Anders Melin

Johannes Sjöstrand

Bohr-Sommerfeld quantization condition for non-selfadjoint operators in dimension 2

Astérisque, tome 284 (2003), p. 181-244

<http://www.numdam.org/item?id=AST_2003__284__181_0>

Astérisque
284, 2003, p. 181–244

BOHR-SOMMERFELD QUANTIZATION CONDITION FOR NON-SELFADJOINT OPERATORS IN DIMENSION 2

by

Anders Melin & Johannes Sjöstrand

Abstract. — For a class of non-selfadjoint h-pseudodifferential operators in dimension 2, we determine all eigenvalues in an h-independent domain in the complex plane and show that they are given by a Bohr–Sommerfeld quantization condition. No complete integrability is assumed, and as a geometrical step in our proof, we get a KAM–type theorem (without small divisors) in the complex domain.

Résumé **(Condition de quantification de Bohr-Sommerfeld pour des opérateurs non-autoadjoints en dimension 2)**
Pour une classe d'opérateurs h-pseudodifférentiels non-autoadjoints, nous déterminons toutes les valeurs propres dans un domaine complexe indépendant de h et nous montrons que ces valeurs propres sont données par une condition de quantification de Bohr-Sommerfeld. Aucune condition d'integrabilité complète est supposée, et une étape géométrique de la démonstration est donnée par un théoreme du type KAM dans le complexe (sans petits dénominateurs).

0. Introduction

In [**MeSj**] we developed a variational approach for estimating determinants of pseudodifferential operators in the semiclassical setting, and we obtained many results and estimates of some aesthetical and philosophical value. The original purpose of the present work was to continue the study in a somewhat more special situation (see [**MeSj**], section 8) and show in that case, that our methods can lead to optimal results. This attempt turned out to be successful, but at the same time the results below are of independent interest, so the relation to the preceding work, will only be hinted upon here and there.

Let $p(x,\xi)$ be bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4 = \mathbf{C}^2_x \times \mathbf{C}^2_\xi$. (The assumptions near ∞ will be of importance only in the quantized case,

2000 ***Mathematics Subject Classification.*** — 31C10, 35P05, 37J40, 37K05, 47J20, 58J52.
Key words and phrases. — Bohr, Sommerfeld, eigenvalue, torus, Cauchy-Riemann equation.

and can then be be varied in many ways.) Assume that

(0.1) $$\mathbf{R}^4 \cap p^{-1}(0) \neq \varnothing \text{ is connected,}$$

(0.2) $$\text{on } \mathbf{R}^4 \text{ we have } |p(x,\xi)| \geqslant \frac{1}{C}, \text{ for } |(x,\xi)| \geqslant C,$$

for some $C > 0$,

(0.3) $d\,\mathrm{Re}\,p(x,\xi)$, $d\,\mathrm{Im}\,p(x,\xi)$ are linearly independent for all $(x,\xi) \in p^{-1}(0) \cap \mathbf{R}^4$.

It follows that $p^{-1}(0) \cap \mathbf{R}^4$ is a compact (2-dimensional) surface. Also assume that

(0.4) $$|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}| \text{ is sufficiently small on } p^{-1}(0) \cap \mathbf{R}^4.$$

Here

$$\{a,b\} = \sum_1^2 \Big(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}\Big) = H_a(b)$$

is the Poisson bracket, and we adopt the following convention: We assume that p varies in some set of functions that are uniformly bounded in some fixed tube as above and satisfy (0.2), (0.3) uniformly. Then we require $|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}|$ to be bounded on $p^{-1}(0) \cap \mathbf{R}^4$ by some constant > 0 which only depends on the class.

If we strengthen (0.4) to requiring that $\{\mathrm{Re}\,p, \mathrm{Im}\,p\} = 0$ on $p^{-1}(0) \cap \mathbf{R}^4$, then the latter manifold becomes Lagrangian and will carry a complex elliptic vector field $H_p = H_{\mathrm{Re}\,p} + iH_{\mathrm{Im}\,p}$. It is then a well-known topological fact (and reviewed from the point of view of analysis in appendix B of section 1) that $p^{-1}(0) \cap \mathbf{R}^4$ is (diffeomorphic to) a torus. If we only assume (0.1)–(0.4), then H_p is close to being tangent to $p^{-1}(0) \cap \mathbf{R}^4$ and the orthogonal projection of this vector field to $p^{-1}(0) \cap \mathbf{R}^4$ is still elliptic. So in this case, we have still a torus, which in general is no more Lagrangian.

In section 1 we will establish the following result:

Theorem 0.1. — *There exists a smooth 2-dimensional torus $\Gamma \subset \mathbf{C}^4$, close to $p^{-1}(0) \cap \mathbf{R}^4$ such that $\sigma_{|_\Gamma} = 0$ and $I_j(\Gamma) \in \mathbf{R}$, $j = 1,2$. Here $I_j(\Gamma) = \int_{\gamma_j} \xi \cdot dx$ are the actions along the two fundamental cycles $\gamma_1, \gamma_2 \subset \Gamma$, and $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ is the complex symplectic $(2,0)$-form.*

If we form

$$L = \{\exp \widehat{tH_p}(\rho);\ \rho \in \Gamma,\ t \in \mathbf{C},\ |t| < 1/C\},$$

where $\widehat{tH_p} = tH_p + \overline{tH_p}$ is the real vector field associated to tH_p, then, as we shall see, L is a complex Lagrangian manifold $\subset p^{-1}(0)$ and L will be uniquely determined near $p^{-1}(0) \cap \mathbf{R}^4$ contrary to Γ. As a matter of fact, we will show that there is a smooth family of 2-dimensional torii $\Gamma_a \subset p^{-1}(0)$ with $\sigma_{|_{\Gamma_a}} = 0$, depending on a complex parameter a, such that the corresponding L_a form a holomorphic foliation of $p^{-1}(0)$ near $p^{-1}(0) \cap \mathbf{R}^4$. The L_a depend holomorphically on a and so do the corresponding actions $I_j(\Gamma_a)$. We can even take one of the actions to be our complex parameter a.

It then turns out that $\mathrm{Im}\,\frac{dI_2}{dI_1} \neq 0$, and this implies the existence of a unique value of a for which $I_j(\Gamma_a) \in \mathbf{R}$ for $j = 1, 2$.

Theorem 0.1 can be viewed as a complex version of the KAM theorem, in a case where no small denominators are present. As pointed out to us by D. Bambusi and S. Graffi, the absence of small divisors for certain dynamical systems in the complex has been exploited by Moser [**Mo**], Bazzani–Turchetti [**BaTu**] and by Marmi–Yoccoz.

The proof we give in section 1 finally became rather simple. Using special real symplectic coordinates, we reduce the construction of the Γ_a to that of multivalued functions with single-valued gradient (from now on grad-periodic functions) on a torus, that satisfy a certain Hamilton-Jacobi equation. In suitable coordinates, this becomes a Cauchy-Riemann equation with small non-linearity and can be solved in non-integer C^m-spaces by means of a straight-forward iteration.

The fact that $I_j(\Gamma) \in \mathbf{R}$ implies that there exists an IR-manifold $\Lambda \subset \mathbf{C}^4$ (i.e. a smooth manifold for which $\sigma_{|\Lambda}$ is real and non-degenerate) which is close to $\mathbf{R}^4$ and contains Γ. The reality of the actions $I_j(\Gamma)$ is an obvious necessary condition and the sufficiency will be established in section 1. When $p(x,\xi) \to 1$ sufficiently fast at ∞, Λ will be a critical point of the functional

$$\Lambda \longmapsto I(\Lambda) := \int_\Lambda \log|p(x,\xi)|\mu(d(x,\xi)), \tag{0.5}$$

where μ is the symplectic volume element on Λ. This was discussed in [**MeSj**] and in section 8 of that paper we also discussed the linearized problem corresponding to finding such a critical point. The reason for studying the functional (0.5) is that $I(\Lambda)$ enters in a general asymptotic upper bound on the determinant of an h-pseudodifferential operator with symbol p. Our quantum result below implies that this bound is essentially optimal.

Now let $p(x,\xi,z)$ be a uniformly bounded family of functions as above, depending holomorphically on a parameter $z \in \mathrm{neigh}\,(0,\mathbf{C})$ (some neighborhood of 0 in $\mathbf{C}$). Let $P(z) = p^w(x,hD,z)$ be the corresponding h-Weyl quantization of p, given by

$$p^w(x,hD,z)u(x) = \frac{1}{(2\pi h)^2}\iint e^{\frac{i}{h}(x-y)\cdot\theta}p\Big(\frac{x+y}{2},\theta,z\Big)u(y)dyd\theta. \tag{0.6}$$

It is well known (see for instance [**DiSj**]) that $P(z)$ is bounded: $L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, uniformly with respect to (z,h). Moreover, the ellipticity near infinity, imposed by (0.2), implies that it is a Fredholm operator (of index 0 as will follow from the contructions below). Let us say that z is an eigen-value if $p^w(x,hD,z)$ is not bijective. The main result of our work is that the eigen-values are given by a Bohr-Sommerfeld quantization condition. We here state a shortened version (of Theorem 6.3). Let $I(z) = (I_1(z), I_2(z))$, where $I_j(z) = I_j(\Gamma(z)) \in \mathbf{R}$ and $\Gamma(z) \subset p^{-1}(0,z)$ is given by Theorem 0.1. $I(z)$ depends smoothly on z, since $\Gamma(z)$ can be chosen with smooth z-dependence.

Theorem 0.2. — *Under the above assumptions, there exists* $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ *and* $\theta(z;h) \sim \theta_0+\theta_1(z)h+\theta_2(z)h^2+\cdots$ *in* $C^\infty(\mathrm{neigh}\,(0,\mathbf{C});\mathbf{R}^2)$, *such that for* z *in an* h*-independent neighborhood of* 0 *and for* $h>0$ *sufficiently small, we have*
1) z *is and eigen-value iff we have*

$$\text{(BS)} \qquad \frac{I(z)}{2\pi h} = k-\theta(z;h), \ \textit{for some } k\in\mathbf{Z}^2.$$

2) When I *is a local diffeomorphism, then the eigen-values are simple (in a natural sense) and form a distorted lattice.*

Classically, the Bohr-Sommerfeld quantization condition describes the eigen-values of self-adjoint operators in dimension 1. See for instance [**HeRo**], [**GrSj**] exercise 12.3. In higher dimension Bohr-Sommerfeld conditions can still be used in the (quantum) completely integrable case for self-adjoint operators and can give all eigen-values in some interval independent of h. See for instance [**Vu**] and further references given there. This case is also intimately related to the development of Fourier integral operator theory in the version of Maslov's canonical operator theory, [**Mas**].

When dropping the integrability condition, one can still justify the BS condition and get families of eigen-values for self-adjoint operators by using quantum and classical Birkhoff normal forms, sometimes in combination with the KAM theorem, but to the authors' knowledge, no result so far describes all the eigen-values in some h-independent non-trivial interval in the self-adjoint case. See Lazutkin [**La**], Colin de Verdière [**Co**], [**Sj4**], Bambusi–Graffi–Paul [**BaGrPa**] Kaidi-Kerdelhué [**KaKe**], Popov [**Po1, Po2**]. It therefore first seems that Theorem 0.2 (6.3) is remarkable in that it describes all eigen-values in an h-independent domain and that the non-self-adjoint case (for once!) is easier to handle than the self-adjoint one. The following philosophical remark will perhaps make our result seem more natural: In dimension 1, the BS-condition gives a sequence of eigen-values that are separated by a distance $\sim h$. In higher dimension $n \geqslant 2$, this cannot hold in the self-adjoint case, since an h-independent interval will typically contain $\sim h^{-n}$ eigen-values by Weyl asymptotics, so the average separation between eigen-values is $\sim h^n$. In dimension 2 however, we can get a separation of $\sim h$ between neighboring eigen-values for non-self-adjoint operators, since the number of eigen-values in some bounded open h-independent complex domain can be bounded from above by $\mathcal{O}(h^{-2})$ by general methods.

In section 7, we study resonances of a Schrödinger operator, generated by a saddle point of the potential and apply Theorem 6.3 and its proof. In this case, the resonances in a disc of radius Ch around the corresponding critical value of the potential were determined in [**Sj2**] for every fixed $C>0$, and this result was extended by Kaidi–Kerdelhué [**KaKe**] to a description of all resonances in a disc of radius h^δ, with $\delta>0$ arbitrary but independent of h. We show that the description of [**KaKe**] extends to give all resonances in some h-independent domain.

To prove Theorem 6.3, we use the machinery of FBI (here Bargman-) transformations and the corresponding calculus of pseudodifferential operators and Fourier integral operators on weighted L^2-spaces of holomorphic functions (see [**Sj1, Sj3**], [**HeSj**], [**MeSj**]). This allows us to define spaces $H(\Lambda)$ when Λ is an IR-manifold close to $\mathbf{R}^4$ in such a way that $H(\mathbf{R}^4)$ becomes the usual $L^2(\mathbf{R}^2)$ with the usual norm. Viewing p^w as an operator: $H(\Lambda) \to H(\Lambda)$, the corresponding leading symbol becomes $p_{|\Lambda}$. We apply this to the IR-manifold $\Lambda(z)$ which contains $\Gamma(z)$ and get a reduction to the case when the characteristics of p (in $\Lambda(z)$) is a Lagrangian torus.

Contents of the paper

Acknowledgements. — We thank M. Zworski, S. Graffi, D. Bambusi, M. Rouleux, R. Perez-Marco and L. Stolovitch for useful discussions and informations.

1. Construction of complex Lagrangian torii in $p^{-1}(0)$ in dimension 2

We shall work in $\mathbf{R}^4 = T^*\mathbf{R}^2$ and its complexification $\mathbf{C}^4$, equipped with the standard symplectic form $\sigma = \sum_{j=1}^2 d\xi_j \wedge dx_j$. Let $\Gamma \subset \mathbf{R}^4$ be a smooth two-dimensional manifold, and assume that there exist real-valued real-analytic functions p_1 and p_2 defined in some tubular real neighborhood of Γ, which vanish on Γ and have linearly independent differentials at every point of Γ. We shall assume that

$$\sigma_{|\Gamma} \text{ is small,} \tag{1.1}$$

in the sense that $|\langle \sigma, t \wedge s\rangle| \leqslant \varepsilon$ for all $\rho \in \Gamma$ and all $t, s \in T_\rho(\Gamma)$ with $|t|, |s| \leqslant 1$, where $\varepsilon > 0$ is sufficiently small. Here we use the standard norm on $\mathbf{R}^4$. It is tacitly assumed that nothing else degenerates when ε tends to 0; the tubular neighborhood is independent of ε, and p_j and all their derivatives satisfy uniform bounds there. Moreover

$|p_1| + |p_2|$ is bounded from below by a strictly positive constant near the boundary of the tubular neighborhood and we have a fixed positive lower bound on $|\lambda_1 dp_1 + \lambda_2 dp_2|$ uniformly in λ_1, λ_2 with $|\lambda_1|^2 + |\lambda_2|^2 = 1$. Under these additional uniformity assumptions, (1.1) is equivalent to saying that the Poisson bracket $\{p_1, p_2\} = \langle \sigma, H_{p_1} \wedge H_{p_2} \rangle$ is small ($\mathcal{O}(\varepsilon)$) on Γ. Indeed, if $\rho \in \Gamma$, then the symplectic orthogonal space to $T_\rho\Gamma$ is the space spanned by H_{p_1}, H_{p_2} and to say that the Poisson bracket is very small is equivalent to saying that the tangent space and its symplectic orthogonal are close to each other. (Alternatively, we may notice that there is a new symplectic form σ_ε in a tubular neighborhood of Γ with $\sigma_\varepsilon - \sigma = \mathcal{O}(\varepsilon)$, $\sigma_{\varepsilon|_\Gamma} = 0$.) In what follows we extend p_1 and p_2 to holomorphic functions in a complex neighborhood of Γ and complexify Γ (the complexification is sometimes denoted $\Gamma_{\mathbf{C}}$). Then $\sigma_{|\Gamma_{\mathbf{C}}} = \mathcal{O}(\varepsilon)$ in a full complex neighborhood of the original real manifold and with a new ε that we can take equal to the square root of the previous one. Since the complex vector field $H_p = H_{p_1} + iH_{p_2}$ is close to be tangent to Γ and H_{p_1}, H_{p_2} are linearly independent, it can be projected to an elliptic vector field on Γ. It is then a well-known fact (that we recall in Appendix B) that Γ is (diffeomorphic to) a torus.

We shall say that a multi-valued smooth function is grad-periodic if its differential is single-valued. Let x_1, x_2 be grad-periodic, real and real-analytic on Γ such that (x_1, x_2) induces an identification between the original torus and $\mathbf{R}^2/L$ for some lattice $L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$. Extend x_1 to a real-analytic, grad-periodic (and real-valued) function in a tubular neighborhood of Γ in $\mathbf{R}^4$ in such a way that dx_1 vanishes on the orthogonal plane of $T_\rho\Gamma$ (w.r.t. the standard scalar product on $\mathbf{R}^4$) at every point $\rho \in \Gamma$. (We could even get a unique extension by requiring that x_1 be constant on the sets L_ρ of points in the (small) tubular neighborhood, which are closer to $\rho \in \Gamma$ than to any other point in Γ.) If $\sigma_{|\Gamma}$ is sufficiently small, then $|H_{p_1}x_1| + |H_{p_2}x_1| \neq 0$, so H_{x_1} is transversal to Γ. Let $H \subset \mathbf{R}^4$ be a real-analytic closed hypersurface in a tubular neighborhood of Γ which contains Γ and is everywhere transversal to H_{x_1}. Extend x_2 real-analytically first to a grad periodic function on H, and then to a full tubular neighborhood in $\mathbf{R}^4$, by requiring that

$$\{x_1, x_2\} = 0. \tag{1.2}$$

We further extend x_1 and x_2 to grad-periodic holomorphic functions in a complex neighborhood of Γ. This will allow us to identify $\Gamma_{\mathbf{C}}$ with a complex neighborhood in $\mathbf{C}^2/L$ of $\mathbf{R}^2/L$. We notice that $\sigma_{|\Gamma_{\mathbf{C}}} = f(x)dx_1 \wedge dx_2$, where $f(x) = \mathcal{O}(\varepsilon)$ and f is holomorphic in a full complex neighborhood of $\mathbf{R}^2/L$ in $\mathbf{C}^2/L$. Since σ is exact and $\mathcal{O}(\varepsilon)$ when restricted to Γ, there are real-analytic functions γ_1 and γ_2 on Γ, with values in $\mathbf{R}$ (hence single-valued) such that

$$\sigma_{|\Gamma} = d(\gamma_1 dx_1 + \gamma_2 dx_2), \quad \gamma_1, \gamma_2 = \mathcal{O}(\varepsilon), \tag{1.3}$$

in the C^∞-sense. Since the Hamilton fields H_{x_1} and H_{x_2} commute in view of (1.2) and Jacobi's identity and span a space transversal to Γ at every point of Γ, we may find real-valued and real-analytic functions ξ_1 and ξ_2 in a neighborhood of Γ in $\mathbf{R}^4$ such that

$$\xi_{j|_\Gamma} = \gamma_j, \; H_{x_j}\xi_k = -\delta_{jk}. \tag{1.4}$$

Proposition 1.1. — *(x,ξ) are symplectic coordinates for $\mathbf{R}^4$ in a neighborhood of Γ.*

Proof. — Locally we may find $(\widetilde{\xi_1},\widetilde{\xi_2})$ such that $(x,\widetilde{\xi})$ are symplectic coordinates. Since $H_{x_j}\widetilde{\xi_k} = -\delta_{jk} = H_{x_j}\xi_k$, it follows that $\xi_j - \widetilde{\xi_j} = g_j(x)$ is a function of x only. Then

$$\sum_1^2 d\xi_j \wedge dx_j - \sum_1^2 d\widetilde{\xi_j} \wedge dx_j = \sum_1^2 d(g_j(x)) \wedge dx_j. \tag{1.5}$$

Since the restriction to Γ of the left-hand side vanishes in view of (1.3) and (1.4) it follows that $\sum_1^2 d(g_j(x)) \wedge dx_j = 0$. Hence $\sum_1^2 d\xi_j \wedge dx_j = \sigma$. Since we know already that (x_1,x_2) is a coordinate system for Γ it follows that (x,ξ) is a coordinate system in a tubular neighborhood. $\square$

In the coordinates (x,ξ), Γ takes the form

$$\xi = \gamma(x), \; \gamma = \mathcal{O}(\varepsilon), \; x \in \mathbf{R}^2/L, \tag{1.6}$$

where we view γ also as an L-periodic function in $\mathbf{R}^2$. Considering $p = p_1 + ip_2$ as a function in the new coordinates we get

$$\begin{aligned} p(x,\xi) &= p_1(x,\xi) + ip_2(x,\xi) \\ &= \sum_1^2 a_j(x)(\xi_j - \gamma_j(x)) + \sum_{j,k} b_{j,k}(x,\xi)(\xi_j - \gamma_j(x))(\xi_k - \gamma_k(x)) \\ &= \sum_1^2 a_j(x)\xi_j + \mathcal{O}(|\xi - \gamma(x)|^2) - r(x), \quad r(x) = \sum a_j(x)\gamma_j(x) = \mathcal{O}(\varepsilon) \end{aligned} \tag{1.7}$$

in the sense of holomorphic functions in a fixed tubular complex neighborhood of $\mathbf{R}^2_x \times \{\xi = 0\}$. With this point of view p is L-periodic in x.

We look for torii Γ_ϕ in a complex neighborhood of Γ of the form

$$\Gamma_\phi : \xi = \phi'(x), \quad x \in \mathbf{R}^2/L, \tag{1.8}$$

where ϕ is complex-valued and grad-periodic with $\nabla\phi \in \mathbf{C}^m$ for some $0 < m \in \mathbf{R}\setminus\mathbf{N}$. We want $\Gamma_\phi \subset p^{-1}(0)$, so ϕ has to satisfy the Hamilton-Jacobi equation

$$p(x,\phi'(x)) = 0. \tag{1.9}$$

Using (1.7) we can write this as

$$Z\phi + F(x,\phi'(x) - \gamma(x)) - r(x) = 0, \tag{1.10}$$

where $Z = \sum_1^2 a_j(x)\frac{\partial}{\partial x_j}$ and $F(x,\xi) = \mathcal{O}(\xi^2)$, $r = \mathcal{O}(\varepsilon)$. Look for ϕ in the form

$$\phi = \widetilde{\varepsilon}\psi, \quad \varepsilon \ll \widetilde{\varepsilon} \ll 1. \tag{1.11}$$

Then ψ has to solve

$$Z\psi + \frac{1}{\widetilde{\varepsilon}}F(x, \widetilde{\varepsilon}(\psi' - \frac{\gamma}{\widetilde{\varepsilon}})) - \frac{r(x)}{\widetilde{\varepsilon}} = 0. \tag{1.12}$$

We look for solutions ψ with $\psi' = \mathcal{O}(1)$, and we rewrite (1.12) as

$$Z\psi + \widetilde{\varepsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} = 0, \tag{1.13}$$

where

$$G(x, \xi; \widetilde{\varepsilon}) = \frac{1}{\widetilde{\varepsilon}^2}F(x, \widetilde{\varepsilon}\xi) \tag{1.14}$$

is holomorphic and uniformly bounded with respect to $\widetilde{\varepsilon}$ when $|\operatorname{Im} x|, |\xi| = \mathcal{O}(1)$.

Changing the x-coordinates and L conveniently, we may (by applying Theorem B.6), assume that

$$Z = A(x)\frac{\partial}{\partial \overline{x}}, \quad x = x_1 + ix_2, \tag{1.15}$$

where A is real-analytic and non-vanishing. (We now view L as a lattice in $\mathbf{C}$.) After division by $A(x)$, (1.13) becomes

$$\frac{\partial \psi}{\partial \overline{x}} + \widetilde{\varepsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} = 0 \tag{1.16}$$

with new functions $G = G_{\text{new}}$, $r = r_{\text{new}}$, obtained from the earlier ones by division by $A(x)$ (and therefore satisfying the same estimates as before).

We look for solutions ψ of the form

$$\psi = \psi_{\text{per}} + ax + b\overline{x}, \tag{1.17}$$

where ψ_{per} is periodic with respect to L and $a, b \in \mathbf{C}$. We shall apply an iteration procedure and get a corresponding solution for each a in the unit disc $D(0,1)$. So, let $u(x) = \psi_{\text{per}} + b\overline{x}$ belong to the space of grad-periodic functions on $\mathbf{C}/L$ with antiholomorphic linear part. Then (1.16) becomes

$$\frac{\partial u}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} = 0, \tag{1.18}$$

where

$$G_a(x, \xi; \widetilde{\varepsilon}) = G(x, \xi + adx; \widetilde{\varepsilon}),$$

and dx denotes the complex cotangent vector given by the differential of x. Notice that G_a depends holomorphically on a.

Fix $m \in \mathbf{R}_+ \setminus \mathbf{N}$, and solve (1.18) for $u' \in C^m$ by the natural iteration procedure $u_0 = 0$,

$$\frac{\partial u_{j+1}}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} = 0, \quad j \geqslant 0. \tag{1.19}$$

Write $u_j(x) = u_{j,\mathrm{per}}(x) + b_j\overline{x}$. If u_j has already been determined, then considering the Fourier series expansion of $u_{j+1,\mathrm{per}}$, we see that

$$b_{j+1} = -\mathcal{F}(\widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}})(0), \tag{1.20}$$

where $\mathcal{F}v(0)$ denotes the 0:th Fourier coefficient of the function v with respect to L. We see that $u_{j+1,\mathrm{per}}$ is uniquely determined modulo a constant through the equation

$$\frac{\partial u_{j+1,\mathrm{per}}}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} + b_{j+1} = 0. \tag{1.21}$$

For $j = 0$, we get $b_1 = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}})$. Applying a basic result about the boundedness in $C^m(\mathbf{R}^2/L)$ of Calderon–Zygmund operators (see [**BeJoSc**]) and considering also Fourier expansions, we get the bound

$$\|u_{1,\mathrm{per}}'\|_{C^m} \leqslant \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}).$$

For $j \geqslant 1$, we write

$$b_{j+1} - b_j + \widetilde{\varepsilon}\mathcal{F}(G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - G_a(x, u_{j-1}' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}))(0) = 0 \tag{1.22}$$

and

$$\frac{\partial}{\partial \overline{x}}(u_{j+1,\mathrm{per}} - u_{j,\mathrm{per}}) + \widetilde{\varepsilon}(G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - G_a(x, u_{j-1}' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon})) + (b_{j+1} - b_j) = 0. \tag{1.23}$$

From (1.22) we get

$$|b_{j+1} - b_j| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{j,\mathrm{per}}' - u_{j-1,\mathrm{per}}'\|_{C^m} + |b_j - b_{j-1}|),$$

and using this in (1.23) together with (1.22), we get

$$\|u_{j+1,\mathrm{per}}' - u_{j,\mathrm{per}}'\|_{C^m} + |b_{j+1} - b_j| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{j,\mathrm{per}}' - u_{j-1,\mathrm{per}}'\|_{C^m} + |b_j - b_{j-1}|). \tag{1.24}$$

So, if $\widetilde{\varepsilon}$ (and ε) is small enough, our procedure converges to a solution

$$u = u_{\mathrm{per}} + b\overline{x} \tag{1.25}$$

of (1.18) with

$$\|u_{\mathrm{per}}'\|_{C^m} + |b| = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}). \tag{1.26}$$

Summing up we have for a given m:

Proposition 1.2. — *Let $C \geqslant 1$ be large enough. For $0 < \varepsilon \ll \widetilde{\varepsilon}$ small enough and for $|a| < 1$, the equation (1.18) has a solution u of the form (1.25) with $|b| + \|u_{\mathrm{per}}'\|_{C^m} \leqslant 1/C$. This solution is unique modulo constants and satisfies (1.26).*

Proof of the uniqueness. — Let $u_{\mathrm{per}} + b\overline{x}$ and $\widetilde{u} = \widetilde{u}_{\mathrm{per}} + \widetilde{b}\overline{x}$ be two solutions of (1.18). Then as above, we have

$$\|u_{\mathrm{per}}' - \widetilde{u}_{\mathrm{per}}'\|_{C^m} + |b - \widetilde{b}| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{\mathrm{per}}' - \widetilde{u}_{\mathrm{per}}'\|_{C^m} + |b - \widetilde{b}|),$$

and the uniqueness follows. □

This means that we have solved (1.9) with

$$\phi = \widetilde{\varepsilon}(u_{\mathrm{per}} + ax + b\overline{x}), \quad 0 < \varepsilon \ll \widetilde{\varepsilon} \ll 1, \tag{1.27}$$

where $|a| < 1$, and b, u_{per} depend on the choice of a (and of $\widetilde{\varepsilon}$). In (1.27) it is further assumed that x_1, x_2 are chosen so that (1.15) holds.

We next show that ϕ' depends holomorphically on a, and for that we again consider (1.18), where we recall that G_a depends holomorphically on a. This is actually immediate because the preceding iteration argument trivially extends to the case of functions of a: $u = u_{\mathrm{per}}(x,a) + b(a)\overline{x}$, with

$$D(0,1) \ni a \longmapsto (u'_{\mathrm{per}}(\cdot\,,a), b(a)) \in C^m \times \mathbf{C} \tag{1.28}$$

holomorphic. Hence (after imposing the extra condition that $\mathcal{F}u_{\mathrm{per}}(0) = 0$) we have

Proposition 1.3. — *u_{per}, b and hence ϕ depend holomorphically on a.*

Now let $p = p_z$ depend holomorphically on a spectral parameter $z \in D(0,1)$ and assume that $p_z = \mathcal{O}(1)$ uniformly in some fixed tubular neighborhoood of $\mathbf{R}^4$. Assume that p_0 fulfills the assumptions of p above. Choose coordinates (x,ξ) as above for $p = p_0$. We now look for $\Gamma_\phi \subset p_z^{-1}(0)$ of the form (1.8), and (1.10) becomes:

$$Z\phi + F(x, \phi'(x) - \gamma(x); z) - r(x,z) = 0, \tag{1.29}$$

where $F(x,\xi;z)$, $r(x,z)$ depend holomorphically on z. If we restrict the attention to $|z| < \mathcal{O}(\widetilde{\varepsilon})$, then the previous considerations go through and we get a solution

$$\phi = \phi_a = \phi_{a,z}(x) = \widetilde{\varepsilon}(u_{\mathrm{per}}(x,z,a) + ax + b(z,a)\overline{x}) \tag{1.30}$$

depending holomorphically on z, a with $|z| < \frac{\widetilde{\varepsilon}}{C}$, $|a| < \frac{1}{C}$, and

$$\|u'_{\mathrm{per}}(\cdot\,,z,a)\|_{C^m} + |b| = \mathcal{O}(1). \tag{1.31}$$

We shall now extend ϕ to the complex domain in x. Let $\widetilde{\phi}(x) \in C^{k+1}(\mathbf{C}^2)$ denote an almost holomorphic extension of ϕ, where k is a positive integer and m has been chosen larger than k. (Here we consider $\widetilde{\phi}$ as a grad-periodic function in $\mathbf{R}^4$.) Then $p(x, \partial_x\widetilde{\phi}(x))$ vanishes to the order k on $\mathbf{R}^2$, and the corresponding manifold $\Lambda_{\widetilde{\phi}} = \{(x, \partial_x\widetilde{\phi}(x));\ x \in \mathbf{C}^2\}$ is to that order a complex Lagrangian manifold at the points of intersection with $\mathbf{R}^2_x \times \mathbf{C}^2_\xi$. This intersection is nothing else but Γ_ϕ in (1.8).

The complex Hamilton field H_p is transversal to $\mathbf{R}^2_x \times \mathbf{C}^2_\xi$ at the points of Γ_ϕ and we form the flow out

$$\Lambda_\phi = \{\exp \widehat{tH_p}(\rho);\ \rho \in \Gamma_\phi,\ t \in \mathbf{C},\ |t| < 1/\mathcal{O}(1)\}. \tag{1.32}$$

Here $\widehat{tH_p} = tH_p + \overline{tH_p}$ is the real vectorfield (in the complex domain) which has the same action as tH_p as differential operators acting on holomorphic functions. Since $\widehat{tH_p}$ is tangential to $\Lambda_{\widetilde{\phi}}$ to the order k at Γ_ϕ, we see that Λ_ϕ is tangential to $\Lambda_{\widetilde{\phi}}$ there. In particular $T_\rho\Lambda_\phi$ is a complex Lagrangian space for every $\rho \in \Gamma_\phi$. Since $\exp \widehat{tH_p}$ are complex canonical transformations, the same fact is true for the tangent spaces

$T_{\exp t\widehat{H_p}(\rho)}\Lambda_\phi = (\exp t\widehat{H_p})_* T_\rho \Lambda_\phi$ at an arbitrary point $\exp t\widehat{H_p}(\rho) \in \Lambda_\phi$. Hence Λ_ϕ is a complex Lagrangian manifold. Restricting the size of $|t|$ in (1.32) we see also that the projection $\Lambda_\phi \ni (x,\xi) \mapsto x$ is a holomorphic diffeomorphism, so Λ_ϕ is of the form $\{\xi = \phi'(x);\ |\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}\}$ for a function ϕ which is a holomorphic extension of the previously constructed one.

Let $\Lambda \subset p^{-1}(0)$ be a relatively closed complex Lagrangian manifold in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$ and assume that Λ contains a torus Γ which is $\widehat{\varepsilon}$-close to $p^{-1}(0) \cap \mathbf{R}^4$ in C^1, for $\varepsilon \ll \widehat{\varepsilon} \ll 1$. Let (x,ξ) be the coordinates constructed above. If $\rho \in \Gamma$, we know that $T_\rho\Gamma$ is $\widehat{\varepsilon}$-close to $\mathbf{R}^2_x \times \{\xi = 0\}$, so $T_\rho\Lambda$ is $\widehat{\varepsilon}$-close to $\mathbf{C}^2_x \times \{\xi = 0\}$. Using that Λ is locally invariant under the $\widehat{\mathbf{C}H_p}$-flow, we see that Λ is of the form $\{(x,\phi'(x));\ |\mathrm{Im}\, x| < \frac{1}{C}\}$ in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$, where ϕ is holomorphic and grad-periodic with $\phi' = \mathcal{O}(\widetilde{\varepsilon})$, $\widetilde{\varepsilon} = \mathcal{O}(\widehat{\varepsilon}^{1/2})$. Moreover, we have the eiconal equation $p(x,\phi'(x)) = 0$ and restricting it to $\mathbf{R}^2$, we get (1.9), and Proposition 1.2 shows that $\phi = \phi_a$ for some a. Hence in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$, Λ coincides with Λ_ϕ in (1.32).

The parameter dependence of ϕ in (1.27) behaves as expected: Clearly the holomorphic extension $\phi(x,a,z)$ depends in a C^1-fashion of a (and possibly z), and we know that it is holomorphic in a and z when x is real. Then $\frac{\partial\phi}{\partial\overline{a}}$, $\frac{\partial\phi}{\partial\overline{z}}$ are holomorphic in x and vanish for real x. Consequently they vanish for all x. Summing up we have shown:

Proposition 1.4. — *The function ϕ in (1.27) depends holomorphically on (x,a,z) in a domain*

$$|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)},\ |\alpha| < \frac{1}{C},\ |z| < \frac{\varepsilon}{\mathcal{O}(1)}.$$

We shall next show (in the z-independent case) that the Λ_{ϕ_a} form a complex fibration of $p^{-1}(0)$ in a region where $|\xi| < \frac{\widetilde{\varepsilon}}{\mathcal{O}(1)}$, $|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}$. Let first x be real. From Propositions 1.2 and 1.3, we see that

$$\frac{\partial}{\partial a}u'_{\mathrm{per}},\ \frac{\partial}{\partial a}b = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}), \tag{1.33}$$

and consequently for ϕ in (1.27), we get for the x-differential $\phi'_x = d_x\phi$:

$$\frac{\partial}{\partial a}\phi'_x = \widetilde{\varepsilon}dx + \mathcal{O}(\varepsilon + \widetilde{\varepsilon}^2). \tag{1.34}$$

In order to treat the case of complex x, we notice that the geometric arguments leading to Proposition 1.4 together with the form $\sum a_j(x)\xi_j + \widetilde{\varepsilon}^{-1}F(x, \xi - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - r(x)/\widetilde{\varepsilon}$ for the Hamiltonian for ψ, show that $u'_{\mathrm{per}} = \mathcal{O}(\widetilde{\varepsilon} + \varepsilon/\widetilde{\varepsilon})$ also in the complex domain, and hence by the Cauchy inequality (in a) that (1.34) holds also for $|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}$. This shows that

$$a \longmapsto \phi'_x \in (p(x,\cdot))^{-1}(0) \tag{1.35}$$

is a local diffeomorphism and hence that the Λ_{ϕ_a} form a foliation of $p^{-1}(0)\cap\{(x,\xi):|\xi|<\frac{\widetilde{\varepsilon}}{\mathcal{O}(1)},\ |\mathrm{Im}\,x|<\frac{1}{\mathcal{O}(1)}\}$ in the natural sense. (Recall that $\widetilde{\varepsilon}$ can be close to a fixed constant so we get a foliation in $\{(x,\xi):|\xi|<\frac{1}{\mathcal{O}(1)},\ |\mathrm{Im}\,x|<\frac{1}{\mathcal{O}(1)}\}$.)

We next consider the actions associated to a torus. Let $\gamma_j(a)$ be a closed curve in Γ_{ϕ_a} (assuming $\widetilde{\varepsilon}>0$ fixed) which corresponds to e_j in the natural way, where we recall that $Z=A(x)\frac{\partial}{\partial\overline{x}}$, $x\in\mathbf{C}/L$, and take $L=\mathbf{Z}e_1\oplus\mathbf{Z}e_2$. If ω is a $(1,0)$-form with holomorphic coefficients, such that $d\omega=\sigma$ near Γ_{ϕ_a}, then we define the actions

$$I_j(\Gamma_{\phi_a},\omega)=\int_{\gamma_j(a)}\omega. \tag{1.36}$$

These only depend on the homotopy class of $\gamma_j(a)$ in Γ_{ϕ_a}, and we can even deform $\gamma_j(a)$ from this set into the complex, provided that we stay inside the complex Lagrangian manifold Λ_{ϕ_a}. Also notice that if $\widetilde{\omega}$ is another $(1,0)$-form with the same properties, then

$$\int_\gamma\omega-\int_\gamma\widetilde{\omega}$$

only depends on the homotopy class of γ as a closed curve in the intersection of the domains of definition of ω and $\widetilde{\omega}$. In particular,

$$I_j(\Gamma_{\phi_a},\omega)-I_j(\Gamma_{\phi_a},\widetilde{\omega})=C_j \tag{1.37}$$

is a constant which is independent of a (and of z if we let p depend holomorphically on z). If ω and $\widetilde{\omega}$ are both real in the real domain then C_j in (1.37) is real.

For the special x-coordinates above, we let ξ be the corresponding coordinates constructed in the beginning of this section and we choose

$$\widetilde{\omega}=\sum_1^2\xi_j dx_j. \tag{1.38}$$

Then

$$I_j(\Gamma_{\phi_a},\widetilde{\omega})=\phi_a(e_j)-\phi_a(0)$$

depends holomorphically on a, and from (1.11), (1.17) and Proposition 1.2 we get

$$I_j(\Gamma_{\phi_a},\widetilde{\omega})=\widetilde{\varepsilon}(ae_j+b\overline{e_j})=\widetilde{\varepsilon}ae_j+\mathcal{O}(\varepsilon+\widetilde{\varepsilon}^2). \tag{1.39}$$

For ω we can choose the fundamental 1-form in the original coordinates on $\mathbf{R}^4$ (formally given by the right-hand side of (1.38) for these original coordinates (x,ξ)). Thus

$$I_j(\Gamma_{\phi_a},\omega)=C_j+\widetilde{\varepsilon}ae_j+\mathcal{O}(\varepsilon+\widetilde{\varepsilon}^2). \tag{1.40}$$

From this we see that we can use, say, $I_1(\Gamma_{\phi_a},\omega)\in C_j+D(0,\widetilde{\varepsilon}/\mathcal{O}(1))$ as a new holomorphic parameter instead of a. In the z-dependent case, we can replace the parameters (a,z) by $(I_1,z)=(I_1(\Gamma_{\phi_a},\omega),z)$ and the correspondence $(a,z)\mapsto(I_1,z)$ is biholomorphic.

The advantage of using I_1 instead of a as a parameter, is that the family Λ_{ϕ_a} is now independent of the way we choose the coordinates (x,ξ) in the beginning of this section, so we get an intrinsic parametrisation. From (1.40) it follows that

$$\frac{dI_2}{dI_1}=\frac{e_2}{e_1}+\mathcal{O}(\widetilde{\varepsilon}+\varepsilon/\widetilde{\varepsilon}), \tag{1.41}$$

so $\operatorname{Im}\frac{dI_2}{dI_1}\neq 0$, and we have a unique value $a=\mathcal{O}(\widetilde{\varepsilon}+\varepsilon/\widetilde{\varepsilon})$ for which I_1 and I_2 are both real.

There are two related reasons why we want to select Γ_{ϕ_a}, with both I_1 and I_2 real. The first reason is geometric: Γ_{ϕ_a} is a small deformation of a real torus $\Gamma\subset\mathbf{R}^4$ and we want to find an I-Lagrangian manifold $\Lambda\subset\mathbf{C}^4$ which is a small deformation of $\mathbf{R}^4$ and which contains Γ_{ϕ_a}. If we have such a Λ, the cycles $\gamma_j(\Gamma_{\phi_a})$, $j=1,2$ become boundaries of some 2-dimensional discs $D_j\subset\Lambda$ and we get

$$I_j(\Gamma_{\phi_a},\omega)=\int_{\gamma_j}\omega=\int_{D_j}\sigma\in\mathbf{R},$$

since $\sigma_{|\Lambda}$ is real.

Conversely, let $\widetilde{\Gamma}$ be a two-dimensional torus which is a small perturbation of Γ with

$$\sigma_{|\widetilde{\Gamma}}=0,\ \operatorname{Im}I_j(\widetilde{\Gamma},\omega)=0,\ j=1,2. \tag{1.42}$$

We can construct an I-Lagrangian manifold $\Lambda\supset\widetilde{\Gamma}$ as a small perturbation of $\mathbf{R}^4$ in the following way: After applying a complex linear canonical transformation, we may replace $\mathbf{R}^4$ by Λ_{Φ_0}: $\xi=\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)$, $x\in\mathbf{C}^2$, where Φ_0 is a strictly plurisubharmonic quadratic form (see [**Sj1, Sj3**]), so that $\widetilde{\Gamma}$ becomes a small perturbation of a torus $\Gamma\subset\Lambda_{\Phi_0}$. The canonical 1-form ω is now transformed into some other globally defined 1-form $\widetilde{\omega}$ with holomorphic coefficients satisfying $d\widetilde{\omega}=\sigma$, but the actions $I_j(\widetilde{\Gamma},\widetilde{\omega})$ do not change if we replace $\widetilde{\omega}$ by $\xi\cdot dx$, so

$$\int_{\gamma_j(\widetilde{\Gamma})}\xi\cdot dx\in\mathbf{R},\ j=1,2. \tag{1.43}$$

We can write this as

$$\int_{\gamma_j(\widetilde{\Gamma})}(-\operatorname{Im}(\xi\cdot dx))=0, \tag{1.44}$$

where $-\operatorname{Im}\xi\cdot dx$ is a primitive of $-\operatorname{Im}\sigma$, so $-\operatorname{Im}\xi\cdot dx_{|\widetilde{\Gamma}}$ is closed. (1.44) then implies that it is exact:

$$-\operatorname{Im}\xi\cdot dx_{|\widetilde{\Gamma}}=d\phi, \tag{1.45}$$

where ϕ is a smooth real-valued function on $\widetilde{\Gamma}$. We now view ϕ as a function on the x-space projection $\pi_x(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$, which is also a smooth torus and represent $\widetilde{\Gamma}$ by $\xi=\widetilde{\xi}(x)$, $x\in\pi_x(\widetilde{\Gamma})$. Then with the obvious identifications, (1.45) becomes

$$-\operatorname{Im}(\widetilde{\xi}(x)\cdot dx)_{|\pi_x(\widetilde{\Gamma})}=d\phi,\ \text{on } \pi_x(\widetilde{\Gamma}). \tag{1.46}$$

We can find real smooth extensions Φ of ϕ to $\mathbf{C}_x^2$ with an arbitrary prescription of the conormal part of the derivative, so we can choose Φ satisfying

$$-\mathrm{Im}\,(\widetilde{\xi}(x)\cdot dx) = d\Phi(x),\ \forall x \in \pi_x(\widetilde{\Gamma}). \tag{1.47}$$

This means that

$$-\frac{1}{2i}\widetilde{\xi}(x)dx + \frac{1}{2i}\overline{\widetilde{\xi}}(x)\overline{dx} = d\Phi,\ x \in \pi_x(\widetilde{\Gamma}),$$

or that

$$\widetilde{\xi}(x) = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x),\ x \in \pi_x(\widetilde{\Gamma}). \tag{1.48}$$

Since $\widetilde{\Gamma}$ is close to Γ, $\frac{\partial\Phi}{\partial x}(x)$ is close to $\frac{\partial\Phi_0}{\partial x}$ on $\pi_x(\widetilde{\Gamma})$, and we may choose the extension Φ so that $\frac{\partial\Phi}{\partial x} - \frac{\partial\Phi_0}{\partial x}$ is small everywhere. The I-Lagrangian manifold $\Lambda = \Lambda_\Phi$ given by $\xi = \frac{2}{i}\frac{\partial\Phi}{\partial x}$ then has the desired properties when $\mathbf{R}^4$ is replaced by Λ_{Φ_0}, and applying the inverse of the above mentioned complex linear canonical transformation, we get the desired Λ in terms of the original problem.

The second reason, why we want $I_1(\Gamma_{\phi_a},\omega)$ and $I_2(\Gamma_{\phi_a},\omega)$ to be real comes from the Bohr-Sommerfeld, Einstein, Keller, Maslov quantization condition. The actions $I_j(\Gamma_{\phi_a},\omega)$ coincide with the corresponding actions $I_j(\Lambda_{\phi_a},\omega)$, and if we want Λ_{ϕ_a} to correspond to an eigenstate of some pseudodifferential operator with leading symbol p and eigenvalue $o(h)$, it is natural to impose a quantization condition of the type

$$I_j(\Lambda_{\phi_a},\omega) = 2\pi k_j h,\ k_j \in \mathbf{Z}, \tag{1.49}$$

where we choose to ignore the Maslov indices, and where $h > 0$ is the small semi-classical parameter. Since Λ_{ϕ_a} are not real Lagrangian manifolds (even after introducing Λ as a new real phase space), the quantization condition (1.49) will need an entirely new justification.

Consider the case when p depends on z and choose $w = I_1(\Lambda_{\phi_{a,z}},\omega)$ so that we can use the simplified notation $\Lambda_{(z,w)}$ for $\Lambda_{\phi_{a,z}}$. Also write $\nu = (z,w)$. Recall that

$$\mathrm{Im}\,\frac{dI_2}{dI_1} \neq 0 \tag{1.50}$$

when z is kept constant. It follows that there is a unique smooth function $z \mapsto w(z) \in \mathbf{C}$ such that $I_j(z,w(z))$ are real for $j = 1,2$, where we write $I_j(z,w) = I_j(\Lambda_{(z,w)},\omega)$. We will be interested in the property

$$z \longmapsto I(z,w(z)) = (I_1(z,w(z)), I_2(z,w(z))) \in \mathbf{R}^2 \text{ is a local diffeomorphism.} \tag{1.51}$$

This is equivalent to the property

$$\nu \longmapsto (I_1(\Lambda_\nu), I_2(\Lambda_\nu)) \in \mathbf{C}^2 \text{ is locally biholomorphic.} \tag{1.52}$$

In fact, if $\delta_z \in \mathbf{C}$ belongs to the kernel of the differential of the map (1.51) at some point, then (δ_z,δ_w) with $\delta_w = \frac{\partial w}{\partial z}\delta_z + \frac{\partial w}{\partial \overline{z}}\overline{\delta_z}$ will belong to the kernel of the differential of (1.52) at the corresponding point. Conversely if (δ_z,δ_w) is in the kernel of the differential of (1.52) at some point (z, w) with w real (so that $w = w(z)$), then necessarily $\delta_w = \frac{\partial w}{\partial z}\delta_z + \frac{\partial w}{\partial \overline{z}}\overline{\delta_z}$ for some δ_z in the kernel of the differential of (1.51).

Example. — Let $p = p_1(x_1, \xi_1) + ip_2(x_2, \xi_2)$, where p_j is real with $p_j^{-1}(0)$ being a closed curve in $\mathbf{R}^2$ on which $dp_j \neq 0$. For E in a small complex neighborhood of 0, we put

$$A_j(E) = \int_{p_j^{-1}(E)} \xi_j dx_j$$

and notice that these one dimensional actions are real for real E and that $A_j'(E) \neq 0$. With $z, w \in \mathbf{C}$ close to 0, we get the complex fibration

$$\Lambda_{z,w} = \{(x, \xi) \in \mathbf{C}^4; p_1(x_1, \xi_1) = w,\ ip_2(x_2, \xi_2) = z - w\}.$$

Then

$$I_1(\Lambda_{z,w}) = A_1(w),\ I_2(\Lambda_{z,w}) = A_2\Big(\frac{z-w}{i}\Big),$$

and we see that (1.51) and (1.52) hold.

Appendix A: Reduction of elliptic vector fields on a torus

Let Z be a smooth complex elliptic vector field on $\mathbf{T}^2 = (\mathbf{R}/\mathbf{Z})^2$. After left multiplication by a non-vanishing function and possibly reversal of one of the coordinates, we may assume that with $z = x_1 + ix_2$:

$$Z = \frac{\partial}{\partial \overline{z}} + g\frac{\partial}{\partial z},\ \|g\|_\infty < 1,\ g \in C^\infty. \tag{A.1}$$

Let

$$\mathcal{H}^1 = \{u = a\overline{z} + v;\ a \in \mathbf{C},\ v \in H^1_{\mathrm{per}},\ \widehat{v}(0) = 0\}, \tag{A.2}$$

where

$$H^k_{\mathrm{per}} = \{v \in H^k_{\mathrm{loc}}(\mathbf{R}^2); v(x+\gamma) = v(x), \forall \gamma \in \mathbf{Z}^2\}$$

and $\widehat{v}(k)$ is the kth Fourier coefficient and H^s is the standard Sobolev space. Let $\mathcal{H}^0 = H^0_{\mathrm{per}}$, and let $\|\cdot\|$ denote the L^2 norm on the torus (i.e. the H^0_{per} norm) if nothing else is specified. We choose the norm in $\mathcal{H}^1$ with

$$\|u\|^2_{\mathcal{H}^1} = |a|^2 + \Big\|\frac{\partial v}{\partial z}\Big\|^2 = |a|^2 + \Big\|\frac{\partial v}{\partial \overline{z}}\Big\|^2, \tag{A.3}$$

for $u = a\overline{z} + v \in \mathcal{H}^1$. Since $\frac{\partial}{\partial \overline{z}}(a\overline{z} + v) = a + \frac{\partial v}{\partial \overline{z}}$ (orthogonal sum), we see that

$$\Big\|\frac{\partial u}{\partial \overline{z}}\Big\|_{\mathcal{H}^0} = \|u\|_{\mathcal{H}^1}. \tag{A.4}$$

Moreover $\frac{\partial}{\partial \overline{z}} : \mathcal{H}^1 \to \mathcal{H}^0$ is surjective, so in view of (A.4) it is unitary. It is also clear that $\frac{\partial}{\partial z} : \mathcal{H}^1 \to \mathcal{H}^0$ is of norm 1:

$$\Big\|\frac{\partial u}{\partial z}\Big\|_{\mathcal{H}^0} \leqslant \|u\|_{\mathcal{H}^1}. \tag{A.5}$$

Since $\|g\frac{\partial}{\partial z}\|_{\mathcal{H}^1\to\mathcal{H}^0} < 1$, we see that $Z : \mathcal{H}^1 \to \mathcal{H}^0$ is bijective with inverse Z^{-1} satisfying

$$\|Z^{-1}\|_{\mathcal{H}^0\to\mathcal{H}^1} \leqslant \frac{1}{1-\|g\|_\infty}.$$

Consider the function

$$u = z - Z^{-1}(g) \in z + \mathcal{H}^1. \tag{A.6}$$

It is clear that

$$Zu = 0, \tag{A.7}$$

and u is the unique function in $z+\mathcal{H}^1$ which is annihilated by Z. It follows that the kernel of Z, acting on $\{u = \text{linear function}+v;\ v \in H^1_{\text{per}},\ \widehat{v}(0) = 0\}$, is of dimension 1.

Lemma A.1. — *$\overline{Z}u \neq 0$ everywhere.*

Proof. — $\overline{Z}u$ cannot be identically zero since otherwise we would have both $Zu = 0$ and $\overline{Z}u = 0$, implying that u is constant; which is impossible.

We have

$$[Z,\overline{Z}] = \overline{a}Z - a\overline{Z} \tag{A.8}$$

for some $a \in C^\infty_{\text{per}}$. Then $Z\overline{Z}u = -a\overline{Z}u$, so

$$(Z+a)(\overline{Z}u) = 0. \tag{A.9}$$

It is well known that if v is a null solution of a 1st order elliptic equation on a connected domain and v is not identically zero, then v cannot vanish to infinite order at any point, and (by looking at Taylor expansions) the zeros are all isolated. We can apply this to $v = \overline{Z}u$. We also see that the argument variation of $\overline{Z}u$, along a small positively oriented circle around a zero is equal to $2\pi k$ for some finite integer $k > 0$. Let $\Gamma = \partial\Omega$, where $\Omega = z_0 + (]0,1[+i]0,1[)$ and z_0 is chosen so that $\overline{Z}u$ has no zeros on Γ. If $\overline{Z}u$ has at least one zero in $\mathbf{T}^2$, then it has a zero in Ω and $\operatorname{var}\arg_\Gamma(\overline{Z}u) > 0$. This is in contradiction with the fact that $\overline{Z}u$ is periodic and hence that $\operatorname{var}\arg_\Gamma(\overline{Z}u) = 0$. It follows that

$$\overline{Z}u \neq 0. \tag{A.10}$$

□

If we view u as a map $\mathbf{R}^2 \to \mathbf{R}^2$, it follows from (A.7,10), that the corresponding Jacobian is everywhere $\neq 0$. It follows that $u = u_1 + iu_2$ is a diffeomorphism from $\mathbf{C}$ to $\mathbf{C}$. Let

$$u(z+1) - u(z) =: e_1,\ u(z+i) - u(z) =: e_2. \tag{A.11}$$

Then e_1, e_2 are $\mathbf{R}$-linearly independent, and we let $L = \mathbf{Z}e_1+\mathbf{Z}e_2$ be the corresponding lattice. Using that $u : \mathbf{C} \to \mathbf{C}$ is a diffeomorphism, we see that the induced map $[u] : \mathbf{T}^2 \to \mathbf{C}/L$ is bijective. (Only the injectivity needs to be checked: Let $x, y \in \mathbf{T}^2$ with $[u](x) = [u](y) := u_0$. We can find corresponding points $\widetilde{x}, \widetilde{y}, \widetilde{u}_0 \in \mathbf{C}$, such that

$u(\widetilde{x}) = \widetilde{u}_0$, $u(\widetilde{y}) = \widetilde{u}_0 + k_1 e_1 + k_2 e_2$, $k_j \in \mathbf{Z}$. Then $u(\widetilde{y} - k_1 - k_2 i) = \widetilde{u}_0$, so by the injectivity of u, we have $\widetilde{x} = \widetilde{y} - k_1 - k_2 i$ and hence $x = y$.)

If $f(w)$ is a C^1 function on $\mathbf{C}$, then

$$Z(f(u(z))) = \frac{\partial f}{\partial w} Zu + \frac{\partial f}{\partial \overline{w}} Z(\overline{u}) = Z(\overline{u}) \frac{\partial f}{\partial \overline{w}}.$$

In other words, if we let lower $*$ indicate push forward of vector fields, then

$$u_*(Z) = Z(\overline{u}) \frac{\partial}{\partial \overline{w}}, \ [u]_*(Z) = Z(\overline{u}) \frac{\partial}{\partial \overline{w}}. \tag{A.12}$$

Conversely, if $\widetilde{L}$ is some lattice and $[t] : \mathbf{T}^2 \to \mathbf{C}/\widetilde{L}$ a diffeomorphism corresponding to a grad periodic function t with

$$t_*(Z) = F \frac{\partial}{\partial \overline{w}}, \ F \neq 0 \text{ everywhere}, \tag{A.13}$$

then

$$Z(t) = 0. \tag{A.14}$$

Since $t \in \mathbf{C}z + \mathcal{H}^1$, we know that $\exists\, 0 \neq \alpha \in \mathbf{C}$ such that $t = \alpha u$. Consequently,

$$\widetilde{L} = \alpha L. \tag{A.15}$$

Actually, we can see this more directly, by considering the biholomorphic map $[u][t]^{-1}$.

It follows from our constructions that if Z depends smoothly (real-analytically) on an additional parameter w, then so does u.

Appendix B: 2-dimensional manifolds with elliptic vector fields

Let M be a smooth compact connected 2-dimensional manifold with an elliptic (complex) vector field Z. We shall see that M is diffeomorphic to a torus $\mathbf{C}/L$ in such a way that Z maps to a multiple of $\frac{\partial}{\partial \overline{z}}$. Clearly $Z : H^1(M) \to H^0(M)$ is a Fredholm operator. Let $\operatorname{ind} Z = \dim \mathcal{N}(Z) - \operatorname{codim} \mathcal{R}(Z) = \dim \mathcal{N}(Z) - \dim \mathcal{N}(Z^*)$ be the index, where Z^* denotes the adjoint of Z with respect to some positive density on M. Recall that the kernels $\mathcal{N}(Z)$, $\mathcal{N}(Z^*)$ are contained in $C^\infty(M)$, since Z and Z^* are elliptic.

Lemma B.1. — $\operatorname{ind} Z = 0$.

Proof. — Clearly $\operatorname{ind} Z^* = -\operatorname{ind} Z$. On the other hand $Z^* = -\overline{Z} + f$ for some $f \in C^\infty(M)$ and since the index is stable under changes of the lower order part:

$$\operatorname{ind} Z^* = \operatorname{ind}(-\overline{Z}) = \operatorname{ind} \overline{Z} = \operatorname{ind} Z.$$

Here the last equality follows from the fact that $\mathcal{N}(\overline{Z}) = \overline{\mathcal{N}(Z)}$, $\mathcal{R}(\overline{Z}) = \overline{\mathcal{R}(Z)}$. Then $\operatorname{ind} Z = -\operatorname{ind} Z^* = -\operatorname{ind} Z$, and hence $\operatorname{ind} Z = 0$. □

Because of the ellipticity, there is a unique $a \in C^\infty(M)$, such that

$$[Z, \overline{Z}] = \overline{a} Z - a \overline{Z}. \tag{B.1}$$

Lemma B.2. — *$P := -(Z+a)\overline{Z}$ is a real differential operator.*

Proof. — $\overline{P}-P=(Z+a)\overline{Z}-(\overline{Z}+\overline{a})Z=[Z,\overline{Z}]-(\overline{a}Z-a\overline{Z})=0.$ □

Let us identify M with the zero section in T^*M and let $p=p_1+ip_2$ be the principal symbol of Z. Then p_j are linear in ξ and dp_1, dp_2 are independent at the points of $M\subset T^*M$. Let $\lambda(dx)$ be the Liouville measure on M induced by p_1,p_2, so that

$$\lambda(dx)\wedge dp_1\wedge dp_2=dxd\xi \text{ at the points of } M, \tag{B.2}$$

where $dxd\xi$ denotes the symplectic volume. The principal symbol of $\overline{Z}$ is $\overline{p(x,-\xi)}=-\overline{p(x,\xi)}$, so if we take the principal symbols of (B.1), we get

$$\{p,\overline{p}\}=\overline{ia}p-ia\overline{p}. \tag{B.3}$$

We use this to compute the Lie derivative $\mathcal{L}_{H_p}(\lambda(dx))$: Since $\mathcal{L}_{H_p}(dxd\xi)=0$, we get from (B.2), (B.3) at $\xi=0$:

$$\mathcal{L}_{H_p}(\lambda)\wedge dp\wedge d\overline{p}+\lambda\wedge dp\wedge\mathcal{L}_{H_p}d\overline{p}=0,$$

$$\mathcal{L}_{H_p}(\lambda)\wedge dp\wedge d\overline{p}+\lambda\wedge dp\wedge d\{p,\overline{p}\}=0,$$

$$\mathcal{L}_{H_p}(\lambda)\wedge dp\wedge d\overline{p}-ia\lambda\wedge dp\wedge d\overline{p}=0.$$

Hence

$$\mathcal{L}_{H_p}(\lambda)=ia\lambda \text{ on } \xi=0. \tag{B.4}$$

But the restriction of H_p to $\xi=0$, can be identified with iZ, so (B.4) gives

$$\mathcal{L}_Z(\lambda)=a\lambda \text{ on } M. \tag{B.5}$$

Let A^* and ${}^tA$ denote the adjoint and the transpose of A in $L^2(M,\lambda(dx))$. From (B.5), we get

Lemma B.3. — *$Z^*=-(\overline{Z}+\overline{a})$, ${}^tZ=-(Z+a)$.*

Proof. — We start with the general fact that

$$\int_M \mathcal{L}_Z(u\lambda(dx))=0,$$

for all $u\in C^\infty(M)$. Using (B.5), we get

$$\int_M (Z+a)u\lambda(dx)=0. \tag{B.6}$$

Replace u by uv:

$$\int_M ((Zu)v+u(Z+a)v)\lambda(dx)=0. \tag{B.7}$$

It follows that ${}^tZ=-(Z+a)$, $Z^*=\overline{{}^tZ}=-(\overline{Z}+\overline{a})$. □

We also have $\overline{Z}^* = -(Z+a)$. Lemma B.2 gave us the real operator

$$P = -(Z+a)\overline{Z} = -(\overline{Z}+\overline{a})Z. \tag{B.8}$$

Lemma B.3 shows that the operator is self-adjoint and $\geqslant 0$:

$$P = \overline{Z}^*\overline{Z} = Z^*Z. \tag{B.9}$$

Moreover it is an elliptic 2nd order operator. From (B.9) it is easy to see that

$$\mathcal{N}(P) = \mathcal{N}(Z) = \mathcal{N}(\overline{Z}) = \mathbf{C}1. \tag{B.10}$$

The last equality follows from the other equalities since $Zu = 0$, $\overline{Z}u = 0$ implies that u is constant.

By a more direct argument, we have

Proposition B.4. — *Let $f \in C^\infty(M)$. If $u \not\equiv 0$, $(Z+f)u = 0$, then $u(x) \neq 0$ for every $x \in M$. We have* $\dim \mathcal{N}(Z+f) \leqslant 1$.

Proof. — Applying a classical result of Aronsjajn about the strong uniqueness of nullsolutions of second order elliptic equations, we know that u cannot vanish to ∞ order at any point. Let x_0 be a zero and choose local coordinates x_1, x_2 centered at x_0, such that

$$Z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2}\Big) + \mathcal{O}(|x|)\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big).$$

Let m be the order of vanishing of u at x_0, so that $u(x) = u_m(x) + \mathcal{O}(|x|^{m+1})$, where $u_m(x)$ is a homogeneous polynomial of degree m. Then we get

$$\frac{\partial u_m}{\partial \overline{z}} = 0, \text{with } z = x_1 + ix_2,$$

so $u_m(x) = Cz^m$ for some $C \neq 0$. Hence x_0 is an isolated zero. Moreover, $\operatorname{var}\arg_\gamma u = 2\pi m$, if γ is a simple closed loop around x_0 (contained in the coordinate neighborhood) which is positively oriented with respect to the directions $(\operatorname{Re} Z, \operatorname{Im} Z)$. We can now triangulate M in such a way that every zero of u is in the interior of one of the triangles. If Δ is one of the triangles, then $\operatorname{var}\arg_{\partial\Delta} u \geqslant 0$ with strict inequality precisely when D contains a zero of u. Since every boundary segment is common to two different triangles, but with opposite orientations, we see that

$$\sum_\Delta \operatorname{var}\arg_{\partial\Delta} u = 0,$$

when we sum over all the triangles in the triangulation. It follows that u cannot have any zeros.

The second statement is now clear: Let $0 \neq u_0 \in \mathcal{N}(Z+f)$, so that u_0 is everywhere different from 0. Let $u \in \mathcal{N}(Z+f)$ and let $x_0 \in M$. Then $v(x) := u(x) - \frac{u(x_0)}{u_0(x_0)}u_0(x)$ belongs to $\mathcal{N}(Z+f)$ and vanishes at one point (x_0). The first part of the proposition implies that v vanishes identically, and hence that u is a constant multiple of u_0. This shows that the dimension of $\mathcal{N}(Z+f)$ is at most equal to 1. $\square$

Proposition B.5. — *There exists a non-vanishing function $b \in C^\infty(M)$ such that* $[\bar{b}Z, b\overline{Z}] = 0$.

Proof. — We develop the commutation relation to solve and get:

$$\begin{aligned}0 &= \bar{b}b[Z,\overline{Z}] + \bar{b}[Z,b]\overline{Z} + b[\bar{b},\overline{Z}]Z\\ &= \bar{b}b(\bar{a}Z - a\overline{Z}) + \bar{b}Z(b)\overline{Z} - b\overline{Z(b)}Z\\ &= (\bar{b}b\bar{a} - b\overline{Z(b)})Z - (\bar{b}ba - \bar{b}Z(b))\overline{Z}\\ &= b\overline{(ab - Z(b))}Z - \bar{b}(ab - Z(b))\overline{Z},\end{aligned}$$

so b should solve

$$(Z - a)b = 0. \tag{B.11}$$

Notice that if (B.11) holds for some non-vanishing b, then

$$Z\frac{1}{b} = -\frac{1}{b^2}Z(b) = -a\frac{1}{b},$$

so

$$(Z + a)\frac{1}{b} = 0, \text{ i.e. } Z^*c = 0,\ c = \frac{1}{\bar{b}}. \tag{B.12}$$

Conversely, (B.12) implies (B.11).

We have seen that Z has index 0 and has a 1-dimensional kernel. Then the same holds for Z^* and Proposition B.4 shows that $\mathcal{N}(Z^*)$ is generated by a non-vanishing function c. It suffices to take $b = 1/\bar{c}$. □

Theorem B.6. — *There exists a diffeomorphism $\kappa : \mathbf{C}/L \to M$ such that $\bar{b}Z$ corresponds to $\frac{\partial}{\partial \bar{z}}$. Here $L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ is a lattice (so that $e_1, e_2 \in \mathbf{C}$ are $\mathbf{R}$-linearly independent).*

Proof. — Write $\bar{b}Z = \frac{1}{2}(\nu_1 + i\nu_2)$, where ν_1, ν_2 are real *commuting* vector fields which are pointwise linearly independent. Fix a point $x_0 \in M$ and consider the map

$$K : \mathbf{C} \simeq \mathbf{R}^2 \ni x \longmapsto \exp(x_1\nu_1 + x_2\nu_2)(x_0) \in M.$$

Notice that $\exp(x_1\nu_1 + x_2\nu_2) = \exp(x_1\nu_1) \circ \exp(x_2\nu_2) = \exp(x_2\nu_2) \circ \exp(x_1\nu_1)$ by commutativity. Let

$$L = \{x \in \mathbf{R}^2; K(x) = x_0\}.$$

L is a discrete Abelian subgroup of $\mathbf{R}^2$ and hence of the form 0, $\mathbf{Z}e$ with $e \neq 0$, or a lattice $\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ with e_1, e_2 $\mathbf{R}$-linearly independent. K induces a diffeomorphism $\kappa : \mathbf{R}^2/L \to M$, so $\mathbf{R}^2/L$ must be compact and hence L is a lattice. Clearly the inverse image of $\bar{b}Z$ is $\frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial \bar{z}}$ with $z = x_1 + ix_2$. □

2. Review of Fourier integral operators between H_Φ spaces

We shall not review all the aspects of Fourier integral operator calculus (see [**MeSj**] for a similar discussion), and for simplicity, we restrict the attention to the Toeplitz (or Bergman projection) point of view. Let Φ be a smooth real-valued function defined near some point $x_0 \in \mathbf{C}^n$. Assume that Φ is strictly plurisubharmonic (s.pl.s.h.). Then

$$\Lambda_\Phi := \{(x, \tfrac{2}{i}\tfrac{\partial\Phi}{\partial x}(x));\ x \in \operatorname{neigh}(x_0, \mathbf{C}^n)\} \tag{2.1}$$

is I-Lagrangian and R-symplectic. Assume that $\Gamma \subset \Lambda_\Phi$ is a smooth Lagrangian submanifold (i.e. Lagrangian for the real symplectic form $\sigma_{|\Lambda_\Phi}$). If we identify Γ with its projection $\pi_x\Gamma$ in $\mathbf{C}^n$ then on Γ the fundamental 1-form $\xi \cdot dx$ can be identified with $\omega = \frac{2}{i}\partial\Phi_{|\Gamma}$ and hence this is a closed one-form in Γ. Here

$$\operatorname{Im}\omega = -d\Phi, \tag{2.2}$$

so $\operatorname{Im}\omega$ is exact. We notice that $\pi_x\Gamma$ is totally real. In fact, if $u \in \mathbf{C}^n$ and u, iu are both tangential to $\pi_x\Gamma$ at a point x, then

$$U_1 = (u, \tfrac{2}{i}(\Phi''_{xx}(x)u + \Phi''_{x\overline{x}}(x)\overline{u})) \text{ and } U_2 = (iu, \tfrac{2}{i}(\Phi''_{xx}(x)iu + \Phi''_{x\overline{x}}(x)\overline{iu}))$$

are both tangential to Γ at $(x, \frac{2}{i}\frac{\partial\Phi(x)}{\partial x})$. It follows that

$$0 = \sigma(U_1, U_2) = \sigma(U_1, U_2 - iU_1) = 4\langle\Phi''_{x\overline{x}}(x)\overline{u}, u\rangle, \tag{2.3}$$

which implies that $u = 0$. Locally in $\pi_x\Gamma$ we may then find a primitive ϕ of ω and extend $\phi(x)$ to an almost analytic function in $\mathbf{C}^n$ so that $\overline{\partial}\phi(x) = \mathcal{O}(\operatorname{dist}(x, \pi_x\Gamma)^\infty)$. Then at the points of $\pi_x\Gamma$, we have $d\phi = \frac{2}{i}\partial\Phi$, so at those points, we get

$$d\operatorname{Im}\phi = \tfrac{1}{2i}(\tfrac{2}{i}\partial\Phi + \tfrac{2}{i}\overline{\partial}\Phi) = -d\Phi.$$

After modifying ϕ by an imaginary constant (assuming Γ connected) we have that $\operatorname{Im}\phi + \Phi$ vanishes to the second order on Γ. Since this function is s.pl.s.h. it follows that

$$\Phi(x) + \operatorname{Im}\phi(x) \sim \operatorname{dist}(x, \pi_x(\Gamma))^2 \text{ near } \pi_x(\Gamma). \tag{2.4}$$

Let $\widetilde{\Phi}(y)$ be a second smooth s.pl.s.h function defined near $y_0 \in \mathbf{C}^n$. Let $\xi_0 = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)$, $\eta_0 = \frac{2}{i}\frac{\partial\widetilde{\Phi}(y)}{\partial y}(y_0)$, and let $\kappa : \operatorname{neigh}((y_0, \eta_0), \Lambda_{\widetilde{\Phi}}) \to \operatorname{neigh}((x_0, \xi_0), \Lambda_\Phi)$ be a smooth canonical transformation (with Λ_Φ, $\Lambda_{\widetilde{\Phi}}$ considered as real symplectic manifolds).

On $\mathbf{C}^{2n}_{x,\xi} \times \mathbf{C}^{2n}_{y,\eta}$, we choose the complex structure for which holomorphic functions are holomorphic in $(x, \xi; \overline{y}, \overline{\eta})$ in the usual sense. A corresponding "holomorphic" symplectic form is then given by

$$d\xi \wedge dx - d\overline{\eta} \wedge d\overline{y}. \tag{2.5}$$

We notice that the form (2.5) and the more standard form $d\xi \wedge dx - d\eta \wedge dy$ have the same restriction to $\Lambda_\Phi \times \Lambda_{\widetilde{\Phi}}$, since $d\eta \wedge dy_{|\Lambda_{\widetilde{\Phi}}}$ is real. The manifold $\Lambda_\Phi \times \Lambda_{\widetilde{\Phi}}$ is

I-Lagrangian and R-symplectic for the form (2.5), and we can view it as a "Λ_F" for our non-standard structure, with $F = \Phi(x) + \widetilde{\Phi}(y)$, since it can be represented as

$$\xi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x), \ -\overline{\eta} = \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}}(y).$$

The earlier discussion for Lagrangian manifolds can then be applied with Γ equal to graph$\,(\kappa)$, and we conclude that there is a function $\psi(x,y)$ such that

$$\partial_{\overline{x},y}\psi \text{ vanishes to infinite order on } \pi_{x,y}(\Gamma), \tag{2.6}$$

$$\partial_x\psi(x,y) = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x),\ \partial_{\overline{y}}\psi(x,y) = \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}},\ \text{for } (x,y)\in \pi_{x,y}(\Gamma), \tag{2.7}$$

$$\Phi(x) + \widetilde{\Phi}(y) + \operatorname{Im}\psi(x,y) \sim \operatorname{dist}\left((x,y), \pi_{x,y}(\Gamma)\right)^2. \tag{2.8}$$

When $\widetilde{\Phi} = \Phi$ and $\kappa = \mathrm{id}$ is the identity, we can choose $\psi(x,y)$ to be the unique function (up to $\mathcal{O}(|x-y|^\infty)$), which satisfies (2.6) and $\psi(x,x) = \frac{2}{i}\Phi(x)$. In the general case, we deduce from (2.6), (2.7) that on $\pi_{x,y}(\Gamma)$:

$$d\psi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)dx + \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}}d\overline{y}. \tag{2.9}$$

If we restrict ψ to $\pi_{x,y}(\Gamma)$ and identify it with a function on Γ, we get

$$d(\psi_{|_\Gamma}) = \xi dx - \overline{\eta}d\overline{y},\ (x,\xi;y,\eta)\in\Gamma. \tag{2.10}$$

Since ξdx and $\overline{\eta}d\overline{y}$ are primitives of $\sigma_{|_{\Lambda_\Phi}}$ and $\sigma_{|_{\Lambda_{\widetilde{\Phi}}}}$ respectively, we can interpret (2.10) as stating that $\psi_{|_\Gamma}$ is a generating function for κ. For the moment, we make a local discussion and all our domains can be assumed to be simply connected. Later this will no more be the case and we have to consider what happens when we follow the locally defined function ψ around a closed loop in Γ, of the form $\widehat{\gamma} = \{(\kappa(\rho),\rho);\ \rho\in\gamma\}$, where γ is a closed loop in the domain of κ in $\Lambda_{\widetilde{\Phi}}$. We have

$$\operatorname{Im}(\xi dx)_{|_{\Lambda_\Phi}} = \operatorname{Im}\left(\tfrac{2}{i}\partial\Phi\right) = -d\Phi,$$

which is exact, since we will always require Φ and $\widetilde{\Phi}$ to be single valued. Similarly $\operatorname{Im}(\overline{\eta}d\overline{y})_{|_{\Lambda_{\widetilde{\Phi}}}}$ is exact. Hence

$$\int_{\widehat{\gamma}} d\psi = \int_{\kappa\circ\gamma}\operatorname{Re}(\xi dx) - \int_\gamma \operatorname{Re}(\eta dy). \tag{2.11}$$

So the undeterminacy in ψ is real (as can also be seen from (2.8)) and following ψ around a closed loop as above, ψ changes by a real constant, which is the difference of two real actions.

The implementation of Fourier integral operators is now fairly routine, and we will not go into all the details. (See [**Sj1**].) Formally such an operator is of the form

$$Au(x) = h^{-n}\int e^{\frac{i}{h}\psi(x,y)}a(x,y;h)u(y)e^{-\frac{2}{h}\widetilde{\Phi}(y)}L(dy), \tag{2.12}$$

where $L(dy)$ is the Lebesgue measure and a is a symbol of order m in $1/h$:

$$\nabla^k_{x,y} a = \mathcal{O}_k(1) h^{-m}, \tag{2.13}$$

$$\partial_{\overline{x}} a,\ \partial_y a = \mathcal{O}(h^{-m} \operatorname{dist}((x,y), \pi_{x,y}(\Gamma))^\infty + h^\infty). \tag{2.14}$$

See also section 3 of [**MeSj**].

3. Formulation of the problem in H_Φ and reduction to a neighborhood of $\xi = 0$ in $T^*\Gamma_0$

Let Φ_0 be a s.pl.s.h. quadratic form on $\mathbf{C}^n$. Let $P(x,\xi;h)$ be holomorphic and bounded in a tubular neighborhood V of Λ_{Φ_0} and assume that

$$|P(x,\xi;h)| \geqslant \frac{1}{C},\ (x,\xi) \in V,\ |(x,\xi)| > C. \tag{3.1}$$

Also assume (for simplicity) that

$$P \sim \sum_0^\infty h^k p_k(x,\xi), \tag{3.2}$$

in the space of bounded holomorphic functions on V. Then $|p_0(x,\xi)| \geqslant 1/C$, $(x,\xi) \in V$, $|(x,\xi)| > C$.

If we take the Weyl quantization, we know (see [**Sj3**], [**MeSj**]), that

$$P^w(x, hD_x; h) = \mathcal{O}(1) : H_{\Phi_0} \longrightarrow H_{\Phi_0}, \tag{3.3}$$

where

$$H_{\Phi_0} := \operatorname{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0/h} L(dx)), \tag{3.4}$$

and $\operatorname{Hol}(\mathbf{C}^n)$ denotes the space of holomorphic (entire) functions on $\mathbf{C}^n$.

Since Φ_0 is a quadratic form, we can infer (3.3) solely from the fact that P is a symbol of class S^0 on Λ_{Φ_0}, i.e. from the fact that $\nabla^k(P_{|\Lambda_{\Phi_0}}) = \mathcal{O}_k(1)$ for every $k \in \mathbf{N}$. However the fact that P is bounded and holomorphic in a tubular neighborhood of Λ_{Φ_0} allows us to consider other weights as well. Let $\Phi \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$ (the space of C^1 functions with Lipschitz gradient) with $\Phi - \Phi_0$ bounded and $\sup |\frac{\partial \Phi}{\partial x} - \frac{\partial \Phi_0}{\partial x}|$ small enough. Then,

$$P^w(x, hD_x; h) = \mathcal{O}(1) :\ H_\Phi \longrightarrow H_\Phi, \tag{3.5}$$

where H_Φ is defined as in (3.4). In fact, in the standard formula,

$$P^w(x, hD_x; h) u = \frac{1}{(2\pi h)^n} \iint_{(\frac{x+y}{2},\xi) \in \Lambda_{\Phi_0}} e^{\frac{i}{h}(x-y)\xi} P\Big(\frac{x+y}{2}, \xi; h\Big) u(y)\, dy\, d\xi, \tag{3.6}$$

we deform to the contour

$$\xi = \frac{2}{i} \int_0^1 \frac{\partial \Phi}{\partial x}(tx + (1-t)y)\, dt + \frac{i}{C} \frac{\overline{x-y}}{\langle x-y \rangle},\ \langle x \rangle = (1+|x|^2)^{1/2}. \tag{3.7}$$

In the following, we also assume for simplicity that $\Phi \in C^\infty$, that $\nabla^k\Phi$ is bounded for every $k \geqslant 2$, and that Φ is uniformly s.pl.s.h. We also assume that $n = 2$ and that there is a smooth Lagrangian torus $\Gamma \subset \Lambda_\Phi$, such that $p_\Phi = p_{0|_{\Lambda_\Phi}}$ satisfies

$$p_\Phi^{-1}(0) = \Gamma, \tag{3.8}$$

$$dp_\Phi, d\overline{p_\Phi} \text{ are independent at every point of } \Gamma. \tag{3.9}$$

Let $\Gamma_0 = (\mathbf{R}/2\pi\mathbf{Z})^2$ be the standard 2 torus and view Γ_0 as a maximally totally real submanifold of $X := \Gamma_0 + i\mathbf{R}^2$. In $X \times \mathbf{C}^2$ (equipped with the standard symplectic form) we consider

$$\Lambda_{\Phi_1} : \xi = \frac{2}{i}\frac{\partial \Phi_1}{\partial x}, \; \Phi_1(x) = \frac{1}{2}(\operatorname{Im} x)^2. \tag{3.10}$$

Φ_1 is s.pl.s.h. so Λ_{Φ_1} is I-Lagrangian and R-symplectic. According to section 1 and the beginning of section 2, there is a smooth "real" canonical transformation:

$$\kappa : \operatorname{neigh}(\Gamma, \Lambda_\Phi) \longrightarrow \operatorname{neigh}(\Gamma_0 \times \{0\}, \Lambda_{\Phi_1}), \tag{3.11}$$

mapping Γ onto $\Gamma_0 \times \{0\}$. Let $\psi(x, y)$ be a corresponding function defined as in section 2 for (x, y) in a neighborhood of $\pi_{x,y}(\operatorname{graph}(\kappa))$. Strictly speaking, it is clear how to define ψ locally up to a constant and up to a function which vanishes to infinite order on $\pi_{x,y}(\operatorname{graph}(\kappa))$. To see that we can get a corresponding grad-periodic function, we first define ψ on the projection of the graph of κ with $d\psi$ single-valued, then let α denote a single-valued almost holomorphic extension of this differential. For $(x, y) \in \operatorname{neigh}(\pi_{x,y}(\operatorname{graph}(\kappa)))$, let $\gamma_{x,y} : [0, 1] \to \mathbf{C}^4$ be the shortest segment from a point $\gamma_{x,y}(0)$ in the projection of the graph to $\gamma_{x,y}(1) = (x, y)$, and put $\psi(x, y) = \psi(\gamma_{x,y}(0)) + \int_{\gamma_{x,y}} \alpha$. Then ψ is grad-periodic and $\operatorname{Im}\psi$ is single-valued.

Let γ_j, $j = 1, 2$ be fundamental cycles in Γ, so that $\kappa \circ \gamma_j$ are fundamental cycles in $\Gamma_0 \times \{0\}$. Define $\widehat{\gamma}_j = \{(\kappa(\rho), \rho);\, \rho \in \gamma_j\}$. Then (2.11) is applicable and gives:

$$\int_{\widehat{\gamma}_j} d\psi = -\int_{\gamma_j} \operatorname{Re}(\eta dy) = -I_j(\Gamma), \tag{3.12}$$

where the last equality defines the action $I_j(\Gamma)$, which does not depend on the choice of global primitive of $\sigma_{|_{\Lambda_\Phi}}$, since Λ_Φ is diffeomorphic to $\mathbf{R}^4$. Here as in (2.11) we view ψ as a function on graph(κ). Since $d\psi$ is single valued, this means that if we start from a point (x, y) close to some point $(x_0, y_0) \in (\Gamma_0 \times \pi_y(\Gamma)) \cap \pi_{x,y}(\operatorname{graph}(\kappa))$, and follow a closed curve $[0, 1] \ni t \mapsto (x(t), y(t))$ which remains close to $\pi_{x,y}(\operatorname{graph}(\kappa))$ and with $x(t)$ close to a fundamental cycle $\gamma_{0,j}$ in Γ_0, then we get a new value of $\psi(x, y)$: "$\psi(x(1), y(1))$" satisfying

$$\psi(x(1), y(1)) = \psi(x(0), y(0)) - I_j(\Gamma). \tag{3.13}$$

We now implement κ by a Fourier integral operator of the form (2.12) with a of class $S^0_{\mathrm{cl}}(\mathrm{neigh}(\pi_{x,y}(\mathrm{graph}(\kappa))))$:

$$a(x,y;h) \sim \sum_0^\infty a_j(x,y)h^j \text{ in } C^\infty(\mathrm{neigh}(\pi_{x,y}(\mathrm{graph}(\kappa)))), \tag{3.14}$$

with a_j of class C^∞, and

$$\partial_{\overline{x}}a_j,\ \partial_y a_j = \mathcal{O}(\mathrm{dist}((x,y),\pi_{x,y}(\mathrm{graph}(\kappa)))^\infty). \tag{3.15}$$

We also choose a elliptic, i.e. with a_0 non-vanishing. (Notice that unlike ψ, a is single valued.)

Let $U \subset \Lambda_\Phi$, $V \subset \Lambda_{\Phi_1}$ be small neighborhoods of Γ and $\Gamma_0 \times \{0\}$ respectively, with $\kappa(U) = V$. Then putting a suitable cutoff in (2.12) (equal to 1 near the projection of the graph of κ and replacing $\widetilde{\Phi}$ by Φ), we get an operator

$$A = \mathcal{O}(1) : L^2(\pi(U); e^{-2\Phi/h}L(dy)) \longrightarrow L^2_h(\pi(V); e^{-2\Phi_1/h}L(dx)),$$

where the subscript h indicates that we have a space of multi-valued Floquet periodic functions v:

$$v(x(1)) = e^{-iI_j(\Gamma)/h}v(x(0)), \tag{3.16}$$

if $[0,1] \ni t \mapsto x(t)$ is a closed curve which is close to the jth fundamental cycle in Γ_0. We also see that $\|\overline{\partial}Au\|_{L^2_h} \leqslant \mathcal{O}(h^\infty)\|u\|_{L^2}$.

The complex adjoint A^* will be a Fourier integral operator associated to κ^{-1} by the same general procedure, and it is a routine matter to see that a can be chosen so that A^*A, AA^* are formally the orthogonal projections

$$L^2(\pi(U); e^{-2\Phi/h}L(dy)) \to H(\pi(U),\Phi), \quad L^2_h(\pi(V); e^{-2\Phi_1/h}L(dx)) \to H_h(\pi(V)),\Phi_1),$$

where $H(\pi(U),\Phi) := \mathrm{Hol}(\pi(U)) \cap L^2(\pi(U); e^{-2\Phi/h}L)$ and H_h is defined similarly. (See [**MeSj**].) This implies that if $u \in H(\pi(U),\Phi)$ and $\widetilde{U} \subset\subset \pi(U)$, then

$$\|A^*Au - u\|_{L^2(\widetilde{U},e^{-2\Phi/h}L(dy))} = \mathcal{O}(h^\infty)\|u\|_{L^2(\pi(U),e^{-2\Phi/h}L(dy))},$$

and similarly for AA^*.

We also have Egorov's theorem which permits us to find $Q \in S^0_{\mathrm{cl}}(V)$ such that if q_0 is the leading symbol, then

$$q_0 \circ \kappa = p_0, \tag{3.17}$$

$$Q^w(x,hD_x)A \equiv AP^w,\ A^*Q^w \equiv P^wA^*, \tag{3.18}$$

in the sense that

$$\|(Q^wA - AP^w)u\|_{L^2_h(\widetilde{V};e^{-2\Phi/h}L(dx))} \leqslant \mathcal{O}(h^\infty)\|u\|_{H(U;\Phi)},$$

when $\widetilde{V} \subset\subset \pi(V)$, and similarly for the second relation. Here Q^w is defined as in (3.6) after replacing Q by $(\chi Q)(\frac{x+y}{2},\xi;h)$, where χ is suitable cut-off, and where we identify $\Gamma_0 + i\mathbf{R}^2$ with $\mathbf{C}^2/(2\pi\mathbf{Z}^2)$.

Finally we shall take a unitary transform

$$B:\ H(\Gamma_0+i\mathbf{R}^2,\Phi_1)\longrightarrow L^2(\Gamma_0), \tag{3.19}$$

and similarly on the corresponding spaces of Floquet-periodic functions, that will be the inverse of a Bargman transform. Since Φ_1 only depends on $\operatorname{Im} z$, we may view this function also as a function on $\mathbf{C}^2$. We recall that the Bargman transform

$$Tu(z;h)=C_2h^{-3/2}\int e^{-\frac{1}{2h}(z-y)^2}u(y)dy=\int k(z-y;h)u(y)dy, \tag{3.20}$$

(with the last equality defining the kernel k in the obvious way) is unitary: $L^2(\mathbf{R}^2)\to H(\mathbf{C}^2,\Phi_1)$ for a suitable $C_2>0$. The inverse is given by $T^{-1}=T^*$, with

$$\begin{aligned}T^*v(x;h)&=C_2h^{-3/2}\int e^{-\frac{1}{2h}(\overline{z}-\overline{x})^2-\frac{2}{h}\Phi_1(z)}v(z)L(dz)\\&=\int\overline{k(z-x;h)}e^{-\frac{2}{h}\Phi_1(z)}v(z)L(dz).\end{aligned} \tag{3.21}$$

If we identify $L^2_h(\Gamma_0)$ with the θ-Floquet periodic locally square integrable functions, for $\theta=(I_1(\Gamma)/2\pi h, I_2(\Gamma)/2\pi h)$ on $\mathbf{R}^2$, and view $H_h(\Gamma_0+i\mathbf{R}^2,\Phi_1)$ similarly, we see that T generates an operator B^* from $L^2_h(\Gamma_0)$ to θ-Floquet periodic holomorphic functions on $\mathbf{C}^2$, given by

$$B^*u(z)=\int_{\mathbf{R}^2}k(z-y;h)u(y)dy=\int_{y\in E}\sum_{\nu\in(2\pi\mathbf{Z})^2}k(z-y+\nu)e^{i\langle\theta,\nu\rangle}u(y)dy, \tag{3.22}$$

where $E\subset\mathbf{R}^2$ is a fundamental domain for $(2\pi\mathbf{Z})^2$ (so $u(z+\nu)=e^{-i\langle\theta,\nu\rangle}u(z)$, $\nu\in(2\pi\mathbf{Z})^2$). Put

$$\ell(z,y)=\sum_{\nu\in(2\pi\mathbf{Z})^2}k(z-y+\nu)e^{i\langle\theta,\nu\rangle} \tag{3.23}$$

so that

$$\ell(z+\nu,y)=e^{-i\langle\theta,\nu\rangle}\ell(z,\nu),\ \ell(z,y+\nu)=e^{i\langle\theta,\nu\rangle}\ell(z,y).$$

The adjoint B is given by
(3.24)

$$Bv(x)=\int_{z\in E+i\mathbf{R}^2}\overline{\ell(z,x)}e^{-2\Phi_1(z)/h}v(z)L(dz)=\int\overline{k(z-x)}v(z)e^{-2\Phi_1(z)/h}L(dz),$$

so B coincides with T^*. Recall that $T^*T=1$ on $L^2(\mathbf{R}^2)$. It is easy to see that this relation extends to $L^2_h(\Gamma_0)$ and we get

$$BB^*=1. \tag{3.25}$$

To check that B^*B is also the identity on $H_h(\Gamma_0+i\mathbf{R}^2,\Phi_1)$, we first recall that TT^* is the identity on $H(\mathbf{C}^2,\Phi_1)$ and when we compute TT^* in a straight forward manner, we get the orthogonal projection: $L^2(\mathbf{C}^2, e^{-2\Phi_1/h}L(dz)) \to H(\mathbf{C}^2, \Phi_1)$:

$$\begin{aligned} TT^*v(z) &= \iint k(z-y;h)\overline{k(w-y;h)}v(w)e^{-2\Phi_1(w)/h}L(dw)dy \\ &= \widetilde{C}h^{-2}\int e^{\frac{2}{h}\psi_1(z,w)}v(w)e^{-\frac{2}{h}\Phi_1(w)}L(dw), \end{aligned}$$

where

$$\psi_1(z,w) = -\frac{1}{8}(z-\overline{w})^2 \tag{3.26}$$

is the unique function which is holomorphic in z, antiholomorphic in w and satisfies $\psi_1(z,z) = \Phi_1(z)$. Recall that $-\Phi_1(z) + 2\mathrm{Re}\,\psi_1(z,w) - \Phi_1(w) \sim -|z-w|^2$, so TT^* is a bounded operator on $H_h(\Gamma_0 + i\mathbf{R}^2, \Phi_1)$. If u is a normalized element of this space, then by solving a correcting d-bar problem for $\chi(x/R)u(x)$, we see that there is a sequence of functions $u_R \in H(\mathbf{C}^2, \Phi_1)$, $R = 1, 2, \ldots$, with $\|u_R\|_{H(\mathbf{C}^2,\Phi_1)} = \mathcal{O}_h(1)R^{1/2}$, such that

$$\|u - u_R\|_{L^2(B(0,R/2), e^{-2\Phi_1/h}L(dx))} \leqslant \mathcal{O}_h(1)e^{-R/C_0h},$$

for some $C_0 > 0$. Using that $TT^*u_R = u_R$, we see that $TT^*u = u$. Hence $B^*B = 1$. We have then checked that $B^*B = 1$, $BB^* = 1$, so B is unitary.

We recall that B is associated to a canonical transformation from Λ_{Φ_1} to $T^*(\Gamma_0)$. This allows us to view the previously defined κ also from a neighborhood of Γ in Λ_Φ to a neighborhood of $\Gamma_0 \times \{0\}$ in $T^*\Gamma_0$. We therefore have a Egorov's theorem and using $U := BA$, we get an equivalence between classical h-pseudodifferential operators acting in $H(\pi(U), \Phi)$ and h-pseudodifferential operators microlocally defined near $\xi = 0$ in $T^*\Gamma_0$, acting on Floquet periodic functions $u(x)$, satisfying:

$$u(x + e_j) = e^{-iI_j(\Gamma)/h}u(x), \tag{3.27}$$

where $e_1 = (2\pi, 0)$, $e_2 = (0, 2\pi)$.

Let $L^2_\theta(\Gamma_0)$ be the subspace of $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ of θ-Floquet periodic functions: $u(x-k) = e^{i\theta\cdot k}u(x)$, $k \in (2\pi\mathbf{Z})^2$, $\theta \in (\mathbf{R}/\mathbf{Z})^2$.

Proposition 3.1. — *Let $P^w = P^w(x, hD_x; h) : H_\Phi \to H_\Phi$ be defined as in the beginning of this section and assume that $\Gamma \subset \Lambda_\Phi$ is a Lagrangian torus satisfying (3.8), (3.9). Then there exists a smooth canonical diffeomorphism*

$$\kappa : \mathrm{neigh}\,(\Gamma, \Lambda_\Phi) \longrightarrow \mathrm{neigh}\,(\Gamma_0 \times \{0\}, T^*\Gamma_0)$$

with $\kappa(\Gamma) = \Gamma_0$, where $\Gamma_0 = (\mathbf{R}/2\pi\mathbf{Z})^2$ is the standard torus.

Moreover, there exists an operator $U : H_\Phi \to L^2_{I/2\pi h}(\Gamma_0), I = (I_1(\Gamma), I_2(\Gamma))$, with the following properties:

1) $\|U\|_{\mathcal{L}(H_\Phi, L^2_{I/2\pi h}(\Gamma_0))} = \mathcal{O}(1)$, uniformly, when $h \to 0$.

2) U is concentrated to $\overline{\mathrm{graph}\,(\kappa)}$ in the sense that if $N \in \mathbf{N}$ and $\chi_1 \in S(T^*\Gamma_0, 1)$, $\chi_2 \in C_b^\infty(\mathbf{C}^2)$ are independent of h and

$$\mathrm{supp}\,\chi_1 \times \mathrm{supp}\,\chi_2 \cap \overline{\{\kappa(y,\eta), y);\ (y,\eta) \in \mathrm{neigh}\,(\Gamma, \Lambda_\Phi)\}} = \varnothing,$$

then

$$\langle hD\rangle^N \chi_1^w(x,hD)\circ U\circ \Pi_\Phi\circ\chi_2=\mathcal{O}(h^\infty):L^2(e^{-2\Phi/h}L(dx))\longrightarrow L^2_{I/2\pi h}(\Gamma_0).$$

Here Π_Φ *is the orthogonal projection* $L^2(e^{-2\Phi/h}L(dx))\to H_\Phi$ *(see* [**MeSj**]*).*
3) U *is microlocally unitary: For every* $\chi_2\in C_0^\infty(\mathrm{neigh}\,(\pi_x(\Gamma),\mathbf{C}^2))$*, independent of* h*, we have* $(U^*U-1)\Pi_\Phi\chi_2=\mathcal{O}(h^\infty):L^2(e^{-2\Phi/h}L(dy))\to L^2(e^{-2\Phi/h}L(dy))$*. For every* $\chi_1\in C_0^\infty(\mathrm{neigh}\,(\Gamma_0\times\{0\},T^*\Gamma_0))$*, independent of* h*, we have* $(UU^*-1)\chi_1^w(x,hD)=\mathcal{O}(h^\infty):L^2_{I/2\pi h}(\Gamma_0)\to L^2_{I/2\pi h}(\Gamma_0)$*.*
4) We have a Egorov's theorem: $\exists Q(x,\xi;h)\sim q_0(x,\xi)+hq_1(x,\xi)+\cdots\in S(T^*\Gamma_0,1)$*, with* $q_0\circ\kappa=p_0$ *in* $\mathrm{neigh}\,(\Gamma,\Lambda_\Phi)$*, such that* $Q^wU=UP^w$ *microlocally, i.e.* $(Q^wU-UP^w)\Pi_\Phi\chi_2=\mathcal{O}(h^\infty)$*,* $\chi_1^w(Q^wU-UP^w)=\mathcal{O}(h^\infty)$*, for* χ_1,χ_2 *as in 3).*
5) If P*,* Φ *depend smoothly on* $z\in\mathrm{neigh}\,(0,\mathbf{C})$*, then we can find* U,κ *depending smoothly on* z *in a possibly smaller neighborhood of* 0*.*

4. Spectrum of elliptic first order differential operators on Γ_0

Let $P=Z+q$ be a first order elliptic differential operator on Γ_0 with smooth coefficients, Z denoting the vector field part. After applying a diffeomorphism, we may assume that

$$P=A(x)\frac{\partial}{\partial\overline{x}}+q(x),\tag{4.1}$$

on $\mathbf{C}/L$, $L=\mathbf{Z}e_1\oplus\mathbf{Z}e_2$, where e_1,e_2 are $\mathbf{R}$ linearly independent and $A\in C^\infty(\mathbf{C}/L)$ is non-vanishing. Further, $q\in C^\infty(\mathbf{C}/L)$, and this function will later depend on a spectral parameter. It will be convenient to introduce $B(x)=1/A(x)$. The equation $Pu=v$ becomes

$$\Big(\frac{\partial}{\partial\overline{x}}+Bq\Big)u=Bv.\tag{4.2}$$

Let $\phi\in C^\infty(\mathbf{C}/L)$ and conjugate by e^ϕ:

$$e^{-\phi}\Big(\frac{\partial}{\partial\overline{x}}+Bq\Big)e^\phi e^{-\phi}u=Be^{-\phi}v,$$

i.e.

$$\Big(\frac{\partial}{\partial\overline{x}}+\Big(\frac{\partial\phi}{\partial\overline{x}}+Bq\Big)\Big)(e^{-\phi}u)=Be^{-\phi}v.\tag{4.3}$$

Let ϕ be the periodic solution (unique up to a constant) of

$$\frac{\partial\phi}{\partial\overline{x}}+Bq=\widehat{Bq}(0),\tag{4.4}$$

where $\widehat{Bq}$ is the Fourier transform, defined on the dual lattice

$$L^*=\mathbf{Z}e_1^*\oplus\mathbf{Z}e_2^*,\ \langle e_j^*,e_k\rangle_{\mathbf{R}^2}=2\pi\delta_{j,k}.\tag{4.5}$$

Then (4.3) becomes

$$\Big(\frac{\partial}{\partial \overline{x}} + \widehat{Bq}(0)\Big)(e^{-\phi}u) = Be^{-\phi}v. \tag{4.6}$$

We want to solve (4.2) ($\Leftrightarrow$ (4.6)) in the space of θ-Floquet periodic functions, where $\theta \in \mathbf{C}/L^*$, that is in the space of functions satisfying the condition

$$u(x-\ell) = e^{i\langle \ell,\theta\rangle_{\mathbf{R}^2}}u(x),\ \forall \ell \in L. \tag{4.7}$$

Writing $\theta \equiv \theta_1 e_1^* + \theta_2 e_2^* \bmod L^*$, we get

$$u(x-e_j) = e^{2\pi i\theta_j}u(x), \tag{4.8}$$

so the relation between θ in (4.7) and the $I_j(\Gamma)$ in (3.27) is given by

$$\theta_j \equiv \frac{I_j(\Gamma)}{2\pi h} \bmod \mathbf{Z}. \tag{4.9}$$

Let $H^k_\theta(\mathbf{C}/L)$ denote the space of θ-Floquet periodic functions on $\mathbf{C}$, which are of class H^k_{loc} (standard Sobolev spaces). The Fourier series representation of such a function (with convergence at least in the sense of distributions) becomes

$$f(x) = \sum_{\nu\in L^*-\theta} \widehat{f}(\nu)e^{i\langle \nu,x\rangle_{\mathbf{R}^2}} = \sum_{\nu\in L^*-\theta} \widehat{f}(\nu)e^{\frac{i}{2}(\overline{\nu}x+\nu\overline{x})}, \tag{4.10}$$

where we used that $\langle \nu,x\rangle_{\mathbf{R}^2} = \mathrm{Re}\,\overline{\nu}x$ in the last step. The corresponding expression for $\partial f/\partial\overline{x}$ becomes:

$$\frac{\partial f}{\partial \overline{x}} = \sum_{\nu\in L^*-\theta} \frac{i}{2}\nu\widehat{f}(\nu)e^{\frac{i}{2}(\overline{\nu}x+\nu\overline{x})}. \tag{4.11}$$

We now consider (4.2), (4.6) for $u \in H^1_\theta$, $v \in H^0_\theta$, and identify Fourier coefficients,

$$(\tfrac{i}{2}\nu + \widehat{Bq}(0))(\widehat{e^{-\phi}u})(\nu) = \mathcal{F}(Be^{-\phi}v)(\nu),\ \nu \in L^*-\theta, \tag{4.12}$$

where we write $\mathcal{F}u = \widehat{u}$. We get,

Proposition 4.1
(a) If $\frac{2}{i}\widehat{Bq}(0) - \theta \notin L^*$, then P in (4.1) is bijective $H^1_\theta \to H^0_\theta$.
(b) If $\frac{2}{i}\widehat{Bq}(0) - \theta \in L^*$, then P in (4.1) is a Fredholm operator of index 0 with one-dimensional kernel given by

$$\mathrm{Ker}\,(P) = \mathbf{C}\exp[(\overline{\widehat{Bq}}(0)x - \widehat{Bq}(0)\overline{x}) + \phi(x)],$$

where ϕ solves (4.4).

Before continuing, let us compute e_1^*, e_2^*. We have

$$\begin{pmatrix}\overline{e}_1 & e_1\\ \overline{e}_2 & e_2\end{pmatrix}\begin{pmatrix}e_1^* & e_2^*\\ \overline{e}_1^* & \overline{e}_2^*\end{pmatrix} = 4\pi I,$$

so

$$\begin{pmatrix}e_1^* & e_2^*\\ \overline{e}_1^* & \overline{e}_2^*\end{pmatrix} = 2\pi\frac{2i}{\overline{e}_1e_2 - e_1\overline{e}_2}\frac{1}{i}\begin{pmatrix}e_2 & -e_1\\ -\overline{e}_2 & \overline{e}_1\end{pmatrix} = \frac{2\pi}{\mathrm{Im}\,(\overline{e}_1e_2)}\frac{1}{i}\begin{pmatrix}e_2 & -e_1\\ -\overline{e}_2 & \overline{e}_1\end{pmatrix}.$$

Hence

$$e_1^* = \frac{2\pi}{i\mathrm{Im}\,(\overline{e}_1 e_2)} e_2,\ e_2^* = -\frac{2\pi}{i\mathrm{Im}\,(\overline{e}_1 e_2)} e_1. \tag{4.13}$$

Next we introduce a complex spectral parameter z and let q be of the form

$$q(x,z) = q_0(x) + zr(x). \tag{4.14}$$

The z dependence is chosen to be linear, since the situation we examine in this section is the linearized case. Let us call the spectrum of P, the set of values z for which P is not invertible (case (b) in the proposition). Then the spectrum of P is the set of values z that satisfy

$$\frac{2}{i}\widehat{Bq_0}(0) + z\frac{2}{i}\widehat{Br}(0) - \theta \in L^*, \tag{4.15}$$

or equivalently

$$\frac{2}{i}\widehat{Bq}(0,z) \in \theta + L^*, \tag{4.15$'$}$$

and we get a non-degenerate (affine) lattice precisely when

$$\widehat{Br}(0) \neq 0. \tag{4.16}$$

5. Grushin problem near $\xi = 0$ in $T^*\Gamma_0$

In the original problem, we shall restrict the spectral parameter z to some small disc. Performing the reduction of section 3, we are led to the operator

$$Q = Q_z = Q_z^w(x, hD) = Q^w(x, hD, z)$$

on $\Gamma_0 = \mathbf{T}^2$ with semiclassical Weyl symbol:

$$Q(x,\xi,z;h) \sim q_0(x,\xi,z) + hq_1(x,\xi,z) + h^2 q_2(x,\xi,z) + \cdots,\ |\xi| \leqslant \mathcal{O}(1), \tag{5.1}$$

with

$$q_0(x,0,z) = 0, \tag{5.2}$$

and $q_0, q_1, q_2, \ldots$ depend smoothly on z. Further, we have the ellipticity property:

$$|q_0(x,\xi,z)| \sim |\xi|. \tag{5.3}$$

In the region $|\xi| \in]h^\delta, \mathcal{O}(1)]$, for $\delta > 0$ close to 0, we shall invert Q_z^w by ellipticity. In the region $|\xi| \leqslant h^\delta$, we shall use 2nd microlocalization, which here only amounts to considering our operators in the "$h = 1$" quantization, after a cosmetic multiplication by h^{-1}. The corresponding symbol (for the $h = 1$ quantization) is then

$$\frac{1}{h}Q(x,h\xi,z;h) \sim Q_0(x,\xi,z) + hQ_1(x,\xi,z) + h^2 Q_2(x,\xi,z) + \cdots, \tag{5.4}$$

where the RHS is obtained by Taylor expanding at $\xi = 0$ and regrouping terms according to powers of h. We get

$$Q_0(x,\xi,z) = \sum_{j=1}^{2} \frac{\partial q_0}{\partial \xi_j}(x,0,z)\xi_j + q_1(x,0,z), \tag{5.5}$$

while the higher Q_j will involve higher order Taylor expansions. Q_j is a polynomial of degree at most $j+1$ in ξ, and in particular,

$$Q_j \in S_{1,0}^{j+1}(T^*\Gamma_0). \tag{5.6}$$

The expression (5.4) shall be considered only in the region $|h\xi| \leqslant h^\delta$, i.e. for $|\xi| \leqslant h^{\delta-1}$, so (5.4) is a well-defined asymptotic sum for $h \to 0$ of symbols in $S_{1,0}^1$. The operator $Q_0(x, D_x, z)$ is precisely of the type studied in the preceding section, the ellipticity follows from (5.3).

From section 4 and Appendix A of section 1 we recall that $Q_0(x, D_x, z)$ can be reduced by a change of variable to $A(x,z)\frac{\partial}{\partial \overline{x}} + q(x,z)$ on $\mathbf{C}/L(z)$, where A, q, L depend smoothly on z, and that this operator: $H_\theta^1 \to H_\theta^0$ is invertible when $\theta \notin \frac{2}{i}\mathcal{F}(B(\cdot,z)q(\cdot,z))(0) + L^*(z)$ (with $B = 1/A$) and otherwise it has one dimensional kernel and cokernel. It will also be useful to recall that Q_0 can be further simplified by conjugation to

$$Q_0(x, D_x, z) = Q_0 = \frac{\partial}{\partial \overline{x}} + \theta_0(z), \tag{5.7}$$

where

$$\theta_0(z) = \frac{2}{i}\widehat{Bq}(0,z). \tag{5.8}$$

A simplified version of the discussion below shows that $\frac{1}{h}Q_z : H_\theta^1 \to H_\theta^0$ is invertible (microlocally in $|\xi| \leqslant \mathcal{O}(1)$), when $\operatorname{dist}(\theta, \theta_0(z) + L^*(z)) \geqslant 1/\mathcal{O}(1)$. We concentrate on the more interesting case when this distance is small. Since θ is really defined only modulo $L^*(z)$, we decide to think of θ as a complex number close to $\theta_0(z)$.

Let $e_\theta(x) = ce^{-i\theta\cdot x}$ with $\cdot$ indicating that we take the $\mathbf{R}^2$ scalar product, and $c = c(z)$ is chosen to normalize $e_\theta(x)$ in $H_\theta^0(\mathbf{C}/L(z))$. Then

$$\mathcal{Q}_0(\theta,z) = \begin{pmatrix} Q_0(z) & R_{-,\theta} \\ R_{+,\theta} & 0 \end{pmatrix} : H_\theta^1 \times \mathbf{C} \longrightarrow H_\theta^0 \times \mathbf{C} \tag{5.9}$$

is bijective, where

$$R_{+,\theta}u = (u|e_\theta), \ R_{-,\theta}u_- = u_- e_\theta. \tag{5.10}$$

We denote the inverse by

$$\mathcal{E}_0(\theta,z) = \begin{pmatrix} E^0(\theta,z) & E_+^0(\theta,z) \\ E_-^0(\theta,z) & E_{-+}^0(\theta,z) \end{pmatrix}. \tag{5.11}$$

This depends smoothly on z and analytically on θ. By Beals' lemma, we know that

$$E^0 \in \mathrm{Op}_1(S_{1,0}^{-1}). \tag{5.12}$$

Moreover,

$$E_+^0(\theta,z)v_+ = v_+e_+^0(\theta,z),\ E_-^0(\theta,z)v = (v|e_-^0(\theta,z)), \tag{5.13}$$

where $e_\pm^0 \in C_\theta^\infty$, and $E_{-+}^0 \in \mathbf{C}$ with

$$|E_{-+}^0(\theta,z)| \sim |\theta-\theta_0(z)|, \tag{5.14}$$

and with $\theta_0(z)$ defined in (5.8). More explictily, using (5.7), we have $e_+^0 = e_-^0 = e_\theta$, $E_{-+}^0(\theta,z) = \frac{i\theta}{2} - \widehat{Bq}(0)$. Recall or notice that $Q_0(z) : H_\theta^1 \to H_\theta^0$ is invertible precisely for $\theta \neq \theta_0$ and that the inverse is given by $E^0(\theta,z) - E_+^0(\theta,z)E_{-+}^0(\theta,z)^{-1}E_-(\theta,z)$.

Now put

$$\mathcal{Q}(\theta,z) = \begin{pmatrix} \frac{1}{h}Q_z(x,hD_x;h) & R_{-,\theta} \\ R_{+,\theta} & 0 \end{pmatrix} \tag{5.15}$$

formally as an operator $H_\theta^1 \times \mathbf{C} \to H_\theta^0 \times \mathbf{C}$, so that (in view of (5.4))

$$\mathcal{Q}(\theta,z) \sim \sum_0^\infty h^j \mathcal{Q}_j(\theta,z), \tag{5.16}$$

with

$$\mathcal{Q}_j(\theta,z) = \begin{pmatrix} Q_j(x,D_x,z) & 0 \\ 0 & 0 \end{pmatrix},\ j \geqslant 1. \tag{5.17}$$

For simplicity, we assume that the same conjugation that simplified Q_0 to the form (5.7) has been applied to $h^{-1}Q_z$. We invert $\mathcal{Q}$ formally by the asymptotic Neumann series

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_0 - \mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 + \mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 - \cdots \\ &= \sum_0^\infty (-1)^k \mathcal{E}_0((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \sum_0^\infty (-1)^k (\mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0))^k \mathcal{E}_0. \end{aligned} \tag{5.18}$$

Write $Q_h = \frac{1}{h}Q_z(x,hD_x;h)$. Then

$$(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 = \begin{pmatrix} (Q_h-Q_0)E^0 & (Q_h-Q_0)E_+^0 \\ 0 & 0 \end{pmatrix}, \tag{5.19}$$

and for $k \geqslant 1$:

$$((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \begin{pmatrix} ((Q_h-Q_0)E^0)^k & ((Q_h-Q_0)E^0)^{k-1}(Q_h-Q_0)E_+^0 \\ 0 & 0 \end{pmatrix}. \tag{5.20}$$

The general term in the series (5.18) becomes

$$\begin{aligned} &(-1)^k\mathcal{E}_0((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \\ &\begin{pmatrix} (-1)^k E^0((Q_h-Q_0)E^0)^k & (-1)^k(E^0(Q_h-Q_0))^kE_+^0 \\ (-1)^kE_-^0((Q_h-Q_0)E^0)^k & (-1)^kE_-^0((Q_h-Q_0)E^0)^{k-1}(Q_h-Q_0)E_+^0 \end{pmatrix}. \end{aligned} \tag{5.21}$$

Here $(Q_h-Q_0)E^0$, $E^0(Q_h-Q_0)$ are $(h=1)$ pseudodifferential operators with symbols in $hS_{1,0}^1 + h^2S_{1,0}^2 + \cdots$. $((Q_h-Q_0)E^0)^k$, $(E^0(Q_h-Q_0))^k$ then have their symbols in $h^kS_{1,0}^k + h^{k+1}S_{1,0}^{k+1} + \cdots$. It follows that $E^0((Q_h-Q_0)E^0)^k$ has its symbol in $h^kS_{1,0}^{k-1} + h^{k+1}S_{1,0}^k + \cdots$. Moreover, $(E^0(Q_h-Q_0))^kE_+^0v_+ = v_+e_+^k$, with e_+^k in $h^k C_\theta^\infty + h^{k+1}C_\theta^\infty + \cdots$ and similarly for $E_-^0((Q_h - Q_0)E^0)^k$. Finally $E_-^0((Q_h - Q_0)E^0)^{k-1}(Q_h - Q_0)E_+^0$ belongs to $h^k\mathbf{C} + h^{k+1}\mathbf{C} + \cdots$. Using all this in the asymptotic series (5.18), we get

$$\mathcal{E} = \begin{pmatrix} E(\theta,z) & E_+(\theta,z) \\ E_-(\theta,z) & E_{-+}(\theta,z)\end{pmatrix}, \tag{5.22}$$

where
- $E(\theta,z)$ is a 1-pseudodifferential operator with symbol in $S_{1,0}^{-1} + hS_{1,0}^0 + \cdots$.
- $E_+v_+ = v_+e_+$, $E_-u = (u|e_-)$, with $e_\pm \in C^\infty + hC^\infty + h^2C^\infty + \cdots$.
- $E_{-+}(\theta,z) \in \mathbf{C} + h\mathbf{C} + h^2\mathbf{C} + \cdots$, more explicitly,

$$E_{-+}(\theta,z) \sim E_{-+}^0(\theta,z) + hE_{-+}^1(\theta,z) + \cdots\ . \tag{5.23}$$

Formally, the spectrum of P_z^w (acting on θ-Floquet functions) will be the set of values z for which $E_{-+}(\theta,z) = 0$.

We will now sum up the discussion of this section, and for that it will be convenient to return to the case of the standard torus Γ_0. Then the dual lattice "$L^*(z)$" is simply $\mathbf{Z}^2$ and $Q_0(z)$ in (5.5) will be invertible $H_\theta^1(\Gamma_0) \to H_\theta^0(\Gamma_0)$ precisely when

$$\theta \notin \theta_0(z) + \mathbf{Z}^2, \tag{5.24}$$

where $\theta_0(z) \in \mathbf{R}^2$ depends smoothly on z.

Proposition 5.1. — *Let $C > 0$ be a sufficiently large constant.*
1. For $\operatorname{dist}(\theta, \theta_0(z)+\mathbf{Z}^2) \geqslant 1/C$, $z \in \operatorname{neigh}(0,\mathbf{C})$, *there exists an operator* $F(\theta,z;h) = \mathcal{O}(1) : H_\theta^0 \to H_\theta^0$ *such that:*
1a) F is pseudolocal in the sense that $\langle hD\rangle^N \chi_1(x,hD)F\chi_2(x,hD)\langle hD\rangle^N = \mathcal{O}(h^N) : H_\theta^0 \to H_\theta^0$ *for every $N \in \mathbf{N}$ and all $\chi_j \in C_b^\infty(T^*\Gamma_0)$, $j=1,2$, independent of h with* $(\operatorname{supp}\chi_1 \times \operatorname{supp}\chi_2) \cap (\operatorname{diag}(T^*\Gamma_0)^2 \cup (\Gamma_0 \times \{0\})^2) = \varnothing$.
1b) There is a neighborhood $V \subset T^*\Gamma_0$ of $\Gamma_0 \times \{0\}$ such that

$$\Big(\frac{1}{h}QF - 1\Big)\chi^w,\ \chi^w\Big(\frac{1}{h}QF - 1\Big) = \mathcal{O}(h^\infty) : H_\theta^0 \longrightarrow H_\theta^0,$$

for every $\chi \in C_0^\infty(V)$, independent of h. The same holds with $F\frac{1}{h}Q$ instead of $\frac{1}{h}QF$. (Notice that these compositions are welldefined $\operatorname{mod}\mathcal{O}(h^\infty) : H_\theta^0 \to H_\theta^0$*.)*
2. For $\operatorname{dist}(\theta,\theta_0(z) + \mathbf{Z}^2) \leqslant 1/C$, *we may assume (by $\mathbf{Z}^2$-periodicity in θ) that $\theta \in \mathbf{R}^2$, $|\theta - \theta_0(z)| \leqslant 1/C$. Then we have rank one operators $R_{+,\theta}u = (u|e_{\theta,z})$, $R_{-,\theta}u_- = u_-f_{\theta,z}$, $R_{+,\theta} : H_\theta^0 \to \mathbf{C}$, $R_{-,\theta} : \mathbf{C} \to H_\theta^0$, with $e_{\theta,z}, f_{\theta,z} \in H_\theta^0 \cap C_b^\infty$ depending smoothly on θ, z, independent of h, and a bounded operator*

$$\mathcal{E} = \begin{pmatrix} E(\theta,z;h) & E_+(\theta,z;h) \\ E_-(\theta,z;h) & E_{-+}(\theta,z;h)\end{pmatrix} = \mathcal{O}(1) : H_\theta^0 \times \mathbf{C} \longrightarrow H_\theta^0 \times \mathbf{C},$$

with the following properties:
2a) E is pseudolocal as in 1a.

2b) If $\chi \in C_b^\infty(T^*\Gamma_0)$ is independent of h and $\Gamma_0 \times \{0\} \cap \operatorname{supp}\chi = \varnothing$, then for every $N \in \mathbf{N}$:

$$\langle hD\rangle^N \chi^w E_+ = \mathcal{O}(h^\infty) : \mathbf{C} \longrightarrow H_\theta^0, \quad E_-\chi^w \langle hD\rangle^N = \mathcal{O}(h^\infty) : H_\theta^0 \longrightarrow \mathbf{C}.$$

2c) E_{-+} has the asymptotic expansion (5.23) with $|E_{-+}^0(\theta, z)| \sim |\theta - \theta_0(z)|$.
2d) $\mathcal{E}$ is an inverse of $\mathcal{Q}$ in (5.15) in the sense that

$$(\mathcal{Q}\mathcal{E} - 1)\begin{pmatrix}\chi^w & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}\chi^w & 0\\ 0 & 1\end{pmatrix}(\mathcal{Q}\mathcal{E} - 1) = \mathcal{O}(h^\infty) : H_\theta^0 \times \mathbf{C} \longrightarrow H_\theta^0 \times \mathbf{C},$$

for all $\chi \in C_0^\infty(V)$, independent of h. Here V is as in 1b and we can replace $\mathcal{Q}\mathcal{E}$ by $\mathcal{E}\mathcal{Q}$ in the preceding estimates.

6. The main result

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbf{C}^2$ and let $P(x,\xi) = P(x,\xi,z;h)$ be a bounded holomorphic function in a tubular neighborhood of Λ_{Φ_0}, which depends holomorphically on $z \in \operatorname{neigh}(0,\mathbf{C})$, with the asymptotic expansion

$$P(x,\xi,z;h) \sim \sum_{k=0}^{\infty} p_k(x,\xi,z)h^k, \tag{6.1}$$

in the space of such functions. Later, it will be convenient to assume that the subprincipal symbol vanishes:

$$p_1(x,\xi,z) = 0. \tag{6.2}$$

Also assume ellipticity near infinity:

$$|p(x,\xi,z)| \geqslant 1/C,\ (x,\xi) \in \Lambda_{\Phi_0},\ |(x,\xi)| \geqslant C, \tag{6.3}$$

where $p = p_0$. (The boundedness assumption above could easily be replaced by some other symbol type condition, provided of course that we modify the ellipticity assumption accordingly.)

Assume for $z = 0$, that $\Sigma = p^{-1}(0) \cap \Lambda_{\Phi_0}$ is smooth, connected and that

$$dp_{\Lambda_{\Phi_0}}, d\overline{p}_{\Lambda_{\Phi_0}} \text{ are linearly independent on } \Sigma. \tag{6.4}$$

Further assume that

$$\{p_{\Lambda_{\Phi_0}}, \overline{p}_{\Lambda_{\Phi_0}}\} \text{ is small on } \Sigma, \tag{6.5}$$

where we adopt the convention of section 1, that we have uniformity in the other assumptions. Recall also from section 1, that this implies that Σ is a smooth torus. Notice that the assumptions above will also be fulfilled for $p = p(\cdot, z)$ when z is close enough to 0.

In section 1, we showed that $p(\cdot, z)^{-1}(0)$ contains a smooth torus $\Gamma(z)$, which is close to Σ and such that

$$\sigma_{|\Gamma(z)} = 0, \tag{6.6}$$

$$I_j(\Gamma(z),\omega) \in \mathbf{R},\ j=1,2, \tag{6.7}$$

where $\omega = \xi_1 dx_1 + \xi_2 dx_2$ and $I_j(\Gamma(z),\omega)$ is the corresponding action along the jth fundamental cycle in $\Gamma(z)$. (Any other global primitive of σ gives the same actions.) $\Gamma(z)$ is not unique, but thanks to (6.7) its image in the quotient space $\mathcal{M}(z)$ of $p(\cdot,z)^{-1}(0)$ by the action of $H_{p(\cdot,z)}$, is unique. The full preimage of this image is a complex Lagrangian manifold $\widetilde{\Lambda}(z)$ which is also uniquely determined and which can be viewed as a complexification of the totally real manifold $\Gamma(z)$. This is Λ_ϕ in (1.32).

It is easy to see that $\Gamma(z)$ can be chosen to depend smoothly on z. Also thanks to (6.7), we have $\Gamma(z) \subset \Lambda_z$, where $\Lambda_z = \Lambda_{\Phi_z}$ is an IR-manifold close to Λ_{Φ_0} and we can view $\Gamma(z)$ as a Lagrangian submanifold of this real symplectic manifold. Λ_z can also be chosen to depend smoothly on z, and we may assume that $\Phi_z - \Phi_0 = \mathcal{O}(1)$

Let $I_j(z) = I_j(\Gamma(z),\omega)$, $I(z) = (I_1(z), I_2(z)) \in \mathbf{R}^2$. Let $P(z) = P^w(z) = P^w(x, hD_x, z; h)$ be the corresponding Weyl quantization which acts on H_{Φ_z}. Let $U = U(z)$, $Q = Q^w(z)$ be as in Proposition 3.1, and depend smoothly on $z \in \mathrm{neigh}\,(0,\mathbf{C})$. Then $Q_0 = Q_0(x, D_x, z) : H^1_\theta(\Gamma_0) \to H^0_\theta(\Gamma_0)$ (c.f. (5.4)) is invertible precisely when $\theta \notin \theta_0(z) + \mathbf{Z}^2$, where $\theta_0 \in \mathbf{R}^2$ depends smoothly on z (cf. (5.14)). We also recall from section 3, that we will naturally have $\theta = I(z)/(2\pi h)$. We first consider the case when

$$\mathrm{dist}\left(\frac{I(z)}{2\pi h}, \theta_0(z) + \mathbf{Z}^2\right) \geqslant \frac{1}{C}. \tag{6.8}$$

Let $\chi_2 \in C_0^\infty(\mathrm{neigh}\,(\pi_x(\Gamma(0)), \mathbf{C}^2))$ be equal to 1 in a neighborhood of $\pi_x\Gamma(0)$. As an approximate right inverse to $h^{-1}P^w(z)$, we take

$$J := h\Pi_\Phi G \Pi_\Phi (1-\chi_2) + \Pi_\Phi U^* F U \Pi_\Phi \chi_2, \tag{6.9}$$

where $F = F(z)$ is given by Proposition 5.1, with $\theta = I(z)/(2\pi h)$, and $G = G(z)$ is an asymptotic inverse to P^w away from $\pi_x(\Gamma)$ in the sense of Töplitz operators in section 3 of [**MeSj**], and Π_Φ is the orthogonal projection $L^2(e^{-2\Phi/h}L(dx)) \to H_\Phi$. Then

$$P^w \Pi_\Phi G \Pi_\Phi (1-\chi_2) = \Pi_\Phi(1-\chi_2) + \mathcal{O}(h^\infty) : H_\Phi \longrightarrow H_\Phi.$$

On the other hand, if we use local unitarity of U, the pseudolocality of F and 4) of Proposition 3.1, we get

$$\begin{aligned}\frac{1}{h}P^w \Pi_\Phi U^* F U \Pi_\Phi \chi_2 &\equiv \Pi_\Phi U^* \frac{1}{h} Q^w F U \Pi_\Phi \chi_2 \\ &\equiv \Pi_\Phi U^* U \Pi_\Phi \chi_2 \equiv \Pi_\Phi \chi_2 \mod \mathcal{O}(h^\infty) : H_\Phi \longrightarrow H_\Phi.\end{aligned}$$

It follows that

$$\frac{1}{h}P^w(z) J \equiv \Pi_\Phi = 1 \mod \mathcal{O}(h^\infty) : H_\Phi \longrightarrow H_\Phi. \tag{6.10}$$

(Most of our operators as well as Φ depend on z, and this dependence is always smooth.)

In the same way we can show that

$$K = \Pi_\Phi(1-\chi_2)\Pi_\Phi hG + \Pi_\Phi\chi_2\Pi_\Phi U^* FU \tag{6.11}$$

satisfies

$$K\frac{1}{h}P^w \equiv 1 \bmod \mathcal{O}(h^\infty) : H_\Phi \longrightarrow H_\Phi. \tag{6.12}$$

We conclude that under the assumption (6.8), the operator $\frac{1}{h}P^w : H_\Phi \to H_\Phi$ has an inverse which is uniformly bounded, when $h \to 0$.

We now consider the case when

$$\operatorname{dist}\left(\frac{I}{2\pi h}, \theta_0(z) + \mathbf{Z}^2\right) \leqslant \frac{1}{C}, \tag{6.13}$$

for some large fixed $C > 0$. Let k be the point in $\mathbf{Z}^2$ such that

$$\left|k + \frac{I}{2\pi h} - \theta_0(z)\right| \leqslant \frac{1}{C}.$$

We apply the second part of Proposition 5.1 with $\theta = k + I/(2\pi h)$. Let $\mathcal{E}, R_+, R_-$ be as there. Consider

$$\mathcal{P}(z) = \begin{pmatrix} \frac{1}{h}P(z) & \Pi_\Phi \widetilde{R}_-(z) \\ \widetilde{R}_+(z) & 0 \end{pmatrix} : H_{\Phi_z} \times \mathbf{C} \longrightarrow H_{\Phi_z} \times \mathbf{C}, \tag{6.14}$$

with

$$\widetilde{R}_+ = R_+ U, \ \widetilde{R}_- = U^* R_-. \tag{6.15}$$

As an approximate right inverse to $\mathcal{P}$, we take (with $E, E_\pm, E_{-+}$ as in (5.22))

$$\widetilde{\mathcal{E}}_r = \begin{pmatrix} h\Pi_\Phi G\Pi_\Phi(1-\chi_2) + \Pi_\Phi U^* EU\Pi_\Phi\chi_2 & \Pi_\Phi U^* E_+ \\ E_- U & E_{-+} \end{pmatrix} =: \begin{pmatrix} \widetilde{E} & \widetilde{E}_+ \\ \widetilde{E}_- & \widetilde{E}_{-+} \end{pmatrix}. \tag{6.16}$$

We need to check that

$$\begin{cases} \frac{1}{h}P\widetilde{E} + \Pi_\Phi \widetilde{R}_-\widetilde{E}_- \equiv 1, \ \frac{1}{h}P\widetilde{E}_+ + \Pi_\Phi\widetilde{R}_-\widetilde{E}_{-+} \equiv 0, \\ \widetilde{R}_+\widetilde{E} \equiv 0, \ \widetilde{R}_+\widetilde{E_+} \equiv 1, \end{cases} \tag{6.17}$$

modulo terms that are $\mathcal{O}(h^\infty)$ in operator norm:

$$\begin{aligned} \frac{1}{h}P\widetilde{E} + \Pi_\Phi\widetilde{R}_-\widetilde{E}_- &\equiv \Pi_\Phi(1-\chi_2) + \frac{1}{h}P\Pi_\Phi U^* EU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^*\frac{1}{h}QEU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^*\frac{1}{h}QEU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U\Pi_\Phi\chi_2 \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^* U\Pi_\Phi\chi_2 \equiv \Pi_\Phi \\ &= 1, \end{aligned}$$

$$\begin{aligned}\frac{1}{h}P\widetilde{E}_+ + \Pi_\Phi \widetilde{R}_- \widetilde{E}_{-+} &\equiv \frac{1}{h}P\Pi_\Phi U^* E_+ + \Pi_\Phi U^* R_- E_{-+}\\ &\equiv \Pi_\Phi U^*(\frac{1}{h}QE_+ + R_-E_{-+}) \equiv \Pi_\Phi U^* 0 = 0,\end{aligned}$$

$$\begin{aligned}\widetilde{R}_+\widetilde{E} &\equiv R_+U\Pi_\Phi\chi_2(h\Pi_\Phi G\Pi_\Phi(1-\chi_2) + \Pi_\Phi U^* EU\Pi_\Phi\chi_2)\\ &\equiv 0 + R_+UU^*EU\Pi_\Phi\chi_2 \equiv R_+EU\Pi_\Phi\chi_2 \equiv 0,\end{aligned}$$

$$\widetilde{R}_+\widetilde{E}_+ \equiv R_+U\Pi_\Phi U^* E_+ \equiv R_+E_+ \equiv 1.$$

So,

$$\mathcal{P}\widetilde{\mathcal{E}}_r = 1 + \mathcal{O}(h^\infty). \tag{6.18}$$

Similarly, we check that

$$\widetilde{\mathcal{E}}_\ell\mathcal{P} = 1 + \mathcal{O}(h^\infty), \tag{6.19}$$

where

$$\widetilde{\mathcal{E}}_\ell = \begin{pmatrix}\Pi_\Phi(1-\chi_2)\Pi_\Phi hG + \Pi_\Phi\chi_2\Pi_\Phi U^*EU & \Pi_\Phi U^* E_+\\ E_-U & E_{-+}\end{pmatrix}. \tag{6.20}$$

We sum up the discussion so far:

Proposition 6.1. — *Under the preceding assumption, there exists a smooth map* $\mathrm{neigh}\,(0,\mathbf{C}) \mapsto \theta_0(z) \in \mathbf{R}^2$, *such that if we fix* $C > 0$ *large enough:*
1) For $\mathrm{dist}\,(I(z)/(2\pi h), \theta_0(z) + \mathbf{Z}^2) \geqslant (2C)^{-1}$, $h^{-1}P^w(z) : H_{\Phi_z} \to H_{\Phi_z}$ *has a uniformly bounded inverse.*
2) For

$$\mathrm{dist}(I(z)/(2\pi h), \theta_0(z) + \mathbf{Z}^2) < 1/C, \tag{6.21}$$

the operator $\mathcal{P}(z)$ *in (6.14) has a uniformly bounded inverse*

$$\mathcal{F}(z) = \begin{pmatrix}F(z) & F_+(z)\\ F_-(z) & F_{-+}(z)\end{pmatrix} : H_{\Phi_z}\times\mathbf{C} \longrightarrow H_{\Phi_z}\times\mathbf{C}. \tag{6.22}$$

Modulo terms that are $\mathcal{O}(h^\infty)$ *in operator norm, we have*

$$\begin{aligned}F_+(z) &\equiv U^*(z)E_+\Big(k + \frac{I(z)}{2\pi h}, z; h\Big),\\ F_-(z) &\equiv E_-\Big(k + \frac{I(z)}{2\pi h}, z; h\Big)U(z),\\ F_{-+}(z) &\equiv E_{-+}\Big(k + \frac{I(z)}{2\pi h}, z; h\Big),\end{aligned} \tag{6.23}$$

where $k \in \mathbf{Z}^2$ *is the point with* $|k - \theta_0(z) + I(z)/(2\pi h)| < 1/C$, *and* E_+, E_-, E_{-+} *are given in Proposition 5.1.*

From (6.23) and (5.23) we get the following asymptotic expansion in case 2) of the proposition:

$$F_{-+}(z;h) \sim E^0_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big)+hE^1_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big)+\cdots, \tag{6.24}$$

valid in the sense that

$$|R_N(z;h)| \leqslant C_N h^{N+1}, \tag{6.25}$$

where

$$R_N(z;h)=F_{-+}(z;h)-\sum_0^N h^j E^j_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big). \tag{6.26}$$

We shall next see that (6.24) can be differentiated with respect to z in the natural sense. Indeed, it is clear that $\nabla_z^j\mathcal{P}(z)=\mathcal{O}(h^{-j})$ in operator norm for $j=0,1,2,\dots$, so if we use that

$$\nabla_z\mathcal{F}(z)=-\mathcal{F}(z)\nabla_z\mathcal{P}(z)\mathcal{F}(z) \tag{6.27}$$

and similar more elaborate expressions for $\nabla_z^j\mathcal{F}(z)$, we see that

$$\nabla^j\mathcal{F}(z)=\mathcal{O}(h^{-j}) \tag{6.28}$$

in operator norm for $j=0,1,2,\dots$. In particular,

$$\nabla_z^j F_{-+}(z;h)=\mathcal{O}(h^{-j}), \tag{6.29}$$

and the same estimate holds for each of the terms in (6.24). It follows that

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(1). \tag{6.30}$$

Now combine (6.25,30) with elementary convexity estimates for the derivatives to conclude that

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(h^{N+1-\varepsilon}),$$

for every $\varepsilon>0$ (after an arbitrarily small increase of the constant C in (6.21)). Since

$$R_N(z;h)=h^{N+1}E^{N+1}_{-+}\Big(k-\frac{I(z)}{2\pi h},z\Big)+R_{N+1}(z;h),$$

we get

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(h^{N+1}), \tag{6.31}$$

for every $j=0,1,2,\dots$. So we have proved that (6.24) can be differentiated with respect to z as many times as we want, in the natural way.

In this context, it may be of some interest to notice that F_{-+} is holomorphic in z after multiplication by a non-vanishing factor. Indeed, from (6.27) and the fact that $\partial_{\overline{z}}P^w(z)=0$, we get

$$\partial_{\overline{z}}F_{-+}+F_-(\partial_{\overline{z}}\Pi_\Phi\widetilde{R}_-(z))F_{-+}+F_{-+}(\partial_{\overline{z}}\widetilde{R}_+)F_+=0.$$

Since F_{-+} is scalar, this simplifies to

$$(\partial_{\overline{z}}+v(z))F_{-+}(z)=0,\ v(z)=F_-(\partial_{\overline{z}}\Pi_\Phi\widetilde{R}_-(z))+(\partial_{\overline{z}}\widetilde{R}_+(z))F_+. \tag{6.32}$$

If $\partial_{\overline{z}}V(z) = v(z)$ (and this equation can always be solved after increasing C in (6.21)), we get

$$\partial_{\overline{z}}(e^{V(z)}F_{-+}) = 0. \tag{6.33}$$

Since $\nabla_z^j v = \mathcal{O}(h^{-1-j})$, $j \geqslant 0$, we can restrict the attention to some disc of radius ch (after fixing k after (6.23)) and get

$$\nabla_z^j V = \mathcal{O}(h^{-j}),\ j \geqslant 0. \tag{6.34}$$

(Make the change of variable: $z = z_0 + hw$.)

We recall a general fact about Grushin problems, namely that $P^w(z)$ is invertible precisely when $F_{-+}(z)$ is. We will say that $z = z_0$ is an eigen-value of $z \mapsto P^w(z)$ if $P^w(z_0)$ is non-invertible. For such an eigen-value, we define the corresponding multiplicity $m(z_0)$ to be the order of z_0 as a zero of the holomorphic function $e^V F_{-+}$. In the appendix A to this section we show that this multiplicity does not depend on the way we construct the Grushin problem and also that it is the order of z_0 as a zero of $\det P^w(z)$ in case $P(z) - 1$ is of trace class.

We shall next use the assumption (6.2) and show that we have $\theta_0(z) = \text{Const.} \in (\frac{1}{2}\mathbf{Z})^2$. We shall do this by studying Floquet periodic WKB solutions in a neighborhood of $\pi_x(\Sigma)$, and we start by reviewing some facts for such solutions when working with the Weyl quantization for the corresponding pseudodifferential operators. (Cf. Appendix a in [**HeSj2**].)

Recall that the Weyl quantization of a symbol p on $\mathbf{R}^{2n}$ is given by:

$$p^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta} p\Big(\frac{x+y}{2}, \theta\Big)u(y)dyd\theta. \tag{6.35}$$

Let $\phi(x)$ be a smooth and real function. (The adaptation to the complex environment will be quite immediate.) Then

$$\begin{aligned} &e^{-\frac{i}{h}\phi(x)}p^w(x, hD_x)e^{\frac{i}{h}\phi(x)}u(x) \\ &\qquad = \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}((x-y)\cdot\theta-(\phi(x)-\phi(y)))} p\Big(\frac{x+y}{2}, \theta\Big)u(y)dyd\theta. \end{aligned} \tag{6.36}$$

Employ the Kuranishi trick: $\phi(x) - \phi(y) = (x - y) \cdot \Phi(x, y)$, with

$$\Phi(x, y) = \int_0^1 \frac{\partial\phi}{\partial x}(tx + (1 - t)y)dt,$$

and notice that

$$\Phi(x, y) = \frac{\partial\phi}{\partial x}\Big(\frac{x+y}{2}\Big) + \mathcal{O}((x - y)^2).$$

Then,

$$(6.37)\quad \begin{aligned} &e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}u \\ &\quad= \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot(\theta-\Phi(x,y))}p\Big(\frac{x+y}{2},\theta\Big)u(y)dyd\theta \\ &\quad= \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}p\Big(\frac{x+y}{2},\theta+\Phi(x,y)\Big)u(y)dyd\theta. \end{aligned}$$

Here

$$(6.38)\quad p\Big(\frac{x+y}{2},\theta+\Phi(x,y)\Big) = p\Big(\frac{x+y}{2},\theta+\frac{\partial\phi}{\partial x}\Big(\frac{x+y}{2}\Big)\Big) + \mathcal{O}((x-y)^2),$$

and it follows easily (by double integration by parts with respect to θ for the contribution from the remainder) that the h-Weyl symbol of $e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}$ is equal to $p(x,\theta+\frac{\partial\phi}{\partial x}(x))+\mathcal{O}(h^2)$.

Suppose that ϕ solves the eikonal equation

$$(6.39)\quad p\Big(x,\frac{\partial\phi}{\partial x}(x)\Big)=0.$$

We look for a smooth function $a(x)$, independent of h, such that

$$(6.40)\quad e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}a(x)=\mathcal{O}(h^2),$$

and get

$$(6.41)\quad p_\phi^w(x,hD_x)a(x)=\mathcal{O}(h^2),$$

where

$$(6.42)\quad p_\phi(x,\xi)=p\Big(x,\xi+\frac{\partial\phi}{\partial x}(x)\Big).$$

Write

$$(6.43)\quad p_\phi(x,\xi)=\sum_1^n\frac{\partial p_\phi}{\partial\xi_j}(x,0)\xi_j+\mathcal{O}(\xi^2).$$

The remainder will give an $\mathcal{O}(h^2)$ contribution to (6.41) and the Weyl quantization of the sum is

$$(6.44)\quad \frac{1}{2}\sum_1^n\Big(\frac{\partial p_\phi}{\partial\xi_j}(x,0)\circ hD_{x_j}+hD_{x_j}\circ\frac{\partial p_\phi}{\partial\xi_j}(x,0)\Big)=\frac{h}{i}\Big(\nu\Big(x,\frac{\partial}{\partial x}\Big)+\frac{1}{2}\mathrm{div}\,(\nu)\Big),$$

where $\nu(x,\frac{\partial}{\partial x})=\sum\frac{\partial p_\phi}{\partial\xi_j}(x,0)\frac{\partial}{\partial x_j}$ can be identified with the restriction of H_p to Λ_ϕ: $\xi=\phi'(x)$. The equation (6.41) therefore boils down to the transport equation

$$(6.45)\quad \Big(\nu\Big(x,\frac{\partial}{\partial x}\Big)+\frac{1}{2}\mathrm{div}\,(\nu)\Big)a=0.$$

As in [**DuHo**] the last equation can also be written in terms of the Lie derivative of ν acting on a half density:

$$(6.46)\quad \mathcal{L}_\nu(a(x)(dx_1\cdots dx_n)^{1/2})=0.$$

Recall that $\widetilde{\Lambda}(z) \subset p(\cdot, z)^{-1}(0)$ is a complex Lagrangian manifold which can be viewed as a complexification of $\Gamma(z)$. We can represent $\widetilde{\Lambda}(z)$ by

$$\xi = \frac{\partial \phi}{\partial x}(x, z), \ x \in \text{neigh}\,(\pi_x(\Sigma)), \tag{6.47}$$

where ϕ is grad periodic, smooth in both variables, holomorphic in z and (cf. (2.4)) satisfies

$$\Phi(x, z) + \operatorname{Im} \phi(x, z) \sim \text{dist}\,(x, \pi_x\Gamma(z))^2, \tag{6.48}$$

where $\Phi(\cdot, z) = \Phi_z$, $\Lambda_z = \Lambda_{\Phi_z}$. If $\gamma_1, \gamma_2 \subset \pi_x(\Gamma(z))$ are two fundamental cycles, we also have

$$\text{var}_{\gamma_j} \phi(\cdot, z) = I_j(z), \ p(x, \frac{\partial \phi}{\partial x}(x, z), z) = 0, \tag{6.49}$$

with $I_j(z) = I_j(\Gamma(z), \omega)$. We look for a multivalued holomorphic symbol $a(x) = a(x, z)$ (being the leading term in an asymptotic expansion) such that

$$P^w(x, hD_x, z; h)(\frac{1}{h} a(x, z) e^{i\phi(x,z)/h}) = \mathcal{O}(h) e^{i\phi(x,z)/h}. \tag{6.50}$$

As reviewed above, (6.50) is equivalent to the transport equation

$$\mathcal{L}_\nu(a(x)(dx_1 \wedge dx_2)^{1/2}) = 0, \tag{6.51}$$

where $\nu \simeq H_{p|_{\widetilde{\Lambda}(z)}}$. We only want to solve (6.51) to infinite order on $\pi_x(\Gamma(z))$ which is maximally totally real, so we can restrict (6.51) to this torus by interpreting $\nu \simeq H_p$ as a complex vector field here. Once (6.51) is solved on the submanifold, we get it to infinite order there, by taking almost holomorphic extensions.

Recall from section 1 that there is a diffeomorphism

$$Q : \Gamma(z) \longrightarrow \mathbf{C}/L(z), \tag{6.52}$$

depending smoothly on z such that

$$\nu \simeq H_p = A \frac{\partial}{\partial \overline{Q}}, \tag{6.53}$$

where $A = A(Q, z)$ is smooth and non-vanishing.

Write $a(x)(dx_1 \wedge dx_2)^{1/2} = b(Q)(dQ_1 \wedge dQ_2)^{1/2}$, $Q = Q_1 + iQ_2$. We notice that

$$\frac{(dx_1 \wedge dx_2)^{1/2}}{(dQ_1 \wedge dQ_2)^{1/2}}$$

is not necessarily single valued, but θ_1-Floquet periodic for some $\theta_1 \in \frac{1}{2}L^*$. Then (6.51) becomes

$$\mathcal{L}_{A\frac{\partial}{\partial \overline{Q}}}(b(dQ_1 \wedge dQ_2)^{1/2}) = 0,$$

and more explicitly

$$A \frac{\partial}{\partial \overline{Q}} b + \frac{1}{2} \frac{\partial}{\partial \overline{Q}}(A) b = 0, \tag{6.54}$$

since $\operatorname{div} A\frac{\partial}{\partial \overline{Q}} = \frac{\partial}{\partial \overline{Q}} A$. (6.54) can also be written

$$\frac{\partial}{\partial \overline{Q}}(A^{1/2}b) = 0, \tag{6.55}$$

where we notice that $A^{1/2}$ is α-Floquet periodic for some $\alpha \in \frac{1}{2}L^*$.

We restrict the attention to solutions $u = h^{-1/2}ae^{i\phi/h}$ of (6.50) which are multi-valued but ω-Floquet periodic in the sense that

$$u(\Gamma_j^{-1}(x), z) = e^{2\pi i\omega_j}u(x, z),\ j = 1, 2,\ \omega = (\omega_1, \omega_2) \in \mathbf{R}^2/\mathbf{Z}^2,$$

where Γ_j is the natural action of the fundamental cycle γ_j on the covering space of $\operatorname{neigh}(\pi_x(\Gamma(z)), \mathbf{C}^2)$. Then,

$$a(\Gamma_j^{-1}(x), z) = e^{i(2\pi h\omega_j + I_j(z))/h}a(x, z),$$

so the restriction of $a(\cdot, z)$ to $\pi_x(\Gamma(z))$ is $\omega + I(z)/(2\pi h)$ Floquet periodic if we identify $\pi_x(\Gamma(z))$ with the standard torus Γ_0. Then $b(Q, z)$ is $\omega + \frac{I(z)}{2\pi h} + \theta_1$ Floquet periodic (as a function on Γ_0) and hence $A^{1/2}b$ is $\omega + \frac{I(z)}{2\pi h} - \theta_2$ Floquet periodic for some $\theta_2 \in (\frac{1}{2}\mathbf{Z})^2$. We now require that a be non-vanishing. Then from (6.55), we see that $A^{1/2}b$ is periodic and hence $\omega + \frac{I(z)}{2\pi h} - \theta_2 \equiv 0$, mod $\mathbf{Z}^2$:

$$\omega = -\frac{I(z)}{2\pi h} + \theta_2 \text{ in } \mathbf{R}^2/\mathbf{Z}^2. \tag{6.56}$$

Since $U(z)$ is pseudolocal, we can define $U(z)u$ mod $\mathcal{O}(h^\infty)$ as a θ_2-Floquet periodic function on Γ_0 which is microlocally concentrated to a small neighborhood of the zero-section of $T^*\Gamma_0$ with the property that $\|U(z)u\|_{H_{\theta_2}} \sim 1$. From (6.50), we get

$$Q^w(x, hD_x, z; h)(U(z)u) = \mathcal{O}(h^2) \text{ in } H_{\theta_2}.$$

This implies that we are not in the case 1) of Proposition 5.1 for any $C > 0$ and consequently (since θ_2, $\theta(z)$ are independent of h), that $\theta_2 \equiv \theta(z) \bmod \mathbf{Z}^2$. We have proved under the assumptions above, in particular (6.2):

Proposition 6.2. — *θ_0 in Proposition 6.1 is independent of z and belongs to $(\frac{1}{2}\mathbf{Z})^2$.*

We have proved most of our main theorem below. The result will be most complete, under the additional assumption (1.51):

$$z \longmapsto (I_1(z), I_2(z)) \in \mathbf{R}^2 \text{ is a local diffeomorphism.} \tag{6.57}$$

Theorem 6.3. — *Let $P^w(z) : H_{\Phi_0} \to H_{\Phi_0}$ satisfy (6.1–5), where Φ_0 is a strictly plurisubharmonic quadratic form on $\mathbf{C}^2$, and define $I(z) = (I_1(z), I_2(z))$ as after (6.7). Let $\theta_0 \in \frac{1}{2}\mathbf{Z}^2$ be defined as above. There exists $\theta(z; h) \sim \theta_0 + \theta_1(z)h + \theta_2(z)h^2 + \cdots$ in $C^\infty(\operatorname{neigh}(0, \mathbf{C}); \mathbf{R}^2)$, such that for z in an h-independent neighborhood of 0 and for $h > 0$ sufficiently small, we have:*

1) z is an eigen-value (i.e. P^w is non-bijective) iff we have

$$\frac{I(z)}{2\pi h} = \theta(z; h) - k, \text{ for some } k \in \mathbf{Z}^2. \tag{6.58}$$

2) If I is a local diffeomorphism then the eigenvalues form a distorted lattice and they are of the form $z(k;h) = z_0(k;h) + \mathcal{O}(h^2)$, $k \in \mathbf{Z}^2$, where $z_0(k;h)$ is the solution of the approximate BS-condition:

$$\frac{I(z_0(k;h))}{2\pi h} = \theta_0 - k. \tag{6.59}$$

These eigen-values have multiplicity 1 *as defined after (6.34).*

Let

$$Z_k = \{z \in \operatorname{neigh}(0, \mathbf{C});\ |\tfrac{I(z)}{2\pi h} + k - \theta_0| < 1/3\},\ \ k \in \mathbf{Z}^2,$$

so that the Z_k are mutually disjoint and all eigen-values have to belong to the union of the Z_k and so that every eigen-value in Z_k has to be a solution of (6.58) with the same value of k. Let $\widetilde{Z}_k$ be a connected component of Z_k.

3) Assume (for a given sufficiently small h) that not every point of $\widetilde{Z}_k$ is an eigen-value. Then the set of eigen-values in $\widetilde{Z}_k$ is discrete and the multiplicity of such an eigen-value z (solving (6.58)) is equal to $\operatorname{var}\arg_\gamma(\frac{I(w)}{2\pi h} + k - \theta(w;h)) \in \{1, 2, \dots\}$, where γ is the oriented boundary of a sufficiently small disc centered at z. Here the orientation in the I is obtained from identifying the θ-plane with $\mathbf{C}$ so that we have the expression for $E^0_{-+}(\theta, z)$ after (5.14).

Proof. — For $k \in \mathbf{Z}^2$, let

$$\Omega_k(h) = \{z \in \operatorname{neigh}(0, \mathbf{C});\ |\tfrac{I(z)}{2\pi h} + k - \theta_0| < 1/C\},$$

for some fixed and sufficiently large $C > 0$. Then according to Proposition 6.1, all eigen-values of $P^w(z)$ are contained in the union of the $\Omega_k(h)$. Moreover the $\Omega_k(h)$ are mutually disjoint, and for $k \neq \ell$, we have that $\operatorname{dist}(\Omega_k(h), \Omega_\ell(h)) \geqslant c|k - \ell|h$, for some constant $c > 0$.

From (6.24) and the fact that this also holds in the C^∞-sense, we see that there exists a smooth function

$$E_{-+}(\theta, z; h) \sim E^0_{-+}(\theta, z) + hE^1_{-+}(\theta, z) + \cdots,\ \ h \longrightarrow 0,$$

defined for $\theta \in \operatorname{neigh}(\theta_0, \mathbf{C})$, such that

$$F_{-+}(z;h) = E_{-+}(k + \tfrac{I(z)}{2\pi h}, z; h),\ \ z \in \Omega_k(h). \tag{6.60}$$

As remarked after (5.14), we may assume, with a suitable identification of the I-plane and $\mathbf{C}$, that

$$E^0_{-+}(\theta, z) = \frac{i}{2}(\theta - \theta_0), \tag{6.61}$$

where $\theta_0 = \theta_0(z)$ now denotes the complex number which is identified with the previous θ_0. We equip the I-plane with the corresponding orientation.

Let $\theta(z;h)$ be the unique zero close to θ_0, of the function $\theta \mapsto E_{-+}(\theta, z; h)$. Then θ is smooth in z and has an asymptotic expansion as in the theorem. Clearly $z \in \Omega_k(h)$ is an eigen-value iff $k + \frac{I(z)}{2\pi h} = \theta(z;h)$, i.e. iff (6.58) holds. This proves 1).

The implicit function theorem gives everything in the statement 2) except perhaps that the eigen-values are simple. From (6.60) it is clear however that the eigen-values $z(k;h)$ must be simple zeros of the holomorphic function $e^{V(z;h)}F_{-+}(z;h)$ in (6.33), so 2) holds.

We now make the assumptions of 3) and identify the I-plane with $\mathbf{C}$ as in (6.61). In view of (6.60), and Taylor's formula for $\theta \mapsto E_{-+}(\theta, z; h)$, we get for $w \in \widetilde{Z}_k \cap \Omega_k$:

$$\begin{aligned} F_{-+}(w;h) &= E_{-+}(k + \tfrac{I(w)}{2\pi h}, w; h) - E_{-+}(\theta(w;h), w; h) \\ &= A(w;h)(k + \tfrac{I(w)}{2\pi h} - \theta(w;h)) + B(w;h)\overline{(k + \tfrac{I(w)}{2\pi h} - \theta(w;h))}, \end{aligned} \tag{6.62}$$

where A, B are smooth in w with bounded derivatives to all orders. Moreover $|A| \sim 1$, $|B| \ll |A|$. Let $z \in \widetilde{Z}_k$ be an eigen-value (necessarily in Ω_k, and let γ be as in 3). From (6.62) and the fact that A dominates over B, it follows that F_{-+} and $k + \frac{I(w)}{2\pi h} - \theta(w;h)$ have the same argument variation along γ, and 3) follows. □

We next compute the differential and the Jacobian of the map $z \mapsto (I_1(z), I_2(z))$ and show that (6.57) ((1.51)) is equivalent to the property (4.16). We fix some value of z, say $z = 0$. Choose grad-periodic coordinates Q_1, Q_2 on $\Gamma(0)$, so that

$$H_p = A(Q)\frac{\partial}{\partial \overline{Q}} \text{ on } \Gamma(0) \simeq \mathbf{C}/L,\ L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2, \tag{6.63}$$

where $A(Q) \neq 0\ \forall\, Q$. Extend Q_1, Q_2 to grad-periodic functions in a neighborhood of $\Gamma(0)$ in $\Lambda_{\Phi_{z=0}}$, and let P_1, P_2 be corresponding "dual" coordinates, vanishing on $\Gamma(0)$, so that $(Q_1, Q_2; P_1, P_2)$ are symplectic coordinates near $\Gamma(0)$.

Then

$$p = \frac{1}{2}A(Q)(P_1 + iP_2) + zr(Q) + \mathcal{O}(P^2) + \mathcal{O}(z^2),$$

where $r = \frac{\partial p}{\partial z}(\cdot, 0)$. $\Gamma(z)$ can be represented by

$$P = \nabla_Q g(Q, z),$$

where $g = \mathcal{O}(z)$ is grad-periodic and

$$p(Q, \nabla_Q g, z) = 0,$$

so that

$$A(Q)\frac{\partial g}{\partial \overline{Q}} + zr(Q) = \mathcal{O}(z^2). \tag{6.64}$$

Let $J_j(z)$ be the actions in $\Gamma(z)$ with respect to $P_1 dQ_1 + P_2 dQ_2$. By Stokes' formula, $I_j - J_j$ is independent of z and since the difference is real for $z = 0$, we know that $J_j(z)$ are real. From this and (6.64) we see that

$$g = \overline{b(z)}Q + b(z)\overline{Q} + g_{\mathrm{per}} + \mathcal{O}(z^2), \tag{6.65}$$

where g_{per} is periodic and

$$b(z) = -\widehat{zr/A}(0), \tag{6.66}$$

where the hat denotes Fourier transform on $\mathbf{C}/L(0)$: $\widehat{r/A} = \mathcal{F}(r/A)$. It follows that

$$J_j(z) = \overline{b(z)}e_j + b(z)\overline{e_j} + \mathcal{O}(z^2). \tag{6.67}$$

The map in (6.57) has the same differential as that of the map $z \mapsto (J_1(z), J_2(z))$, and we get for $z = 0$:

$$dI_1 \wedge dI_2 = (e_1 d\bar{b} + \bar{e}_1 db) \wedge (e_2 d\bar{b} + \bar{e}_2 db) = (e_1\bar{e}_2 - \bar{e}_1 e_2) d\bar{b} \wedge db,$$

so for $z = 0$:

$$\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)} = 2i(e_1\bar{e}_2 - \bar{e}_1 e_2)|\mathcal{F}(\partial_z p/A)(0)|^2. \tag{6.68}$$

The equivalence of (6.57) and (4.16) follows.

For $z = 0$, let $\lambda_{p,0}$ be the Liouville measure on $\Gamma(0)$ defined by

$$\lambda_{p,0} \wedge d\mathrm{Re}\, p \wedge d\mathrm{Im}\, p = \mu,$$

where $\mu = \frac{1}{2}\sigma^2$ is the symplectic volume element on $\Lambda_{\Phi_{z=0}}$. In our special coordinates, we have

$$p = \frac{1}{2}A(Q)(P_1 + iP_2) + \mathcal{O}(P^2),$$

for $z = 0$, and the Liouville measure becomes $\lambda_{p,0} = 4|A|^{-2}L(dQ)$. The Hamilton field of H_p on $\Gamma(0)$ is $H_p = A(Q)\frac{\partial}{\partial \overline{Q}}$, which has the adjoints

$$H_p^* = -\frac{\partial}{\partial \overline{Q}} \circ \overline{A}(Q),\ H_p^\dagger = -|A|^2 \frac{\partial}{\partial \overline{Q}} \circ \frac{1}{A},$$

with respect to the measures $L(dQ)$ and $\lambda_{p,0}(dQ)$ respectively. Using that the volume of $\mathbf{C}/L = \Gamma(0)$ with respect to $L(dQ)$ is equal to $|\frac{i}{2}(e_1\bar{e}_2 - \bar{e}_1 e_2)|$, we see that the 1-dimensional kernel of $H_p^\dagger$ in $L^2(\Gamma(0), \lambda_{p,0}(d\rho))$ is spanned by the normalized element

$$f := |2i(e_1\bar{e}_2 - \bar{e}_1 e_2)|^{-1/2} A,$$

and a straight forward calculation from (6.68) gives for $z = 0$

$$\Big|\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)}\Big| = \Big|\int \partial_z p \overline{f} \lambda_{p,0}(d\rho)\Big|^2 = \int |(1 - \Pi)\partial_z p|^2 \lambda_{p,0}(d\rho), \tag{6.69}$$

where in the last expression we used the notation of (8.38) in **[MeSj]**, so that $1 - \Pi$ is the orthogonal projection onto the kernel of $H_p^\dagger$ in $L^2(\lambda_{p,0})$.

Assuming (6.57), the density of eigenvalues, given in 2) of the theorem, is

$$\frac{1}{(2\pi h)^2}\Big(\Big|\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)}\Big| + o(1)\Big),\ h \longrightarrow 0.$$

Assume that $P(\cdot, z) \to 1$ sufficiently fast at ∞, so that $\det P^w$ is well defined. Since the eigenvalue $z(k; h)$ is a simple zero of this determinant and $\partial_z \partial_{\bar{z}} \log|z| = \frac{\pi}{2}\delta$, $z(k; h)$ will give the contribution $\frac{\pi}{2}\delta(z - z(k; h))$ to $\partial_z\partial_{\bar{z}} \log|\det P^w(z)|$ and hence in the sense of distributions (or even the weak measure sense), we have

$$\partial_z\partial_{\bar{z}} \log|\det P^w(z)| = \frac{1}{(2\pi h)^2}\Big(\frac{\pi}{2}\Big|\int_{p^{-1}(\cdot,z)(0)} \partial_z p \overline{f(z)} \lambda_{p,0}(d\rho)\Big|^2 + o(1)\Big), \tag{6.70}$$

where we now let f vary with z in the obvious sense. This is in perfect agreement with (8.38) of [**MeSj**], where we computed $\partial_z\partial_{\bar{z}}I(z)$ for an (infinitesimal) majorant $(2\pi h)^{-2}(I(z)+o(1))$ of $\log|\det P^w(z)|$.

Appendix A: Remark on multiplicities

Let $\Omega\subset\mathbf{C}$ be open and simply connected. Let $\mathcal{H}$ be a complex Hilbert space and let

$$\mathcal{P}(z)=\begin{pmatrix}P(z) & R_-(z)\\ R_+(z) & 0\end{pmatrix}:\mathcal{H}\times\mathbf{C}^N\longrightarrow\mathcal{H}\times\mathbf{C}^N$$

depend smoothly on $z\in\Omega$ and be bijective for all z. Assume that

$$dP(z)=\frac{\partial P}{\partial\mathrm{Re}\,z}d\mathrm{Re}\,z+\frac{\partial P}{\partial\mathrm{Im}\,z}d\mathrm{Im}\,z$$

is of trace class locally uniformly in z. Write

$$\mathcal{P}(z)^{-1}=\mathcal{E}(z)=\begin{pmatrix}E(z) & E_+(z)\\ E_-(z) & E_{-+}(z)\end{pmatrix}.$$

Recall that $P(z)$ is invertible precisely when $E_{-+}(z)$ is and that we have

$$P(z)^{-1}=E(z)-E_+(z)E_{-+}(z)^{-1}E_-(z). \tag{A.1}$$

Proposition. — *Let $\gamma\subset\Omega$ be a closed C^1-curve along which $P(z)$ (or equivalently $E_{-+}(z)$) is invertible. Then*

$$\mathrm{tr}\left(\frac{1}{2\pi i}\int_\gamma P(z)^{-1}dP(z)\right)=\mathrm{tr}\left(\frac{1}{2\pi i}\int_\gamma E_{-+}(z)^{-1}dE_{-+}(z)\right). \tag{A.2}$$

Proof. — From $d\mathcal{E}=-\mathcal{E}d\mathcal{P}\mathcal{E}$, we get

$$\begin{aligned}-dE&=EdPE+E_+dR_+E+EdR_-E_-,\\ -dE_+&=EdPE_++E_+dR_+E_++EdR_-E_{-+},\\ -dE_-&=E_-dPE+E_{-+}dR_+E+E_-dR_-E_-,\\ -dE_{-+}&=E_-dPE_++E_{-+}dR_+E_++E_-dR_-E_{-+}.\end{aligned} \tag{A.3}$$

We get,

$$\mathrm{tr}\,P^{-1}dP=\mathrm{tr}\,(EdP)-\mathrm{tr}\,(E_+E_{-+}^{-1}E_-dP).$$

Here by the cyclicity of the trace and the last equation in (A.3):

$$\begin{aligned}&-\mathrm{tr}\,(E_+E_{-+}^{-1}E_-dP)=-\mathrm{tr}\,(E_{-+}^{-1}E_-dPE_+)\\ &=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\mathrm{tr}\,(E_{-+}^{-1}E_{-+}dR_+E_+)+\mathrm{tr}\,(E_{-+}^{-1}E_-dR_-E_{-+})\\ &=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\mathrm{tr}\,(dR_+E_+)+\mathrm{tr}\,(E_-dR_-).\end{aligned}$$

It follows that

$$\mathrm{tr}\,(P^{-1}dP)=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\omega, \tag{A.4}$$

with

$$\omega = \mathrm{tr}\,(EdP) + \mathrm{tr}\,(dR_+E_+) + \mathrm{tr}\,(E_-dR_-). \tag{A.5}$$

If we assume that $\mathcal{P}$ is holomorphic, then $\mathcal{E}$ will be holomorphic and ω will be a (1,0)-form with holomorphic coefficients, hence closed, and the Proposition follows, since Ω is simply connected.

In the general case it suffices to verify that ω is still closed, as we shall now do. Using the obvious calculus of differential forms with operator coefficients, we get:

$$d\omega = \mathrm{tr}\,(dE \wedge dP) - \mathrm{tr}\,(dR_+ \wedge dE_+) + \mathrm{tr}\,(dE_- \wedge dR_-).$$

Use (A.3):

$$\begin{aligned} -d\omega =& \mathrm{tr}\,(EdPE \wedge dP) + \mathrm{tr}\,(E_+dR_+E \wedge dP) + \mathrm{tr}\,(EdR_-E_- \wedge dP) \\ &-\mathrm{tr}\,(dR_+ \wedge EdPE_+) - \mathrm{tr}\,(dR_+ \wedge E_+dR_+E_+) - \mathrm{tr}\,(dR_+ \wedge EdR_-E_{-+}) \\ &+\mathrm{tr}\,(E_-dPE \wedge dR_-) + \mathrm{tr}\,(E_{-+}dR_+E \wedge dR_-) + \mathrm{tr}\,(E_-dR_-E_- \wedge dR_-). \end{aligned} \tag{A.6}$$

The cyclicity of the trace implies that if μ is an operator 1-form, with trace class coefficients, then $\mathrm{tr}\,\mu \wedge \mu = 0$. It follows that the 1st, 5th and 9th terms of the right hand side of (A.6) vanish:

$$\begin{aligned} \mathrm{tr}\,(EdPE \wedge dP) &= \mathrm{tr}\,(EdP \wedge EdP) = 0, \\ \mathrm{tr}\,(dR_+ \wedge E_+dR_+E_+) &= \mathrm{tr}\,(dR_+E_+ \wedge dR_+E_+) = 0, \\ \mathrm{tr}\,(E_-dR_-E_- \wedge dR_-) &= \mathrm{tr}\,(E_-dR_- \wedge E_-dR_-) = 0. \end{aligned}$$

The terms no 2 and 4, no 3 and 7 as well as no 6 and 8 cancel each other mutually, becauce the cyclicity of the trace implies that $\mathrm{tr}\,(\mu_1 \wedge \mu_2) = -\mathrm{tr}\,(\mu_2 \wedge \mu_1)$ for operator 1-forms with one factor of trace class, and hence

$$\begin{aligned} \mathrm{tr}\,(E_+dR_+E \wedge dP) &= \mathrm{tr}\,(E_+dR_+ \wedge EdP) = \mathrm{tr}\,(dR_+ \wedge EdPE_+) \\ \mathrm{tr}\,(EdR_- \wedge E_-dP) &= -\mathrm{tr}\,(E_-dP \wedge EdR_-), \\ \mathrm{tr}\,(dR_+ \wedge EdR_-E_{-+}) &= \mathrm{tr}\,(E_{-+}dR_+ \wedge EdR_-). \end{aligned}$$

Thus $d\omega = 0$ and we get the proposition in the general case. □

Now drop the assumption that $dP(z)$ be of trace class, but assume that there exists an invertible operator $Q(z)$ which depends smoothly on z such that $d(Q(z)P(z))$ is locally uniformly of trace class. Then we have the invertible Grushin operator:

$$\begin{pmatrix} Q(z)P(z) & Q(z)R_-(z) \\ R_+(z) & 0 \end{pmatrix}, \text{ with inverse } \begin{pmatrix} EQ^{-1} & E_+ \\ E_-Q^{-1} & E_{-+} \end{pmatrix}.$$

The equation (A.2) then holds, if we replace P by QP in the left hand side. Notice that if we add the assumption that $Q(z)P(z) - 1$ be of trace class, then (A.2) (with QP replacing P) gives

$$\mathrm{var}\arg_\gamma \det(Q(z)P(z)) = \mathrm{var}\arg_\gamma(\det(E_{-+}(z)). \tag{A.7}$$

Assume that $P(z)$ is invertible for $z_0 \neq z \in \operatorname{neigh}(z_0, \mathbf{C})$, but that $P(z_0)$ is not invertible. Then it is easy to see that there exists an operator K of finite rank such that $P(z_0) + K$ is invertible, and hence also that $P(z) + K$ is invertible for z in a small neighborhood of z_0. Put $Q(z) = (P(z) + K)^{-1}$. Then $Q(z)P(z) - 1 = -Q(z)K$ is of finte rank and hence of trace class, so (A.7) applies. Let

$$m(z_0) = \frac{1}{2\pi i} \operatorname{var} \arg_\gamma(\det Q(z)P(z)), \tag{A.8}$$

where γ is the oriented boundary of a small disc centered at z_0. (A.7) shows that this integer is independent both of the choice of Q and of the Grushin problem, and by the definition this will be the multiplicity of z_0 as an "eigen-value" of $z \mapsto P(z)$. In the main text, $P(z)$ depends holomorphically on z and then have $m(z_0) \geqslant 1$.

Appendix B: Modified $\overline{\partial}$-equation for $(I_1(z), I_2(z))$

We recall from section 1, that we have a holomorphic map

$$\operatorname{neigh}((0,0), \mathbf{C}^2) \ni (z, w) \longmapsto I(z, w) = (I_1(z, w), I_2(z, w)) \in \mathbf{C}^2, \tag{B.1}$$

with $I(0,0) \in \mathbf{R}^2$ and with

$$\operatorname{Im}(\partial_w I_1 \overline{\partial_w I_2}) \neq 0. \tag{B.2}$$

Let $(f_1(z, w), f_2(z, w))$ be holomorphic, non-vanishing such that

$$f_1(z, w)\partial_w I_1(z, w) + f_2(z, w)\partial_w I_2(z, w) = 0. \tag{B.3}$$

This implies that

$$f_1(z, w) dI_1 + f_2(z, w) dI_2 = g(z, w) dz, \tag{B.4}$$

where $g(z, w)$ is holomorphic.

From (B.2) it follows (as we saw in section 1) that there is a unique smooth function: $\operatorname{neigh}(0, \mathbf{C}) \ni z \mapsto z(w) \in \operatorname{neigh}(0, \mathbf{C})$, such that

$$I(z, w(z)) \in \mathbf{R}^2. \tag{B.5}$$

Indeed, this follows from the implicit function theorem, for if we formally make infinitesimal increments to z, w, we get

$$\partial_z I_j \, \delta_z + \partial_w I_j \, \delta_w = \overline{\partial_z I_j} \, \overline{\delta_z} + \overline{\partial_w I_j} \, \overline{\delta_w},$$

$$\begin{cases} \partial_w I_1 \, \delta_w - \overline{\partial_w I_1} \, \overline{\delta_w} = -\partial_z I_1 \, \delta_z + \overline{\partial_z I_1} \, \overline{\delta_z}, \\ \partial_w I_2 \, \delta_w - \overline{\partial_w I_2} \, \overline{\delta_w} = -\partial_z I_2 \, \delta_z + \overline{\partial_z I_2} \, \overline{\delta_z}, \end{cases} \tag{B.6}$$

and notice that

$$\det \begin{pmatrix} \partial_w I_1 & -\overline{\partial_w I_1} \\ \partial_w I_2 & -\overline{\partial_w I_2} \end{pmatrix} = -2i \operatorname{Im}(\partial_w I_1 \overline{\partial_w I_2}) \neq 0.$$

Treating $\delta_w, \overline{\delta_w}$ as independent variables, we see that (B.6) has a unique solution $(\delta_w, \overline{\delta_w}) \in \mathbf{C}^2$ for a given $\delta_z \in \mathbf{C}$, and it is easy to see that $\overline{\delta_w}$ has to be the complex conjugate of δ_w. The existence of the smooth function $w(z)$ in (B.5) therefore follows from the implicit function theorem.

Let $J_j(z) = I_j(z, w(z))$. (In the main text, we simply write $I_j(z) = I_j(z, w(z))$.) Restricting (B.4) to the submanifold, given by $w = w(z)$, we get

$$f_1 dJ_1 + f_2 dJ_2 = g dz, \tag{B.7}$$

with $f_j = f_j(z, w(z))$, $g = g(z, w(z))$. Taking the antilinear part of this relation, we get

$$f_1 \overline{\partial} J_1 + f_2 \overline{\partial} J_2 = 0, \ \overline{\partial} = \partial_{\overline{z}}. \tag{B.8}$$

This can also we written

$$\overline{\partial}(f_1(J_1 - J_1^0) + f_2(J_2 - J_2^0)) - ((\overline{\partial} f_1)(J_1 - J_1^0) + (\overline{\partial} f_2)(J_2 - J_2^0)) = 0, \tag{B.9}$$

where J_j^0 are arbitrary real constants. Put

$$u = f_1(J_1 - J_1^0) + f_2(J_2 - J_2^0). \tag{B.10}$$

Using (B.2) and (B.3), we see that the two real functions J_1, J_2 can be recovered from u by means of the formula,

$$\begin{cases} J_1 - J_1^0 = \frac{1}{2i\mathrm{Im}\,(f_1 \overline{f_2})}(\overline{f_2} u - f_2 \overline{u}), \\ J_2 - J_2^0 = \frac{1}{2i\mathrm{Im}\,(f_1 \overline{f_2})}(-\overline{f_1} u + f_1 \overline{u}). \end{cases} \tag{B.12}$$

(Notice that $(f_1, f_2) = a(\partial_w I_2, -\partial_w I_1)$ for some non-vanishing a, so that $\mathrm{Im}\,(f_1 \overline{f_2}) \neq 0$.) Then (B.9) gives

$$\overline{\partial} u + a u + b \overline{u} = 0, \tag{B.13}$$

for some smooth (and even real-analytic) functions a, b.

It follows from a classical result by Carleman **[Ca]** that if u solves (B.13) in a complex domain and vanishes to infinite order at some point then u vanishes identically. Since a and b are real-analytic we can show this differently: Treating u and $u^* = \overline{u}$ as independent functions, we get

$$\begin{cases} \overline{\partial} u + a(z) u + b(z) u^* = 0, \\ \partial u^* + \overline{b(z)} u + \overline{a(z)} u^* = 0, \end{cases} \tag{B.14}$$

which is an elliptic system with real-analytic coefficients. Hence u is real-analytic and cannot vanish to infinite order at any point without vanishing identically.

If u is not identically 0, let z_0 be a zero of u and write the Taylor expansion as

$$u(z) = p_m(z - z_0) + \mathcal{O}(|z - z_0|^{m+1}),$$

where $p_m \neq 0$ is a homogeneous polynomial of degree m. Substitution into (B.13) shows that p_m is holomorphic, so

$$u(z) = C(z - z_0)^m + \mathcal{O}(|z - z_0|^{m+1}), \; C \neq 0. \tag{B.15}$$

This means that the map

$$\operatorname{neigh}(0, \mathbf{C}) \longmapsto J(z) = (J_1(z), J_2(z)) \in \mathbf{R}^2 \tag{B.16}$$

is either constant or takes any given value J^0 only at isolated points, and if z_0 is such a point, then

$$|J(z) - J^0| \sim |z - z_0|^m. \tag{B.17}$$

Write (B.12) as

$$J(z) - J^0 = F(z) \begin{pmatrix} \operatorname{Re} u \\ \operatorname{Im} u \end{pmatrix}, \tag{B.18}$$

where F is a smooth invertible 2×2-matrix. Then from (B.15), we get

$$\frac{\partial J(z)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} = F(z_0) \frac{\partial(\operatorname{Re} u, \operatorname{Im} u)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} + \mathcal{O}(|z - z_0|^m), \tag{B.19}$$

where the first term to the right is $\mathcal{O}(|z - z_0|^{m-1})$ and has an inverse which is $\mathcal{O}(|z - z_0|^{1-m})$. From this, we see that the critical points of J are isolated if J is not identically constant, and that

$$\det \frac{\partial J(z)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} \text{ is either } \geqslant 0, \forall z, \text{ or } \leqslant 0, \forall z. \tag{B.20}$$

This means that we can introduce a natural orientation on the (J_1, J_2)-plane such that the differential of J becomes orientation preserving. We can then define the multiplicity of a solution z_0 of $J(z) = J^0$ by

$$m(z_0) = \frac{1}{2\pi} \operatorname{var} \arg_\gamma (J(z) - J^0), \tag{B.21}$$

where γ is the positively oriented boundary of a small disc centered at z_0.

In the main text of section 6, we write $I_j(z)$ instead of $J_j(z)$. It is also clear from our discussion, that the orientation of the J-plane is the same as the one we got in the proof of Theorem 6.3 from (6.61).

7. Saddle point resonances

Consider the operator

$$P = -\frac{h^2}{2} \Delta + V(x), \; x \in \mathbf{R}^2, \tag{7.1}$$

where V is a real-valued analytic potential, which extends holomorphically to a set $\{x \in \mathbf{C}^2; \, |\operatorname{Im} x| < \frac{1}{C} \langle \operatorname{Re} x \rangle\}$, with $V(x) \to 0$, when $x \to \infty$ in that set. The resonances of P can be defined in an angle $\{z \in \mathbf{C}; \, -2\theta_0 < \arg z \leqslant 0\}$ for some fixed $\theta_0 > 0$

as the eigen-values of $P_{|_{e^{i\theta_0}\mathbf{R}^n}}$. In [**HeSj**], they were also defined as the eigen-values of $P : H(\Lambda_G, 1) \to H(\Lambda_G, 1)$ with domain $H(\Lambda_G, \langle\xi\rangle^2)$, and below we shall have the occasion to recall some more about that approach. (Such a space consists of the functions u such that a suitable FBI-transformation Tu belongs to a certain exponentially weighted L^2 space.)

Let $E_0 > 0$. Let $p(x, \xi) = \xi^2 + V(x)$. We assume that the union of trapped H_p-trajectories in $p^{-1}(E_0) \cap \mathbf{R}^4$ (see [**GeSj**]) is reduced to a single point (x_0, ξ_0). Necessarily, $\xi_0 = 0$ and after a translation, we may also assume that $x_0 = 0$. (Recall for instance from [**GeSj**] that a trapped trajectory is a maximally extended trajectory which is contained in a bounded set.) It follows that 0 is a critical point for V and that $V(0) = E_0$. Assume,

(7.2) 0 is a non-degenerate critical point of V, of signature $(1, -1)$.

After a linear change of coordinates in x and a corresponding dual one in ξ, we may assume that

$$p(x, \xi) - E_0 = \frac{\lambda_1}{2}(\xi_1^2 + x_1^2) + \frac{\lambda_2}{2}(\xi_2^2 - x_2^2) + \mathcal{O}((x, \xi)^3), \ (x, \xi) \longrightarrow 0. \tag{7.3}$$

Under the assumptions above, but without any restriction on the dimension and without the assumption on the signature in (7.3), the second author ([**Sj2**]) determined all resonances in a disc $D(E_0, Ch)$ for any fixed $C > 0$, when $h > 0$ is small enough. (See also [**BrCoDu**] for the barrier top case.) Under the same assumptions plus a diophantine one on the eigen-values of $V''(0)$, Kaidi and Kerdelhué [**KaKe**] determined all resonances in a disc $D(E_0, h^\delta)$ for any fixed $\delta > 0$ and for $h > 0$ small enough. In the two dimensional case, their diophantine condition follows from (7.2), and we recall their result in that case.

Theorem 7.1 ([**KaKe**]). — *Under the assumptions from (7.1) to (7.2), let $\lambda_j > 0$ be defined in (7.3). Fix $\delta > 0$. Then for $h > 0$ small enough, the resonances in $D(E_0, h^\delta)$ are all simple and coincide with the values in that disc, given by:*

$$z = E_0 + f(2\pi h(k - \theta_0); h), \ k \in \mathbf{N}^2, \tag{7.4}$$

where $\theta_0 = (-\frac{1}{2}, \frac{1}{2}) \in (\frac{1}{2}\mathbf{Z})^2$ is fixed, and $f(\theta; h)$ is a smooth function of $\theta \in$ neigh $(0, \mathbf{R}^2)$, *with*

$$f(\theta; h) \sim f_0(\theta) + hf_1(\theta) + h^2 f_2(\theta) + \cdots, \ h \longrightarrow 0, \tag{7.5}$$

in the space of such functions. Further,

$$f_0(\theta) = \frac{1}{2\pi}(\lambda_1\theta_1 - i\lambda_2\theta_2) + \mathcal{O}(\theta^2).$$

The purpose of this section is to show that the description (7.4) extends to all resonances in a fixed disc $D(E_0, r_0)$ with $r_0 > 0$ small but independent of h, provided that we avoid arbitrarily small angular neighborhoods of $]0; +\infty[$ and $-i]0, +\infty[$. The main ingredient of the proof will be Theorem 6.3, that we will be able to apply after some reductions, using [**HeSj**], [**KaKe**].

As in [**KaKe**], we choose an escape function (in the sense of [**HeSj**]) G which is equal to $x_2\xi_2$ in a neighborhood of $(x,\xi) = (0,0)$ and such that $H_pG > 0$ on $p^{-1}(E_0) \setminus \{(0,0)\}$. Then for a small fixed $t > 0$, we take an FBI-transformation as in [**HeSj**] which is isometric:

$$T : \begin{cases} H(\Lambda_{tG}, 1) \to L^2(\mathbf{C}^2; e^{-2\phi/h}L(dx)), \\ H(\Lambda_{tG}, \langle\xi\rangle^2) \to L^2(\mathbf{C}^2, m^2e^{-2\phi/h}L(dx)). \end{cases} \tag{7.6}$$

Here $m \geqslant 1 - \mathcal{O}(h)$ is a weight which is independent of h to leading order and $m \sim 1$ in any fixed compact set. Moreover, ϕ is a smooth real-valued function. For possibly only technical reasons, T has to take its values in $L^2(\cdots)\otimes\mathbf{C}^3$ rather than in $L^2(\cdots)$, but as noticed in [**KaKe**], we may modify the definition of T in such a way that the last two components of Tu vanish identically in a neighborhood Ω of $0 \in \mathbf{C}^2$, the point corresponding to $(x,\xi) = (0,0)$, and so that the first component of $Tu(x)$ is given by a standard Bargman transformation in that neighborhood and is consequently a holomorphic function of x. We can also arrange so that ϕ is a strictly plurisubharmonic quadratic form in Ω. Hence $Tu \in H_\phi(\Omega) := L^2_\phi(\Omega) \cap \mathrm{Hol}\,(\Omega)$, where $\mathrm{Hol}\,(\Omega)$ is the space of holomorphic functions on Ω and $L^2_\phi(\Omega) = L^2(\Omega; e^{-2\phi(x)/h}L(dx))$.

Kaidi and Kerdelhué showed that there exists a uniformly bounded operator

$$V : H_\phi(\Omega) \longrightarrow H_\psi(\widetilde{\Omega}),$$

which is a metaplectic operator, i.e. a Fourier integral operator as in [**Sj1**] with quadratic phase and constant amplitude, with an almost inverse (the lack of exactness being due to the fact that we do not work on all of $\mathbf{C}^2$ and consequently get cutoff errors) $U = \mathcal{O}(1) : H_\psi(\widetilde{\Omega}) \to H_\phi(\Omega)$ with the following properties:

(1) $\widetilde{\Omega}$ is a neighborhood of 0 and ψ is a strictly plurisubharmonic quadratic form.

(2) If $\phi_- \leqslant \phi \leqslant \phi_+$ are smooth, and $\phi_\pm$ are sufficiently close to ϕ in C^2 and equal to ϕ outside some neighborhood of 0, then there exist $\psi_- \leqslant \psi \leqslant \psi_+$ with analogous properties, such that

$$\begin{cases} 1 - UV = \mathcal{O}(1) : H_{\phi_+}(\Omega) \to H_{\phi_-}(\Omega), \\ 1 - VU = \mathcal{O}(1) : H_{\psi_+}(\widetilde{\Omega}) \to H_{\psi_-}(\widetilde{\Omega}). \end{cases} \tag{7.7}$$

(3) If we choose $\phi_\pm$ with $\phi_-(0) < \phi(0) < \phi_+(0)$, then $\psi_-(0) < \psi(0) < \psi_+(0)$.

(4) There exists an analytic h-pseudodifferential operator

$$Q^w(x, hD_x; h) = H_{\psi/\psi_+/\psi_-} \longrightarrow H_{\psi/\psi_+/\psi_-}$$

with symbol $Q(x,\xi;h) \sim q_0(x,\xi) + hq_1(x,\xi) + \cdots$, holomorphic in a neighborhood of the closure of $\{(x, \frac{2}{i}\frac{\partial\psi}{\partial x}(x));\ x \in \widetilde{\Omega}\}$ such that

$$Q^wVT - VTP = \mathcal{O}(1) : H(\Lambda_{G_+}, \langle\xi\rangle^2) \longrightarrow H_{\psi_-}(\widetilde{\Omega}). \tag{7.8}$$

Here we extend $\phi_\pm$ to be equal to ϕ outside Ω and define

$$H(\Lambda_{G_\pm}, \langle\xi\rangle^2) = \{u \in H(\Lambda_G, \langle\xi\rangle^2);\ Tu \in L^2(\mathbf{C}^2; m^2 e^{-2\phi_\pm/h} L(dx)).$$

The spaces $H(\Lambda_{G_\pm}, 1)$ are defined similarly. For simplicity, we have also introduced a new G; $G_{\text{new}} = tG_{\text{old}}$, so that $t = 1$ from now on. Q^w is realized by means of choices of "good" integration contours as in [**Sj1**].

(5) We have

$$q_0(x,\xi) = i\lambda_1 x_1\xi_1 + \lambda_2 x_2 \xi_2 + \mathcal{O}((x,\xi)^3). \tag{7.9}$$

Later on we shall also use that we have a local quasi-inverse S to T with

$$S = \mathcal{O}(1) : H_{\phi/\phi_+/\phi_-}(\Omega) \longrightarrow H(\Lambda_{G/G_+/G_-}, \langle\xi\rangle^2), \tag{7.10}$$

$$1 - TS = \mathcal{O}(1) : H_{\phi_+}(\Omega) \longrightarrow H_{\phi_-}(\Omega). \tag{7.11}$$

(6) A last feature of the reduction in [**KaKe**] is that there exists a strictly plurisubharmonic smooth function $\widetilde{\phi}$ on $\widetilde{\Omega}$, equal to ψ outside any previously given fixed neighborhood of 0, with

$$\widetilde{\phi}(x) = \frac{1}{2}|x|^2 \text{ in some neighborhood of } 0, \tag{7.12}$$

$$q_0\Big(x, \frac{2}{i}\frac{\partial\widetilde{\phi}}{\partial x}\Big) \neq 0,\ x \in \widetilde{\Omega} \setminus \{0\}. \tag{7.13}$$

Moreover, $\widetilde{\phi}$ can be chosen with $\psi - \widetilde{\phi}$ arbitrarily small in C^1-norm.

Notice that for x in a region, where (7.12) holds, we have

$$q_0\Big(x, \frac{2}{i}\frac{\partial\widetilde{\phi}}{\partial x}\Big) = \lambda_1|x_1|^2 - i\lambda_2|x_2|^2 + \mathcal{O}(|x|^3). \tag{7.14}$$

We shall next discuss the invertibility of $Q^w - z$ for $|z|$ small, by applying Theorem 6.3. For that, it will be convenient to globalize the problem. We recall that $\widetilde{\phi} = \psi = \psi_+ = \psi_- =$ a quadratic form in $\widetilde{\Omega} \setminus \operatorname{neigh}(0)$, and we extend these functions to all of $\mathbf{C}^2$, so that they keep the same properties. Extend Q to a symbol in $S^0(\Lambda_{\widetilde{\phi}}) = C_b^\infty(\Lambda_{\widetilde{\phi}})$ with the asymptotic expansion

$$Q(x,\xi;h) \sim q_0(x,\xi) + hq_1(x,\xi) + \cdots \tag{7.15}$$

in that space, and so that

$$|q_0(x,\xi)| \geqslant \frac{1}{C}, \tag{7.16}$$

outside a small neighborhood of $(0,0)$.

Let $\chi \in C_b^\infty(\mathbf{C}^2 \times \mathbf{C}^2)$ with $1_{|x-y|\leqslant\frac{1}{2C}} \prec \chi \prec 1_{|x-y|\leqslant\frac{1}{C}}$, where we write $f \prec g$ for two functions f, g, if $\operatorname{supp} f \cap \operatorname{supp}(1-g) = \varnothing$. Put

$$Q_\chi^w(x, hD_x; h)u = \frac{1}{(2\pi h)^2}\iint e^{\frac{i}{h}(x-y)\cdot\theta} Q\Big(\frac{x+y}{2}, \theta; h\Big)\chi(x,y)u(y)\,dy\,d\theta, \tag{7.17}$$

where we integrate over a contour of the form

$$\theta = \frac{2}{i}\frac{\partial \widetilde{\phi}}{\partial x}\Big(\frac{x+y}{2}\Big) + iC(x)\overline{(x-y)},$$

where $C(x) \geqslant 0$ is a smooth function which is > 0 near $x = 0$ and with compact support in $\widetilde{\Omega}$. Then,

$$\begin{aligned} Q_\chi^w &= \mathcal{O}(1) : H_{\widetilde{\phi}/\psi/\psi_+/\psi_-}(\mathbf{C}^2) \longrightarrow L^2_{\widetilde{\phi}/\psi/\psi_+/\psi_-}(\mathbf{C}^2), \\ \overline{\partial} Q_\chi^w &= \mathcal{O}(h^\infty) : H_{\psi_+} \longrightarrow L^2_{\psi_-}. \end{aligned} \tag{7.18}$$

Let $\Pi_{\psi_-} = (1 - \overline{\partial}^*(\Delta^{(1)}_{\psi_-})^{-1}\overline{\partial}) : L^2_{\psi_-}(\mathbf{C}^2) \to H_{\psi_-}(\mathbf{C}^2)$ be the orthogonal projection (see [**MeSj**], [**Sj3**]), and put

$$Q^w = \Pi_{\psi_-} Q_\chi^w. \tag{7.19}$$

Then $Q^w = \mathcal{O}(1) : H_{\widetilde{\phi}/\psi/\psi_+/\psi_-} \to H_{\widetilde{\phi}/\psi/\psi_+/\psi_-}$, $Q^w - Q_\chi^w = \mathcal{O}(h^\infty) : H_{\psi_+} \to L^2_{\psi_-}$.

Consider the change of variables $x = \mu\widetilde{x}$, $h^\delta \leqslant \mu \leqslant 1$, for $0 < \delta < \frac{1}{2}$. Formally, we get

$$\frac{1}{\mu^2} Q^w(x, hD_x; h) = \frac{1}{\mu^2} Q^w(\mu(\widetilde{x}, \widetilde{h}D_{\widetilde{x}}); h), \ \widetilde{h} = \frac{h}{\mu^2}. \tag{7.20}$$

The corresponding new symbol is

$$\frac{1}{\mu^2} Q(\mu(\widetilde{x},\widetilde{\xi}); h) \sim \frac{1}{\mu^2} q_0(\mu(\widetilde{x},\widetilde{\xi})) + \widetilde{h} q_1(\mu(\widetilde{x},\widetilde{\xi})) + \mu^2\widetilde{h}^2 q_2(\mu(\widetilde{x},\widetilde{\xi})) + \cdots . \tag{7.21}$$

Write $\widetilde{\phi}(x)/h = \widetilde{\phi}_\mu(\widetilde{x})/\widetilde{h}$, with

$$\widetilde{\phi}_\mu(\widetilde{x}) = \frac{1}{\mu^2}\widetilde{\phi}(x) = \frac{1}{\mu^2}\widetilde{\phi}(\mu\widetilde{x}). \tag{7.22}$$

It follows that,

$$\Lambda_{\widetilde{\phi}_\mu} = \{\mu^{-1}(x,\xi);\ (x,\xi) \in \Lambda_{\widetilde{\phi}}\}. \tag{7.23}$$

The same change of variables in (7.17) gives (with $x = \mu\widetilde{x}$)

$$\begin{aligned} &Q_\chi^w(x, hD_x; h)u \\ &\quad = \frac{1}{(2\pi\widetilde{h})^2}\iint e^{\frac{i}{\widetilde{h}}(\widetilde{x}-\widetilde{y})\cdot\widetilde{\theta}} Q\Big(\mu(\frac{\widetilde{x}+\widetilde{y}}{2}, \widetilde{\theta}); h\Big)\chi(\mu(\widetilde{x},\widetilde{y}))u(\mu\widetilde{y})d\widetilde{y}d\widetilde{\theta}, \end{aligned} \tag{7.24}$$

where the integration is now along the contour

$$\widetilde{\theta} = \frac{2}{i}\frac{\partial\widetilde{\phi}_\mu}{\partial\widetilde{x}}\Big(\frac{\widetilde{x}+\widetilde{y}}{2}\Big) + iC(\mu\widetilde{x})\overline{(\widetilde{x}-\widetilde{y})}.$$

Recall from (7.9) that q_0 vanishes to the 2nd order at $(0,0)$ and let $q_0(x,\xi) = q_{0,2}(x,\xi) + q_{0,3}(x,\xi) + \cdots$ be the Taylor expansion at $(0,0)$, so that $q_{0,j}$ is a homogenous polynomial of degree j. Then for $(\widetilde{x},\widetilde{\xi})$ in a μ-independent neighborhood of $(0,0)$, we get

$$\frac{1}{\mu^2} q_0(\mu(\widetilde{x},\widetilde{\xi})) = q_{0,2}(\widetilde{x},\widetilde{\xi}) + \mu q_{0,3}(\widetilde{x},\widetilde{\xi}) + \mu^2 q_{0,4}(\widetilde{x},\widetilde{\xi}) + \cdots . \tag{7.25}$$

This expansion actually holds in a μ^{-1}-neighborhood of (0,0), and outside such a neighborhood, we know that $\mu^{-2}|q_0(\mu\cdot)|$ is of the order of μ^{-2}, while $\nabla^k(\mu^{-2}q_0(\mu\cdot)) = \mathcal{O}(\mu^{-2+k})$. The sum of the other terms in the right hand side of (7.21) is $\mathcal{O}(\widetilde{h})$ together with all its derivatives.

From (7.22), we see that $\nabla^2\widetilde{\phi}_\mu$ varies in a bounded set in C_b^∞, when $\mu \to 0$, and in view of (7.12), we know that

$$\widetilde{\phi}_\mu(\widetilde{x}) = \frac{1}{2}|\widetilde{x}|^2, \tag{7.26}$$

for $\mu\widetilde{x}$ in a neighborhood of $(0,0)$. Consider the restriction of $q_{0,2}$ to $\Lambda_{\widetilde{\phi}_\mu} \cap \operatorname{neigh}((0,0))$. Let $w \in \mathbf{C}$ with

$$\frac{1}{2} < |w| < 2,\ -\frac{\pi}{2} + \varepsilon_0 < \arg w < -\varepsilon_0, \tag{7.27}$$

for some small but fixed $\varepsilon_0 > 0$. Then if $p_0 = q_{0,2}|_{\Lambda_{\widetilde{\phi}_\mu}}$, we see from (7.14) that

$$p_0(\widetilde{x},\widetilde{\xi}) - w = 0 \Longrightarrow \begin{cases} d\operatorname{Re} p_0, d\operatorname{Im} p_0 \text{ are independent,} \\ \text{and } \{\operatorname{Re} p_0, \operatorname{Im} p_0\} = 0, \text{ at } (\widetilde{x},\widetilde{\xi}). \end{cases} \tag{7.28}$$

Here the bracket is the Poisson bracket on the IR-manifold $\Lambda_{\widetilde{\phi}_\mu}$ and the linear independence is uniform with respect to μ.

Let $p = \mu^{-2}q_0(\mu(\cdot))|_{\Lambda_{\widetilde{\phi}_\mu}}$. Then from (7.25), (7.28), we get

$$p(\widetilde{x},\widetilde{\xi}) - w = 0 \Longrightarrow \begin{cases} d\operatorname{Re} p, d\operatorname{Im} p \text{ are independent,} \\ \text{and } \{\operatorname{Re} p, \operatorname{Im} p\} = \mathcal{O}(\mu), \text{ at } (\widetilde{x},\widetilde{\xi}). \end{cases} \tag{7.29}$$

Again the independence is uniform with respect to μ.

This means that we can apply Theorem 6.3 to $\mu^{-2}Q^w(x, hD_x; h) - w$, when μ is small and we use $\widetilde{h}$ as the new semi-classical parameter. Indeed, all the assumptions are then fulfilled in a fixed neighborhood of $(0,0)$. Outside such a neighborhood, the symbol is only defined on $\Lambda_{\widetilde{\phi}_\mu}$, but elliptic and of a sufficiently good class to guarantee invertibility there. We also need to recall how Theorem 6.3 is connected to a Grushin problem. (To have a better notational agreement with Theorem 7.1, we replaced θ, k by $-\theta, -k$ in (6.58).)

Proposition 7.2. — *For w in the domain (7.27), $\mu^{-2}Q^w(x, hD_x; h) - w : H_{\widetilde{\phi}_\mu} \to H_{\widetilde{\phi}_\mu}$ is non-invertible precisely when*

$$w = K(2\pi\widetilde{h}(k - \theta_0), \mu; \widetilde{h}), \tag{7.30}$$

for some $k \in \mathbf{Z}^2$. Here $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ is fixed.

$$K(\theta, \mu; \widetilde{h}) \sim K_0(\theta, \mu) + \widetilde{h}^2 K_2(\theta, \mu) + \widetilde{h}^3 K_3(\theta, \mu) + \cdots, \tag{7.31}$$

where $K_0(\cdot, \mu)$ is the inverse of the action map

$$w \longmapsto I_0(w, \mu) \tag{7.32}$$

which is a diffeomorphism from a neighborhood of the closure of the domain (7.27) onto a neighborhood of its image. K_j depend smoothly on (θ,μ).

If

$$\operatorname{dist}(w, K(2\pi\widetilde{h}(\mathbf{Z}^2-\theta_0),\mu;\widetilde{h})) \geqslant \frac{\widetilde{h}}{2C}, \tag{7.33}$$

then the inverse of $\mu^{-2}Q^w - w$ is of norm $\leqslant \widetilde{h}^{-1}e^{\mathcal{O}(\mu)/\widetilde{h}}$. More precisely, we can define weights $\widetilde{\psi}_\mu$, with $\widetilde{\psi}_\mu - \widetilde{\phi}_\mu$ of uniformly compact support in $\mathbf{C}^2\setminus\{0\}$, and $=\mathcal{O}(\mu)$ in C^∞, depending smoothly on μ (and also on w), such that

$$(\tfrac{1}{\mu^2}Q^w - w)^{-1} = \mathcal{O}(1/\widetilde{h}) : H_{\widetilde{\psi}_\mu} \to H_{\widetilde{\psi}_\mu},$$

when (7.33) holds.

If

$$|w - K(2\pi\widetilde{h}(k-\theta_0),\mu;\widetilde{h})| < \frac{\widetilde{h}}{C}, \text{ for some } k\in\mathbf{Z}^2, \tag{7.34}$$

then there exist operators

$$R_+(w,\mu:\widetilde{h}) : H_{\widetilde{\psi}_\mu} \longrightarrow \mathbf{C},\ R_-(w,\mu;\widetilde{h}) : \mathbf{C} \longrightarrow H_{\widetilde{\psi}_\mu}, \tag{7.35}$$

depending smoothly on w,μ, such that the corresponding norms of $\nabla_w^j R_\pm$ are $\mathcal{O}(\widetilde{h}^{-j})$ and such that

$$\begin{pmatrix} \frac{1}{\widetilde{h}}(\frac{1}{\mu^2}Q^w - w) & R_- \\ R_+ & 0 \end{pmatrix} : H_{\widetilde{\psi}_\mu}\times\mathbf{C} \longrightarrow H_{\widetilde{\psi}_\mu\times\mathbf{C}} \tag{7.36}$$

has a uniformly bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}. \tag{7.37}$$

Here $E_{-+}(w,\mu;\widetilde{h})$ has an asymptotic expansion as in (6.24) where $\theta\mapsto E^0_{-+}(\theta,w,\mu)$ has a simple zero at 0.

We notice that the eigen-values w are even functions of μ (if we make the change of variables also for negative μ) and to infinite order in $\widetilde{h}$, they are smooth in μ. Hence

$$K(2\pi h(k-\theta_0),\mu;h) = K(2\pi h(k-\theta_0),-\mu;h) + \mathcal{O}(h^\infty),$$

from which we deduce that $K_j(\theta,\mu) = K_j(\theta,-\mu)$, $j=0,1,2,\dots$.

Introduce the Taylor expansion in μ:

$$K_j(\theta,\mu) \sim \sum_{\ell=0}^{\infty} K_{j,\ell}(\theta)\mu^{2\ell}. \tag{7.38}$$

In (7.20) we put $\widetilde{x} = \lambda\widetilde{y}$, and obtain the isospectral operator

$$\lambda^2\frac{1}{(\mu\lambda)^2}Q^w(\mu\lambda(\widetilde{y},\tfrac{\widetilde{h}}{\lambda^2}D_{\widetilde{y}});h). \tag{7.39}$$

The eigen-values are given by

$$\lambda^2 K(2\pi\tfrac{\widetilde{h}}{\lambda^2}(k-\theta_0),\lambda\mu;\tfrac{\widetilde{h}}{\lambda^2})+\mathcal{O}(h^\infty),$$

so we get

$$K(2\pi\widetilde{h}(k-\theta_0),\mu;\widetilde{h})=\lambda^2 K(2\pi\tfrac{\widetilde{h}}{\lambda^2}(k-\theta_0),\lambda\mu;\tfrac{\widetilde{h}}{\lambda^2})+\mathcal{O}(\widetilde{h}^\infty), \tag{7.40}$$

for $\lambda\sim 1$, $|\mu|\leqslant 1$, $k\in\mathbf{Z}^2$, and $w=K(2\pi\widetilde{h}(k-\theta_0),\mu;\widetilde{h})$ in the region (7.27). Combining this with (7.31), we get successively for $j=0,1,2,\dots$:

$$\widetilde{h}^j K_j(\theta,\mu)=\lambda^2 K_j(\theta/\lambda^2,\lambda\mu)\big(\widetilde{h}/\lambda^2\big)^j,$$

for θ in a domain with $K_0(\theta,\mu)$ in the domain (7.27). Dividing by $\widetilde{h}^j$, we get

$$K_j(\theta,\mu)=(\lambda^2)^{1-j}K_j(\theta/\lambda^2,\lambda\mu). \tag{7.41}$$

This relation can be used to extend the definition to a domain

$$0\leqslant|\theta|\mu^2\leqslant\frac{1}{C},\ |\theta|\neq 0, \tag{7.42}$$

with $K_0(\theta,0)$ in the domain (7.27). Indeed, if (θ,μ) satisfies (7.42), then we can take $\lambda\sim|\theta|^{1/2}$ and notice that $|\lambda\mu|\leqslant 1$. Also notice that

$$K_j(\theta,1)=\mu^{2(1-j)}K_j(\theta/\mu^2,\mu). \tag{7.43}$$

Combining (7.38), (7.41), we get

$$K_{j,\ell}(\theta)=\lambda^{2(1-j+\ell)}K_{j,\ell}(\theta/\lambda^2), \tag{7.44}$$

so $K_{j,\ell}$ is positively homogeneous of degree $1-j+\ell$.

The scaling argument above allows us to describe all eigen-values z of $Q^w(x,hD_x;h)$ in a domain

$$h^\delta<|z|<\frac{1}{C_1},\ -\frac{\pi}{2}+\varepsilon_0<\arg z<-\varepsilon_0, \tag{7.45}$$

for $0<\delta<1/2$, by

$$z=\mu^2K(2\pi\tfrac{h}{\mu^2}(k-\theta_0),\mu;\tfrac{h}{\mu^2})+\mathcal{O}(h^\infty), \tag{7.46}$$

where we choose $\mu>0$ with $|z|/\mu^2\sim 1$.

We now return to the operator P in (7.1). Let z be as in (7.45) and consider the most interesting case when

$$\Big|\frac{z}{\mu^2}-K(2\pi\widetilde{h}(k-\theta_0),\mu;\widetilde{h})\Big|\leqslant\frac{\widetilde{h}}{C},\ \text{for some } k\in\mathbf{Z}^2, \tag{7.47}$$

where μ is given as after (7.46) and $\widetilde{h}=h/\mu^2$. We shall need the Grushin problem evocated in Proposition 7.2, but now for simplicity for the unscaled operator

$$\frac{1}{h}(Q^w-z)=\frac{1}{\widetilde{h}}\Big(\frac{1}{\mu^2}Q^w-w\Big),\ w=\frac{z}{\mu^2}:$$

$$\begin{pmatrix} \frac{1}{h}(Q^w - z) & R_- \\ R_+ & 0 \end{pmatrix} : H_{\widetilde{\psi}} \times \mathbf{C} \longrightarrow H_{\widetilde{\psi}} \times \mathbf{C}. \tag{7.48}$$

This is the same as in (7.36) except that we work in the original unscaled variables $x = \mu\widetilde{x}$, so $\widetilde{\psi}(x) = \widetilde{\psi}_\mu(\widetilde{x})$. Then $\widetilde{\psi} = \widetilde{\phi} + \mathcal{O}(\mu^3)$ with $\widetilde{\psi} = \widetilde{\phi}$ outside a μ-neighborhood of 0. Recall that $\widetilde{\phi} = \psi$ outside a fixed neighborhood of 0. Also recall that $\widetilde{\psi}$ is a small perturbation of ψ and that $\widetilde{\psi} = \psi$ outside a small neighborhood of 0.

From the fact that (7.48) is globally bijective with a bounded inverse, we deduce that if $u \in H_{\widetilde{\psi}}(\widetilde{\Omega})$, $u_-, v_+ \in \mathbf{C}$ and

$$\frac{1}{h}(Q^w - z)u + R_- u_- = v, \text{ in } \widetilde{\Omega}, \ R_+ u = v_+, \tag{7.49}$$

then

$$\|u\|_{H_{\widetilde{\psi}}(\widetilde{\Omega}_2)} + |u_-| \leqslant \mathcal{O}(1)(\|v\|_{H_{\widetilde{\psi}}(\widetilde{\Omega}_3)} + |v_+|) + \mathcal{O}(e^{-1/Ch})(\|u\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + |u_-|). \tag{7.50}$$

Here we let

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$$

be neighborhoods of 0 and $\widetilde{\Omega}_j$ be the corresponding neighborhoods of 0 in $\mathbf{C}^2$ such that $\widetilde{\Omega}_j = \pi_x \kappa_V(\pi_x^{-1}\Omega_j \cap \Lambda_\phi)$, where κ_V is the canonical transformation associated to V. We may assume that $\widetilde{\psi}, \widetilde{\phi}, \psi$ coincide outside $\widetilde{\Omega}_1$. In (7.50) it is understood that we realize Q^w on $H_{\widetilde{\psi}(\widetilde{\Omega})}$ (see **[Sj1]**) and the last term in (7.50) takes into account the corresponding boundary effects.

We let $\widetilde{H}(1)$ be the space $H(\Lambda_G, 1)$ equipped with the norm

$$\|u\|_{\widetilde{H}(1)} = \|VTu\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + \|Tu\|_{L^2_\phi(\mathbf{C}^2 \setminus \Omega_1)}. \tag{7.51}$$

We will see that this is a norm and that we get a uniformly equivalent norm if we replace Ω_1 by Ω_0 or Ω_2. We define $\widetilde{H}(\langle\xi\rangle^2)$ analogously.

We shall study the global Grushin problem

$$\begin{cases} \frac{1}{h}(P - z)u + SUR_- u_- = v \\ R_+ VTu = v_+, \end{cases} \tag{7.52}$$

for $u_-, v_+ \in \mathbf{C}$, $u \in \widetilde{H}(\langle\xi\rangle^2)$, $v \in \widetilde{H}(1)$.

Apply VT to the first equation,

$$\begin{cases} \frac{1}{h}(Q^w - z)VTu + R_- u_- = VTv + w, \\ R_+ VTu = v_+, \end{cases} \tag{7.53}$$

where

$$w = \frac{1}{h}(Q^w VT - VTP)u + (1 - VU)R_- u_- + V(1 - TS)UR_- u_-. \tag{7.54}$$

Here we notice that we may assume that

$$|\widetilde{\psi} - \psi| \leqslant \varepsilon_0, \tag{7.55}$$

for $\varepsilon_0 > 0$ fixed and arbitrarily small, provided that we restrict the spectral parameter to a sufficiently small h-independent disc. Combining (7.54) and the earlier estimates on $Q^w VT - VTP$, $1 - VU$, $1 - TS$, U, V, we see that

$$\|w\|_{H_\psi(\widetilde{\Omega}_3)} \leqslant \mathcal{O}(1) e^{-1/Ch}(\|Tu\|_{H_\phi(\Omega)} + |u_-|), \tag{7.56}$$

where C does not depend on ε_0 in (7.55). Applying this and (7.50) (with u, v replaced by VTu, VTv) to (7.53), we get the "interior" estimate

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_2)} + |u_-| \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_3)} + |v_+| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega)}). \tag{7.57}$$

On the other hand, if we restrict the the spectral parameter to a sufficiently small (h-independent) disc, we get from [**HeSj**]:

$$\begin{aligned}\|m^2 Tu\|_{L^2_\phi(\mathbf{C}^2 \setminus \Omega_1)} \\ \leqslant \mathcal{O}(1)(\|Tv\|_{H_\phi(\mathbf{C}^2\setminus\Omega_0)} + e^{-1/Ch}|u_-| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega_0)}).\end{aligned} \tag{7.58}$$

Indeed, we can apply the [**HeSj**] theory to the space $H(\langle\xi\rangle^2, \Lambda_{G_+})$, defined to be $H(\Lambda_G, \langle\xi\rangle^2)$ as a space, and with the norm $\|m^2 Tu\|_{L^2_{\phi_+}}$, where $\phi_+ - \phi \geqslant 0$ is small in C^∞, strictly positive on $\overline{\Omega_0}$ and equal to 0 in a neighborhood of $\mathbf{C}^2 \setminus \Omega_1$. We then see that $P - z$ is elliptic in this space away from a small neighborhood of $(0, 0)$, and (7.58) follows.

If we use

$$Tu = UVTu + (1 - UV)Tu,$$

we get

$$\|Tu\|_{H_\phi(\Omega_1)} \leqslant \mathcal{O}(1)(e^{\varepsilon_0/h}\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega)}). \tag{7.59}$$

Here the last term can be replaced by $e^{-1/Ch}\|Tu\|_{H_\phi(\Omega\setminus\Omega_1)}$ when h is small. Moreover, it is clear that

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}\setminus\widetilde{\Omega}_2)} = \|VTu\|_{H_\psi(\widetilde{\Omega}\setminus\widetilde{\Omega}_2)} \leqslant \mathcal{O}(1)\|Tu\|_{H_\phi(\Omega\setminus\Omega_1)}. \tag{7.60}$$

It follows that

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_2)} + \|Tu\|_{H_\phi(\Omega\setminus\Omega_1)} \sim \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tu\|_{H_\phi(\Omega\setminus\Omega_1)}, \tag{7.61}$$

and similarly with $\|Tu\|_{H_\phi(\Omega\setminus\Omega_1)}$ replaced by $\|m^2 Tu\|_{H_\phi(\Omega\setminus\Omega_1)}$. Now add (7.57), (7.58) and use (7.61):

$$\begin{aligned}\|m^2 Tu\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_1)} + \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + |u_-| \\ \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_2)} + |v_+| + e^{-1/Ch}\|Tu\|_{L^2_\phi(\mathbf{C}^2)}).\end{aligned} \tag{7.62}$$

Here we can absorb the contribution from $\|Tu\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_1)}$ to the last term, and get

$$\begin{aligned}\|m^2 Tu\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_1)} + \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + |u_-| \\ \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_2)} + |v_+| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega_1)}).\end{aligned} \tag{7.63}$$

Now use (7.59) to estimate the last term. We can assume that $\varepsilon_0 < 1/2C$ and get with a new constant C:

$$\begin{aligned}(7.64)\qquad \|m^2Tu\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_1)} + \|VTu\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + |u_-|\\ \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_2)} + |v_+|).\end{aligned}$$

We have then proved:

Proposition 7.3. — *Let z be in the region (7.47) and (7.45) with $|z| < r$ and $r > 0$ small enough. Then the problem (7.52) has a unique solution $(u, u_-) \in \widetilde{H}(\langle\xi\rangle^2) \times \mathbf{C}$ for every $(v, v_+) \in \widetilde{H} \times \mathbf{C}$, satisfying*

$$(7.65)\qquad \|u\|_{\widetilde{H}(\langle\xi\rangle^2)} + |u_-| \leqslant \mathcal{O}(1)(\|v\|_{\widetilde{H}(1)} + |v_+|).$$

Indeed, it is clear that (7.52) is Fredholm of index 0 and (7.64) implies injectivity. (7.65) is just an equivalent form of (7.64).

Proposition 7.4. — *Under the assumptions of Proposition 7.3, let*

$$\mathcal{F} = \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix}$$

be the inverse of

$$\begin{pmatrix} \frac{1}{h}(P-z) & SUR_- \\ R_+VT & 0 \end{pmatrix}.$$

Then

$$(7.66)\qquad \nabla_z^k(F_{-+} - E_{-+}) = \mathcal{O}(h^\infty) \text{ for every } k \in \mathbf{N}.$$

Proof. — It is easy to see that $\nabla_z^k F_{-+}$, $\nabla_z^k E_{-+}$ are $\mathcal{O}(h^{-N(k)})$, for every $k \in \mathbf{N}$ with some $N(k) \geqslant 0$, so it suffices to verify (7.66) for $k = 0$. Let $\widetilde{u} = E_+v_+$, $u_- = E_{-+}v_+$, $|v_+| = 1$, so that

$$(7.67)\qquad \frac{1}{h}(Q-z)\widetilde{u} + R_-u_- = 0,\ R_+\widetilde{u} = v_+.$$

Put $u = SU\widetilde{u}$. Then

$$\frac{1}{h}(P-z)u + SUR_-u_- = \frac{1}{h}(PSU - SUQ)\widetilde{u}.$$

Here in analogy with (7.8), we have

$$(7.68)\qquad PSU - SUQ = \mathcal{O}(1) : H_{\psi_+}(\widetilde{\Omega}) \longrightarrow H(\Lambda_{G_-}) \longrightarrow \widetilde{H}(1),$$

and $\widetilde{u}$ is exponentially small in $H_{\psi_+}(\widetilde{\Omega})$, so

$$(7.69)\qquad \frac{1}{h}(P-z)u + SUR_-u_- = v,\ \|v\|_{\widetilde{H}(1)} = \mathcal{O}(e^{-1/Ch}).$$

Similarly,

$$(7.70)\qquad R_+VTu = v_+ + R_+(VTSU - 1)\widetilde{u} =: v_+ + w_+$$

and $(VTSU-1)\widetilde{u}$ is exponentially small in $H_{\widetilde{\psi}}(\widetilde{\Omega})$, so $|w_+| = \mathcal{O}(e^{-1/Ch})$. It follows form this and Proposition 7.3 that

$$u_- = F_{-+}v_+ + F_{-+}w_+ + F_-v = \mathcal{O}(e^{-1/Ch}),$$

and the proposition follows since $u_- = E_{-+}v_+$. □

It is now clear that (7.46) describes all eigen-values of P in the domain (7.45).

If we further restrict the attention to

$$h^{\delta_2} < |z| < h^{\delta_1},\ -\frac{\pi}{2} + \varepsilon_0 < \arg z < -\varepsilon_0, \tag{7.71}$$

with $0 < \delta_1 < \delta_2 < 1/2$, then μ in (7.46) is $\mathcal{O}(h^{\delta_1/2})$ and we can apply the Taylor expansion (7.38). Then (7.46) becomes

$$z \sim \mu^2 \sum_1^\infty \sum_{\ell=0}^\infty \Big(\frac{h}{\mu^2}\Big)^j K_{j,\ell}(2\pi \tfrac{h}{\mu^2}(k-\theta_0))\mu^{2\ell}. \tag{7.72}$$

Now use that $K_{j,\ell}$ is homogeneous of degree $1-j+\ell$ to get the eigen-values in (7.71) on the form

$$z \sim \sum_{j=0}^\infty \sum_{\ell=0}^\infty K_{j,\ell}(2\pi h(k-\theta_0))h^j,\ k \in \mathbf{Z}^2. \tag{7.73}$$

From Theorem 7.1 (of [**KaKe**]) we know on the other hand that the eigen-values in (7.71) are given by

$$z \sim \sum_{j=0}^\infty h^j f_j(2\pi h(\widetilde{k} - \theta_0)),\ \widetilde{k} \in \mathbf{Z}^2, \tag{7.74}$$

where $f_j \in C^\infty(\mathrm{neigh}\,(0, \mathbf{R}^2))$ (with the same neighborhood for every j. Here $\widetilde{k}$ is not necessarily equal to k for the same eigen-value but if we start with some fixed small h and then let $h \to 0$, we see tht $\widetilde{k} = k + k_0$, where k_0 is constant. Approximating $f_j(\theta)$ for $\theta = 2\pi h(\widetilde{k} - \theta_0)$ by the Taylor expansion at $\theta = 2\pi h(k-\theta_0)$, we get a representation (7.74) with new f_js for $j \geqslant 1$, where we may assume that $\widetilde{k} = k$.

If we introduce the Taylor expansion of each f_j at 0, we see that (7.74) takes the form

$$z \sim \sum_{j=0}^\infty \sum_{\ell=0}^\infty h^j \widetilde{K}_{j,\ell}(2\pi h(k-\theta_0)), \tag{7.75}$$

where $\widetilde{K}_{j,\ell}$ is a homogeneous polynomial of degree $1-j+\ell$ (which vanishes for $1-j+\ell < 0$).

Let $F_{j,\ell} = K_{j,\ell} - \widetilde{K}_{j,\ell}$, so that $F_{j,\ell}(\theta)$ is smooth and positively homogeneous of degree $1-j+\ell$ in the angle V, defined by $-\frac{\pi}{2} + \varepsilon_0 < \arg F_{0,0}(\theta) < -\varepsilon_0$. We then know that

$$\sum_{j=0}^\infty \sum_{\ell=0}^\infty h^j F_{j,\ell}(2\pi h(k-\theta_0)) = \mathcal{O}(h^\infty), \tag{7.76}$$

for $2\pi h(k-\theta_0)\in V$ with $h^{\delta_2}<|2\pi h(k-\theta_0)|<h^{\delta_1}$. We restrict the attention to the domain $|2\pi h(k-\theta_0)|\sim h^{\delta}$, where we are free to choose δ in $]0,\frac{1}{2}[$, and let $h\to 0$ for each fixed δ. We shall show that $F_{j,\ell}=0$ by induction in alphabetical order in (j,ℓ). Assume that we already know that $F_{j,\ell}=0$ for $j<j_0$ and for $j=j_0$, $\ell<\ell_0$. Here $(j_0,\ell_0)\in\mathbf{N}^2$. Then (7.76) gives

$$F_{j_0,\ell_0}(2\pi h(k-\theta_0))=\mathcal{O}(1)\max(h^{\delta(2-j_0+\ell_0)},h^{1-\delta j_0}),\tag{7.77}$$

for $k\in\mathbf{Z}^2$ with $\theta:=2\pi h(k-\theta_0)$ in V and $|\theta|\sim h^{\delta}$. In this region $\nabla F_{j_0,\ell_0}=\mathcal{O}(1)h^{\delta(-j_0+\ell_0)}$ and (7.77) implies that

$$F_{j_0,\ell_0}(\theta)=\mathcal{O}(1)\max(h^{1+\delta(\ell_0-j_0)},h^{2\delta+\delta(\ell_0-j_0)},h^{1-\delta j_0})=\mathcal{O}(1)h^{\delta(2+\ell_0-j_0)},\tag{7.78}$$

if $\delta>0$ is small enough depending on (ℓ_0,j_0), and for $|\theta|\sim h^{\delta}$, $\theta\in V$. Since F_{j_0,ℓ_0} is homogeneous of degree $1+\ell_0-j_0$, we see that $F_{j_0,\ell_0}=0$ in V. Consequently, we have

Proposition 7.5. — *$K_{j,\ell}(\theta)$ is a homogeneous polynomial of degree $1+\ell-j$ (equal to 0 for $1+\ell-j<0$).*

Using this, we get

Proposition 7.6. — *$K_j(\theta,1)$ extends to a smooth function in a j-independent neighborhood of 0.*

Proof. — We study the asymptotics when $V\ni\theta\to 0$, using (7.38), (7.41) and get with $\mu^2\sim|\theta|$:

$$\begin{aligned}K_j(\theta,1)=\mu^{2(1-j)}K_j(\theta/\mu^2,\mu)&\sim\sum_{\ell\geqslant\max(0,j-1)}\mu^{2(1-j)}K_{j,\ell}(\theta/\mu^2)\mu^{2\ell}\\&\sim\sum_{\ell\geqslant\max(0,j-1)}K_{j,\ell}(\theta),\ \theta\longrightarrow 0.\end{aligned}$$

This expansion is also valid after differentiation and since $K_{j,\ell}$ are polynomials, we see that (7.38) is the Taylor expansion of a smooth function in a neighborhood of 0. □

We now return to the description (7.46) of the resonances of P in (7.45) and use (7.41):

$$z\sim\sum_{j=0}^{\infty}\mu^2K_j\Big(\frac{2\pi h(k-\theta_0)}{\mu^2},\mu\Big)\frac{h^j}{\mu^{2j}}=\sum_{j=0}^{\infty}K_j(2\pi h(k-\theta_0),1)h^j.\tag{7.79}$$

With $f_j(\theta)=K_j(\theta,1)$, we get from this, Theorem 7.1 and the identification of the different ks in (7.73), (7.74):

Theorem 7.7. — *The description of the resonances in Theorem 7.1 extends to the set of z in (7.45), provided that C_1 there is sufficiently large as a function of $\varepsilon_0>0$ and that $h>0$ is small enough.*

References

[BaGrPa] D. Bambusi, S. Graffi, T. Paul, *Normal forms and quantization formulae,* Comm. Math. Phys. 207 (1) (1999), 173–195.

[BaTu] A. Bazzani, G. Turchetti, *Singularities of normal forms and topology of orbits in area-preserving maps,* J. Phys. A, 25 (8) (1992), 427–432.

[BeJoSc] L. Bers, F. John, M. Schechter, *Partial differential equations,* Reprint of the 1964 original. Lectures in Applied Mathematics, AMS, Providence, R.I., 1979.

[BrCoDu] P. Briet, J.M. Combes, P. Duclos, *On the location of resonances for Schrödinger operators in the semiclassical limit. I. Resonance free domains,* J. Math. Anal. Appl. 126 (1) (1987), 90–99.

[Ca] Carleman, *Sur les systémes linéaires aux dérivées partielles de premier ordre à deux variables,* C.R. Acad. Sci., vol 197 (1933), 471-474.

[Co] Y. Colin de Verdière, *Quasi-modes sur les variétés Riemanniennes,* Inv. Math., 43 (1) (1977), 15–52.

[DiSj] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit,* London Math. Soc. Lecture Notes Series 269, Cambridge University Press 1999.

[DuHo] J.J. Duistermaat, L. Hörmander, *Fourier integral operators. II,* Acta Math. 128 (1972), 183–269.

[GeSj] C. Gérard, J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type,* Comm. Math. Phys., 108 (1987), 391–421.

[GrSj] A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators, an introduction,* London Math. Soc. Lecture Notes Series 196, Cambridge Univ. Press, 1994.

[HeRo] B. Helffer, D. Robert, *Puits de potentiel généralisés et asymptotique semiclassique,* Ann. Henri Poincaré, Phys. Th., 41 (3) (1984), 291–331.

[HeSj] B. Helffer, J. Sjöstrand, *Résonances en limite semi-classique,* Mém. Soc. Math. France (N.S.) No. 24-25, (1986).

[HeSj2] B. Helffer, J. Sjöstrand, *Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum,* Mém. Soc. Math. France (N.S.) No. 39 (1989), 1–124.

[KaKe] N. Kaidi, Ph. Kerdelhué, *Forme normale de Birkhoff et résonances,* Asympt. Anal. 23 (2000), 1–21.

[La] V.F. Lazutkin, *KAM theory and semiclassical approximations to eigenfunctions.* With an addendum by A. I. Shnirelman. Ergebnisse der Mathematik und ihrer Grenzgebiete, 24. Springer-Verlag, Berlin, 1993.

[Mas] V.P. Maslov, *Théorie des perturbations et méthodes asymptotiques,* (translated by J. Lascoux et R. Seneor), Dunod (Paris) (1972).

[MeSj] A. Melin, J. Sjöstrand, *Determinants of pseudodifferential operators and complex deformations of phase space,* `http://xxx.lanl.gov/abs/math.SP/0111292`, Methods and Applications of Analysis, to appear.

[Mo] J. Moser, *On the generalization of a theorem of A. Liapounoff,* Comm. Pure Appl. Math. 11 (1958), 257–271.

[Po1] G. Popov, *Invariant tori, effective stability, and quasimodes with exponentially small error terms. I. Birkhoff normal forms,* Ann. Henri Poincaré, Phys. Th., 1 (2) (2000), 223–248.

[Po2] G. Popov, *Invariant tori, effective stability, and quasimodes with exponentially small error terms. II. Quantum Birkhoff normal forms,* Ann. Henri Poincaré, Phys. Th., 1 (2) (2000), 249–279.

[Sj1] J. Sjöstrand, *Singularités analytiques microlocales,* Astérisque, 95 (1982).

[Sj2] J. Sjöstrand, *Semiclassical resonances generated by a non-degenerate critical point,* Springer LNM, 1256, 402-429.

[Sj3] J. Sjöstrand, *Function spaces associated to global I-Lagrangian manifolds,* pages 369-423 in Structure of solutions of differential equations, Katata/Kyoto, 1995, World Scientific 1996.

[Sj4] J. Sjöstrand, *Semi-excited states in nondegenerate potential wells,* Asymptotic Analysis, 6 (1992), 29–43.

[Vu] S. Vu-Ngoc, *Invariants symplectiques et semi-classiques des systèmes intégrables avec singularité,* Séminaire e.d.p., Ecole Polytechnique, 23 janvier, 2000–2001.

A. MELIN, Dept. of Mathematics, University of Lund, Box 118, S-22100 Lund, Sweden
E-mail : `melin@maths.lth.se`
Url : `http://www.maths.lth.se/matematiklu/personal/melin/`

J. SJÖSTRAND, Centre de Mathématiques, École polytechnique, F-91128 Palaiseau cedex, France
E-mail : `johannes@math.polytechnique.fr`
Url : `http://daphne.math.polytechnique.fr/~sjoestrand`