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MICROLOCAL ANALYSIS, BILINEAR ESTIMATES AND CUBIC QUASILINEAR WAVE EQUATION

by

Hajer Bahouri & Jean-Yves Chemin

Abstract. — In this paper, we study the local wellposedness of a cubic quasilinear wave equation. The Strichartz estimate used for the solutions of linear variable coefficients wave equations are not relevant here. We prove bilinear estimates for solutions of linear wave equations with variable coefficients. The main tools are Bony's paradifferential calculus and the microlocalization in the sense of Weyl-Hörmander calculus.

***Résumé* (Analyse microlocale et équation d'onde quasilinéaire cubique). —** Dans cet article, nous étudions l'existence et l'unicité locale de solutions pour une équation d'onde quasilinéaire cubique. Les classiques estimations de Strichartz ne sont pas adaptées dans ce cas. Nous démontrons des estimations bilinéaires pour des solutions d'équations d'ondes à coefficients variables. Les deux outils principaux sont le calcul paradifférentiel de Bony et la microlocalisation au sens du calcul pseudodifférentiel de Weyl-Hörmander.

Introduction

In this paper, our interest is to prove local solvability for equations of the type

$$(EC)\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u = 0 \\ \qquad\qquad\qquad\qquad \Delta g^{j,k} = Q_{j,k}(\partial u, \partial u) \\ \qquad\qquad\qquad (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

where $Q_{j,k}$ are quadratic forms on $\mathbf{R}^{d+1}$. In all this work, we shall state, for a real valued function u on $[0,T]\times\mathbf{R}^d$,

$$\nabla u \stackrel{\text{def}}{=} (\partial_1 u, \cdots, \partial_d u)\,,\ \partial u \stackrel{\text{def}}{=} (\partial_t u, \partial_1 u, \cdots, \partial_d u) \quad \text{and} \quad g\cdot\nabla^2 u \stackrel{\text{def}}{=} \sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u.$$

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When no confusion is possible, we shall also state

$$\gamma \stackrel{\text{def}}{=} (\nabla u_0, u_1).$$

This problem of course is a model one. The general problem consists in considering equations of the type

$$\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u = \sum_{1\leqslant j,k\leqslant d} \widetilde{Q}_{j,k}(\partial g^{j,k}, \partial u) \\ \Delta g^{j,k} = Q_{j,k}(\partial u, \partial u) \\ (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

where $\widetilde{Q}_{j,k}$ are quadratic form on $\mathbf{R}^{d+1}$ and where all the quadratic forms are supposed to be smooth functions of u. This simply complicates a little the estimates without any relevant new phenomenon. In the frame work of equation (EC), it makes sense to work with small data and this simplifies the proofs.

Energy methods allow to prove local wellposedness for initial data (u_0, u_1) in $H^{\frac{d}{2}+\frac{1}{2}} \times H^{\frac{d}{2}-\frac{1}{2}}$. More precisely, we have the following theorem.

Theorem 0.1. — *If $d \geqslant 3$, let (u_0, u_1) be in $H^{\frac{d}{2}+\frac{1}{2}} \times H^{\frac{d}{2}-\frac{1}{2}}$ such that $\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists in $C([0,T]; H^{\frac{d}{2}+\frac{1}{2}}) \cap C^1([0,T]; H^{\frac{d}{2}-\frac{1}{2}})$. Moreover, a constant C exists (which of course does not depend on the initial data) such that*

$$T \geqslant C\|\gamma\|^{-2}_{\dot{H}^{\frac{d}{2}-\frac{1}{2}}}.$$

Let us recall that H^s is the usual Sobolev space on $\mathbf{R}^d$ and that $\dot{H}^s$ is the homogeneous one and we shall state

$$\|f\|_s^2 \stackrel{\text{def}}{=} \int_{\mathbf{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi.$$

This is an Hilbert space when $s < d/2$.

The goal of this paper is to go below the regularity $H^{d/2+1/2}$ for the initial data. Let us have a look to the scaling properties of equation (EC). If u is a solution of (EC), then $u_\lambda(t,x) \stackrel{\text{def}}{=} u(\lambda t, \lambda x)$ is also a solution of (EC). The space which is invariant under this scaling is $\dot{H}^{d/2}$. So the above theorem appears to require $1/2$ derivative more than the scaling. The goal of this work is to try to go as closed as possible to the scaling invariant regularity.

Some results in that direction have been proved by the authors (see [**4**] and [**5**]) and also by D. Tataru (see [**27**] and [**28**]) for quasilinear wave equations of the following type

$$(E) \begin{cases} \partial_t^2 u - \Delta u - G(u)\cdot\nabla^2 u = F(u)Q(\partial u, \partial u) \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) \end{cases}$$

where G is a smooth function vanishing at 0 and with value in K such that $\mathrm{Id}+K$ is a convex compact subset of the set of positive symmetric matrices. Let us recall this results. Let us notice that the scaling of the two equations (E) and (EC) is the same.

Theorem 0.2. — *If $d \geqslant 3$, let (u_0, u_1) be in $H^s \times H^{s-1}$ for $s > s_d$ with $s_d = \frac{d}{2}+\frac{1}{2}+\frac{1}{6}\cdot$ Then, a positive time T exists such that a unique solution u exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2([0,T]; L^\infty).$$

Moreover, a constant C exists such that

$$T^{\frac{2}{3}+(s-s_d)} \geqslant C\|\gamma\|^{-1}_{\dot{H}^{s-1}}.$$

This theorem has been proved with $1/4$ instead than $1/6$ in [4] and then improved a little bit in [5] and proved with $1/6$ by D. Tataru in [28]. Strichartz estimates for quasilinear equations are the key point of the proofs. Recently, S. Klainerman and S. Rodnianski have announced a better index. Their proof is based on very different methods. In this case, the energy methods give the classical index $s > d/2+1$ and

$$T \geqslant C\|\gamma\|^{-1}_{H^{s-1}}.$$

The goal of this work is to do the analogous in the case of Equation (EC). The result will be the following.

Theorem 0.3. — *If $d \geqslant 5$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > \frac{d}{2}+\frac{1}{6}$ such that $\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$$

where $\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}$ denotes the Besov space defined in Definition 1.1. Moreover, for any positive α, a constant C_α exists such that

$$T^{\frac{1}{6}+\alpha} \geqslant C_\alpha\|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}.$$

The case of dimension 4 is a little bit different. The theorem is the following.

Theorem 0.4. — *If $d=4$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > 2+\frac{1}{6}$ such that $\|\gamma\|_{\dot{H}^1}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{1/6}_{6,2}) \quad \text{and} \quad \partial g \in L^1_T(L^\infty)$$

where $\dot{B}^{\frac{d}{6}-\frac{1}{2}}_{6,2}$ denotes the Besov space defined in Definition 1.1. Moreover, for any positive α, a constant C_α exists such that

$$T^{\frac{1}{6}+\alpha} \geqslant C_\alpha\|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}.$$

Remarks

– If we think in term of small data (i.e. of initial data of the type $\varepsilon(u_0, u_1)$), then energy methods give a life span in ε^{-2}. The above theorem gives a life span of order $\varepsilon^{-6+\alpha}$ for any positive α.

– As we shall see, the case when $d \geqslant 5$ can be treated only with Strichartz estimates simply because laws of product in Besov spaces imply that if ∂u belongs to $L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$ then ∂g is in $L^1_T(L^\infty)$.

– The case when $d = 4$ requires bilinear estimates. This fact appears in the statement of Theorem 0.4 through the following phenomenon: the fact that ∂u is in $L^2_T(\dot{B}^{1/6}_{6,2})$ does not imply that the time derivative of g belongs to $L^1_T(L^\infty)$. Of course this condition is crucial in particular to get the basic energy estimate. But we have been unable to exibit a Banach space $\mathcal{B}$ which contains the solution u and such that if a function a is contained in $\mathcal{B}$, then $\partial\Delta^{-1}(a^2)$ belongs to $L^1_T(L^\infty)$.

– In all that follows, the dimension d will supposed to be greater or equal than 4.

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1. Method of the proof and structure of the paper

As we shall use Littlewood-Paley theory all along this work, let us begin by recalling some basic facts and definitions related to it.

1.1. Some basic facts in Littlewood-Paley theory. — Let us denote by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and of big radius 8/3. Let us choose two non negative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0, 4/3))$ and $\mathcal{D}(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q\in\mathbf{N}} \varphi(2^{-q}\xi) = \sum_{q\in\mathbf{Z}} \varphi(2^{-q}\xi) = 1, \tag{1}$$

$$|p - q| \geqslant 2 \Rightarrow \operatorname{Supp}\varphi(2^{-q}\cdot) \cap \operatorname{Supp}\varphi(2^{-p}\cdot) = \varnothing, \tag{2}$$

$$q \geqslant 1 \Rightarrow \operatorname{Supp}\chi \cap \operatorname{Supp}\varphi(2^{-q}\cdot) = \varnothing, \tag{3}$$

and if $\widetilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|p - q| \geqslant 5 \Rightarrow 2^p\widetilde{\mathcal{C}} \cap 2^q\mathcal{C} = \varnothing. \tag{4}$$

Notations

$$h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q u = \varphi(2^{-q}D)u = 2^{qd}\int h(2^q y)u(x-y)dy,$$

$$S_q u = \sum_{p\leqslant q-1} \Delta_p u = \chi(2^{-q}D)u = 2^{qd}\int \widetilde{h}(2^q y)u(x-y)dy.$$

We shall often denote $\Delta_q u$ by u_q. Let us recall the definition of Besov spaces.

Definition 1.1. — Let s be a real number, and (p,r) in $[1,\infty]^2$. Let us state

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left\|\left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{q\in\mathbf{Z}}\right\|_{\ell^r(\mathbf{Z})}.$$

If $s < d/p$ then the closure of the compactly smooth functions with respect to this norm is a Banach space and we have that $\dot{H}^s = \dot{B}^s_{2,2}$ and the norm $\|\cdot\|_{\dot{B}^s_{2,2}}$ is equivalent to $\|\cdot\|_s$.

Notation. — We shall also state

$$\|a\|_s \stackrel{\text{def}}{=} \|a\|_{\dot{B}^s_{2,2}}, \quad \|b\|_{L^p_I(E)} \stackrel{\text{def}}{=} \|b\|_{L^p(I;E)}, \quad \|b\|_{L^p_T(E)} \stackrel{\text{def}}{=} \|b\|_{L^p([0,T];E)}$$
$$\text{and}\quad \|b\|_{T,s} \stackrel{\text{def}}{=} \|b\|_{L^\infty_T(\dot{B}^s_{2,2})}.$$

Here we want to explain the problems we have to solve in order to prove Theorem 0.4. As in the case of Equation (E), the basic fact is energy estimates. This implies the control of

$$\int_0^T \|\partial g(t,\cdot)\|_{L^\infty} dt.$$

In the case of Equation (E), it is obtained by Strichartz estimates. This will be the case here when $d \geqslant 5$ but this will not be the case when $d=4$. Let us have a look on a model problem to understand this difficulty. Here we essentially follow ideas of S. Klainerman and D. Tataru (see [**22**]).

Let us assume that u is the solution of the constant coefficient wave equation and let us estimate

$$\int_0^T \|\partial\Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big)\|_{L^\infty} dt.$$

As

$$\partial_t\Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big) = \Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big) + \Delta^{-1}\big(\partial_j u\partial_t\partial_k u(t,\cdot)\big),$$

we have to control expression of the type

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty} dt.$$

When $d \geqslant 3$, we have (see Lemma 2.1) that

$$\|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{\dot{B}^{d/2}_{2,1}} \leqslant C\|\partial u(t,\cdot)\|^2_{\frac{d}{2}-\frac{1}{2}}.$$

So we get that

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty} dt \leqslant T\|\partial u\|^2_{T,\frac{d}{2}-\frac{1}{2}}.$$

Then the proof of Theorem 0.1 is routine. If we want to go below this $H^{\frac{d}{2}+\frac{1}{2}}$ regularity of the initial data, we shall use Strichartz estimates. Let us introduce Bony's decomposition which consists in writing

$$ab = \sum_q S_{q-1}a\Delta_q b + \sum_q S_{q-1}b\Delta_q a + \sum_{\substack{-1\leqslant j\leqslant 1\\ q}} \Delta_q a\Delta_{q-j} b.$$

When $d \geqslant 4$, we have

$$\|\partial^k u_q\|_{L^2_T(L^\infty)} \leqslant C 2^{q\left(\frac{d}{2}-\frac{1}{2}+k-1\right)}\|\gamma_q\|_{L^2}.$$

Then it is not difficult to prove that

$$\Big\|\Delta^{-1}\Big(\sum_q S_{q-1}\partial^2 u\partial u_q\Big)\Big\|_{L^1_T(L^\infty)} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

The symmetric term can be treated exactly along the same lines. The so called remainder term

$$\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q}} \partial^2 u_q\partial u_{q-j}\Big)$$

is much more difficult to treat particulary in dimension 4. The reason why is the following. When d is greater or equal to 5, the Strichartz estimates tells us that

$$\|\partial^k u_q\|_{L^2_T(L^4)} \leqslant 2^{q\left(\frac{d}{4}-\frac{1}{2}+k-1\right)}\|\gamma_q\|_{L^2}.$$

So thanks to Bernstein inequality, we infer that

$$\begin{aligned}\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} \Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} &\leqslant C2^{p\left(\frac{d}{2}-2\right)}\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} 2^{qd/2}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}\\ &\leqslant C\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} 2^{-(q-p)\left(\frac{d}{2}-2\right)}2^{q(d-2)}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.\end{aligned}$$

Convolution and Cauchy-Schwarz inequalities implies that

$$\|\Delta^{-1}(\partial^2 u\partial u)\|_{L^1_T(L^\infty)} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

The case of dimension 4 is much more delicate. In dimension 4, the Strichartz estimate is

$$\|\partial^k u_q\|_{L^2_T(L^6)} \leqslant 2^{q\left(\frac{4}{3}-\frac{1}{2}+k-1\right)}\|\gamma_q\|_{L^2}.$$

So the series $\partial^2 u_q\partial u_{q-j}$ does not converge in $L^1_T(L^3)$ because the only estimate we have is

$$\begin{aligned}\|\partial^2 u_q\partial u_{q-j}\|_{L^1_T(L^3)} &\leqslant C2^{q8/3}\|\gamma_q\|^2_{L^2}\\ &\leqslant C2^{q2/3}d_q\|\gamma\|^2_1 \quad\text{with}\quad \sum_q d_q = 1.\end{aligned}$$

To overcome this difficulty, we follow an idea of S. Klainerman and D. Tataru: the precised Strichartz estimate which will allow to prove bilinear estimates.

1.2. Bilinear estimates and precised Strichartz estimates. — To explain the basic ideas of bilinear estimates, let us consider the case of constant coefficient case. In this paragraph, we essentially follow the ideas of [**22**]. What a bilinear estimates lookes like is described by the following proposition.

Proposition 1.1. — *Let u_1 and u_2 two solutions of*

$$\begin{cases} \partial_t^2 u_j - \Delta u_j = 0 \\ (\partial u_j)_{|t=0} = \gamma_j. \end{cases}$$

Then, if $d \geqslant 4$, we have

$$\|\partial\Delta^{-1}Q(\partial u_1 \partial u_2)\|_{L^1_T(L^\infty)} \leqslant C_{\varepsilon,T}\|\gamma_1\|_{\frac{d}{2}-1+\varepsilon}\|\gamma_2\|_{\frac{d}{2}-1+\varepsilon}.$$

Remark. — We find a gain of half a derivative about the regularity of the initial data compared with purely Strichartz methods.

The precised Strichartz estimates is described by the following proposition proved in [**22**].

Proposition 1.2. — *A constant C exists such that for any T and any $h \leqslant 1$, if* $\operatorname{Supp}\widehat{u}_j$ *and* $\operatorname{Supp}\mathcal{F}(\square\, u(t,\cdot))$ *are included in a ball of radius h and in the ring $\mathcal{C}$, we have*

$$\|u\|_{L^2_T(L^\infty)} \leqslant C\big(h^{d-2}\log(e+T)\big)^{1/2}\big(\|u(0)\|_{L^2} + \|\partial_t u(0)\|_{L^2} + \|\square\, u\|_{L^1_T(L^2)}\big).$$

To prove Proposition 1.1, let us recall that we want to estimate the

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} \Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)}.$$

With a rescaling of the equation, we can assume that $q=1$ and let us state $h = 2^{p-q}$. Let us define $(\phi_\nu)_{1\leqslant\nu\leqslant N_h}$ a partition of unity of the ring $\mathcal{C}$ such that

$$\operatorname{Supp}\phi_\nu \subset B(\xi_\nu, h).$$

Then, using the fact that the support of the Fourier transform of the product of two functions is included in the sum of the supports of their Fourier transform, a family of function $(\widetilde{\phi}_\nu)_{1\leqslant\nu\leqslant N_h}$ exists such that $\operatorname{Supp}\widetilde{\phi}_\nu \subset B(-\xi_\nu, 2h)$ and

$$\chi(h^{-1}D)(\partial^2 v\partial v) = \sum_{\nu=1}^{N_h} \chi(h^{-1}D)\big(\partial^2\widetilde{\phi}_\nu(D)v\partial\phi_\nu(D)v\big). \tag{5}$$

Applying Proposition 1.2 gives

$$\|\chi(h^{-1}D)(\partial^2 v\partial v)\|_{L^1_T(L^\infty)} \leqslant Ch^{d-2}\log(e+T)\sum_{\nu=1}^{N_h}\|\widetilde{\phi}_\nu(D)\gamma\|_{L^2}\|\phi_\nu(D)\gamma\|_{L^2}.$$

The Cauchy Schwarz inequality implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v)\|_{L^1_T(L^\infty)} \leqslant Ch^{d-2}\log(e+T)\Big(\sum_{\nu=1}^{N_h}\|\widetilde{\phi}_\nu(D)\gamma\|^2_{L^2}\Big)^{1/2}\Big(\sum_{\nu=1}^{N_h}\|\phi_\nu(D)\gamma\|^2_{L^2}\Big)^{1/2}.$$

The almost orthogonality of $(\widetilde{\phi}_\nu(D)\gamma_1)_{1\leqslant\nu\leqslant N_h}$ and $(\phi_\nu(D)\gamma_2)_{1\leqslant\nu\leqslant N_h}$ implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v)\|_{L^1_T(L^\infty)} \leqslant Ch^{d-2}\log(e+T)\|\gamma\|_{L^2}\|\gamma\|_{L^2}. \tag{6}$$

So after rescaling, we get that

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leqslant 2^{p(d-4)}\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\log(e+2^qT)2^{2q}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.$$

If $\gamma\in\dot{H}^{\frac{d}{2}-1+\varepsilon}$ then we have

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leqslant (2^pT)^{-\varepsilon}\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}2^{-(q-p)(d-4+\varepsilon)} \times 2^{q(\frac{d}{2}-1)}(2^qT)^\varepsilon\|\gamma_q\|_{L^2}2^{(q-j)(\frac{d}{2}-1)}(2^qT)^\varepsilon\|\gamma_{q-j}\|_{L^2}.$$

So the series convergences in $L^1_T(L^\infty)$ for large p. The case when p is small (low frequencies) is nothing but Sobolev embeddings.

The real problem we have to solve in this work is to prove this bilinear estimate in the context of quasilinear wave equation. To do this, we follow the lines of [**4**] and [**5**].

As we shall use geometrical optics technics, we need to deal with smooth functions in time also. This leads to the following iterative scheme introduced in [**5**]. Let us define the sequence $(u^{(n)})_{n\in\mathbf{N}}$ by the first term $u^{(0)}$ satisfying

$$\begin{cases}\partial_t^2 u^{(0)}-\Delta u^{(0)}=0\\ (u^{(0)},\partial_t u^{(0)})_{|t=0}=(S_0u_0,S_0u_1),\end{cases}$$

and by the following induction

$$(\mathcal{R}_n)\begin{cases}\partial_t^2 u^{(n+1)}-\Delta u^{(n+1)}-G_{n,T}\cdot\nabla^2 u^{(n+1)}=0\\ \qquad (u^{(n+1)},\partial_t u^{(n+1)})_{|t=0}=(S_{n+1}u_0,S_{n+1}u_1)\end{cases}$$

with

$$G_{n,T}\stackrel{\text{def}}{=}\theta(T^{-1})G_n\quad\text{with}\quad G_n^{j,k}\stackrel{\text{def}}{=}\Delta^{-1}Q_{j,k}(\partial u^{(n)},\partial u^{(n)}).$$

where θ is a function of $\mathcal{D}(]-1,1[)$ whose value is 1 near 0. Let us point out that the sequence $(u^{(n)})_{n\in\mathbf{N}}$ does depend on T. We introduce some notations which will be used all along this work. If α is a (small) positive number, let us define

$$s_\alpha \stackrel{\text{def}}{=} \frac{d}{2} + \frac{1}{6} + \alpha \quad \text{and} \quad N_T^\alpha(\gamma) \stackrel{\text{def}}{=} T^{\frac{1}{6}+\alpha}\|\gamma\|_{s_\alpha-1}.$$

Let us introduce the assertions we are going to prove by induction.

– If $d \geqslant 5$,

$$(\mathcal{P}_n)\begin{cases} \|\partial u^{(n)}\|_{L^2([0,T];\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})} \leqslant C_0 N_T^\alpha(\gamma) \\ \|\partial u^{(n)}\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s \in \Big[s_\alpha - 1, \frac{d}{2} + \frac{1}{2}\Big]; \end{cases}$$

– if $d = 4$,

$$(\mathcal{P}_n)\begin{cases} \|\partial u^{(n)}\|_{L^2([0,T];\dot{B}^{\frac{d}{6}-\frac{1}{2}}_{6,2})} \leqslant C_0 N_T^\alpha(\gamma) \\ \|\partial G_{n,T}\|_{L^1([0,T];L^\infty)} \leqslant 2 \\ \|\partial u^{(n)}\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s \in \Big[\frac{3}{2} + \alpha, \frac{d}{2} + \frac{1}{2}\Big]. \end{cases}$$

All what follows in this paper consists in proving that if

$$\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}} + N_T^\alpha(\gamma)$$

is small enough, $(\mathcal{P}_0)$ is true and $(\mathcal{P}_n)$ implies $(\mathcal{P}_{n+1})$. Then the proof of Theorems 0.3 and 0.4 is pure routine of non linear partial differential equations.

To do this, we shall localize in frequency and transform equation $\mathcal{R}_n$ into an equation where the space-time frequencies of the metric which defines the d'Alembertian are very small with respect to the level frequencies we work with. This is the purpose of the second section.

In the third section, we show how the proof can be reduced to "microlocal" Strichartz and bilinear estimates. By microlocal estimates, we mean estimates that are valid only a time interval whose length depends on the size of the frequencies we work with. To prove the complete estimates (with a loose of course), we use D. Tataru's version of the method we introduced in [**4**] which consists in a decomposition of the interval $[0, T]$ on intervals where microlocal estimates are true.

In the forth section, we recall the method of approximation of solutions of (variable coefficients) wave equation by the method of geometrical optics. This is the opportunity to study precisely the link between the solutions of the Hamilton-Jacobi equation

$$\begin{cases} \partial_\tau \Phi(\tau, y, \eta) = F(\tau, y, \partial_y \Phi(\tau, y, \eta)) \\ \Phi(0, y, \eta) = (y|\eta) \end{cases}$$

and the flow of H_F and also properties of this flow which will be useful in the seventh section.

The fifth section is devoted to the following problem: in the proof of the equivalent of Inequality (6), we use the fact that the support of the Fourier transform is preserved by the flow of the constant coefficient wave equation; this is no longer true in the variable coefficient case. So this information is not relevant because it is not preserved by the flow of the equation. The purpose of this fifth section is to define the concept of microlocalized function near a point $X = (x, \xi)$ of the cotangent space $T^\star \mathbf{R}^d$ (the cotangent space of $\mathbf{R}^d$). This notion is due to J.-M. Bony ([**7**]) and means that the function is concentrated in space near the point x and in frequency near the point ξ with of course the limit on the uncertainty principle. The good framework of this is a simplified version of Weyl-Hörmander calculus which is also presented in this section. Properties of the product of microlocalized functions is also studied.

In the sixth section, we prove that for solutions of a variable coefficients wave equation, microlocalization properties propagates nicely along the Hamiltonian flows related to the wave operator.

In the seventh section, we apply the three previous sections to prove the microlocal bilinear estimates. This proof consists in a second microlocalization, which means that we have to decompose again the interval on which we work. The reason why is that interaction in the product and propagation of microlocalization are badly related.

2. Littlewood-Paley theory and Paralinearization of the equation

All along this work, we shall need to study the quadratic operator $\Delta^{-1}((Du)^2)$. Let us summarize now some basic properties of this operator in the following lemma.

Lemma 2.1. — *A constant C exists such that*

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{B}^{d/2}_{2,1}} \leqslant C\|\partial a\|_{\dot{H}^{\frac{d}{2}-1}}\|\partial b\|_{\dot{H}^{\frac{d}{2}-1}} \quad \textit{and}$$
$$\|\nabla\Delta^{-1}(\partial a \partial b)\|_{\dot{B}^{d/4}_{4,1}} \leqslant C\|\partial a\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\|\partial b\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}.$$

Moreover, for any σ greater than $3/2$, a constant C exists such that

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leqslant C\big(\|\partial a\|_{\dot{C}^{-1/2}}\|\partial b\|_{\dot{H}^{\sigma-1}} + \|\partial a\|_{\dot{H}^{\sigma-1}}\|\partial b\|_{\dot{C}^{-1/2}}\big).$$

And, for any σ greater than $\dfrac{3}{2} - \dfrac{d}{4}$, a constant C exists such that

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leqslant C\big(\|\partial a\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\|\partial b\|_{\dot{H}^{\sigma-1}} + \|\partial a\|_{\dot{H}^{\sigma-1}}\|\partial b\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\big).$$

From this lemma, we give the following corollary.

Corollary 2.1. — *A constant C exists such that, if $(\mathcal{P}_n)$ holds, then*

$$\|G_{n,T}\|_{L^\infty} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

Moreover, if $d \geqslant 5$, then

$$\|\partial G_{n,T}\|_{L^1_T(L^\infty)} \leqslant C N^\alpha_T(\gamma)^2.$$

The proof of this lemma and its corollary is an exercice on Littlewood-Paley theory and we omit it.

Theorem 2.1. — *For any $s > 3/2$, a constant C exists which satisfies the following properties. Let us consider two functions u and v whose partial derivatives belong to the space $L^\infty_T(\dot H^{s-1}) \cap L^2_T(\dot C^{-1/2})$ and a function F in $L^1_T(\dot H^{s-1})$. Let us assume that*

$$G^{j,k}_{v,T} \stackrel{def}{=} \theta(T^{-1}\cdot)\Delta^{-1}Q_{j,k}(\partial v, \partial v) \in L^1_T(L^\infty)$$

and that

$$\partial_t^2 u - \Delta u - G_{v,T}\cdot\nabla^2 u = F.$$

Then we have

$$\partial_t^2 u_q - \Delta u_q - S_{q-1}G_{v,T}\nabla^2 u_q = R_q(\nabla u, \partial v) + F_q$$

with

$$\begin{aligned}\|R_q(\nabla u(t), \partial v(t))\|_{L^2} \leqslant C c_q(t) 2^{-q(s-1)}\big(&\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1}\\ &+ \|\partial v(t)\|_{s-1}\|\partial v(t)\|_{\dot C^{-1/2}}\|\nabla u(t)\|_{\dot C^{-1/2}}\big).\end{aligned}$$

with as in all that follows $\sum_q c_q^2(t) = 1$.

To prove this theorem, we use paradifferential calculus. More precisely, we apply Bony's decomposition which consists in writing

$$\begin{aligned} G_{v,T}(t)\nabla^2 u(t) &= \mathcal{R}_1(t) + \mathcal{R}_2(t) \quad \text{with}\\ \mathcal{R}_1(t) &\stackrel{\text{def}}{=} \sum_{q'} S_{q'-1}G_{v,T}\nabla^2 u_{q'} \quad \text{and}\\ \mathcal{R}_2(t) &\stackrel{\text{def}}{=} \sum_{q'} S_{q'+2}\nabla^2 u \Delta_{q'} G_{v,T}.\end{aligned} \tag{7}$$

The first term $\mathcal{R}_1(t)$ is easy to estimate. As the support of the Fourier transform of the function $S_{q'-1}G_{v,T}\nabla^2 u_{q'}$ is included in a ring of type $2^{q'}\widetilde{\mathcal{C}}$, we have

$$\begin{aligned}\Delta_q \mathcal{R}_1(t) &= \sum_{|q-q'|\leqslant N_1} \Delta_q\big(S_{q'-1}G_{v,T}\nabla^2 u_{q'}\big)\\ &= S_{q-1}G_{v,T}\nabla^2 u_q + \sum_{|q-q'|\leqslant N_1}\big[\Delta_q, S_{q'-1}G_{v,T}\big]\nabla^2 u_{q'}\\ &\qquad + \sum_{|q-q'|\leqslant N_1}(S_{q'-1}G_{v,T} - S_{q-1}G_{v,T})\nabla^2\Delta_q u_{q'}.\end{aligned} \tag{8}$$

As for instance in [4], we have

$$\big\|\big[\Delta_q, S_{q'-1}G_{v,T}\big]\nabla^2 u_{q'}\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1} \quad \text{and}$$
$$\big\|(S_{q'-1}G_{v,T} - S_{q-1}G_{v,T})\nabla^2 u_{q'}\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1}.$$

So it turns out that

$$(9) \qquad \big\|\Delta_q \mathcal{R}_1(t) - S_{q-1}G_{v,T}\nabla^2 u_q\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1}.$$

The second term is a little bit more delicate to estimate. Because the support of the Fourier transform of $S_{q'+1}\nabla^2 u \Delta_{q'} G_{v,T}$ is included in a ball of center 0 and radius $C2^{q'}$, we have that

$$\Delta_q \mathcal{R}_2(t) = \sum_{q' \geqslant q-N_1} \Delta_q\big(S_{q'+2}\nabla^2 u \Delta_{q'} G_{v,T}\big)$$

By definition of $\dot{C}^{1/2}$ and using Bernstein inequalities, it is obvious that

$$\|S_{q'+1}\nabla^2 u\|_{L^\infty} \leqslant 2^{3q'/2}\|\nabla u(t)\|_{\dot{C}^{-1/2}}.$$

Using Lemma 2.1, we get that

$$\|\Delta_{q'} G_{v,T}\|_{L^2} \leqslant Cc_{q'}(t)2^{-q'(s+\frac12)}\|\partial v(t)\|_{\dot{C}^{-1/2}}\|\partial v(t)\|_{\dot{H}^{s-1}}$$

when s is greater than $3/2$. So the theorem is proved.

Now we are going to state two corollaries of this theorem.

Corollary 2.2. — *If $(\mathcal{P}_n)$ is satisfied, then for any $s \in]3/2, s_\alpha]$, a constant C exists such that*

$$\|\partial u^{(n+1)}\|_{T,s-1} \leqslant e^2\|\gamma\|_{s-1}\Big(1 + CC_0 N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot{C}^{-1/2})}\Big).$$

To prove it, let us first deduce by standard energy estimates from Theorem 2.1 above applied with $u = u^{(n+1)}$ and $v = u^{(n)}$ that

$$\frac{d}{dt}\|\partial u_q^{(n+1)}(t)\|_{L^2}^2 \leqslant Cc_q^2(t)2^{-2q(s-1)}\Big(\|\partial G_{n,T}(t)\|_{L^\infty}\|\partial u^{(n+1)}(t)\|_{s-1}^2$$
$$+ C\|\gamma\|_{s-1}\|\partial u^{(n)}(t)\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t)\|_{\dot{C}^{-1/2}}\|\partial u^{(n+1)}(t)\|_{s-1}\Big).$$

By multiplication by $2^{2q(s-1)}$ and summation we have that

$$\frac{d}{dt}\|\partial u^{(n+1)}(t)\|_{s-1}^2 \leqslant C\Big(\|\partial G_{n,T}(t)\|_{L^\infty}\|\partial u^{(n+1)}(t)\|_{s-1}^2$$
$$+ C\|\gamma\|_{s-1}\|\partial u^{(n)}(t)\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t)\|_{\dot{C}^{-1/2}}\|\partial u^{(n+1)}(t)\|_{s-1}\Big).$$

Using Gronwall lemma, it turns out that

$$\|\partial u^{(n+1)}(t)\|_{s-1}\exp\Big(-C\int_0^t \|\partial G_{n,T}(t')\|_{L^\infty}dt'\Big)$$
$$\leqslant \|\gamma\|_{s-1} + C\|\gamma\|_{s-1}\int_0^t \|\partial u^{(n)}(t')\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t')\|_{\dot{C}^{-1/2}}dt'.$$

Using Cauchy-Schwarz inequality, we get

$$\|\partial u^{(n+1)}(t)\|_{s-1}\exp\Big(-C\int_0^t\|\partial G_{n,T}(t')\|_{L^\infty}dt'\Big)$$
$$\leqslant \|\gamma\|_{s-1}+C\|\gamma\|_{s-1}\|\partial u^{(n)}\|_{L^2_T(\dot C^{-1/2})}\|\nabla u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}.$$

Using $(\mathcal{P}_n)$, we get that

$$\|\partial u^{(n+1)}(t)\|_{s-1}\exp\Big(-C\int_0^t\|\partial G_{n,T}(t')\|_{L^\infty}dt'\Big)$$
$$\leqslant \|\gamma\|_{s-1}+CC_0N_T^\alpha(\gamma)\|\gamma\|_{s-1}\|\nabla u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}.$$

The fact that

$$\partial_{t'}G_{n,T}(t')=\frac{1}{T}\theta'\left(\frac{t'}{T}\right)G_n+\theta\partial_{t'}G_n$$

together with induction hypothesis and corollary 2.1 implies the result.

The second corollary treats the case of low frequencies.

Corollary 2.3. — *A constant C exists such that under the hypothesis $(\mathcal{P}_n)$, we have, for any $r\geqslant 2$,*

$$\|\partial S_q u^{(n+1)}\|_{L^2_T(\dot B^{\frac{d}{r}-\frac12}_{r,2})}\leqslant C(2^qT)^{\frac13-\alpha}N_T^\alpha(\gamma)\big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

Using Bernstein inequalities and Corollary 2.2, we get that

$$\begin{aligned}2^{2p(\frac{d}{r}-\frac12)}\|\partial u_p^{(n+1)}\|^2_{L^2_T(L^r)}&\leqslant CT2^{p(d-1)}\|\partial u_p^{(n+1)}\|^2_{L^\infty_T(L^2)}\\&\leqslant C(2^pT)^{\frac23-2\alpha}T^{\frac13+2\alpha}\|\partial u^{(n+1)}\|^2_{T,s_\alpha-1}\\&\leqslant C(2^pT)^{\frac23-2\alpha}N_T^\alpha(\gamma)^2\big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big)^2.\end{aligned}$$

Thus as

$$\|\partial S_q u^{(n+1)}\|^2_{L^2_T(\dot B^{\frac{d}{r}-\frac12}_{r,2})}\leqslant C\sum_{p\leqslant q-1}2^{2p(\frac{d}{r}-\frac12)}\|\partial u_p^{(n+1)}\|^2_{L^2_T(L^r)},$$

we have proved the corollary.

Let us now do a precised paralinearization in the spirit of [**4**].

Theorem 2.2. — *A constant C exists which satisfies the following properties. Let us consider two functions u and v whose partial derivatives belong to $L^\infty_T(\dot H^{s_\alpha-1})\cap L^2_T(\dot C^{-1/2})$ and a function F in $L^1_T(\dot H^{s-1})$. Let us assume that $\partial G_{v,T}$ belongs to $L^1_T(L^\infty)$ and that*

$$\partial_t^2u-\Delta u-G_{v,T}\cdot\nabla^2u=F.$$

Then for any $\delta\in[0,1]$, we have

$$\partial_t^2u_q-\Delta u_q-S_q^\delta(G_{v,T})\cdot\nabla^2u_q=R_q^\delta(\nabla u,\partial v)+F_q$$

with

$$S_q^\delta b\overset{def}{=}S^{(1+d)}_{q\delta-(1-\delta)\log_2T-N_0}b\quad\textit{and}$$

$$\|R_q^\delta(\nabla u,\partial v)\|_{L^1_T(L^2)} \leqslant C2^{-q(s-1)}(1+(2^qT)^{1-\delta})\big(\|\partial G_{v,T}\|_{L^1_T(L^\infty)}\|\nabla u\|_{L^\infty_T(\dot H^{s-1})} + \|\partial v\|_{L^\infty_T(\dot H^{s-1})}\|\partial v\|_{L^2_T(\dot C^{-1/2})}\|\nabla u\|_{L^2_T(\dot C^{-1/2})}\big).$$

The proof of this theorem is based essentially on Theorem 2.1 and Corollary 2.2. Using Theorem 2.1, it is obvious that

$$R_q^\delta(\nabla u,\partial v) = R_q(\nabla u,\partial v) + (S_q^\delta - S_{q-1})(G_{v,T})\cdot\nabla^2 u_q.$$

As we have

$$\|(S_q^\delta - S_{q-1})G_{v,T}\|_{L^1_T(L^\infty)} \leqslant C2^{-q}(2^qT)^{1-\delta}\|\partial G_{v,T}\|_{L^1_T(L^\infty)},$$

we get the theorem applying Theorem 2.1.

As a corollary, we have

Corollary 2.4. — *A constant C exists such that under the hypothesis $(\mathcal{P}_n)$ we have for any δ in the interval $[0,1]$,*

$$(ECP_{T,q})\begin{cases} \partial_t^2 u_q^{(n+1)} - \Delta u_q^{(n+1)} - S_q^\delta(G_{n,T})\cdot\nabla^2 u_q^{(n+1)} = R_q^\delta(n) \\ \partial u_q^{(n+1)}{}_{|t=0} = \gamma_q^{(n+1)} \end{cases}$$

with

$$S_q^\delta b \stackrel{def}{=} S^{(1+d)}_{q\delta-(1-\delta)\log_2 T - N_0} b \quad \textit{and}$$

$$\|R_q^\delta(n)\|_{L^1_T(L^2)} \leqslant C2^{-q(\frac{d}{2}-1)}(2^qT)^{-\frac16-\alpha}(1+(2^qT)^{1-\delta})N_T^\alpha(\gamma) \times\Big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\Big).$$

3. Reduction to microlocalized estimates

By microlocalization of the estimates, we mean that we shall prove estimates that are valid on time intervals whose length depend on the frequency parameter. These techniques have been introduced in [**4**] and used in [**5**] and improved by D. Tataru in [**28**]. For technical reasons, we prefer to work at frequencies of size 1.

3.1. The statement of the microlocal estimates. — In all that follows, we shall consider a family of smooth functions $\mathcal{G} = (G_\Lambda)_{\Lambda\geqslant\Lambda_0}$ defined on $I_\Lambda\times\mathbf{R}^d$ such that G_Λ is small enough and such that, for any $k\geqslant 0$, the following quantities

$$(10)\qquad \|\mathcal{G}\|_0 \stackrel{\text{def}}{=} \sup_{\Lambda\geqslant\Lambda_0}\|\nabla G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} + |I_\Lambda|\,\|\nabla^2 G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad\text{and}$$

$$(11)\qquad \|\mathcal{G}\|_k \stackrel{\text{def}}{=} \sup_{\Lambda\geqslant\Lambda_0}|I_\Lambda|\Lambda^k\|\nabla^{k+2}G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad\text{for}\quad k\geqslant 1.$$

are finite. Let us denote by P_Λ the operator

$$P_\Lambda v \stackrel{\text{def}}{=} \partial_\tau^2 v - \Delta v - \sum_{k,\ell} G_\Lambda^{k,\ell}\partial_k\partial_\ell v.$$

Theorem 3.1. — *Let $\mathcal{C}$ be a ring of $\mathbf{R}^d$ and ε_0 a positive real number. Let us consider two families of smooth metrics $\mathcal{G}^{(j)} \stackrel{def}{=} (G_\Lambda^{(j)})_{\Lambda\geqslant\Lambda_0}$ such that for any k, $\|\mathcal{G}^{(j)}\|_k$ is finite and such that $\|\mathcal{G}^{(j)}\|_0$ is small enough. For any positive real number $\varepsilon \leqslant \varepsilon_0$, a constant C_ε exists which satisfies the following properties. Let f_1 and f_2 be two functions in $L^1_{I_\Lambda}(L^2)$ and γ_1 and γ_2 two functions of L^2; let us assume that the Fourier transform of those functions have their support included in $\mathcal{C}$. Let us assume that*

$$|I_\Lambda| \leqslant \Lambda^{2-\varepsilon}.$$

Then if $v_{1,\Lambda}$ and $v_{2,\Lambda}$ are solutions of

$$(E_\Lambda)\begin{cases} P_\Lambda^{(j)} v_{j,\Lambda} & = f_j \\ \nabla v_{j,\Lambda|T=0} & = \gamma_j \end{cases}$$

we shall have the following properties:

– *if $d \geqslant 5$, we have*

$$\|v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^4)} \leqslant C(\|\gamma_j\|_{L^2} + \|f_j\|_{L^1_{I_\Lambda}(L^2)}).$$

– *if $d = 4$,*

$$\|v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^6)} \leqslant C(\|\gamma_j\|_{L^2} + \|f_j\|_{L^1_{I_\Lambda}(L^2)}).$$

– *if $d \geqslant 4$, then we have, for any $h \leqslant 1$ and any $\varepsilon > 0$,*

$$\begin{aligned}\|\chi(h^{-1}D)(v_{1,\Lambda}v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} \leqslant C_\varepsilon h^{d-2-\varepsilon}\log(e+|I_\Lambda|)& \\ \times \big(\|\gamma_1\|_{L^2} + \|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2} + \|f_2\|_{L^1_{I_\Lambda}(L^2)}\big)&.\end{aligned}$$

Let us point out at this step that when h is small enough, this estimate is nothing but the Sobolev embedding. Using Bernstein inequality, we can write

$$\begin{aligned}\|\chi(h^{-1}D)(v_{1,\Lambda}v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} &\leqslant h^d\|v_{1,\Lambda}v_{2,\Lambda}\|_{L^1_{I_\Lambda}(L^1)} \\ &\leqslant h^d|I_\Lambda|\,\|v_{1,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)}\|v_{2,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)} \\ &\leqslant h^d|I_\Lambda|\,\big(\|\gamma_1\|_{L^2} + \|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2} + \|f_2\|_{L^1_{I_\Lambda}(L^2)}\big).\end{aligned}$$

So when $h^d|I_\Lambda| \leqslant h^{d-2-\varepsilon}$, the inequality of above Theorem 3.1 is proved. In all that follows, we shall assume that

$$|I_\Lambda| \geqslant h^{-2-\varepsilon}. \tag{12}$$

The proof of this theorem will be the purpose of sections 4 to 7 and this is in fact the core of this work.

3.2. The local estimates. — From this microlocal statement, let us deduce now the following local result.

Theorem 3.2. — *Let $(G^{(j)})_{1\leqslant j\leqslant 2}$ be two metrics such that $\|\partial G^{(j)}\|_{L^1_T(L^\infty)} \leqslant C_0$. For any ε, a constant C_ε exists (which of course depends on d) such that if $u_{j,q}$ are functions whose Fourier transform is supported in a ring $2^q\mathcal{C}$ and are solutions of*

$$(E_\Lambda)\begin{cases} \partial_t^2 u_q^{(j)} - \Delta u_q^{(j)} - \widetilde{G}^{(j)}\cdot\nabla^2 u_q^{(j)} = f_{j,q} \\ \nabla u_q^{(j)}{}_{|t=0} = \gamma_{j,q}\end{cases}$$

where $\widetilde{G}^{(j)} \stackrel{def}{=} S_q^{2/3}G^{(j)}$ and where $\gamma_{j,q}$ and $f_{j,q}$ have Fourier transform supported in a ring $2^q\mathcal{C}$, then we have

– *if $d\geqslant 5$,*

$$2^{q\left(\frac{d}{4}-\frac{1}{2}-k\right)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^4)} \leqslant C_\varepsilon 2^{q\left(\frac{d}{2}-1\right)}(2^qT)^{\frac{1}{6}+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big);$$

– *if $d=4$,*

$$2^{q\left(\frac{1}{6}-k\right)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^6)} \leqslant C_\varepsilon 2^q(2^qT)^{\frac{1}{6}+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big);$$

– *if $d\geqslant 4$, for any $p\leqslant q$,*

$$\begin{aligned}\big\|\chi(2^{-p}D)\big(\partial^{1+k}u_q^{(1)}\partial u_q^{(2)}\big)\big\|_{L^1_T(L^\infty)} &\leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^{\frac{1}{3}+\varepsilon}\big(\|\partial u_q^{(1)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{1,q}\|_{L^1_T(L^2)}\big)\\ &\quad\times\big(\|\partial u_q^{(2)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{2,q}\|_{L^1_T(L^2)}\big).\end{aligned}$$

To start with, let us observe that after a rescaling of the above Theorem 3.1, we get that, for any subinterval $I=(t^-,t^+)$ of $[0,T]$ such that

$$|I|\leqslant T(2^qT)^{1-2\delta-\varepsilon} \quad\text{and}\quad \|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)} + |I|\,\|\nabla^2 G_\delta^{(j)}\|_{L^1_I(L^\infty)} \leqslant \varepsilon_0\,, \tag{13}$$

we have

– if $d\geqslant 5$,

$$\|\partial^{1+k}u_q^{(j)}\|_{L^2_I(L^4)} \leqslant C2^{q\left(\frac{d}{4}-\frac{1}{2}+k\right)}\big(\|\partial u_q^{(j)}(t^-)\|_{L^2} + \|f_{j,q}\|_{L^1_I(L^2)}\big). \tag{14}$$

– if $d=4$,

$$\|\partial^{1+k}u_q^{(j)}\|_{L^2_I(L^6)} \leqslant C2^{q\left(\frac{5}{6}+k\right)}\big(\|\partial u_q^{(j)}(t^-)\|_{L^2} + \|f_{j,q}\|_{L^1_I(L^2)}\big). \tag{15}$$

– if $d\geqslant 3$, for any $p\leqslant q$ and any $\varepsilon>0$,

$$\begin{aligned}\big\|\chi(2^{-p}D)\big(\partial^{1+k}u_q^{(1)}\cdot\partial u_q^{(2)}\big)\big\|_{L^1_I(L^\infty)} &\leqslant C_\varepsilon 2^{p(d-2)}(2^qT)^{\varepsilon}2^{q(1+k)}\\ &\times\big(\|\partial u_q^{(1)}(t^-)\|_{L^2} + \|f_{1,q}\|_{L^1_I(L^2)}\big)\big(\|\partial u_q^{(2)}(t^-)\|_{L^2} + \|f_{2,q}\|_{L^1_I(L^2)}\big).\end{aligned} \tag{16}$$

Let us observe that in the case when $2^pT \leqslant 1$, the above inequality (16) is obtained by Bernstein inequality.

Then the method consists in a decomposition of the interval $[0,T]$ in subintervals I on which the above microlocalized estimates are true. The key point is a careful counting of the number of such intervals. This method has been introduced by the authors in [**4**] and improved by D. Tataru in [**28**].

Let us state $G_\delta^{(j)} \stackrel{\text{def}}{=} S_q^\delta G^{(j)}$. Using the fact that

$$\|\nabla^2 G_\delta^{(j)}\|_{L^1_I(L^\infty)} \leqslant \frac{1}{T}(2^qT)^\delta \|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)},$$

Condition (13) becomes

$$|I| \leqslant T(2^qT)^{1-2\delta-\varepsilon} \quad \text{and} \quad \frac{|I|}{T}(2^qT)^\delta \, \|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)} \leqslant \varepsilon_0. \tag{17}$$

But as seen in Corollary 2.4, there is a loose on the remainder. The decomposition is the opportunity to compensate this loose. To do so, let us consider a parameter λ in the interval $[0,1]$ which will be determined later on. We impose on the interval I that

$$\|f_{j,q}\|_{L^1_I(L^2)} \leqslant \lambda \|f_{j,q}\|_{L^1_T(L^2)}.$$

This constraint joint to the condition (17) can be sum up by

$$\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}} \int_I dt + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_I \|f_{j,q}(t)\|_{L^2} dt + \frac{|I|}{T}(2^qT)^\delta \int_I \|\nabla G_\delta^{(j)}(t)\|_{L^\infty} dt \leqslant \varepsilon_0. \tag{18}$$

We shall prove that such a finite decomposition exists (and also control the number of intervals) by induction. Let us assume that an increasing sequence $(t_j)_{0\leqslant j\leqslant k}$ of points of $[0,T]$ such that $t_n < T$ and, for any $j \leqslant n-1$,

$$\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j) + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_{t_j}^{t_{j+1}} \|f_{j,q}(t)\|_{L^2} dt + \frac{t_{j+1}-t_j}{T}(2^qT)^\delta \int_{t_j}^{t_{j+1}} \|\nabla G_\delta^{(j)}(t)\|_{L^\infty} dt = \varepsilon_0$$

As the function

$$F_k(t) \stackrel{\text{def}}{=} \frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t-t_k) + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_{t_k}^t \|f_{j,q}(t')\|_{L^2} dt' + \frac{t-t_k}{T}(2^qT)^\delta \int_{t_k}^t \|\nabla G_\delta^{(j)}(t')\|_{L^\infty} dt'$$

is a increasing function on the interval $[t_k,T]$, either the interval $[t_k,T]$ satisfies Condition (18), or a unique t_{k+1} exists in the interval $]t_k,T[$ sucht that $F_k(t_{k+1}) = \varepsilon_0$.

Now let us estimate the number of intervals. At least one of the three terms of the left inside of the above inequality is greater or equal to $\varepsilon_0/3$. So either

$$\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j) \geqslant \frac{\varepsilon_0}{3},$$

or

$$\frac{1}{\lambda\|f_{j,q}\|_{L^1_T(L^2)}}\int_{t_j}^{t_{j+1}}\|f_{j,q}(t)\|_{L^2}dt \geqslant \frac{\varepsilon_0}{3},$$

or

$$\frac{t_{j+1}-t_j}{T}(2^qT)^{\delta}\int_{t_j}^{t_{j+1}}\|\nabla G_\delta^{(j)}(t)\|_{L^\infty}dt \geqslant \frac{\varepsilon_0}{3}\cdot$$

In the third case, we get that for any positive real number A,

$$\frac{3}{\varepsilon_0}(t_{j+1}-t_j)A + (2^qT)^{\delta}\frac{1}{AT}\int_{t_j}^{t_{j+1}}\|\nabla G_\delta^{(j)}(t')\|_{L^\infty}dt' \geqslant 1.$$

It turns out that in any case,

$$\begin{aligned}\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j) &+ \frac{1}{\lambda\|f_{j,q}\|_{L^1_T(L^2)}}\int_{t_j}^{t_{j+1}}\|f_{j,q}(t)\|_{L^2}dt\\ &+ (t_{j+1}-t_j)A + (2^qT)^{\delta}\frac{\varepsilon_0}{3AT}\int_{t_j}^{t_{j+1}}\|\nabla G_\delta^{(j)}(t)\|_{L^\infty}dt \geqslant \frac{\varepsilon_0}{3}\cdot\end{aligned}$$

So by summation we infer that the number N of intervals is finite and that

$$N \leqslant \frac{C}{\varepsilon_0}(2^qT)^{2\delta-1+\varepsilon} + \frac{1}{\lambda\varepsilon_0} + \frac{3AT}{\varepsilon_0^2} + (2^qT)^{\delta}\frac{1}{AT}\|\nabla G_\delta^{(j)}\|_{L^1_T(L^\infty)}\cdot$$

As usual, the best choice in the above inequality is the one that ensures that all the terms are (almost) equivalent. So here, we choose

$$AT = (2^qT)^{\delta/2},\ \lambda = (2^qT)^{-\delta/2} \quad\text{and}\quad \delta = \frac{2}{3}\cdot$$

So the number of intervals N is less than $C(2^qT)^{\frac{1}{3}+\varepsilon}$. So let us denote by $(I_{q,\ell})_{1\leqslant\ell\leqslant N}$ the partition of the interval $[0,T]$ and state $I_{q,\ell} = (t_{q,\ell}, t_{q,\ell+1})$. Using (18) and (14), we can write

$$\begin{aligned}2^{2q(\frac{d}{4}-\frac{1}{2}-k)}\|\partial^{1+k}u_q^{(j)}\|^2_{L^2_T(L^4)} &= C2^{2q(\frac{d}{4}-\frac{1}{2}-k)}\sum_{\ell=1}^{N}\|\partial^{1+k}u_q^{(j)}\|^2_{L^2_{I_{q,\ell}}(L^4)}\\ &\leqslant C2^{2q(\frac{d}{2}-1)}N\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big)^2\end{aligned}$$

As N is less than $C(2^qT)^{\frac{1}{3}+\varepsilon}$, we have, when $d\geqslant 5$,

$$\begin{aligned}&2^{q(\frac{d}{4}-\frac{1}{2}-k)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^4)}\\ &\qquad\leqslant C_\varepsilon 2^{q(\frac{d}{2}-1)}(2^qT)^{\frac{1}{6}+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)} + (2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big)\cdot\end{aligned}$$

The case when $d = 4$ can be treated exactly along the same lines and is thus omitted. In order to prove the bilinear estimate, let us write, using (16) and (18),

$$\begin{aligned}\|\chi(2^{-p}D)(\partial^{1+k}u_q^{(1)}u_q^{(2)})\|_{L^1_T(L^\infty)} &\leqslant \sum_{\ell=1}^{N}\|\chi(2^{-p}D)(\partial^{1+k}u_q^{(1)}u_q^{(2)}\|_{L^1_{I_{q,\ell}}(L^\infty)}\\ &\leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^\varepsilon N\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\qquad\times\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^{\frac13+2\varepsilon}\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\qquad\times\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big).\end{aligned}$$

So Theorem 3.2 is proved. Let us state the following corollary.

Corollary 3.1. — *If $N_T^\alpha(\gamma)$ is small enough and C_0 large enough, then assertion $(\mathcal{P}_n)$ implies assertion $(\mathcal{P}_{n+1})$.*

Let us first investigate the case when $d \geqslant 5$. Assertion $(\mathcal{P}_n)$ and Theorem 3.2 imply that

$$\begin{aligned}2^{q(\frac d4-\frac12)}\|\partial u_q^{(n+1)}\|_{L^2_T(L^4)} &\leqslant C_\varepsilon 2^{q(\frac d2-1)}(2^qT)^{\frac16+\varepsilon}\\ &\qquad\times\big(\|\partial u_q^{(n+1)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|R_q^\delta(n)\|_{L^1_T(L^2)}\big).\end{aligned}$$

Corollaries 2.2 and 2.4 imply that, if $2^qT \geqslant C_1$,

$$2^{q(\frac d4-\frac12)}\|\partial u_q^{(n+1)}\|_{L^2_T(L^4)} \leqslant C_\varepsilon(2^qT)^{\varepsilon-\alpha}N_T^\alpha(\gamma)\big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

When $2^qT \leqslant C_1$, we use Corollary 2.3 to write that

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B_{4,2}^{\frac d4-\frac12})} \leqslant CN_T^\alpha(\gamma)\big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

As the space $\dot B_{4,2}^{\frac d4-\frac12}$ is continuously embedded in $\dot C^{-1/2}$, we have, if $N_T^\alpha(\gamma)$ is small enough and C_0 large enough,

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B_{4,2}^{\frac d4-\frac12})} \leqslant C_0N_T^\alpha(\gamma)$$

and so using Corollary 2.3 we get $(\mathcal{P}_{n+1})$ for $d \geqslant 5$.

In the case $d = 4$, following exactly the same lines we obtain that

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B_{6,2}^{1/6})} \leqslant C_0N_T^\alpha(\gamma). \tag{19}$$

We have to control $\|\partial G_{n+1,T}\|_{L^1_T(L^\infty)}$. Let us use Bony's decomposition as in the introduction. We get

$$\begin{aligned}
\Delta_p \partial G_{n+1,T} &= \Delta_p \Delta^{-1}(\partial \nabla u^{(n+1)} \nabla u^{(n+1)}) \\
&= \sum_{j=1}^{3} \Delta_p^{(j)} \quad \text{with} \\
\Delta_p^{(1)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_q S_{q-1} \partial \nabla u^{(n+1)} \nabla u_q^{(n+1)}, \\
\Delta_p^{(2)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_q S_{q-1} \nabla u^{(n+1)} \partial \nabla u_q^{(n+1)} \quad \text{and} \\
\Delta_p^{(3)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_{\substack{q \\ -1 \leqslant \ell \leqslant 1}} \partial \nabla u_q^{(n+1)} \nabla u_{q-\ell}^{(n+1)}.
\end{aligned}$$

To estimate the norm $\|\cdot\|_{L^1_T(L^\infty)}$ of $\Delta_p^{(1)}$, let us observe that we have, for $k \in \{0,1\}$,

$$\begin{aligned}
\|S_{q-1} \partial^k \nabla u^{(n+1)}\|_{L^2_T(L^\infty)} &\leqslant \sum_{q' \leqslant q-2} 2^{q'k} \|\Delta_{q'} \nabla u^{(n+1)}\|_{L^2_T(L^\infty)} \\
&\leqslant \sum_{q' \leqslant q-2} 2^{q'(k+\frac{2}{3})} \|\Delta_{q'} \nabla u^{(n+1)}\|_{L^2_T(L^6)}.
\end{aligned}$$

So by convolution inequality on the series, we get that

$$2^{-q(k+\frac{1}{2})} \|S_{q-1} \partial^k \nabla u^{(n+1)}\|_{L^2_T(L^\infty)} \in \ell^2(\mathbf{Z}).$$

As the support of the Fourier transform of $S_{q-1} \partial \nabla u^{(n+1)} \nabla u_q^{(n+1)}$ is included in a ring of the type $2^q \widetilde{C}$, we get that

$$\sum_p \|\Delta_p^{(1)}\|_{L^1_T(L^\infty)} \leqslant C C_0 N_T^\alpha(\gamma)^2.$$

The term $\Delta_p^{(2)}$ can be estimated exactly in the same way. As seen in the introduction, the remainder term will required the use of bilinear estimates. Using the fact that the support of the Fourier transform of $\partial \nabla u_q^{(n+1)} \nabla u_{q-\ell}^{(n+1)}$ is included in ball of the type $2^q B$, we have that

$$\|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} \leqslant C_\varepsilon (2^p T)^{-2(\alpha-\varepsilon)} N_T^\alpha(\gamma)^2 \sum_{q \geqslant p - N_0} (2^{q-p} T)^{-2(\alpha-\varepsilon)}.$$

So choosing for instance $\varepsilon = \alpha/2$, we have that

$$\sum_{p \,/\, 2^p T \geqslant C} \|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} \leqslant C_\alpha N_T^\alpha(\gamma)^2.$$

But, for low frequencies in p, we simply observe that, by Bernstein inequality and Corollary 2.2, we have

$$\begin{aligned}\|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} &\leqslant CT2^{2p}\sum_{q\geqslant p-N_0}\|\partial\nabla u_q^{(n+1)}\|_{L^\infty_T(L^2)}\|\nabla u_{q-\ell}^{(n+1)}\|_{L^\infty_T(L^2)}\\ &\leqslant CT2^{2p}\|\gamma\|^2_{s_\alpha-1}\sum_{q\geqslant p-N_0}2^{-q\left(1+\frac{1}{3}+2\alpha\right)}\\ &\leqslant C(2^pT)^{\frac{2}{3}-2\alpha}N_T^\alpha(\gamma)^2.\end{aligned}$$

So we have

$$\|\partial G_{n+1,T}\|_{L^1_T(L^\infty)}\leqslant CN_T^\alpha(\gamma)^2$$

and Corollary 3.1 is proved.

Now the proof of Theorem 0.3 (i.e. the case of dimension greater or equal to 5) is pure routine of non linear hyperbolic partial differential equations.

3.3. The existence and uniqueness when $d=4$. — The case of dimension 4 requires some attention. Let us first assume that γ belongs to $H^{\frac{d}{2}-\frac{1}{2}}$. So it is clear that on an interval $[-T,T]$ the length of which depends only on $\|\gamma\|_{\dot H^{\frac{d}{2}-1}}$ and $\|\gamma\|_{\dot H^{\frac{d}{2}-1+\frac{1}{6}+\alpha}}$, the sequence $(\partial u^{(n)})_{n\in\mathbf{N}}$ is bounded in $L^\infty_T(H^{\frac{d}{2}-\frac{1}{2}})$. So energy methods (because the initial data is more regular) allow to claim that a solution u exists on $[-T,T]$ such that

$$\partial u\in L^\infty_T(H^{\frac{d}{2}-\frac{1}{2}}).$$

Moreover, we have on this interval the following estimates:

$$\begin{aligned}&\|\partial u\|_{L^2([0,T];\dot B^{\frac{d}{6}-\frac{1}{2}}_{6,2})}\leqslant C_0N_T^\alpha(\gamma)\\ &\|\partial G_u\|_{L^1([0,T];L^\infty)}\leqslant 2\\ &\|\partial u\|_{T,s-1}\leqslant e^3\|\gamma\|_{s-1}\quad\text{for any}\quad s\in\Big[\frac{3}{2}+\alpha,2+\frac{1}{6}+\alpha\Big].\end{aligned}$$

This solution is unique because of the result based on energy methods. Now let us consider intial data (u_0,u_1) which satisfy the hypothesis of Theorem 0.4. So if we consider initial data (S_nu_0,S_nu_1), a solution $\widetilde u^{(n)}$ associated to (S_nu_0,S_nu_1), exists on an interval $[-T,T]$ such that

$$\|\partial\widetilde u^{(n)}\|_{L^2([0,T];\dot B^{\frac{d}{6}-\frac{1}{2}}_{6,2})}\leqslant C_0N_T^\alpha(\gamma)\tag{20}$$

$$\|\partial G_{\widetilde u^{(n)}}\|_{L^1([0,T];L^\infty)}\leqslant 2\tag{21}$$

$$\|\partial\widetilde u^{(n)}\|_{T,s-1}\leqslant e^3\|\gamma\|_{s-1}\quad\text{for any}\quad s\in\Big[\frac{3}{2}+\alpha,2+\frac{1}{6}+\alpha\Big].\tag{22}$$

In order to prove that $(\widetilde u^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence and thus the uniqueness part of Theorem 0.4, we shall prove the following lemma which clearly concludes the proof.

Lemma 3.1. — *Let $u^{(j)}$ be two solutions of (EC) on the interval $[-T_0, T_0]$ such that*

$$\partial u^{(j)} \in C([-T_0, T_0]; H^{s_\alpha - 1}) \cap L^2_{T_0}(\dot{B}^{1/6}_{6,2}) \quad \textit{and} \quad \partial g_{u^{(j)}} \in L^1_{T_0}(L^\infty).$$

Then if T is small enough, we have that

$$\|\partial u^{(1)} - \partial u^{(2)}\|_{L^\infty_T(\dot{H}^{s_\alpha-2})} \leqslant 2\|\gamma^{(1)} - \gamma^{(2)}\|_{\dot{H}^{s_\alpha-2}}.$$

As in the iterative scheme, let us introduce a time cut-off. Let θ be a smooth function such that $\operatorname{Supp}\theta \subset]-2, 2[$ and θ has value 1 near $[-1, 1]$. So on the interval $[-T, T]$, the function $u^{(j)}$ is the solution of

$$(EC) \begin{cases} \partial_t^2 u^{(j)} - \Delta u^{(j)} - \displaystyle\sum_{1 \leqslant k,\ell \leqslant d} G^{k,\ell}_{u^{(j)},T} \partial_k \partial_\ell u^{(j)} = 0 \\ \qquad\qquad\qquad\qquad (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

with

$$G^{k,\ell}_{u^{(j)},T} = \theta\Big(\frac{t}{T}\Big) g^{k,\ell}_{u^{(j)}} \quad \text{with} \quad \Delta g^{k,\ell}_{u^{(j)}} = Q_{k,\ell}(\partial u^{(j)}, \partial u^{(j)}).$$

From now on in this section, we shall always work in the interval $[-T, T]$ with $2T \leqslant T_0$. Let us define $w = u^{(1)} - u^{(2)}$. Then on the interval $[-T, T]$, w is the solution of

$$\begin{cases} \partial_t^2 w - \Delta w - \displaystyle\sum_{1 \leqslant k,\ell \leqslant d} G^{k,\ell}_{u^{(1)},T} \partial_k \partial_\ell w = F_{1,2} \\ \qquad\qquad (w, \partial_t w)_{|t=0} = (u_0^{(1)} - u_0^{(2)}, u_1^{(1)} - u_1^{(2)}) \end{cases}$$

with

$$F_{1,2} \stackrel{\text{def}}{=} (G_{u^{(2)}} - G_{u^{(1)}}) \cdot \nabla^2 u^{(2)}.$$

We shall use the fact all the time in this paragraph that the two solutions $u^{(j)}$ satisfies

$$\|\partial u^{(j)}\|_{L^2_T(\dot{C}^{-1/2})} \leqslant C\|\partial u^{(j)}\|_{L^2_T(\dot{B}^{1/6}_{6,2})} \leqslant N^\alpha_T(\gamma^{(j)}) \quad \text{and} \quad \|\partial G_{u^{(j)},T}\|_{L^1_T(L^\infty)} \leqslant C_0.$$

Moreover, we state

$$\Gamma \stackrel{\text{def}}{=} \|\gamma^{(1)}\|_{s_\alpha - 1} + \|\gamma^{(2)}\|_{s_\alpha-1}\,, \ \Gamma_T \stackrel{\text{def}}{=} N^\alpha_T(\gamma^{(1)}) + N^\alpha_T(\gamma^{(2)}) \quad \text{and} \quad \underline{\gamma} \stackrel{\text{def}}{=} \gamma^{(1)} - \gamma^{(2)}.$$

Let us use computations done during the proof of the paralinearization theorem 2.1. Thanks to Formulas (7) and (8), we get that the function $w_q = \Delta_q w$ is solution of

$$\partial_t^2 w_q - \Delta w_q - S_{q-1} G_{u^{(1)},T} \nabla^2 w_q = R_q(t)$$

with

$$
\begin{aligned}
R_q &= \Delta_q F_{1,2}(t) + \sum_{j=1}^{4} R_q^{(j)} \quad \text{where} \\
R_q^{(1)} &\stackrel{\text{def}}{=} \sum_{|q-q'|\leqslant N_1} \big[\Delta_q, S_{q'-1}G_{u^{(1)},T}\big]\nabla^2 w_{q'} \\
R_q^{(2)} &\stackrel{\text{def}}{=} \sum_{|q-q'|\leqslant N_1} (S_{q'-1}G_{u^{(1)},T} - S_{q-1}G_{u^{(1)},T})\nabla^2 \Delta_q u_{q'} \\
R_q^{(3)} &\stackrel{\text{def}}{=} \Delta_q \sum_{\substack{q'\geqslant q-N_1\\ \ell\in\{-1,0,1\}}} \nabla^2 w_{q'}\Delta_{q'+\ell}G_{u^{(1)},T} \\
R_q^{(4)} &\stackrel{\text{def}}{=} \Delta_q \sum_{|q'-q|\leqslant N_1} S_{q'-1}\nabla^2 w_{q'}\Delta_{q'}G_{u^{(1)},T}.
\end{aligned}
$$

It is obvious that, if $s_\alpha - 2 > 1$, we have for any $j \in \{1,2,3\}$,

$$
\|R_q^{(j)}(t)\|_{L^2} \leqslant c_q(t) C 2^{-q(s_\alpha-2)} \|\nabla G_{u^{(1)},T}(t)\|_{L^\infty} \|\partial w(t)\|_{s_\alpha-2}.
$$

Using Lemma 2.1, we have

$$
\|\Delta_{q'} G_{u^{(1)},T}(t)\|_{L^2} \leqslant C c_{q'}(t) 2^{-q'(s_\alpha-\frac{3}{2})} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} \|\partial u^{(1)}(t)\|_{\dot H^{s_\alpha-1}}.
$$

Thus

$$
\|R_q^{(4)}(t)\|_{L^2} \leqslant C c_q(t) 2^{-q(s_\alpha-2)} \|\partial w(t)\|_{\dot C^{-3/2}} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} \|\partial u^{(1)}(t)\|_{s_\alpha-1}.
$$

Using the properties of $u^{(j)}$ on the interval $[-T,T]$ imply that

$$
\begin{aligned}
(23)\quad \|R_q(t)\|_{L^2} \leqslant c_q(t) C 2^{-q(s_\alpha-2)} \big(&\|\nabla G_{u^{(1)},T}(t)\|_{L^\infty} \|\partial w(t)\|_{s_\alpha-2} \\
&+ \|\gamma^{(1)}\|_{s_\alpha-1} \|\partial w(t)\|_{\dot C^{-3/2}} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} + \|F_{1,2}(t)\|_{s_\alpha-2}\big).
\end{aligned}
$$

So using Gronwall lemma, we infer that for any t in $[-T,T]$,

$$
\begin{aligned}
\|\partial w\|_{L^\infty_T(\dot H^{s_\alpha-2})} \leqslant \big(&\|\underline{\gamma}\|_{s_\alpha-2} + \|\gamma^{(1)}\|_{s_\alpha-1} \|\partial w\|_{L^2_T(\dot C^{-3/2})} \|\partial u^{(1)}\|_{L^2_T(\dot C^{-1/2})} \\
&+ \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big) \exp\big(\|\partial G_{u^{(1)},T}\|_{L^1_T(L^\infty)}\big).
\end{aligned}
$$

So the properties of the solution $u^{(1)}$ imply that

$$
(24)\quad \|\partial w\|_{L^\infty_T(\dot H^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T \|\partial w\|_{L^2_T(\dot C^{-3/2})} + \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big) \quad \text{and}
$$

$$
(25)\quad \|R_q\|_{L^1_T(L^2)} \leqslant C 2^{-q(s_\alpha-2)} \big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T \|\partial w\|_{L^2_T(\dot C^{-3/2})} + \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big).
$$

Because the L^2 norm in time with value in $\dot C^{-3/2}$ of w appears in the right side of the above inequality, we have to use the Strichartz estimates. Applying Theorem 2.2 with $\delta = 2/3$, and (25), it turns out that w_q is solution of

$$
\partial_t^2 w_q - \Delta w_q - \widetilde G_{u^{(1)},T} \nabla^2 w_q = \widetilde R_q(t)
$$

with $\widetilde{G}_{u^{(1)},T} \stackrel{\text{def}}{=} S_q^{2/3} G_{u^{(1)},T}$ and (dropping the case of low frequencies)

$$(26)\quad \|\widetilde{R}_q\|_{L^1_T(L^2)} \leqslant C2^{-q(s_\alpha-2)}(2^qT)^{1/3}\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T\|\partial w\|_{L^2_T(\dot{C}^{-3/2})} + \|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

Now thanks to Theorem 3.2 applied with $\varepsilon = \alpha/2$, we infer that

$$2^{-q5/6}\|\partial w_q\|_{L^2_T(L^6)} \leqslant (2^qT)^{-\alpha/2}\big(T^{\frac16+\alpha}\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma_T^2\|\partial w\|_{L^2_T(\dot{C}^{-3/2})} + T^{\frac16+\alpha}\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

As $2^{-q3/2}\|\partial w_q\|_{L^2_T(L^\infty)} \leqslant 2^{-q5/6}\|\partial w_q\|_{L^2_T(L^6)}$ it turns out that, if Γ_T is small enough, (dropping the case of low frequencies),

$$(27)\qquad \|\partial w\|_{L^2_T(\dot{C}^{-3/2})} \leqslant CT^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2} + C\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big)$$

and thus with (24),

$$(28)\qquad \|\partial w\|_{L^\infty_T(\dot{H}^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

The estimate of the term $F_{1,2}$ is more delicate than the others. Using Bony's decomposition, we get that

$$\begin{aligned}
F_{1,2} &= \sum_{j=1}^4 F^{(j)} \quad \text{with}\\
F^{(1)} &\stackrel{\text{def}}{=} T_{\nabla^2 u^{(2)}}\Delta^{-1}Q(\partial w, \partial u^{(1)} + \partial u^{(2)}),\\
F^{(2)} &\stackrel{\text{def}}{=} R\big(\nabla^2 u^{(2)}, \Delta^{-1}Q(\partial w, \partial u^{(1)} + \partial u^{(2)}\big),\\
F^{(3)} &\stackrel{\text{def}}{=} T_{\Delta^{-1}Q(T_{\partial w},(\partial u^{(1)}+\partial u^{(2)}))+Q(T_{\partial u^{(1)}+\partial u^{(2)}},\partial w)}\nabla^2 u^{(2)} \quad \text{and}\\
F^{(4)} &\stackrel{\text{def}}{=} T_{\Delta^{-1}(QR(\partial w,\partial u^{(1)})+QR(\partial w,\partial u^{(2)}))}\nabla^2 u^{(2)}.
\end{aligned}$$

The terms $F^{(j)}$ with $j \leqslant 3$ will require only Strichartz inequalities to be controled. So law of product in Besov spaces implies that

$$\begin{aligned}
\|Q(\partial w, \partial u^{(1)} + \partial u^{(2)})(t)\|_{s_\alpha-\frac52} &\leqslant C\big(\|\partial w(t)\|_{\dot{B}^{-5/6}_{6,\infty}}(\|\partial u^{(1)}(t)\|_{s_\alpha-1}\\
&\quad + \|\partial u^{(2)}(t)\|_{s_\alpha-1}) + \|\partial w(t)\|_{s_\alpha-2}(\|\partial u^{(1)}(t)\|_{\dot{B}^{1/6}_{6,\infty}} + \|\partial u^{(2)}(t)\|_{\dot{B}^{1/6}_{6,\infty}})\big)
\end{aligned}$$

Using the properties of $u^{(1)}$, we get that

$$\begin{aligned}
\|F^{(1)}(t)\|_{s_\alpha-2} &\leqslant C\Gamma_T\|\partial w(t)\|_{\dot{B}^{-5/6}_{6,\infty}}\|\partial u^{(2)}(t)\|_{\dot{B}^{1/6}_{6,\infty}}\\
&\quad + C\|\partial w(t)\|_{s_\alpha-2}\big(\|\partial u^{(1)}(t)\|^2_{\dot{B}^{-5/6}_{6,\infty}} + \|\partial u^{(2)}(t)\|^2_{\dot{B}^{-5/6}_{6,\infty}}\big).
\end{aligned}$$

By integration, using the properties of the two solutions and (24), (27) and (28), we get that, if T is small enough,

$$(29)\qquad \|F^{(1)}\|_{L^1_T(\dot{H}^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma_T\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

The term $F^{(2)}$ is estimated exactly along the same lines. The term $F^{(3)}$ is the analog to the paraproduct term in the first section. Let us write that

$$\begin{aligned}\|T_{\partial w}(\partial u^{(1)}+\partial u^{(2)})\|_{\dot B^{-2}_{\infty,1}} &\leqslant C\|\partial w\|_{\dot B^{-3/2}_{\infty,2}}(\|\partial u^{(1)}\|_{\dot B^{-1/2}_{\infty,2}}+\|\partial u^{(2)}\|_{\dot B^{-1/2}_{\infty,2}})\\ &\leqslant C\|\partial w\|_{\dot B^{-5/6}_{6,2}}(\|\partial u^{(1)}\|_{\dot B^{1/6}_{6,2}}+\|\partial u^{(2)}\|_{\dot B^{1/6}_{6,2}}).\end{aligned}$$

As the same estimate is true for $T_{\partial u^{(1)}+\partial u^{(2)}}\partial w$, using the estimate (27), it turns out after times integration, and if T is small enough, that

$$(30)\qquad \|F^{(3)}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2}+\Gamma_T\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

The estimate of the term $F^{(4)}$ requires the use of the bilinear estimate stated in Theorem 3.2. The key point is obviously to estimate

$$\Delta_{p,q}\overset{\text{def}}{=}\|\Delta_p\Delta^{-1}(\partial w_q\partial u_{q-j})\|_{L^1_T(L^\infty)}.$$

Theorem 3.2 applied with $\varepsilon=\alpha$ and $f=\widetilde R_q$ implies that

$$\begin{aligned}\Delta_{p,q}\leqslant C_\alpha 2^q&\Big((2^qT)^{\frac16+\frac\alpha2}\|\partial w_q\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|\widetilde R_q\|_{L^1_T(L^2)}\Big)\\ &\times\Big((2^qT)^{\frac16+\frac\alpha2}\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|R_q^{\frac23-\frac\varepsilon2}(u^{(j)})\|_{L^1_T(L^2)}\Big).\end{aligned}$$

As $s_\alpha=2+\dfrac16+\alpha$, Theorem 2.2 and properties of the solution $u^{(1)}$ imply that

$$(2^qT)^{\frac16+\frac\alpha2}\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|R_q^{\frac23-\frac\alpha2}(u^{(j)})\|_{L^1_T(L^2)}\leqslant C_\alpha 2^{-q}(2^qT)^{-\alpha/2}\Gamma_T.$$

Theorem 2.2 and estimation (28) imply that

$$\begin{aligned}(2^qT)^{\frac16+\frac\alpha2}\|\partial w_q\|_{L^\infty_T(L^2)}&+(2^qT)^{-\frac16+\frac\alpha2}\|\widetilde R_q\|_{L^1_T(L^2)}\\ &\leqslant C(2^qT)^{-\alpha/2}(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}).\end{aligned}$$

So it turns out that

$$\Delta_{p,q}\leqslant C_\varepsilon(2^qT)^{-\alpha}\Gamma_TT^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

So dropping the case of low frequencies (treated exactly along the same lines as in the proof of Corollary 2.3), we get that

$$(31)\qquad \|\Delta^{-1}R(\partial w,\partial u)\|_{L^1_T(L^\infty)}\leqslant C\Gamma_TT^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

Using the properties of the solution $u^{(2)}$ and the properties of the action of the paraproduct, we deduce from the above inequality that

$$\|F^{(4)}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\Gamma_T\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

Together with the inequalities (29) and (30), we get that

$$\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\|\underline{\gamma}\|_{s_\alpha-2}+C\Gamma_T\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}.$$

So if Γ_T is small enough, we have

$$\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})} \leqslant C\|\underline{\gamma}\|_{s_\alpha-2}.$$

Plugging this estimate into (28) implies that

$$\|\partial w\|_{L^\infty_T(\dot{H}^{s_\alpha-2})} \leqslant C\|\underline{\gamma}\|_{s_\alpha-2}.$$

So uniqueness (and in fact stability) is proved.

4. Approximation of the solution and geometrical optics

4.1. The Hamilton-Jacobi equation. — The following proposition (and its proof) is a small modification of Proposition 6.1 of [**4**].

Proposition 4.1. — *Let F be a real valued smooth function on $\mathbf{R}^d\times\mathbf{R}^N$ bounded as all its derivatives such that*

$$F(\zeta,G) = \pm\big(|\zeta|^2 + G(\zeta,\zeta)\big)^{1/2} \quad \textit{for all} \quad \zeta\in\widetilde{\mathcal{C}}.$$

For any positive real number ε, a positive real number α exists such that, if $\|\mathcal{G}\|_0\leqslant\alpha$ and $\Lambda\geqslant\alpha^{-1}$, for any η, a solution Φ_Λ of the equation

$$(\widetilde{HJ_\Lambda})\begin{cases}\partial_\tau\Phi_\Lambda(\tau,y,\eta) = F_\Lambda(\tau,y,\partial_y\Phi_\Lambda(\tau,y,\eta))\\ \Phi_\Lambda(0,y,\eta) \;= (y|\eta)\end{cases}\quad \textit{with } F_\Lambda(\tau,z,\zeta)\stackrel{def}{=}F(\zeta,G_\Lambda(\tau,z)).$$

exists and is smooth on $I_\Lambda\times\mathbf{R}^d\times\mathbf{R}^d$. Moreover, the family defined by $\Phi\stackrel{def}{=}(\Phi_\Lambda)_{\Lambda\geqslant\Lambda_0}$ satisfies the following properties: for any couple of integer (k,ℓ), a constant $C_{k,\ell}$ (independent of ε) exists such that

$$\sup_{\Lambda\geqslant\Lambda_0}\|(\partial_y\partial_\eta\Phi_\Lambda-\mathrm{Id})\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})}\leqslant C\varepsilon\,, \tag{32}$$

$$\sup_{\Lambda\geqslant\Lambda_0}|I_\Lambda|\Lambda^k\|\partial_\eta^\ell\nabla^{2+k}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})}\leqslant C_{k,\ell}\,\varepsilon \quad \textit{and} \tag{33}$$

$$\sup_{\Lambda\geqslant\Lambda_0}\|\partial_\eta^{\ell+2}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})}\leqslant C|I_\Lambda|. \tag{34}$$

In section 6, we shall use the link between the solution of the above Hamilton-Jacobi equation and the Hamiltonian flow of the function $-F_\Lambda$ on $T^\star\mathbf{R}^d$. This link is classical but here we need precise estimates with respect to the metric G_Λ. It is described by the two following lemmas.

Lemma 4.1. — *Let Φ_Λ be the solution of the above Hamilton-Jacobi equation $(\widetilde{HJ_\Lambda})$ and Ψ_Λ the Hamiltonian flow of $-F_\Lambda(\tau,Y)$ i.e. the solution of*

$$\begin{cases}\dfrac{d\Psi_\Lambda}{d\tau}(\tau,y,\eta) = -H_{F_\Lambda}(\tau,\Psi_\Lambda(Y))\\ \Psi_\Lambda(0,y,\eta) = (y,\eta).\end{cases}$$

Then we have

$$(\partial_\eta \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) = y \quad \textit{and}$$
$$(\partial_y \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) = \Psi^\eta_\Lambda(\tau, y, \eta).$$

To prove this, we have simply to remember that by construction of the solution of Hamilton-Jacobi equations (see for instance [**3**]), we have

$$\Big\{\big(\tau, \Psi^y_\Lambda(\tau, y, \eta), \Psi^\eta_\Lambda(\tau, y, \eta)\big)\ ,\ \tau \in I_\Lambda\Big\} = \Big\{\big(\tau, \widetilde{y}, (\partial_{\widetilde{y}} \Phi_\Lambda)(\tau, \widetilde{y}, \eta)\big)\ ,\ \tau \in I_\Lambda\Big\}.$$

So we deduce immediately that

$$(\partial_y \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) = \Psi^\eta_\Lambda(\tau, y, \eta). \tag{35}$$

Now let us compute

$$A_j \overset{\text{def}}{=} \frac{d}{d\tau}\big((\partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta)\big).$$

The chain rule implies that

$$A_j = (\partial_\tau \partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) + \sum_{k=1}^d (\partial_{\eta_j} \partial_{y_k} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) \frac{d\Psi^{y_k}_\Lambda}{d\tau}(\tau, y, \eta).$$

By differentiation of $(\widetilde{HJ_\Lambda})$ with respect to η, we get that, for any $\widetilde{y} \in \mathbf{R}^d$,

$$\partial_\tau \partial_{\eta_j} \Phi_\Lambda(\tau, \widetilde{y}, \eta) = \sum_{k=1}^d (\partial_{\zeta_k} F_\Lambda)(\tau, \widetilde{y}, \partial_{\widetilde{y}} \Phi_\Lambda(\tau, \widetilde{y}, \eta)) \partial_{\widetilde{y}_k} \partial_{\eta_j} \Phi_\Lambda(\tau, \widetilde{y}, \eta).$$

Applying this identity with $\widetilde{y} = \Psi^y_\Lambda(\tau, y, \eta)$, we get

$$\begin{aligned}(\partial_\tau \partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) = \sum_{k=1}^d (\partial_{\zeta_k} F_\Lambda)\big(\tau, \Psi^y_\Lambda(\tau, y, \eta), (\partial_{\widetilde{y}} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta)\big)\\ \times (\partial_{\widetilde{y}_k} \partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta).\end{aligned}$$

Using identity (35), we infer that

$$\begin{aligned}(\partial_\tau \partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) = \sum_{k=1}^d (\partial_{\zeta_k} F_\Lambda)\big(\tau, \Psi^y_\Lambda(\tau, y, \eta), \Psi^\eta_\Lambda(\tau, y, \eta)\big)\\ \times (\partial_{\widetilde{y}_k} \partial_{\eta_j} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta).\end{aligned}$$

Then we deduce that

$$A_j = \sum_{k=1}^d (\partial_{\eta_j} \partial_{y_k} \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau, y, \eta), \eta) \bigg(\frac{d\Psi^{y_k}_\Lambda}{d\tau}(\tau, y, \eta) + (\partial_{\zeta_k} F_\Lambda)(\tau, \Psi_\Lambda(\tau, y, \eta))\bigg).$$

As for $\tau = 0$, we have $\partial_\eta \Phi_\Lambda(0, \Psi^y_\Lambda(0, y, \eta), \eta) = \partial_\eta \Phi_\Lambda(0, y, \eta) = \partial_\eta(y|\eta) = y$, the first lemma is proved.

The second lemma is more technical and is related to properties of the hamiltonian flow with respect to a large class of metrics (i.e. of positive quadratic forms) on $T^\star \mathbf{R}^d$. It will be crucial in section 6.

Lemma 4.2. — *A constant C_0 exists such that for any couple of positive numbers (r,h) such that $|I_\Lambda| \geqslant h^{-2}$ we have the following properties. If*

$$g_a(dy^2, d\eta^2) \stackrel{def}{=} \frac{dy^2}{K^2} + \frac{d\eta^2}{h^2} \quad \textit{with} \quad K = C|I_\Lambda| h$$

then, provided we choose C large enough, we have:

– *for any couple (Y,Z) and for any $\tau \in I_\Lambda$, we have*

$$\frac{1}{C_0}\, g_a(Y-Z) \leqslant g_a(\Psi_\Lambda(\tau,Y) - \Psi_\Lambda(\tau,Z)) \leqslant C_0\, g_a(Y-Z); \tag{36}$$

– *for any couple of points (Y_0, Z_τ) of $T^\star \mathbf{R}^d$ such that*

$$g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} \geqslant C_0 r$$

if $(z,\eta) \in B_{g_a}(Y_0, r)$ and if $(y,\zeta) \in B_{g_a}(Z_\tau, r)$ then

$$g_a\big(\nabla_\eta \Phi_\Lambda(\tau,y,\eta) - z, \nabla_y \Phi_\Lambda(\tau,y,\eta) - \zeta\big) \geqslant \frac{1}{C_0} g_a\big(Z_\tau - \Psi_\Lambda(\tau,Y_0)\big).$$

Remark. — The choice of the metric g_a will become clearer in section 6. But anyway, it is essentially the only choice of a metric such that the above inequalities are true.

Let us prove the first point of this lemma. By differentiation of the equation of the Hamiltonian flow, we have

$$\frac{d}{d\tau}\big(D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\big) = -DH_{F_\Lambda} \cdot \big(D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\big) - DH_{F_\Lambda}.$$

By Gronwall lemma, we get, for any $\tau \in I_\Lambda$,

$$\begin{aligned}\|D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} \leqslant \int_{I_\Lambda} \sup_{Y \in T^\star \mathbf{R}^d} \|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} d\tau \\ \times \exp \int_{I_\Lambda} \sup_{Y \in T^\star \mathbf{R}^d} \|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} d\tau\end{aligned}$$

where

$$\|A\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} \stackrel{\text{def}}{=} \sup_{\substack{Z \in T^\star \mathbf{R}^d \\ g_a(Z) \leqslant 1}} g_a(A \cdot Z)^{1/2}.$$

By definition of the Hamiltonian of F_Λ and of the metric g_a, we infer that, if $Z=(z,\zeta)$,

$$\begin{aligned} g_a(DH_{F_\Lambda}(\tau,Y)\cdot Z) &\leqslant \frac{C}{K^2}\big(\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|z|^2+|\zeta|^2\big)\\ &\qquad + \frac{C}{h^2}\big(\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|z|^2+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|\zeta|^2\big)\\ &\leqslant \frac{C|z|^2}{K^2}\Big(\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}+\frac{K^2}{h^2}\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\Big)\\ &\qquad + \frac{C|\zeta|^2}{h^2}\Big(\frac{h^2}{K^2}+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\Big)\\ &\leqslant \Big(\frac{h^2}{K^2}+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}+\frac{K^2}{h^2}\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\Big)g_a(Z). \end{aligned}$$

So it turns out that

$$\sup_{Y\in T^\star\mathbf{R}^d}\|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}\leqslant C\Big(\frac{h}{K}+\|\nabla G_\Lambda(\tau,\cdot)\|_{L^\infty}+\frac{K}{h}\|\nabla^2 G_\Lambda(\tau,\cdot)\|_{L^\infty}\Big).$$

By integration and by definition (10) of $\|\mathcal{G}\|_0$, we get that

$$\int_{I_\Lambda}\sup_{Y\in T^\star\mathbf{R}^d}\|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}d\tau\leqslant C\Big(\frac{|I_\Lambda|h}{K}+\|\mathcal{G}\|_0\Big(1+\frac{K}{h|I_\Lambda|}\Big)\Big)\cdot$$

If ε is any positive real number, let us choose

$$K=\frac{4C}{\varepsilon}|I_\Lambda|h\quad\text{and } \|\mathcal{G}\|_0 \text{ such that}\quad \|\mathcal{G}\|_0\Big(1+\frac{4C}{\varepsilon}\Big)\leqslant\frac{\varepsilon}{4C}\cdot \tag{37}$$

Then, we have, for ε small enough,

$$\sup_{(\tau,Y)\in I_\Lambda\times T^\star\mathbf{R}^d}\|D\Psi_\Lambda(\tau,Y)-\mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}\leqslant\varepsilon.$$

Using Taylor formula, we write that

$$\begin{aligned} g_a\big(\psi_\Lambda(\tau,Y)-Y-\psi_\Lambda(\tau,Z)+Z\big)^{1/2}&\\ \leqslant \sup_{\substack{Y\in T^\star\mathbf{R}^d\\ \tau\in I_\Lambda}}&\|D\Psi_\Lambda(\tau,Y)-\mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}g_a(Y-Z)^{1/2}\\ \leqslant{}& \varepsilon g_a(Y-Z)^{1/2}. \end{aligned}$$

Using the inequality of the triangle and choosing $\varepsilon=1/2$, we get that, for any $\tau\in I_\Lambda$, any couple (Y,Z) of points of $T^\star\mathbf{R}^d$, we have

$$\frac{1}{2}g_a(Y-Z)^{1/2}\leqslant g_a\big(\Psi_\Lambda(\tau,Y)-\Psi_\Lambda(\tau,Z)\big)^{1/2}\leqslant\frac{3}{2}g_a(Y-Z)^{1/2}.$$

To prove the second point of this lemma let us write, with of course the obvious notation $Y_0=(y_0,\eta_0)$ and $Z_\tau=(z_\tau,\zeta_\tau)$, that

$$\begin{aligned} \frac{1}{K}\big|\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)\big|&\\ \leqslant \frac{1}{K}\|\nabla_z\nabla_\eta\Phi_\Lambda\|_{L^\infty}&|y-z_\tau|+\frac{1}{K}\|\nabla^2_\eta\Phi_\Lambda\|_{L^\infty}|\eta-\eta_0|. \end{aligned}$$

Estimates (33) and (34) imply that

$$\begin{aligned}\frac{1}{K}\big|\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)\big| &\leqslant \frac{C|y-z_\tau|}{K}+\frac{|I_\Lambda|}{K}|\eta-\eta_0|\\ &\leqslant Cr.\end{aligned}$$

Along the same lines, we have

$$\frac{1}{h}|\nabla_y\Phi_\Lambda(\tau,y,\eta)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)|\leqslant Cr.$$

So using the inequality of the triangle and the fact that (z,η) is in $B_{g_a}(Y_0,r)$ and (y,ζ) in $B_{g_a}(Z_\tau,r)$, we infer that

$$\begin{aligned}(38)\quad & g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\\ &\qquad\geqslant g_a\big(\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)-y_0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\zeta_\tau\big)^{1/2}-4r.\end{aligned}$$

Let us define $Z_0\stackrel{\text{def}}{=}\Psi_\Lambda^{-1}(\tau,Z_\tau)=(z_0,\zeta_0)$ and let us assume that

$$g_a(0,\zeta_0-\eta_0)\leqslant\beta^2 g_a(Z_0-Y_0)$$

for some β in the interval $]0,1[$ that will determine later on. Then, using estimates (32)–(34) as above, we obtain that

$$\begin{aligned}& g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\\ &\quad\geqslant g_a\big(\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\zeta_0)-y_0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)-\zeta_\tau\big)^{1/2}-Cr-C\beta g_a(Z_0-Y_0)^{1/2}.\end{aligned}$$

Using Lemma 4.1, we infer that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant g_a(z_0-y_0,0)^{1/2}-Cr-C\beta g_a(Z_0-Y_0)^{1/2}.$$

But, as $g_a(z_0-y_0,0)\geqslant(1-\beta^2)g_a(Z_0-Y_0)$, we get that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\big((1-\beta^2)^{1/2}-C\beta\big)g_a(Z_0-Y_0)^{1/2}-Cr.$$

Let us choose for instance β so small that

$$(1-\beta^2)^{1/2}-C\beta\geqslant\frac{1}{2}\cdot$$

Then, if $g_a(Z_0-Y_0)^{1/2}\geqslant C_0 r$ with C_0 large enough, we have that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\frac{1}{4}g_a(Z_0-Y_0)^{1/2}.$$

Now let us assume that

$$g_a(0,\zeta_0-\eta_0)\geqslant\beta^2 g_a(Z_0-Y_0).$$

Going back to Inequality (38) and using Lemma 4.1, we claim that

$$\begin{aligned}& g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\\ &\qquad\geqslant g_a\big(0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)\big)^{1/2}-Cr.\end{aligned}$$

Using estimate (32) and choosing ε small enough in it, we have that

$$\begin{aligned}\left|\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)\right| &\geqslant \left(1-\|\nabla_y\nabla_\eta\Phi_\Lambda-\mathrm{Id}\,\|_{L^\infty(I_\Lambda\times T^\star\mathbf{R}^d)}\right)|\zeta_0-\eta_0|\\ &\geqslant \frac{1}{2}|\zeta_0-\eta_0|.\end{aligned}$$

So by definition of the metric g_a it turns out that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\frac{\beta}{2}g_a(Z_0-Y_0)^{1/2}-Cr.$$

This concludes the proof of the lemma if C_0 is large enough. To be able to handle interactions between pair of points of type (x,ξ)–$(x,-\xi)$, we shall need to control the time variation of the Hamiltonian flow. This will be crucial in section 7. The following lemma determines the subintervals of I_Λ such that the flow does vary very few.

Lemma 4.3. — *Let J be any subinterval of I_Λ. Then, we have*

$$\sup_{\substack{(\tau,\tau')\in J^2\\ Y\in T^\star\mathbf{R}^d}} g_a(\Psi_\Lambda(\tau,Y)-\Psi_\Lambda(\tau',Y))^{1/2}\leqslant C\bigg(\frac{|J|}{h|I_\Lambda|}+\frac{1}{h}\|\nabla G_\Lambda\|_{L^1_J(L^\infty)}\bigg).$$

To prove this, let us observe that by definition of the Hamiltonian flow, we have

$$\Psi_\Lambda(\tau',Y)-\Psi_\Lambda(\tau,Y)=-\int_\tau^{\tau'}\mathcal{H}_{F_\Lambda}(\tau'',\Psi_\Lambda(\tau'',Y))d\tau''.$$

So we immediately get that, for any $(\tau,\tau')\in J$,

$$g_a(\Psi_\Lambda(\tau',Y)-\Psi_\Lambda(\tau,Y))^{1/2}\leqslant\int_J\sup_{Y\in T^\star\mathbf{R}^d}g_a(\mathcal{H}_{F_\Lambda}(\tau'',Y))^{1/2}d\tau''.$$

But by definition of the Hamiltonian vector field and the metric g_a, we have

$$g_a(\mathcal{H}_{F_\Lambda}(\tau,Y))=\frac{1}{K^2}|\partial\eta F_\Lambda(\tau,Y)|^2+\frac{1}{h^2}|\partial_y F_\Lambda(\tau,Y)|^2.$$

By definition of F_Λ and of K, we infer that

$$g_a(\mathcal{H}_{F_\Lambda}(\tau,Y))^{1/2}\leqslant C\bigg(\frac{1}{h|I_\Lambda|}+\frac{1}{h}\|\nabla G_\Lambda(\tau,\cdot)\|_{L^\infty}\bigg).$$

So an immediat integration concludes the proof of the lemma.

4.2. The approximation of the solution.

— Before stating the theorem, let us recall the concept of symbols we introduced in [**4**].

Definition 4.1. — Let us denote by S^{-N} the set of families of functions $\sigma=(\sigma_\Lambda)_{\Lambda\geqslant\Lambda_0}$ such that

– the function σ_Λ is smooth on $I_\Lambda\times\mathbf{R}^d\times\mathcal{C}$ in $\mathbf{C}$;

– for any integer k, the quantity defined by

$$q_k^{(N)}(\sigma) \stackrel{\text{def}}{=} \sup_{\substack{j+j'\leqslant k \\ \Lambda\geqslant\Lambda_0}} \Lambda^{N+j}\|\partial_\eta^{j'}\nabla^j\sigma_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^d\times\mathcal{C})}$$

is finite.

– An element of S^{-N} is a symbol of order $-N$.

Now we are able to state the approximation theorem.

Theorem 4.1. — *Let us assume that $\|\mathcal{G}\|_0$ is small enough. Then, for any integer N, two symbols $\sigma^\pm$ (with value in $\mathbf{R}^2$) belonging to S^0 and a constant C exists such that the following properties are satisfied.*

Let $(v_\Lambda)_{\Lambda\geqslant\Lambda_0}$ be the family of solutions of (E_Λ) with $f=0$ and with initial data $\gamma=(\gamma^0,\gamma^1)$; if we state

$$\mathcal{I}_\Lambda^\pm(\gamma) \stackrel{\text{def}}{=} \int_{\mathbf{R}^d} e^{i\Phi_\Lambda^\pm(\tau,y,\eta)}\sigma_\Lambda^\pm(\tau,y,\eta)\cdot\widehat{\gamma}_\pm(\eta)d\eta, \tag{39}$$

then, if

$$|I_\Lambda|\leqslant\Lambda^{2-\varepsilon} \tag{40}$$

we have

$$\|\nabla(v_\Lambda-\mathcal{I}_\Lambda^+(\gamma)-\mathcal{I}_\Lambda^-(\gamma))\|_{L^\infty_{I_\Lambda}(L^2)}\leqslant C\Lambda^{-N}\|\gamma\|_{L^2}. \tag{41}$$

The proof of this is done in [**4**] and [**5**].

4.3. The precised Strichartz estimate. — The theorem is the following.

Theorem 4.2. — *Let $\mathcal{C}$ be a ring of $\mathbf{R}^d$ and let us assume that $\|\mathcal{G}\|_0$ is small enough. For any positive real number ε, a constant C_ε exists which satisfies the following properties. Let f be a function in $L^1_{I_\Lambda}(L^2)$ and γ a function of L^2; let us assume that those two functions have their support included in $\mathcal{C}$ and of diameter less than h. Let us assume*

$$|I_\Lambda|\leqslant\Lambda^{2-\varepsilon}.$$

Then if v_Λ is the solution of

$$(E_\Lambda)\begin{cases} P_\Lambda v_\Lambda &= f\\ \partial v_{\Lambda|\tau=0} &= \gamma.\end{cases}$$

we have

$$\|\nabla v_\Lambda\|_{L^2_{I_\Lambda}(L^\infty)}\leqslant Ch^{(d-2)/2}\big(\log(e+|I_\Lambda|)\big)^{1/2}\big(\|\gamma\|_{L^2}+\|f\|_{L^1_{I_\Lambda}(L^2)}\big).$$

To prove this theorem, we shall use the classical $TT^\star$ method. Following [**5**] and using the fact that the support of the Fourier transform of $\widehat{\gamma}$ is included in the ball of center ξ_0 and radius h denoted by $B(\xi_0, h)$, let us write that, for any $f \in \mathcal{D}(I_\Lambda \times \mathbf{R}^d)$, we have

$$\begin{aligned}\langle \mathcal{I}_\Lambda(\gamma), f\rangle &= \langle \widehat{\gamma}, A_\Lambda f\rangle \quad \text{with}\\ A_\Lambda f &\stackrel{\text{def}}{=} \int e^{i\Phi_\Lambda(\tau,x,\xi)}\sigma_\Lambda(\tau,x,\xi)\chi\Big(\frac{\xi-\xi_0}{h}\Big)f(\tau,x)d\tau dx.\end{aligned}$$

where χ is a function of $\mathcal{D}(\mathbf{R}^d)$.

$$\langle \mathcal{I}_\Lambda(\gamma), f\rangle \leqslant \|\gamma\|_{L^2}\|A_\Lambda f\|_{L^2(B(\xi_0,h))}.$$

By definition of A_Λ we have

$$|A_\Lambda f(\xi)|^2 = \int e^{i(\Phi_\Lambda(\tau,x,\xi)-\Phi_\Lambda(\tau',y,\xi))}\widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,\xi)f(\tau,x)f(\tau',y)d\tau d\tau' dxdy$$

where

$$\widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,\xi) \stackrel{\text{def}}{=} \sigma_\Lambda(\tau,x,\xi)\overline{\sigma}_\Lambda(\tau',y,\xi)\chi\Big(\frac{\xi-\xi_0}{h}\Big)\overline{\chi}\Big(\frac{\xi-\xi_0}{h}\Big).$$

First, let us decompose A_Λ as follows

$$\begin{aligned}|A_\Lambda f(\xi)|^2 &= B_\Lambda f(\xi) + C_\Lambda f(\xi) \quad \text{with}\\ B_\Lambda f(\xi) &\stackrel{\text{def}}{=} \int_{|\tau-\tau'|h^2\geqslant 1} e^{i(\Phi_\Lambda(\tau,x,\xi)-\Phi_\Lambda(\tau',y,\xi))}\widetilde{\sigma}_\Lambda(\tau,\tau',x,y,\xi)f(\tau,x)f(\tau',y)d\tau d\tau' dxdy.\end{aligned}$$

The estimate about $C_\Lambda f$ is very easy. As the support of $C_\Lambda f$ is included in the ball $B(\xi_0, h)$ we have

$$\begin{aligned}\int |C_\Lambda f(\xi)|d\xi &\leqslant Ch^d \sup_\xi |C_\Lambda f(\xi)|\\ &\leqslant Ch^d \int_{|\tau-\tau'|h^2\leqslant 1} \|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ &\leqslant Ch^{d-2}\int \frac{h^2}{1+(\tau-\tau')^2h^4}\|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ &\leqslant Ch^{d-2}\|f\|^2_{L^2_{I_\Lambda}(L^1(\mathbf{R}^d))}.\end{aligned}$$

Now we shall assume that $|\tau-\tau'|h^2 \geqslant 1$. Let us follow [**5**]. Using Taylor formula, we write that

$$\Phi_\Lambda(\tau,x,\xi) - \Phi_\Lambda(\tau',y,\xi) = (x-y)\Theta_\Lambda(\tau,\tau',x,y,\xi) + (\tau-\tau')\widetilde{\Psi}_\Lambda(\tau,\tau',x,y,\xi) \text{ with}$$

$$\widetilde{\Psi}_\Lambda(\tau,\tau',x,y,\xi) = \int_0^1 \frac{\partial\Phi_\Lambda}{\partial\tau}(\tau'+t(\tau-\tau'),y+t(x-y),\xi)dt \tag{42}$$

$$\Theta_\Lambda(\tau,\tau',x,y,\xi) = \int_0^1 \frac{\partial\Phi_\Lambda}{\partial x}(\tau'+t(\tau-\tau'),y+t(x-y),\xi)dt.$$

Stating the change of variables

$$\eta = \Theta_\Lambda(\tau,\tau',x,y,\xi),$$

we get, denoting by F_Λ the inverse of the above diffeomorphism,

$$\widetilde{K}_\Lambda(\tau,\tau',x,y) = \int e^{i(x-y)\eta} e^{i(\tau-\tau')\Psi_\Lambda(\tau,\tau',x,y,\eta)} \sigma_{\Lambda,h}(\tau,\tau',x,y,\eta)d\eta \quad \text{with}$$

$$\sigma_{\Lambda,h}(\tau,\tau',x,y,\eta) \stackrel{\text{def}}{=} \widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta))J_\Lambda(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta)))$$

and

$$\Psi_\Lambda(\tau,\tau',x,y,\eta) = \widetilde{\Psi}_\Lambda(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta)).$$

where J_Λ denotes the Jacobian of this change of variables.

Now let us change the variable

$$\eta = \eta_0 + h\zeta \quad \text{with} \quad \eta_0 \stackrel{\text{def}}{=} \Theta_\Lambda(\tau,\tau',x,y,\xi_0).$$

Then we have

$$\begin{aligned}\widetilde{K}_\Lambda(\tau,\tau',x,y) &= h^d e^{ih(x-y)\eta_0} K_\Lambda(\tau,\tau',x,y) \quad \text{with}\\ K_\Lambda(\tau,\tau',x,y) &\stackrel{\text{def}}{=} \int e^{ih(x-y)\zeta} e^{i(\tau-\tau')\Psi_\Lambda(\tau,\tau',x,y,\eta_0+h\zeta)} \sigma_{\Lambda,h}(\tau,\tau',x,y,\zeta)d\zeta\end{aligned}$$

where all derivatives of $\sigma_{\Lambda,h}$ are bounded with respect to ζ. Let us study of the form of the function Ψ_Λ. Using Taylor formula, we can write (dropping the fact that Ψ_Λ depends on τ, τ', x and y)

$$\psi_\Lambda(\eta_0+h\zeta) = \psi_\Lambda(\eta_0) + h(\nabla_\eta\Psi_\Lambda(\eta_0)|\zeta) + h^2\int_0^1 D^2\Psi_\Lambda(\eta_0+sh\zeta)ds(\zeta,\zeta).$$

Using the inequalities (32) and (33) it turns out that, for any s, h and ζ, we have

$$\forall\,\theta\in\mathbf{R}^d,\ \left|D^2\Psi_\Lambda(\eta_0+sh\zeta)(\theta,\theta) - |p_{(\eta_0+sh\zeta)^\perp}\theta|^2\right| \leqslant \varepsilon|\theta|^2.$$

As ζ belongs to the unit ball of $\mathbf{R}^d$, and h can be choosen small enough, we have that the quadratic form

$$Q(h\zeta) \stackrel{\text{def}}{=} \int_0^1 D^2\Psi_\Lambda(\eta_0+sh\zeta)ds$$

is a non negative quadratic form of rank greater or equal to $d-1$. Then stating $x-y = (\tau-\tau')z$, we can write the phase

$$i(\tau-\tau')h(z+\nabla\Psi_\Lambda(\eta_0)|\zeta) + i(\tau-\tau')h^2Q(h\zeta)(\zeta,\zeta).$$

Then we can choose coordinates such that the phase function is

$$i(\tau-\tau')h(z+\nabla\Psi_\Lambda(\eta_0))\zeta_1 + i(\tau-\tau')h^2\sum_{j=1}^d a_j(h\zeta)\zeta_j^2$$

where for any $j \geqslant 2$, the functions a_j are smooth with bounded derivatives and $1/2 \leqslant a_j \leqslant 2$ except possibily one of them. Then following the basic proof of the stationnary phase theorem we get that

$$\begin{aligned}|\widetilde{K}_\Lambda(\tau,\tau',x,y)| &\leqslant C\frac{h^d}{(|\tau-\tau'|h^2)^{(d-2)/2}}\\ &\leqslant C\frac{h^2}{|\tau-\tau'|^{(d-2)/2}}\cdot\end{aligned}$$

As $d \geqslant 4$, we have that

$$\begin{aligned}\int |B_\Lambda f(\xi)|d\xi &\leqslant C\int_{|\tau-\tau'|h^2\geqslant 1}\frac{h^2}{|\tau-\tau'|^{(d-2)/2}}\|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ &\leqslant Ch^{d-2}\log(e+|I_\Lambda|)\|f\|^2_{L^2_{I_\Lambda}(L^1(\mathbf{R}^d))}\cdot\end{aligned}$$

Thus Theorem 4.2 holds.

5. The concept of microlocalized functions

In this section, we present the concept of microlocalized functions introduced by J.-M. Bony in [**7**]. This concept is related to the Weyl-Hörmander calculus (see [**11**], [**9**]). But the problem we investigate here allows us to use a simplified version of it.

5.1. A simplified version of pseudo-differential calculus. — In this paragraph, we shall consider a positive quadratic form g on $T^\star\mathbf{R}^d$ such that the symplectic conjugate quadratic form g^σ defined by

$$g^\sigma(T) \overset{\text{def}}{=} \sup_{W\neq 0}\frac{[T,W]^2}{g(W)}$$

satisfies the uncertainty principle

$$g^\sigma \geqslant g.$$

Here $[\cdot,\cdot]$ denotes the basic symplectic form on $T^\star\mathbf{R}^d$ defined by

$$[(x,\xi),(y,\eta)] = \sum_{j=1}^d(\xi^j y_j - \eta^j x_j).$$

In all this paper, we are going to be in the case when

$$g(dx,d\xi) = \frac{dx^2}{K^2}+\frac{d\xi^2}{h^2}\cdot$$

In this case, we have

$$g^\sigma = \lambda^2 g \quad \text{with} \quad \lambda = Kh.$$

The uncertainty principle means that $\lambda \geqslant 1$.

We shall mesure the length of derivatives of smooth functions on $T^\star \mathbf{R}^d$ with respect to this metric g. More precisely, let us define, for any smooth function φ on $T^\star \mathbf{R}^d$,

$$\|\varphi\|_{j,g} \stackrel{\text{def}}{=} \sup_{\substack{k\leqslant j \\ X\in T^\star \mathbf{R}^d}} \sup_{\substack{(T_\ell)_{1\leqslant \ell\leqslant k} \\ g(T_\ell)\leqslant 1}} |D^k\varphi(X)(T_1,\cdot,T_k)|.$$

Now, to a function φ in $\mathcal{D}(T^\star \mathbf{R}^d)$, we associate the operator φ^D defined by

$$(\varphi^D u)(x) = (2\pi)^{-d} \int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)} \varphi(y,\xi) u(y) dy d\xi.$$

The choice of this quantization process makes the computation of section 6 simpler. Let us remark that if the function $\varphi(x,\xi)$ is equal to $\varphi_1(x)\varphi_2(\xi)$, then

$$\varphi^D u = \mathcal{F}^{-1}\big(\varphi_2(\mathcal{F}(\varphi_1 u)\big).$$

Moreover we have

$$\mathcal{F}(\varphi^D u)(\xi) = \int_{\mathbf{R}^d} e^{-i(y|\xi)} \varphi(y,\xi) u(y) dy.$$

Later on in this paper we shall need to decompose L^2 functions whose Fourier transform is supported in the ring $\mathcal{C}$ using these operators φ^D. Let us state the following lemma which will be useful.

Lemma 5.1. — *A sequence $(X_\nu)_{\nu\in\mathcal{Z}}$ exists such that two sequencies $(\varphi_\nu)_{\nu\in\mathcal{Z}}$ and $(\psi_\nu)_{\nu\in\mathcal{Z}}$ exist which satisfy the following properties.*

– *the support of φ_ν is included in a ball $B_\nu \stackrel{\text{def}}{=} B_g(X_\nu, r)$,*

– *A sequence $(C_j)_{j\in\mathbf{N}}$ exists (which depends only on r and not in the parameters K and h) such that*

$$\forall\, \nu\in\mathcal{Z},\ \|\varphi_\nu\|_{j,g} \leqslant C_j,$$

– *the functions ψ_ν are not supported in B_ν but confined, which means that a sequence $(C_N)_{N\in\mathbf{N}}$ exists such that*

$$\forall\, \nu\in\mathcal{Z},\ \|\psi_\nu\|_{N,g,X} \stackrel{\text{def}}{=} \sup_{\substack{k\leqslant N \\ X\in T^\star \mathbf{R}^d}} (1+\lambda^2 g(X-B_\nu))^N \sup_{\substack{(T_\ell)_{1\leqslant \ell\leqslant k} \\ g(T_\ell)\leqslant 1}} |D^k\psi_\nu(X)(T_1,\cdot,T_k)| \leqslant C_N,$$

– *For any function u of L^2 whose Fourier transform has a support included in $\mathcal{C}$, we have*

$$\sum_{\nu\in\mathcal{Z}} \varphi_\nu^D \psi_\nu^D u = \sum_{\nu\in\mathcal{Z}} \varphi_\nu^D u = u.$$

Such partitions of unity are "compatible" with L^2 in the following sense.

Lemma 5.2. — *A constant C exists such that*

$$C^{-1}\|u\|_{L^2}^2 \leqslant \sum_\nu \|\varphi_\nu^D u\|_{L^2}^2 \leqslant C\|u\|_{L^2}^2 \quad \textit{and} \quad \sum_\nu \|\psi_\nu^D u\|_{L^2}^2 \leqslant C\|u\|_{L^2}^2.$$

Those two lemmas are proved in [**8**].

Lemma 5.3. — *For any N, a constant C_N and an integer k_N exist which satisfy the following properties. Let ϕ and $\widetilde{\phi}$ be two functions on $\mathcal{S}(T^\star \mathbf{R}^d)$ and Y and $\widetilde{Y}$ two points of $T^\star \mathbf{R}^d$. Then a function θ exists in $\mathcal{S}(T^\star \mathbf{R}^d)$ such that*

$$\theta^D = \phi^D \widetilde{\phi}^D \quad \textit{and} \quad \|\theta\|_{N,g,Y} + \|\theta\|_{N,g,\widetilde{Y}} \leqslant C_N \|\phi\|_{N,g,Y} \|\widetilde{\phi}\|_{N,g,\widetilde{Y}}$$

This lemma is proved in [**9**].

Of course, the operators φ^D does not completely fit with any L^p space when $p \neq 2$. Nevertheless we have the following lemma.

Lemma 5.4. — *Let φ be a function of $\mathcal{S}(T^\star \mathbf{R}^d)$. The operator φ^D maps L^p into L^p for any p in $[1, \infty]$. More precisely, a constant C and an integer N exists such that*

$$\forall X_0 \in T^\star \mathbf{R}^d, \ \|\varphi^D a\|_{L^p} \leqslant C \|\varphi\|_{N,g,X_0} \|a\|_{L^p}.$$

This lemma can be seen as a corollary of Lemma 4.3 of [**10**]. For the convenience of the reader, we give here a self contained proof based of course on integrations by part. We have

$$\begin{aligned}\varphi^D a(x) &= \int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)} \varphi(y,\xi) a(y) dy d\xi \\ &= \int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)} \big(1 + h^2|x-y|^2\big)^{-d} (\mathrm{Id} - h^2 \Delta_\xi)^d \varphi(y,\xi) a(y) dy d\xi.\end{aligned}$$

So it turns out that

$$\begin{aligned}|\varphi^D a(x)| \leqslant C \|\varphi\|_{4d,g,X_0} \bigg(&\int_{\{(y,\xi)\,/\,\frac{|\xi-\xi_0|}{h} \leqslant r\}} \big(1 + h^2|x-y|^2\big)^{-d} |a(y)| dy d\xi \\ &+ \int_{T^\star \mathbf{R}^d} \big(1 + h^2|x-y|^2\big)^{-d} \big(1 + K^2|\xi - \xi_0|^2\big)^{-d} |a(y)| dy d\xi \bigg).\end{aligned}$$

So the lemma is proved, as thanks to the uncertainty principle, Kh is greater or equal to 1.

Remark. — The points X_ν are exactly the points of the lattice

$$\mathcal{Z} \stackrel{\text{def}}{=} (c_d r K\, \mathbf{Z}^d) \times (c_d r h\, \mathbf{Z}^d \cap \mathcal{C}). \tag{43}$$

Now we can defined the concept of microlocalized function.

Definition 5.1. — Let X_0 be a point of $T^\star \mathbf{R}^d$ and (C_0, r) a couple of positive real numbers. A function u in $L^2(\mathbf{R}^d)$ is said to be (C_0,r)-microlocalized in X_0 if a sequence of integers $(k_N)_{N \in \mathbf{N}}$ exists such that, for any integer N, the quantities

$$\mathcal{M}^{C_0,r}_{X_0,N}(u) \stackrel{\text{def}}{=} \sup_{g(X-X_0)^{1/2} \geqslant C_0 r} \lambda^{2N} g(X - X_0)^N \sup_{\substack{\varphi \in \mathcal{D}(B_g(X,r)) \\ \|\varphi\|_{k_N,g} \leqslant 1}} \|\varphi^D u\|_{L^2}$$

are finite. Here, $B_g(X,r)$ denotes as in all that follows the set of points of $T^\star \mathbf{R}^d$ such that $g(Y-X)^{1/2} \leqslant r$.

A basic example of microlocalized functions is given by the following proposition.

Proposition 5.1. — *A sequence of integers $(k_N)_{N\in\mathbf{N}}$ and a sequence of positive real numbers $(C_N)_{N\in\mathbf{N}}$ exist such that the following properties are satisfied. Let X_0 be a point of $T^\star\mathbf{R}^d$, φ_0 a function in $\mathcal{D}(B_g(X_0,r))$ and u a function of $L^2(\mathbf{R}^d)$. Then the function $\varphi_0^D u$ is $(3,r)$-microlocalized in X_0 and, for any N, we have*

$$\mathcal{M}^{3,r}_{X_0,N,g}(\varphi_0^D u) \leqslant C_N\|\varphi_0\|_{k_N,g}\|u\|_{L^2}.$$

This proposition can be seen as an immediat corollary of the general theory of Weyl-Hörmander calculus, for instance as a corollary of Theorem 2.2.1. of [**9**]. But as a warm up for the next section, we are going to give a proof of it in our particular situation.

By definition of φ_0^D, we have, for any function φ belonging to $\mathcal{D}(B_g(X,r))$,

$$\mathcal{F}(\varphi^D\varphi_0^D u)(\xi) = (2\pi)^{-d}\int_{\mathbf{R}^{3d}} e^{-i(y|\xi-\eta)-i(z|\eta)}\varphi(y,\xi)\varphi_0(z,\eta)u(z)dzdyd\eta.$$

Let us do some integrations by part with respect to some derivatives of g-length less than 1. It is obvious that

$$(K^2\Delta_y + h^2\Delta_\eta)\big(e^{-i(y|\xi-\eta)-i(z|\eta)}\big) = -\lambda^2 g(y-z,\xi-\eta)e^{-i(y|\xi-\eta)-i(z|\eta)}.$$

Using the fact that derivatives in (y,η) of g-length less than 1 of $g(y-z,\xi-\eta)$ is less than $g(y-z,\xi-\eta)^{1/2}$ it turns out that

$$\mathcal{F}(\varphi^D\varphi_0^D u)(\xi) = (2\pi)^{-d}\int_{\mathbf{R}^{3d}} e^{-i(y|\xi-\eta)-i(z|\eta)}\mathcal{K}(y,z,\xi,\eta)u(z)dzdyd\eta \quad \text{with}$$

$$|\mathcal{K}(y,z,\xi,\eta)| \leqslant C_{r,N}\big(1+\lambda^2 g(X-X_0)\big)^{-N}\|\varphi\|_{2N+N_0}\|\varphi_0\|_{2N+N_0} \times \frac{1}{\big(r^2+h^2|y-z|^2+K^2|\xi-\eta|^2\big)^{N_0}}\cdot$$

Using the fact that $\lambda = Kh \geqslant 1$ and convolution inequalities, we get that

$$\|\mathcal{F}(\varphi^D\varphi_0^D u)\|_{L^2} \leqslant C_{r,N}\big(1+\lambda^2 g(X-X_0)\big)^{-N}\|\varphi\|_{2N+N_0}\|\varphi_0\|_{2N+N_0}\|u\|_{L^2}.$$

This concludes the proof of the proposition.

In all that follows, the concept of uniformly microlocalized families of functions will be a basic tool.

Definition 5.2. — Let $g \stackrel{\text{def}}{=} (g_a)_{a\in A}$ be a family of metrics, $\mathcal{X} \stackrel{\text{def}}{=} (X_a)_{a\in A}$ a family of points of $T^\star\mathbf{R}^d$ and (C_0,r) a pair of positive real numbers. A family of functions $U \stackrel{\text{def}}{=} (u_a)_{a\in A}$ in $L^2(\mathbf{R}^d)$ is said to be uniformly (C_0,r)-microlocalized in $\mathcal{X}$ with respect to g if, for any integer N,

$$\mathcal{M}^{C_0,r}_{N,\mathcal{X},g}(U) \stackrel{\text{def}}{=} \sup_{a\in A}\mathcal{M}^{C_0,r}_{X_a,N,g_a}(u_a) < \infty.$$

5.2. A lemma about the product. — We want here to study the interaction between two (typical examples of) microlocalized functions. More precisely we are going to prove the following lemma.

Lemma 5.5. — *A constant C_0 exists such that, for any integer N, a constant C_N and an integer k_N exist which satisfy the following properties.*

If u_1 and u_2 are two L^2 functions on $\mathbf{R}^d$, if χ is a function of $\mathcal{D}(\mathbf{R}^d)$ supported in an euclidian ball of radius r, if φ_1 and φ_2 are two functions of $\mathcal{D}(T^\star\mathbf{R}^d)$ respectively supported in $B_g(Y_1,r)$ and in $B_g(Y_2,r)$, then if

$$g(\check{Y}_1 - Y_2)^{1/2} \geqslant C_0 r,$$

for any N, we have

$$\begin{aligned}&\big\|\chi(h^{-1}D)(\varphi_1^D u_1 \varphi_2^D u_2)\big\|_{L^1}\\ &\qquad\leqslant C_N \|\varphi_1\|_{k_N,g}\|\varphi_2\|_{k_N,g}\big(1+\lambda^2 g(\check{Y}_1 - Y_2)\big)^{-N}\|u_1\|_{L^2}\|u_2\|_{L^2}\end{aligned}$$

where $\check{Y} \overset{def}{=} (y,-\eta)$ if $Y=(y,\eta)$.

Let us suppose first that

$$\frac{|\eta_1+\eta_2|}{h} \geqslant \frac{1}{2} g(\check{Y}_1 - Y_2)^{1/2}.$$

By definition of the operator φ_j^D, the support of the Fourier transform of $\varphi_j^D u_j$ is included in the (euclidian) ball of center η_j and radius rh. So, it is clear that, if C_0 is large enough,

$$\operatorname{Supp}\mathcal{F}(\varphi_1^D u_1 \varphi_2^D u_2) \subset \{\eta\in\mathbf{R}^d \ / \ |\eta| \geqslant 2rh\}.$$

So it turns out that

$$\chi(h^{-1}D)(\varphi_1^D u_1 \varphi_2^D u_2) = 0.$$

Now we have to study the case when

$$\frac{|y_1-y_2|}{K} \geqslant \frac{1}{2} g(\check{Y}_1 - Y_2)^{1/2}.$$

By definition of the operator φ_j^D, we have

$$(44)\quad (\varphi_1^D u_1 \varphi_2^D u_2)(x) = (2\pi)^{-2d}\int_{B_g(Y_1,r)\times B_g(Y_2,r)} e^{i(x-y|\eta)+i(x-z|\zeta)} \times \varphi_1(Y)\varphi_2(Z)u_1(y)u_2(z)dYdZ.$$

The fact that

$$(\operatorname{Id} - h^2\Delta_\eta)e^{i(x-y|\eta)} = (1+h^2|x-y|^2)e^{i(x-y|\eta)}$$

So by repeated integration by parts, we get

$$\big|(\varphi_1^D u_1 \varphi_2^D u_2)(x)\big| \leqslant \int_{B_g(Y_1,r)\times B_g(Y_2,r)} (1+h^2|x-y|^2)^{-N}(1+h^2|x-z|^2)^{-N}$$
$$|(\mathrm{Id}-h^2\Delta_\eta)^N\varphi_1(Y)|\,|(\mathrm{Id}-h^2\Delta_\zeta)^N\varphi_2(Z)|\,|u_1(y)u_2(z)|dYdZ.$$

The inequality of the triangle implies that

$$|x-y_1|+|x-y_2| \geqslant |y_1-y_2| \quad \text{and} \quad |x-y|+|x-z| \geqslant |y_1-y_2|-2rK.$$

So, if C_0 is greater than 12, we have that

$$|x-y|+|x-z| \geqslant \frac{1}{2}|y_1-y_2|+rK.$$

So we infer that, for any N,

$$\big|(\varphi_1^D u_1\varphi_2^D u_2)(x)\big| \leqslant C_N(1+h^2|y_1-y_2|^2)^{-N}\|\varphi_1\|_{2N+2d}\|\varphi_2\|_{2N+2d}\,\mathcal{I}(x) \quad \text{with}$$
$$\mathcal{I}(x) \stackrel{\text{def}}{=} C_d h^{2d}\int (1+h^2|x-y|^2)^{-d}(1+h^2|x-z|^2)^{-d}|u_1(y)u_2(z)|dydz.$$

By Cauchy-Schwarz inequality, we have that

$$|\mathcal{I}(x)|^2 \leqslant C_d h^{4d}\prod_{j=1}^{2}\int (1+h^2|x-y|^2)^{-d}(1+h^2|x-z|^2)^{-d}|u_j(z)|^2 dydz$$

So using again Cauchy-Schwarz inequality, we get that

$$\|\mathcal{I}\|_{L^1(\mathbf{R}^d)} \leqslant C_d\|u_1\|_{L^2}\|u_2\|_{L^2}.$$

So the lemma is proved.

6. The propagation theorem

One of the important point of this study is that the (approximate) flow of the operator P_Λ preserves the microlocalization of functions. The aim of this section is to state and prove a theorem of propagation of microlocalization.

Theorem 6.1. — *A constant C_0 exists which satisfies the following property.*

Let us consider a point $Y_0=(y_0,\eta_0)$ of $T^\star\mathbf{R}^d$ such that η_0 belongs to $\mathcal{C}$, a smooth function ϕ supported in $B_{g_a}(Y_0,r)$ and a function γ of L^2. Then $\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot)$ is (C_0,r)-microlocalized near $\Psi_\Lambda^\pm(\tau,Y_0)$. Moreover, for any integer N, a constant C and an integer k exist (which depend only on N) such that

$$\mathcal{M}^{C_0,r}_{\Psi_\Lambda^\pm(\tau,Y_0),N,g_a}\big(\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot)\big) \leqslant C\,\|\phi\|_{k,g_a}\|\gamma\|_{L^2}.$$

In the following proof of this theorem, we shall drop the exponant $\pm$ for sake of simplicity of the notations. By definition of the microlocalized functions, we have to estimate the following quantity

$$\mathcal{J} \stackrel{\text{def}}{=} \mathcal{F}\big(\varphi_{Z_\tau}^D\mathcal{I}_\Lambda(\phi^D\gamma)(\tau,\cdot)\big)$$

where Z_τ is a point of $T^\star \mathbf{R}^d$ such that $g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} \geqslant C_0 r$. By definition, we have

$$\begin{aligned}\mathcal{J}(\zeta) &= \int_{\mathbf{R}^d} \mathcal{K}(\zeta, z)\gamma(z)dz \quad \text{with}\\ \mathcal{K}(\zeta, z) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^{2d}} e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)dyd\eta.\end{aligned}$$

The proof consists in integrations by parts in the above integral. Let us define the vector Θ by

$$\Theta = (\Theta^y, \Theta^\eta) \stackrel{\text{def}}{=} \Big(|I_\Lambda|^{-1/2}(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z), |I_\Lambda|^{1/2}(\nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta)\Big)$$

and the vector field $\mathcal{L}$ by

$$\mathcal{L}f \stackrel{\text{def}}{=} \frac{1}{1+|\Theta|^2}\Big(f - i|I_\Lambda|^{-1/2}\Theta_y \partial_\eta f - i|I_\Lambda|^{1/2}\Theta_\eta \partial_y f\Big).$$

It is obvious that

$$\mathcal{L}\big(e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\big) = e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}.$$

So as usual, we have, for any integer N,

$$\mathcal{K}(\zeta, z) = \int_{\mathbf{R}^{2d}} e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)dyd\eta.$$

Let us state the following technical lemma which will allow us to estimate the repeated action of the differential operator ${}^t\mathcal{L}$.

Lemma 6.1. — *For any integer N, a family of functions $(L_{\alpha,N})_{|\alpha|\leqslant N}$ exists such that $L_{\alpha,N}(Y,\mathcal{Y})$ is a smooth function from $T^\star \mathbf{R}^d \times (T^\star \mathbf{R}^d)^{M_N}$ and such that*

$$\|\partial_Y^\beta L_{\alpha,N}(Y,\cdot)\|_{L^\infty((T^\star \mathbf{R}^d)^{M_N})} \leqslant C_{N,|\beta|}(1+|Y|^2)^{-(N+|\beta|)/2}. \tag{45}$$

Moreover, they satisty

$$({}^t\mathcal{L})^N f = \sum_{|\alpha|\leqslant N} L_{\alpha,N}(\Theta, (\partial^\beta\Theta)_{|\beta|\leqslant N})\widetilde{\partial}^\alpha f$$

where $\widetilde{\partial}$ denotes differentiation of length 1 for the metric $\widetilde{g}_a$ defined by

$$\widetilde{g}_a(dy^2, d\eta^2) \stackrel{def}{=} |I_\Lambda|^{-1}dy^2 + |I_\Lambda|d\eta^2 = \lambda g_a(dy^2, d\eta^2).$$

The metric $\widetilde{g}_a$ is the interpolation between g_a and $g_a^\sigma = \lambda^2 g_a$. To prove this lemma, let us notice that the two vectors

$$|I_\Lambda|^{-1/2}\partial_\eta \quad \text{and} \quad |I_\Lambda|^{1/2}\partial_y$$

are of $\widetilde{g}_a$-length 1. Proposition 4.1 and the fact that $|I_\Lambda| \leqslant \Lambda^{2-\varepsilon}$ implies that, for any positive integer k, a constant c_k (which depends only on constants of Proposition 4.1) such that

$$\|\widetilde{\partial}^k\Theta\|_{L^\infty(I_\Lambda \times T^\star \mathbf{R}^d)} \leqslant c_k. \tag{46}$$

Now, we write that ${}^t\mathcal{L}f = \mathcal{L}f + L_0 f$ with

$$L_0 \stackrel{\text{def}}{=} i\Big(|I_\Lambda|^{-1/2}\sum_{j=1}^{d}\partial_{\eta_j}\Big(\frac{\Theta_{y_j}}{1+|\Theta|^2}\Big) + |I_\Lambda|^{1/2}\sum_{j=1}^{d}\partial_{y_j}\Big(\frac{\Theta_{\eta_j}}{1+|\Theta|^2}\Big)\Big)\cdot$$

So thanks to (46), we have

$$\|L_0(\Theta,\cdot)\|_{L^\infty} \leqslant \frac{C}{1+|\Theta|^2}\cdot$$

But, by definition of $\mathcal{L}$, it is obvious that $\mathcal{L}$ is of the form

$$\sum_{|\alpha|\leqslant 1} L_{\alpha,1}(\Theta)\widetilde{\partial}^\alpha f$$

where $\mathcal{L}_{\alpha,1}$ satisfy (45) for $N=1$. So the lemma is proved for $N=1$. The lemma follows by an omitted (and straightforward) induction.

Now, let us go back to the proof of the propagation theorem. The point is to prove that derivatives of $\widetilde{g}_a$-length 1 of

$$\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)$$

are bounded uniformly to the involved parameters. Thanks to Leibnitz formula, we have

$$\begin{aligned}\widetilde{\partial}_y^\alpha\widetilde{\partial}_\eta^\beta&\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\\ &= \sum_{\substack{\alpha_1\leqslant\alpha\\ \beta_1\leqslant\beta}} C_{\alpha,\beta}^{\alpha_1,\beta_1}\widetilde{\partial}_y^{\alpha-\alpha_1}\varphi_{Z_\tau}(y,\zeta)\widetilde{\partial}_y^{\alpha_1}\widetilde{\partial}_\eta^{\beta_1}\sigma_\Lambda(\tau,y,\eta)\widetilde{\partial}_\eta^{\beta-\beta_1}\phi(z,\eta).\end{aligned}$$

The metric $\widetilde{g}_a$ is choosen such that it is greater than the metric g_a and the metric g_Λ defined by

$$g_\Lambda(dy^2,d\eta^2) \stackrel{\text{def}}{=} \frac{dy^2}{\Lambda^2} + d\eta^2.$$

Then, it is obvious that, for any integer k, we have

$$\sup_{\substack{|\alpha+\beta|\leqslant k\\ (\tau,Y,Z)\in I_\Lambda\times(T^\star\mathbf{R}^d)^2}} \big|\widetilde{\partial}_y^\alpha\widetilde{\partial}_\eta^\beta\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\big| \leqslant C_k.$$

So thanks to Lemma 6.1, it turns out that, for any N, a constant C_N exists such that

$$\Big\|({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\Big\| \leqslant \frac{C_N}{(1+|\Theta|^2)^{N/2}}\cdot$$

So by definition of Θ, we infer that, for any integer N, a constant C_N exists such that

$$|\mathcal{K}(\zeta,z)| \leqslant C_N\int_{\substack{|y-z_\tau|\leqslant rK\\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1+\widetilde{g}_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)\Big)^N},$$

with of course

$$|z-y_0|\leqslant rK \quad\text{and}\quad |\zeta-\zeta_\tau|\leqslant rh.$$

But, as $K = C|I_\Lambda|h$ and $\lambda = Kh$, we have

$$\widetilde{g}_a(dy^2, d\eta^2) \geqslant c\lambda g_a(dy^2, d\eta^2).$$

So we have that

$$|\mathcal{K}(\zeta, z)| \leqslant C_N \int_{\substack{|y-z_\tau|\leqslant rK \\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}.$$

Now let us apply Lemma 4.2. As $g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0))^{1/2}$ is greater than $C_0 r$, as (z, η) belongs to $B_{g_a}(Y_0, r)$ and (y, ζ) to $B_{g_a}(Z_\tau, r)$ then

$$g_a(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta) \geqslant \frac{1}{C_0} g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0)).$$

So we have, if $g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0))^{1/2}$ is greater than $C_0 r$,

$$\begin{aligned} |\mathcal{K}(\zeta, z)| \leqslant & \frac{C_N}{\big(1 + \lambda g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))\big)^N} \\ & \times \int_{\substack{|y-z_\tau|\leqslant rK \\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}. \end{aligned}$$

As (y, ζ) belongs to $B_{g_a}(Z_\tau, r)$, we have that

$$\begin{aligned} g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} & \geqslant g_a((y, \zeta) - \Psi_\Lambda(\tau, Y_0))^{1/2} - r \\ & \geqslant \frac{|\zeta - \Psi_\Lambda^\eta(\tau, Y_0)|}{h} - r. \end{aligned}$$

Stating $Z_0 \stackrel{\text{def}}{=} \Psi_\Lambda^{-1}(\tau, Z_\tau)$, we have, thanks to the assertion (36) of Lemma 4.2 and as (z, η) belongs to $B_{g_a}(Y_0, r)$,

$$\begin{aligned} g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} & \geqslant C g_a(Z_0 - Y_0)^{1/2} \\ & \geqslant C\frac{|z - z_0|}{K} - Cr. \end{aligned}$$

So it turns out that

$$\begin{aligned} |\mathcal{K}(\zeta, z)| \leqslant C_N & \Big(1 + \lambda g_a\big(Z_\tau - \Psi_\Lambda(\tau, Y_0)\big)\Big)^{-N} \Big(1 + \lambda g_a\big((z, \zeta) - (z_0, \Psi_\Lambda^\eta(\tau, Y_0))\big)\Big)^{-N} \\ & \times \int_{\substack{|y-z_\tau|\leqslant rK \\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}. \end{aligned}$$

Let us state the change of variables

$$\begin{cases} y' = |I|^{-1/2}(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z) \\ \eta' = |I|^{1/2}(\nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta). \end{cases}$$

Using estimates (32), we infer that the jacobian of this change of variables is closed to 1. Then it turns out that

$$|\mathcal{K}(\zeta, z)| \leqslant C_N \Big(1 + \lambda g_a\big(Z_\tau - \Psi_\Lambda(\tau, Y_0)\big)\Big)^{-N} \Big(1 + \lambda g_a\big((z, \zeta) - (z_0, \Psi_\Lambda^\eta(\tau, Y_0))\big)\Big)^{-N}.$$

But Schur's lemma implies that

$$\|\mathcal{J}\|_{L^2}^2 \leqslant \Big(\sup_\zeta \int |\mathcal{K}(\zeta,z)|dz\Big)\Big(\sup_z \int |\mathcal{K}(\zeta,z)|d\zeta\Big)\|\gamma\|_{L^2}^2.$$

Immediate integrations imply that

$$\int |\mathcal{K}(\zeta,z)|dz \leqslant C_N\big(1+\lambda g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0))\big)^{-N}|I|^{d/2} \quad \text{and}$$

$$\int |\mathcal{K}(\zeta,z)|d\zeta \leqslant C_N\big(1+\lambda g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0))\big)^{-N}|I|^{-d/2}.$$

So, for any N, we have

$$\|\mathcal{J}\|_{L^2} \leqslant C_N\big(1+\lambda g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0))\big)^{-N}\|\gamma\|_{L^2}.$$

As $g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0))^{1/2}$ is greater that C_0r, then

$$\lambda g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0)) \geqslant C_0 r\lambda g_a(Z_\tau - \Psi_\Lambda(\tau,Y_0))^{1/2}.$$

So Theorem 6.1 is proved.

In the next section, the following corollary will be useful.

Corollary 6.1. — *A constant C_0 exists which satisfies the following property.*

Let us consider a point $Y_0=(y_0,\eta_0)$ of $T^\star\mathbf{R}^d$ such that η_0 belongs to $\mathcal{C}$, a smooth function ϕ supported in $B_{g_a}(Y_0,r)$ and a function γ of L^2.

For any integer N, a constant C and an integer k exist (which depend only on N) such that, for any a, if $g_a(\Psi_\Lambda(\tau,Y_0)-Y)\geqslant C_0r$, for any function ψ in $\mathcal{S}(T^\star\mathbf{R}^d)$, we have

$$\|\psi^D(\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot))\|_{L^2} \leqslant C\lambda^{-N}\big(1+g_a(\Psi_\Lambda(\tau,Y_0)-Y)\big)^{-N}\|\psi\|_{k,g,Y}\|\phi\|_{k,g_a}\|\gamma\|_{L^2}.$$

The proof of this corollary is a simple combination of Theorem 6.1 and Lemmas 5.3 and 5.4.

7. The conclusion of the proof

This section is the conclusion of the proof of theorem 3.1. The strategy is the following. First, we apply Lemma 5.5 about the product and the propagation theorem 6.1 to concentrate on real interaction (see the proof in the constant coefficient case). Because of the fact that variable coefficents do not respect the localization in frequency space, we need at this step of the proof to decompose the interval I_Λ.

In this section, we shall state

$$\mathcal{J}(\tau,y) \stackrel{\text{def}}{=} \chi(h^{-1}D)\big(\mathcal{I}_\Lambda^{(1)}(\gamma_1)(\tau,y)\mathcal{I}_\Lambda^{(2)}(\gamma_2)(\tau,y)\big).$$

The equivalent of Identity (5) that appears in the constant coefficient case is the following lemma.

Lemma 7.1. — *Let $J = (\tau_J, \tau_J^+)$ be a subinterval of I_Λ such that*

$$|J| \leqslant h|I_\Lambda| \quad \textit{and} \quad \|\nabla G_\Lambda^{(j)}\|_{L^1_J(L^\infty)} \leqslant h\|\nabla G_\Lambda^{(j)}\|_{L^1_{I_\Lambda}(L^\infty)}.$$

Then two families (ϕ_μ) and (θ_μ) of confined symbols exist such that, for any integer N,

$$\forall\, \mu,\ \|\phi_\mu\|_{N,g_a,\Psi_\Lambda^{(1)}(\tau_J,Y_\mu)} + \|\theta_\mu\|_{N,g_a,\check\Psi_\Lambda^{(2)}(\tau_J,Y_\mu)} \leqslant C_N$$

and, for any N, a constant C_N exists such that

$$\|\mathcal{J} - \underline{\mathcal{J}}\|_{L^1_J(L^\infty)} \leqslant C_N h\lambda^{-N}(|I_\Lambda|h^2)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.$$

with

$$\underline{\mathcal{J}}(\tau) \stackrel{\text{def}}{=} \sum_{\substack{\mu,\mu' \\ \mu'\in A_\mu}} \chi(h^{-1}D)\Big(\phi_\mu^D \mathcal{I}_\Lambda^{(1)}(\varphi_\mu^D\psi_\mu^D\gamma_1)(\tau,\cdot) \times \theta_\mu^D \mathcal{I}_\Lambda^{(2)}(\varphi_{\mu'}^D\psi_{\mu'}^D\gamma_2)(\tau,\cdot)\Big) \quad \textit{and}$$

$$A_\mu \subset \big\{\mu' \,/\, g_a\big(\Psi_\Lambda^{(2)}(\tau_J, Y_{\mu'}) - \check\Psi_\Lambda^{(1)}(\tau_J, Y_\mu)\big)^{1/2} \leqslant Cr\big\}.$$

Let us admit this lemma for a while. Let us simply notice that the number of elements of A_μ is finite and bounded indepently on μ and J.

Now we shall decompose the interval I_Λ on subintervals J such that the above lemma can be applied. To do this let us introduce the following function on the interval I_Λ

$$H(\tau) \stackrel{\text{def}}{=} \Big(\sum_\mu \|\mathcal{I}_\Lambda^{(1)}(\varphi_\mu^D\psi_\mu^D\gamma_1)(\tau,\cdot)\|^2_{L^\infty}\Big)^{1/2}\Big(\sum_\mu \|\mathcal{I}_\Lambda^{(2)}(\varphi_\mu^D\psi_\mu^D\gamma_2)(\tau,\cdot)\|^2_{L^\infty}\Big)^{1/2}.$$

Using precised Strichartz estimates, we get that

$$\|\mathcal{I}_\Lambda^{(j)}(\varphi_\mu^D\psi_\mu^D\gamma_j)\|_{L^2_{I_\Lambda}(L^\infty)} \leqslant C\big(\log(e+|I_\Lambda|)\big)^{1/2}h^{(d-2)/2}\|\psi_\mu^D\gamma_j\|_{L^2}.$$

So, using Cauchy-Schwartz inequality, we get that

$$\int_{I_\Lambda} H(\tau)d\tau \leqslant C\big(\log(e+|I_\Lambda|)\big)h^{d-2}\Big(\sum_\mu \|\psi_\mu^D\gamma_1\|^2_{L^2}\Big)^{1/2}\Big(\sum_\mu \|\psi_\mu^D\gamma_2\|^2_{L^2}\Big)^{1/2}.$$

Lemma 5.2 implies that

$$\int_{I_\Lambda} H(\tau)d\tau \leqslant C\big(\log(e+|I_\Lambda|)\big)h^{d-2}\|\gamma_1\|^2_{L^2}\|\gamma_2\|^2_{L^2}.$$

As in section 3, we decompose I_Λ in intervals J such that

$$|J| \leqslant h|I_\Lambda|,\ \|\nabla G_\Lambda^{(j)}\|_{L^1_J(L^\infty)} \leqslant h\|\nabla G_\Lambda^{(j)}\|_{L^1_{I_\Lambda}(L^\infty)} \quad \text{and} \quad \int_J H(\tau)d\tau \leqslant h\int_{I_\Lambda} H(\tau)d\tau.$$

Let us estimate $\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)}$. Lemma 5.4 implies that

$$\|\underline{\mathcal{J}}(\tau)\|_{L^\infty} \leqslant \sum_{\substack{\mu,\mu' \\ \mu'\in A_\mu}} \big\|\mathcal{I}_\Lambda^{(1)}(\varphi_\mu^D\psi_\mu^D\gamma_1)(\tau,\cdot)\big\|_{L^\infty}\big\|\mathcal{I}_\Lambda^{(2)}(\varphi_{\mu'}^D\psi_{\mu'}^D\gamma_2)(\tau,\cdot)\big\|_{L^\infty}.$$

By Cauchy-Schwarz inequality, we infer that

$$\|\underline{\mathcal{J}}(\tau)\|_{L^\infty} \leqslant H(\tau).$$

So by construction of J, we get that

$$\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)} \leqslant Ch\big(\log(e+|I_\Lambda|)\big)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.$$

Exactly along the same lines as in section 3, the number of intervals J is less than Ch^{-1}. As $|I_\Lambda| \geqslant h^{-2+\varepsilon}$ and $\lambda = |I_\Lambda|h^2$, the theorem is proved if we apply Lemma 7.1 with N large enough.

But we have to prove Lemma 7.1. First, let us write

$$\mathcal{J}(\tau) = \sum_{\nu,\nu',\mu,\mu'} \mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau) \quad \text{with}$$

$$\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau) \stackrel{\text{def}}{=} \chi(h^{-1}D)\Big(\varphi^D_\nu\psi^D_\nu\mathcal{I}^{(1)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_1)(\tau,\cdot)\varphi^D_{\nu'}\psi^D_{\nu'}\mathcal{I}^{(2)}_\Lambda(\varphi^D_{\mu'}\psi^D_{\mu'}\gamma_2)(\tau,\cdot)\Big).$$

Propagation theorem 6.1 and its corollary 6.1 imply that, if

$$g_a\big(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J, Y_\mu)\big)^{1/2} \geqslant C_0 r,$$

then

$$\|\varphi^D_\nu\psi^D_\nu\mathcal{I}^{(j)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_j)(\tau)\|_{L^2} \leqslant C_N\lambda^{-N}\big(g_a(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J,Y_\mu)\big)^{-d-1}\|\psi^D_\mu\gamma_j\|_{L^2}.$$

So using Bernstein inequality and integrating on the interval J, if

$$g_a\big(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J, Y_\mu)\big)^{1/2} \geqslant C_0 r,$$

we get that

$$\text{(47)}\quad \|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)} \leqslant C_N h(|I_\Lambda|h^2)h^{d-2}\lambda^{-N}\big(1+g_a(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J,Y_\mu))\big)^{-d-1} \times \|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.$$

Lemma 5.5 implies that if $g_a(\check{Y}_\nu - Y_{\nu'})^{1/2} \geqslant C_0 r$, then for any N we have

$$\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau)\|_{L^1(\mathbf{R}^d)} \leqslant C_N\lambda^{-N}\big(1+g_a(\check{Y}_\nu - Y_{\nu'})\big)^{-d-1}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.$$

Using Bernstein inequality, we get by integration that

$$\text{(48)}\qquad \|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau)\|_{L^\infty} \leqslant C_N\lambda^{-N}h^d\big(1+g_a(\check{Y}_\nu - Y_{\nu'})\big)^{-d-1}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.$$

Let us define

$$\Delta_{g_a}(X) \stackrel{\text{def}}{=} \begin{cases} 1+g_a(X) & \text{if } g_a(X)^{1/2} \geqslant Cr \\ 1 & \text{if } g_a(X)^{1/2} \leqslant Cr. \end{cases} \quad \text{and}$$

$$A \stackrel{\text{def}}{=} \big\{(\nu,\nu',\mu,\mu') \,/\, g_a(Y_\nu - \Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{1/2} \leqslant Cr \quad\text{and}\quad g_a(Y_{\nu'} - \Psi^{(2)}_\Lambda(\tau_J,Y_{\mu'}))^{1/2} \leqslant Cr \quad\text{and}\quad g_a(\check{Y}_\nu - Y_{\nu'})^{1/2} \leqslant Cr\big\}.$$

Thanks to Inequality (36) of Lemma 4.2, and thanks to the fact that the point (X_ν) are the points of the lattice $\mathcal{Z}$ defined in (43), the number of indices ν such that

$$g_a(Y_\nu - \Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{1/2} \leqslant Cr$$

is finite and independent of the interval J. So plugging the estimates (47) and (48) together, we get, if $(\nu,\nu',\mu,\mu')\notin A$,

$$\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)} \leqslant K_N(\nu,\nu',\mu,\mu')\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2} \quad \text{with}$$

$$K_N(\nu,\nu',\mu,\mu') \stackrel{\text{def}}{=} C_N\Delta_{g_a}(Y_\nu-\Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{-d-1}\Delta_{g_a}(Y_{\nu'}-\Psi^{(2)}_\Lambda(\tau_J,Y_{\mu'}))^{-d-1} \times\Delta_{g_a}(\check{Y}_\nu-Y_{\nu'})^{-d-1}.$$

But we have that

$$\sup_X\sum_\nu\Delta_{g_a}(X-Y_\nu)^{-d-1}<\infty.$$

Applying Schur's lemma and then Lemma 5.2, it turns out that

$$\begin{aligned}\Big\|\sum_{(\nu,\nu',\mu,\mu')\notin A}\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\Big\|_{L^1_J(L^\infty)} &\leqslant \sum_{(\nu,\nu',\mu,\mu')\notin A}\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)}\\ &\leqslant C_N\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\Big(\sum_\mu\|\psi^D_\mu\gamma_1\|^2_{L^2}\Big)^{1/2}\Big(\sum_\mu\|\psi^D_\mu\gamma_2\|^2_{L^2}\Big)^{1/2}\\ &\leqslant C_N\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.\end{aligned}$$

Now let us state

$$\underline{\mathcal{J}}\stackrel{\text{def}}{=}\sum_{(\nu,\nu',\mu,\mu')\in A}\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}$$

and check that it satisfies the conclusions of the lemma. Let us define

$$\begin{aligned}B^{(j)}_\mu &\stackrel{\text{def}}{=}\big\{\nu\,/\,g_a\big(Y_\nu-\Psi^{(j)}_\Lambda(\tau_J,Y_\mu)\big)^{1/2}\leqslant Cr\big\},\\ C_\mu &\stackrel{\text{def}}{=}\big\{\nu'\,/\exists\nu\in B^{(1)}_\mu\,/g_a(\check{Y}_\nu-Y_{\nu'})^{1/2}\leqslant Cr\big\},\\ A_\mu &\stackrel{\text{def}}{=}\big\{\mu',/\exists\nu'\in C_\mu\cap B^{(2)}_{\mu'}\big\}.\end{aligned}$$

Let us notice that, thanks to Inequality (36) of Lemma 4.2, we have

$$A_\mu\subset\big\{\mu'\,/\,g_a\big(\Psi^{(1)}_\Lambda(\tau_J,Y_\mu)-\check{\Psi}^{(2)}_\Lambda(\tau_J,Y_{\mu'})\big)^{1/2}\leqslant Cr\big\}.$$

Now let us state

$$\phi^D_\mu\stackrel{\text{def}}{=}\sum_{\nu\in B^{(1)}_\mu}\varphi^D_\nu\psi^D_\nu \quad\text{and}\quad \theta^D_\mu\stackrel{\text{def}}{=}\sum_{\nu'\in C_\mu}\varphi^D_{\nu'}\psi^D_{\nu'}.$$

We apply Lemma 5.3 to conclude the proof.

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