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TORSION ÉTALE AND CRYSTALLINE COHOMOLOGIES

by

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"This cohomology should also, most importantly, explain torsion phenomena, and in particular p-torsion"
A. Grothendieck, Crystals and the de Rham cohomology of schemes.

Abstract. — Following our two courses at the Centre Émile Borel of the I.H.P. during the Semestre p-adique of 1997, we present a survey of the Fontaine-Laffaille and Fontaine-Messing theories and (with more details) of their extension by one of us to the semi-stable setting. We also very quickly discuss some ℓ-adic analogues of Nakayama. We take advantage to include a few proofs which are not in the literature and raise several remaining open questions.

1. Introduction

This article is both a resume of our two courses at the Centre Émile Borel of the I.H.P. during the Semestre p-adique and a survey of the papers [**31**], [**32**], [**9**], [**10**]. These courses were, from the outset, coordinated. Indeed, the course of the second author was largely foundational and was viewed as preparatory for the course of the first author, a Cours Peccot, devoted to his generalization to the semi-stable situation, via log-syntomic methods, of some of the results of [**32**]. We concentrate here primarily on [**9**], [**10**], adopting a strictly utilitarian point of view and, hopefully, making then the article more useful to number theorists or algebraic geometers who are not specialists in p-adic theories. Nevertheless, to keep the text to a reasonable length, we have found it necessary to assume the reader has some awareness of crystalline and semi-stable p-adic Galois representations and the corresponding comparison theorems. Certainly, an acquaintance with log-schemes would also be helpful, although we recall their definition.

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We do not intend to review, even in the most cursory fashion, the history of what has become a somewhat intricate and still evolving complex of theories and techniques all ultimately intended to clarify the relationships between the diverse p-adic objects which are cohomologically associated to appropriate algebraic varieties. These objects are either the cohomology groups or are the "coefficients" which serve as input for or arise as the output from such cohomology groups. We refer the reader to [**30**], [**25**] for discussion of the comparison conjectures and to [**41**], [**71**] for surveys of the comparison theorems in the $\mathbf{Q}_p$-coefficient context. The proofs, with varying degree of detail, are given in [**18**], [**19**], [**22**], [**32**], [**44**], [**46**], [**59**], [**70**].

The case of torsion coefficients has had itself a long gestation. The dictionary relating unramified representations and "unit root F-crystals" goes back to Artin, Hasse and (especially) Witt during the thirties. The extension of classical Dieudonné theory from the case of smooth (commutative) formal groups to finite connected or unipotent group schemes over a perfect field k is due to Gabriel ([**67**]). The analogous results over $W(k)$ are due to Fontaine ([**29**]). Grothendieck stressed both in [**34**] and in [**35**] the geometric importance of understanding p-torsion phenomena in the Picard scheme and also in higher cohomological contexts. Important examples and results were given by Mumford and Raynaud ([**55**], [**56**], [**62**]). To the best of our knowledge it was Grothendieck who, in his Algerian letter to Deligne ([**36**]), first explicitly raised the question of understanding the relation between the torsion invariants in the p-adic étale cohomology (or equivalently the Betti cohomology) of the geometric generic fiber and in the "p-adic cohomology" of the special fiber. Shortly after with the creation of crystalline cohomology ([**35**], [**3**]), it was possible to attach precise meaning to this last term. In fact, the situation is subtle as examples, due to Ekedahl ([**17**]), show that for V a complete discrete valuation ring of unequal characteristic and residue field k and X/V proper and smooth, the π-torsion invariants for $H^*_{\mathrm{dR}}(X/V)$ are not necessarily those of $H^*_{\mathrm{cris}}(X_k/W) \otimes_W V$ (where $W = W(k)$). Even today there remains much to understand concerning torsion in the (very) ramified case.

The approach we discuss in the text for studying torsion phenomena is via the use of log-syntomic methods (section 6). Although he made no application of it, it was Mazur who first discussed the syntomic topology ([**51**]). Fontaine and the second author showed in 1982 that $\mathcal{O}_n^{\mathrm{cris}}$ is a sheaf for the syntomic topology and subsequently made systematic use of syntomic methods to establish the crystalline conjecture for $e = 1$ and in degree $< p$. Using Kato's K-theoretic calculations of the nearby cycles they established the equality of the torsion invariants in the same context (see section 3). It is the extension of these results to the semi-stable situation and the log-syntomic generalization of these methods which is the subject of this survey.

In the semi-stable situation, even when working over $K_0 = \mathrm{Frac}(W)$, it is useful to introduce the larger ring $S_{K_0} = S \otimes_W K_0$ where $S = \widehat{W\langle u\rangle}$ is the p-adic completion of the divided power polynomial ring in the variable u. The (ϕ, N)-filtered modules D of the semi-stable theory do not satisfy Griffiths' transversality, but it is shown in [**8**] that $D \mapsto D \otimes_{K_0} S_{K_0}$ establishes an equivalence with a category of S_{K_0}-modules (equipped with additional structure) whose objects now do satisfy Griffiths' transversality (see section 4). It is the torsion analogue of this last category which generalizes in the semi-stable context the filtered module category of Fontaine and Laffaille ([**31**]). This is discussed in detail in the text. Suffice it here to say that for each r with $0 \leqslant r \leqslant p-2$, we define such a category, $\underline{\mathcal{M}}^r$ (see section 5).

The categories $\underline{\mathcal{M}}^r$ are interesting for two reasons. The first reason is that they allow one to get a handle on new and interesting phenomena in the semi-stable situation which don't arise in the analogous crystalline situation. For instance, irreducible 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with distinct Hodge-Tate weights in $\{0, \dots, p-2\}$ are all irreducible modulo p whereas this is far from being the case with irreducible 2-dimensional semi-stable representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with distinct Hodge-Tate weights in $\{0, \dots, p-2\}$ (and here the reduction modulo p is very interesting to study, see [**9**] and the last section). The second reason is that these categories are related to geometry. Let X/W be proper and semi-stable. Endow it with its canonical log-structure (cf. section 2), denote by X_n its reduction modulo p^n and consider the log-crystalline cohomology of X_n relative to the base $E_n = \mathrm{Spec}(S/p^n S)$. This is also the log-syntomic cohomology of X with coefficients in the sheaf $\mathcal{O}_n^{\mathrm{st}}$ (which plays here the role of the classical $\mathcal{O}_n^{\mathrm{cris}}$). Then one proves that, for $0 \leqslant i \leqslant r \leqslant p-2$, the corresponding H^i (equipped with its Fil^r, ϕ_r, N) is an object of the category $\underline{\mathcal{M}}^r$ (see section 7) and, using Hyodo-Kato-Tsuji's K-theoretic calculations of the nearby cycles in the semi-stable situation, that the torsion Galois representation associated to it by the generalized Fontaine-Laffaille theory is the étale cohomology of the geometric generic fiber $X_{\overline{K}}$ with coefficients in $\mathbf{Z}/p^n\mathbf{Z}$ (see section 8).

We discuss applications of these results and related open questions in the last section. In particular we explain how to recover in the above situation the torsion invariants of the étale cohomology of the geometric generic fiber.

The reader will note that we frequently refer to the literature for the proofs. However we give proofs, or at least sketches of proofs, when a result does not have an otherwise published proof (as for instance in section 6) or when we think that the proof gives insight into the result discussed or into the techniques we use.

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2. The ℓ-torsion case

We set up the notations which we will keep throughout: p is a prime, k a perfect field of characteristic p, W the Witt vectors $W(k)$, $K_0 = \mathrm{Frac}(W)$, K a finite totally ramified extension of K_0, $\mathcal{O}_K$ its ring of integers, $\overline{K}$ an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ its ring of integers, $\overline{k}$ the corresponding algebraic closure of k, and $G_K \subset G_{K_0}$ the Galois groups $\mathrm{Gal}(\overline{K}/K) \subset \mathrm{Gal}(\overline{K}/K_0)$. For any prime ℓ, recall that an ℓ-adic representation of G_K or G_{K_0} is a continuous linear representation in a finite dimensional $\mathbf{Q}_\ell$-vector space and that a (finite) ℓ-torsion representation is a continuous (and hence finite) representation of G_K or G_{K_0} in a finite length $\mathbf{Z}_\ell$-module.

2.1. Good reduction. — Let $\ell \neq p$ be another prime. As is well known, an ℓ-adic or ℓ-torsion representation of G_K that has "good reduction" is just an unramified continuous representation. One of the first and most important results of étale cohomology is certainly:

***Theorem 2.1.1* (SGA$_4$ IX.2.2 + XVI.2.2).** — *Let X be a proper smooth scheme over $\mathcal{O}_K$. For $n \in \mathbf{N}$ and $i \in \mathbf{N}$, the specialization map induces isomorphisms compatible with the action of G_K:*

$$H^i((X \times_{\mathcal{O}_K} \overline{k})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} H^i((X \times_{\mathcal{O}_K} \overline{K})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}).$$

Notice that we compare something living on the geometric special fiber of X to something living on the geometric generic fiber. In particular, the étale cohomology of the geometric generic fiber is unramified (as a G_K-module). Till the end of this paper, we will keep this philosophy of comparing in various situations (torsion) Galois representations coming from the geometric special fiber to (torsion) Galois representations coming from the geometric generic fiber. In each case, the comparison will yield deep properties of the latter.

2.2. Semi-stable reduction. — We want to consider now the more general situation of a smooth proper K-scheme admitting a proper semi-stable model X over $\mathcal{O}_K$, that is X is regular and its special fiber is a reduced divisor with normal crossings in X. Equivalently, this means there exists an étale covering (U_i) of X such that each U_i is étale over an affine scheme of the form $\mathcal{O}_K[X_1, \ldots, X_s]/(X_1X_2\ldots X_r - \pi_K)$ $(1 \leqslant r \leqslant s)$ where π_K is an uniformizer of $\mathcal{O}_K$. We want an analogue of theorem 2.1.1 and consequently have to find a candidate to replace $H^i((X \times_{\mathcal{O}_K} \overline{k})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ that is still related to $X \times_{\mathcal{O}_K} k$ and that contains enough information to recover the étale cohomology of the generic fiber $X \times_{\mathcal{O}_K} K$. There is little hope the singular scheme $X \times_{\mathcal{O}_K} k$ alone will now be sufficient. What we need is some extra information related to the generic fiber, together with $X \times_{\mathcal{O}_K} k$, that is rich enough to give back the cohomology of the geometric generic fiber. It turns out that this extra information will be the log-structure (defined by Fontaine and Illusie) canonically attached to the model X (see 2.2.1.2 below). The idea is then to replace the étale cohomology of the scheme $X \times_{\mathcal{O}_K} \overline{k}$ by the *log*-étale cohomology of the *log*-scheme $X \times_{\mathcal{O}_K} \overline{k}$.

2.2.1. We rapidly recall some facts concerning log-schemes. The main reference is [**43**]. The monoids that are considered are all commutative with a unit element and will be usually written additively (this turns out to be more convenient in many situations). If M is a monoid, we denote by M^* its group of invertible elements and M^{gp} the group that it generates ([**43**, 1]).

***Definition 2.2.1.1* (Fontaine-Illusie).** — A pre-log-structure on a scheme X is a sheaf of monoids $\mathcal{M}_X$ on $X_{\text{ét}}$ together with a morphism of sheaves of monoids on $X_{\text{ét}}$, $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$, where $\mathcal{O}_X$ is viewed as a sheaf of multiplicative monoids. A pre-log-structure is a log-structure if $\alpha_X^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*$. A scheme endowed with a log-structure is called a log-scheme.

To a pre-log-structure $\mathcal{M}_X$, one can associate in a canonical way a log-structure by taking the push-out of $\mathcal{O}_X^* \leftarrow \alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{M}_X$ in the category of sheaves of monoids on $X_{\text{ét}}$. A monoid M is called integral if $a + b = a + c \Rightarrow b = c$ in M. A log-structure is called integral if it is a sheaf of integral monoids, and fine if locally on $X_{\text{ét}}$ it is associated to a pre-log-structure $\alpha : M \to \mathcal{O}_X$ where M is an integral monoid of finite type viewed as a constant sheaf. All the log-schemes of this paper are integral and most of them are fine. A morphism of log-schemes is a morphism of schemes together with a morphism of sheaves of monoids such that the obvious diagram is commutative [**43**, 1.1]. If $f : X \to Y$ is a morphism of schemes and if $\mathcal{M}_Y$ is a log-structure on Y, by definition the induced log-structure on X is the log-structure associated to $f^{-1}(\mathcal{M}_Y) \to \mathcal{O}_X$. Any scheme has a trivial log-structure (with $\mathcal{M}_X = \mathcal{O}_X^*$) and hence the category of schemes is a full subcategory of the category of log-schemes. If $(X, \mathcal{M}_X)$ is a log-scheme, we will refer to X itself as the underlying scheme. If we consider a log-scheme associated to a pre-log-structure $\alpha : M \to A$ where A is a commutative ring and M an integral monoid (that is, the underlying scheme is $\mathrm{Spec}(A)$), we will just write (A, M) and call this pair a log-ring. With additive notations on M, recall then that $0 \in M$ maps to $1 \in A$.

Example 2.2.1.2. — Let X be a scheme flat over $\mathrm{Spec}(\mathcal{O}_K)$, then:

$$\mathcal{M}_X = \{f \in \mathcal{O}_X \text{ such that } f|_{X \times_{\mathcal{O}_K} K} \in \mathcal{O}^*_{X \times_{\mathcal{O}_K} K}\}$$

is easily checked to be an integral log-structure on X. It is called the *canonical* log-struture associated to X. If $X = \mathcal{O}_K$, one finds $\mathcal{O}_K \smallsetminus \{0\} \to \mathcal{O}_K$ which is also the log-scheme associated to $(\mathbf{N} \to \mathcal{O}_K, 1 \mapsto \pi_K)$ where π_K is any uniformizer of $\mathcal{O}_K$. If X is semi-stable over $\mathcal{O}_K$ (cf. previously), one finds an étale covering (U_i) of X with induced log-structures such that each U_i is étale (with induced log-structure) over a log-scheme associated to:

$$\begin{array}{ccc} \mathbf{N}^r & \longrightarrow & \dfrac{\mathcal{O}_K[X_1,\ldots,X_s]}{(X_1X_2\cdots X_r-\pi_K)} \\ \uparrow & & \uparrow \\ \mathbf{N} & \longrightarrow & \mathcal{O}_K \end{array}$$

where $1 \leqslant r \leqslant s$, $\mathbf{N} \to \mathbf{N}^r$ is the diagonal embedding and $(0,\ldots,1,\ldots,0) \in \mathbf{N}^r$ maps to X_i if 1 is in position i. This semi-stable example is the main reason why one (usually) uses the étale site and not the Zariski site.

We stop here our brief review of log-schemes. In the sequel, we refer without comment to [**43**] or to [**71**, 3] in this volume for the definition of log-étale and log-smooth morphisms, exact morphisms, integral morphisms, closed immersions of log-schemes, ...

2.2.2. The semi-stable ℓ-adic or ℓ-torsion representations of G_K are the continuous representations such that the inertia acts unipotently (and consequently through its tame quotient). Let Σ_K be the log-scheme $\mathcal{O}_K \smallsetminus \{0\} \to \mathcal{O}_K$ and $\Sigma_{\overline{k}}$ the integral (not fine) log-scheme $\mathcal{O}_{\overline{K}} \smallsetminus \{0\} \to \overline{k}$. If X is a fine log-scheme over Σ_K, we denote by $X \times_{\Sigma_K} \Sigma_{\overline{k}}$ the fiber product in the category of integral log-schemes which is also, in this case, the fiber product in the category of all log-schemes (in particular the underlying scheme is just $X \times_{\mathcal{O}_K} \overline{k}$).

Theorem 2.2.1 **([58**, 4.2]). — *Let X be a proper semi-stable scheme over $\mathcal{O}_K$ and endow it with its canonical log-structure (2.2.1.2). For $n \in \mathbf{N}$ and $i \in \mathbf{N}$, there are isomorphisms compatible with the action of G_K:*

$$H^i((X \times_{\Sigma_K} \Sigma_{\overline{k}})_{\mathrm{log-\acute{e}t}}, \mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} H^i((X \times_{\mathcal{O}_K} \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/\ell^n\mathbf{Z}).$$

Here, the left hand side is the log-étale cohomology of the log-scheme $X \times_{\Sigma_K} \Sigma_{\overline{k}}$ defined by Nakayama ([**57**]) and the map is also induced by a specialization map (see [**58**]). One can show this implies $(g-\mathrm{Id})^{i+1}=0$ on $H^i((X \times_{\mathcal{O}_K} \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/\ell^n\mathbf{Z})$ for g in the inertia subgroup ([**58**, 3.7]), and so the representation $H^i((X \times_{\mathcal{O}_K} \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/\ell^n\mathbf{Z})$ is semi-stable. This result was already known in this situation by work of Rapoport-Zink ([**63**]), but the above theorem can be extended to a much more general situation. For details, see [**58**] and Illusie's nice surveys [**39**], [**40**].

In the sequel, we will consider the case $\ell = p$. The theory here becomes more involved and it turns out that it's *not* convenient to describe directly the action of Galois on $H^i((X \times_{\mathcal{O}_K} \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/p^n\mathbf{Z})$. Fortunately, one has instead explicit objects, living in the realm of linear algebra, that can (and will) be used to state comparison theorems between this p-torsion étale cohomology and a cohomology theory related to the special fiber, at least (so far) if one restricts to $K = K_0$ and H^i's with $p > i$. The case of arbitrary K is still under investigation for p-torsion ([**21**], [**11**]), although there is probably a nice theory if $p > i[K:K_0]$ (see [**11**]). The case $i \geqslant p$ (or even $i[K:K_0] \geqslant p$) is still largely open. For these reasons, we will now assume from § 3 to § 8 that $K = K_0$ and consider only those cohomology groups H^i for i not too big.

3. The p-torsion case: good reduction and Fontaine-Laffaille-Messing theory

Recall that a p-adic representation V of G_{K_0} that has "good reduction" is a crystalline representation i.e. such that $\dim_{K_0}(B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_{K_0}} = \dim_{\mathbf{Q}_p} V$ ([**25**, 5]). Here B_{cris} is a K_0-algebra that only depends on $\overline{K}/K_0$ and will be defined in (3.1.2). Our aim is to recall briefly the Fontaine-Laffaille theory of [**31**], that basically describes torsion subquotients of some crystalline representations, and the Fontaine-Messing theory [**32**] that applies the work of Fontaine and Laffaille to the study of p-torsion (and so p-adic) étale cohomology of varieties with good reduction over W.

3.1. Review of the Fontaine-Laffaille theory

3.1.1. To any crystalline representation, Fontaine associates in [**25**, 5] a weakly admissible filtered ϕ-module. We explain briefly what this is. A filtered ϕ-module D is a finite dimensional K_0-vector space endowed with a decreasing filtration by sub-K_0-vector spaces $\mathrm{Fil}^i D$ such that $\mathrm{Fil}^i D = D$ if $i \ll 0$, $\mathrm{Fil}^i D = 0$ if $i \gg 0$ and an injective K_0-semi-linear map $\phi : D \to D$ (the "Frobenius"). To such a D, we associate:

$$t_H(D) = \sum_{i \in \mathbf{Z}} (\dim_{K_0} \mathrm{gr}^i D) i$$

$$t_N(D) = \sum_{\alpha \in \mathbf{Q}} (\dim_{K_0} D_\alpha) \alpha$$

where $\alpha \in \mathbf{Q}$ and D_α is the sub-K_0-vector space of D of slope α for ϕ (see [**2**] and [**14**, 3.2]). We say D is weakly admissible if $t_H(D) = t_N(D)$ and $t_H(D') \leqslant t_N(D')$ for any sub-K_0-vector space $D' \subset D$ stable under ϕ with $\mathrm{Fil}^i D' = \mathrm{Fil}^i D \cap D'$. By the main result of [**14**], there is an *equivalence of categories* between weakly admissible filtered ϕ-modules and crystalline representations of G_{K_0}. Hence it's natural, if one wants integral or torsion crystalline representations, to look for integral structures first on the filtered module side. The following definition was inspired by the work of Mazur ([**49**], [**50**]) and Berthelot-Ogus [**4**, 8] on the Katz conjecture.

For $r \in \mathbf{N}$, define $\underline{MF}^{f,r}_{\mathrm{tor}}$ to be the category of W-modules of finite length M endowed with a decreasing filtration by sub-W-modules $(\mathrm{Fil}^i M)_{i \in \mathbf{Z}}$ such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$, and semi-linear maps (with respect to the Frobenius on W) $\phi_i : \mathrm{Fil}^i M \to M$ such that $\phi_i|_{\mathrm{Fil}^{i+1} M} = p\phi_{i+1}$ and $\sum_{i=0}^r \phi_i(\mathrm{Fil}^i M) = M$ (in the notation, "f" stands for "finite", since the modules are of finite length). Morphisms are the W-linear maps that send Fil^i to Fil^i and commute with ϕ_i. One thinks of ϕ_i

as "ϕ/p^i". Clearly $\underline{MF}_{\mathrm{tor}}^{f,r}$ is a full sub-category of $\underline{MF}_{\mathrm{tor}}^{f,r+1}$. More importantly, one has the surprising result:

Proposition 3.1.1.1. — *Let $f : M \to N$ be a morphism in $\underline{MF}_{\mathrm{tor}}^{f,r}$. Then:*
1) f is strict with respect to the filtration, i.e. for all i, $f(\mathrm{Fil}^i M) = \mathrm{Fil}^i N \cap f(M)$
2) if M' is the kernel of the underlying linear map, $\mathrm{Fil}^i M' = M' \cap \mathrm{Fil}^i M$ and ϕ_i : $\mathrm{Fil}^i M' \to M'$ the restriction of $\phi_i : \mathrm{Fil}^i M \to M$, we have $\sum_{i=0}^r \phi_i(\mathrm{Fil}^i M') = M'$.

See [**31**, 1.10, (b)] for the proof of 1) and [**31**, 1.10, (a)] for the proof of 2). Notice also that each $\mathrm{Fil}^i M$ is a direct factor of $\mathrm{Fil}^{i-1} M$: we say the M's are "filtered free" (this terminology is due to Faltings).

Corollary 3.1.1.2. — *The category $\underline{MF}_{\mathrm{tor}}^{f,r}$ is abelian. More precisely, if f is as in (3.1.1.1), we have:*

$$\mathrm{Ker}(f) = (M', M' \cap \mathrm{Fil}^i M, \phi_i)$$
$$\mathrm{Coker}(f) = (N/f(M), \mathrm{Fil}^i N/f(\mathrm{Fil}^i M), \phi_i).$$

Since the underlying W-modules have finite length, $\underline{MF}_{\mathrm{tor}}^{f,r}$ is also artinian. It is also of interest to consider the "without p-torsion" counterpart:

Definition 3.1.1.3. — A strongly divisible module is a free W-module M of finite type equipped with a decreasing filtration by sub-W-modules $(\mathrm{Fil}^i M)_{i\in\mathbf{Z}}$ such that $\mathrm{Fil}^0 M = M$, $\mathrm{Fil}^i M = 0$ for i big enough, $M/\mathrm{Fil}^i M$ has no p-torsion, and a semi-linear map $\phi : M \to M$ such that $\phi(\mathrm{Fil}^i M) \subset p^i M$ and $\sum_{i\in\mathbf{Z}} \frac{\phi}{p^i}(\mathrm{Fil}^i M) = M$.

If M is a strongly divisible module, $M/p^n M$ is in an obvious way an object of $\underline{MF}_{\mathrm{tor}}^{f,r}$ for $r \gg 0$ by defining $\phi_i = \frac{\phi}{p^i}|_{\mathrm{Fil}^i} \bmod p^n$.

3.1.2. Let us recall the cohomological definition of A_{cris} (see also [**32**, I.1.3-1.5] or [**75**, 2.1]). The brutal formula is $A_{\mathrm{cris}} = \varprojlim H^0_{\mathrm{cris}}((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})/W_n)$. The right hand side naturally appears as one of the components of a Künneth formula (see [**32**, III.1.3]). Either by a de Rham computation as in [**27**, 3.2] or by noticing that the crystalline site $(\mathrm{Spec}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})/W_n)_{\mathrm{cris}}$ has a final object as in [**32**, II.1.4], one can prove:

$$H^0_{\mathrm{cris}}((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})/W_n) \simeq W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})^{\mathrm{DP}}$$

where "DP" means that we take the divided power envelope "compatible with the divided powers on (p) ([**4**, 3.19])" with respect to the kernel of the surjection θ_n : $W_n(\mathcal{O}_{\overline{K}}/p) \to \mathcal{O}_{\overline{K}}/p^n$ defined by $\theta_n(a_0, \ldots, a_{n-1}) = \widehat{a}_0^{p^n} + p\widehat{a}_1^{p^{n-1}} + \cdots + p^{n-1}\widehat{a}_{n-1}^p$ ($\widehat{a}_i$ = any lifting of a_i in $\mathcal{O}_{\overline{K}}/p^n$). So $A_{\mathrm{cris}} \simeq \varprojlim W_n(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}}$, the projective system being taken with respect to the maps $W_n(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}} \to W_{n-1}(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}}$ induced by $(a_0, \ldots, a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-2}^p)$. Fontaine shows ([**27**, 3.1]) that A_{cris} is p-torsion free and that the projection on $W_n(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}}$ induces an isomorphism $A_{\mathrm{cris}}/p^n A_{\mathrm{cris}} \simeq W_n(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}}$. Because there is a Frobenius ϕ on the Witt vectors and because $\phi(\mathrm{Ker}(\theta_n)) \subset \mathrm{Ker}(\theta_n) + p(\mathcal{O}_{\overline{K}}/p^n)$, the Frobenius extends to A_{cris}. Let

J_n^{cris} be the kernel of the surjection $W_n(\mathcal{O}_{\overline{K}}/p)^{\mathrm{DP}} \to \mathcal{O}_{\overline{K}}/p^n$ induced by θ_n, $J_n^{\mathrm{cris},[i]}$ its i^{th} divided power ([**4**, 3.24]) and $\mathrm{Fil}^i A_{\mathrm{cris}} = \varprojlim J_n^{\mathrm{cris},[i]}$, then $(\mathrm{Fil}^i A_{\mathrm{cris}})_{i\in\mathbf{N}}$ is a decreasing filtration on A_{cris} such that $\mathrm{Fil}^0 A_{\mathrm{cris}} = A_{\mathrm{cris}}$ and $\phi(\mathrm{Fil}^i A_{\mathrm{cris}}) \subset p^i A_{\mathrm{cris}}$ if $0 \leqslant i \leqslant p-1$ (look at the action of ϕ on J_n^{cris}). For $i \leqslant p-1$ let $\phi_i = \frac{\phi}{p^i}|_{\mathrm{Fil}^i A_{\mathrm{cris}}}$. Since $A_{\mathrm{cris}}/\mathrm{Fil}^i A_{\mathrm{cris}}$ has no p-torsion, $\mathrm{Fil}^i A_{\mathrm{cris}}/p^n \mathrm{Fil}^i A_{\mathrm{cris}}$ injects into $A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}$ and this defines a filtration on $A_{\mathrm{cris}}/p^n A_{\mathrm{cris}}$ to which we can extend ϕ_i for $0 \leqslant i \leqslant p-1$. Finally there is by functoriality a continuous action of G_{K_0} on A_{cris} that preserves the filtration and commutes with the Frobenius. For completeness, we recall that $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/\log([\varepsilon])] = A_{\mathrm{cris}}[1/p, 1/\log([\varepsilon])]$ where $\varepsilon = (\varepsilon_n)_n$ is a compatible system of primitive $p^{n^{\mathrm{th}}}$-roots of unity in $\mathcal{O}_{\overline{K}}$, $[\overline{\varepsilon}_n] \in W_n(\mathcal{O}_{\overline{K}}/p)$ the Teichmüller representative of the reduction modulo p of ε_n, and $[\varepsilon] = ([\overline{\varepsilon}_n])_n$ the corresponding element of A_{cris}.

3.1.3. Let $r \in \{0, \ldots, p-1\}$ and $M \in \underline{MF}^{f,r}_{\mathrm{tor}}$. Choose $n \in \mathbf{N}$ such that $p^n M = 0$ and define:

$$T^*_{\mathrm{cris}}(M) = \mathrm{Hom}_{W,\mathrm{Fil}^{\cdot},\phi_{\cdot}}(M, A_{\mathrm{cris}}/p^n A_{\mathrm{cris}})$$

where the subscript means we take the W-linear maps that send Fil^i to Fil^i and commute with ϕ_i. The $\mathbf{Z}_p$-module $T^*_{\mathrm{cris}}(M)$ is independent of the choice of n such that $p^n M = 0$ and is endowed with an action of G_{K_0} given by $g(f)(x) = g(f(x))$ if $x \in M, f \in T^*_{\mathrm{cris}}(M)$. We thus have a functor from $\underline{MF}^{f,r}_{\mathrm{tor}}$ to representations of G_{K_0}. The main result of the Fontaine-Laffaille theory is:

Theorem 3.1.3.1. — *For $0 \leqslant r \leqslant p-1$, the functor T^*_{cris} is exact and faithful. For $0 \leqslant r \leqslant p-2$, it is fully faithful.*

The proof reduces by dévissage to the case $pM = 0$ (M in $\underline{MF}^{f,r}_{\mathrm{tor}}$) and the fully faithfulness uses the classification of the simple objects of $\underline{MF}^{f,r}_{\mathrm{tor}}$. There is a nice variant in [**75**, 2] that avoids this classification. Actually, the full faithfulness extends to $r = p-1$ if one restricts to appropriate subcategories of $\underline{MF}^{f,p-1}_{\mathrm{tor}}$ ([**31**, 0.9]). As a corollary of (3.1.3.1), we get that for $0 \leqslant r \leqslant p-1$ the invariant factors of M and $T^*_{\mathrm{cris}}(M)$ coincide and in particular that $T^*_{\mathrm{cris}}(M)$ is a finite representation. The link to crystalline representations is provided by the following theorem, which is proved by a limit argument:

Theorem 3.1.3.2 ([**31**, 8.4]). — *Let M be a strongly divisible module of rank d such that $\mathrm{Fil}^p M = 0$, then $\mathrm{Hom}_{W,\mathrm{Fil}^{\cdot},\phi}(M, A_{\mathrm{cris}})$ is a $\mathbf{Z}_p$-lattice in a d-dimensionnal crystalline representation of G_{K_0} with Hodge-Tate weights between 0 and $p-1$.*

Using [**48**, 3.2] this even shows that we get like this all the crystalline representations of G_{K_0} with Hodge-Tate weights between 0 and $p-1$. Define a *torsion* crystalline representation of weight $\leqslant r$ ($r \in \mathbf{N}$) to be any finite representation of G_{K_0} that can be written T/T' where $T' \subset T$ are Galois stable lattices in a crystalline representation of G_{K_0} with Hodge-Tate weights $\in \{0, \ldots, r\}$. Using (3.1.3.1), (3.1.3.2) and [**48**, 3.2] together with the fact any object of $\underline{MF}_{\mathrm{tor}}^{f,r}$ can be lifted as a strongly divisible module (easy), we finally get:

Theorem 3.1.3.3. — *For $0 \leqslant r \leqslant p-2$, the functor T^*_{cris} induces an anti-equivalence of categories between $\underline{MF}_{\mathrm{tor}}^{f,r}$ and torsion crystalline representations of G_{K_0} of weight $\leqslant r$.*

Let us end this subsection with the description of the covariant version of T^*_{cris} which turns out to be more convenient for the application to geometry. Let M be in $\underline{MF}_{\mathrm{tor}}^{f,r}$ and for simplicity assume $0 \leqslant r \leqslant p-2$ (we will only need that case in the sequel). Define $\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_W M) = \sum_{i=0}^r \mathrm{Fil}^{r-i} A_{\mathrm{cris}} \otimes_W \mathrm{Fil}^i M$ and $\phi_r = \sum_{i=0}^r \phi_{r-i} \otimes \phi_i$.

Lemma 3.1.3.4. — *With the above hypothesis, there is a canonical isomorphism of G_{K_0}-modules: $\mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_W M)^{\phi_r=1} \xrightarrow{\sim} T^*_{\mathrm{cris}}(M)^\wedge(r)$ where the exponent "$\phi_r = 1$" on the left hand side means "kernel of $\phi_r - \mathrm{Id}$", where "(r)" denotes twisting by the r^{th} power of the cyclotomic character of G_{K_0} and where the exponent $\wedge$ on the right hand side means the Pontryagin dual with respect to $\mathbf{Q}_p/\mathbf{Z}_p$.*

For a proof, see for instance [**10**, 3.2.1.7]. In the sequel, we write:

$$T_{\mathrm{cris}}(M) = T^*_{\mathrm{cris}}(M)^\wedge \simeq \mathrm{Fil}^r(A_{\mathrm{cris}} \otimes_W M)^{\phi_r=1}(-r).$$

3.2. Review of the Fontaine-Messing results. — Let X be a proper smooth scheme over $\mathrm{Spec}(W)$. The Fontaine-Messing theory shows the functor T_{cris} above sends the torsion crystalline, or de Rham, cohomology of X to the torsion étale cohomology of $X \times_W \overline{K_0}$. We just give here a brief overview of the results of [**32**], since more details will be given in the sequel in the log-case.

Let $X_n = X \times_W W_n$ and $\sigma_{\geqslant j}\Omega^{\cdot}_{X_n} = 0 \to \cdots \to 0 \to \Omega^j_{X_n} \to \Omega^{j+1}_{X_n} \to \cdots$ the truncated classical de Rham complex. By Berthelot's comparison theorem ([**4**, 7.2]), $H^i(X_n, \sigma_{\geqslant j}\Omega^{\cdot}_{X_n}) \simeq H^i((X_n/W_n)_{\mathrm{cris}}, \mathcal{J}^{[j]}_{X_n/W_n})$ where $\mathcal{J}_{X_n/W_n} = \mathrm{Ker}(\mathcal{O}_{X_n/W_n} \to \mathcal{O}_{X_n})$ (here $\mathcal{O}_{X_n/W_n}$ is the structure sheaf on $(X_n/W_n)_{\mathrm{cris}}$ and $\mathcal{O}_{X_n}$ the classical structure sheaf on X_n, see [**4**, 5.2] for details). In particular, there is a Frobenius ϕ (the "crystalline Frobenius") on $H^i(X_n, \Omega^{\cdot}_{X_n})$. Working with the syntomic interpretation of the groups $H^i((X_n/W_n)_{\mathrm{cris}}, \mathcal{J}^{[j]}_{X_n/W_n})$ ([**32**, II.2.2]), it is also possible to define for $0 \leqslant j \leqslant p-1$ semi-linear maps $\phi_j = $ "ϕ/p^j" $: H^i(X_n, \sigma_{\geqslant j}\Omega^{\cdot}_{X_n}) \to H^i(X_n, \Omega^{\cdot}_{X_n})$ such that $\phi_0 = \phi$.

Theorem 3.2.1 ([**32**, II.2.7]). — *Let X be a proper smooth scheme over W. For $n \in \mathbf{N}$ and $0 \leqslant i \leqslant r \leqslant p-1$, the data:*

$$\left(H^i(X_n, \Omega^{\cdot}_{X_n}), (H^i(X_n, \sigma_{\geqslant j}\Omega^{\cdot}_{X_n}))_{0\leqslant j\leqslant r}, (\phi_j)_{0\leqslant j\leqslant r}\right)$$

define an object of the category $\underline{MF}_{\mathrm{tor}}^{f,r}$. That is to say the maps:

$$H^i(X_n, \sigma_{\geqslant j}\Omega^{\cdot}_{X_n}) \longrightarrow H^i(X_n, \Omega^{\cdot}_{X_n})$$

induced by the canonical injection of complexes are injective and $\sum_{j=0}^{r}\mathrm{Im}(\phi_j) = H^i(X_n, \Omega^{\cdot}_{X_n})$.

The main ingredient of the proof is an isomorphism which is now called the Deligne-Illusie isomorphism (because its construction was simplified and generalized in [**16**]).

Remark 3.2.2. — One can also define the Frobenius maps by purely de Rham considerations using local liftings of Frobenius (see [**45**, 1] or [**16**]).

Theorem 3.2.3 ([**32**, III.6.4]). — *Let* X *be a proper smooth scheme over* W. *For* $n \in \mathbf{N}$ *and* $0 \leqslant i \leqslant r \leqslant p-2$, *there are isomorphisms compatible with the action of* G_{K_0}:

$$T_{\mathrm{cris}}\Big(H^i(X_n,\Omega^{\cdot}_{X_n}),(H^i(X_n,\sigma_{\geqslant j}\Omega^{\cdot}_{X_n}))_{0\leqslant j\leqslant r},(\phi_j)_{0\leqslant j\leqslant r}\Big)\simeq H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z}).$$

It doesn't seem to be known in general whether (3.2.3) extends to $i = r = p-1$. As in the ℓ-torsion case, this theorem compares something living on the special fiber of X with something living on the geometric generic fiber. The strategy to prove (3.2.3) is *first* to define a third Galois representation called the "syntomic" cohomology (this uses the syntomic sheaves $\mathcal{S}_n^r$ of [**32**, III.3] and their cohomology), *secondly* to show this syntomic cohomology maps isomorphically and compatibly with Galois to $H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z})(r)$ (this relies heavily on computations of Bloch-Kato [**6**] and Kato [**45**] on the sheaves of nearby cycles), *thirdly* to show this syntomic cohomology also maps isomorphically and compatibly with Galois to $T_{\mathrm{cris}}(H^i(X_n,\Omega^{\cdot}_{X_n}))(r)$ (this uses (3.2.1) and properties of A_{cris} together with Künneth formulas, see [**32**, III.1-2]). The isomorphism (3.2.3) gives deep information about the action of G_{K_0} on $H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z})$ for $p-1>i$. For example, one can give an upper bound for the valuation of the different of the finite extension of K_0 cut out by this finite representation, or a lower bound for the index of the ramification subgroups of G_{K_0} (in the upper numbering) that act trivially on $H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z})$ (see 9.2.2). Also, one can deduce that the weights of the action of the tame inertia (of G_{K_0}) on the semi-simplification of the reduction modulo p of $H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z})$ are between 0 and i ([**31**, 5.3]).

Remark 3.2.4. — If one is only interested in the $\mathbf{Q}_p$-version of (3.2.3), there is a way to obtain it without using Kato's computations by first building a map from the syntomic cohomology to $H^i((X\times_W\overline{K_0})_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z})(r)$ using the so-called "syntomic-étale" site of $X\times_W\mathcal{O}_{\overline{K}}$ and then showing it is an isomorphism after taking the inverse limit and tensoring by $\mathbf{Q}_p$ using Poincaré duality. See [**32**, III.4-III.6]. In [**72**], Tsuji has sketched an extension of this approach using a "log-syntomic-étale" site.

4. Semi-stable reduction: why $W_n\langle u\rangle$ -modules?

The p-adic semi-stable representations are defined similarly to the crystalline representations by using B_{st} instead of B_{cris} (see [**24**], [**25**]; non canonically B_{st} may be defined as $B_{\text{cris}}[v]$, a polynomial algebra, on which the Galois action extends that on B_{cris} and acts on v via $g(v) = \log[\varepsilon(g)] + v$ where $\varepsilon : G_{K_0} \to \varprojlim \mu_{p^n}(\overline{K})$ is the 1-cocycle associated to a choice of a compatible system of $p^{n^{\text{th}}}$-roots of a uniformizer of K_0; this ring is closely connected to the ring $\widehat{A_{\text{st}}}$ of (5.2.1)). Our aim is to generalize to this situation the previous integral theories. So there are two tasks: the first is to find good categories of torsion objects of linear algebra that can be related to semi-stable representations, the second is to apply this theory to the cohomology of varieties with semi-stable reduction. We start with the first.

4.1. As in the crystalline case, one can associate to a semi-stable representation a weakly admissible filtered (ϕ, N)-module, that is to say a filtered ϕ-module D as in (3.1.1), but endowed with a K_0-linear endomorphism $N : D \to D$ (the "monodromy") satisfying $N\phi = p\phi N$ and such that the previous weakly admissibility conditions hold, except that one considers only those D' which are preserved both by ϕ and N in the second condition (see [**25**, 5]). Thanks to [**14**], one also has an equivalence of categories between weakly admissible filtered (ϕ, N)-modules and semi-stable representations (some cases here were known by work of the first author, see for instance (9.1.1.1)). An important point is that there are no direct relations between N and the filtration other than those coming from the weakly admissibility conditions.

If one wants to mimic the definition of (3.1), one is naturally led to introduce an operator N on the objects of $\underline{MF}_{\text{tor}}^{f,r}$ $(0 \leqslant r \leqslant p-2)$. The problem is that the only reasonnable translation of $N\phi = p\phi N$ is $N\phi_i = \phi_{i-1}N$ $(0 \leqslant i \leqslant r)$ since ϕ_i is morally ϕ/p^i. But such a relation would make sense only if $N(\text{Fil}^i) \subset \text{Fil}^{i-1}$ for all $i \in \{0, \ldots, r\}$: this is called the "Griffiths transversality condition" (because similar conditions were found by Griffiths for the Gauss-Manin connection and the logarithm of monodromy in the classical case). Nevertheless, one can add this transversality condition and consider objects M of $\underline{MF}_{\text{tor}}^{f,r}$ together with a W-linear endomorphism $N : M \to M$ such that $N(\text{Fil}^i M) \subset \text{Fil}^{i-1} M$ and $N\phi_i = \phi_{i-1}N$ for $0 \leqslant i \leqslant r$. One ends up again with an abelian category and it is essentially routine to extend the previous Fontaine-Laffaille theory to this new context (see [**54**, I.3]). The only serious point is that one has to replace A_{cris} by the integral version of B_{st} (with the notation of (5.2.1) below, this is $A_{\text{st}} = A_{\text{cris}}[\log(1 + X_\pi)]$). These modules are called "naive" in [**9**, 5] as they correspond to a naive extension of the Fontaine-Laffaille theory (this terminology is due to Fontaine).

Of course, this approach only gives a small part of the picture of semi-stable representations, since the condition $N(\text{Fil}^i) \subset \text{Fil}^{i-1}$ is not required in general on a weakly admissible module. It is sufficient in some cases, for example $\text{Fil}^2 = 0$. For instance, using such naive modules, one can build (up to twist) the "très ramifiées" local representations of Serre in [**65**, 2.4] (the "peu ramifiées" ones corresponding to $N = 0$ i.e. classical objects of $\underline{MF}_{\text{tor}}^{f,1}$).

4.2. In [**44**, 3], Kato defines a W-algebra $\varprojlim P_n$, which is called $\widehat{A_{\mathrm{st}}}$ in [**9**], by mimicing in the logarithmic context the crystalline construction of B_{cris} of [**27**] and [**32**] (see 5.2.1). This algebra naturally lives over the p-adic completion S of $W\langle u\rangle = \{\sum_{i=0}^{n} w_i u^i/i! \mid w_i \in W, n \in \mathbf{N}\}$ where u is an indeterminate whose image in $\widehat{A_{\mathrm{st}}}$ depends on the choice of an uniformizer in W (other authors have used t or T). More importantly, $\widehat{A_{\mathrm{st}}}$ is endowed with a filtration, a Frobenius, a monodromy operator and the above Griffiths transversality *is satisfied*, whereas it is certainly not the case on A_{st} (viewed in $B_{\mathrm{st}} \subset B_{\mathrm{dR}}$, cf. [**8**, 7]). This suggests working with S-modules instead of W-modules, and imposing on these the Griffiths transversality condition.

Choose an uniformizer π of W and define on S a filtration by $\mathrm{Fil}^i S = p$-adic completion of the ideal generated by $\{(u-\pi)^j/j!,\ j \geqslant i\}$, a (lifting of) Frobenius by $\phi(\sum w_i\, u^i/i!) = \sum \phi(w_i) u^{pi}/i!$ (here $\phi(w_i)$ is the classical Frobenius on the Witt vectors) and a W-linear derivation N by $N(\sum w_i\, u^i/i!) = \sum (-1)^i i w_i\, u^i/i!$ (i.e. $N(u) = -u$: the reason for the minus sign is explained in (6.2.3.3)). Let $S_{K_0} = K_0 \otimes_W S$ and extend in the obvious way these structures to S_{K_0}. Let $\underline{MF}^+_{K_0}(\phi, N)$ be the category of filtered (ϕ, N)-modules D of (4.1) such that $\mathrm{Fil}^0 D = D$ (morphisms being the K_0-linear maps that preserve the filtration and commute with the operators). Let $\underline{\mathcal{MF}}^+_{K_0}(\phi, N)$ be the category of finitely generated free S_{K_0}-modules $\mathcal{D}$ equipped with:
(i) a decreasing filtration by sub-S_{K_0}-modules $\mathrm{Fil}^i \mathcal{D}$ such that

$$\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}, \quad \mathrm{Fil}^j S_{K_0} \mathrm{Fil}^i \mathcal{D} \subset \mathrm{Fil}^{j+i} \mathcal{D} \quad \text{and}$$
$$\mathrm{Fil}^i \mathcal{D} = \mathrm{Fil}^1 S_{K_0} \mathrm{Fil}^{i-1} \mathcal{D} + \mathrm{Fil}^i S_{K_0} \mathcal{D} \text{ if } i \gg 0$$

(ii) an S_{K_0}-semi-linear map $\phi : \mathcal{D} \to \mathcal{D}$ such that $\det(\phi) \in S^*_{K_0}$ (in one, or equivalently any, basis of $\mathcal{D}$ over S_{K_0})
(iii) a K_0-linear map $N : \mathcal{D} \to \mathcal{D}$ such that $N(sx) = N(s)x + sN(x)$ ($s \in S_{K_0}, x \in \mathcal{D}$), $N\phi = p\phi N$ and $N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$.
The morphisms in this category are the S_{K_0}-linear maps compatible with the structures. Let:

$$\begin{aligned} f_\pi : S_{K_0} &\longrightarrow K_0 \\ \textstyle\sum w_i\, u^i/i! &\longmapsto \textstyle\sum w_i\, \pi^i/i!. \end{aligned}$$

We define a functor $\underline{MF}^+_{K_0}(\phi, N) \to \underline{\mathcal{MF}}^+_{K_0}(\phi, N)$ as follows: to D, we associate $\mathcal{D} = S_{K_0} \otimes_{K_0} D$ with $\phi = \phi \otimes \phi$, $N = N \otimes \mathrm{Id} + \mathrm{Id} \otimes N$ and $\mathrm{Fil}^i \mathcal{D}$ defined inductively by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and $\mathrm{Fil}^i \mathcal{D} = \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^{i-1} \mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^i D\}$.

Theorem 4.2.1 ([**8**, 6]). — *The above functor induces an equivalence of categories between* $\underline{MF}^+_{K_0}(\phi, N)$ *and* $\underline{\mathcal{MF}}^+_{K_0}(\phi, N)$.

The proof uses an argument of iteration of Frobenius which goes back to Berthelot-Ogus ([**5**]) (and which was actually rediscovered independently). The last condition in (i) above corresponds to the fact that the filtrations on objects in $\underline{MF}^+_{K_0}(\phi, N)$ are separated. Because of this theorem, we can try to look for integral structures inside the $\mathcal{D}$'s instead of inside the D's. The fact one could try to work with S-modules instead of W-modules had also been noticed independently by Faltings ([**20**]), Tsuzuki ([**73**]), and Quiros (in a related context, see [**60**]).

4.3. First, notice that for $0 \leqslant i \leqslant p-1$, $\phi(\mathrm{Fil}^i S) \subset p^i S$, and define $\phi_i = \frac{\phi}{p^i}|_{\mathrm{Fil}^i S}$. Since $\mathrm{Fil}^i S \cap pS = p\,\mathrm{Fil}^i S$, there is a filtration on $S/p^n S$ defined by $\mathrm{Fil}^i(S/p^n S) = \mathrm{Fil}^i S/p^n \mathrm{Fil}^i S$ and we can extend ϕ_i to $\mathrm{Fil}^i(S/p^n S)$ for $0 \leqslant i \leqslant p-1$. Also $\phi_1(u-\pi) = (u^p - \phi(\pi))/p \in S^*$. Starting from a filtered (ϕ, N)-module D, we have now another module where the Griffiths transversality is satisfied. Thinking about the naive case of (4.1), it is then natural to look for torsion S-modules $\mathcal{M}$ which are isomorphic to, say, $\oplus_{i\in I}(S/p^i S)^{d_i}$ where I is a finite set of integers and $d_i \in \mathbf{N}$, and which are endowed with:

(i) a filtration $\mathrm{Fil}^i \mathcal{M}$ such that

$$\mathrm{Fil}^0 \mathcal{M} = \mathcal{M}, \quad \mathrm{Fil}^j S\,\mathrm{Fil}^i \mathcal{M} \subset \mathrm{Fil}^{j+i} \mathcal{M},$$
$$\mathrm{Fil}^i \mathcal{M} = \mathrm{Fil}^1 S\,\mathrm{Fil}^{i-1} \mathcal{M} + \mathrm{Fil}^i S\mathcal{M} \text{ if } i \geqslant r+1 \text{ (for, say, an } r \in \{0,\dots,p-2\})$$

(ii) for $0 \leqslant i \leqslant r$ maps $\phi_i : \mathrm{Fil}^i \mathcal{M} \to \mathcal{M}$ such that

$$\phi_{j+i}(sx) = \phi_j(s)\phi_i(x) \ (s \in \mathrm{Fil}^j S, x \in \mathrm{Fil}^i \mathcal{M}), \quad \phi_i|_{\mathrm{Fil}^{i+1} S} = p\phi_{i+1} \quad \text{and}$$
$$\textstyle\sum_{i\geqslant 0} \phi_i(\mathrm{Fil}^i \mathcal{M}) \text{ generates } \mathcal{M} \text{ over } S$$

(iii) a W-linear map $N : \mathcal{M} \to \mathcal{M}$ such that

$$N(sx) = N(s)x + sN(x) \ (s \in S, x \in \mathcal{M}),\ N\phi_i = \phi_{i-1}N \text{ and } N(\mathrm{Fil}^i \mathcal{M}) \subset \mathrm{Fil}^{i-1} \mathcal{M}.$$

Moreover, thinking about the objects of $\underline{MF}^{f,r}_{\mathrm{tor}}$ (3.1.1), it is tempting at first to consider only those modules which are "filtered free" in the following sense: we assume we can write:

$$\mathcal{M} = \oplus_{j=1}^{d} S/p^{n_j} S e_j \text{ and } \mathrm{Fil}^i \mathcal{M} = \oplus_{j\geqslant d_i}^{d} S/p^{n_j} S e_j + \mathrm{Fil}^1 S\,\mathrm{Fil}^{i-1} \mathcal{M} + \mathrm{Fil}^i S\mathcal{M}$$

for some integers $1 = d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{r+1} = d+1$. Remember that our aim is to define an abelian (or even artinian) category of such objects. As in the Fontaine-Laffaille case, this is reasonnable only if the morphisms in this hypothetical category are *strict* with respect to the filtration, i.e. if $f(\mathrm{Fil}^i \mathcal{M}) = \mathrm{Fil}^i \mathcal{N} \cap f(\mathcal{M})$ for any morphism $f : \mathcal{M} \to \mathcal{N}$ and any i. But consider the following example:

Example 4.3.1. — Consider the filtered free S/pS-modules $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ defined by: $\mathcal{M} = S/pSe_1 \oplus S/pSe_2$ with

$$\mathrm{Fil}^1 \mathcal{M} = S/pS(e_1 + ue_2) + \mathrm{Fil}^1(S/pS)\mathcal{M},$$
$$\mathrm{Fil}^2 \mathcal{M} = S/pS(e_1 + ue_2) + \mathrm{Fil}^1(S/pS)\,\mathrm{Fil}^1 \mathcal{M} + \mathrm{Fil}^2(S/pS)\mathcal{M},$$
$$\mathrm{Fil}^i \mathcal{M} = \mathrm{Fil}^1(S/pS)\,\mathrm{Fil}^{i-1} \mathcal{M} + \mathrm{Fil}^i(S/pS)\mathcal{M} \text{ if } i \geqslant 3,$$

$\phi_0(e_2) = e_1$, $\phi_2(e_1 + ue_2) = e_2$, $N(e_2) = -\phi_1(u)e_1$, $N(e_1) = 0$.

$\mathcal{M}' = S/pSe_1$ with

$$\begin{aligned}&\operatorname{Fil}^1 \mathcal{M}' = \mathcal{M}',\\ &\operatorname{Fil}^i \mathcal{M}' = \operatorname{Fil}^1(S/pS) \operatorname{Fil}^{i-1} \mathcal{M}' + \operatorname{Fil}^i(S/pS)\mathcal{M}' \text{ if } i \geqslant 2,\end{aligned}$$

$\phi_1(e_1) = -\phi_1(u)e_1$, $N(e_1) = 0$.

$\mathcal{M}'' = S/pSe_2$ with

$$\begin{aligned}&\operatorname{Fil}^1 \mathcal{M}'' = \mathcal{M}'',\\ &\operatorname{Fil}^i \mathcal{M}'' = \operatorname{Fil}^1(S/pS) \operatorname{Fil}^{i-1} \mathcal{M}'' + \operatorname{Fil}^i(S/pS)\mathcal{M}'' \text{ if } i \geqslant 2,\end{aligned}$$

$\phi_1(e_2) = \phi_1(u)^{-1}e_2$, $N(e_2) = 0$.

One checks there are morphisms compatible with all the structures $\mathcal{M}' \to \mathcal{M}, e_1 \mapsto e_1$ and $\mathcal{M} \to \mathcal{M}'', e_1 \mapsto 0, e_2 \mapsto e_2$ and that the sequences of S-modules $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ and $0 \to \operatorname{Fil}^i \mathcal{M}' \to \operatorname{Fil}^i \mathcal{M} \to \operatorname{Fil}^i \mathcal{M}'' \to 0$ for $i \geqslant 2$ are exact. However, the sequence $0 \to \operatorname{Fil}^1 \mathcal{M}' \to \operatorname{Fil}^1 \mathcal{M} \to \operatorname{Fil}^1 \mathcal{M}'' \to 0$ is *not* exact since $e_2 \in \operatorname{Fil}^1 \mathcal{M}''$ cannot be lifted in $\operatorname{Fil}^1 \mathcal{M}$. In particular the morphism $\mathcal{M} \to \mathcal{M}''$ is *not* strict.

The above example suggests one should give up the full data of a filtration and keep only the "last step" $\operatorname{Fil}^r \mathcal{M}$ (Fil^2 above) in order to get (hopefully) strict morphisms. Moreover, using the fact $\phi_1(u - \pi)$ is a unit, it is possible to give analogues of all the above conditions on an object $\mathcal{M}$ in terms of $\operatorname{Fil}^r \mathcal{M}$ only: for instance $\sum_{i\geqslant 0} \phi_i(\operatorname{Fil}^i \mathcal{M})$ generates $\mathcal{M}$ over S if and only if $\phi_r(\operatorname{Fil}^r \mathcal{M})$ does. Hence working with "just" Fil^r and ϕ_r may not be a bad idea.

We explain in the next section that this idea indeed works, and yields nice artinian categories of torsion S-modules.

5. A generalization of the Fontaine-Laffaille theory

Because the maps ϕ_i on S are only defined for $0 \leqslant i \leqslant p-1$, we have to make at once a restriction on the length of the filtration, contrary to what we did in (3.1.1) with the Fontaine-Laffaille objects. This is not very important since, anyway, these Fontaine-Laffaille objects could only be used in (3.1.3) under this restriction (and are apparently *not* the right objects when the filtration goes further). Moreover, the theory in [**9**] has only been worked out when this length is actually strictly smaller than $p-1$. So in the sequel, we only look at modules with a "filtration" between 0 and r for a fixed integer r between 0 and $p-2$ (although the theory probably extends to $r = p-1$ if one restricts to appropriate subcategories, as in [**31**]). We let $c = \phi_1(u - \pi) \in S^*$.

5.1. Definition of the categories

5.1.1. Recall we have fixed an uniformizer π of W. Define $\underline{\mathcal{M}}_\pi^r$ to be the following category. An object is the data of:
(i) an S-module $\mathcal{M}$ abstractly isomorphic to $\oplus_{i\in I}(S/p^iS)^{d_i}$ where I is a finite set of integers and $d_i \in \mathbf{N}$
(ii) a sub-S-module $\mathrm{Fil}^r\mathcal{M}$ containing $\mathrm{Fil}^r S\cdot\mathcal{M}$
(iii) a map $\phi_r : \mathrm{Fil}^r\mathcal{M}\to\mathcal{M}$ semi-linear with respect to the Frobenius on S and such that $c^r\phi_r(sx) = \phi_r(s)\phi_r((u-\pi)^r x)$ ($s\in\mathrm{Fil}^r S$, $x\in\mathcal{M}$) and $\phi_r(\mathrm{Fil}^r\mathcal{M})$ generates $\mathcal{M}$ over S
(iv) a map $N:\mathcal{M}\to\mathcal{M}$ such that

$$N(sx) = N(s)x + sN(x)\ (s\in S,\ x\in\mathcal{M}),\quad (u-\pi)N(\mathrm{Fil}^r\mathcal{M})\subset\mathrm{Fil}^r\mathcal{M}\quad\text{and}\quad cN\circ\phi_r = \phi_r\circ(u-\pi)N|_{\mathrm{Fil}^r\mathcal{M}}$$

and morphisms are the S-linear maps that send Fil^r to Fil^r and commute with ϕ_r and N. If $r+1\leqslant p-2$, there is a fully faithful functor $\underline{\mathcal{M}}_\pi^r\to\underline{\mathcal{M}}_\pi^{r+1}$ ([**9**, 2.1.2.1]).

Theorem 5.1.1.1. — *Let $f:\mathcal{M}\to\mathcal{N}$ be a morphism in $\underline{\mathcal{M}}_\pi^r$. Then:*
1) $f(\mathrm{Fil}^r\mathcal{M}) = \mathrm{Fil}^r\mathcal{N}\cap f(\mathcal{M})$
2) if $\mathcal{M}'$ is the kernel of the underlying S-linear map, $\mathrm{Fil}^r\mathcal{M}' = \mathrm{Fil}^r\mathcal{M}\cap\mathcal{M}'$, ϕ_r : $\mathrm{Fil}^r\mathcal{M}'\to\mathcal{M}'$ *the restriction of* $\phi_r:\mathrm{Fil}^r\mathcal{M}\to\mathcal{M}$ *and* $N:\mathcal{M}'\to\mathcal{M}'$ *the restriction of* $N:\mathcal{M}\to\mathcal{M}$, *we have* $\mathcal{M}'\simeq\oplus_{i\in I'}(S/p^iS)^{d_i'}$ *and* $\phi_r(\mathrm{Fil}^r\mathcal{M}')$ *generates* $\mathcal{M}'$ *over* S
3) $\mathcal{N}/f(\mathcal{M})\simeq\oplus_{i\in I''}(S/p^iS)^{d_i''}$.

The proof is by a dévissage that reduces to the case where $\mathcal{M}$, $\mathcal{N}$ are killed by p and then uses (5.1.2.1) below. See [**9**, 2.1.2.2] for details.

Corollary 5.1.1.2. — *The category $\underline{\mathcal{M}}_\pi^r$ is abelian. More precisely, if f is as in (5.1.1.1), we have:*

$$\mathrm{Ker}(f) = (\mathcal{M}',\mathrm{Fil}^r\mathcal{M}',\phi_r,N)$$
$$\mathrm{Coker}(f) = (\mathcal{N}/f(\mathcal{M}),\mathrm{Fil}^r\mathcal{N}/f(\mathrm{Fil}^r\mathcal{M}),\phi_r,N).$$

Since all the modules are of the form $\oplus_{i\in I}(S/p^iS)^{d_i}$, $\underline{\mathcal{M}}_\pi^r$ is artinian. There is a natural functor $\mathcal{F}_\pi^r:\underline{MF}_{\mathrm{tor}}^{f,r}\to\underline{\mathcal{M}}_\pi^r$ that associates to M the object $\mathcal{F}_\pi^r(M) = S\otimes_W M$ with $\mathrm{Fil}^r\mathcal{F}^r(M) = \sum_{j=0}^r\mathrm{Fil}^{r-j}S\otimes_W\mathrm{Fil}^j M$, $\phi_r = \sum_{j=0}^r\phi_{r-j}\otimes\phi_j$ and $N = N\otimes\mathrm{Id}$.

Proposition 5.1.1.3 ([**9**, 2.4]). — *The functor $\mathcal{F}_\pi^r$ is exact and fully faithful. Via $\mathcal{F}_\pi^r$, the categories $\underline{MF}_{\mathrm{tor}}^{f,r}$ and $\underline{\mathcal{M}}_\pi^r$ have the same simple objects.*

The same statement is true if one replaces $\underline{MF}_{\mathrm{tor}}^{f,r}$ by the "naive" corresponding objects (i.e. adding a N on the objects of $\underline{MF}_{\mathrm{tor}}^{f,r}$, see (4.1)). In fact, for $r\leqslant 1$, there is even an equivalence of categories between $\underline{\mathcal{M}}_\pi^r$ and these naive objects ([**10**, 4.4.1]).

As in the classical case, we can define the "without p-torsion" version of $\underline{\mathcal{M}}_\pi^r$:

Definition 5.1.1.4. — A strongly divisible module of weight $\leqslant r$ is a free S-module $\mathcal{M}$ of finite type equipped with a sub-S-module $\mathrm{Fil}^r \mathcal{M}$ containing $\mathrm{Fil}^r S \cdot \mathcal{M}$ and such that $\mathcal{M}/\mathrm{Fil}^r \mathcal{M}$ has no p-torsion, a semi-linear map $\phi : \mathcal{M} \to \mathcal{M}$ such that $\phi(\mathrm{Fil}^r \mathcal{M}) \subset p^r \mathcal{M}$ and $\frac{\phi}{p^r}(\mathrm{Fil}^r \mathcal{M})$ generates $\mathcal{M}$ over S, and a map $N : \mathcal{M} \to \mathcal{M}$ such that $N(sx) = N(s)x + sN(x)$ $(s \in S, x \in \mathcal{M})$, $N\phi = p\phi N$ and $(u - \pi)N(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r \mathcal{M}$.

If $\mathcal{M}$ is a strongly divisible module of weight $\leqslant r$, $\mathcal{M}/p^n\mathcal{M}$ is in an obvious way an object of $\underline{\mathcal{M}}_\pi^r$ by defining $\phi_r = \frac{\phi}{p^r}|_{\mathrm{Fil}^r} \bmod p^n$ and $\mathcal{M}$ is also of weight $\leqslant r+1$ (if $r + 1 < p - 1$). Finally, we claim the categories $\underline{\mathcal{M}}_\pi^r$ (and the categories of strongly divisible modules) do not depend on the choice of π:

Proposition 5.1.1.5. — *For each choice of $w \in W^*$, there is a canonical equivalence of categories $\underline{\mathcal{M}}_\pi^r \xrightarrow{\approx} \underline{\mathcal{M}}_{\pi w}^r$ such that the composite $\underline{MF}_{\mathrm{tor}}^{f,r} \xrightarrow{\mathcal{F}_\pi^r} \underline{\mathcal{M}}_\pi^r \xrightarrow{\approx} \underline{\mathcal{M}}_{\pi w}^r$ is $\mathcal{F}_{\pi w}^r$.*

Proof. — We give a proof, since it's not in the literature. If $r = 0$, this is trivially true since in that case $\mathcal{F}_\pi^0$ above is actually an equivalence of categories and $\underline{MF}_{\mathrm{tor}}^{f,0}$ doesn't depend on any choice. So assume $1 \leqslant r \leqslant p-2$ and let $\pi' = \pi w$ with $w \in W^*$. Assume first that $w = [\kappa]$ for a κ in k^* (Teichmüller representative). Then the map $[\kappa^{-1}] : S \to S, \gamma_i(u) \mapsto \gamma_i(u[\kappa^{-1}])$ commutes with ϕ. To any object $\mathcal{M}_\pi \in \underline{\mathcal{M}}_\pi^r$ we associate $\mathcal{M}_{\pi'} = \mathcal{M}_{\pi[\kappa]} = (S \otimes_{[\kappa^{-1}],S} \mathcal{M}_\pi, S \otimes_{[\kappa^{-1}],S} \mathrm{Fil}^r \mathcal{M}_\pi, \phi \otimes \phi_r, N \otimes \mathrm{Id} + \mathrm{Id} \otimes N)$: this clearly defines an equivalence of categories $\underline{\mathcal{M}}_\pi^r \xrightarrow{\sim} \underline{\mathcal{M}}_{\pi'}^r$. In general, write $w = [\kappa]\omega$ with $\omega \in 1 + pW$ and define:

$$\nu : \mathcal{M}_{\pi[\kappa]} \longrightarrow \mathcal{M}_{\pi[\kappa]}, \quad x \longmapsto \exp(N(\log(\omega^{-1})))(x) = \sum_{i \geqslant 0} \frac{(-\log \omega)^i}{i!} N^i(x)$$

which makes sense since $p \geqslant 3$. To $\mathcal{M}_{\pi[\kappa]}$ we associate:

$$\mathcal{M}_{\pi'} = \Big(\mathcal{M}_{\pi[\kappa]}, \nu(\mathrm{Fil}^r \mathcal{M}_{\pi[\kappa]}), \exp(N(\frac{\log(\phi(\omega^{-1}))}{p})) \circ \phi_r \circ \nu^{-1}, N\Big)$$

which also makes sense since $N^i \circ \phi_r(x) \in p^{i-r}\mathcal{M}_{\pi[\kappa]}$ if $i \geqslant r$. One checks the functor $\mathcal{M}_\pi \mapsto \mathcal{M}_{\pi'}$ satisfies the required properties. □

5.1.2. We describe here in more detail the case when $\mathcal{M}$ is killed by p which turns out to be simpler. Denote by $\underline{\mathcal{M}}_{k,\pi}^r$ the full subcategory of $\underline{\mathcal{M}}_\pi^r$ of objects killed by p. For $\mathcal{M} \in \underline{\mathcal{M}}_{k,\pi}^r$, let $\mathrm{Fil}^{r+1} \mathcal{M} = u \, \mathrm{Fil}^r \mathcal{M} + \mathrm{Fil}^p(S/pS)\mathcal{M}$.

Lemma 5.1.2.1. — *Let $\mathcal{M}$ in $\underline{\mathcal{M}}_{k,\pi}^r$.*
1) $\mathrm{Id} \otimes \phi_r$ induces an isomorphism $S/pS \otimes_{(\phi),k} \mathrm{Fil}^r \mathcal{M}/\mathrm{Fil}^{r+1} \mathcal{M} \xrightarrow{\sim} \mathcal{M}$
2) the natural map $S/pS \otimes_k \phi_r(\mathrm{Fil}^r \mathcal{M}) \to \mathcal{M}$ is an isomorphism.

See [**9**, 2.2.2.2] for the proof of this easy lemma. The isomorphism 1) above is called the Faltings Isomorphism Condition in [**9**] because a variant was already considered in [**20**] (but with different categories). Using 1), it is not difficult to show the category

$\underline{\mathcal{M}}^r_{k,\pi}$ is abelian. Another advantage of $\underline{\mathcal{M}}^r_{k,\pi}$ is that it can be described without divided powers:

Let $S_1 = S/pS$, $\widetilde{S}_1 = k[u]/u^p$ and $s : S_1 \to \widetilde{S}_1$ the surjection that sends u^i to u^i and $\gamma_i(u)$ to 0 if $i \geqslant p$. Define $\mathrm{Fil}^i \widetilde{S}_1 = s(\mathrm{Fil}^i S_1) = u^i \widetilde{S}_1$ and $\widetilde{\phi}_i$, $\widetilde{N}$ to be the image of ϕ_i, N. Let $\widetilde{\underline{\mathcal{M}}}^r_{k,\pi}$ the category of finitely generated free $\widetilde{S}_1$-modules $\widetilde{\mathcal{M}}$ endowed with a sub-$\widetilde{S}_1$-module $\mathrm{Fil}^r \widetilde{\mathcal{M}}$ containing $u^r\widetilde{\mathcal{M}}$, a semi-linear map $\widetilde{\phi}_r : \mathrm{Fil}^r \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ such that $\widetilde{\phi}_r(\mathrm{Fil}^r \widetilde{\mathcal{M}})$ generates $\widetilde{\mathcal{M}}$ and an additive map $\widetilde{N} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ such that $\widetilde{N}(sx) = \widetilde{N}(s)x + s\widetilde{N}(x)$, $u\widetilde{N}(\mathrm{Fil}^r \widetilde{\mathcal{M}}) \subset \mathrm{Fil}^r \widetilde{\mathcal{M}}$ and $s(c)\widetilde{N} \circ \widetilde{\phi}_r = \widetilde{\phi}_r \circ u\widetilde{N}|_{\mathrm{Fil}^r}$. To $\mathcal{M}$ in $\underline{\mathcal{M}}^r_{k,\pi}$ we associate $\widetilde{\mathcal{M}} = (\widetilde{S}_1 \otimes_{s,S_1} \mathcal{M}, \widetilde{S}_1 \otimes_{s,S_1} \mathrm{Fil}^r \mathcal{M}, \widetilde{\phi} \otimes \phi_r, \widetilde{N} \otimes \mathrm{Id} + \mathrm{Id} \otimes N)$ (it is easily checked that everything is well defined). This construction is functorial and we have:

Proposition 5.1.2.2 ([**9**, 2.2.2]). — *The functor* $\underline{\mathcal{M}}^r_{k,\pi} \to \widetilde{\underline{\mathcal{M}}}^r_{k,\pi}$ *that sends* $\mathcal{M}$ *to* $\widetilde{\mathcal{M}}$ *is an equivalence of categories.*

5.2. Definition of T^*_{st} and T_{st}

5.2.1. We now introduce Kato's ring $\widehat{A_{\mathrm{st},\pi}}$ which was first defined in [**44**, 3] (see also [**8**, 2], [**9**, 3.1.1], [**70**, 1.6] for more details). Let $(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{\log}$ (resp. $(\mathcal{O}_L/p\mathcal{O}_L)_{\log}$ for any finite extension L of K_0) the log-version of $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ (resp. $\mathcal{O}_L/p\mathcal{O}_L$), that is to say the log-scheme associated to $\mathcal{O}_{\overline{K}} \smallsetminus \{0\} \to \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ (resp. $\mathcal{O}_L \smallsetminus \{0\} \to \mathcal{O}_L/p\mathcal{O}_L$), and $(S/p^nS)_{\log}$ the log-scheme associated to $(\mathbf{N} \to S/p^nS, 1 \mapsto u)$ (it will be denoted E_n in (6.2.2), following Kato's original notation). We define a morphism of log-schemes $(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{\log} \to (S/p^nS)_{\log}$ by sending u to the image of π. Although the log-structure of $(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{\log}$ is integral, but not fine, we can still define its log-crystalline site relative to $(S/p^nS)_{\log}$ and a conceptual definition of $\widehat{A_{\mathrm{st},\pi}}$ is:

$$\begin{aligned}\widehat{A_{\mathrm{st},\pi}} &= \varprojlim H^0_{\mathrm{cris}}((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})_{\log}/(S/p^nS)_{\log}) \\ &\simeq \varprojlim \Big(\varinjlim H^0_{\mathrm{cris}}((\mathcal{O}_L/p\mathcal{O}_L)_{\log}/(S/p^nS)_{\log}) \Big)\end{aligned}$$

where the inverse limit is over n, the direct limit over all the finite extensions L of K_0 in $\overline{K}$, and where H^0_{cris} is the global sections of log-crystalline cohomology (see [**38**, 2.14] and [**44**, 2.4]; a technical argument shows that the individual H^0_{cris} terms in the first line are canonically isomorphic to the $\varinjlim H^0_{\mathrm{cris}}$ terms in the second). The origin of this definition is essentially the Künneth formula (see for instance 8.2.1). The ring $\widehat{A_{\mathrm{st},\pi}}$ is an S-algebra and from the above definition, it can be endowed with a filtration, a Frobenius, a monodromy operator and a Galois action. We now make them more explicit. Either by a de Rham computation (see for instance [**70**, 1.6.5]) or by noticing the log-crystalline site of $(\mathcal{O}_{\overline{K}}/p)_{\log}$ over the base $(S/p^n)_{\log}$ has a final object (see [**44**, prop.3.3] or [**7**, 5.1.1]), one can define a non canonical isomorphism of S-algebras:

$$H^0_{\mathrm{cris}}((\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}_0})_{\log}/(S/p^nS)_{\log}) \simeq W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})^{\mathrm{DP}}\langle X_\pi\rangle$$
$$= \left\{\sum_{i=0}^{n} w_i \frac{X_\pi^i}{i!} \mid w_i \in W_n(\mathcal{O}_{\overline{K}_0}/p)^{\mathrm{DP}}, n \in \mathbf{N}\right\}$$

where $W_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})^{\mathrm{DP}}$ is as in (3.1.2), X_π is an indeterminate related to the choice of a $p^{n^{\mathrm{th}}}$-root π_n of π in $\mathcal{O}_{\overline{K}}$ and $u = [\overline{\pi}_n](1+X_\pi)^{-1}$ (compare with (6.2.2.3, 2)). Choosing a compatible system of such π_n (i.e. $\pi_n^p = \pi_{n-1}$) and denoting by $[\underline{\pi}]$ the corresponding "Teichmüller" element in A_{cris} (see 3.1.2), one can thus identify $\widehat{A_{\mathrm{st},\pi}}$ with the p-adic completion of $A_{\mathrm{cris}}\langle X_\pi\rangle$ and u with $[\underline{\pi}](1+X_\pi)^{-1}$. The Frobenius ϕ on $\widehat{A_{\mathrm{st},\pi}}$ extends that on A_{cris}, is continuous, commutes with divided powers and is such that $\phi(X_\pi) = (1+X_\pi)^p - 1$. The filtration is:

$$\mathrm{Fil}^i \widehat{A_{\mathrm{st},\pi}} = \left\{\sum_{j=0}^{\infty} a_j \frac{X_\pi^j}{j!} \mid a_j \in \mathrm{Fil}^{i-j} A_{\mathrm{cris}}, a_j \to 0\right\}.$$

The monodromy operator is the continuous A_{cris}-derivation N determined by $N(X_\pi) = 1 + X_\pi$. The Galois action is continuous, extends the action on A_{cris}, commutes with divided powers and is such that $g(X_\pi) = [\varepsilon(g)]X_\pi + [\varepsilon(g)] - 1$, where $\varepsilon : G_{K_0} \to \varprojlim \mu_{p^n}(\overline{K})$ is the (continuous) 1-cocycle determined by our choice of a compatible system of $p^{n^{\mathrm{th}}}$ roots of π. Note the divided powers on $A_{\mathrm{cris}}\langle X_\pi\rangle$ are automatically compatible with those on $\mathrm{Fil}^1 A_{\mathrm{cris}}$ and $[\varepsilon(g)] - 1$ belongs to this ideal. This action preserves the filtration and commutes with ϕ and N. As for A_{cris}, one has $\phi(\mathrm{Fil}^i \widehat{A_{\mathrm{st},\pi}}) \subset p^i \widehat{A_{\mathrm{st},\pi}}$ if $0 \leqslant i \leqslant p-1$ and one defines $\phi_i = \frac{\phi}{p^i}|_{\mathrm{Fil}^i}$ for such i. All these structures extend obviously to $\widehat{A_{\mathrm{st},\pi}}/p^n\widehat{A_{\mathrm{st},\pi}}$ endowed with the filtration $\mathrm{Fil}^i \widehat{A_{\mathrm{st},\pi}}/p^n \mathrm{Fil}^i \widehat{A_{\mathrm{st},\pi}} \hookrightarrow \widehat{A_{\mathrm{st},\pi}}/p^n\widehat{A_{\mathrm{st},\pi}}$.

5.2.2. Let $\mathcal{M} \in \underline{\mathcal{M}}^r_\pi$. Choose $n \in \mathbf{N}$ such that $p^n\mathcal{M} = 0$ and define:

$$T^*_{\mathrm{st},\pi}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \widehat{A_{\mathrm{st},\pi}}/p^n\widehat{A_{\mathrm{st},\pi}})$$

where the subscript means we take the S-linear maps that send Fil^r to Fil^r and commute with ϕ_r and N. The $\mathbf{Z}_p$-module $T^*_{\mathrm{st},\pi}(M)$ is independent of the choice of n such that $p^n\mathcal{M} = 0$ and of the choice of r such that $\mathcal{M} \in \underline{\mathcal{M}}^r_\pi$ (see 5.1.1). It is endowed with a action of G_{K_0} given by $g(f)(x) = g(f(x))$ if $x \in \mathcal{M}, f \in T^*_{\mathrm{st},\pi}(\mathcal{M})$. We thus have a functor from $\underline{\mathcal{M}}^r_\pi$ to representations of G_{K_0}.

Theorem 5.2.2.1 ([**9**, 3.2-3.3]). — *For $0 \leqslant r \leqslant p-2$, the functor $T^*_{\mathrm{st},\pi}$ is exact and fully faithful.*

By dévissage, one is reduced to checking this for $\underline{\mathcal{M}}^r_{k,\pi}$ (5.1.2). The exactness and faithfulness are proved by the same techniques as for (3.1.3.1) using (5.1.1.3). The full faithfulness is more subtle (its proof is inspired by the proof of [**20**, 5]).

Corollary 5.2.2.2. — *If* $\mathcal{M} \simeq \oplus_{i\in I}(S/p^iS)^{d_i}$ *as an* S*-module, then* $T^*_{\mathrm{st},\pi}(\mathcal{M}) \simeq \oplus_{i\in I}(\mathbf{Z}/p^i\mathbf{Z})^{d_i}$ *as a* $\mathbf{Z}_p$*-module.*

As in (3.1.1), the link to semi-stable representations is:

Theorem 5.2.2.3 (**[9**, 4.1.2.1]). — *Let* $\mathcal{M}$ *be a strongly divisible module of weight* $\leqslant r$ *and rank* d*, then* $\mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \widehat{A_{\mathrm{st},\pi}})$ *is a* $\mathbf{Z}_p$*-lattice in a* d*-dimensionnal semi-stable representation of* G_{K_0} *with Hodge-Tate weights between* 0 *and* r.

The proof uses (4.2.1). As in the crystalline case, one obtains in this manner all the semi-stable representations of G_{K_0} with Hodge-Tate weights between 0 and $p-2$ (see 9.1.1).

Let us now give the covariant version of $T^*_{\mathrm{st},\pi}$. For $\mathcal{M}$ in $\underline{\mathcal{M}}^r_\pi$, let:

$$\mathrm{Fil}^i\,\mathcal{M} = \{x \in \mathcal{M} \mid (u-\pi)^{r-i}x \in \mathrm{Fil}^r\,\mathcal{M}\} \quad (0 \leqslant i \leqslant r)$$

and:

$$\mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}) = \sum_{i=0}^{r} \mathrm{Fil}^{r-i}\,\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathrm{Fil}^i\,\mathcal{M} \subset \widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}.$$

One can prove that the maps ϕ_i on $\mathrm{Fil}^i\,\widehat{A_{\mathrm{st},\pi}}$ and ϕ_r on $\mathrm{Fil}^r\,\mathcal{M}$ give rise to a map $\phi_r : \mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}) \to \widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}$ (see [**10**, 3.2.1]). The operators N on $\widehat{A_{\mathrm{st},\pi}}$ and $\mathcal{M}$ give an operator $N = N \otimes \mathrm{Id} + \mathrm{Id} \otimes N$ on $\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}$. Let:

$$\mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M})^{\phi_r=1}_{N=0} = \{x \in \mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M}) \mid N(x) = 0,\ \phi_r(x) = x\}.$$

Proposition 5.2.2.4 (**[10**, 3.2.1.7]). — *There is a canonical isomorphism of* G_{K_0}*-modules:* $\mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M})^{\phi_r=1}_{N=0} \xrightarrow{\sim} T^*_{\mathrm{st},\pi}(\mathcal{M})^\wedge(r)$ *where the exponent* $\wedge$ *means the Pontryagin dual with respect to* $\mathbf{Q}_p/\mathbf{Z}_p$.

In the sequel, we write:

$$T_{\mathrm{st},\pi}(\mathcal{M}) = T^*_{\mathrm{st},\pi}(\mathcal{M})^\wedge \simeq \mathrm{Fil}^r(\widehat{A_{\mathrm{st},\pi}} \otimes_S \mathcal{M})^{\phi_r=1}_{N=0}(-r).$$

One can check the equivalence $\underline{\mathcal{M}}^r_\pi \xrightarrow{\sim} \underline{\mathcal{M}}^r_{\pi'}$ of (5.1.1.5) commutes with the functors $T_{\mathrm{st},\pi}$ and $T_{\mathrm{st},\pi'}$ (or their dual version), that is to say $T_{\mathrm{st},\pi}(\mathcal{M}_\pi) \simeq T_{\mathrm{st},\pi'}(\mathcal{M}_{\pi'})$ if $\mathcal{M}_{\pi'}$ is associated to $\mathcal{M}_\pi$. So ultimately nothing depends on π; we choose in the sequel $\pi = p$ for simplicity and drop the subscript π. We also choose a compatible system of $p^{n^{\mathrm{th}}}$-roots of p in $\mathcal{O}_{\overline{K}}$ which enables us to write $\widehat{A_{\mathrm{st}}}$ as the p-adic completion of $A_{\mathrm{cris}}\langle X\rangle$ where $X = X_p$ and $u = [\underline{p}](1+X)^{-1}$ (see 5.2.1).

We end this section with an open question. Define a *torsion* semi-stable representation of weight $\leqslant r$ ($r \in \mathbf{N}$) to be any finite representation of G_{K_0} that can be written T/T' where $T' \subset T$ are Galois stable lattices in a semi-stable representation of G_{K_0} with Hodge-Tate weights $\in \{0, \dots, r\}$. Using (9.1.1.1), one can prove that the functor T^*_{st} establishes an anti-equivalence between a full subcategory of $\underline{\mathcal{M}}^r$ and the category of torsion semi-stable representations of G_{K_0} of weight $\leqslant r$, so it's natural to ask:

Question 5.2.2.5. — For $0 \leqslant r \leqslant p-2$, does the functor T^*_{st} actually induce an anti-equivalence of categories between $\underline{\mathcal{M}}^r$ and torsion semi-stable representations of G_{K_0} of weight $\leqslant r$?

To answer positively this question, it would be enough to prove that any object of $\underline{\mathcal{M}}^r$ can be written $\mathcal{M}/\mathcal{M}'$ where $\mathcal{M}' \subset \mathcal{M}$ are two strongly divisible modules as in (5.1.1.4) of the same rank (this implies $\mathcal{M}' \otimes_W K_0 \simeq \mathcal{M} \otimes_W K_0$ using (4.2.1)) or equivalently can be written $\mathrm{Ker}(\mathcal{M}' \otimes (\mathbf{Q}_p/\mathbf{Z}_p) \to \mathcal{M} \otimes (\mathbf{Q}_p/\mathbf{Z}_p))$ (T^*_{st} being contravariant).

6. Log-syntomic morphisms and topology: a review

Following [**32**], we want to apply the previous theory to the case of a proper smooth K_0-scheme admitting a proper semi-stable model X on W. As in the ℓ-torsion case, one has to find a candidate to replace $H^i_{\mathrm{dR}}(X_n) = H^i_{\mathrm{cris}}(X_n/W_n)$ that is still related to X_n and that contains enough information to recover the étale cohomology of the geometric generic fiber $X \times_W \overline{K}$. Once again, the extra information will be the log-structure canonically attached to X in (2.2.1.2). As we also look for S/p^nS-modules, the idea is then to replace the crystalline cohomology of the scheme X_n with respect to the base W_n by the *log*-crystalline cohomology of the *log*-scheme X_n with respect to the *log*-base $(S/p^nS)_{\log}$ (as for $\widehat{A_{\mathrm{st}}}$: see (5.2.1)). To do this, following the Fontaine-Messing method, we first define log-syntomic morphisms and log-syntomic sites.

6.1. Log-syntomic morphisms

6.1.1. Classical syntomic morphisms were introduced by Grothendieck in (EGA IV, 19.3.6) where they were called flat relative complete intersection morphisms. The terminology "syntomic" itself is due to Mazur ([**51**]), who also noted the syntomic topology had interesting properties. A morphism $X \to \Sigma$ between classical schemes is syntomic if it is flat, locally of finite presentation and if locally on X (for the Zariski or equivalently étale topology), there is a factorization $X \overset{i}{\hookrightarrow} Y \overset{h}{\longrightarrow} \Sigma$ where h is smooth and i is a regular closed immersion (in the sense of (SGA$_6$, VII.1.4); since X/Σ is flat this is, by (EGA IV, 11.3.8), equivalent to requiring that the ideal defining i is locally generated by a regular sequence). This definition was generalized by Kato to the log-setting:

Definition 6.1.1.1 ([**44**, 2.5]). — Let $f : X \to \Sigma$ be a morphism of fine log-schemes. One says f is log-syntomic if it satisfies the following conditions:
(i) it is integral,
(ii) the underlying morphism of schemes is flat and locally of finite presentation,
(iii) étale locally on X, there is a factorization $X \overset{i}{\hookrightarrow} Y \overset{h}{\longrightarrow} \Sigma$ where h is log-smooth and i is an exact closed immersion which is regular on the underlying schemes (as in the classical case).

We give now the four main properties of log-syntomic morphisms with brief proofs.

Proposition 6.1.1.2. — *If there is another factorization of* $f\colon X \overset{i'}{\hookrightarrow} Y' \xrightarrow{h'} \Sigma$ *with* h' *log-smooth and* i' *an exact closed immersion, then* i' *is also regular.*

Proof. — Consider the fiber product $Y \times_\Sigma Y'$ (in the category of fine log-schemes). The closed immersion $X \overset{i\times i'}{\hookrightarrow} Y \times_\Sigma Y'$ is no longer necessarily exact but can be factored, étale locally on X, as $X \overset{i''}{\hookrightarrow} Y'' \xrightarrow{g} Y \times_\Sigma Y'$ where i'' is an exact closed immersion and g is log-étale ([**43**, 4.10]). Let $x \in X$, replacing Y'' by an étale neighbourhood around $i''(x)$, one can assume $M_{Y''} \xrightarrow{\sim} \pi^* M_Y$ and $M_{Y''} \xrightarrow{\sim} \pi'^* M_{Y'}$ where π, π' are the maps $Y'' \xrightarrow{\pi} Y$ and $Y'' \xrightarrow{\pi'} Y'$ obtained by composing g with the two projections from $Y \times_\Sigma Y'$. Then π and π' are classically smooth ([**43**, 3.8]). One finishes by applying to $Z = Y$ and $Z = Y'$ the following classical fact (cf. SGA$_6$, VII.1.3): if one has a commutative diagram of (classical) schemes:

$$\begin{array}{ccc} X & \overset{i''}{\hookrightarrow} & Y'' \\ \| & & \downarrow{\scriptstyle \pi_Z} \\ X & \overset{i_Z}{\hookrightarrow} & Z \end{array}$$

with π_Z smooth and i'', i_Z closed immersions, i'' is regular if and only if i_Z is regular. □

Proposition 6.1.1.3. — *Log-syntomic morphisms are stable by base change.*

Proof. — Let X/Σ be log-syntomic and $\Sigma' \to \Sigma$ be the base change morphism. After étale localization on X and Σ, we may assume we have $X \overset{i}{\hookrightarrow} Y$ an exact, regular closed immersion where Y/Σ is log-smooth and integral. Denote with a prime the result of base change to Σ' in the category of all log-schemes. As $X \to \Sigma$, $Y \to \Sigma$ are integral, X', Y' are automatically fine, $X' \overset{i'}{\hookrightarrow} Y'$ is an exact closed immersion and Y'/Σ' is log-smooth. It follows from (EGA IV, 19.2.7 (ii)) that i' is regular. □

Proposition 6.1.1.4. — *Log-syntomic morphisms are stable by composition.*

Proof. — If Y/Σ is log-smooth, integral and $\Sigma \hookrightarrow Z$ is an exact closed immersion, one can always find, étale locally on Z and Y, a log-smooth integral morphism $W \to Z$ such that $Y = W \times_Z \Sigma$. (By using the local description of log-smooth morphisms ([**43**, 3.5]), one can always find such a W at least log-smooth over Z. Since the closed immersions are all exact, by localizing on W around Y, one can assume $W \to Z$ to be integral.) Hence on the scheme level, $W \to Z$ is also flat and, if $\Sigma \hookrightarrow Z$ is a regular closed immersion, then so is $Y \hookrightarrow W$ by (EGA IV, 19.1.5 (ii)). Now, let $X \to \Sigma$ and $\Sigma \to T$ be two log-syntomic morphisms and choose factorizations $X \hookrightarrow Y \to \Sigma$, $\Sigma \hookrightarrow Z \to T$ as in (6.1.1.1) but with $Y \to \Sigma$ integral (see the previous proof). By taking W as above, one has $X \overset{i_1}{\hookrightarrow} Y \overset{i_2}{\hookrightarrow} W \xrightarrow{h_1} Z \xrightarrow{h_2} T$ with i_1, i_2 regular exact closed immersions and h_1, h_2 log-smooth. So $X \hookrightarrow W \to T$ is a factorization as in (6.1.1.1). □

Proposition 6.1.1.5. — *Let X'/Σ' log-syntomic and $\Sigma' \hookrightarrow \Sigma$ an exact closed immersion. Then étale locally on X', one can find a log-syntomic morphism $X \to \Sigma$ such that $X' = \Sigma' \times_\Sigma X$.*

Proof. — Choose a factorization $X' \hookrightarrow Y' \to \Sigma'$ as in (6.1.1.1) with Y'/Σ' integral log-smooth and, as in the previous proof, choose Y/Σ integral log-smooth such that $Y' = Y \times_\Sigma \Sigma'$. By lifting to $\mathcal{O}_Y$ a transversally regular sequence in $\mathcal{O}_{Y'}$, it is easy to find an exact closed immersion $X \hookrightarrow Y$ such that $X' = X \times_\Sigma \Sigma'$. Moreover, standard arguments (see for instance EGA 0_{IV}, 15.1.16) show that X/Σ is flat and $X \hookrightarrow Y$ transversally regular (with respect to Σ) at each point of X coming from X'. By (EGA IV, 19.2.4), $X \hookrightarrow Y$ is regular in a (Zariski) neighbourhood of such a point. Thus, we get the desired X by localizing for the Zariski topology and taking the induced log-structure. □

6.1.2. In the classical case, a syntomic morphism is described locally as a flat morphism $A \to A[X_1, \dots, X_s]/(f_1, \dots, f_t)$ where $f_1, \dots, f_t$ is a regular sequence. In the log-case, there are several (equivalent) local descriptions due to the fact that there are several ways of writing the charts. We give here a description which turns out to be quite convenient for local computations on the log-syntomic site (see for instance 6.2.2.3). Consider an integral and locally of finite type morphism X/Σ of fine log-schemes. It is easy to see one can find (locally) a chart:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

where $M \to N$ is an integral morphism of fine monoids. Since N is of finite type, there is an $r \in \mathbf{N}$ and a surjection $M \oplus \mathbf{N}^r \to N$. Denoting by G the kernel of the induced map $M^{\mathrm{gp}} \oplus \mathbf{Z}^r \to N^{\mathrm{gp}}$ and by $(M \oplus \mathbf{N}^r) + G$ the submonoid of $M^{\mathrm{gp}} \oplus \mathbf{Z}^r$ generated by $M \oplus \mathbf{N}^r$ and G, one gets an exact morphism of monoids $(M \oplus \mathbf{N}^r) + G \to N$ by sending G to 0. Notice that $M \to (M \oplus \mathbf{N}^r) + G$ is still integral. One can also find $s \in \mathbf{N}$ and a surjection $A \otimes_{\mathbf{Z}[M]} \mathbf{Z}[(M \oplus \mathbf{N}^r) + G][X_1, \dots, X_s] \to B$ where $(M \oplus \mathbf{N}^r) + G \to B$ factorizes through N. Hence, we have a factorization:

$$\begin{array}{ccccc} M & \longrightarrow & (M \oplus \mathbf{N}^r) + G & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A \otimes_{\mathbf{Z}[M]} \mathbf{Z}[(M \oplus \mathbf{N}^r) + G][X_1, \dots, X_s] & \longrightarrow & B \end{array}$$

where the first morphism of (the corresponding) log-schemes is clearly log-smooth and the second is an exact closed immersion. Now, if we start with X/Σ log-syntomic, (6.1.1.2) tells us that, up to further Zariski localization, the ideal of the closed immersion on the right is generated by a regular sequence. Hence, any log-syntomic morphism can be locally written as:

$$\begin{array}{ccc} M & \longrightarrow & (M\oplus \mathbf{N}^r)/G = N \\ \downarrow & & \downarrow \\ A & \longrightarrow & \dfrac{A\otimes_{\mathbf{Z}[M]}\mathbf{Z}[(M\oplus \mathbf{N}^r)+G][X_1,\dots,X_s]}{(f_1,\dots,f_t)} = B \end{array}$$

where G is a subgroup of $M^{\mathrm{gp}}\oplus\mathbf{Z}^r$, $(M\oplus\mathbf{N}^r)/G$ the image of $M\oplus\mathbf{N}^r$ in $(M^{\mathrm{gp}}\oplus\mathbf{Z}^r)/G$, $f_1,\dots,f_t$ a transversally regular sequence with respect to A such that $(f_1,\dots,f_t)$ contains $[g]-1$ for $g\in G$ and where $M\to(M\oplus\mathbf{N}^r)+G$ is injective and integral.

Example 6.1.2.1. — Very important among log-syntomic morphisms are those which correspond to extracting $p^{n^{\mathrm{th}}}$-roots both on the sheaf of monoids and the scheme (classical case $= A\to\frac{A[X_1,\dots,X_s]}{(X_1^{p^n}-a_1,\dots,X_s^{p^n}-a_s)}$). Let $r,s,n\in\mathbf{N}$, $m_1,\dots,m_r\in M$ and $a_1,\dots,a_s\in A$. These morphisms are obtained by taking as G above the subgroup G_n of $M^{\mathrm{gp}}\oplus\mathbf{Z}^r$ generated by $g_i=-m_i\oplus(0,\dots,p^n,\dots 0)$, $1\leqslant i\leqslant r$ (p^n in position i) and $(f_1,\dots,f_t)=([g_1]-1,\dots,[g_r]-1,X_1^{p^n}-a_1,\dots,X_s^{p^n}-a_s)$. In particular, if (A,M) itself is log-syntomic over, let's say, $(\mathbf{N}\to W, 1\mapsto p)$ (a situation we'll have to deal with very soon) then, locally, we can write $M=(\mathbf{N}\oplus\mathbf{N}^{r'})/G$ (= image of $\mathbf{N}\oplus\mathbf{N}^{r'}$ in $(\mathbf{Z}\oplus\mathbf{Z}^{r'})/G$) and $A=\frac{W\otimes_{\mathbf{Z}[\mathbf{N}]}\mathbf{Z}[(\mathbf{N}\oplus\mathbf{N}^{r'})+G][X_1,\dots,X_s]}{(f_1,\dots,f_t)}$, and we can take $r=1+r'$, $m_i=$ image of $(0,\dots,1,\dots,0)\in\mathbf{N}\oplus\mathbf{N}^{r'}$ in M (1 in position i, $1\leqslant i\leqslant r$) and $a_j=X_j$ ($1\leqslant j\leqslant s$). We get:

$$\begin{aligned} M_n &= (M\oplus\mathbf{N}^r)/G_n \simeq \Big(\mathbf{N}\frac{1}{p^n}\oplus(\mathbf{N}\frac{1}{p^n})^{r'}\Big)\Big/G \\ A_n &= \frac{A\otimes_{\mathbf{Z}[M]}\mathbf{Z}[(M\oplus\mathbf{N}^r)+G_n][Y_1,\dots,Y_s]}{([g_i]-1, Y_j^{p^n}-X_j)} \\ &\simeq \frac{W\otimes_{\mathbf{Z}[\mathbf{N}]}\mathbf{Z}[(\mathbf{N}\frac{1}{p^n}\oplus(\mathbf{N}\frac{1}{p^n})^{r'})+G][X_1^{p^{-n}},\dots,X_s^{p^{-n}}]}{(f_1,\dots,f_t)} \end{aligned}$$

There are obvious injective morphisms of log-rings $(A_n,M_n)\to(A_{n+1},M_{n+1})$ induced by $X_j^{p^{-n}}\mapsto(X_j^{p^{-(n+1)}})^p$, $(0,\dots,\frac{1}{p^n},\dots,0)\mapsto p(0,\dots,\frac{1}{p^{n+1}},\dots,0)$ and we denote in the sequel $M_\infty=\varinjlim M_n\simeq(\mathbf{N}\frac{1}{p^\infty}\oplus(\mathbf{N}\frac{1}{p^\infty})^{r'})/G$ and $A_\infty=\varinjlim A_n$. The log-ring (A_∞,M_∞) is still integral (but not fine!). Note that the Frobenius on M_∞, i.e. the multiplication by p map, and the Frobenius on A_∞/pA_∞ are both surjective.

6.2. The log-syntomic topology

6.2.1. Let Σ be a fine log-scheme. One defines Σ_{syn} to be the category of all fine log-schemes which are log-syntomic over Σ. This category is endowed with the

Grothendieck topology generated by log-syntomic morphisms which are surjective on the underlying schemes and sheaves are defined in the obvious way. It is frequently useful to consider also the big log-syntomic site Σ_{SYN}. Its underlying category consists of all fine log-schemes over Σ, its topology is defined as for Σ_{syn}. A sheaf $\mathcal{F}$ on Σ_{SYN} has a restriction $\mathcal{F}_{\mathrm{syn}}$ on Σ_{syn} and, if abelian, the cohomology of $\mathcal{F}$ coincides with that of $\mathcal{F}_{\mathrm{syn}}$. An advantage of the big site is that it is functorial in Σ while the small site is not. An important property of the small site is that various sheaves of rings or modules defined on the big site have, when restricted to the small site, good flatness properties (see 6.2.2.4). Technically, this is a key point. An indication that this log-syntomic topology is reasonable is due to:

Lemma 6.2.1.1. — *Let $X \to \Sigma$ be a morphism of fine log-schemes, then the functor $Y \mapsto \mathrm{Hom}_\Sigma(Y, X)$ ($Y \in \Sigma_{\mathrm{SYN}}$) is a sheaf for the log-syntomic topology.*

Proof (Sketch). — Let $\pi : Y' \to Y$ be a log-syntomic covering and $Y'' = Y' \times_Y Y'$ with π_1, π_2 the two projections onto Y'. One has to prove:
1) If $f_1, f_2 \in \mathrm{Hom}_\Sigma(Y, X)$ and $f_1 \circ \pi = f_2 \circ \pi$, then $f_1 = f_2$,
2) If $f' : Y' \to X$ is such that $f' \circ \pi_1 = f' \circ \pi_2$, then there is an unique $f : Y \to X$ such that $f' = f \circ \pi$.
Everything is clear if one forgets the log-structures since the covering is flat on the underlying schemes. Using the fact that $Y' \to Y$ is integral, flat and surjective, one can show there exist local charts of π:

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A' \end{array}$$

where $M \to M'$ is an injective, integral and exact morphism of monoids. If $M' \oplus_M M'$ denotes the inductive limit of the diagram $M' \leftarrow M \to M'$, one has an exact sequence $M \hookrightarrow M' \rightrightarrows M' \oplus_M M'$ and it is not difficult to deduce 1) and 2) from this. □

6.2.2. Now let Σ be the log-base $(\mathbf{N} \to W, 1 \mapsto p)$. This base is very important since it's the one that naturally arises in geometry (see 2.2.1.2). We will simply write $\mathrm{Spec}(W)$ when considering this scheme as equipped with its trivial log-structure (i.e. $M = W^*$) and E the log-scheme associated to $(\mathbf{N} \to S, 1 \mapsto u)$ where S is the ring of (4.2) and (5). One has a commutative diagram:

$$\begin{array}{ccc} \Sigma & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \mathrm{Spec}(W) & = & \mathrm{Spec}(W) \end{array}$$

where $\Sigma \hookrightarrow E$ is the DP-thickening obtained by sending u to p and the two vertical maps are the two obvious log-syntomic coverings. Notice that Σ_{syn} is a sub-site of $\mathrm{Spec}(W)_{\mathrm{syn}}$. The following proposition is useful and straightforward:

Proposition 6.2.2.1. — *Let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a sequence of abelian sheaves on* $\mathrm{Spec}(W)_{\mathrm{syn}}$ *or* Σ_{syn}. *If, for all (A, M) in Σ_{syn} and (A_∞, M_∞) as in (6.1.2.1), one has exact sequences of abelian groups:*

$$0 \longrightarrow \mathcal{F}'(A_\infty, M_\infty) \longrightarrow \mathcal{F}(A_\infty, M_\infty) \longrightarrow \mathcal{F}''(A_\infty, M_\infty) \longrightarrow 0$$

where we set $\mathcal{G}(A_\infty, M_\infty) = \varinjlim \mathcal{G}(A_n, M_n)$ if $\mathcal{G}$ is a sheaf, then $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on $\mathrm{Spec}(W)_{\mathrm{syn}}$ *or* Σ_{syn}.

Proof. — The morphisms of log-schemes associated to $(A_n, M_n) \to (A_{n+1}, M_{n+1})$ in (6.1.2.1) are obvious log-syntomic coverings. □

Log-crystalline cohomology was first defined in [**43**] by mimicing the classical theory of Berthelot ([**3**], [**4**]) and we refer to [**43**, 5] or to [**71**, 4] in this volume for its definition and properties. If X is any fine log-scheme over W, we write X_n for the log-scheme $X \times_W W_n$ with the induced log-structure from X. For X a log-scheme in Σ_{syn} and $r \in \mathbf{N}$, define:

$$\mathcal{O}_n^{\mathrm{st}}(X) = H^0((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$$
$$\mathcal{J}_n^{\mathrm{st},[r]}(X) = H^0((X_n/E_n)_{\mathrm{cris}}, \mathcal{J}_{X_n/E_n}^{[r]})$$

where $\mathcal{O}_{X_n/E_n}$ is the structure sheaf and $\mathcal{J}_{X_n/E_n} = \mathrm{Ker}(\mathcal{O}_{X_n/E_n} \to \mathcal{O}_{X_n})$. Notice that $\mathcal{O}_n^{\mathrm{st}}(X)$ is an S-algebra. For X a log-scheme in $\mathrm{Spec}(W)_{\mathrm{syn}}$ and $r \in \mathbf{N}$, define in the same way:

$$\mathcal{O}_n^{\mathrm{cris}}(X) = H^0((X_n/\mathrm{Spec}(W_n))_{\mathrm{cris}}, \mathcal{O}_{X_n/\mathrm{Spec}(W_n)})$$
$$\mathcal{J}_n^{\mathrm{cris},[r]}(X) = H^0((X_n/\mathrm{Spec}(W_n))_{\mathrm{cris}}, \mathcal{J}_{X_n/\mathrm{Spec}(W_n)}^{[r]})$$

Set $\mathcal{J}_n^{\mathrm{st},[r]} = \mathcal{O}_n^{\mathrm{st}}$ if $r \leqslant 0$ (resp. with "cris"). For X an object of Σ_{syn}, there are morphisms $\mathcal{J}_n^{\mathrm{cris},[r]}(X) \to \mathcal{J}_n^{\mathrm{st},[r]}(X)$ by the functoriality of the log-crystalline topos ([**43**, 5.9]). Using property (6.1.1.5) and the key log-syntomic morphisms (6.1.2.1) together with the de Rham computation of log-crystalline cohomology ([**43**, 6]), it is a standard matter to generalize the results of [**32**, II.1.3] and prove (cf. [**7**], 3.2.3 and 3.3 for the case $r = 0$):

Proposition 6.2.2.2
1) For $r \in \mathbf{Z}$, the presheaves $\mathcal{J}_n^{\mathrm{st},[r]}$ (resp. $\mathcal{J}_n^{\mathrm{cris},[r]}$) are sheaves on Σ_{syn} (resp. $\mathrm{Spec}(W)_{\mathrm{syn}}$*).*
2) For $r \in \mathbf{Z}$ and $i \in \mathbf{N}$, there are canonical and functorial isomorphisms:

$$H^i(X_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]}) \simeq H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{J}_{X_n/E_n}^{[r]}) \text{ if } X \in \Sigma_{\mathrm{syn}}$$
$$H^i(X_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{cris},[r]}) \simeq H^i((X_n/\mathrm{Spec}(W_n))_{\mathrm{cris}}, \mathcal{J}_{X_n/\mathrm{Spec}(W_n)}^{[r]}) \text{ if } X \in \mathrm{Spec}(W)_{\mathrm{syn}}.$$

For a general X in Σ_{syn}, one doesn't know explicitly $\mathcal{J}_n^{\mathrm{st},[r]}(X)$ or $\mathcal{J}_n^{\mathrm{cris},[r]}(X)$. However, using (6.2.2.1), one can usually restrict to $\mathcal{J}_n^{\mathrm{st},[r]}(A_\infty, M_\infty)$ (resp. with "cris") which can be described explicitly:

Lemma 6.2.2.3. — *Let (A, M) and (A_∞, M_∞) be as in (6.1.2.1) and $M_\infty + \frac{1}{p^n}G$ be the sub-monoid of M_∞^{gp} generated by M_∞ and the image of $\frac{1}{p^n}G = \{x \in \mathbf{Z}\oplus\mathbf{Z}^{r'} \mid p^n x \in G\}$ (see 6.1.2.1), which maps to A_∞ through the composite $M_\infty + \frac{1}{p^n}G \xrightarrow{p^n} M_\infty \longrightarrow A_\infty$.*
1) There is a canonical isomorphism:

$$\Big(W_n(A_\infty/pA_\infty) \otimes_{\mathbf{Z}[M_\infty]} \mathbf{Z}\big[M_\infty + \frac{1}{p^n}G\big]\Big)^{\mathrm{DP}} \xrightarrow{\sim} \mathcal{O}_n^{\mathrm{cris}}(A_\infty, M_\infty)$$

where we take the divided power envelope (compatible with the divided powers on (p)) with respect to the kernel of the map to A_∞/p^nA_∞ that maps $M_\infty + \frac{1}{p^n}G$ as above and $(a_0, \dots, a_{n-1}) \in W_n(A_\infty/p)$ to $\widehat{a}_0^{p^n} + p\widehat{a}_1^{p^{n-1}} + \cdots + p^{n-1}\widehat{a}_{n-1}^p$ ($\widehat{a}_i$ lifting a_i in A_∞). It induces isomorphisms between $\mathcal{J}_n^{\mathrm{cris},[r]}(A_\infty, M_\infty)$ on the right and the r^{th} divided power of the tautological DP ideal on the left.
2) To each choice of an $h \in M_\infty + \frac{1}{p^n}G$ such that $p^n h = (1, 0, \dots, 0) \in M_\infty$, there is an element $X_h \in \mathcal{O}_n^{\mathrm{st}}(A_\infty, M_\infty)$ and an isomorphism:

$$\mathcal{O}_n^{\mathrm{cris}}(A_\infty, M_\infty)\langle X_h\rangle \xrightarrow{\sim} \mathcal{O}_n^{\mathrm{st}}(A_\infty, M_\infty)$$

such that $[h](1 + X_h)^{-1} \mapsto u$ which induces isomorphisms:

$$\sum_{s=0}^{\infty} \mathcal{J}_n^{\mathrm{cris},[r-s]}(A_\infty, M_\infty)\frac{X_h^s}{s!} \xrightarrow{\sim} \mathcal{J}_n^{\mathrm{st},[r]}(A_\infty, M_\infty).$$

For more details, see [**10**, appendix D]. Using this description, one can prove for instance ([**10**, 2.1.2]):

Proposition 6.2.2.4
1) The sheaf of S_n-algebras $\mathcal{O}_n^{\mathrm{st}}$ is flat over S_n.
2) For $r \in \mathbf{Z}$, the sheaves $\mathcal{J}_n^{\mathrm{st},[r]}$ and $\mathcal{J}_n^{\mathrm{cris},[r]}$ are flat over W_n.
3) For $r \in \mathbf{Z}$ and $* =$ "st" or "cris", there are short exact sequences:

$$0 \longrightarrow \mathcal{J}_m^{*,[r]} \xrightarrow{p^n} \mathcal{J}_{n+m}^{*,[r]} \longrightarrow \mathcal{J}_n^{*,[r]} \longrightarrow 0.$$

6.2.3. We defined in (4.2) operators ϕ_r on $\mathrm{Fil}^r S$ $(0 \leqslant r \leqslant p-1)$ and N on S. We want to extend them to the above sheaves and their cohomology. Because one has a Frobenius on $E_n = \mathrm{Spec}(S_n)$ and $\mathrm{Spec}(W_n)$, one gets the usual crystalline Frobenius on $H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$ and $H^i((X_n/\mathrm{Spec}(W_n))_{\mathrm{cris}}, \mathcal{O}_{X_n/\mathrm{Spec}(W_n)})$, so in particular on $\mathcal{O}_n^{\mathrm{st}}$, $\mathcal{O}_n^{\mathrm{cris}}$ and their cohomology groups. It is formal, if one uses the big log-crystalline and log-syntomic sites instead of the small ones, to check that the isomorphisms in (6.2.2.2) for $r = 0$ are then compatible with the Frobeniuses. Moreover:

Lemma 6.2.3.1. — *For* $0 \leqslant r \leqslant p-1$, $\phi(\mathcal{J}_n^{\mathrm{st},[r]}) \subset p^r \mathcal{O}_n^{\mathrm{st}}$ *and* $\phi(\mathcal{J}_n^{\mathrm{cris},[r]}) \subset p^r \mathcal{O}_n^{\mathrm{cris}}$.

Proof. — One easily reduces to the case $r = 1$. Then, the result is just due to the fact that the sheaves of ideals $\mathcal{J}_n^{\mathrm{st}}$ and $\mathcal{J}_n^{\mathrm{cris}}$ are endowed with divided powers. □

If x is a section of $\mathcal{J}_n^{\mathrm{st},[r]}$ with $0 \leqslant r \leqslant p-1$ and $\widehat{x}$ a local lifting in $\mathcal{J}_{n+r}^{\mathrm{st},[r]}$ (using (6.2.2.4, 3)), then $\phi(\widehat{x}) \in p^r \mathcal{O}_{n+r}^{\mathrm{st}}$ (locally) and because $p^r \mathcal{O}_{n+r}^{\mathrm{st}} \simeq \mathcal{O}_n^{\mathrm{st}}$ (still 6.2.2.4), the image of $\phi(\widehat{x})/p^r$ in $\mathcal{O}_n^{\mathrm{st}}$ doesn't depend on the lifting. This gives a global map $\phi_r : \mathcal{J}_n^{\mathrm{st},[r]} \to \mathcal{O}_n^{\mathrm{st}}$. The same thing applies to $\mathcal{J}_n^{\mathrm{cris},[r]}$, giving a commutative diagram:

$$\begin{array}{ccc} \mathcal{J}_n^{\mathrm{cris},[r]} & \xrightarrow{\phi_r} & \mathcal{O}_n^{\mathrm{cris}} \\ \downarrow & & \downarrow \\ \mathcal{J}_n^{\mathrm{st},[r]} & \xrightarrow{\phi_r} & \mathcal{O}_n^{\mathrm{st}}. \end{array}$$

Using the local description (6.2.2.3, 2), one can check that $\phi_r(X_h^r) = \left(\frac{(1+X_h)^p-1}{p}\right)^r$.

Using the de Rham computation of log-crystalline cohomology, Hyodo and Kato define in [**38**, 3.6] a W-linear derivation N_{HK} on $\mathcal{O}_n^{\mathrm{st}}(X) = H^0((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$ called the (p-adic) monodromy operator (actually they assume X log-smooth but it is not used in the definition [**38**, 3.6]). We define $N = -N_{HK} : \mathcal{O}_n^{\mathrm{st}} \to \mathcal{O}_n^{\mathrm{st}}$. One thus gets an operator N on $H^i(X_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}})$. In terms of the local description (6.2.2.3, 2), N is the unique $\mathcal{O}_n^{\mathrm{cris}}(A_\infty, M_\infty)$-linear map such that $N(X_h^s/s!) = (1+X_h)X_h^{s-1}/(s-1)!$.

Proposition 6.2.3.2. — *1) For* $0 \leqslant r \leqslant p-1$, *one has* $N\phi_r = \phi_{r-1}N$.
2) For $r \in \mathbf{Z}$, *there are exact sequences of sheaves on* Σ_{syn}:

$$0 \longrightarrow \mathcal{J}_n^{\mathrm{cris},[r]} \longrightarrow \mathcal{J}_n^{\mathrm{st},[r]} \xrightarrow{N} \mathcal{J}_n^{\mathrm{st},[r-1]} \longrightarrow 0.$$

Proof. — Straightforward from (6.2.2.1) and (6.2.2.3) with the above expressions of ϕ_r and N. □

Remark 6.2.3.3. — The reason we take $-N_{HK}$ and not N_{HK} is because there is another, purely syntomic, way to define a monodromy operator on $\mathcal{O}_n^{\mathrm{st}}$ (see [**7**, 6.1]) and one can show this operator is precisely $-N_{HK}$.

Remark 6.2.3.4. — Hyodo-Kato's definition of N_{HK} also extends to higher cohomology groups $H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$ for $i \geqslant 1$. The authors *do not know* if the isomorphisms in (6.2.2.2) are compatible with the operators N and $-N_{HK}$ for $i \geqslant 1$, although this is probable. This won't be very important in the sequel where we use N only.

7. A generalization of the Deligne-Illusie-Fontaine-Messing isomorphism

7.1. Preliminaries. — From now on, we fix X/Σ log-smooth, proper and such that X_1/Σ_1 is a morphism of Cartier type. This last and somewhat technical condition is explained in [**43**, 4.8] and turns out to be necessary for computations. Let us just say that it is automatically satisfied in the semi-stable case, although the above situation is much more general than the semi-stable one (for instance, the log-structure on X_{K_0} need not be trivial). The aim of this section is to prove that in this general situation and for $0 \leqslant i \leqslant r \leqslant p-2$ and $n \in \mathbf{N}$:

$$\left(H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n}), H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{J}^{[r]}_{X_n/E_n}), \phi_r, N\right)$$

is an object of the category $\underline{\mathcal{M}}^r$ of (5.1.1) (described as $(\mathcal{M}, \mathrm{Fil}^r \mathcal{M}, \phi_r, N)$). It is easy to see that the only non formal facts to prove are:

1) $H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n}) \simeq \oplus_{i\in I}(S/p^iS)^{d_i}$ as an S-module (I finite)

2) the map $H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{J}^{[r]}_{X_n/E_n}) \to H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$ induced by the injection $\mathcal{J}^{[r]}_{X_n/E_n} \hookrightarrow \mathcal{O}_{X_n/E_n}$ is injective

3) the image of ϕ_r generates $H^i((X_n/E_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/E_n})$ over S.

As we mentioned in (3.2), the main tool to prove the analogous statements in the good reduction case over $\mathrm{Spec}(W)$ was the Deligne-Illusie isomorphism (compare [**32**, II.2.5] and [**16**, 2.1]). So the first task is to find an analogous isomorphism, but involving the two bases Σ and E. We then explain briefly how this result is used to prove the above statements 1)-3) for $n = 1$. The general case is finally deduced by dévissage. We, of course, work over the site Σ_{syn} with the log-syntomic interpretation (6.2.2.2) of the above cohomology groups and, for brevity, we write $H^i(\mathcal{F})$ instead of $H^i(X_{\mathrm{syn}}, \mathcal{F})$ whenever $\mathcal{F}$ is a sheaf on Σ_{syn}.

7.2. Generalization of the DIFM isomorphism. — We saw in (5.1.2.1) that any object $\mathcal{M}$ of $\underline{\mathcal{M}}^r$ $(0 \leqslant r \leqslant p-2)$ that is killed by p is such that the map $\mathrm{Id} \otimes \phi_r$ induces an isomorphism:

$$S_1 \otimes_{(\phi),k} \frac{\mathrm{Fil}^r \mathcal{M}}{\mathrm{Fil}^{r+1} \mathcal{M}} \xrightarrow{\sim} \mathcal{M}.$$

where $\mathrm{Fil}^{r+1} \mathcal{M} = \mathrm{Fil}^p S_1 \cdot \mathcal{M} + \mathrm{Fil}^1 S_1 \cdot \mathrm{Fil}^r \mathcal{M}$. Thus, if $(H^i(\mathcal{O}_1^{\mathrm{st}}), H^i(\mathcal{J}_1^{\mathrm{st},[r]}), \phi_r, N)$ is in $\underline{\mathcal{M}}^r$, we should hopefully have some cohomology group $H^i(?)$, probably related to the sheaf $\mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]}$, with a map $\phi_r : H^i(?) \to H^i(\mathcal{O}_1^{\mathrm{st}})$ such that $S_1 \otimes_{(\phi),k} H^i(?) \xrightarrow{\sim} H^i(\mathcal{O}_1^{\mathrm{st}})$. Indeed, on Σ_{syn}, $\phi_r(\mathcal{J}_1^{\mathrm{st},[r+1]}) = 0$ (since $r \leqslant p-2$) so there is a map of sheaves:

$$\mathrm{Id} \otimes \phi_r : S_1 \otimes_{(\phi),k} \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}} \longrightarrow \mathcal{O}_1^{\mathrm{st}}$$

but, unfortunately, it is not an isomorphism in general.

Recall that the Frobenius on a log-scheme in characteristic p is just the usual Frobenius on the underlying scheme and the multiplication by p map on the sheaf of monoids (with additive notations). For any (fine) log-scheme Y over $\Sigma_1(\hookrightarrow E_1)$, denote by Y' the pullback of Y by F_{E_1} where F_{E_1} is the Frobenius on E_1. Then the relative Frobenius $F_{Y/E_1} : Y \to Y'$ can be factored in a unique way as: $Y \xrightarrow{F''} Y'' \xrightarrow{F'} Y'$ where F' is log-étale and F'' is exact (see [**43**, 4.9] for all this). One defines a presheaf $\mathcal{O}_1^{\mathrm{car}}$ on Σ_{syn} ("car" for Cartier) by $\mathcal{O}_1^{\mathrm{car}}(U) = \Gamma(U_1'', \mathcal{O}_{U_1''})$ (recall $U_1 = U \times_W k$). It turns out $\mathcal{O}_1^{\mathrm{car}}$ is in fact a sheaf on Σ_{syn} ([**10**, 2.2.1.1]) and that one has a canonical injection $\mathcal{O}_1^{\mathrm{car}} \hookrightarrow \mathcal{O}_1^{\mathrm{st}}$. Notice that if there are no log-structures (only classical schemes), $\mathcal{O}_1^{\mathrm{car}}$ is just $S_1 \otimes_{(\phi),k} \mathcal{O}_1$ where $\mathcal{O}_1(U) = \Gamma(U_1, \mathcal{O}_{U_1})$.

Let $r \in \mathbf{N}$ and x a local section of $\mathcal{O}_{r+1}^{\mathrm{st}}$. Following [**32**, II.2.3], whenever $\phi(x) \in p^r \mathcal{O}_{r+1}^{\mathrm{st}}$ (locally), define $f_r(x) \in \mathcal{O}_1^{\mathrm{st}}$ such that $\phi(x) = p^r \widehat{f_r(x)}$, where $\widehat{f_r(x)}$ is a (local) lifting of $f_r(x)$. Then f_r is a homomorphism and we denote by $F_r\mathcal{O}_1^{\mathrm{st}}$ its image in $\mathcal{O}_1^{\mathrm{st}}$. Finally, let $F_r^{\mathrm{car}}\mathcal{O}_1^{\mathrm{st}}$ be the image of $\mathcal{O}_1^{\mathrm{car}} \otimes_k F_r\mathcal{O}_1^{\mathrm{st}}$ in $\mathcal{O}_1^{\mathrm{st}}$.

Theorem 7.2.1. — *For $0 \leqslant r \leqslant p-2$, the map $\mathrm{Id} \otimes \phi_r$ induces isomorphisms of sheaves on Σ_{syn}:*

$$\mathcal{O}_1^{\mathrm{car}} \otimes_{(\phi),k} \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}} \xrightarrow{\sim} F_r^{\mathrm{car}}\mathcal{O}_1^{\mathrm{st}}.$$

Remark 7.2.2. — The previous map:

$$\mathrm{Id} \otimes \phi_r : S_1 \otimes_{(\phi),k} \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}} \longrightarrow \mathcal{O}_1^{\mathrm{st}}$$

is injective, but not surjective (in general).

As usual, the proof is reduced to the case (A_∞, M_∞) where everything is made explicit: see [**10**, 2.2.2] for details. Now, what we would like (see the previous discussion) are isomorphisms $H^i(F_r^{\mathrm{car}}\mathcal{O}_1^{\mathrm{st}}) \xrightarrow{\sim} H^i(\mathcal{O}_1^{\mathrm{st}})$ and $S_1 \otimes_{(\phi),k} H^i(\mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]}) \xrightarrow{\sim} H^i(\mathcal{O}_1^{\mathrm{car}} \otimes_{(\phi),k} \mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]})$. This is certainly false for general X, but if X is log-smooth over Σ with X_1/Σ_1 of Cartier type as we assumed (the properness is not even necessary here) and if α denotes the projection: (sheaves on X_{syn})$\longrightarrow$ (sheaves on $X_{\mathrm{\acute{e}t}}$) (= small classical étale site with induced log-structures), then:

Theorem 7.2.3 **([10, 2.2.3]).** — *For $0 \leqslant r \leqslant p-2$, there are isomorphisms in the derived category of complexes of sheaves on $X_{\mathrm{\acute{e}t}}$:*

1) $S_1 \otimes_{(\phi),k} R\alpha_* \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}} \xrightarrow{\sim} R\alpha_* \left(\mathcal{O}_1^{\mathrm{car}} \otimes_{(\phi),k} \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}}\right)$

2) $\tau_{\leqslant r} R\alpha_*(F_r^{\mathrm{car}}\mathcal{O}_1^{\mathrm{st}}) \xrightarrow{\sim} \tau_{\leqslant r} R\alpha_*\mathcal{O}_1^{\mathrm{st}}$.

Remark 7.2.4. — Of course, all the above sheaves on $X_{\mathrm{\acute{e}t}}$ in fact have support contained in the special fiber.

Remark 7.2.5. — The assertion 2) is false in general if one replaces $F_r^{\mathrm{car}}\mathcal{O}_1^{\mathrm{st}}$ by $F_r\mathcal{O}_1^{\mathrm{st}}$ (compare with [**32**, II.2.5]). This is one of the reasons why one has to deal with the sheaf $\mathcal{O}_1^{\mathrm{car}}$.

Note that by a de Rham computation, $R^i\alpha_*\Big(\dfrac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}}\Big) = 0$ if $i \geqslant r+1$. From (7.2.1) and (7.2.3), we get:

Corollary 7.2.6

1) (Generalization of "Deligne-Illusie") For $0 \leqslant r \leqslant p-2$, *the map* $\mathrm{Id}\otimes\phi_r$ *induces isomorphisms in the derived category of complexes of sheaves on* $X_{\text{ét}}$*:*

$$S_1 \otimes_{(\phi),k} R\alpha_* \frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}} \xrightarrow{\sim} \tau_{\leqslant r} R\alpha_* \mathcal{O}_1^{\mathrm{st}}.$$

2) For $0 \leqslant i \leqslant r \leqslant p-2$, *the map* $\mathrm{Id}\otimes\phi_r$ *induces isomorphisms:*

$$S_1 \otimes_{(\phi),k} H^i\Big(\frac{\mathcal{J}_1^{\mathrm{st},[r]}}{\mathcal{J}_1^{\mathrm{st},[r+1]}}\Big) \xrightarrow{\sim} H^i(\mathcal{O}_1^{\mathrm{st}}).$$

This already implies statement 1) of (7.1) in the case $n = 1$.

Remark 7.2.7. — In a different log-context, Kato gave another generalization of the DIFM-isomorphism (see [**43**, 4.12]).

7.3. Application

7.3.1. Thanks to (7.2.6), the statement 3) in (7.1) is now equivalent to the surjectivity of the map $H^i(\mathcal{J}_1^{\mathrm{st},[r]}) \to H^i(\mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]})$ for $0 \leqslant i \leqslant r \leqslant p-2$. In (5.1.2), we saw that we could easily get rid of the divided powers of S_1 when dealing with objects killed by p. Here is the cohomological counterpart: let $\widetilde{S}_1 = k[u]/u^p$ as in (5.1.2) and define a log-scheme $\widetilde{E}_1 = (\mathbf{N} \to \widetilde{S}_1, 1 \mapsto u)$. There are "stupid" divided powers on $\widetilde{S}_1$ given by $\gamma_i(u) = \frac{u^i}{i!}$ if $0 \leqslant i \leqslant p-1$ and $\gamma_i(u) = 0$ otherwise. The map $\Sigma_1 \hookrightarrow E_1$ factors through DP-thickenings $\Sigma_1 \hookrightarrow \widetilde{E}_1 \hookrightarrow E_1$ and we define presheaves $\widetilde{\mathcal{O}}_1^{\mathrm{st}}$ and $\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}$ on Σ_{syn} as before by $\widetilde{\mathcal{O}}_1^{\mathrm{st}}(U) = H^0((U_1/\widetilde{E}_1)_{\mathrm{cris}}, \mathcal{O}_{U_1/\widetilde{E}_1})$ and $\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}(U) = H^0((U_1/\widetilde{E}_1)_{\mathrm{cris}}, \mathcal{J}^{[r]}_{U_1/\widetilde{E}_1})$. We write $\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]} = \widetilde{\mathcal{O}}_1^{\mathrm{st}}$ if $r \leqslant 0$. As in (6.2.2.2), all these are sheaves and we have functorial isomorphisms $H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}) \simeq H^i((X_1/\widetilde{E}_1)_{\mathrm{cris}}, \mathcal{J}^{[r]}_{X_1/\widetilde{E}_1})$. The advantage of $\widetilde{E}_1$ is that the k-vector spaces $H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]})$ are now finite dimensional ([**10**, 2.2.6.1]). The functoriality of the crystalline topos gives natural maps of sheaves for $r \in \mathbf{Z}$: $\mathcal{J}_1^{\mathrm{st},[r]} \to \widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}$ which are surjective and induce isomorphisms for $0 \leqslant r+s \leqslant p$: $\mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+s]} \xrightarrow{\sim} \widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}/\widetilde{\mathcal{J}}_1^{\mathrm{st},[r+s]}$ ([**10**, 2.2.4.1]).

Lemma 7.3.1. — *Assume* $0 \leqslant i \leqslant r \leqslant p-2$.

1) The map $H^i(\mathcal{J}_1^{\mathrm{st},[r]}) \to H^i(\mathcal{O}_1^{\mathrm{st}})$ *is injective if and only if the map* $H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}) \to H^i(\widetilde{\mathcal{O}}_1^{\mathrm{st}})$ *is injective.*

2) The map $H^i(\mathcal{J}_1^{\mathrm{st},[r]}) \to H^i(\mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]})$ *is surjective if and only if the map* $H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}) \to H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}/\widetilde{\mathcal{J}}_1^{\mathrm{st},[r+1]})$ *is surjective.*

Proof. — 1) Diagram chase using the long exact sequences associated to:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}_1^{[r]} & \longrightarrow & \mathcal{O}_1^{\mathrm{st}} & \longrightarrow & \mathcal{O}_1^{\mathrm{st}}/\mathcal{J}_1^{\mathrm{st},[r]} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \widetilde{\mathcal{J}}_1^{[r]} & \longrightarrow & \widetilde{\mathcal{O}}_1^{\mathrm{st}} & \longrightarrow & \widetilde{\mathcal{O}}_1^{\mathrm{st}}/\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]} & \longrightarrow & 0\,. \end{array}$$

2) Diagram chase using the long exact sequences associated to:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}_1^{[r+1]} & \longrightarrow & \mathcal{J}_1^{\mathrm{st},[r]} & \longrightarrow & \mathcal{J}_1^{\mathrm{st},[r]}/\mathcal{J}_1^{\mathrm{st},[r+1]} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \widetilde{\mathcal{J}}_1^{\mathrm{st},[r+1]} & \longrightarrow & \widetilde{\mathcal{J}}_1^{\mathrm{st},[r]} & \longrightarrow & \widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}/\widetilde{\mathcal{J}}_1^{\mathrm{st},[r+1]} & \longrightarrow & 0\,. \end{array}$$

□

So to prove 2) and 3) in (7.1) for $n = 1$, it remains to prove the above two assertions in the "tilda" case. We won't give details here: the ingredients are a careful study of the long exact sequences associated to the short exact sequences:

$$0 \longrightarrow u^k\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]} \longrightarrow u^l\widetilde{\mathcal{J}}_1^{\mathrm{st},[s]} \longrightarrow u^l\widetilde{\mathcal{J}}_1^{\mathrm{st},[s]}/u^k\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]} \longrightarrow 0$$

where $0 \leqslant l \leqslant k \leqslant p-1$ and $l+s \leqslant k+r$, together with dimension arguments (which make sense now), the de Rham computation of log-crystalline cohomology and suitable variants of (7.2.6). The proofs are a bit technical and not very illuminating; for the details, we refer the reader to [**10**, 2.2.5-2.2.6].

Remark 7.3.2. — One can in fact define an object $\widetilde{\mathcal{M}} = (H^i(\widetilde{\mathcal{O}}_1^{\mathrm{st}}), H^i(\widetilde{\mathcal{J}}_1^{\mathrm{st},[r]}), \widetilde{\phi}_r, \widetilde{N})$ of the category $\underline{\widetilde{\mathcal{M}}}_k^r$ of (5.1.2) and show it corresponds to $\mathcal{M} = (H^i(\mathcal{O}_1^{\mathrm{st}}), H^i(\mathcal{J}_1^{\mathrm{st},[r]}), \phi_r, N)$ under the equivalence (5.1.2.2).

7.3.2. Finally, we deduce the result for any n from the result for $n = 1$. Using the flatness and the exact sequences of (6.2.2.4), we have long sequences for $i, r \in \mathbf{N}$:

$$\cdots \to H^{i-1}(\mathcal{J}_{n-1}^{\mathrm{st},[r]}) \to H^i(\mathcal{J}_1^{\mathrm{st},[r]}) \to H^i(\mathcal{J}_n^{\mathrm{st},[r]}) \to H^i(\mathcal{J}_{n-1}^{\mathrm{st},[r]}) \to H^{i+1}(\mathcal{J}_1^{\mathrm{st},[r]}) \to \cdots$$

Assume $0 \leqslant i \leqslant r \leqslant p-2$. By induction on n, we can assume that the data $(H^i(\mathcal{O}_{n-1}^{\mathrm{st}}), H^i(\mathcal{J}_{n-1}^{\mathrm{st},[r]}), \phi_r, N)$ is in $\underline{\mathcal{M}}^r$ (i.e. satisfies 1), 2) and 3) of (7.1)). As $\underline{\mathcal{M}}^r$ is abelian (5.1.1.2), we end up (using the case $n = 1$) with a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^r\mathcal{M}' & \longrightarrow & H^i(\mathcal{J}_n^{\mathrm{st},[r]}) & \longrightarrow & \mathrm{Fil}^r\mathcal{M}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}' & \longrightarrow & H^i(\mathcal{O}_n^{\mathrm{st}}) & \longrightarrow & \mathcal{M}'' & \longrightarrow & 0 \end{array}$$

where $\mathcal{M}', \mathcal{M}''$ are in $\underline{\mathcal{M}}^r$ and $p\mathcal{M}' = 0$, and where the two vertical maps on the right and on the left are injective (caution: one has to be a bit careful for $i = r$

since this case involves $H^{r+1}(\mathcal{J}_1^{\mathrm{st},[r]})$, see [**10**, 2.3.2]). Thus one has an injection $H^i(\mathcal{J}_n^{\mathrm{st},[r]}) \hookrightarrow H^i(\mathcal{O}_n^{\mathrm{st}})$.

Lemma 7.3.3. — *Let $\mathcal{M}$ be an S-module satisfying all the conditions of (5.1.1) EXCEPT perhaps the two conditions "$\mathcal{M} \simeq \oplus_{i\in I}(S/p^iS)^{d_i}$" and "$\phi_r(\mathrm{Fil}^r\,\mathcal{M})$ generates $\mathcal{M}$". Assume we have an exact sequence of S-modules: $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$, with $\mathcal{M}', \mathcal{M}'' \in \underline{\mathcal{M}}^r$, inducing an exact sequence on the* Fil^r *and commuting with ϕ_r and N. Then $\mathcal{M}$ is in $\underline{\mathcal{M}}^r$, i.e. the two above conditions are automatically satisfied.*

The first condition is the hardest, see [**10**, 2.3.1.2]. Applying this lemma to $\mathcal{M} = (H^i(\mathcal{O}_n^{\mathrm{st}}), H^i(\mathcal{J}_n^{\mathrm{st},[r]}), \phi_r, N))$ and $\mathcal{M}'$, $\mathcal{M}''$ as previously, we finally obtain as a conclusion:

Theorem 7.3.4. — *Let X be a fine and proper log-scheme which is log-smooth over Σ and such that X_1/Σ_1 is of Cartier type. For $n \in \mathbf{N}$ and $0 \leqslant i \leqslant r \leqslant p-2$, the data:*

$$\Big(H^i(X_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}}), H^i(X_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]}), \phi_r, N\Big)$$

define an object of the category $\underline{\mathcal{M}}^r$.

8. The log-syntomic cohomology

We keep the same notations as in (7) but we now assume X/Σ is semi-stable (and proper) as in (2.2.1.2). In that case, the geometric generic fiber $X \times_W K_0$ has a trivial log-structure and is (classically) smooth over Spec(K_0). We also fix two integers i, r such that $0 \leqslant i \leqslant r \leqslant p-2$. Now that we know $(H^i(X_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}}), H^i(X_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]}), \phi_r, N)$ is in $\underline{\mathcal{M}}^r$, we can compute its associated representation of G_{K_0} as in (5.2.2) using T_{st}^*, or rather its dual version T_{st}. Our aim is to prove this representation is isomorphic to $H^i((X \times_W \overline{K})_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z})$ as in the smooth case. The main tool for this is an intermediate cohomology called the "log-syntomic" cohomology (a log-analogue of the cohomology mentioned in (3.2)) that we introduce now.

8.1. For $n \in \mathbf{N}$, define $\mathcal{S}_n^r = \mathrm{Ker}(\phi_r - \mathrm{Id} : \mathcal{J}_n^{\mathrm{cris},[r]} \to \mathcal{O}_n^{\mathrm{cris}})$ where Id is the natural injection $\mathcal{J}_n^{\mathrm{cris},[r]} \hookrightarrow \mathcal{O}_n^{\mathrm{cris}}$.

Proposition 8.1.1. — *There are exact sequences of sheaves on* Spec$(W)_{\mathrm{syn}}$*:*

$$0 \longrightarrow \mathcal{S}_n^r \longrightarrow \mathcal{J}_n^{\mathrm{cris},[r]} \xrightarrow{\phi_r - \mathrm{Id}} \mathcal{O}_n^{\mathrm{cris}} \longrightarrow 0.$$

One has to prove the surjectivity. By flatness (6.2.2.4), one is easily reduced to the case $n = 1$, but it should be noticed that here the proof *can't* be reduced to the case (A_∞, M_∞) of (6.1.2.1), i.e. the map $\mathcal{J}_1^{\mathrm{cris},[r]}(A_\infty, M_\infty) \xrightarrow{\phi_r - \mathrm{Id}} \mathcal{O}_1^{\mathrm{cris}}(A_\infty, M_\infty)$ is *not* surjective in general. One has to use other log-syntomic coverings than just those of (6.1.2.1), namely coverings of the form $A \to A[X]/(X^p - aX - b)$ with induced log-structure $(a, b \in A)$. See [**10**, 3.1.4].

For L a finite extension of K_0 in $\overline{K}$, denote by Σ_L the log-scheme $\mathcal{O}_L \setminus \{0\} \to \mathcal{O}_L$ and $X_{\Sigma_L} = X \times_\Sigma \Sigma_L$ (fiber product in the category of fine log-schemes or all log-schemes), one checks $X_{\Sigma_L} \to X$ is log-syntomic, so X_{Σ_L} is in Σ_{syn}.

Definition 8.1.2. — We define the torsion log-syntomic cohomology of X (resp. the absolute torsion log-syntomic cohomology of X) to be the groups $\varinjlim H^i((X_{\Sigma_L})_{\mathrm{syn}}, \mathcal{S}_n^r)$ (resp. $H^i(X_{\mathrm{syn}}, \mathcal{S}_n^r)$) where the direct limit is taken over the finite extensions L of K_0 in $\overline{K}$.

For brevity, we write $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r)$ instead of $\varinjlim H^i((X_{\Sigma_L})_{\mathrm{syn}}, \mathcal{S}_n^r)$. These last groups are endowed with a natural action of G_{K_0} and all the groups of (8.1.2) can be computed on the étale site of the special fiber just because $H^i((X_{\Sigma_L})_{\mathrm{syn}}, \mathcal{S}_n^r) = H^i((X_{\Sigma_L})_{\text{ét}}, R\alpha_*\mathcal{S}_n^r)$ where α_* is as in (7.2). One wants to relate these groups to the étale cohomology of the geometric generic fiber. In the smooth case, this was done in two steps. First, in [**6**], Bloch-Kato(-Gabber) computed the sheaves of nearby cycles $i^*R^qj_*\mathbf{Z}/p^n\mathbf{Z}(q)$ for $0 \leqslant q \leqslant p-2$ where $i : X \times_W k \hookrightarrow X$ and $j : X \times_W K_0 \hookrightarrow X$. Secondly, in [**45**] and [**47**], Kato and Kurihara related these computations to the sheaves $\mathcal{S}_n^r$ of [**32**]. The computations of [**6**] have been extended by Hyodo to the semi-stable case ([**37**]) and in [**69**], Tsuji finally used Hyodo's computations to generalize Kato's results to the above $\mathcal{S}_n^r$. All these computations work in fact over any finite extension L of K_0. As a consequence:

Theorem 8.1.3 (**[44**, 5.5], [**69**]). — *Let X be a proper semi-stable scheme over W and endow it with its canonical log-structure (2.2.1.2). For $n \in \mathbf{N}$ and $0 \leqslant i \leqslant r \leqslant p-2$, there are canonical isomorphisms:*

$$H^i(X_{\mathrm{syn}}, \mathcal{S}_n^r) \xrightarrow{\sim} H^i((X \times_W K_0)_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z}(r))$$
$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r) \xrightarrow{\sim} H^i((X \times_W \overline{K})_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z}(r)).$$

The second isomorphism is compatible with the actions of G_{K_0}.

Remark 8.1.4. — Although we won't need the isomorphism for the $H^i(X_{\mathrm{syn}}, \mathcal{S}_n^r)$ in the sequel, it should be noticed that if the log-structure on X is induced from the one on Σ (i.e. X is proper smooth over W), the groups $H^i(X_{\mathrm{syn}}, \mathcal{S}_n^r)$ are *not always* equal to the syntomic cohomology groups defined by Fontaine-Messing forgetting the log-structures: compare (8.1.3) with the main theorem of [**47**]. This difference, however, disappears when one looks at $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r)$.

8.2. As in (3.2), we have now to relate $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r)$ to $T_{\mathrm{st}}(H^i(\mathcal{O}_n^{\mathrm{st}}), H^i(\mathcal{J}_n^{\mathrm{st},[r]}), \phi_r, N)$ (recall our notation $H^i(\mathcal{F}) = H^i(X_{\mathrm{syn}}, \mathcal{F})$). As for $\mathcal{S}_n^r$, define for any sheaf $\mathcal{F}$ on Σ_{syn}: $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{F}) = \varinjlim H^i((X_{\Sigma_L})_{\mathrm{syn}}, \mathcal{F})$. Recall from (5.2.2) that we have chosen an isomorphism between $\widehat{A_{\mathrm{st}}}$ and the p-adic completion of $A_{\mathrm{cris}}\langle X\rangle$ (do not confuse this X with the log-scheme X!). We first relate $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]})$ to the groups $H^i(\mathcal{J}_n^{\mathrm{st},[*]})$ by the following Künneth formula (see [**10**, 3.2.2]):

Lemma 8.2.1. — *For $s \in \mathbf{N}$ let* $\mathrm{Fil}^s_X \widehat{A_{\mathrm{st}}} = \{\sum_{i=s}^m a_i \frac{X^i}{i!}, a_i \in A_{\mathrm{cris}}, m \in \mathbf{N}\} \subset \mathrm{Fil}^s \widehat{A_{\mathrm{st}}}$. *There are short exact sequences:*

$$0 \longrightarrow \bigoplus_{s=1}^{r} \mathrm{Fil}^s_X \widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{J}_n^{\mathrm{st},[r+1-s]}) \longrightarrow \bigoplus_{s=0}^{r} \mathrm{Fil}^s_X \widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{J}_n^{\mathrm{st},[r-s]}) \longrightarrow H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]}) \longrightarrow 0.$$

The reason we use $\mathrm{Fil}^s_X \widehat{A_{\mathrm{st}}}$ is because it is a flat S-module (which is not the case of $\mathrm{Fil}^s \widehat{A_{\mathrm{st}}}$). One should notice that this lemma is *exactly* the place where the ring $\widehat{A_{\mathrm{st}}}$ appears.

Define:

$$\mathrm{Fil}^r_X(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}})) = \sum_{s=0}^{r} \mathrm{Fil}^s_X \widehat{A_{\mathrm{st}}} \otimes_S \mathrm{Im}\Big(H^i(\mathcal{J}_n^{\mathrm{st},[r-s]}) \to H^i(\mathcal{O}_n^{\mathrm{st}})\Big) \subset \widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}).$$

The operators N on $\widehat{A_{\mathrm{st}}}$ and $H^i(\mathcal{O}_n^{\mathrm{st}})$ give an operator $N = N \otimes \mathrm{Id} + \mathrm{Id} \otimes N$ on $\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}})$.

Proposition 8.2.2. — *The short sequences of (8.2.1) induce isomorphisms:*

$$(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}))_{N=0} \xrightarrow{\sim} H^i(\overline{X}_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}})_{N=0}$$
$$\mathrm{Fil}^r_X(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}))_{N=0} \xrightarrow{\sim} H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]})_{N=0}$$

where "$N = 0$" means "kernel of N" (in particular we have that $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]})_{N=0} \hookrightarrow H^i(\overline{X}_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}})_{N=0}$).

The first isomorphism is a consequence of (8.2.1) with $r = 0$. The second is derived from a careful study of the action of N on the exact sequences in (8.2.1) together with the fact that the maps $H^i(\mathcal{J}_n^{\mathrm{st},[s]}) \to H^i(\mathcal{O}_n^{\mathrm{st}})$ coming from the natural injections of sheaves are injective for $s = 0$ (trivial) and for $s = r$ (7.3.4). For proofs, see [**10**, 3.2.3.2-3.2.3.4].

Recall that in (5.2.2), we have defined $\mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})$ for any object $\mathcal{M}$ of $\underline{\mathcal{M}}^r$ (and so using only $\mathrm{Fil}^r \mathcal{M}$). Let $\mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})_{N=0} = \{x \in \mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M}) \mid N(x) = 0\}$. Fortunately, we have ([**10**, 3.2.1.4 and 3.2.1.2]):

Lemma 8.2.3. — *There are isomorphisms:*

$$\mathrm{Fil}^r_X(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}))_{N=0} \xrightarrow{\sim} \mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}))_{N=0}$$

where the right hand side is defined by viewing $H^i(\mathcal{O}_n^{\mathrm{st}})$ as an object of $\underline{\mathcal{M}}^r$ (7.3.4).

Now we want to make more explicit the groups:

$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}})_{N=0} \quad \text{and} \quad H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]})_{N=0}.$$

Recall there is an exact sequence (6.2.3.2):

$$0 \longrightarrow \mathcal{J}_n^{\mathrm{cris},[r]} \longrightarrow \mathcal{J}_n^{\mathrm{st},[r]} \xrightarrow{N} \mathcal{J}_n^{\mathrm{st},[r-1]} \longrightarrow 0.$$

Combining its associated long exact sequences with (8.2.1), (7.3.4) and a dévissage in the category $\underline{\mathcal{M}}^r$, we obtain:

Lemma 8.2.4 ([**10**, 3.2.3.1]). — *The long cohomology sequences associated to the short exact sequence of (6.2.3.2) yield isomorphisms:*

$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{cris}}) \xrightarrow{\sim} H^i(\overline{X}_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}})_{N=0}$$
$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{cris},[r]}) \xrightarrow{\sim} H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]})_{N=0}.$$

Taking the kernel of $\phi_r - \mathrm{Id}$ on both sides of the second isomorphism of (8.2.2) and using (8.2.4) and (8.2.3), we see that what remains to prove, in order to relate $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^{\mathrm{cris},[r]})$ to $T_{\mathrm{st}}(H^i(\mathcal{O}_n^{\mathrm{st}}))$, is:

Proposition 8.2.5. — *The long cohomology sequences associated to the short exact sequence of (8.1.1) yield isomorphisms: $H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r) \xrightarrow{\sim} H^i(\overline{X}_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{cris},[r]})^{\phi_r=1}$ where the exponent on the right hand side means "kernel of $\phi_r - \mathrm{Id}$".*

Proof. — Take the direct limit over L on the long exact sequences associated to (8.1.1) and use (8.2.4), (8.2.2) and (8.2.3) together with the surjectivity of $\phi_r - \mathrm{Id}$: $\mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})_{N=0} \to (\widehat{A_{\mathrm{st}}} \otimes_S \mathcal{M})_{N=0}$ for any object $\mathcal{M}$ of $\underline{\mathcal{M}}^r$. See [**10**, 3.2.4.4] for more details. □

To sum up, the theory of section 5 together with (7.3.4) and the above results finally furnish Galois equivariant isomorphisms:

$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r) \xrightarrow{\sim} \mathrm{Fil}^r(\widehat{A_{\mathrm{st}}} \otimes_S H^i(\mathcal{O}_n^{\mathrm{st}}))_{N=0}^{\phi_r=1} = T_{\mathrm{st}}(H^i(\mathcal{O}_n^{\mathrm{st}}))(r).$$

Hyodo-Kato-Tsuji's theory of nearby cycles in the semi-stable reduction case also furnishes Galois equivariant isomorphisms (8.1.3):

$$H^i(\overline{X}_{\mathrm{syn}}, \mathcal{S}_n^r) \xrightarrow{\sim} H^i((X \times_W \overline{K})_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z})(r).$$

In conclusion:

Theorem 8.2.6. — *Let X be a proper semi-stable scheme over W and endow it with its canonical log-structure (2.2.1.2). For $n \in \mathbf{N}$ and $0 \leqslant i \leqslant r \leqslant p-2$, there are isomorphisms compatible with the action of G_{K_0}:*

$$T_{\mathrm{st}}(H^i(X_{\mathrm{syn}}, \mathcal{O}_n^{\mathrm{st}}), H^i(X_{\mathrm{syn}}, \mathcal{J}_n^{\mathrm{st},[r]}), \phi_r, N) \simeq H^i((X \times_W \overline{K})_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z}).$$

9. Applications and open problems

We give four applications and suggest four open questions.

9.1. Applications

9.1.1. If V is a p-adic semi-stable representation of G_{K_0} with negative Hodge-Tate weights, one can show $(\widehat{A_{\rm st}} \otimes_{\mathbf{Z}_p} V)^{G_{K_0}}$ is in a natural way an object of the category $\underline{\mathcal{MF}}^+_{S_{K_0}}(\phi, N)$ of (4.2) and that its associated D given by the equivalence (4.2.1) is a weakly admissible filtered module (4.1). In fact, this D is nothing else than $(B^+_{\rm st} \otimes_{\mathbf{Q}_p} V)^{G_{K_0}}$ by [**8**, 8.2] (here $B^+_{\rm st} = B^+_{\rm cris}[v]$, cf. introduction of section 4). Fontaine conjectured in [**25**, 5.4.4] that the above functor $V \mapsto D$ is an equivalence of categories between semi-stable representations of G_{K_0} with negative Hodge-Tate weights (or positive if one dualizes) and weakly admissible filtered (ϕ, N)-modules D such that $\mathrm{Fil}^0 D = D$. This conjecture has recently been proven by him and Colmez in [**14**]. However, their result doesn't yield information about the lattices. If D is an object of $\underline{MF}^+_{K_0}(\phi, N)$ such that $\mathrm{Fil}^{p-1} D = 0$ and $\mathcal{D} = S_{K_0} \otimes_{K_0} D$ the corresponding object of $\underline{\mathcal{MF}}^+_{S_{K_0}}(\phi, N)$ given by (4.2.1), define a *strongly divisible lattice* in $\mathcal{D}$ to be any finitely generated free sub-S-module $\mathcal{M}$ of $\mathcal{D}$ stable by ϕ, N such that $\mathcal{M}[1/p] = \mathcal{D}$ and $\phi(\mathcal{M} \cap \mathrm{Fil}^{p-2} \mathcal{D})$ generates $p^{p-2}\mathcal{M}$ over S. The theory of section 5 gives only a small piece of the Colmez-Fontaine theorem, but describes the lattices:

Theorem 9.1.1.1
1) There is an anti-equivalence of categories between weakly admissible filtered (ϕ, N)-modules D such that $\mathrm{Fil}^0 D = D$ *and* $\mathrm{Fil}^{p-1} D = 0$ *and semi-stable representations V of G_{K_0} with Hodge-Tate weights between 0 and $p-2$.*
2) There is an anti-equivalence of categories between strongly divisible lattices in $S_{K_0} \otimes_{K_0} D$ for a given D as in 1) and Galois stable lattices in the corresponding V.

See [**11**] for a proof of 1) and [**12**] for a proof of 2).

9.1.2. The second application is of course the recovery, in the situation we consider, of the "usual" comparison theorem with $\mathbf{Q}_p$-coefficients as conjectured by Fontaine-Jannsen ([**44**, 1.1]). We won't insist on this topic because our main interest here is not $\mathbf{Q}_p$-coefficients and because there are now different proofs of this $\mathbf{Q}_p$-comparison theorem in its full generality ([**70**], [**71**], [**22**]). So let us just describe the main steps.
1) Fix X/Σ proper semi-stable and $i \in \{0, \dots, p-2\}$. Let $\mathcal{D} = \mathbf{Q}_p \otimes \varprojlim H^i(X_{\rm syn}, \mathcal{O}^{\rm st}_n)$ and $\mathrm{Fil}^j \mathcal{D} = \mathbf{Q}_p \otimes \varprojlim H^i(X_{\rm syn}, \mathcal{J}^{{\rm st},[j]}_n)$ for $j \in \mathbf{Z}$. Then $(\mathcal{D}, (\mathrm{Fil}^j \mathcal{D})_j, \phi, N)$ is an object of $\underline{\mathcal{MF}}^+_{S_{K_0}}(\phi, N)$ and its associated filtered D given by (4.2.1) can be identified with $H^i_{\rm dR}(X \times_W K_0)$ endowed with the Hodge filtration (cf. [**10**, 4.3.2], an important argument here is due to Kato ([**44**, 6.4.2])).
2) There are canonical Galois equivariant isomorphisms (with obvious notations):
$\mathrm{Fil}^i(\widehat{A_{\rm st}} \otimes_S \mathcal{D})^{\phi=p^i}_{N=0} \simeq \mathbf{Q}_p \otimes \varprojlim T_{\rm st}(H^i(X_{\rm syn}, \mathcal{O}^{\rm st}_n))$ ([**10**, 4.3.2.2] with $r = i$)
$\mathrm{Fil}^i(\widehat{A_{\rm st}} \otimes_S \mathcal{D})^{\phi=p^i}_{N=0} \simeq \mathrm{Fil}^i(B_{\rm st} \otimes_{K_0} H^i_{\rm dR}(X \times_W K_0))^{\phi=p^i}_{N=0}$ ([**8**, 8.1.2]).
3) By (8.2.6), we thus obtain a Galois equivariant isomorphism:

$$\mathrm{Fil}^i(B_{\rm st} \otimes_{K_0} H^i_{\rm dR}(X \times_W K_0))^{\phi=p^i}_{N=0}(-i) \simeq \mathbf{Q}_p \otimes \varprojlim H^i((X \times_W \overline{K})_{\rm \acute{e}t}, \mathbf{Z}/p^n\mathbf{Z})$$

which we can rewrite $B_{\rm st} \otimes_{K_0} H^i_{\rm dR}(X \times_W K_0) \simeq B_{\rm st} \otimes_{\mathbf{Q}_p} H^i_{\rm \acute{e}t}(X \times_W \overline{K}, \mathbf{Q}_p)$.

9.1.3. The third application concerns the invariant factors in the torsion of the étale cohomology. Fix X/Σ proper semi-stable and $i \in \{0,\dots,p-2\}$. Let $H^i_{\mathrm{st}}(X) = \varprojlim H^i(X_{\mathrm{syn}}, \mathcal{O}^{\mathrm{st}}_n)$ and $H^i_{\mathrm{\acute{e}t}}(X \times_W \overline{K}, \mathbf{Z}_p) = \varprojlim H^i((X \times_W \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/p^n\mathbf{Z})$. Using the previous results, one has $H^i_{\mathrm{st}}(X) \simeq S^d \oplus \oplus_{i\in I}(S/p^iS)^{d_i}$ and $H^i_{\mathrm{\acute{e}t}}(X \times_W \overline{K}, \mathbf{Z}_p) \simeq \mathbf{Z}_p^d \oplus \oplus_{i\in I}(\mathbf{Z}/p^i\mathbf{Z})^{d_i}$ (same d and d_i, see [**10**, 4.1-4.2]). Let Σ^{HK} be the log-scheme associated to $(\mathbf{N} \mapsto W, 1 \mapsto 0)$. Notice that $\Sigma_1 = \Sigma_1^{HK}$ so one has morphisms $X_1 \to \Sigma_1^{HK} \hookrightarrow \Sigma_n^{HK}$ (recall our notation $U_n = U \times_W W_n$ with the induced log-structure). Let $H^i_{HK}(X) = \varprojlim H^i((X_1/\Sigma_n^{HK})_{\mathrm{cris}}, \mathcal{O}_{X_1/\Sigma_n^{HK}})$ (HK for Hyodo-Kato: this cohomology is of high importance in [**38**]) and $H^i_{\mathrm{dR-log}}(X)$ be the de Rham cohomology of X with logarithmic poles at the singular locus ([**37**, 1.5], [**38**, 2.5], it coincides with the classical de Rham cohomology when X/W is smooth), one can show:

$$H^i_{\mathrm{dR-log}}(X) \simeq \varprojlim H^i_{\mathrm{dR-log}}(X_n) \simeq \varprojlim H^i((X_n/\Sigma_n)_{\mathrm{cris}}, \mathcal{O}_{X_n/\Sigma_n})$$

(the first isomorphism is a consequence of (EGA III.3.2.3+4.1.7) and the fact the E^1-terms in the spectral sequence Hodge $\Rightarrow$ de Rham (log version) satisfy in that case the Mittag-Leffler conditions, the second isomorphism is an application of [**43**, 6.4]). Using (7.3.4), one proves ([**10**, 4.3.1.3]):

$$H^i_{\mathrm{st}}(X) \otimes_{S,f_0} W \xrightarrow{\sim} H^i_{HK}(X) \quad \text{and} \quad H^i_{\mathrm{st}}(X) \otimes_{S,f_p} W \xrightarrow{\sim} H^i_{\mathrm{dR-log}}(X)$$

where $f_0 : S \to W$, $\sum w_i u^i/i! \mapsto w_0$ (i.e. $u \mapsto 0$) and f_p is as in (4.2) (i.e. $u \mapsto p$). We sum up:

Theorem 9.1.3.1 ([**10**, 4.3.1.5]). — *Let X be a proper semi-stable scheme over W. For $i \in \{0,\dots,p-2\}$, the invariant factors of $H^i_{\mathrm{\acute{e}t}}(X \times_W \overline{K}, \mathbf{Z}_p)$, $H^i_{HK}(X)$ and $H^i_{\mathrm{dR-log}}(X)$ coincide.*

9.1.4. Let $h \in \mathbf{N} \setminus \{0\}$, $\pi \in \mathcal{O}_{\overline{K}}$ such that $\pi^{p^h-1} = p$ and $\theta_h : I_{K_0} \to \mu_{p^h-1}(\mathcal{O}_{\overline{K}})$, $g \mapsto g(\pi)/\pi$ where I_{K_0} is the inertia subgroup of G_{K_0} (θ_h is Serre's fundamental character of level h, see [**64**, 1.7]). As a last application, we get:

Theorem 9.1.4.1 ([**10**, 3.2.5.1]). — *Let X be a proper semi-stable scheme over W and fix $n \in \mathbf{N}$, $i \in \mathbf{N}$. Let T be either $H^i((X \times_W \overline{K})_{\mathrm{\acute{e}t}}, \mathbf{Z}/p^n\mathbf{Z})$ or a G_{K_0}-stable lattice in $H^i_{\mathrm{\acute{e}t}}(X \times_W \overline{K}, \mathbf{Q}_p)$ and $\widetilde{T}$ the semi-simplification of the G_{K_0}-module T/pT. Then the action of I_{K_0} (through its tame quotient) on $\widetilde{T}$ is given by characters of the form $\theta_h^{-(i_0+pi_1+\cdots+p^{h-1}i_{h-1})}$ with $0 \leqslant i_j \leqslant i$.*

This is essentially derived from [**31**, 5.3], (5.1.1.3) and (8.2.6)+(5.2.2.4) (and is of course automatic if $i \geqslant p-1$). This theorem answers part of a question of Serre ([**64**, 1.13]) and still holds if T is replaced by any Galois stable lattice in any semi-stable representation of G_{K_0} with Hodge-Tate weights between $-i$ and 0 ([**11**, 1.2]).

9.2. Open problems. — As we mentioned at the end of (2.2.2), the first open problem is of course to remove the restrictions $K = K_0$ and $i < p-1$ (and we have already mentioned that there should be a good theory for $i[K : K_0] < p-1$, see appendix of [**11**]). As this doesn't seem to be an easy task in the general case, we only suggest in the sequel questions that we view as interesting even *under* the assumptions $K = K_0$ and $i \leqslant p-2$.

9.2.1. In [**19**], Faltings extended the Fontaine-Laffaille-Messing theory to a much more general situation: he allowed non constant coefficients and treated the case of open varieties over W with smooth normal crossings compactifications (he also treated the relative case). The generalized torsion comparison theorems he obtained could be applied for instance in [**42**] and [**23**] to the study of Galois representations modulo p arising from eigenforms of weight k on $\Gamma_1(N)$ with $(N,p)=1$ and $p>k$. Following Faltings, is it possible to extend the previous theory to deal with non constant coefficients and open varieties over W with "good" compactifications? This could be useful for the study of Galois representations modulo p arising from eigenforms of weight k on $\Gamma_1(pN)$ with $(N,p)=1$, $p>k$ and Dirichlet character of conductor dividing N. Of course, if one wants to follow the "syntomic" method, this would also mean extending to these situations the computations of [**69**] (actually, in *loc.cit.*, some categories of non constant coefficients are already considered).

9.2.2. The finite representations of G_{K_0} built in (3.1.3) and (5.2.2) via the categories $\underline{MF}^{f,r}_{\mathrm{tor}}$ and $\underline{\mathcal{M}}^r$ are in general wildly ramified. There are several (related) ways to measure this wild ramification. One is to compute the maximal power of p that divides the different $\mathcal{D}_{F/K_0}$ where F is the finite Galois extension of K_0 cut out by the corresponding finite representation. Another is to determine which higher ramification subgroups of the inertia $I_{K_0} = G^0_{K_0}$ have non trivial image in $\mathrm{Gal}(F/K_0)$. For the objects of $\underline{MF}^{f,r}_{\mathrm{tor}}$, this was done by Abrashkin and Fontaine:

Theorem 9.2.2.1 **([26], [28], [1]).** — *Let $n \in \mathbf{N}$, $r \in \{0,\dots,p-2\}$, M an object of $\underline{MF}^{f,r}_{\mathrm{tor}}$ such that $p^nM = 0$ and F the finite Galois extension of K_0 cut out by the finite representation $T^*_{\mathrm{cris}}(M)$. Then:*
1) $val_p(\mathcal{D}_{F/K_0}) < n + \frac{r}{p-1}$
2) if $\nu > n-1+\frac{r}{p-1}$, then $\mathrm{Gal}(F/K_0)^\nu = \{1\}$.

Here val_p is the p-adic valuation normalized by $\mathrm{val}_p(p)=1$ and $\mathrm{Gal}(F/K_0)^\nu$ is the upper numbering as in [**66**, IV.3]. Using the Fontaine-Messing results of (3.2), this implies $G^\nu_{K_0}$ acts trivially on $H^i((X\times_W \overline{K})_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z})$ (and any subquotient killed by p^n of $H^i_{\text{ét}}(X\times_W \overline{K}, \mathbf{Q}_p)$) whenever $\nu > n-1+\frac{i}{p-1}$ if X is proper smooth over W and $i\in\{0,\dots,p-2\}$. What is this lower bound if X is only proper semi-stable over W? More generally, what is the bound for the representations coming from $\underline{\mathcal{M}}^r$? In [**33**], Gross suggests an upper bound for $\mathrm{val}_p(\mathcal{D}_{F/K_0})$ in a special case of some modulo p ordinary representations of $\mathrm{Gal}(F/\mathbf{Q}_p)$. One can show using (9.2.2.1) that Gross' bound is actually valid for *naive* objects (see 4.1) killed by p (which do not necessarily correspond to ordinary representations):

Proposition 9.2.2.2 **([13]).** — *Let $r \in \{0, \ldots, p-2\}$ and M an object of $\underline{MF}_{\mathrm{tor}}^{f,r}$ killed by p and endowed with a linear endomorphism N such that $N(\mathrm{Fil}^i M) \subset \mathrm{Fil}^{i-1} M$ and $N\phi_i = \phi_{i-1} N$ for any i. Let F be the finite Galois extension of K_0 cut out by the finite representation $T^*_{\mathrm{st}}(S_1 \otimes_k M)$ where $S_1 \otimes_k M$ is viewed as an object of $\underline{\mathcal{M}}^r$ in the obvious way. Then:*
1) $val_p(\mathcal{D}_{F/K_0}) < 2 + \frac{r}{p(p-1)}$
2) if $\nu > 1 + \frac{1}{p-1}$, then $\mathrm{Gal}(F/K_0)^{\nu} = \{1\}$.

Of course, the bounds here are not as good as in (9.2.2.1) with $n = 1$ (which corresponds to the case $N = 0$). However, the bound in 2) is optimal as can be easily seen by looking at $F = K_0(\mu_p, p^{1/p})$. What are the bounds that work for any object $\mathcal{M}$ of $\underline{\mathcal{M}}^r$ such that $p\mathcal{M} = 0$, or more generally such that $p^n\mathcal{M} = 0$?

9.2.3. We assume here $K_0 = \mathbf{Q}_p$. Fix a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$, and a continuous (hence finite) representation:

$$\overline{\rho} : G_{\mathbf{Q}_p} \to GL_2(\mathbf{F})$$

such that $\mathrm{End}_{\mathbf{F}[G_{\mathbf{Q}_p}]}(\overline{\rho}) = \mathbf{F}$. It is known that the continuous deformations $\rho : G_{\mathbf{Q}_p} \to GL_2(A)$ of $\overline{\rho}$ in the sense of Mazur ([**52**, 1.1]) where A is a local noetherian complete $\mathcal{O}$-algebra are parametrized by a local noetherian complete $\mathcal{O}$-algebra $R_{\overline{\rho},\mathcal{O}}$ of residue field $\mathbf{F}$ ([**52**], [**68**]). Suppose that $\overline{\rho}$, viewed as a $\mathbf{F}_p$-representation of $G_{\mathbf{Q}_p}$, is in the essential image of $\underline{MF}_{\mathrm{tor}}^{f,r}$ via the functor T^*_{cris} of (3.1.3) for an $r \in \{0, \ldots, p-2\}$ and consider only those deformations ρ such that, for each n, $\rho_n : G_{\mathbf{Q}_p} \to GL_2(A) \to GL_2(A/\mathfrak{m}_A^n)$ comes from $\underline{MF}_{\mathrm{tor}}^{f,r}$ (via T^*_{cris}) when viewed as a $\mathbf{Z}/p^n\mathbf{Z}$-representation. These deformations are parametrized by a quotient $R_{\overline{\rho},\mathcal{O}}(\underline{MF}_{\mathrm{tor}}^{f,r})$ of $R_{\overline{\rho},\mathcal{O}}$ which turns out to be isomorphic to a power series ring $\mathcal{O}[[T_1, T_2]]$ ([**61**, 5.1]). Since any representation of $G_{\mathbf{Q}_p}$ coming from $\underline{MF}_{\mathrm{tor}}^{f,r}$ can be lifted as a (Galois stable) lattice of a crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights in $\{0, \ldots, r\}$ (3.1.3.2), and since any such lattice comes from a strongly divisible module (this is an easy consequence of the Fontaine-Laffaille theory, see [**12**, 2.1]), $R_{\overline{\rho},\mathcal{O}}(\underline{MF}_{\mathrm{tor}}^{f,r})$ is also isomorphic to $R_{\overline{\rho},\mathcal{O}}/(\cap\mathfrak{p})$, the intersection being over all prime ideals $\mathfrak{p}$ of $R_{\overline{\rho},\mathcal{O}}$ such that $G_{\mathbf{Q}_p} \to GL_2(R_{\overline{\rho},\mathcal{O}}) \to GL_2(R_{\overline{\rho},\mathcal{O}}/\mathfrak{p})$ is the representation corresponding to a lattice in a crystalline representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $\in \{0, \ldots, r\}$.

The question is: what happens if we replace $\underline{MF}_{\mathrm{tor}}^{f,r}$ by $\underline{\mathcal{M}}^r$ and the word crystalline by the word semi-stable? Of course, in that case, the two analogous quotients of $R_{\overline{\rho},\mathcal{O}}$ are not isomorphic in general since an object of $\underline{\mathcal{M}}^r$ cannot always be lifted as a strongly divisible module of weight $\leqslant r$. However, since any Galois stable lattice in a semi-stable representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights between 0 and r comes from such a strongly divisible module (see 9.1.1.1), one has a canonical surjection $R_{\overline{\rho},\mathcal{O}}(\underline{\mathcal{M}}^r) \to R_{\overline{\rho},\mathcal{O}}/(\cap\mathfrak{p})$ where $R_{\overline{\rho},\mathcal{O}}(\underline{\mathcal{M}}^r)$ is the quotient parametrizing all liftings in

$\underline{\mathcal{M}}^r$ and $\mathfrak{p}$ is such that $G_{\mathbf{Q}_p} \to GL_2(R_{\overline{\rho},\mathcal{O}}/\mathfrak{p})$ is a representation corresponding to a lattice in a semi-stable representation of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights in $\{0,\dots,r\}$. Can one describe these two rings and the kernel of the surjection (which contains the torsion part of $R_{\overline{\rho},\mathcal{O}}(\underline{\mathcal{M}}^r)$) in terms of generators and relations? For instance, the minimal number of generators of $R_{\overline{\rho},\mathcal{O}}(\underline{\mathcal{M}}^r)$ should be obtained by computing extension groups (Ext^1) in the abelian category $\underline{\mathcal{M}}^r$ and a possible description of $R_{\overline{\rho},\mathcal{O}}/(\cap\mathfrak{p})$ (when non zero) could be obtained by looking for suitable families of strongly divisible lattices. This is related to the computations of [**9**, 6] and to the next, and last, question.

9.2.4. Let V be a Hodge-Tate representation of G_{K_0} with Hodge-Tate weights between 0 and $p-2$. By definition, the tame inertia weights on the semi-simplification of the reduction modulo p of V are also between 0 and $p-2$ (see 9.1.4.1). If V is crystalline, by using [**31**, 5.3] together with the fact that the morphisms in $\underline{MF}_{\mathrm{tor}}^{f,p-2}$ are strict with respect to the filtration (3.1.1.1), one gets these two lists of figures are the same. If V is semi-stable, this is not always true anymore, as was first shown by Ribet using an example coming from modular forms (see the correction to [**20**]). In [**9**, 6.1.1.2], using the categories $\underline{\mathcal{M}}^r$, the difference between the two lists is computed for all 2-dimensional (over $\mathbf{Q}_p$) semi-stable representations of G_{K_0} with the above restriction on the Hodge-Tate weights and involves a number $\mathcal{L}(V)$ which only exists when V is semi-stable non crystalline. Is there a general statement which would allow the comparison of the two lists in any dimension? Can one build some kind of polygon out of the tame inertia weights, and compare it with the usual Newton and Hodge polygons (so, in the crystalline case, that polygon would just be the Hodge polygon)? It was noticed by many people (Ulmer, Mazur, Conrad, Diamond, Taylor, ...) that similar phenomena also happen when one deals with (2-dimensional) potentially crystalline (non crystalline) representations of $G_{\mathbf{Q}_p}$ (see [**74**, 1.10] and [**15**, 1.2.1-1.2.3]) and one wonders, with Mazur ([**53**]), what is the general rule behind this.

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