and in fact the flexibility of the choice is very much related to the freedom one has in the construction of tubular families. So this was a major breakthrough at the time and still is, in my opinion, a major contribution, that came quite early in his career.

This had two main corollaries. The first one was that

– an open dense subset of the set of gradient systems on any manifold consists of structurally stable vector fields;

Even more, the time-one maps of such vector fields are structurally stable diffeomorphisms. That's much stronger. Indeed, as we know, the usual equivalence relation for vector fields is homeomorphisms taking orbits to orbits. A stronger equivalence relation is conjugacy, actual one parameter group conjugacy. And structural stability for the time-one maps gives stability under this stronger equivalence relation, for gradient flows. So, as an extension of this, the problem of the existence of structural stability was solved in a very positive way:

– every manifold has structurally stable vector fields and diffeomorphisms.

The Stability Conjectures

Around this time, in the late 60's, having proved that structurally stable systems are not dense, Smale was looking for a more general kind of system, that would still have some good structure and have the chance to form a dense subset in the space of all dynamical systems. And so he formulated what was called the Ω-stability theorem.

Our system is Ω-stable if when you take a C^1 perturbation of it you have a conjugacy from the non-wandering set of the first system to the non-wandering set of the second one (not a global conjugacy on the whole manifold, as in the definition of structural stability). He studied special systems, the so-called Axiom A diffeomorphisms, where the non-wandering sets are hyperbolic sets, and the periodic points are dense in the non-wandering set. He also assumed an additional property, the no-cycle property, that gives the ability to construct so-called filtrations for the system, that is, to isolate the recurrent orbits in individual indecomposable pieces. And he proved the theorem that Axiom A and the no-cycle property implied that the diffeomorphism was Ω-stable.

Around the same time, Jacob proved that if you have an Axiom A system and it has a cycle, then it is not Ω-stable. And that led to the Stability Conjectures, which were also present in the Palis and Smale paper of 1969 [**42**]:

(1) a diffeomorphism $f \in \mathcal{D}^r(M)$ is structurally stable if and only if it satisfies the Axiom A and the so-called strong transversality condition: stable and unstable manifolds are in general position at each point wherever they meet;

(2) and $f \in \mathcal{D}^r(M)$ is Ω-stable if and only if it satisfies the Axiom A and the no-cycle property.

And they made analogous conjectures for flows.

Let me mention a little personal anecdote in connection with this theorem and the formulation of these conjectures. For those who were around that time, you remember that the first formulation of the Ω-stability theorem had another stronger condition, called Axiom B. Axiom B said that if you have two basic sets and the unstable manifold of one accumulates on the other, then there is a periodic point in the first whose unstable manifold has a transversal intersection with the stable manifold of the other. And the first formulation of the Ω-stability theorem, in fact the formulation that is in the Bulletin paper [**65**], says: Axiom A plus Axiom B implies Ω-stability, or something to that effect.

I remember Smale giving a lecture in the seminar in Berkeley in 1966 or maybe 1967. I was a new graduate student just sort of going to this seminar from time to time, but it was a very active and energetic seminar, many questions, comments, discussions. I remember Charles Pugh was there, and Mike Shub, Morris Hirsch, Jacob Palis. As a young graduate student we look around at all those famous people in the room, and just watch what they were doing. Well, Terry Wall had just come in from England and was interested, so he went to the seminar. In fact, he was under jet-lag so he was asleep in a large part of the talk. So, Smale was doing the construction of the local conjugacy of the Ω-stability for the basic sets. Then, with Axiom B, he constructed this partial order on the basic sets, and hence a filtration to isolate each piece, so that one can get the global conjugacy. And, suddenly, Terry woke up and looked and said, quietly: "Is all you need, the partial order relation, in order to get the stability?" This was an agitated seminar with many people. Steve turned and said: "Well, maybe, I'm not sure about that, I'm not sure."

At that instant, I didn't know who Jacob Palis was, but he became very animated and said: "That's right, that's it, that is all you need!" And the next day, as I recall, he proved that if you had a cycle then you had Ω-explosions, and so, in fact, this no-cycle condition was necessary for stability. Later on, in the paper that actually appears in the proceedings of the symposium [**42**], you see Axiom A and no-cycle condition, not Axiom A and Axiom B, Axiom B disappeared. So, as part of this discussion, Jacob had a significant part in the formulation of the Ω-stability theorem as it now sits.

From Hyperbolicity to Stability

How does one go beyond toward more general stability theorems and proving these conjectures? What did people know at that time? They knew that the Morse-Smale systems were structurally stable. They knew that Axiom A and no-cycle property implies Ω-stability. How does one to get more general structurally stable systems? One idea at the time was to take Jacob's tubular family construction and extend it to Axiom A systems. That is, to get an invariant foliation on neighborhoods of complicated hyperbolic sets. It turned out to be quite a complicated thing to do and, in fact, this is still not known in general, it's not known how to do that for high dimensional systems. But that program did succeed for two-dimensional diffeomorphisms, with the thesis of Welington de Melo in 1971.

The next progress came in what might seem a curious way. Jürgen Moser gave a second proof of the stability of Anosov systems, using the so-called infinitesimal methods. His idea was the following: you want to solve the equation $h \circ f = g \circ h$ for a homeomorphism h. You rewrite this as

$$f^{-1} \circ h \circ f = f^{-1} \circ g \circ h.$$

Then you take a Riemannian metric on your manifold, and try to find h as the exponential of some continuous vector field v, which should be C^0-small so that the homeomorphism is close to the identity. So, writing $h = \exp(v)$, and also $f^{-1} \circ g = \exp(w)$ for a C^1-small vector field w, you get

$$f^{-1} \circ \exp(v) \circ f = \exp(w) \circ \exp(v).$$

Linearizing this equation (or using infinitesimal methods, which is the term I use), you get

$$\exp(Df^{-1} \circ v \circ f) = \exp(w + v),$$

up to a small error. So, taking $\exp^{-1}$ in the previous relation, it becomes

$$Df^{-1} \circ v \circ f + s(v, w) = w + v,$$

where $s(v, w)$ is small. Denoting $F\,v = Df^{-1} \circ v \circ f$, this may be rewritten as

$$(\mathrm{I} - F)v = v - Df^{-1} \circ v \circ f = s(v, w) - w.$$

So, we know w, which is a C^1-small vector field, and we are looking for v, a small continuous vector field. Moser realized that if you could invert this operator $(\mathrm{I} - F)$ on the space of continuous vector fields, then you could solve this functional relation for v, using the contraction mapping theorem. And, in fact, the Anosov condition was precisely the condition you need to make $(\mathrm{I} - F)$ invertible.

So, he was able to give a new proof of the stability of Anosov systems using vector field methods, infinitesimal methods, whereas Anosov's proof made strong use of the existence of integral manifolds for the expanding and contracting distributions, the stable and unstable manifolds. Well, at the time this was interesting because it made Anosov's proof understandable to people in the West, there was no published English version of it. And also I think it was thought of as a useful addition, a curious new proof of a known result. One thing that came out of it is that you get unique solutions near the the identity, which you can also prove by other methods.

There is an other development that I should mention. In the group of people who were in Berkeley and in the West at the time, the way that Moser's methods became known was through an implicit function theorem argument that John Mather produced. It turned out that, in detail, Mather's argument was actually incorrect, because differentiability assumptions were not satisfied. What the method gave you was a continuous solution to the functional equation, it didn't prove that the solution was a homeomorphism. But the arguments could be fixed up. I think it was Mike Shub who observed, and was well-known in the Soviet Union as well, that Anosov systems were expansive, and you can use that to show that solutions which are C^0-close to the identity actually have to be one-to-one. So you got the proof anyway, even if the implicit function theorem didn't work.

Far away, in the middle of the United States, Joel Robbin was learning about those things, and I think he shocked everybody by announcing that he could prove that, in the C^2 case, Axiom A diffeomorphisms satisfying the strong transversality condition are structurally stable. Well, how did he do it? He used infinitesimal adaptations of the tubular families constructions. Basically, the conjugacies were not unique, they involved choices, and he used the fact that Moser's transformation $(\mathrm{I} - F)$ had a continuous right inverse. You can see Jacob's influence again, even at that level: at the end of the paper [**60**] there's a ratio that says

$$(\text{Moser}) : (\text{Anosov}) = (\text{Robbin}) : (\text{Palis} - \text{Smale}).$$

The idea being that Moser produced an infinitesimal proof of the structural stability, removing the necessity of integrating the invariant subbundles for the construction, and Robbin produced an infinitesimal proof for Axiom A systems, removing the necessity of tubular families.

For technical reasons Robbin needed the C^2 assumption, not for the perturbations, but for the original diffeomorphisms. That was ultimately improved by Clark Robinson, who proved the general structural stability theorem, that Axiom A C^1 diffeomorphisms satisfying the strong transversality condition are structurally stable [**62**], and he also proved it for the vector field case [**61**]. Concerning Ω-stability, in Smale's paper [**65**] where he proves his Ω-stability theorem, he makes the statement that, presumably, similar methods can be used for flows. It was a highly non-trivial extension required to do it for flows, and it was carried out by Charles Pugh and Mike Shub [**57**]. So, at this stage, which I suppose is the mid-70's, we had general sufficient conditions for structural stability and Ω-stability, both for diffeomorphisms and for flows.

From Stability Back to Hyperbolicity

Remember the stability conjecture had a converse as well. So there was a lot of activity focussed on the converse. The initial efforts involved changing the definition of stability, to include conditions about dependence of the solution on the perturbation (whether it is continuous, whether is Lipschitz), and a number of people contributed with interesting works in that direction. John Franks [**14**] had a notion of time-dependent stability, with which he was able to characterize Axiom A and strong transversality systems. John Guckenheimer [**16**] had a notion of absolute stability, and so on. And then the full problem itself was treated in some special cases in low dimensions, by Liao [**21**], Mañé [**23, 24**], Pliss [**55**], and Sannami [**64**].

But the major breakthrough came in 1986, when Ricardo Mañé, one of Jacob's early graduate students, completely solved the problem! He proved what was the main remaining part, that is, that structurally stable systems had to satisfy the Axiom A [**25**].

Curiously enough, although this is a substantial result which uses much information about the non-wandering set, Ricardo was not able to prove the Ω-stability converse, he only proved the structural stability statement. It took some other intricate knowledge, and a fair amount of effort, for Jacob to prove that converse, and so complete the Ω-stability conjecture for diffeomorphisms, again around 1986. For the flow case, neither of the statements was known at the time, they were resolved only recently, by Shuhei Hayashi [**19**] in 1994.

So, in the development of this very important concept and theory, a period of almost 25 years was needed to accomplish what is now one of the crown jewels in the field of Dynamical Systems, the complete characterization of structurally stable systems. And as you saw, Jacob Palis played a very central role in that.

That's what I wanted to say about stability, the global stability issue. Now I want to go toward bifurcation theory.

Bifurcation Theory

In 1970 or so, I had the privilege to come to IMPA for two years, and to begin our program in bifurcation theory with Jacob. We started to work on the problem of understanding the structure of how hyperbolicity breaks down when you start with a Morse-Smale system. Basically, what we wanted to study was the so-called accessible part of the boundary of the Morse-Smale systems. The idea is the following. Let $\{\xi_\mu\}_\mu$ be an arc (a curve) of diffeomorphisms starting at a Morse-Smale system ξ_0. See Figure 2. You look at the first value $\mu = b$ of the parameter where the system

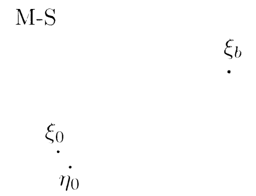

FIGURE 2. Bifurcations along parametrized families

fails to be structurally stable, the so-called first bifurcation point, and you want to describe the structure of such systems ξ_b.

Some ideas and problems were motivated by work done by Jorge Sotomayor [**66**] for one-parameter families of vector fields on surfaces, and also by a general periodic point description for one-parameter families of diffeomorphisms, which was obtained by Pavel Brunovsky [**6**]. In addition, there were mathematicians in the Soviet Union studying similar problems, Gavrilov and Shilnikov [**15**], although we didn't know that at the time, we only became aware of their work somewhat later.

During that period I wrote two papers with Jacob, [**35**] and [**36**], in which we basically proved the following. Assuming that at the first bifurcation point the limit set (the closure of the α- and ω-limit sets of the system) consists of a finite number of orbits, we completely described the structure at the bifurcation for generic arcs of diffeomorphisms. We also studied other issues related to stability as you move along the parameter, that I'll talk a bit more about later. But the main contents of the first paper [**35**] was this description at the bifurcation in the case when the limit set has finitely many orbits.

In the second paper [**36**] we considered systems where at the bifurcation point the limit set was actually hyperbolic, it stayed hyperbolic, but structural stability or Ω-stability failed all the same, because of the creation of a cycle. We studied the situation where the cycle was equidimensional, that is, the stable manifolds of all the periodic points in the cycle have the same dimension. We were able to prove that in that situation the bifurcation map ξ_b was accumulated by Axiom A, non Morse-Smale diffeomorphisms. That is,

– there existed parameter values $\mu_1 > \mu_2 > \cdots > \mu_i > \cdots$ converging to the first bifurcation point b, such that the diffeomorphisms ξ_{μ_i} satisfied the Axiom A and the strong transversality condition, and the non-wandering sets $\Omega(\xi_{\mu_i})$ were infinite.

Moreover, the non-wandering sets were all topologically distinct, so that ξ_{μ_i} could not be Ω-conjugate to each other. In fact, we proved that ξ_μ satisfies the Axiom A and the strong transversality condition for most parameters $\mu > b$ near b, in the sense that such parameters are a fraction close to 1, in measure, of small intervals $(\mu, \mu + \varepsilon)$.

Later, in a paper with Floris Takens and Jacob [**37**], we completely characterized the so-called stable arcs of diffeomorphisms, under the assumption that the limit set have finitely many orbits for each parameter value. An arc $\{\xi_\mu\}_\mu$ of diffeomorphisms is called stable if, given any perturbation $\{\eta_\mu\}_\mu$, as represented in Figure 2, then

(1) every diffeomorphism ξ_μ in the arc is conjugate to a diffeomorphism η_ν in the perturbed one, with a nearby parameter ν,

(2) and the conjugacy varies continuously with the parameter.

That's the condition of stability for arcs of diffeomorphisms. In [**37**] we characterized this condition and, as part of that work, a number of new concepts and ideas were introduced. In particular, a notion of rotation interval for circle endomorphisms was introduced. Strong rigidity for saddle-node bifurcations also came up in this work. One consequence of this strong rigidity phenomenon for saddle-node bifurcations is that the strong-stable and strong-unstable manifolds have to be preserved under conjugacy that varies continuously with the parameter (in general, topological conjugacies don't preserve strong-stable and strong-unstable manifolds).

Then these works were extended in a very significant way by Palis and Takens [**43**], who proved in 1983 that

– an open dense set of one-parameter families of gradients systems on any manifold were stable in the sense I've just described (continuous variation of the conjugacy with the parameter).

And somewhat later, in 1990, Mário Jorge Dias Carneiro and Jacob [**8**] proved that one can extend that to two-parameter families: an open and dense subset of families of gradient systems depending on two parameters are stable.

One might have hoped, in fact the hope around that time and earlier was that k-parameter families of gradient systems in a dense open set would be stable. That was shown to be false by Takens, who proved that for 8 or more parameters the stable families of gradient systems are not dense. I don't know how far down one has got yet, I think the conjecture still is that for k less than or equal to 4 the stable families should form an open and dense subset in the space of gradient systems.

In these constructions, the geometric freedom of tubular families and how you bring them up is, again, of fundamental importance. It's interesting to point out that at the time people discussed whether infinitesimal maps could be used for this theorems, but, as far as I know, they never managed to work. So far, infinitesimal methods have only been useful for the general structural stability theorem.

Homoclinic Bifurcations

Bifurcation theory continued to be one of Jacob's major projects during the 80's and afterwards. Initially, the goal was to extend some of these results, especially from [**36**], to the case where the limit set may have infinitely many orbits. In particular, now you want to consider more general arcs of systems starting inside the Axiom A, not just the Morse-Smale systems. But this also led to some very interesting new problems and ideas related, for instance, to fractal dimensions.

To explain this, let me consider a situation as described in Figure 3, a surface diffeomorphism with a non-transverse intersection between the stable and the unstable

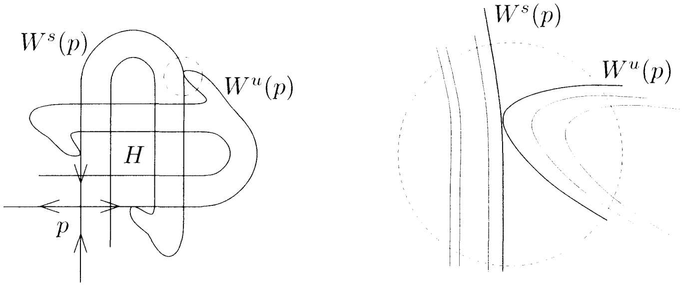

FIGURE 3. Homoclinic tangency associated to a hyperbolic set

manifold of a periodic saddle point p. We call that intersection a homoclinic tangency. And the periodic point p is contained in an infinite hyperbolic set H of the diffeomorphism, a horseshoe. This means that the homoclinic tangency is accumulated by a pair of laminations, or partial foliations, formed by the stable and unstable manifolds of all the points in H.

A diffeomorphism like this may be obtained as a first bifurcation ξ_b of an arc $\{\xi_\mu\}$ starting at an Axiom A system. The map ξ_b itself is not Axiom A, the homoclinic tangency implies that the non-wandering set is not hyperbolic. Then, as you increase the parameter, the stable and the unstable laminations move with respect to each other and, whenever there is a tangency between a leaf of one and a leaf of the other, the diffeomorphism can not be Axiom A.

Since these are just laminations, not full foliations of open sets, you might expect that such tangencies should be easy to avoid, taking advantage of the gaps between the leaves. However, I showed in my thesis [**32**] that it is not true in general. In fact,

– if the laminations are transversely thick, that is, if the gaps are relatively small, it is impossible to avoid tangencies between leaves of the two laminations, they exist for a whole open set of diffeomorphisms.

I'll call this phenomenon *persistent homoclinic tangencies*. Later, in [**34**], I proved that this phenomenon occurs near any surface diffeomorphism with a homoclinic tangency:

– there always exist open sets in the parameter space arbitrarily close to the bifurcation, that correspond to persistent tangencies.

And then Clark Robinson [**63**] deduced a version of this result for arcs of diffeomorphisms.

Palis and Takens wanted to understand this issue in more detail, and they came to establish a deep connection between homoclinic bifurcations and fractal dimensions of hyperbolic sets. Let me explain this.

In the paper [**36**], that I mentioned before, Jacob and I had shown that tangencies between the stable and the unstable laminations were, essentially, the only thing one has to worry about. We showed that if there were no tangencies and, in fact, the map was not too close to having a tangency, then the non-wandering set was hyperbolic. So this was a kind of converse to the fact that tangencies are an obstruction to hyperbolicity.

In the setting we were dealing with the limit set was finite, and we were able to show that parameters for which the map *is* too close to a tangency have small relative measure near the bifurcation. That's how we proved that hyperbolicity (Axiom A and strong transversality) prevails near these homoclinic tangencies, in terms of measure in parameter space. And the arguments suggested that it might be possible to avoid tangencies for most parameter values in more general situations, provided the laminations were not too thick.

Now, Palis and Takens realized that this should be formulated in terms of the transverse fractal dimensions of the laminations. The condition they required was that the sum of the transverse Hausdorff dimensions of the stable and unstable laminations should be less than 1. By definition, the transverse Hausdorff dimension is the Hausdorff dimension of the intersection of the lamination with some cross-section. It can be shown, in this context, that the definition doesn't depend on the choice of the cross-section.

It turns out that the sum of these transverse Hausdorff dimensions is equal to the Hausdorff dimension of the hyperbolic set H. So, their theorem, proved around 1984, has a very elegant statement [**45**]:

– if the Hausdorff dimension $HD(H)$ of the hyperbolic set involved in the tangency is less than 1, then ξ_μ is hyperbolic (Axiom A and strong transversality) for most nearby parameters $\mu > b$:

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} m\left(\{\mu \in (b, b+\varepsilon) : \xi_\mu \text{ is hyperbolic }\}\right) = 1, \tag{1}$$

where $m(\cdot)$ is Lebesgue measure.

At about the same time they proved a similar result for the heteroclinic case [**44**], where the tangency is between stable and unstable manifolds of different periodic points. Actually, in those papers they used another notion of dimension, called limit capacity, or box dimension, instead of Hausdorff dimension. But then it became clear that the two notions of fractal dimension coincide for hyperbolic sets of surface diffeomorphisms. This is discussed in their book [**46**, Chapters 4-5], where they also explain why (1) can always be stated with the full limit, initially in the heteroclinic case they only had a lim sup.

Then, in a paper [**51**] that was published in 1994, Jacob and Jean-Christophe Yoccoz proved that the condition in the previous theorem is, in fact, optimal:

– if the Hausdorff dimension of H is larger than 1, then the conclusion (1) above no longer holds.

This statement and, to some extent, the proof itself were inspired on a result of John Marstrand [**26**] about arithmetic differences

$$K_1 - \lambda K_2 = \{a_1 - \lambda\, a_2 : a_1 \in K_1 \text{ and } a_2 \in K_2\}$$

of Cantor sets in the real line: if the sum $HD(K_1) + HD(K_2)$ is larger than 1 then the difference has positive Lebesgue measure, for almost every λ. So, at this point it was already clear that there was an important relation between this part of Dynamics and other topics, like Geometric Theory of Dimension and Harmonic Analysis.

Cantor sets and Fractal Invariants

Motivated by this, Jacob started asking several questions about arithmetic differences of Cantor sets, with an eye on their applications to Dynamical Systems and other areas. In particular, he conjectured that for generic regular Cantor sets K_1 and K_2, the arithmetic difference either has zero Lebesgue measure or contains some interval. A Cantor set is called regular if it is generated by a smooth expanding map. The set of such Cantor sets comes with a natural topology, inherited from the corresponding maps.

Well, this conjecture was proved by Carlos Gustavo Moreira and Yoccoz [**30**], around the beginning of 1995. Actually, they proved a rather strong version of the conjecture. Their result applied to an open and dense set of regular Cantor sets that has "full probability", in some natural sense. Moreover, they get stable intersections, which is much stronger than just having an interval contained in the arithmetic difference. Then, they proved the following substantial extension of the previous results about homoclinic tangencies [**31**]: for generic arcs of diffeomorphisms $\{\xi_\mu\}_\mu$ with a homoclinic tangency at $\mu = b$,

– for most parameters $\mu > b$ close to b, in the sense of (1), either ξ_μ is hyperbolic or μ is in some interval with persistent homoclinic tangencies.

In other words, if $PT + AT$ is the union of all the intervals of persistent tangencies with those parameters for which the map satisfies the Axiom A and the strong transversality condition, then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} m\big(PT + AT \cap (b, b+\varepsilon)\big) = 1.$$

The theorem of Palis and Takens says that if the Hausdorff dimension of the horseshoe H is less than 1 then we have the same result already for the set of parameters corresponding to hyperbolic maps. So, the main novelty of this result is when the Hausdorff dimension is larger than 1.

There is a very natural question that arises, which is, what can we say about the dynamics when it's not hyperbolic. Well, Jacob has some recent joint work with Yoccoz [**52**] about this, that Yoccoz will talk about later in this Conference, so I won't discuss in any detail.[(2)] But the point is that they define so-called non-uniformly hyperbolic sets, or non-uniformly hyperbolic horseshoes, that are an extension of the hyperbolic sets that still have several nice properties. And they were able to show that if the Hausdorff dimension of the original hyperbolic set H is not much larger than 1 (they have a precise technical condition), then the diffeomorphisms ξ_μ are non-uniformly hyperbolic for most parameters $\mu > b$ near b. That is, if NUH is the set of parameters such that the non-wandering set is a non-uniformly hyperbolic set, then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} m\big(NUH \cap (b, b+\varepsilon)\big) = 1,$$

as long as the Hausdorff dimension is not much larger than 1.

Now let me say a few words about the higher dimensional case. Most of this has been proved for surface diffeomorphisms, and there are several serious difficulties that appear in higher dimensions. The main reason is that the stable and unstable laminations need not be transversely smooth. So, in general, it's not even known whether the transverse Hausdorff dimension is well defined. In fact, the geometry of hyperbolic sets in high dimensions is much less understood than in the surface case.

(2) Abstracts of talks given at the Conference are available at www.impa.br/~dsconf/.

In general, the Hausdorff dimension and the limit capacity are not equal, and they do not vary continuously with the dynamical system.

However, and this is a development near my heart, Jacob and Marcelo Viana were able to overcome some of these difficulties and, around 1989, prove the higher dimensional extension of the result about persistent homoclinic tangencies. The result was published in [**47**].

And they have very recent results together with Moreira, as we heard in Moreira's talk in this Conference, which show that the relation between fractal dimensions and abundance of hyperbolicity in parameter space stays valid for families of diffeomorphisms in arbitrary dimension.

Non-Hyperbolic Systems

The study of bifurcations, and these results that I mentioned, are part of an effort to go beyond the hyperbolic systems and understand very general dynamical systems. I think that, from the beginning, Jacob was convinced that bifurcation theory was the right way to do that or, at least, an essential part of trying to understand systems that are not hyperbolic, that are not structurally stable. And as the theory of homoclinic bifurcations developed, he became more and more convinced that they should play a key role in this.

By 1989 there was a paper of Benedicks and Carleson [**4**] where they proved that non-uniformly hyperbolic dynamics is frequent in the so-called Hénon family of plane maps

$$h(x, y) = (1 - ax^2 + y, bx).$$

That is, for a set of values of the parameters a and b with positive Lebesgue measure, the maps have a non-uniformly hyperbolic attractor. This was a striking extension of a very important pioneering work of Jakobson [**20**], back in the late seventies, where he had obtained a similar result for the family of quadratic real maps $q(x) = 1 - ax^2$.

Even before their paper appeared, Palis suggested that this result should be true, more generally, for generic arcs $\{\xi_\mu\}$ of surface diffeomorphisms with a homoclinic tangency. You see, it was known that returns maps of ξ_μ to certain regions near the tangency look like the Hénon model, so that was the idea. So, he proposed this problem to two of his students at the time, Leonardo Mora and Marcelo Viana. And Mora and Viana [**27**] were able to show that the approach of Benedicks and Carleson extended to more general dissipative systems, that are called Hénon-like maps, and from this they could prove Jacob's conjecture, in 1990.

These kinds of results, there are many others, relating homoclinic tangencies to other types of complicated dynamics, convinced Jacob that homoclinic tangencies might be some sort of unifying notion for understanding non-hyperbolic systems, at least in low dimensions. So he made the following conjecture:

– the union of Axiom A diffeomorphisms with those that have a homoclinic tangency is dense in $\mathcal{D}^r(M)$, if M is a surface.

In other words, every C^r surface diffeomorphism that is not in the closure of the Axiom A systems is approximated by other diffeomorphisms that have homoclinic tangencies.

As you probably know, this conjecture was proved a couple of years ago by two other former students of Jacob, Enrique Pujals and Martin Sambarino, in the case $r = 1$. Their paper has just appeared [**58**]. In fact, the result had been announced by Araújo and Mañé in the early 90's, but they never provided a proof. As a consequence of their methods, Pujals and Sambarino also got another most interesting result [**59**]:

– any arc of surface diffeomorphisms such that the topological entropy is not constant on it must contain a homoclinic tangency.

There is a version of the previous conjecture for high dimensions, that says that the union of Axiom A diffeomorphisms with those that have a homoclinic tangency or a heterodimensional cycle should be dense in $\mathcal{D}^r(M)$. A cycle is called heterodimensional if the stable manifolds of the periodic points involved in the cycle are not all of the same dimension. It seems that several groups of people have made progress in the direction of this high dimensional conjecture, indeed there will be a couple of talks on this subject in this Conference, but a complete proof is not yet available.

Back in the late eighties, Jacob suggested the study of heterodimensional cycles to Lorenzo Díaz, as his thesis problem. The idea was to complement our own results in [**36**], as I said before, we studied the equidimensional case. Now, Díaz found out that the conclusions are quite different for heterodimensional cycles: most of the times the bifurcating diffeomorphism ξ_b is not accumulated by hyperbolic ones, in fact, there is a whole interval $(b, b+\varepsilon)$ such that ξ_μ is not hyperbolic, not structurally stable, for any parameter μ in this interval. These results appeared in his thesis [**11**] and were much developed in a series of joint papers with Jorge Rocha, another former student of Jacob. See for instance [**13**].

And, sometime later, it became clear that heterodimensional cycles also have an important connection with the phenomenon of robust non-hyperbolic attractors, which I'll mention again in a little while.

A Unifying View of Dynamics

By 1995, Jacob had put several ideas and conjectures together to form a coherent picture of what might be the typical kinds of behavior of non-hyperbolic systems. This appeared in a preprint that was published in Douady's volume of Astérisque [**41**]. The main point is a conjecture that every system can be approximated by another having only finitely many attractors, whose basins of attraction contain almost all points. In fact these systems should have large probability in parameter space, in some natural sense. And the attractors should have nice properties, such as the existence of so-called Sinai-Ruelle-Bowen measures.

It is interesting to observe that the idea that most dynamical systems should have a finite number of attractors goes back to René Thom, in the sixties, although he didn't make precise what "most" was supposed to mean. Certainly, he was motivated by Smale's ideas in hyperbolic theory at the time[(3)], where the point of view was, primarily, topological. Maybe because of this, it was widely understood that Thom had in mind a residual (second category of Baire) subset of all dynamical systems and, in this form, the finiteness statement turned out to be false [**33**]. So, Jacob's conjecture is a very interesting revival of this classical idea, in a new and more probabilistic framework. A key novelty in Palis' approach is to allow the existence of cycles occupying a small volume in the dynamical space. Indeed, cycles have been a main obstruction to the realization of previous global scenarios for Dynamics.

So far, it is known that this conjecture holds for quadratic maps of interval, as a consequence of work done by Lyubich, Martens, and Nowicki. See [**22**]. And both Misha Lyubich and Artur de Melo will speak in this conference about their recent work with Welington de Melo, where they extended this to general analytic families of unimodal maps.

In higher dimensions, there have been some very interesting results that, I believe, were at least partially motivated by Jacob's questions and conjectures.

There is the work of Díaz, Pujals, Ures, and Bonatti [**12, 5**] where they characterized the robust sets of diffeomorphisms in any dimension. An invariant set is robust if it is transitive and remains transitive under any C^1 small perturbation of the system. They proved that robust sets must have a so-called dominated splitting, which is a decomposition of the tangent space into two continuous distributions such that one is more expanding than the other at every point, by a definite factor. In dimension 3 at least one of the distribution is hyperbolic, either expanding or contracting. This is called partial hyperbolicity.

Moreover, Alves, Bonatti, and Viana proved existence and finiteness of ergodic attractors, or Sinai-Ruelle-Bowen measures, for certain types of partially hyperbolic systems, in a paper [**1**] that has just appeared.

And there is also very important work of Carlos Morales, Maria José Pacifico, and Enrique Pujals [**28, 29**], characterizing the robust sets of arbitrary flows in 3 dimensions. Robust sets containing only regular orbits must be hyperbolic, so the more interesting case is when the set contains some singularity. They proved that any robust set that contains a singularity is a Lorenz-like attractor, or repeller, meaning that it has all the main features of the geometric Lorenz models of Guckenheimer-Williams [**17**].

[(3)] "Toutefois, selon certaines idées récentes de S. Smale, si la varieté est compacte, presque tout champ X présenterait un nombre fini d'attracteurs isolément structurellement stables" [**67**, p. 56]

Many Other Results

There are many other important contributions that Palis has done. For instance, there is his work on moduli invariants, that is, characterizing systems with the property that the number of topological types of perturbations depends on a finite number of real parameters. In [**40**], he discovered a smooth invariant for topological conjugacy

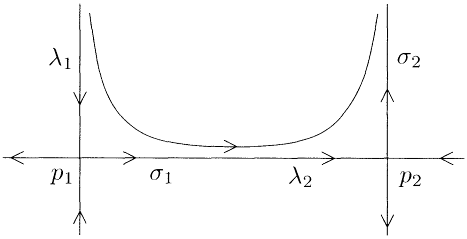

FIGURE 4. Moduli of conjugacy in saddle-connections

between flows with a saddle connection as in Figure 4. In fact, two such flows are conjugated if and only if they have the same ratio of eigenvalues

$$\frac{\lambda_1}{\sigma_2}$$

And, together with Welington de Melo and Sebastian van Strien [**9, 10**], he obtained a characterization of such systems with mild recurrence, in a wide variety of situations.

As a part of the development of moduli theory there was a description of typical holomorphic vectors fields, the topological types of linear holomorphic vector fields in CP^n, which was done by César Camacho, Nicolaas Kuiper, and Jacob in [**7**].

I should also mention his series of papers with Yoccoz, where they study rigidity of centralizers of diffeomorphisms, that are the sets of diffeomorphisms which commute with a given diffeomorphism. In a series of papers [**48, 49, 50**], they prove that, generically, the centralizer is trivial for a hyperbolic diffeomorphism, it just contains the iterates of the map.

Actually, even back in his thesis, Jacob had been interested in a related problem: how frequently diffeomorphisms embed in flows. He observed that there were open sets of diffeomorphisms where the natural topological conditions that you would need to embed in a flow were not sufficient: there were open sets of such diffeomorphisms that did not embed in flows. And, somewhat later, in [**39**], he was able to prove that, C^1 generically, diffeomorphisms do not embed in flows.

If you look at Jacob's list of scientific works attached to this paper, you'll see that I could still go on for a long time. So, let me just conclude with some personal remarks.

Conclusion

It's interesting to note that up to 1993 Jacob had 16 graduate students, whose theses appeared up to that year. He's been Director of IMPA since around 1993, and as of 2000 he has 35 graduate students. So one might conclude that administration is not so bad for someone with the talents of Jacob Palis...

In any event, he has exhibited leadership, as I indicated, direction and scope in formulating conjectures and stimulating many people throughout the world. The scope has increased dramatically as we get evidence of collaboration with Yoccoz, with Viana, with many other people, and of much activity, many interesting results, going deeply into the study of dynamical systems.

So, on the occasion of his 60th birthday, we all look forward to continued development for many, many years.

References

[1] J.F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140(2):351–398, 2000.

[2] A. Andronov and L. Pontryagin. Systèmes grossiers. *Dokl. Akad. Nauk. USSR*, 14:247–251, 1937.

[3] D.V. Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90:209, 1967.

[4] M. Benedicks and L. Carleson. The dynamics of the Hénon map. *Ann. of Math. (2)*, 133(1):73–169, 1991.

[5] C. Bonatti, L.J. Díaz, and E.R. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. Preprint, 1999.

[6] P. Brunovský. On one-parameter families of diffeomorphisms. II. Generic branching in higher dimensions. *Comment. Math. Univ. Carolinae*, 12:765–784, 1971.

[7] C. Camacho, N.H. Kuiper, and J. Palis. The topology of holomorphic flows with singularity. *Inst. Hautes Études Sci. Publ. Math.*, 48:5–38, 1978.

[8] M.J. Dias Carneiro and J. Palis. Bifurcations and global stability of families of gradients. *Inst. Hautes Études Sci. Publ. Math.*, 70:103–168, 1990.

[9] W. de Melo and J. Palis. Moduli of stability for diffeomorphisms. In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, pages 318–339. Springer, Berlin, 1980.

[10] W. de Melo, J. Palis, and S.J. van Strien. Characterising diffeomorphisms with modulus of stability one. In *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, pages 266–285. Springer, Berlin, 1981.

[11] L.J. Díaz. Robust nonhyperbolic dynamics and heterodimensional cycles. *Ergodic Theory Dynam. Systems*, 15(2):291–315, 1995.

[12] L.J. Díaz, E.R. Pujals, and R. Ures. Partial hyperbolicity and robust transitivity. *Acta Math.*, 183(1):1–43, 1999.

[13] L.J. Díaz and J. Rocha. Large measure of hyperbolic dynamics when unfolding heteroclinic cycles. *Nonlinearity*, 10(4):857–884, 1997.

[14] J.M. Franks. Time dependent stable diffeomorphisms. *Invent. Math.*, 24:163–172, 1974.

[15] N.K. Gavrilov and L.P. Šil'nikov. Three-dimensional dynamical systems that are close to systems with a structurally unstable homoclinic curve. I. *Mat. Sb. (N.S.)*, 88(130):475–492, 1972.

[16] J. Guckenheimer. Absolutely Ω-stable diffeomorphisms. *Topology*, 11:195–197, 1972.

[17] J. Guckenheimer and R.F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, 50:59–72, 1979.

[18] C. Gutiérrez. Structural stability for flows on the torus with a cross-cap. *Trans. Amer. Math. Soc.*, 241:311–320, 1978.

[19] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows. *Ann. of Math. (2)*, 145(1):81–137, 1997.

[20] M.V. Jakobson. Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.*, 81(1):39–88, 1981.

[21] S.-T. Liao. On the stability conjecture. *Chinese Ann. Math.*, 1(1):9–30, 1980.

[22] M. Lyubich. Almost every real quadratic map is either regular or stochastic. *Annals of Math.* To appear.

[23] R. Mañé. Contributions to the stability conjecture. *Topology*, 17(4):383–396, 1978.

[24] R. Mañé. An ergodic closing lemma. *Ann. of Math. (2)*, 116(3):503–540, 1982.

[25] R. Mañé. A proof of the C^1 stability conjecture. *Inst. Hautes Études Sci. Publ. Math.*, 66:161–210, 1988.

[26] J.M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954.

[27] L. Mora and M. Viana. Abundance of strange attractors. *Acta Math.*, 171(1):1–71, 1993.

[28] C. A. Morales, M.J. Pacifico, and E.R. Pujals. Robust transitive singular sets for 3-flows are partially hyperbolic attractors and repellers. Preprint, 1999.

[29] C.A. Morales, M.J. Pacífico, and E.R. Pujals. On C^1 robust singular transitive sets for three-dimensional flows. *C. R. Acad. Sci. Paris Sér. I Math.*, 326(1):81–86, 1998.

[30] C.G. Moreira and J.-C. Yoccoz. Stable intersections of regular Cantor sets with large Hausdorff dimension. Preprint 1998.

[31] C.G. Moreira and J.-C. Yoccoz. Tangences homocliniques stables pour les ensembles hyperboliques de grande dimension fractale. Preprint 2000.

[32] S.E. Newhouse. Nondensity of Axiom A(a) on S^2. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 191–202. Amer. Math. Soc., Providence, R.I., 1970.

[33] S.E. Newhouse. Diffeomorphisms with infinitely many sinks. *Topology*, 13:9–18, 1974.

[34] S.E. Newhouse. The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.*, 50:101–151, 1979.

[35] S.E. Newhouse and J. Palis. Bifurcations of Morse-Smale dynamical systems. In *Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, pages 303–366, New York, 1973. Academic Press.

[36] S.E. Newhouse and J. Palis. Cycles and bifurcation theory. *Astérisque*, 31:43–140, 1976.

[37] S.E. Newhouse, J. Palis, and F. Takens. Bifurcations and stability of families of diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.*, 57:5–71, 1983.

[38] J. Palis. On Morse-Smale dynamical systems. *Topology*, 8:385–404, 1968.

[39] J. Palis. Vector fields generate few diffeomorphisms. *Bull. Amer. Math. Soc.*, 80:503–505, 1974.

[40] J. Palis. A differentiable invariant of topological conjugacies and moduli of stability. *Astérisque*, 51:335–346, 1978.

[41] J. Palis. A global view of dynamics and a conjecture on the denseness of finitude of attractors. *Astérisque*, 261:xiii–xiv, 335–347, 2000. Géométrie complexe et systèmes dynamiques (Orsay, 1995).

[42] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 223–231. Amer. Math. Soc., Providence, R.I., 1970.

[43] J. Palis and F. Takens. Stability of parametrized families of gradient vector fields. *Ann. of Math. (2)*, 118(3):383–421, 1983.

[44] J. Palis and F. Takens. Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms. *Invent. Math.*, 82(3):397–422, 1985.

[45] J. Palis and F. Takens. Hyperbolicity and the creation of homoclinic orbits. *Ann. of Math. (2)*, 125(2):337–374, 1987.

[46] J. Palis and F. Takens. *Hyperbolicity and sensitive-chaotic dynamics at homoclinic bifurcations.* Cambridge University Press, 1993.

[47] J. Palis and M. Viana. High dimension diffeomorphisms displaying infinitely many periodic attractors. *Ann. of Math. (2)*, 140(1):207–250, 1994.

[48] J. Palis and J.-C. Yoccoz. Centralizers of Anosov diffeomorphisms on tori. *Ann. Sci. École Norm. Sup. (4)*, 22(1):99–108, 1989.

[49] J. Palis and J.-C. Yoccoz. Rigidity of centralizers of diffeomorphisms. *Ann. Sci. École Norm. Sup. (4)*, 22(1):81–98, 1989.

[50] J. Palis and J.-C. Yoccoz. Differentiable conjugacies of Morse-Smale diffeomorphisms. *Bol. Soc. Brasil. Mat. (N.S.)*, 20(2):25–48, 1990.

[51] J. Palis and J.-C. Yoccoz. Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension. *Acta Math.*, 172(1):91–136, 1994.

[52] J. Palis and J. C. Yoccoz. Nonuniformily hyperbolic horseshoes unleashed by homoclinic bifurcations and zero density of attractors. *C. R. Ac. Sc. Paris*, 2000. To appear.

[53] M.M. Peixoto. On structural stability. *Ann. of Math. (2)*, 69:199–222, 1959.

[54] M.M. Peixoto. Structural stability on two dimensional manifolds. *Topology*, 1:101-120, 1962.

[55] V.A. Pliss. Analysis of the necessity of the conditions of Smale and Robbin for structural stability for periodic systems of differential equations. *Differencial'nye Uravnenija*, 8:972–983, 1972.

[56] C. Pugh. The closing lemma. *Amer. J. Math.*, 89:956–1009, 1967.

[57] C. Pugh and M. Shub. The Ω-stability theorem for flows. *Invent. Math.*, 11:150–158, 1970.

[58] E.R. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms: a conjecture of Palis. *Annals of Math.*, 2000. To appear.

[59] E.R. Pujals and M. Sambarino. On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and varation of entropy. *Nonlinearity*, 13(3):921–926, 2000.

[60] J.W. Robbin. A structural stability theorem. *Ann. of Math. (2)*, 94:447–493, 1971.

[61] C. Robinson. Structural stability of vector fields. *Ann. of Math. (2)*, 99:154–175, 1974.

[62] C. Robinson. Structural stability of C^1 diffeomorphisms. *J. Differential Equations*, 22(1):28–73, 1976.

[63] C. Robinson. Bifurcation to infinitely many sinks. *Comm. Math. Phys.*, 90(3):433–459, 1983.

[64] A. Sannami. The stability theorems for discrete dynamical systems on two-dimensional manifolds. *Nagoya Math. J.*, 90:1–55, 1983.

[65] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[66] J. Sotomayor. Generic one-parameter families of vector fields on two-dimensional manifolds. *Inst. Hautes Études Sci. Publ. Math.*, 43:5–46, 1974.

[67] R. Thom. *Stabilité structurelle et morphogénèse.* W. A. Benjamin, Inc., Reading, Mass., 1972. Essai d'une théorie générale des modèles, Mathematical Physics Monograph Series.

[68] R.F. Williams. The "DA″ maps of Smale and structural stability. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 329–334. Amer. Math. Soc., Providence, R.I., 1970.

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