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SEMI-STABLE CONJECTURE OF FONTAINE-JANNSEN: A SURVEY

by

Takeshi Tsuji

Abstract. — We give an outline of the proof of the semi-stable conjecture of J.-M. Fontaine and U. Jannsen by O. Hyodo, K. Kato and the author. This conjecture compares the two p-adic cohomologies: p-adic étale cohomology and de Rham cohomology associated to a proper smooth variety over a p-adic field with semi-stable reduction; it especially asserts that these two cohomologies with their additional structures can be reconstructed from each other. Our proof uses syntomic cohomology, which was introduced by J.-M. Fontaine and W. Messing, as a bridge between the two cohomologies. In the appendix, we also show that the semi-stable conjecture implies the de Rham conjecture thanks to the alteration of de Jong.

1. Introduction

In these notes, we will give an outline of the proof in [**HK94**], [**Kat94a**] and [**Tsu99**] of the conjecture of J.-M. Fontaine and U. Jannsen ([**Jan89**] p. 347, [**Fon94b**] § 6) on the p-adic étale cohomology of a proper smooth variety over a p-adic field with semi-stable reduction. Here we note that two other proofs were given by G. Faltings [**Fal**] and then by W. Niziol [**Niz98b**] afterwards. (See after Theorem 1.1 below for more details.) For the history of the p-adic Hodge theory, we refer the readers to the introduction of [**FI93**] and [**Ill90**]. Besides the proof of C_{st}, a theory for p-torsion étale cohomology in the semi-stable reduction case was developed by G. Faltings and C. Breuil ([**Fal92**], [**Bre98a**] and [**Bre98b**]) after [**FI93**] and [**Ill90**] were written. See [**BM**] for a survey.

Let us recall the conjecture of Fontaine-Jannsen. Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$ and let O_K denote the ring of integers of K. Let W be the ring of Witt-vectors with

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coefficients in k and let K_0 denote the field of fractions of W. We choose and fix a uniformizer π of K. Let $\overline{K}$ be an algebraic closure of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. We consider a proper scheme X over O_K with semi-stable reduction, that is, a regular scheme X proper and flat over O_K whose special fiber $Y := X \otimes_{O_K} k$ is a reduced divisor with normal crossings on X.

The conjecture of Fontaine-Jannsen, which is also called the semi-stable conjecture or C_{st} for short, compares the p-adic étale cohomology $H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$ of the geometric generic fiber $X_{\overline{K}} := X \otimes_{O_K} \overline{K}$ of X, which is a finite dimensional vector space over $\mathbb{Q}_p$ naturally endowed with a continuous linear action of G_K, with the log crystalline cohomology $H^m_{\text{log-crys}}(X)$, which is a finite dimensional vector space over K_0 endowed with a semi-linear automorphism φ (called the Frobenius), a linear endomorphism N (called the monodromy operator) satisfying $N\varphi = p\varphi N$ and a descending filtration after $\otimes_{K_0} K$. The log crystalline cohomology with its first two structures φ and N does not depend on the choice of π, but the filtration depends on it. More precisely, the filtration on $H^m_{\text{log-crys}}(X) \otimes_{K_0} K$ is induced by the Hodge filtration on $H^m_{\mathrm{dR}}(X_K/K)$ through the isomorphism ([**HK94**] Theorem (5.1)):

$$\rho_\pi \colon H^m_{\text{log-crys}}(X) \otimes_{K_0} K \xrightarrow{\sim} H^m_{\mathrm{dR}}(X_K/K)$$

depending on the choice of π. If X has a good reduction, $H^m_{\text{log-crys}}(X)$ coincides with the usual crystalline cohomology ([**Ber74**], [**BO78**]) of the special fiber tensored with K_0 over W, the monodromy operator vanishes, and ρ_π (in this case the isomorphism was proven by Berthelot and Ogus [**BO83**]) does not depend on the choice of π. Strictly speaking, when the conjecture was made, the log crystalline cohomology was conjectural and it was constructed afterwards by Hyodo and Kato [**Hyo91**], [**HK94**].

***Theorem 1.1* (Conjecture of Fontaine-Jannsen, C_{st}).** — *With the notations and the assumptions as above, $H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a semi-stable representation of G_K and there exist natural isomorphisms in $MF_K(\varphi, N)$:*

$$D_{\mathrm{st}}(H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^m_{\text{log-crys}}(X) \quad (m \in \mathbb{Z}).$$

See § 2.2 for semi-stable p-adic representations and the filtered φ-N modules in $MF_K(\varphi, N)$ associated to them. Furthermore these isomorphisms are functorial on X and compatible with the product structures and with the Chern classes in $H^m_{\mathrm{\acute{e}t}}$ and H^m_{dR} of a vector bundle on X_K.

Since the functor D_{st} is fully faithful (§ 2.2), the theorem implies that the two cohomology groups with their additional structures can be reconstructed from each other.

This conjecture was studied by many mathematicians [**FM87**], [**Fal89**], [**KM92**], [**HK94**], [**Kat94a**], ... and completely solved by the author in [**Tsu99**]. See Theorem A2 of these notes for the compatibility with the Chern classes. Afterwards, alternative proofs were given by G. Faltings [**Fal**] and then by W. Niziol [**Niz98a**], [**Niz98b**]. In fact, as an easy corollary of the proof of Theorem A1 of these notes, which uses the alteration of de Jong, we further see that the K_0-structure $H^m_{\text{log-crys}}(X)$ with φ and N on $H^m_{\text{dR}}(X_K/K)$ and the isomorphism in Theorem 1.1 are independent of the choice of a semi-stable model X of X_K.

We will explain the ideas of the three proofs of the conjecture. Every proof uses a certain intermediate cohomology or a K-group. Set $V^m := H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, $D^m := H^m_{\text{log-crys}}(X)$ and $D^m_K := K \otimes_{K_0} D^m$ to simplify the notation.

I) *The method of syntomic cohomology and p-adic vanishing cycles*

The syntomic cohomology for X/O_K smooth was first introduced by J.-M. Fontaine and W. Messing [**FM87**] to prove the conjecture in the good reduction case. (See the beginning of § 5 for the idea of the definition of the syntomic cohomology.) To prove the conjecture in general, we use a log version ([**Kat94a**], [**Tsu99**]) $H^m_{\text{log-syn}}(\overline{X}, S^r_{\mathbb{Q}_p})$ $(r, m \geqslant 0)$, from which there are maps to both étale and crystalline cohomologies:

$$V^m(r) \xleftarrow{(A)} H^m_{\text{log-syn}}(\overline{X}, S^r_{\mathbb{Q}_p}) \xrightarrow{(B)} \mathrm{Fil}^r(B_{\text{dR}} \otimes_K D^m_K) \cap (B_{\text{st}} \otimes_{K_0} D^m)^{N=0, \varphi=p^r}.$$

(cf. The proof of Corollary 2.2.9 for the last term).

Theorem 1.2 **([Kur87], [Kat94a] Corollary (5.5), [Tsu99] Theorem 3.3.4)**
The homomorphism (A) above is an isomorphism if $0 \leqslant m \leqslant r$.

By Theorem 1.2, we can invert the homomorphism (A) and obtain Theorem 1.1 using the fact $\dim_{\mathbb{Q}_p} V^m = \dim_{K_0} D^m$ and Poincaré duality for the two cohomology groups. The proof of Theorem 1.2 is based on the description of the p-adic vanishing cycles:

$$i^*_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p\mathbb{Z}(q) \quad (Y := X \otimes_{O_K} k \xrightarrow{i} X \xleftarrow{j} X_K)$$

in terms of the logarithmic differential modules of the special fiber Y (endowed with a natural log structure) by Bloch-Kato and Hyodo [**BK86**], [**Hyo88**]. In the case $r \leqslant p-2$, we have a good integral version of the syntomic cohomology and we can reduce the proof of Theorem 1.2 to the mod p case and use the above description. This was done by K. Kato and M. Kurihara in [**Kur87**], [**Kat94a**], and C_{st} was proved by K. Kato in the case $\dim X_K \leqslant (p-2)/2$. However, in the general case, we don't have a good integral theory so far and the proof of Theorem 1.2 involves much complicated and technical analysis of two kinds of adhoc syntomic complexes, which is the main part of [**Tsu99**].

II) *The method of almost étale extensions*

Associated to a sufficiently small affine open formal subscheme $\mathfrak{U} = \mathrm{Spf}(A)$ of the formal completion $\widehat{X}$ of X along the special fiber, we have a ring B_{crys} with an action of $\pi_1(\mathrm{Spec}(A_K))$. G. Faltings defined an intermediate cohomology $H^m(\mathfrak{X}_{\overline{K}}, B_{\text{crys}})$ by gluing the Galois cohomology $H^m(\pi_1(\mathrm{Spec}(A_{\overline{K}})), B_{\text{crys}})$, to which there are canonical homomorphisms from the two p-adic cohomology groups as follows:

$$B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V^m \xrightarrow{(C)} H^m(\mathfrak{X}_{\overline{K}}, B_{\mathrm{crys}}) \xleftarrow{(D)} (B_{\mathrm{st}} \otimes_{K_0} D^m)^{N=0}.$$

Theorem 1.3 **([Fal] § 3 8. Theorem, § 4 9. Theorem).** — *The homomorphism (C) above is an almost isomorphism.*

Roughly speaking, G. Faltings proved that the ramification along the special fiber of any étale extension of $A_{\overline{K}}$ is "almost" killed by adjoining to $A_{\overline{K}}$ all p-power roots of a coordinate ([**Fal88**] I 3.1. Theorem, [**Fal**] § 2B). This allowed him to reduce the calculation of some Galois cohomology of $\pi_1(\mathrm{Spec}(A_{\overline{K}}))$ to some Galois cohomology of a simple group $\mathbb{Z}_p(1)^d$ ($d = \dim X_K$), and to prove Theorem 1.3.

III) *The method via K-theory*

There are regulator maps from a K-group to the two p-adic cohomology groups as follows:

$$V^m(r) \xleftarrow{(E)} \mathbb{Q} \otimes \varprojlim_n \text{“}F^r_\gamma / F^{r+1}_\gamma K_{2r-m}(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})\text{”}$$
$$\xrightarrow{(F)} \mathrm{Fil}^r(B_{\mathrm{dR}} \otimes_K D^m_K) \cap (B_{\mathrm{st}} \otimes_{K_0} D^m)^{N=0, \varphi=p^r}.$$

The homomorphism (F) is defined as the composite of a regulator map to the syntomic cohomology and the homomorphism (B) above.

Theorem 1.4 **([Niz98b]).** — *The homomorphism (E) is surjective and the kernel of (F) contains the kernel of (E) if r is large enough.*

The proof is based on the comparison theorem of Thomason between algebraic K-theory and étale K-theory.

This paper is organized as follows: In § 2, we review the definition of Hodge-Tate, de Rham, semi-stable and crystalline p-adic representations including the definition and some properties of the rings B_{crys}, B_{st} and B_{dR}. In § 3, we review the theory of log structures in the sense of Fontaine-Illusie. § 4 is devoted to explaining how the usual crystalline cohomology and the comparison theorem of Berthelot-Ogus with de Rham cohomology are extended to the semi-stable reduction case. § 5 and § 6 correspond to the main part of [**Tsu99**]; We survey the proof of the key comparison theorem between syntomic and étale cohomologies. In § 7, we explain how we derive the conjecture of Fontaine-Jannsen from the above key comparison theorem. In the Appendix, we give an argument to derive C_{dR} from C_{st} using the alteration of de Jong. The main references to each section are as follows: § 2 [**Fon82**], [**Fon94a**], [**Fon94b**]. § 3 [**Kat89**]. § 4 [**HK94**]. § 5 and § 6 [**FM87**], [**Kat87**], [**Kur87**], [**Kat94a**], [**Tsu99**], [**BK86**], [**Hyo88**]. § 7 [**FM87**], [**KM92**], [**Kat94a**], [**Tsu99**].

Notation. — Throughout these notes, we fix a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$ and let O_K denote the ring of integers of K. We choose and fix a uniformizer π of K. We denote by

(S, N) the scheme $\mathrm{Spec}(O_K)$ endowed with the log structure defined by the closed point and by (s, N_s) its reduction mod π (see Example 3.1.1). Let W denote the ring of Witt-vectors with coefficients in k and let K_0 denote the field of fractions of W. We denote by σ the Frobenius of k, W and K_0. Let $\overline{K}$ be an algebraic closure of K, and let $\bar{k}$ be the residue field of $\overline{K}$, which is an algebraic closure of k. We set $G_K := \mathrm{Gal}(\overline{K}/K)$. Let C be the completion of $\overline{K}$ with respect to its valuation and let O_C denote its ring of integers. G_K acts continuously on C and O_C. We denote by the subscript n the reduction mod p^n of schemes, log schemes etc.

2. The rings B_{crys}, B_{st}, B_{dR} and p-adic representations

Let l be a prime. An l-adic representation of G_K is a finite dimensional $\mathbb{Q}_l$-vector space V with a continuous and linear action of G_K. Recall $G_K = \mathrm{Gal}(\overline{K}/K)$. It is well-known that $l(\neq p)$-adic representations and p-adic ones have completely different natures. For $l \neq p$, if we assume $[K : \mathbb{Q}_p] < \infty$, every l-adic representation of G_K is quasi-unipotent, that is, after restricting to the Galois group of a suitable finite extension K' of K, it becomes tame and the action of the inertia group becomes unipotent. It is still true without the assumption $[K : \mathbb{Q}_p] < \infty$, if the representation is the l-adic étale cohomology of an algebraic variety over K [**Gro72**]. However a p-adic representation does not have such a simple structure in general; The image of the wild part of the inertia group can have a large image in $\mathrm{GL}(V)$. Furthermore, there are p-adic representations of a type completely different from those realized as p-adic étale cohomology, for instance, ψ^a $(a \in \mathbb{Z}_p \setminus \mathbb{Z})$, where ψ denotes the composite of the cyclotomic character $G_K \to \mathbb{Z}_p^*$ with the projection $\mathbb{Z}_p^* = \mu_{p-1} \times (1 + p\mathbb{Z}_p) \to 1 + p\mathbb{Z}_p$.

Let $\mathrm{Rep}(G_K)$ denote the category of p-adic representations of G_K. In this section, we will briefly review the notions of Hodge-Tate, de Rham, semi-stable and crystalline p-adic representations, whose categories we denote by $\mathrm{Rep}_\bullet(G_K)$ with $\bullet =$ HT, dR, st and crys respectively. The latter implies the former, that is, we have

$$\mathrm{Rep}(G_K) \supset \mathrm{Rep}_{\mathrm{HT}}(G_K) \supset \mathrm{Rep}_{\mathrm{dR}}(G_K) \supset \mathrm{Rep}_{\mathrm{st}}(G_K) \supset \mathrm{Rep}_{\mathrm{crys}}(G_K).$$

The de Rham, semi-stable and crystalline representations were defined by J.-M. Fontaine by introducing the rings B_{dR}, B_{st} and B_{crys}, and they correspond to all, unipotent and unramified representations respectively in the $l(\neq p)$-adic case. We note that the representations ψ^a $(a \in \mathbb{Z}_p \setminus \mathbb{Z})$ mentioned above are not Hodge-Tate. Furthermore, to these kinds of representations, one can also associate K or K_0-vector spaces with some linear or semi-linear structures, from which one can extract some information on the representations.

2.1. The rings B_{crys}, B_{st} and B_{dR}; their structures and properties

In this section, we will list structures and properties of the rings B_{crys}, B_{st} and B_{dR} defined by J.-M. Fontaine [**Fon82**], [**Fon94a**]. We will postpone a construction of them to § 2.3.

$\mathbf{B}_{\mathrm{dR}}$

$(0)_{\mathrm{dR}}$ The ring B_{dR} is a complete discrete valuation field whose residue field is C. We denote by B_{dR}^+ its valuation ring and define a descending filtration on B_{dR} by $\mathrm{Fil}^i B_{\mathrm{dR}} := \{x \in B_{\mathrm{dR}} \mid v(x) \geqslant i\}$ $(i \in \mathbb{Z})$, where v denotes the discrete valuation of B_{dR} normalized by $v_{\mathrm{dR}}(B_{\mathrm{dR}}^*) = \mathbb{Z}$.

$(1)_{\mathrm{dR}}$ The ring B_{dR} is endowed with an action of G_K compatible with the ring structure and the filtration such that the canonical homomorphism $B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+/\mathrm{Fil}^1 B_{\mathrm{dR}} = C$ is G_K-equivariant.

$(2)_{\mathrm{dR}}$ There exists a canonical G_K-equivariant ring homomorphism $\overline{K} \to B_{\mathrm{dR}}^+$ whose composite with $B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+/\mathrm{Fil}^1 B_{\mathrm{dR}} = C$ is the natural embedding. We regard $\overline{K}$ as a subfield of B_{dR}^+ in the following. (See Remark 2.1.2).

$(3)_{\mathrm{dR}}$ There exists a $\mathbb{Q}_p$-linear canonical G_K-equivariant injective homomorphism $\mathbb{Q}_p(1) \hookrightarrow \mathrm{Fil}^1 B_{\mathrm{dR}}$ such that the image of a non-zero element is a uniformizer. Using the field structure of B_{dR}, we obtain injective homomorphisms

$(3.1)_{\mathrm{dR}}$ $\quad \mathbb{Q}_p(r) \hookrightarrow \mathrm{Fil}^r B_{\mathrm{dR}} \quad (r \in \mathbb{Z})$

and isomorphisms

$(3.2)_{\mathrm{dR}}$ $\quad \mathrm{gr}^i B_{\mathrm{dR}} \xleftarrow{\sim} \mathbb{Q}_p(i) \otimes_{\mathbb{Q}_p} \mathrm{gr}^0 B_{\mathrm{dR}} = C(i) \quad (i \in \mathbb{Z})$.

For $r \in \mathbb{Z}$, we regard $\mathbb{Q}_p(r)$ as a submodule of $\mathrm{Fil}^r B_{\mathrm{dR}}$ in the following.

Using $(3.2)_{\mathrm{dR}}$ and the following well-known theorem of Tate, we obtain

$(4)_{\mathrm{dR}}$ $B_{\mathrm{dR}}^{G_K} = \overline{K}^{G_K} = K$.

Theorem 2.1.1 **([Tat67] (3.3) Theorems 1, 2).** — *$H^0(G_K, C(i)) = K$ (if $i = 0$), 0 (otherwise).*

Remark 2.1.2. — We don't have a G_K-equivariant section of $B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+/\mathrm{Fil}^1 B_{\mathrm{dR}} = C$, that is, $B_{\mathrm{dR}} \not\cong C[[t]][t^{-1}]$ $(t \in \mathbb{Q}_p(1), t \neq 0)$.

$\mathbf{B}_{\mathrm{crys}}$

$(0)_{\mathrm{crys}}$ The ring B_{crys} is a G_K-stable subring of B_{dR} containing $\mathbb{Q}_p(r)$ $(r \in \mathbb{Z})$ and $P_0 = \mathrm{Frac}(W(\overline{k}))$.

$(1)_{\mathrm{crys}}$ The ring B_{crys} is endowed with a P_0-semilinear injective endomorphism (called the Frobenius) $\varphi\colon B_{\mathrm{crys}} \to B_{\mathrm{crys}}$ such that

$(1.1)_{\mathrm{crys}}$ $\varphi \circ g = g \circ \varphi$ for all $g \in G_K$

$(1.2)_{\mathrm{crys}}$ $\varphi(t) = p \cdot t$ for $t \in \mathbb{Q}_p(1) \subset B_{\mathrm{crys}} \cap \mathrm{Fil}^1 B_{\mathrm{dR}}$

$(1.3)_{\mathrm{crys}}$ $\mathrm{Fil}^0 B_{\mathrm{dR}} \cap B_{\mathrm{crys}}^{\varphi=1} = \mathbb{Q}_p$.

$(2)_{\mathrm{crys}}$ The canonical homomorphism $K \otimes_{K_0} B_{\mathrm{crys}} \to B_{\mathrm{dR}}$ is injective.

We obtain the following $(3)_{\mathrm{crys}}$ from $(2)_{\mathrm{crys}}$ and $(4)_{\mathrm{dR}}$.

$(3)_{\mathrm{crys}}$ $B_{\mathrm{crys}}^{G_K} = P_0^{G_K} = K_0$.

$(4)_{\mathrm{crys}}$ For a non-zero element $b \in B_{\mathrm{crys}}$ if $\mathbb{Q}_p \cdot b$ $(\subset B_{\mathrm{crys}})$ is G_K-stable, then $b \in P_0 \cdot t^i$ for some $i \in \mathbb{Z}$.

$\mathbf{B}_{\mathrm{st}}$

The rings B_{dR} and B_{crys} do not depend on the choice of a uniformizer π of K, but B_{st} (with their structures) does. See Remark 2.1.3 (2).

$(0)_{\mathrm{st}}$ The ring B_{st} is a B_{crys}-algebra contained in B_{dR} stable under the action of G_K.

$(1)_{\mathrm{st}}$ For each compatible system $s=(s_n)_{n\geqslant 0}$ of p^n-th roots of π in $O_{\overline{K}}$, there is a canonically defined element u_s of B_{st} such that:

$(1.1)_{\mathrm{st}}$ The element u_s is transcendental over B_{crys} and $B_{\mathrm{st}}=B_{\mathrm{crys}}[u_s]$.

$(1.2)_{\mathrm{st}}$ $g(u_s)=u_{g(s)}$ for $g\in G_K$.

$(1.3)_{\mathrm{st}}$ For two systems s and s', if we set $t=(s'_n s_n^{-1})_n\in\mathbb{Z}_p(1)(O_{\overline{K}})\subset B_{\mathrm{crys}}$, we have $u_{s'}=u_s+t$.

$(2)_{\mathrm{st}}$ The ring B_{st} is endowed with an endomorphism (called the Frobenius) $\varphi\colon B_{\mathrm{st}}\to B_{\mathrm{st}}$ extending the Frobenius on B_{crys} and characterized by

$(2.1)_{\mathrm{st}}$ $\varphi(u_s)=p\cdot u_s$ for every s.

By $(1.1)_{\mathrm{crys}}$ and $(1.2)_{\mathrm{st}}$, we have

$(2.2)_{\mathrm{st}}$ $\varphi\circ g=g\circ\varphi$ for all $g\in G_K$.

$(3)_{\mathrm{st}}$ The ring B_{st} is endowed with a B_{crys}-derivation $N\colon B_{\mathrm{st}}\to B_{\mathrm{st}}$ (called the monodromy operator) characterized by (see Remark 2.1.3 (1)):

$(3.1)_{\mathrm{st}}$ $N(u_s)=-1$ for every s.

By definition, it satisfies

$(3.2)_{\mathrm{st}}$ $N\varphi=p\varphi N$.

By $(1.2)_{\mathrm{st}}$, we have

$(3.3)_{\mathrm{st}}$ $N\circ g=g\circ N$ for all $g\in G_K$.

We obtain the following $(4)_{\mathrm{st}}$ from $(1.3)_{\mathrm{crys}}$ and the definition of N above.

$(4)_{\mathrm{st}}$ $B_{\mathrm{st}}^{N=0}=B_{\mathrm{crys}}$ and $\mathrm{Fil}^0 B_{\mathrm{dR}}\cap B_{\mathrm{st}}^{\varphi=1,N=0}=\mathbb{Q}_p$.

$(5)_{\mathrm{st}}$ The canonical homomorphism $K\otimes_{K_0}B_{\mathrm{st}}\to B_{\mathrm{dR}}$ is injective.

From $(4)_{\mathrm{dR}}$ and $(5)_{\mathrm{st}}$, we obtain

$(6)_{\mathrm{st}}$ $B_{\mathrm{st}}^{G_K}=P_0^{G_K}=K_0$.

$(7)_{\mathrm{st}}$ For a non-zero element b of B_{st}, if $\mathbb{Q}_p\cdot b$ $(\subset B_{\mathrm{st}})$ is G_K-stable, then $b\in P_0\cdot t^i$ $(\subset B_{\mathrm{crys}})$ for some $i\in\mathbb{Z}$.

Remark 2.1.3

(1) In [**Fon94a**], the monodromy operator of B_{st} is defined by $N(u_s)=1$, but we change the sign here to make it compatible with the monodromy operator coming from its log crystalline interpretation. (See Proposition 4.4.1.)

(2) The B_{crys}-algebra B_{st} with an action of G_K, φ and N is independent of the choice of π up to canonical isomorphisms. If we choose another uniformizer π', the two embeddings $\iota_\pi,\iota_{\pi'}:B_{\mathrm{st}}\otimes_{K_0}K\hookrightarrow B_{\mathrm{dR}}$ corresponding to π and π' are related by the formula:

$$\iota_{\pi'}=\iota_\pi\circ\exp(\log(\pi'\pi^{-1})\cdot(N\otimes 1_K)).$$

2.2. Hodge-Tate, de Rham, semi-stable and crystalline representations

Let V be a p-adic representation of G_K, that is, a finite dimensional $\mathbb{Q}_p$-vector space endowed with a continuous and linear action of G_K. We define K-vector spaces $D^i_{\mathrm{HT}}(V)$ $(i \in \mathbb{Z})$ by

$$D^i_{\mathrm{HT}}(V) := (C(i) \otimes_{\mathbb{Q}_p} V)^{G_K}$$

and $D_\bullet(V)$ $(\bullet = \mathrm{dR}, \mathrm{st}, \mathrm{crys})$ by

$$D_\bullet(V) := (B_\bullet \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

$D_{\mathrm{dR}}(V)$ is a K-vector space and $D_{\mathrm{st}}(V)$ and $D_{\mathrm{crys}}(V)$ are K_0-vector spaces.

By $(5)_{\mathrm{st}}$, we have canonical injective homomorphisms

$$D_{\mathrm{crys}}(V) \hookrightarrow D_{\mathrm{st}}(V), \tag{2.2.1}$$

$$D_{\mathrm{st}}(V) \otimes_{K_0} K \hookrightarrow D_{\mathrm{dR}}(V). \tag{2.2.2}$$

We define the filtration on $D_{\mathrm{dR}}(V)$ by $(\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ $(i \in \mathbb{Z})$. Then, by $(3.2)_{\mathrm{dR}}$, we have canonical injective homomorphisms

$$\mathrm{gr}^i D_{\mathrm{dR}}(V) \longrightarrow D^i_{\mathrm{HT}}(V) \quad (i \in \mathbb{Z}). \tag{2.2.3}$$

From these facts and Proposition 2.2.6 below (which follows from Theorem 2.1.1 without much difficulty), we obtain

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = 0 \quad (i \gg 0), \quad \mathrm{Fil}^i D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \quad (i \ll 0) \tag{2.2.4}$$

(2.2.5)

$$\dim_{K_0} D_{\mathrm{crys}}(V) \leqslant \dim_{K_0} D_{\mathrm{st}}(V) \leqslant \dim_K D_{\mathrm{dR}}(V) \leqslant \dim_K D_{\mathrm{HT}}(V) \leqslant \dim_{\mathbb{Q}_p}(V),$$

where $D_{\mathrm{HT}}(V)$ denotes the graded module $\oplus_{i \in \mathbb{Z}} D^i_{\mathrm{HT}}(V)$.

***Proposition 2.2.6* ([Ser67] § 2 Proposition 4).** — *The canonical homomorphism*

$$\alpha_{\mathrm{HT}} \colon \bigoplus_{i \in \mathbb{Z}} C(-i) \otimes_K D^i_{\mathrm{HT}}(V) \longrightarrow C \otimes_{\mathbb{Q}_p} V$$

is injective.

Definition 2.2.7. — With the notation above, we say that V is *Hodge-Tate* if $\dim_K D_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p}(V)$. We define *de Rham*, *semi-stable* and *crystalline representations* similarly using $D_{\mathrm{dR}}(V)$, $D_{\mathrm{st}}(V)$ and $D_{\mathrm{crys}}(V)$ respectively instead of $D_{\mathrm{HT}}(V)$.

We define the categories MG_K, MF_K, $MF_K(\varphi, N)$ and $MF_K(\varphi)$, an object of which will be associated to a Hodge-Tate, de Rham, semi-stable and crystalline representation of G_K:

MG_K: The category of finite dimensional K-vector spaces D graded by K-subspaces D^i $(i \in \mathbb{Z})$.

MF_K: The category of finite dimensional K-vector spaces D endowed with exhaustive and separated descending filtrations $\mathrm{Fil}^i D$ $(i \in \mathbb{Z})$ by K-subspaces.

$MF_K(\varphi, N)$: The category of finite dimensional K_0-vector spaces D endowed with K_0-semilinear automorphisms φ, K_0-linear endomorphisms N such that $N\varphi = p\varphi N$, and exhaustive and separated descending filtrations $\mathrm{Fil}^i D_K$ ($i \in \mathbb{Z}$) on $D_K := K \otimes_{K_0} D$ by K-subspaces.

$MF_K(\varphi)$: The full subcategory of $MF_K(\varphi, N)$ consisting of the objects such that $N = 0$.

We have the following commutative diagram of categories and functors.

$$\begin{array}{ccc}
\mathrm{Rep}_{\mathrm{HT}}(G_K) & \xrightarrow{D_{\mathrm{HT}}} & MG_K \\
\cup & & \uparrow \mathrm{gr} \\
\mathrm{Rep}_{\mathrm{dR}}(G_K) & \xrightarrow{D_{\mathrm{dR}}} & MF_K \\
\cup & & \uparrow \\
\mathrm{Rep}_{\mathrm{st}}(G_K) & \xrightarrow{D_{\mathrm{st}}} & MF_K(\varphi, N) \\
\cup & & \cup \\
\mathrm{Rep}_{\mathrm{crys}}(G_K) & \xrightarrow{D_{\mathrm{crys}}} & MF_K(\varphi)
\end{array}$$

Proposition 2.2.8. — *Let V be a p-adic representation of G_K. Then:*

(1) *If V is de Rham, the canonical homomorphism*

$$\alpha_{\mathrm{dR}} \colon B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \longrightarrow B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

is a filtered isomorphism, where we define the filtration on the LHS (resp. the RHS) by $\mathrm{Fil}^i = \sum_{i=i_0+i_1} \mathrm{Fil}^{i_0} \otimes \mathrm{Fil}^{i_1}$ *(resp.* $\mathrm{Fil}^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$*).*

(2) *If V is semi-stable, the canonical homomorphism*

$$\alpha_{\mathrm{st}} \colon B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) \longrightarrow B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism.

Proof

(1) By Proposition 2.2.6, α_{dR} is injective and strict with respect to the filtrations. Since B_{dR} is a field and $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$, α_{dR} is a filtered isomorphism.

(2) By (1), $(5)_{\mathrm{st}}$ and (2.2.2), the homomorphism α_{st} is injective. Choose bases $\{d_i\}_{1\leqslant i\leqslant r}$ and $\{v_i\}_{1\leqslant i\leqslant r}$ of $D_{\mathrm{st}}(V)$ and V respectively and set

$$\alpha_{\mathrm{st}}(1 \otimes d_i) = \sum_{1\leqslant j\leqslant r} b_{ji} \cdot (1 \otimes v_j) \quad (b_{ji} \in B_{\mathrm{st}}).$$

By (1), $\det(b_{ij}) \neq 0$. On the other hand $\mathbb{Q}_p \cdot \det(b_{ij})$ is stable under G_K. Hence, by $(7)_{\mathrm{st}}$, $\det(b_{ij}) \in B_{\mathrm{st}}^*$. □

Corollary 2.2.9. — *The functor $D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathrm{st}}(G_K) \to MF_K(\varphi, N)$ is fully-faithful.*

Proof. — For a semi-stable representation V, by Proposition 2.2.8 and $(4)_{\mathrm{st}}$, α_{st} and α_{dR} induces a G_K-equivariant isomorphism

$$(B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V))^{\varphi\otimes\varphi=1, 1\otimes N+N\otimes 1=0} \cap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)) \xrightarrow{\sim} V. \qquad \square$$

Corollary 2.2.10. — *For a p-adic representation V of G_K and $D \in MF_K(\varphi, N)$, to prove that V is semi-stable and to give an isomorphism $D_{\mathrm{st}}(V) \cong D$ are equivalent to giving a G_K-equivariant B_{st}-linear isomorphism $B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{K_0} D$ preserving the action of G_K, φ, N and the filtration after tensoring with B_{dR} over B_{st}. Here the action of $g \in G_K$ on the LHS (resp. RHS) is $g \otimes g$ (resp. $g \otimes 1$), φ on the LHS (resp. RHS) is $\varphi \otimes 1$ (resp. $\varphi \otimes \varphi$), N on the LHS (resp. RHS) is $N \otimes 1$ (resp. $1 \otimes N + N \otimes 1$); the filtration on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is $\mathrm{Fil}^{\cdot} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$, and the filtration on $B_{\mathrm{dR}} \otimes_K D_K$ is the tensor product of the filtrations on B_{dR} and D_K.*

See [**FI93**] 2.3 for some examples of these kinds of p-adic representations. We will give some in the end of § 2.3.

2.3. The rings B_{crys}, B_{st} and B_{dR}; their construction. — We define the ring R to be the projective limit of

$$O_{\overline{K}}/pO_{\overline{K}} \xleftarrow{\mathrm{Frob}} O_{\overline{K}}/pO_{\overline{K}} \xleftarrow{\mathrm{Frob}} O_{\overline{K}}/pO_{\overline{K}} \xleftarrow{\mathrm{Frob}} \cdots$$

The element of R is a system of elements $a = (a_0, a_1, a_2, \dots)$ of $O_{\overline{K}}/pO_{\overline{K}}$ such that $a_{n+1}^p = a_n$. The absolute Frobenius of R is bijective. Choose a compatible system $s = (s_n)_{n \geqslant 0}$ of p^n-th roots of π in $O_{\overline{K}}$ and define the element $\underline{\pi}$ of R to be $(s_n \mod p)_{n \geqslant 0}$. We have a canonical injective multiplicative homomorphism:

$$\mathbb{Z}_p(1)(O_{\overline{K}}) = \varprojlim_n \mu_{p^n}(O_{\overline{K}}) \hookrightarrow R^\times;\ \varepsilon = (\varepsilon_n)_{n \geqslant 0} \longmapsto \underline{\varepsilon} := (\varepsilon_n \mod p)_{n \geqslant 0}.$$

Roughly speaking, the rings B_{crys}, B_{st} and B_{dR} are constructed as certain modifications of the ring $W(R)$ of Witt-vectors with coefficients in R. We have a canonical surjective ring homomorphism

$$\theta\colon W(R) \longrightarrow O_C$$

characterized by $\theta([a]) = \lim_{n\to\infty} \widetilde{a_n}^{p^n}$, where $a = (a_0, a_1, \dots) \in R$, $\widetilde{a_n}$ denotes a lifting of a_n in $O_{\overline{K}}$ and $[a]$ denotes the Teichmüller representative $(a, 0, 0, \dots) \in W(R)$. We have $\theta([\underline{\pi}]) = \lim_{n\to\infty} s_n^{p^n} = \pi$ and $\theta([\underline{\varepsilon}]) = \lim_{n\to\infty} \varepsilon_n^{p^n} = 1$ for $\varepsilon \in \mathbb{Z}_p(1)(O_{\overline{K}})$. We denote by θ_{O_K} (resp. θ_K) the O_K-linear (resp. K-linear) extension of $\theta\colon O_K \otimes_W W(R) \to O_C$ (resp. $K \otimes_W W(R) \to C$). We have $\theta(1 \otimes [\underline{\pi}] - \pi \otimes 1) = 0$.

Proposition 2.3.1 **([Fon82] 2.4. Proposition).** — *The element $1 \otimes [\underline{\pi}] - \pi \otimes 1$ is a non-zero divisor in $O_K \otimes_W W(R)$ and generates* $\mathrm{Ker}\,(\theta_{O_K})$.

Proof. — Since $O_K \otimes_W W(R)$ and O_C are p-adically complete and separated and p-torsion free, it suffices to prove that the reduction mod π of the sequence

$$0 \longrightarrow O_K \otimes_W W(R) \xrightarrow{1\otimes[\underline{\pi}]-\pi\otimes 1} O_K \otimes_W W(R) \xrightarrow{\theta_{O_K}} O_C \longrightarrow 0$$

is exact, that is,

$$0 \longrightarrow R \xrightarrow{\underline{\pi}} R \xrightarrow{\overline{\theta_{O_K}}} O_C/\pi O_C \longrightarrow 0$$

is exact, where $\overline{\theta_{O_K}}$ is the projection to the first component mod π. This is easy to see. □

We define the ring $B^+_{\mathrm{dR},K}$ by

$$B^+_{\mathrm{dR},K} := \varprojlim_r (K \otimes_W W(R))/(\mathrm{Ker}(\theta_K))^r,$$

which is a complete discrete valuation ring with residue field C by Proposition 2.3.1. We define $B_{\mathrm{dR},K}$ to be the field of fractions of $B^+_{\mathrm{dR},K}$. Since $\theta([\underline{\varepsilon}]) = 1$, for $\varepsilon \in \mathbb{Z}_p(1)(O_{\overline{K}})$, $\log([\underline{\varepsilon}]) = \sum_{i\geqslant 1}(-1)^{i-1}([\underline{\varepsilon}] - 1)^i/i$ converges in B^+_{dR} and we obtain a canonical additive injective homomorphism

$$\mathbb{Z}_p(1) \hookrightarrow \mathrm{Fil}^1 B_{\mathrm{dR},K}; \quad \varepsilon \longmapsto \log([\underline{\varepsilon}]).$$

We can prove that the image of a non-zero element of $\mathbb{Z}_p(1)$ is a uniformizer ([**Fon82**] 2.17. Proposition) and hence, for a finite extension K' of K contained in $\overline{K}$, the canonical homomorphism $B^+_{\mathrm{dR},K'} \to B^+_{\mathrm{dR},K}$ is an isomorphism since these two complete discrete valuation rings have the same residue field and a common uniformizer. This implies $B_{\mathrm{dR},K} \cong B_{\mathrm{dR},K'}$ and we simply write B_{dR} for $B_{\mathrm{dR},K}$ in the following. We have $\overline{K} \subset B_{\mathrm{dR}}$.

We define the ring A_{crys} to be the p-adic completion of

$$W(R)[\xi^n/n!\ (n \geqslant 1)] \subset W(R)[1/p],$$

where $\xi = [\underline{p}] - p \in \mathrm{Ker}(\theta)$. Note that ξ is a generator of $\mathrm{Ker}(\theta)$ (Proposition 2.3.1). For the Frobenius φ on $W(R)$, we have

$$\varphi(\xi) = [\underline{p}]^p - p = (\xi + p)^p - p \in pW(R)[\xi^n/n!\ (n \geqslant 1)].$$

Hence, using $p^n/n! \in p\mathbb{Z}_p$ $(n \geqslant 1)$, we see that the Frobenius φ on $W(R)$ extends to the Frobenius φ on A_{crys}. We see easily that $t = \log([\underline{\varepsilon}])$ converges in A_{crys} for $\varepsilon \in \mathbb{Z}_p(1)(O_{\overline{K}})$, $\varepsilon \neq 0$, and $\varphi(t) = \log([\underline{\varepsilon}]^p) = p \cdot t$. Note $[\underline{\varepsilon}] - 1 \in \mathrm{Ker}(\theta) = \xi \cdot W(R)$. We define B^+_{crys} and B_{crys} to be $A_{\mathrm{crys}}[p^{-1}]$ and $A_{\mathrm{crys}}[t^{-1}, p^{-1}]$. The rings B^+_{st} and B_{st} are defined to be the subrings $B^+_{\mathrm{crys}}[u_s]$ and $B_{\mathrm{crys}}[u_s]$ of B^+_{dR} and B_{dR} respectively, where $u_s = \log((1 \otimes [\underline{\pi}]) \cdot (\pi \otimes 1)^{-1})$. Note $\theta_K((1 \otimes [\underline{\pi}]) \cdot (\pi \otimes 1)^{-1}) = 1$ and hence $\log((1 \otimes [\underline{\pi}]) \cdot (\pi \otimes 1)^{-1})$ converges in B^+_{dR}. For the proof of $(1.3)_{\mathrm{crys}}$, $(2)_{\mathrm{crys}}$, $(7)_{\mathrm{st}}(\Rightarrow (4)_{\mathrm{crys}})$, $(1.1)_{\mathrm{st}}$ and $(5)_{\mathrm{st}}$, see [**Fon82**] 4.12 Théorème (or [**Fon94a**] 5.3.7 Théorème), [**Fon82**] 4.7, [**Fon94b**] 5.1.3 Lemme, [**Fon94a**] 3.1.6 and [**Fon94a**] 4.2.4 Théorème.

Example 2.3.2. — Let $q \in K^*$ and consider an extension $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ defined by the image of q in $(\varprojlim_n (K^*/(K^*)^{p^n})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^1(G_K, \mathbb{Q}_p(1))$. Choose a compatible system $\{q_n\}_{n\geqslant 0}$ of p^n-th roots of q in $\overline{K}$ and define $\tau\colon G_K \to \mathbb{Z}_p(1)$ by $g(q_n) = \tau(g)_n \cdot q_n$, where $\tau(g)_n := \tau(g) \mod p^n \in \mu_{p^n}(\overline{K})$. Then $V = \mathbb{Q}_p(1) \oplus \mathbb{Q}_p$ with the action of G_K: $g(x,y) = (g(x) + y \cdot \tau(g), y)$ $(g \in G_K, x \in \mathbb{Q}_p(1), y \in \mathbb{Q}_p)$.

(1) If $q \in O_K^*$, then V is crystalline: We may assume $q \in 1 + \pi \cdot O_K$. Set $\underline{q} := (q_n \mod p)_n \in R$. Then, for $[\underline{q}] \in W(R)$, $\log([\underline{q}])$ converges in B^+_{crys} and $D :=$

$D_{\mathrm{crys}}(V) = (B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is a K_0-vector space with a base $e_1 = (t^{-1} \otimes t, 0)$, $e_2 = (-\log([q])t^{-1} \otimes t, 1 \otimes 1)$ $(0 \neq t \in \mathbb{Q}_p(1))$. Its filtered φ-module structure is given by $\varphi(e_1) = p^{-1}e_1$, $\varphi(e_2) = e_2$, $\mathrm{Fil}^{-1} D_K = D_K$, $\mathrm{Fil}^0 D_K = K \cdot (\log(q)e_1 + e_2)$, $\mathrm{Fil}^1 D_K = 0$.

(2) If $q = \pi^m \cdot u$ for $0 \neq m \in \mathbb{Z}$ and $u \in O_K^*$, then V is not crystalline but semi-stable: We may assume $u \in 1+\pi O_K$. Choose compatible systems $\{s_n\}$ and $\{u_n\}$ of p^n-th roots of π and u in $O_{\overline{K}}$ and choose $\{s_n^m \cdot u_n\}$ as $\{q_n\}$ above. Set $\underline{u} := (u_n \mod p)_n \in R$ and let u_s be as in the definition of B_{st}. Then $D := D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is a K_0-vector space with a base $e_1 = (t^{-1} \otimes t, 0)$, $e_2 := m^{-1}((-\log([\underline{u}]) - mu_s)t^{-1} \otimes t, 1 \otimes 1)$ $(0 \neq t \in \mathbb{Q}_p(1))$. Its φ-N filtered module structure is given by $\varphi(e_1) = p^{-1}e_1$, $\varphi(e_2) = e_2$, $N(e_1) = 0$, $N(e_2) = e_1$, $\mathrm{Fil}^{-1} D_K = D_K$, $\mathrm{Fil}^0 D_K = K \cdot (m^{-1}\log(u)e_1 + e_2)$, $\mathrm{Fil}^1 D_K = 0$.

3. Logarithmic structures

The theory of logarithmic structures in the sense of Fontaine-Illusie on schemes was established by K. Kato in [**Kat89**] based on an idea of Fontaine and Illusie and it is a useful tool when one wants to generalize a theory concerning smooth schemes to semi-stable schemes or normal crossing varieties. See Example 3.1.1 (2), (3) and Example 3.2.4 (2), (3). We review the theory briefly. See [**Kat89**] for details.

3.1. Definition. — We assume that a monoid is always commutative and has 1 (the unit), and a morphism of monoids preserves 1. We regard $\mathbb{N} = \{0, 1, 2, \ldots\}$ as a monoid by its addition (0 is the unit in this case). For a scheme X, we regard $\mathcal{O}_X$ as a monoid by its multiplication.

A *pre-log structure* on X is a pair (M, α) of a sheaf of monoids M on the étale site $X_{\text{ét}}$ and a morphism of sheaves of monoids $\alpha\colon M \to \mathcal{O}_X$. It is a *log structure* if the canonical homomorphism $\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\alpha} \mathcal{O}_X^*$ is an isomorphism. We define the *log structure* $(M, \alpha)^a$ *associated to a pre-log structure* (M, α) to be the push out of the diagram of sheaves of monoids: $\mathcal{O}_X^* \xleftarrow{\alpha} \alpha^{-1}(\mathcal{O}_X^*) \hookrightarrow M$. A *log scheme* (X, M, α) is a scheme X endowed with a log structure (M, α). We often omit α in the notation of a log structure and a log scheme in the following. We define a morphism of log schemes as a pair of a morphism of schemes and a morphism between the sheaves of monoids compatible with α's in the obvious sense. The monoid $\mathcal{O}_X^*$ with the inclusion into $\mathcal{O}_X$ is a log structure and it is called the *trivial log structure.* The functor from the category of schemes to the category of log schemes which associates $(X, \mathcal{O}_X^*)$ to X is fully faithful. For a morphism of schemes $f\colon Y \to X$ and a log structure M on X, we define the *inverse image* f^*M to be the log structure associated to the pre-log structure $f^{-1}(M) \to f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$.

We say that a monoid P is *integral* if $ac = bc$ implies $a = b$ for $a, b, c \in P$. We say that a log structure M is *fine*, if étale locally on X, M is isomorphic to the log structure associated to a pre-log structure of the form (P_X, β) where P is a finitely generated integral monoid and P_X is the constant sheaf of monoids associated to P. Fiber products are representable in the category of log schemes and also in the category of fine log schemes. We note that in the latter category fiber products are not compatible with fiber products in underlying schemes in general.

For a morphism of log schemes $f\colon (X,M) \to (Y,N)$, we define the *relative differential module* $\Omega^1_{X/Y}(\log(M/N))$ to be the quotient of $\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})$ by the $\mathcal{O}_X$-submodule generated by $(d(\alpha(x)),0) - (0,\alpha(x)\otimes x)$ $(x \in M)$ and $(0, 1\otimes x)$ ($x \in$ the image of $f^{-1}(N^{\mathrm{gp}}) \to M^{\mathrm{gp}}$). We denote by $d\log(x)$ the class of $(0,1\otimes x)$ for $x \in M^{\mathrm{gp}}$. If M and N are fine, then the differential module is quasi-coherent. We can define the de Rham complex $\Omega^{\cdot}_{X/S}(\log(M/N))$ by setting $d(d\log(x)) = 0$ $(x \in M^{\mathrm{gp}})$.

Example 3.1.1

(1) Let X be a regular scheme and D be a reduced divisor with normal crossings on D. Then $M := \mathcal{O}_X \cap j_*\mathcal{O}_U^* \to \mathcal{O}_X$ is a fine log structure, where $U = X \smallsetminus D$ and j denotes $U \hookrightarrow X$. Étale locally on X, we have a decomposition $D = \sum_{1\leqslant i\leqslant r} D_i$ such that D_i is regular and $D_i = \{\pi_i = 0\}$ for $\pi_i \in \Gamma(X, \mathcal{O}_X)$, and M is isomorphic to the log structure associated to $(\mathbb{N}^r)_X \to \mathcal{O}_X; (n_i) \mapsto \prod \pi_i^{n_i}$.

(2) Let A be a discrete valuation ring and let $X \to \operatorname{Spec}(A) = S$ be a morphism of finite type such that étale locally on X, there exists an étale morphism of S-schemes $u\colon X \to \operatorname{Spec}(A[T_1,\ldots,T_d]/(T_1\cdots T_r - \pi))$ for some integers $1 \leqslant r \leqslant d$. Then as in (1), we can define the fine log structures M on X and N on S by the special fiber Y and the closed point s respectively. We have a natural morphism $f\colon (X,M) \to (S,N)$ of log schemes. The relative differential $\Omega^1_{X/S}(\log(M/N))$ is locally free and locally of finite rank. If we have a morphism u as above, we have

$$\Omega^1_{X/S}(\log(M/N)) = (\oplus_{1\leqslant i\leqslant r}\mathcal{O}_X \cdot d\log(\pi_i))/\mathcal{O}_X \cdot d\log(f^{-1}(\pi)) \bigoplus (\oplus_{r+1\leqslant i\leqslant d}\mathcal{O}_X \cdot d\pi_i),$$

where $\pi_i = u^*(T_i)$. Note $d\log(f^{-1}(\pi)) = \sum_{1\leqslant i\leqslant r} d\log(\pi_i)$.

(3) Keep the notation of (2). We denote by M_Y (resp. N_s) the inverse image of M on Y (resp. N on s). We have a natural morphism $g\colon (Y, M_Y) \to (s, N_s)$. If we have a morphism u as in (2) and denote by $\overline{\pi_i}$ $(1 \leqslant i \leqslant r)$ the image of $\pi_i \in M \subset \mathcal{O}_X$ in M_Y, then, for $y \in Y$, we have $M_{Y,\overline{y}} = \mathcal{O}^*_{Y,\overline{y}} \times \prod_{\pi_i \notin \mathcal{O}^*_{X,\overline{x}}} \overline{\pi_i}^{\mathbb{N}}$ and the morphism $M_{Y,\overline{y}} \to \mathcal{O}_{Y,\overline{y}}$ sends $\overline{\pi_i}$ to the image of $\pi_i \in \mathcal{O}_X$ in $\mathcal{O}_{Y,\overline{y}}$. We have $\Omega^1_{Y/s}(\log(M_Y/N_s)) \cong \mathcal{O}_Y \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log(M/N))$.

3.2. Log étale and log smooth morphisms. — Étale morphisms and smooth morphisms of fine log schemes are defined similarly as schemes as follows.

***Definition 3.2.1* ([Kat89] (3.1)).** — We say that a morphism of fine log schemes $i\colon (X,M) \hookrightarrow (Y,N)$ is a *closed immersion* (resp. an *exact closed immersion*) if it is a closed immersion in the underlying schemes and the morphism $i^*N \to M$ is surjective (resp. an isomorphism).

***Definition 3.2.2* ([Kat89] (3.3)).** — We say that a morphism of fine log schemes $f\colon (X,M) \to (Y,N)$ is *étale* (resp. *smooth*) if it is locally of finite presentation in the underlying schemes and, for every commutative diagram of fine log schemes

$$\begin{array}{ccc} (X,M) & \xleftarrow{\ s\ } & (T,M_T) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle i} \\ (Y,N) & \xleftarrow{\ t\ } & (T',M_{T'}) \end{array}$$

such that i is an exact closed immersion and $I^2 = 0$ for the ideal I of $\mathcal{O}_{T'}$ defining T, there exists a unique morphism (resp. a morphism étale locally on T') $g\colon (T',M_{T'}) \to (X,M)$ such that $g \circ i = s$ and $f \circ g = t$.

If $f\colon (X,M) \to (Y,N)$ is étale (resp. smooth), its relative differential module vanishes (resp. is locally free and locally of finite rank) ([**Kat89**] Proposition (3.10)).

A morphism of schemes $f\colon X \to Y$ is smooth if and only if étale locally on X, there exists an étale morphism of Y-schemes $X \to Y[T_1,\dots,T_d]$ for some integer d. We have the following analogue for log schemes.

***Theorem 3.2.3* ([Kat89] Theorem (3.5)).** — *Let $f\colon (X,M) \to (Y,N)$ be a morphism of fine log schemes. Then, f is étale (resp. smooth) if and only if étale locally on X, there exist isomorphisms $N \cong Q^a_Y$, $M \cong P^a_X$, where (Q_Y,β), (P_X,α) are pre-log structures with P, Q finitely generated and integral monoids, and an injective morphism of monoids $h\colon P \to Q$ compatible with f such that the canonical morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ induced by α, β and h, is étale and the cokernel (resp. the torsion part of the cokernel) of h^{gp} is a finite group of order invertible on X.*

If we have $N \cong Q^a_Y$, $M \cong P^a_X$ and h as in the theorem, there is an isomorphism $\Omega^1_{X/Y}(\log(M/N)) \cong \mathcal{O}_X \otimes_{\mathbb{Z}} P^{\mathrm{gp}}/h^{\mathrm{gp}}(Q^{\mathrm{gp}})$.

Example 3.2.4

(1) For a finite extension $A \to A'$ of discrete valuation rings, if we endow $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$ with the log structures M, M' defined by the closed point (Example 3.1.1 (1)), then $(S',M') \to (S,M)$ is étale if and only if A' is tamely ramified over A.

(2) The morphism f in Example 3.1.1 (2) is smooth.

(3) The morphism g in Example 3.1.1 (3) is smooth.

(4) If X is a smooth scheme over a field k with a reduced divisor D with normal crossings relative to k, then (X,M) defined as in Example 3.1.1 (1) is smooth over $\mathrm{Spec}(k)$ endowed with the trivial log structure.

4. Log crystalline cohomology

Let K, k and O_K be as in the Notation in the end of § 1. The étale cohomology of a variety over a field k with coefficients $\mathbb{Z}_l$ or $\mathbb{Q}_l$ (the so called l-adic cohomology) for a prime $l \neq p$ is an analogue of the singular cohomology of a topological space and satisfies good properties such as Poincaré duality. However, if $l = p$, the étale cohomology becomes smaller. For a proper smooth variety Y over k, the crystalline cohomology $H^*_{\text{crys}}(Y/W)$ supplies this lack; It is a finitely generated W-module endowed with a semi-linear automorphism (called the Frobenius).

In these notes, we are especially interested in the case that Y is the special fiber of a proper smooth scheme X over O_K. In this case, the crystalline cohomology tensored with K over W is canonically isomorphic to the de Rham cohomology $H^*_{\text{dR}}(X_K/K)$ of the generic fiber X_K (Berthelot-Ogus [**BO83**]) and it makes $H^*_{\text{dR}}(X_K/K)$ an object of $MF_K(\varphi)$ (§ 2.2).

In this section, we will survey the generalization of the above theory to a proper semi-stable scheme over O_K by O. Hyodo and K. Kato [**HK94**]. In the semi-stable reduction case, the following new phenomena occur: In addition to the Frobenius automorphism, the crystalline cohomology is naturally endowed with a linear endomorphism N called the monodromy operator, which vanishes if X/O_K is smooth. The isomorphism between the crystalline and the de Rham cohomologies depends on the choice of a uniformizer of K.

In the last subsection, we also review a crystalline interpretation of the rings B_{crys}, B_{st} and B_{dR}.

4.1. Log crystalline site. — We assume that the readers are familiar with the usual crystalline site ([**BO78**], [**Ber74**]) and we explain how it is extended to log schemes. In 4.1, S denotes a general scheme and is different from S in the Notation in § 1. Recall that a divided power (or PD for short) structure on an ideal I of a sheaf of rings $\mathcal{A}$ is a set of maps $\{\gamma_m\colon I \to \mathcal{A}\}_{m\in\mathbb{N}}$ indexed by $\mathbb{N} = \{0, 1, 2, \ldots\}$ satisfying the same properties as the operation $x \mapsto x^m/m!$ in characteristic 0 such as $\gamma_m(I) \subset I$ $(m \geqslant 1)$, $\gamma_m(x+y) = \sum_{0\leqslant i\leqslant m} \gamma_i(x)\gamma_{m-i}(y)$. We often write $x^{[m]}$ for $\gamma_m(x)$. By a *PD-thickening of fine log schemes*, we mean an exact closed immersion $(X, M) \hookrightarrow (Y, N)$ of fine log schemes endowed with a PD structure δ on the ideal of $\mathcal{O}_Y$ defining X. We have the following generalization of PD-envelopes.

Proposition and Definition 4.1.1 (**[Kat89] Proposition (5.3)**). — *Let (S, I, γ) be a scheme S endowed with a quasi-coherent PD-ideal (I, γ). Then, for any S-closed immersion $i\colon (X, M) \hookrightarrow (Y, N)$ of fine log schemes over S such that γ extends to X, there exist a PD-thickening $i_D\colon (X, M) \hookrightarrow (D, M_D)$ over S compatible with γ and an S-morphism $p_D\colon (D, M_D) \to (Y, N)$ satisfying $p_D \circ i_D = i$ and the following universal property: For any PD-thickening $i'\colon (X', M') \hookrightarrow (D', M_{D'})$ over S compatible with γ and any S-morphisms $u\colon (X', M') \to (X, M)$, $v\colon (D', M_{D'}) \to (Y, N)$ satisfying*

$v \circ i' = i \circ u$, there exists a unique S-PD-morphism $v_D\colon (D', M_{D'}) \to (D, M_D)$ such that $p_D \circ v_D = v$ and $v_D \circ i' = i_D \circ u$. We call (D, M_D) the PD-envelope of i compatible with γ.

If i admits a factorization $(X, M) \xrightarrow{j} (Y', N') \xrightarrow{k} (Y, N)$ with j an exact closed immersion and k étale, then the PD-envelope of i is the PD-envelope of $X \hookrightarrow Y'$ endowed with the inverse image of N'. In the general case, we take such a factorization étale locally on Y and glue using the universal property.

Definition 4.1.2 **([Kat89] (5.2)).** — Let (S, L, I, γ) be a fine log scheme (S, L) with a quasi-coherent PD-ideal (I, γ) such that $n \cdot \mathcal{O}_S = 0$ for a positive integer n and let (X, M) be a fine log scheme over (S, L) such that γ extends to X. We define the *crystalline site* $((X, M)/(S, L, I, \gamma))_{\mathrm{crys}}$ (or $(X/S)^{\log}_{\mathrm{crys}}$ for short) as follows: The objects of the underlying category are (S, L)-PD-thickenings $(i\colon (U, M|_U) \hookrightarrow (T, M_T), \delta)$ compatible with γ of étale X-schemes U endowed with $M|_U$, which we often abbreviate to $((T, M_T), \delta)$ or (T, M_T). A morphism in the category is a pair of an X-morphism $u\colon U' \to U$ and an (S, L)-PD-morphism $v\colon (T', M_{T'}) \to (T, M_T)$ compatible with i's. We say that a morphism (u, v) is strict étale if v is étale in the underlying scheme, $M_{T'} \cong v^* M_T$ and $U' \xrightarrow{\sim} U \times_T T'$. We say that a family of morphisms $\{(u_\lambda, v_\lambda)\}_{\lambda \in \Lambda}$ is a strict étale covering if each (u_λ, v_λ) is strict étale and $\cup_\lambda u_\lambda(T_\lambda) = T$. We give the above category the topology associated to the pre-topology defined by strict étale coverings.

We define the structure sheaf $\mathcal{O}_{(X,M)/(S,L)}$ by $\Gamma((T, M_T), \mathcal{O}_{(X,M)/(S,L)}) = \Gamma(T, \mathcal{O}_T)$ and the PD-ideal $J_{(X,M)/(S,L)}$ by $\Gamma((T, M_T), J_{(X,M)/(S,L)}) = \Gamma(T, J_T)$, where J_T denotes the PD-ideal of $\mathcal{O}_T$ defining U.

As in the scheme case, the crystalline topos $(X/S)^{\log}_{\mathrm{crys}}$ is functorial on both (X, M) and (S, L, I, γ). We have a canonical morphism of topos

$$u^{\log}_{X/S}\colon (X/S)^{\log\sim}_{\mathrm{crys}} \longrightarrow X^{\sim}_{\text{ét}}$$

defined by $\Gamma(U, u^{\log}_{X/S*}\mathcal{F}) = \Gamma((U/S)^{\log}_{\mathrm{crys}}, \mathcal{F}|_{(U/S)^{\log}_{\mathrm{crys}}})$.

Suppose that there exists a closed immersion $i\colon (X, M) \hookrightarrow (Y, N)$ over (S, L) with $(Y, N)/(S, L)$ smooth and let (D, M_D) be the PD-envelope of i compatible with γ. Let J_D denote the PD-ideal of $\mathcal{O}_D$ defining X. Then, as in the scheme case, we have:

Theorem 4.1.3 **([Kat89] Theorem (6.4)).** — *There exist canonical isomorphisms in $D^+(X_{\text{ét}}, \mathbb{Z})$:*

$$Ru^{\log}_{X/S*}\mathcal{O}_{X/S} \cong \mathcal{O}_D \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{Y/S}(\log(N/L))$$
$$Ru^{\log}_{X/S*}J^{[r]}_{X/S} \cong J_D^{[r-\cdot]} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{Y/S}(\log(N/L)) \quad (r \in \mathbb{Z}),$$

Here, for a PD-ideal (J, δ) of a sheaf of rings $\mathcal{A}$, we denote by $J^{[r]}$ $(r \in \mathbb{Z},\ r \geqslant 1)$ the r-th divided power of J, that is, the ideal generated by $\delta_{m_1}(x_1) \cdots \delta_{m_s}(x_s)$,

$(x_1, \ldots, x_s \in J,\ m_1, \ldots, m_s > 0,\ m_1 + \cdots + m_s \geqslant r)$. We set $J^{[r]} = \mathcal{A}$ for $r \in \mathbb{Z}$, $r \leqslant 0$.

We also have the following invariance property.

***Theorem 4.1.4* (cf. [Ber74] III Théorème 2.3.4).** — *With the notation in Definition 4.1.2, let J be a PD-subideal of I (i.e. $\gamma_m(J) \subset J$ for all $m \geqslant 1$) and let (X', M') be the reduction mod J of (X, M). Then the natural homomorphism*

$$H^m((X/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X/S}) \longrightarrow H^m((X'/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{X'/S})$$

is an isomorphism for any integer $m \geqslant 0$.

4.2. Log crystalline cohomology. — Let K, O_K, k, (S, N) and (s, N_s) be as in the Notation in the end of § 1. We will define a crystalline cohomology of a smooth fine log scheme (Y, M_Y) over (s, N_s) whose underlying scheme Y is proper over s. See [**HK94**] § 3 and [**Tsu99**] § 4.2, § 4.3, § 4.4 for details.

Let N_n^0 denote the log structure on $\mathrm{Spec}(W_n)$ associated to the pre-log structure $\Gamma(s, N_s) \to k \xrightarrow{[\,]} W_n$, where $[\,]$ denotes the Teichmüller representative. Note that if we denote by $\overline{\pi}$ the image of $\pi \in \Gamma(S, N)$ in $\Gamma(s, N_s)$, we have $\Gamma(s, N_s) = k^* \times \overline{\pi}^{\mathbb{N}}$ and the image of $\overline{\pi}$ in k is 0. We have $N_s = N_1^0$ and $\Gamma(\mathrm{Spec}(W_n), N_n^0) = W_n^* \times \overline{\pi}^{\mathbb{N}}$. The multiplication by p on $\Gamma(s, N_s)$ and the Frobenius σ of W_n induce a lifting of Frobenius F on $(\mathrm{Spec}(W_n), N_n^0)$. (The absolute Frobenius $F_{(X,M)}$ of a log scheme (X, M) over $\mathbb{F}_p$ is the absolute Frobenius F_X of X with $F_X^{-1}(M) \cong M \xrightarrow{p} M$).

Remark 4.2.1. — If we endow $\mathrm{Spec}(W)$ with the log structure defined by its closed point, then its reduction mod p^n $(n \geqslant 2)$ does not have a lifting of Frobenius because $\sigma(p) = p$ but p should be sent to $p^p \cdot u$ $(u \in 1 + pW_n)$ in the log structure.

Let γ be the PD-structure on pW_n defined by $\gamma_m(a \mod p^n) = a^m/m! \mod p^n$ $(a \in W)$. Then, for (Y, M_Y) as above, the crystalline cohomology

$$M_n^m := H^m(((Y, M_Y)/(W_n, N_n^0, pW_n, \gamma))_{\mathrm{crys}}, \mathcal{O}_{(Y,M_Y)/(W_n,N_n^0)})$$

is a finitely generated W_n-module endowed with a semi-linear endomorphism φ induced by the absolute Frobenius of (Y, M_Y) and the lifting of Frobenius F on $(\mathrm{Spec}(W_n), N_n^0)$. We set

$$M_\infty^m := \varprojlim_n M_n^m \quad \text{and} \quad D^m := K_0 \otimes_W M_\infty^m.$$

Then M_∞^m and D^m are finitely generated over W and K_0 respectively. If (Y, M_Y) is of Cartier type ([**Kat89**] Definition (4.8)) over (s, N_s), φ on D^m is bijective ([**HK94**] § 3). (The condition "of Cartier type" is necessary for the Cartier isomorphism $\mathcal{H}^q(\Omega^{\cdot}_{Y/s}(\log(M_Y/N_s)) \cong \Omega^q_{Y/s}(\log(M_Y/N_s))$ [**Kat89**] Theorem (4.12) (1).)

M_n^m is endowed with a kind of HPD-stratification with respect to

$$(\mathrm{Spec}(W_n), N_n^0)/W_n$$

as follows: Let (D_n, M_{D_n}) be the PD-envelope of $(\mathrm{Spec}(W_n), N_n^0)$ in the fiber product of two copies of $(\mathrm{Spec}(W_n), N_n^0)$ over W_n, let p_1, p_2 denote the two projections $(D_n, M_{D_n}) \rightrightarrows (\mathrm{Spec}(W_n), N_n^0)$, and let v be the unique element of $\Gamma(D_n, 1 + J_{D_n})$ such that $v \cdot p_2^*(\pi) = p_1^*(\pi)$ in $\Gamma(D_n, M_{D_n})$. Then $\Gamma(D_n, \mathcal{O}_{D_n})$ is a PD-polynomial ring over W_n with its indeterminate $v - 1$, and M_n^m has an "HPD-stratification ":

$$\varepsilon \colon p_1^* M_n^m \xrightarrow{\sim} M_n^m(1) \xleftarrow{\sim} p_2^* M_n^m (= \oplus_{i \geqslant 0} M_n^m \cdot (v-1)^{[i]}),$$

where $M_n^m(1) = H^m((Y, M_Y)/(D_n, M_{D_n}, \mathrm{Ker}(\mathcal{O}_{D_n} \to k), [\,]), \mathcal{O}_{(Y,M_Y)/(D_n,M_{D_n})})$.

We define the monodromy operator $N \colon M_n^m \to M_n^m$ by $N(x) =$ the coefficient of $(v-1)$ in $\varepsilon(p_1^*(x))$. The lifting of Frobenius on $(\mathrm{Spec}(W_n), N_n^0)$ induces that on (D_n, M_{D_n}) and ε becomes compatible with the Frobenius endomorphisms. Hence, from $\varphi(v-1) = v^p - 1 = p(v-1)+$(a term of degree $\geqslant 2$ in $(v-1)$), we obtain

$$N\varphi = p\varphi N.$$

4.3. Comparison with de Rham cohomology. — In § 4.3, we consider a smooth fine log scheme (X, M) over (S, N) whose underlying scheme is proper over S. Let (Y, M_Y) be the special fiber $(X, M) \times_{(S,N)} (s, N_s)$. We assume that (Y, M_Y) is of Cartier type over (s, N_s) ([**Kat89**] Definition (4.8)). A proper scheme X over S with semi-stable reduction endowed with the log structure defined by the special fiber (Example 3.1.1) satisfies this condition (Example 3.2.4, [**Kat89**] Remark after Definition (4.8)).

We define the crystalline cohomology $H^m_{\mathrm{crys}}((X, M))$ of (X, M) to be the crystalline cohomology D^m of (Y, M_Y) defined in § 4.2, which is a K_0-vector space of finite dimension endowed with a σ-semilinear automorphism φ and a K_0-linear endomorphism N satisfying $N\varphi = p\varphi N$.

We define the de Rham cohomology $H^m_{\mathrm{dR}}((X_K, M_K)/K)$ of the generic fiber $(X_K, M_K) := (X, M) \times_{(S,N)} \mathrm{Spec}(K)$ to be

$$\begin{aligned} &H^m(X_K, \Omega^{\cdot}_{X_K/K}(\log(M_K))) \\ &\qquad \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^m(((X_n, M_n)/(S_n, N_n, p\mathcal{O}_{S_n}, \gamma))_{\mathrm{crys}}, \mathcal{O}_{(X_n,M_n)/(S_n,N_n)}), \end{aligned}$$

which is a K-vector space of finite dimension. We write D^m_{dR} for $H^m_{\mathrm{dR}}((X_K, M_K)/K)$ to simplify the notation in the following.

***Theorem 4.3.1* ([HK94] Theorem (5.1), cf. [Tsu99] § 4.4).** — *There exists a canonical isomorphism depending on the choice of the uniformizer π of K:*

$$\rho_\pi \colon K \otimes_{K_0} D^m \xrightarrow{\sim} D^m_{\mathrm{dR}}$$

functorial on X and compatible with the cup products. For another choice of the uniformizer π', we have

$$\rho_{\pi'} = \rho_\pi \circ \exp(\log(\pi'\pi^{-1}) \cdot (1_K \otimes N)).$$

In the rest of § 4.3, we will explain how to construct the map ρ_π. We introduce an intermediate crystalline cohomology $\mathcal{D}^m$ as follows.

Let $\mathcal{L}(T)$ denote the log structure on $\mathrm{Spec}(W_n[T])$ defined by the divisor $\{T = 0\}$ and let $i_{E_n,\pi}\colon (S_n, N_n) \hookrightarrow (E_n, M_{E_n})$ be the PD-envelope compatible with (pW_n, γ) (see § 4.1) of the exact closed immersion $(S_n, N_n) \hookrightarrow (\mathrm{Spec}(W_n[T]), \mathcal{L}(T))$ defined by $T \mapsto \pi$. The scheme E_n is explicitly written as

$$\mathrm{Spec}(W[T, T^{me}/m!\ (m \geqslant 1)] \otimes_W W_n)$$

where $e = [K : K_0]$. We have another exact closed immersion

$$i_{E_n,0}\colon (\mathrm{Spec}(W_n), N_n^0) \longrightarrow (E_n, M_{E_n})$$

defined by $T^{me}/m! \mapsto 0$ $(m \geqslant 1)$ and $T \mapsto \overline{\pi}$ in the log structure. The liftings of Frobenius on $(\mathrm{Spec}(W_n[T]), \mathcal{L}(T))$ defined by $T \mapsto T^p$ and $\sigma\colon W_n \xrightarrow{\sim} W_n$ induces the lifting of Frobenius F_{E_n} on (E_n, M_{E_n}) compatible with the canonical PD-structure $\overline{\delta}$ on $\overline{J}_{E_n} := \mathrm{Ker}(\mathcal{O}_{E_n} \to \mathcal{O}_{S_1})$. The exact closed immersion $i_{E_n,0}$ is compatible with the liftings of Frobenius.

We define the intermediate cohomology $\mathcal{D}^m$ by

$$\begin{aligned}\mathcal{M}_n^m &:= H^m(((X_n, M_n)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta}))_{\mathrm{crys}}, \mathcal{O}_{X_n/E_n})\\ &\cong H^m(((X_1, M_1)/(E_n, M_{E_n}, \overline{J}_{E_n}, \overline{\delta}))_{\mathrm{crys}}, \mathcal{O}_{X_1/E_n}),\\ \mathcal{D}^m &:= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mathcal{M}_n^m.\end{aligned}$$

By the base changes $i_{E_n,0}$ and $i_{E_n,\pi}$, we obtain two homomorphisms

$$D^m \xleftarrow{\mathrm{pr}_0} \mathcal{D}^m \xrightarrow{\mathrm{pr}_\pi} D^m_{\mathrm{dR}}.$$

The absolute Frobenius of (X_1, M_1) and F_{E_n} induces an endomorphism φ on $\mathcal{D}^m$ and pr_0 is compatible with φ.

$\mathcal{M}_n^m$ is endowed with an HPD-stratification with respect to $(S_n, N_n) \hookrightarrow (\mathrm{Spec}(W_n[T]), \mathcal{L}(T))/W_n$ as follows: Let $(E_n(1), M_{E_n(1)})$ be the PD-envelope of (S_n, N_n) in the fiber product of two copies of $(\mathrm{Spec}(W_n[T]), \mathcal{L}(T))$ over W_n, let p_1, p_2 denote the two projections $(E_n(1), M_{E_n(1)}) \rightrightarrows (E_n, M_{E_n})$, and let u denote the unique element of $\Gamma(E_n(1), 1 + J_{E_n(1)})$ such that $u \cdot p_2^*(T) = p_1^*(T)$ in $\Gamma(E_n(1), M_{E_n(1)})$. Then $\Gamma(E_n(1), \mathcal{O}_{E_n(1)})$ is a PD-polynomial ring over $\Gamma(E_n, \mathcal{O}_{E_n})$ with its indeterminate $u - 1$ for either of the two $\Gamma(E_n, \mathcal{O}_{E_n})$-algebra structures, and $\mathcal{M}_n^m$ has an HPD-stratification:

$$\varepsilon\colon p_1^*\mathcal{M}_n^m \xrightarrow{\sim} \mathcal{M}_n^m(1) \xleftarrow{\sim} p_2^*\mathcal{M}_n^m (= \oplus_{i\geqslant 0}\mathcal{M}_n^m \cdot (u-1)^{[i]}),$$

where $\mathcal{M}_n^m(1) = H^m((X_n, M_n)/(E_n(1), M_{E_n(1)}, \overline{J}_{E_n}, \overline{\delta}), \mathcal{O}_{X_n/E_n(1)})$.

We define the monodromy operator N on $\mathcal{M}_n^m$ by $N(x) =$ the coefficient of $(u-1)$ in $\varepsilon(p_1^*(x))$. As in the case of M_n^m, we see $N\varphi = p\varphi N$ on $\mathcal{M}_n^m$. The projection pr_0 is compatible with N, that is, $\mathrm{pr}_0 N = N \mathrm{pr}_0$.

Proposition 4.3.2 **([HK94] Lemma (5.2), [Tsu99] Propositions 4.4.6, 4.4.9)**

There exists a unique K_0-linear section s of pr_0 *compatible with* φ. *The section s is functorial on X and compatible with N and the cup products. It also induces an isomorphism*

$$R_E \otimes_W D^m \xrightarrow{\sim} \mathcal{D}^m,$$

where $R_E = \varprojlim_n \Gamma(E_n, \mathcal{O}_{E_n})$.

The isomorphism ρ_π is the K-linearization of $\mathrm{pr}_\pi \circ s$.

4.4. Crystalline interpretation of B_{crys}, B_{st} and B_{dR}. — We will give a crystalline interpretation of the rings B_{crys}, B_{st} and B_{dR}. We define $H^m((\overline{S},\overline{N})/W, J^{[r]})$ $(m \geqslant 0, r \geqslant 0)$ to be

$$\varprojlim_n \Big(\varinjlim_{K'} H^m(((S'_n, N'_n)/(W_n, pW_n, \gamma))_{\mathrm{crys}}, J^{[r]}_{(S'_n,N'_n)/W_n}) \Big),$$

where W_n is endowed with the trivial log structure, K' ranges over all finite extensions of K contained in $\overline{K}$ and (S', N') denotes the scheme $\mathrm{Spec}(O_{K'})$ endowed with the log structure defined by the closed point. The crystalline cohomology over the base (S,N) or (E, M_E) appearing below is defined similarly. See § 4.3 for the definition of (E_n, M_{E_n}).

By functoriality, the absolute Frobenius of (S'_1, N'_1) and the Frobenii of W_n and (E_n, M_{E_n}) induce the Frobenius endomorphisms φ on $H^m((\overline{S},\overline{N})/W)$ and $H^m((\overline{S},\overline{N})/(E, M_E))$. $H^m((\overline{S},\overline{N})/(E, M_E))$ is naturally endowed with a monodromy operator N satisfying $N\varphi = p\varphi N$ in the same way as $H^m((X_n, M_n)/(E_n, M_{E_n}))$ in § 4.3 (**[Tsu99]** § 4.3). We will denote by the operation $\mathbb{Q}_p \otimes_{\mathbb{Z}_p}$ by the subscript $\mathbb{Q}_p$.

Proposition 4.4.1 **([Fon83] § 3, [Kat94a] § 3, [Tsu99] § 1.6, § 4.6)**

(1) *There exist canonical G_K-equivariant isomorphisms:*

$$\begin{aligned} B^+_{\mathrm{crys}} &\cong H^0((\overline{S},\overline{N})/W)_{\mathbb{Q}_p}, \\ B^+_{\mathrm{st}} &\cong (H^0((\overline{S},\overline{N})/(E, M_E))_{\mathbb{Q}_p})^{N\text{-}nilp}, \\ B^+_{\mathrm{dR}} &\cong \varprojlim_s (H^0((\overline{S},\overline{N})/(S,N), \mathcal{O}/J^{[s]})_{\mathbb{Q}_p}) \\ \cup \quad & \qquad \cup \\ \mathrm{Fil}^r B_{\mathrm{dR}} &\cong \varprojlim_s (H^0((\overline{S},\overline{N})/(S,N), J^{[r]}/J^{[s]})_{\mathbb{Q}_p}) \quad (r \in \mathbb{Z}, r \geqslant 0), \end{aligned}$$

where B^+_{crys} and B^+_{st} are as in § 2.3 *and N-nilp denotes the part where N is nilpotent. The first (resp. the last) isomorphism is compatible with φ (resp. φ and N). Furthermore the pull-backs by $(E_n, M_{E_n}) \to$* $\mathrm{Spec}(W_n)$ *and $i_{E_n,\pi}\colon (S_n, N_n) \hookrightarrow (E_n, M_{E_n})$ in the RHS correspond to the injections $B^+_{\mathrm{crys}} \hookrightarrow B^+_{\mathrm{st}}$ and $\iota_\pi\colon B^+_{\mathrm{st}} \hookrightarrow B^+_{\mathrm{dR}}$ associated to π (see* § 2.3 *and Remark* 2.1.3 (2)).

(2) *The cohomologies $H^m((\overline{S},\overline{N})/W)$, $H^m((\overline{S},\overline{N})/(S,N), J^{[r]}/J^{[s]})$ $(s \geqslant r \geqslant 0)$ and $H^m((\overline{S},\overline{N})/(E, M_E))$ vanish if $m > 0$.*

5. Syntomic complex

In this section, we will survey the definition of the syntomic complexes and their properties proven in [**Tsu99**] § 2.

Syntomic cohomology was first introduced by J.-M. Fontaine and W. Messing as an intermediate cohomology in their proof of C_{crys} (=C_{st} in the good reduction case) in [**FM87**]. We will explain their idea briefly. Assume $K = K_0$, let X be a proper smooth scheme over W, and suppose that C_{crys} is true for X. Then we have the following exact sequence for $r \geqslant m$ ([**FM87**] III 2.4. Proposition and the following remark):

$$0 \longrightarrow H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \longrightarrow \mathrm{Fil}^r(B^+_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(X/W)) \xrightarrow{1-\varphi/p^r} B^+_{\mathrm{crys}} \otimes_W H^m_{\mathrm{crys}}(X/W) \longrightarrow 0.$$

See § 2.3 for the definition of B^+_{crys}. Furthermore, the right two terms are isomorphic to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathrm{crys}}(\overline{X}/W, J^{[r]})$ and $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathrm{crys}}(\overline{X}/W)$ respectively (Künneth formula [**FM87**] III 1.5. Proposition. cf. the crystalline interpretation of B^+_{crys} in Proposition 4.4.1), where $\overline{X} = X \otimes_{O_K} O_{\overline{K}}$. Hence we have a quasi-isomorphism:

$$\text{(5.0.1)} \quad H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(r) \xrightarrow{\sim} [\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathrm{crys}}(\overline{X}/W, J^{[r]}) \xrightarrow{1-\varphi/p^r} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^m_{\mathrm{crys}}(\overline{X}/W)]$$

Fontaine and Messing considered the RHS of (5.0.1) syntomic (= flat and locally complete intersection) locally on $\overline{X}$ and constructed sheaves S^r_n $(n \geqslant 1, r \geqslant 0)$ on the syntomic site of $\overline{X}_s := \overline{X} \otimes \mathbb{Z}/p^s\mathbb{Z}$ $(s \geqslant n+r)$, which we can regard as an analogue of $\mathbb{Z}/p^n\mathbb{Z}(r)$ in characteristic p. The syntomic cohomology $H^m(\overline{X}, S^r_n)$ is defined to be $H^m(\overline{X}_{s,\mathrm{syn}}, S^r_n)$ $(s \geqslant n+r)$. Then Fontaine and Messing constructed canonical maps

$$\text{(5.0.2)} \qquad H^m(\overline{X}, S^r_n) \longrightarrow H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(r))$$

and proved C_{crys} in the case $\dim(X_K) < p$ and $K = K_0$ (see the beginning of § 7).

In [**Kat87**], [**Kur87**], K. Kato and M. Kurihara proved that the above maps are isomorphisms if $m \leqslant r \leqslant p-2$ without assuming $K = K_0$, from which K. Kato and W. Messing derived C_{crys} in the case $\dim(X_K) \leqslant (p-2)/2$ ([**KM92**], see the beginning of § 7). In [**Kat94a**], K. Kato extended these results to the semi-stable reduction case. In their proof, Kato and Kurihara used an étale localization of the RHS of (5.0.1): syntomic complexes (not sheaves) $\mathcal{S}_n(r)$ ($s^{\log}_n(r)$ in the semi-stable reduction case) $(r \leqslant p-1)$ whose étale cohomology gives the syntomic cohomology; it is defined explicitly in terms of certain de Rham complexes (see § 5.2). They compared syntomic complexes with p-adic nearby cycles based on the calculation of the latter by Bloch-Kato [**BK86**] (in the good reduction case) and O. Hyodo [**Hyo88**] (in the semi-stable reduction case).

If $r \geqslant p-1$, unfortunately, the homomorphism (5.0.2) with $m \leqslant r$ does not seem to be an isomorphism in general. In fact, the sheaves S^r_n for $r \geqslant p$ are defined in an adhoc manner compared to the case $r \leqslant p-1$. However, if we allow kernels and cokernels with exponents bounded when n varies, we can remove the restriction $r \leqslant p-2$ ([**Tsu99**] § 2, § 3). We will survey it in § 5 and § 6. We will introduce two complexes of étale sheaves $\mathcal{S}_n^{\sim}(r)$ and $\mathcal{S}'_n(r)$. The first one is canonical but different from $s_n^{\log}(r)$ (defined by K. Kato) when $r \leqslant p-1$. The second one coincides with $s_n^{\log}(r)$ when $r \leqslant p-1$ but depends on some choices when $r \geqslant p$. There is a canonical morphism $\mathcal{S}_n^{\sim}(r) \to \mathcal{S}'_n(r)$ quasi-isomorphic "up to bounded torsion". The complex $\mathcal{S}_n^{\sim}(r)$ is used to prove an invariance (up to bounded torsion) of $\mathcal{H}^q(\mathcal{S}_n^{\sim}(r))$ ($q \leqslant r$) under Tate twists (§ 5.2) and the complex $\mathcal{S}'_n(r)$ is used to compare $\mathcal{H}^q(\mathcal{S}'_n(q))$ with the corresponding p-adic vanishing cycles (§ 5.4, § 6.1, § 6.3). We can also define the syntomic cohomology and the morphism (5.0.2) in the semi-stable reduction case by generalizing the method of Fontaine-Messing [**FM87**] (see [**Bre98b**], [**Tsu98**], [**BM**]), but we won't treat that approach in these notes.

5.1. The complexes $\mathcal{S}_n^{\sim}(r)$. — Let (X,M) be a fine log scheme over W whose underlying scheme X is of finite type over W and let (X_n,M_n) denote $(X,M)\otimes\mathbb{Z}/p^n\mathbb{Z}$. For $r\in\mathbb{Z}$, $r\geqslant 0$, we will define the object $\mathcal{S}_n^{\sim}(r)_{(X,M)}$ of $D^+((X_1)_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$ such that there exists a canonical distinguished triangle

$$\to \mathcal{S}_n^{\sim}(r)_{(X,M)} \longrightarrow Ru_{(X_n,M_n)/W_n*}J^{[r]}_{(X_n,M_n)/W_n} \xrightarrow{p^r-\varphi} Ru_{(X_n,M_n)/W_n*}\mathcal{O}_{(X_n,M_n)/W_n}.$$

Here $u_{(X_n,M_n)/W_n}$ denotes the morphism of topos

$$u_{(X_n,M_n)/W_n}\colon ((X_n,M_n)/(W_n,pW_n,\gamma))^{\sim}_{\text{crys}} \longrightarrow (X_n)^{\sim}_{\text{ét}} = (X_1)^{\sim}_{\text{ét}}$$

and $\operatorname{Spec}(W_n)$ is endowed with the trivial log structure. Note that we cannot take the mapping fiber in a derived category.

First assume that there is a closed immersion i of (X,M) into a smooth fine log scheme (Z,M_Z) over W endowed with a compatible system of liftings of Frobenius $\{F_{Z_n}\}_{n\geqslant 1}$ on $(Z_n,M_{Z_n}) := (Z,M_Z)\otimes\mathbb{Z}/p^n\mathbb{Z}$ (which exists if Z is affine). Set $\omega^q_{Z_n} := \Omega^q_{Z_n/W_n}(\log(M_{Z_n}))$ ($q\geqslant 0$) to simplify the notation. In this case, noting Theorem 4.1.3, we define the syntomic complex $\mathcal{S}_n^{\sim}(r)_{(X,M),(Z,M_Z)}$ to be the mapping fiber of

$$J_{D_n}^{[r-\cdot]}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n} \xrightarrow{p^r-\varphi} \mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n},$$

where (D_n,M_{D_n}) denotes the PD-envelope of $i\otimes\mathbb{Z}/p^n\mathbb{Z}$ compatible with γ and J_{D_n} is the PD-ideal $\operatorname{Ker}(\mathcal{O}_{D_n}\to\mathcal{O}_{X_n})$. Its degree q-part is

$$(J_{D_n}^{[r-q]}\otimes_{\mathcal{O}_{Z_n}}\omega^q_{Z_n})\oplus(\mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega^{q-1}_{Z_n})$$

and its differential map is given by

$$d(x,y) = (dx,(p^r-\varphi)(x)-dy)$$

for $x\in J_{D_n}^{[r-q]}\otimes_{\mathcal{O}_{Z_n}}\omega^q_{Z_n}$ and $y\in\mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega^{q-1}_{Z_n}$. We define a product

$$\mathcal{S}_n^{\sim}(r)_{(X,M),(Z,M_Z)}\otimes\mathcal{S}_n^{\sim}(r')_{(X,M),(Z,M_Z)} \longrightarrow \mathcal{S}_n^{\sim}(r+r')_{(X,M),(Z,M_Z)} \tag{5.1.1}$$

by

$$(x,y)\otimes(x',y')\longmapsto(x\wedge x',(-1)^q p^r x\wedge y'+y\wedge\varphi(x'))$$

$$(x^{(\prime)},y^{(\prime)})\in(J_{D_n}^{[r^{(\prime)}-q^{(\prime)}]}\otimes_{\mathcal{O}_{Z_n}}\omega_{Z_n}^{q^{(\prime)}})\oplus(\mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega_{Z_n}^{q^{(\prime)}-1})$$

and a symbol map

$$M_n^{\mathrm{gp}}\longrightarrow\mathcal{S}_n^{\sim}(1)_{(X,M),(Z,M_Z)}[1] \tag{5.1.2}$$

in $D^+((X_1)_{\text{ét}},\mathbb{Z})$ by

$$M_n^{\mathrm{gp}}[-1]\xleftarrow{\sim}[1+J_{D_n}\longrightarrow M_{D_n}^{\mathrm{gp}}]\longrightarrow\mathcal{S}_n^{\sim}(1)_{(X,M),(Z,M_Z)},$$

where the first quasi-isomorphism is induced by $M_{D_n}^{\mathrm{gp}}/(1+J_{D_n})\xrightarrow{\sim}M_n^{\mathrm{gp}}$ and the second morphism is defined by $\log\colon 1+J_{D_n}\to J_{D_n}$ and

$$\begin{aligned}M_{D_n}^{\mathrm{gp}}&\longrightarrow(\mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega_{Z_n}^1)\oplus\mathcal{O}_{D_n}\\ a&\longmapsto\quad(d\log(a),\log(a^p\varphi_{D_n}(a)^{-1})).\end{aligned}$$

We can also define a homomorphism

$$\mu_{p^n}(\mathcal{O}_{X_n})\longrightarrow\mathcal{H}^0(\mathcal{S}_n^{\sim}(1)_{(X,M),(Z,M_Z)}) \tag{5.1.3}$$

by $\varepsilon\mapsto\log(\widetilde{\varepsilon}^{p^n})$, where $\widetilde{\varepsilon}$ denotes a lifting of ε in $\mathcal{O}_{D_n}^*$. Note $\widetilde{\varepsilon}^{p^n}\in 1+J_{D_n}$ since $\varepsilon^{p^n}=1$.

We define $\mathcal{S}_n^{\sim}(r)_{(X,M)}$ to be the image of $\mathcal{S}_n^{\sim}(r)_{(X,M),(Z,M_Z)}$ in $D^+((X_1)_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$, which is independent of the choice of i and $\{F_{Z_n}\}$ up to canonical isomorphisms ([**Tsu99**] § 2.1).

In the general case, we take an étale hypercovering $X^{\cdot}$ of X and a closed immersion of $(X^{\cdot},M|_{X^{\cdot}})$ into a smooth fine simplicial log scheme $(Z^{\cdot},M_{Z^{\cdot}})$ over W with a compatible system of liftings of Frobenius $\{F_{Z_n^{\cdot}}\}$ and "glue" the above complex associated to each component of the simplicial log schemes using cohomological descent. They still have a product structure and a symbol map. See [**Tsu99**] § 2.1 for details.

Let (X,M) be a fine log scheme over (S,N) whose underlying scheme is of finite type over O_K and let $\overline{Y}$ denote $X\otimes_{O_K}\overline{k}$. For $r\in\mathbb{Z}$, $r\geqslant 0$, we define $\mathcal{S}_n^{\sim}(r)_{(\overline{X},\overline{M})}\in D^+(\overline{Y}_{\text{ét}},\mathbb{Z}/p^n\mathbb{Z})$ to be the "inductive limit" of $\mathcal{S}_n^{\sim}(r)_{(X',M')}|_{\overline{Y}_{\text{ét}}}$, where K' ranges over all finite extensions of K contained in $\overline{K}$, (S',N') is the scheme $\operatorname{Spec}(O_{K'})$ with the log structure defined by the closed point, and $(X',M')=(X,M)\times_{(S,N)}(S',N')$. Precisely speaking, since we cannot take the inductive limit in the derived category, we choose a compatible system $\{(S',N')\hookrightarrow(V',M_{V'}),\{F_{V'_n}\}\}_{K'}$ of a closed immersion of (S',N') into a smooth fine log scheme $(V',M_{V'})$ over W with liftings of Frobenius $\{F_{V'_n}\}$ of its reduction mod p^n, and use the compatible system

$$\{(X^{\cdot},M|_{X^{\cdot}})\times_{(S,N)}(S',N')\hookrightarrow(Z^{\cdot},M_{Z^{\cdot}})\times_W(V',M_{V'}),\{F_{Z_n^{\cdot}}\times F_{V'_n}\}\}_{K'}$$

to describe $\mathcal{S}_n^{\sim}(r)_{(X',M')}$ as explicit complexes. Here $X^{\cdot}$ and $(Z^{\cdot},M_{Z^{\cdot}})$ is the same as in the above definition of $\mathcal{S}_n^{\sim}(r)_{(X,M)}$ in the general case. See [**Tsu99**] § 2.1 for details. We define the syntomic cohomology $H^m((\overline{X},\overline{M}),S^r_{\mathbb{Q}_p})$ to be

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n H^m_{\text{ét}}(\overline{Y},\mathcal{S}^{\sim}_n(r)_{(\overline{X},\overline{M})}).$$

5.2. The complexes $\mathcal{S}'_n(r)$. — Next let us define another complex $\mathcal{S}'_n(r)$ ($r\in\mathbb{Z}$, $r\geqslant 0$). Roughly speaking, we replace $p^r-\varphi$ by $1-$"φ/p^r" in the definition of $\mathcal{S}^{\sim}_n(r)$.

Let (X,M) be a fine log scheme over W whose underlying scheme X is of finite type over W. We assume that X is flat over W and that there exists an exact closed immersion i of (X,M) into a smooth fine log scheme (Z,M_Z) over W endowed with a compatible system of liftings of Frobenius $\{F_{Z_n}\}$ of (Z_n,M_{Z_n}) satisfying the following conditions: i is a regular closed immersion in the underlying scheme (EGA IV Définition (16.9.2)) and there exist global sections $T_1,\ldots,T_d$ of M_Z such that $F_{Z_n}(T_i)=T_i^p$ $(1\leqslant i\leqslant d)$ and $d\log(T_i)$ $(1\leqslant i\leqslant d)$ form a basis of $\omega^1_Z:=\Omega^1_Z(\log M_Z)$. (Any smooth fine log scheme (X,M) over (S,N) satisfies this assumption étale locally on X. See after the statement of Theorem 5.3.2.) Choose such i and $\{F_{Z_n}\}$. Let (D_n,M_{D_n}) be the PD-envelope of $i\otimes\mathbb{Z}/p^n\mathbb{Z}$ compatible with (pW_n,γ) and J_{D_n} be the PD-ideal $\mathrm{Ker}(\mathcal{O}_{D_n}\to\mathcal{O}_{X_n})$. Then, using the assumptions, we can verify:

***Lemma 5.2.1* (cf. [Kat87] I Lemma (1.3)).** — *The sheaves $\mathcal{O}_{D_n}$ and $J^{[r]}_{D_n}$ $(r\in\mathbb{Z})$ are flat over $\mathbb{Z}/p^n\mathbb{Z}$ and the canonical homomorphisms $\mathcal{O}_{D_{n+1}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to\mathcal{O}_{D_n}$ and $J^{[r]}_{D_{n+1}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to J^{[r]}_{D_n}$ are isomorphisms.*

On the other hand, we have

***Lemma 5.2.2* (cf. [Kat87] I Lemma (1.3)).** — *For any integer $0\leqslant r\leqslant p-1$, we have $\varphi(J^{[r]}_{D_n})\subset p^r\mathcal{O}_{D_n}$.*

Proof. — For any $x\in J_{D_n}$, $\varphi(x)$ is described as $x^p+p\cdot y$ $(y\in\mathcal{O}_{D_n})$ and $\varphi(x^{[s]})=p^{[s]}\cdot((p-1)!x^{[p]}+y)^s$ for $s\geqslant 1$. Hence the lemma follows from $p^{[s]}(=p^s/s!)\in p^{\inf\{s,p-1\}}\mathbb{Z}_p$ $(s\geqslant 1)$. □

Hence, for $0\leqslant r\leqslant p-1$, we can define $\varphi_r : J^{[r]}_{D_n}\to\mathcal{O}_{D_n}$ by $\varphi_r(x)=y \mod p^n$, where $\widetilde{x}$ is a lifting of x in $J^{[r]}_{D_{n+r}}$ and $\varphi(\widetilde{x})=p^ry$, $y\in\mathcal{O}_{D_{n+r}}$. For $r\geqslant p$, $\varphi(J^{[r]}_{D_n})\not\subset p^r\mathcal{O}_{D_n}$ in general and we use the modification

$$J^{[r]\prime}_{D_n}:=\{J^{[r]}_{D_{n+r}}\mid\varphi(x)\in p^r\mathcal{O}_{D_{n+r}}\}/p^n\quad(r\in\mathbb{Z},r\geqslant 0).$$

Note $J^{[r]\prime}_{D_n}=J^{[r]}_{D_n}$ $(0\leqslant r\leqslant p-1)$. Using Lemma 5.2.1, we can verify that $J^{[r]\prime}_{D_n}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $J^{[r]\prime}_{D_{n+1}}\otimes\mathbb{Z}/p^n\mathbb{Z}\xrightarrow{\sim}J^{[r]\prime}_{D_n}$ (**[Tsu99]** § 2.1). Hence, we can define $\varphi_r\colon J^{[r]\prime}_{D_n}\to\mathcal{O}_{D_n}$ similarly as above. For $r\leqslant 0$, we set $\varphi_r=p^{-r}\varphi$.

Since $\varphi(\omega^1_{Z_n})\subset p\cdot\omega^1_{Z_n}$ (because $\varphi(\omega^1_{Z_1})=0$ by $\varphi(d\log(b))=d\log(b^p)=p\cdot d\log(b)=0, b\in M_{Z_1}$) and $\omega^q_{Z_n}$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$, we can define the Frobenius "divided by p^q": $\varphi_q\colon\omega^q_{Z_n}\to\omega^q_{Z_n}$ similarly.

We define $\mathcal{S}'_n(r)_{(X,M),(Z,M_Z)}$ to be the mapping fiber of

$$1-\varphi_r\colon J_{D_n}^{[r-\cdot]'}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n}\longrightarrow \mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n},$$

where $\varphi_r = \varphi_{r-q}\otimes\varphi_q$ in degree q. The existence of $T_1,\dots,T_d$ in the assumptions is used here to make $J_{D_n}^{[r-q]'}\otimes_{\mathcal{O}_{Z_n}}\omega^q_{Z_n}$ ($q\geqslant 0$) a complex. (For $x\in J_{D_{n+r}}^{[r]}$, if we set $\nabla(x)=\sum_{1\leqslant i\leqslant d}x_i d\log(T_i)$ ($x_i\in J_{D_{n+r}}^{[r-1]}$), then we have $\nabla(\varphi(x))=\varphi(\nabla(x))=\sum_{1\leqslant i\leqslant d}\varphi(x_i)\cdot pd\log(T_i)$. Hence $\varphi(x)\in p^r\mathcal{O}_{D_{n+r}}$ implies $p\varphi(x_i)\in p^r\mathcal{O}_{D_{n+r}}$ i.e. $\varphi(x_i)\in p^{r-1}\mathcal{O}_{D_{n+r}}$.)

The "multiplication by p^r"

$$\text{“}p^r\text{”}\colon J_{D_n}^{[r-\cdot]'}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n}\longrightarrow J_{D_n}^{[r-\cdot]}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n}$$

and the identity on $\mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\omega^{\cdot}_{Z_n}$ defines a morphism of complexes

$$\text{(5.2.3)}\qquad \mathcal{S}^{\sim}_n(r)_{(X,M),(Z,M_Z)}\longrightarrow \mathcal{S}'_n(r)_{(X,M),(Z,M_Z)}$$

whose kernel and cokernel are killed by p^r.

We can define a product $\mathcal{S}'_n(r)\otimes\mathcal{S}'_n(r')\longrightarrow\mathcal{S}'_n(r+r')$ and a symbol map $M^{\mathrm{gp}}_{n+1}\to\mathcal{S}'_n(1)[1]$ similarly as $\mathcal{S}^{\sim}_n(r)$ in such a way that the following diagrams commute ([**Tsu99**] § 2.1).

$$\begin{array}{ccc} \mathcal{S}^{\sim}_n(r)\otimes\mathcal{S}^{\sim}_n(r') & \longrightarrow & \mathcal{S}^{\sim}_n(r+r') \\ \downarrow & & \downarrow \\ \mathcal{S}'_n(r)\otimes\mathcal{S}'_n(r') & \longrightarrow & \mathcal{S}'_n(r+r'), \end{array}\qquad \begin{array}{ccc} M^{\mathrm{gp}}_n & \xrightarrow{\text{symbol}} & \mathcal{S}^{\sim}_n(1)[1] \\ \uparrow & & \downarrow \\ M^{\mathrm{gp}}_{n+1} & \xrightarrow{p\cdot\text{symbol}} & \mathcal{S}'_n(1)[1]. \end{array}$$

5.3. Invariance of $\mathcal{H}^q(\mathcal{S}^{\sim}_n(r)_{(\overline{X},\overline{M})})$ ($q\leqslant r$) under Tate twists. — Let (X,M) be a smooth fine log scheme over (S,N) whose special fiber $(Y,M_Y):=(X,M)\times_{(S,N)}(s,N_s)$ is of Cartier type over (s,N_s). Choose a generator $t=(\varepsilon_n)_{n\geqslant 0}\in\mathbb{Z}_p(1)(O_{\overline{K}})=\varprojlim_n\mu_{p^n}(O_{\overline{K}})$. Let t_n denote the image of t under $\mathbb{Z}_p(1)(O_{\overline{K}})\to\mu_{p^n}(O_{\overline{K}})\to H^0(\overline{X},\mathcal{S}^{\sim}_n(1))$. See (5.1.3) for the second homomorphism. Then from the product structure of $\mathcal{S}^{\sim}_n(r)_{(\overline{X},\overline{M})}$, we obtain a homomorphism

$$\text{(5.3.1)}\qquad \mathcal{H}^q(\mathcal{S}^{\sim}_n(q)_{(\overline{X},\overline{M})})\longrightarrow\mathcal{H}^q(\mathcal{S}^{\sim}_n(r)_{(\overline{X},\overline{M})});\quad a\longmapsto t_n^{r-q}\cdot a$$

for $0\leqslant q\leqslant r$.

Theorem 5.3.2 **([Tsu99] Theorem 2.3.2).** — *For any integers r and q such that $0\leqslant q\leqslant r$, there exists a positive integer N depending only on p, r and q such that the kernel and the cokernel of the homomorphism* (5.3.1) *are killed by p^N for every $n\geqslant 1$.*

We will explain an outline of the proof of Theorem 5.3.2. Let $\mathcal{L}(T)$ denote the log structure on $\mathrm{Spec}(W[T])$ defined by the divisor $\{T=0\}$ and let i_S denote the exact closed immersion of (S,N) into $(\mathrm{Spec}(W[T]),\mathcal{L}(T))$ defined by $T\mapsto\pi$. Recall that the PD-envelope of the reduction mod p^n of i_S is denoted by (E_n,M_{E_n}) in § 4.2. $(\mathrm{Spec}(W[T]),\mathcal{L}(T))$ and hence (E_n,M_{E_n}) have liftings of Frobenius defined by

$T \mapsto T^p$ and $\sigma\colon W \to W$. Since the question is étale local on X, we may assume that there exist a Cartesian diagram

$$\begin{array}{ccc} (X, M) & \xrightarrow{\ i\ } & (Z, M_Z) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (S, N) & \xrightarrow{\ i_S\ } & (\mathrm{Spec}(W[T]), \mathcal{L}(T)) \end{array}$$

such that g is smooth, a compatible system of liftings of Frobenius $\{F_{Z_n}\}$ of (Z_n, M_{Z_n}) compatible with the lifting of Frobenius of $(\mathrm{Spec}(W[T]), \mathcal{L}(T))$, and $T_1, \dots, T_d \in \Gamma(Z, M_Z)$ such that $F^*_{Z_n}(T_i) = T_i^p$ $(1 \leqslant i \leqslant d)$ and $d\log(T_i)$ $(1 \leqslant i \leqslant d)$ form a basis of $\Omega^1_{Z/W[T]}(\log(M_Z/\mathcal{L}(T)))$ (use Theorem 3.2.3). Choose such a diagram, $\{F_{Z_n}\}$ and T_i.

Choose a compatible system $s = (s_n)_{n\geqslant 0}$ of p^n-th roots of π in $O_{\overline{K}}$ and we regard A_{crys} as a $W[T]$-algebra by the homomorphism $\rho\colon W[T] \to A_{\mathrm{crys}}$; $T \mapsto [(s_n \mod p)_{n\geqslant 0}]$. Note that ρ is not Galois invariant. Set $\omega^{\cdot}_{Z_n/W_n[T]} := \Omega^{\cdot}_{Z_n/W_n[T]}(\log(M_{Z_n}/\mathcal{L}(T)))$ to simplify the notation.

***Proposition 5.3.3* ([Tsu99] Lemma 2.3.4).** — *With the above notation, there exists an isomorphism in $D^+(\overline{Y}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$:*

$$\begin{aligned} &\mathcal{S}_n^{\sim}(r)_{(\overline{X},\overline{M})} \\ &\quad\cong \mathit{fiber}(p^r - \varphi\otimes\varphi\colon \mathrm{Fil}^{r-\cdot} A_{\mathrm{crys}} \otimes_{W[T]} \omega^{\cdot}_{Z_n/W_n[T]} \to A_{\mathrm{crys}} \otimes_{W[T]} \omega_{Z_n/W_n[T]}) \end{aligned}$$

depending on the choice of T_i $(1 \leqslant i \leqslant d)$ and $(s_n)_{n\geqslant 0}$, which is not G_K-equivariant. Furthermore the multiplication by t_n on the LHS corresponds to the multiplication by $t \in \mathbb{Z}_p(1)(O_{\overline{K}}) \subset \mathrm{Fil}^1 A_{\mathrm{crys}}$ on the RHS.

Idea of a proof. — First there is a canonical distinguished triangle in $D^+(\overline{Y}_{\text{ét}}, W_n)$:

$$\longrightarrow Ru^{\log}_{\overline{X}_n/W_n*}\mathcal{O}_{\overline{X}_n/W_n} \longrightarrow Ru^{\log}_{\overline{X}_n/E_n*}\mathcal{O}_{\overline{X}_n/E_n} \xrightarrow{N} Ru^{\log}_{\overline{X}_n/E_n*}\mathcal{O}_{\overline{X}_n/E_n}.$$

Set $P_n := H^0_{\text{log-crys}}(\overline{S}_n/E_n, \mathcal{O}_{\overline{S}_n/E_n}) (\cong R\Gamma_{\text{log-crys}}(\overline{S}_n/E_n, \mathcal{O}_{\overline{S}_n/E_n}))$. This is an $R_{E_n}(:= \Gamma(E_n, \mathcal{O}_{E_n}))$-PD-algebra endowed with a monodromy operator $N_{P_n}\colon P_n \to P_n$ (cf. § 4.4). Then we have a Künneth formula

$$Ru^{\log}_{\overline{X}_n/E_n*}\mathcal{O}_{\overline{X}_n/E_n} \cong P_n \otimes^{\mathbb{L}}_{R_{E_n}} Ru^{\log}_{X_n/E_n*}\mathcal{O}_{X_n/E_n} \cong P_n \otimes_{W_n[T]} \omega^{\cdot}_{Z_n/W_n[T]}.$$

Using T_i $(1 \leqslant i \leqslant d)$, we can define a monodromy operator $N\colon \omega^{\cdot}_{Z_n/W_n[T]} \to \omega^{\cdot}_{Z_n/W_n[T]}$ as $W_n[T]$-modules in such a way that the endomorphism N in the above distinguished triangle is realized as the morphism of complexes $N_{P_n} \otimes 1 + 1 \otimes N$. Using the monodromy operator N on $\omega^{\cdot}_{Z_n/W_n[T]}$ and the PD-structure on P_n, we can change the natural $W_n[T]$-algebra structure of P_n to $\alpha\colon W_n[T] \xrightarrow{\rho} A_{\mathrm{crys}}/p^n =$

$H^0_{\text{log-crys}}(\overline{S}_n/W_n) \to P_n$. Then the monodromy operator $N_{P_n} \otimes 1 + 1 \otimes N$ is replaced by $N_{P_n} \otimes 1$ because $\alpha(W_n[T]) \subset P_n^{N=0}$. Hence, from the exact sequence

$$0 \longrightarrow A_{\text{crys}}/p^n A_{\text{crys}} \longrightarrow P_n \xrightarrow{N} P_n \longrightarrow 0$$

(compare with the above distinguished triangle) and the above distinguished triangle, we obtain

$$(*) \qquad Ru^{\log}_{\overline{X}/W_n *} \mathcal{O}_{\overline{X}_n/W_n} \cong (A_{\text{crys}}/p^n) \otimes_{W_n[T]} \omega^{\cdot}_{Z_n/W_n[T]}.$$

Strictly speaking, we need to describe $Ru^{\log}_{\overline{X}_n/W_n *} \mathcal{O}_{\overline{X}_n/W_n}$ etc. as explicit complexes and construct the relevant maps (especially (*)) as morphisms of complexes. Of course, we also need the filtered version. □

We define the filtration $\operatorname{Fil}^r_p A_{\text{crys}}$ $(r \in \mathbb{Z})$ on A_{crys} by

$$\operatorname{Fil}^r_p A_{\text{crys}} := \{a \in \operatorname{Fil}^r A_{\text{crys}} \mid \varphi(a) \in p^r A_{\text{crys}}\}$$

for $r \geqslant 0$ and $\operatorname{Fil}^r_p A_{\text{crys}} = A_{\text{crys}}$ $(r \leqslant 0)$ (cf. § 5.2). We have $\operatorname{Fil}^r_p A_{\text{crys}} = \operatorname{Fil}^r A_{\text{crys}}$ if $r \leqslant p-1$ and $p^r(\operatorname{Fil}^r A_{\text{crys}}/\operatorname{Fil}^r_p A_{\text{crys}}) = 0$ $(r \geqslant 0)$. Hence we may study the mapping fiber of

$$1 - \varphi_r \colon \operatorname{Fil}^{r-\cdot}_p A_{\text{crys}} \otimes_{W[T]} \omega^{\cdot}_{Z_n/W_n[T]} \longrightarrow A_{\text{crys}} \otimes_{W[T]} \omega^{\cdot}_{Z_n/W_n[T]},$$

which we denote by $C_n(r)$, instead of the RHS of the isomorphism in Proposition 5.3.3. Here $\varphi_r = \frac{\varphi}{p^{r-q}} \otimes \frac{\varphi}{p^q}$ in degree q.

We define the filtration $I^{[s]}A_{\text{crys}}$ on A_{crys} by

$$I^{[s]}A_{\text{crys}} := \{x \in A_{\text{crys}} \mid \varphi^n(x) \in \operatorname{Fil}^s A_{\text{crys}} \text{ for all } n \geqslant 0\}.$$

We also denote by $I^{[\cdot]}$ the induced filtration on $\operatorname{Fil}^r_p A_{\text{crys}}$.

Lemma 5.3.4 **([Fon94a] § 5.2, cf. [Tsu99] Corollary A3.2).** — $t^{p-1} \in pA_{\text{crys}}$.

For an integer $n \geqslant 0$, we set $t^{\{n\}} = t^b(t^{p-1}/p)^{[a]}(= t^n/(p^a a!)) \in A_{\text{crys}}$ where $n = (p-1)a + b$ $(a, b \in \mathbb{Z}, 0 \leqslant b < p-1)$. We verify easily $t^{\{r\}} \in I^{[r]}A_{\text{crys}}$. Let R and $\theta \colon W(R) \to O_C$ be as in § 2.3 and let ξ be a generator of $\operatorname{Ker}(\theta)$ (cf. Proposition 2.3.1). We see easily $\xi^{r-s} \cdot t^{\{s\}} \in I^{[s]}(\operatorname{Fil}^r_p A_{\text{crys}})$ $(0 \leqslant s \leqslant r)$. Set $\pi_\varepsilon := [(\varepsilon_n \bmod p)_{n \geqslant 0}] - 1 \in W(R)$, where $(\varepsilon_n)_{n \geqslant 0}$ is as in the beginning of this subsection.

Proposition 5.3.5 **([Fon94a] 5.3.1 Proposition, 5.3.6 Proposition ii), [Tsu99] § 1.2, A3)**

(1) *For any integer $r \geqslant 0$, we have*

$$I^{[r]}A_{\text{crys}} = \Big\{\sum_{n \geqslant r} a_n t^{\{n\}} \;\Big|\; a_n \in W(R), a_n \text{ converges } p\text{-adically to } 0\Big\}.$$

(2) *For any integer $s \geqslant 0$, there exists an isomorphism:*

$$W(R)/\pi_\varepsilon W(R) \xrightarrow{\sim} \operatorname{gr}^s_I A_{\text{crys}}; x \mapsto x \cdot t^{\{s\}}.$$

(3) *For any integers* $0 \leqslant s < r$,

$$W(R)/\varphi^{-1}(\pi_\varepsilon)W(R) \xrightarrow{\sim} \mathrm{gr}_I^s(\mathrm{Fil}_p^r A_{\mathrm{crys}}); x \mapsto x \cdot \xi^{r-s} \cdot t^{\{s\}}$$

and for any integers $s \geqslant r \geqslant 0$, $I^{[s]}A_{\mathrm{crys}} = I^{[s]}(\mathrm{Fil}_p^r A_{\mathrm{crys}})$.

Let $I^{[\cdot]}C_n(r)$ be the filtration on $C_n(r)$ induced by the filtration $I^{[\cdot]}$ on $\mathrm{Fil}_p^{r-q} A_{\mathrm{crys}}$ and A_{crys}. Then, by Proposition 5.3.5 (2), (3), we see that, for any integers s and r, the complexes $\mathrm{gr}_I^s C_n(r)$ and $\mathrm{gr}_I^{s+r'} C_n(r+r')$ ($r' \geqslant 0$) become isomorphic to the same complex and the multiplication by $t^{\{r'\}}$ from the former to the latter is given by the multiplication by $\alpha \in \mathbb{Z}_p$ defined by $t^{\{s\}} \cdot t^{\{r'\}} = \alpha \cdot t^{\{s+r'\}}$. Hence it suffices to prove:

Lemma 5.3.6 **([Tsu99] Lemma 2.3.19)**

(1) $\mathcal{H}^q(I^{[s]}C_n(r)) = 0$ $(s > r - q + 1)$.

(2) $\mathcal{H}^q(\mathrm{gr}_I^s C_n(r)) = 0$ *if* $s < r - q$.

Sketch of a proof. — We are reduced to the case $n = 1$ easily. Then the morphism

$$1 - \varphi_r \colon I^{[s]}(\mathrm{Fil}^{r-\cdot} A_{\mathrm{crys}} \otimes_{W[T]} \omega^{\cdot}_{Z_1/k[T]}) \longrightarrow I^{[s]}(A_{\mathrm{crys}} \otimes_{W[T]} \omega^{\cdot}_{Z_1/k[T]})$$

becomes the identity maps in degree $q \geqslant r - s + 1$, which implies (1). Next consider gr_I^s of the above morphism. Then, in degree $q \leqslant r - s - 1$, the LHS is isomorphic to $(W(R)/\varphi^{-1}(\pi_\varepsilon)W(R)) \otimes_{W[T]} \omega^q_{Z_1/k[T]}$ with zero differentials, the RHS is isomorphic to $(W(R)/\pi_\varepsilon W(R)) \otimes_{W[T]} \omega^{\cdot}_{Z_1/k[T]}$ and the morphism becomes $\varphi \otimes \varphi/p^q$. Hence, using the Cartier isomorphism (here we use the assumption that (Y, M_Y) is of Cartier type over (s, N_s)), we see that gr_I^s of $1 - \varphi_r$ induces an isomorphism (resp. injective homomorphism) between $\mathcal{H}^q$ if $q \leqslant r-s-2$ (resp. $q = r-s-1$), which implies (2). □

5.4. Calculation of $\mathcal{H}^q(\mathcal{S}'_1(q))$. — Let (X, M) be a smooth fine log scheme over (S, N) whose special fiber $(Y, M_Y) := (X, M) \times_{(S,N)} (s, N_s)$ is of Cartier type over (s, N_s). We assume that there exist a Cartesian diagram and $\{F_{Z_n}\}$ as after the statement of Theorem 5.3.2 and we choose such a diagram and $\{F_{Z_n}\}$. Set $\mathcal{S}'_n(q) := \mathcal{S}'_n(q)_{(X,M),(Z,M_Z)}$ to simplify the notation. We will define a filtration on the sheaf $\mathcal{H}^q(\mathcal{S}'_1(q))$ and give an explicit description of the associated graded quotients.

Define the filtrations $U^{\cdot}$ and $V^{\cdot}$ on $(M_2^{\mathrm{gp}})^{\otimes q}$ $(q \geqslant 0)$ by

$$U^0(M_2^{\mathrm{gp}})^{\otimes 0} := (M_2^{\mathrm{gp}})^{\otimes 0}, \quad U^{m+1}(M_2^{\mathrm{gp}})^{\otimes 0} := V^m(M_2^{\mathrm{gp}})^{\otimes 0} := 0 \ (m \geqslant 0),$$

if $q = 0$,

$$U^0 M_2^{\mathrm{gp}} := M_2^{\mathrm{gp}}, \quad U^m M_2^{\mathrm{gp}} := 1 + \pi^m \mathcal{O}_{X_2} \ (m \geqslant 1),$$
$$V^0 M_2^{\mathrm{gp}} := (1 + \pi\mathcal{O}_{X_2}) \cdot \langle \pi \rangle, \quad V^m M_2^{\mathrm{gp}} := U^{m+1} M_2^{\mathrm{gp}} \ (m \geqslant 1)$$

if $q = 1$, and

$$U^m(M_2^{\mathrm{gp}})^{\otimes q} := \text{the image of } U^m M_2^{\mathrm{gp}} \otimes (M_2^{\mathrm{gp}})^{\otimes(q-1)},$$
$$V^m(M_2^{\mathrm{gp}})^{\otimes q} := \text{the image of } U^m M_2^{\mathrm{gp}} \otimes (M_2^{\mathrm{gp}})^{\otimes(q-2)} \otimes \langle \pi \rangle + U^{m+1}(M_2^{\mathrm{gp}})^{\otimes q}$$

if $q \geqslant 2$. (See **[Hyo88]**(1.4).) Here and hereafter we denote by the same letter π the image of $\pi \in \Gamma(S, N) = O_K \smallsetminus \{0\}$ under the map $\Gamma(S, N) \to \Gamma(X, M)$ and its images in $\Gamma(X_n, M_n)$ $(n \geqslant 1)$.

From the product structure and the symbol map, we obtain a morphism $(M_2^{\mathrm{gp}})^{\otimes q} \to \mathcal{S}'_1(q)[q]$ and then $(M_2^{\mathrm{gp}})^{\otimes q} \to \mathcal{H}^q(\mathcal{S}'_1(q))$ by taking $\mathcal{H}^0$ of both sides. We also call this homomorphism the symbol map. We denote by $\{a_1, \ldots, a_q\}$ the image of $a_1 \otimes \cdots \otimes a_q$ $(a_i \in M_2^{\mathrm{gp}})$ by this map.

We define the filtrations $U^\cdot$ and $V^\cdot$ on $\mathcal{H}^q(\mathcal{S}'_1(q))$ $(q \geqslant 0)$ to be the images of those on $(M_2^{\mathrm{gp}})^{\otimes q}$ defined above under the symbol map. We define gr_0^m and gr_1^m of $\mathcal{H}^q(\mathcal{S}'_1(q))$ by U^m/V^m and V^m/U^{m+1} respectively.

Put $\omega_Y^q := \Omega_{Y/s}^q(\log(M_Y/N_s))$ and define the subsheaves B_Y^q (resp. Z_Y^q) of ω_Y^q to be the image of $d\colon \omega_Y^{q-1} \longrightarrow \omega_Y^q$ (resp. the kernel of $d\colon \omega_Y^q \longrightarrow \omega_Y^{q+1}$). Let $\omega_{Y,\log}^q$ be the subsheaf of abelian groups of ω_Y^q generated by local sections of the form $d\log a_1 \wedge d\log a_2 \wedge \cdots \wedge d\log a_q$, where $a_1, a_2, \cdots, a_q \in M_Y$.

Proposition 5.4.1 **([Tsu99] Proposition 2.4.1, cf. [Kur87] Proposition (4.3))**

If $p = 2$, assume $\sqrt{-1} \in K_{\mathrm{nr}}$, where K_{nr} is the maximal unramified extension of K. Let e be the absolute ramification index of K. Then the sheaf $\mathcal{H}^q(\mathcal{S}'_1(q))$ has the following structure :

(1) $U^0\mathcal{H}^q(\mathcal{S}'_1(q)) = \mathcal{H}^q(\mathcal{S}'_1(q))$.

(2) *If $m = 0$,*

$$\mathrm{gr}_0^0\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \omega_{Y,\log}^q;\ \{a_1, \cdots, a_q\} \mapsto d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_q}$$
$$\mathrm{gr}_1^0\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \omega_{Y,\log}^{q-1};\ \{a_1, \cdots, a_{q-1}, \pi\} \mapsto d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_{q-1}}$$

(3) *If $0 < m < pe/(p-1)$ and $p \nmid m$,*

$$\mathrm{gr}_0^m\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \frac{\omega_Y^{q-1}}{B_Y^{q-1}};\ \{1 + \pi^m x, a_1, \cdots, a_{q-1}\} \mapsto \overline{x}d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_{q-1}}$$
$$\mathrm{gr}_1^m\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \frac{\omega_Y^{q-2}}{Z_Y^{q-2}};\ \{1 + \pi^m x, a_1, \cdots, a_{q-2}, \pi\} \mapsto \overline{x}d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_{q-2}}$$

(4) *If $0 < m < pe/(p-1)$ and $p \mid m$,*

$$\mathrm{gr}_0^m\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \frac{\omega_Y^{q-1}}{Z_Y^{q-1}};\ \{1 + \pi^m x, a_1, \cdots, a_{q-1}\} \mapsto \overline{x}d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_{q-1}}$$
$$\mathrm{gr}_1^m\mathcal{H}^q(\mathcal{S}'_1(q)) \cong \frac{\omega_Y^{q-2}}{Z_Y^{q-2}};\ \{1 + \pi^m x, a_1, \cdots, a_{q-2}, \pi\} \mapsto \overline{x}d\log\overline{a_1} \wedge \cdots \wedge d\log\overline{a_{q-2}}$$

(5) *If $m \geqslant pe/(p-1)$, $U^m\mathcal{H}^q(\mathcal{S}'_1(q)) = 0$.*

Here $a_1, \ldots, a_q \in M_2^{\mathrm{gp}}$, $x \in \mathcal{O}_{X_2}$, $\overline{a_i}$ are the images of a_i in M_Y^{gp}, and $\overline{x}$ is the image of x in $\mathcal{O}_Y$.

Furthermore, (1), (2), (5), and (3) and (4) for $0 < m < e$ are still true when $p = 2$ and $\sqrt{-1} \notin K_{\mathrm{nr}}$.

In degree $\geqslant q-p+2$, $J_{D_1}^{[r]\prime}$ for $r \geqslant p-1$ does not appear in the complex $\mathcal{S}_1'(q)$ and we can prove Proposition 5.4.1 by the same method as [**Kur87**] if $p \geqslant 3$. If $p=2$, we need to introduce a new and more complicated method.

6. Syntomic complexes and p-adic nearby cycles

Let X be a scheme of finite type with semi-stable reduction, that is, X is a regular scheme flat over O_K and its special fiber Y is a reduced divisor with normal crossings on X. We endow X with the log structure M defined by the special fiber (see Example 3.1.1). Then (X, M) is smooth over (S, N) and its special fiber (Y, M_Y) is of Cartier type over (s, N_s) ([**Kat89**] Definition (4.8)). Set $\overline{X} := X \otimes_{O_K} O_{\overline{K}}$, $\overline{Y} := X \otimes_{O_K} \overline{k}$, $X_{\overline{K}} := X \otimes_{O_K} \overline{K}$, and let $\overline{i}$ and $\overline{j}$ denote the canonical morphisms $\overline{i}\colon \overline{Y} \to \overline{X}$ and $\overline{j}\colon X_{\overline{K}} \to \overline{X}$ respectively. For $r \in \mathbb{Z}$, $r \geqslant 0$, let $\mathbb{Z}/p^n\mathbb{Z}(r)'$ denote $(\frac{1}{p^a a!}\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n\mathbb{Z}$, where $r = (p-1)a + b$ $(a, b \in \mathbb{Z}, 0 \leqslant b \leqslant p-2)$. There are natural products $\mathbb{Z}/p^n\mathbb{Z}(r)' \otimes \mathbb{Z}/p^n\mathbb{Z}(s)' \to \mathbb{Z}/p^n\mathbb{Z}(r+s)'$ $(r, s \geqslant 0)$. We will explain an outline of the proof of the following theorem.

Theorem 6.0.1 **([Tsu99] § 3)**

(1) *There exists a canonical G_K-equivariant morphism*

$$\mathcal{S}_n^{\sim}(r)_{(\overline{X},\overline{M})} \longrightarrow \overline{i}^*_{\text{ét}} R\overline{j}_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(r)'$$

in $D^+(\overline{Y}_{\text{ét}}, \mathbb{Z}/p^n\mathbb{Z})$ compatible with the product structures.

(2) *For any integers q, r such that $0 \leqslant q \leqslant r$, there exists $N \geqslant 0$ which depends only on p, q and r such that the kernel and the cokernel of the homomorphism:*

$$\mathcal{H}^q(\mathcal{S}_n^{\sim}(r)_{(\overline{X},\overline{M})}) \longrightarrow \overline{i}^*_{\text{ét}} R^q\overline{j}_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(r)'$$

induced by the morphism in (1) are killed by p^N for every $n \geqslant 1$.

By the proper base change theorem for étale cohomology, we obtain the following corollary.

Corollary 6.0.2. — *Suppose that X is proper over O_K. Then, there exists a canonical G_K-equivariant isomorphisms*

$$H^m((\overline{X}, \overline{M}), \mathcal{S}^r_{\mathbb{Q}_p}) \xrightarrow{\sim} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$$

for $0 \leqslant m \leqslant r$ compatible with the product structures.

6.1. Construction of the maps. — First we consider a smooth fine and saturated log scheme (X, M) over (S, N). (Here a monoid P is called *saturated* if it is integral and if, for any $a \in P^{\text{gp}}$, $a^n \in P$ for some $n \geqslant 1$ implies $a \in P$, and a log structure L on a scheme S is called *saturated* if $L_{\overline{s}}$ are saturated for all $s \in S$ or equivalently if $\Gamma(U, L)$ are saturated for all étale S-schemes U.) We further assume that we are given a closed immersion $(X, M) \hookrightarrow (Z, M_Z)$ and liftings of Frobenius $\{F_{Z_n}\}$ as in the definition of $\mathcal{S}_n'(r)_{(X,M),(Z,M_Z)}$ in § 5.2. (Such a closed immersion and $\{F_{Z_n}\}$ always exist étale locally on X). Let X_{triv} ($\subset X_K := X \otimes_{O_K} K$) denote the locus where the log structure M on X is trivial, which is open dense in X. If X has semi-stable reduction and M is defined by its special fiber, then X_{triv} is precisely the generic fiber. Let i and j denote the morphisms $Y := X \otimes_{O_K} k \to X$ and $X_{\mathrm{triv}} \to X$ respectively. Then we can construct canonical morphisms:

$$\mathcal{S}'_n(r)_{(X,M),(Z,M_Z)} \longrightarrow i^*_{\text{ét}} Rj_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(r)' \quad (r \in \mathbb{Z}, r \geqslant 0) \tag{6.1.1}$$

in the following way. See [**Tsu99**] § 3.1 for details.

For an affine scheme $U = \mathrm{Spec}(A)$ étale over X whose special fiber is connected and non-empty, let A^h denote the p-adic henselization of A, which is a normal domain, choose an algebraic closure $\overline{\mathrm{Frac}(A^h)}$ of the field of fractions $\mathrm{Frac}(A^h)$ of A^h, and let $\overline{A^h}$ denote the integral closure of A^h in the maximal unramified extension of A^h_{triv} in $\overline{\mathrm{Frac}(A^h)}$, where $U^h_{\mathrm{triv}} = \mathrm{Spec}(A^h_{\mathrm{triv}})$ ($\subset \mathrm{Spec}(A^h[1/p])$) denotes the locus where the inverse image of M on $U^h := \mathrm{Spec}(A^h)$ is trivial. We have $\mathrm{Gal}(\mathrm{Frac}(\overline{A^h})/\mathrm{Frac}(A^h)) \cong \pi_1(U^h_{\mathrm{triv}})$ where we use the base point defined by $\overline{\mathrm{Frac}(A^h)}$ in the RHS. Replacing $O_{\overline{K}}$, O_C and R in the definition of A_{crys} (§ 2.3) with $\overline{A^h}$, $\widehat{\overline{A^h}}$ (the p-adic completion of $\overline{A^h}$) and $R_{\overline{A^h}} := \varprojlim_{\mathrm{Frob}} \overline{A^h}/p\overline{A^h}$, we obtain a ring $A_{\mathrm{crys}}(\overline{A^h})$ endowed with an action of $\pi_1(U^h_{\mathrm{triv}})$, a lifting of Frobenius φ and a filtration $\mathrm{Fil}^{\cdot} A_{\mathrm{crys}}(\overline{A^h})$. If we define $\mathrm{Fil}^r_p A_{\mathrm{crys}}(\overline{A^h})$ ($r \in \mathbb{Z}$) in the same way as after the proof of Proposition 5.3.3, then we have the following exact sequences of $\pi_1(U^h_{\mathrm{triv}})$-modules ([**Fon94a**] 5.3.6, [**Tsu99**] § 1.2):

$$0 \longrightarrow \mathbb{Z}_p(r)' \longrightarrow \mathrm{Fil}^r_p A_{\mathrm{crys}}(\overline{A^h}) \xrightarrow{1-\varphi/p^r} A_{\mathrm{crys}}(\overline{A^h}) \longrightarrow 0 \quad (r \in \mathbb{Z},\ r \geqslant 0).$$

Next, for a sufficiently small U, we construct canonical resolutions

$$\mathrm{Fil}^r_p A_{\mathrm{crys}}(\overline{A^h})/p^n \longrightarrow \mathrm{Fil}^{r-\cdot}_p \mathcal{A}_{\mathrm{crys}}(\overline{A^h})/p^n \otimes_{\mathcal{O}_{Z_n}} \omega^{\cdot}_{Z_n} \quad (r \in \mathbb{Z})$$

compatible with the actions of $\pi_1(U^h_{\mathrm{triv}})$ and the Frobenii (divided by p^r) such that there are canonical morphisms:

$$\Gamma(U \otimes_{O_K} k, J^{[r-\cdot]'}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega^{\cdot}_{Z_n}) \longrightarrow (RHS)^{\pi_1(U^h_{\mathrm{triv}})}$$

compatible with the Frobenii (divided by p^r). Let $\overline{\mathcal{S}'_n}(r)_{U,(Z,M_Z)}$ denote the mapping fiber of

$$1 - \frac{\varphi}{p^r} \colon \mathrm{Fil}^{r-\cdot}_p \mathcal{A}_{\mathrm{crys}}(\overline{A^h})/p^n \otimes_{\mathcal{O}_{Z_n}} \omega^{\cdot}_{Z_n} \longrightarrow \mathcal{A}_{\mathrm{crys}}(\overline{A^h})/p^n \otimes_{\mathcal{O}_{Z_n}} \omega^{\cdot}_{Z_n}.$$

Then, regarding discrete $\pi_1(U^h_{\mathrm{triv}})$-modules as étale sheaves on U^h_{triv}, we obtain a series of morphisms in $D^+(\mathbb{Z}/p^n\mathbb{Z})$:

$$\begin{aligned}\Gamma_{\text{ét}}(U\otimes_{O_K}k, \mathcal{S}'_n(r)_{(X,M),(Z,M_Z)}) &\longrightarrow \Gamma_{\text{ét}}(U^h_{\mathrm{triv}}, \overline{\mathcal{S}'_n}(r)_{U,(Z,M_Z)})\\ &\longrightarrow R\Gamma_{\text{ét}}(U^h_{\mathrm{triv}}, \overline{\mathcal{S}'_n}(r)_{U,(Z,M_Z)})\\ &\xleftarrow{\sim} R\Gamma_{\text{ét}}(U^h_{\mathrm{triv}}, \mathbb{Z}/p^n\mathbb{Z}(r)')\\ &\longrightarrow R\Gamma_{\text{ét}}(U^h, i^h_{\text{ét}*}i^{h*}_{\text{ét}}Rj^h_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(r)')\\ &\cong R\Gamma_{\text{ét}}(U, i_{\text{ét}*}i^*_{\text{ét}}Rj_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(r)'),\end{aligned}$$

where i^h and j^h denote the canonical morphisms $U^h\otimes_{O_K}k\ (=U\otimes_{O_K}k)\to U^h$ and $U^h_{\mathrm{triv}}\to U^h$. Describing the above morphisms as morphisms of explicit complexes (using the Godement resolutions) and varying U, we obtain (6.1.1).

Remark. — For an algebraic closure L of $\mathrm{Frac}(A^h)$, let G_L denote the fundamental group of U^h_{triv} with the base point $\mathrm{Spec}(L)\to U^h_{\mathrm{triv}}$. Suppose that we are given the following data: for every algebraic closure L of $\mathrm{Frac}(A^h)$, a discrete G_L-module M_L and, for every isomorphism $s\colon L_1\xrightarrow{\sim}L_2$ over $\mathrm{Frac}(A^h)$, an isomorphism $\iota_s\colon M_{L_1}\xrightarrow{\sim}M_{L_2}$ compatible with the isomorphism $G_{L_1}\xrightarrow{\sim}G_{L_2}$ induced by s, such that $\iota_{s_1\circ s_2}=\iota_{s_1}\circ\iota_{s_2}$ for any composable s_1, s_2 and, for $s\in\mathrm{Gal}(L/\mathrm{Frac}(A^h))$, ι_s is the action of the image of s under the canonical surjection $\mathrm{Gal}(L/\mathrm{Frac}(A^h))\to G_L$. Then, if we denote by $\mathcal{F}_L$ the sheaf on $(U^h_{\mathrm{triv}})_{\text{ét}}$ associated to M_L, then the isomorphism $\mathcal{F}_{L_1}\xrightarrow{\sim}\mathcal{F}_{L_2}$ induced by ι_s for an $s\colon L_1\xrightarrow{\sim}L_2$ is independent of s, and hence, up to canonical isomorphisms, $\mathcal{F}_L$ is independent of the choice of L.

The resolution of $\mathrm{Fil}^r_p\,\mathcal{A}_{\mathrm{crys}}(\overline{A^h})/p^n$ above is constructed as follows. Let $\widetilde{\overline{A^h}}$ be the image of $\theta\colon W(R_{\overline{A^h}})\to\widehat{\overline{A^h}}$ and set $\overline{U}:=\mathrm{Spec}(\widetilde{\overline{A^h}})$. Then

$$\widetilde{\overline{A^h}}\cong\mathcal{A}_{\mathrm{crys}}(\overline{A^h})/\mathrm{Fil}^1\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$$

and hence we have a PD-thickening $\overline{U}\hookrightarrow\overline{D}:=\mathrm{Spec}(\mathcal{A}_{\mathrm{crys}}(\overline{A^h}))$. If U is sufficiently small, the image of A in $\widehat{\overline{A^h}}$ is contained in $\widetilde{\overline{A^h}}$ and hence there exists a canonical morphism $\overline{U}\to U$ ([**Tsu99**] Lemma 1.5.4). Furthermore, if we denote by $M_{\overline{U}}$ the inverse image of M on $\overline{U}$, $M_{\overline{U}}$ lifts to a log structure $M_{\overline{D}}$ on $\overline{D}$ in a canonical way ([**Tsu99**] § 1.4). Thus we obtain a PD-thickening $(\overline{U},M_{\overline{U}})\hookrightarrow(\overline{D},M_{\overline{D}})$ endowed with an action of $\pi_1(U^h_{\mathrm{triv}})$. Let $(\overline{E}_n,M_{\overline{E}_n})$ be the PD-envelope of $(\overline{U}_n,M_{\overline{U}_n})$ in $(\overline{Z}_n,M_{\overline{Z}_n}):=(\overline{D}_n,M_{\overline{D}_n})\times_{W_n}(Z_n,M_{Z_n})$ and set $J_{\overline{E}_n}:=\mathrm{Ker}(\mathcal{O}_{\overline{E}_n}\to\mathcal{O}_{\overline{U}_n})$. We define $\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$ to be $\varprojlim_n\Gamma(\overline{E}_n,\mathcal{O}_{\overline{E}_n})$ and $\mathrm{Fil}^r\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$ to be $\varprojlim_n(\overline{E}_n,J^{[r]}_{\overline{E}_n})$. We define $\mathrm{Fil}^r_p\,\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$ in the same way as $\mathrm{Fil}^r_p\,A_{\mathrm{crys}}$ to obtain $\varphi/p^r\colon\ \mathrm{Fil}^r_p\,\mathcal{A}_{\mathrm{crys}}(\overline{A^h})\to\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$. $\mathcal{A}_{\mathrm{crys}}(\overline{A^h})$ is naturally endowed with a connection $\nabla\colon\mathcal{A}_{\mathrm{crys}}(\overline{A^h})\to\mathcal{A}_{\mathrm{crys}}(\overline{A^h})\otimes_{\mathcal{O}_{\overline{Z}}}\omega^1_{\overline{Z}/\overline{D}}$ satisfying the Griffiths transversality: $\nabla(\mathrm{Fil}^r\mathcal{A}_{\mathrm{crys}}(\overline{A^h}))\subset\mathrm{Fil}^{r-1}\mathcal{A}_{\mathrm{crys}}(\overline{A^h})\otimes_{\mathcal{O}_{\overline{Z}_n}}\omega^1_{\overline{Z}/\overline{D}}$. Note $\omega^1_{\overline{Z}/\overline{D}}=\mathcal{O}_{\overline{Z}}\otimes_{\mathcal{O}_Z}\omega^1_Z$.

Next we will discuss the compatibility with the symbol maps. From the Kummer sequence $0 \to \mathbb{Z}/p^n\mathbb{Z}(1) \to \mathcal{O}^*_{X_{\mathrm{triv}}} \xrightarrow{p^n} \mathcal{O}^*_{X_{\mathrm{triv}}} \to 0$, we obtain a symbol map

$$i^*_{\text{ét}} j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}} \longrightarrow i^*_{\text{ét}} Rj_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(1)[1] \tag{6.1.2}$$

and, then, using the cup products, symbol maps

$$(i^*_{\text{ét}} j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}})^{\otimes q} \longrightarrow i^*_{\text{ét}} R^q j_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(q) \quad (q \in \mathbb{Z}, q \geqslant 0). \tag{6.1.3}$$

By the assumption that M is saturated, we see $j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}} = M^{\mathrm{gp}}$ ([**Kat94b**] Theorem (11.6), [**Tsu99**] Proposition 3.2.1) and hence there is a canonical surjective homomorphism $i^*_{\text{ét}} j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}} \to M^{\mathrm{gp}}_{n+1}$.

Proposition 6.1.4 **([Tsu99] Proposition 3.2.4).** — *The morphisms* (6.1.1) *is compatible with the product structures and the following diagrams commute:*

$$\begin{array}{ccc} M^{\mathrm{gp}}_{n+1} & \xrightarrow{\textit{symbol}} & \mathcal{S}'_n(1)_{(X,M),(Z,M_Z)}[1] \\ \uparrow & & \downarrow{\scriptstyle (6.1.1)} \\ i^*_{\text{ét}} j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}} & \xrightarrow{(6.1.2)} & i^*_{\text{ét}} Rj_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(1)'[1], \end{array}$$

$$\begin{array}{ccc} (M^{\mathrm{gp}}_{n+1})^{\otimes q} & \xrightarrow{\textit{symbol}} & \mathcal{H}^q(\mathcal{S}'_n(q)_{(X,M),(Z,M_Z)}) \\ \uparrow & & \downarrow{\scriptstyle \mathcal{H}^q((6.1.1))} \\ (i^*_{\text{ét}} j_{\text{ét}*}\mathcal{O}^*_{X_{\mathrm{triv}}})^{\otimes q} & \xrightarrow{(6.1.3)} & i^*_{\text{ét}} R^q j_{\text{ét}*}\mathbb{Z}/p^n\mathbb{Z}(q)'. \end{array}$$

Here note that there is a canonical homomorphism $\mathbb{Z}/p^n\mathbb{Z}(r) \to \mathbb{Z}/p^n\mathbb{Z}(r)'$ for $r \in \mathbb{Z}$, $r \geqslant 0$.

Let us return to the special situation in the beginning of this section 6. We define the morphism in Theorem 6.0.1 (1) by "gluing" the composite of (6.1.1) with the canonical map $\mathcal{S}^{\sim}_n(r)_{(X,M),(Z,M_Z)} \to \mathcal{S}'_n(r)_{(X,M),(Z,M_Z)}$ (§ 5.2) and taking the "inductive limit" with respect to finite extensions of K contained in $\overline{K}$. When X is proper over O_K, we define the homomorphisms

$$H^m((\overline{X},\overline{M}), S^r_{\mathbb{Q}_p}) \longrightarrow H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \quad (r, m \geqslant 0) \tag{6.1.5}$$

by multiplying p^{-r} to the homomorphisms induced by the morphism in Theorem 6.0.1 (1) in order to make them compatible with the symbol maps (cf. the diagram in the end of § 5.2).

6.2. Calculation of p-adic vanishing cycles. — We will review the calculation of p-adic vanishing cycles by Bloch-Kato [**BK86**] (in the good reduction case) and by Hyodo [**Hyo88**] (in the semi-stable reduction case).

Keep the notations and assumptions in the beginning of § 6. Let K' be any finite extension of K contained in $\overline{K}$, let $S' := \mathrm{Spec}(O_{K'})$, let N' denote the log structure on S' defined by the closed point and set $(X', M') := (X, M) \times_{(S,N)} (S', N')$. Then (X', M') is smooth over (S', N'), M' is saturated and the special fiber is of Cartier type. Note that M' is trivial on the generic fiber and hence $X'_{\mathrm{triv}} = X'_{K'}$. Let i' and j' denote the morphisms $Y' := X' \otimes_{O_{K'}} k' \to X'$ and $X_{K'} := X' \otimes_{O_{K'}} K' \to X'$. We define the filtrations $U^{\cdot}$ and $V^{\cdot}$ on $(i'^{*}_{\text{ét}} M'^{\text{gp}})^{\otimes q} = (i'^{*}_{\text{ét}} j'_{\text{ét}*} \mathcal{O}^{*}_{X_{K'}})^{\otimes q}$ (cf. § 6.1) and $U^{\cdot}$, $V^{\cdot}$, gr_0^m and gr_1^m of $i'^{*}_{\text{ét}} R^q j'_{\text{ét}*} \mathbb{Z}/p\mathbb{Z}(q)$ in the same way as in § 5.4 using the symbol maps (6.1.3).

***Theorem 6.2.1* (Bloch-Kato-Hyodo).** — *With the notation above, we have*

$$U^0 i'^{*}_{\text{ét}} R^q j'_{\text{ét}*} \mathbb{Z}/p\mathbb{Z}(q) = i'^{*}_{\text{ét}} R^q j'_{\text{ét}*} \mathbb{Z}/p\mathbb{Z}(q)$$

and gr_0^m, gr_1^m *of* $i'^{*}_{\text{ét}} R^q j'_{\text{ét}*} \mathbb{Z}/p\mathbb{Z}(q)$ *have the same description as Proposition* 5.4.1 *without assuming* $\sqrt{-1} \in K'_{\mathrm{nr}}$ *in the case* $p = 2$.

Historically, Theorem 6.2.1 was proven earlier than Proposition 5.4.1.

6.3. Proof of Theorem 6.0.1 (2). — We keep the notation of § 6.2. Comparing Theorem 6.2.1 with Proposition 5.4.1, we will prove the following theorem, from which we can deduce Theorem 6.0.1 (2) easily because the kernel and cokernel of $\mathcal{S}_n^{\sim}(r)_{(X,M),(Z,M_Z)} \to \mathcal{S}'_n(r)_{(X,M),(Z,M_Z)}$ are killed by p^r and $\mathcal{H}^q(\mathcal{S}_n^{\sim}(r)_{(\overline{X},\overline{M})})$ ($q \leqslant r$) are invariant under Tate twists up to bounded torsions (Theorem 5.3.2). We replace (S, N) by (S', N') and omit the prime $'$ from the notation (X', M'), i' etc.

***Theorem 6.3.1* ([Tsu99] Theorem 3.3.2).** — *Let q be a non-negative integer and put $m = v_p(a! p^a)$, where a is the biggest integer which is less than or equal to $q/(p-1)$. Let $n > m$ and assume that the primitive p^n-th roots of unity are contained in K. Assume that there exist a diagram and $\{F_{Z_n}\}$ as after the statement of Theorem* 5.3.2 *and choose such a diagram and $\{F_{Z_n}\}$. Set $\mathcal{S}'_n(q) := \mathcal{S}'_n(q)_{(X,M),(Z,M_Z)}$ to simplify the notation. Then the sequence*

$$\mathcal{H}^q(\mathcal{S}'_n(q)) \xrightarrow{p^{n-m}} \mathcal{H}^q(\mathcal{S}'_n(q)) \longrightarrow \mathcal{H}^q(\mathcal{S}'_{n-m}(q)) \longrightarrow 0$$

is exact, the natural homomorphism

$$i^{*}_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p^{n-m}\mathbb{Z}(q) \longrightarrow i^{*}_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p^n\mathbb{Z}(q)'$$

is injective, and the homomorphism

$$\mathcal{H}^q(\mathcal{S}'_n(q)) \longrightarrow i^{*}_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p^n\mathbb{Z}(q)'$$

induced by (6.1.1) *has a unique factorization*

$$\mathcal{H}^q(\mathcal{S}'_n(q)) \longrightarrow \mathcal{H}^q(\mathcal{S}'_{n-m}(q)) \longrightarrow i^{*}_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p^{n-m}\mathbb{Z}(q) \longrightarrow i^{*}_{\text{ét}} R^q j_{\text{ét}*} \mathbb{Z}/p^n\mathbb{Z}(q)'.$$

Furthermore the middle homomorphism in this factorization is an isomorphism.

Proof. — By Proposition 5.4.1 (1) and the exact sequence $0 \to \mathcal{S}'_N(q) \xrightarrow{p^M} \mathcal{S}'_{N+M}(q) \to \mathcal{S}'_M(q) \to 0$, we see that the symbol maps $(M_{N+1}^{\text{gp}})^{\otimes q} \to \mathcal{H}^q(\mathcal{S}'_N(q))$ ($N \geqslant 1$) are surjective and the first claim holds. Similarly, by the assumption on K and by Theorem 6.2.1, we see that $i^{*}_{\text{ét}} R^{q-1} j_{\text{ét}*} \mathbb{Z}/p^n\mathbb{Z}(q) \to i^{*}_{\text{ét}} R^{q-1} j_{\text{ét}*} \mathbb{Z}/p^m\mathbb{Z}(q)$

is surjective and the second claim is true. Now by the surjectivity of the symbol maps $(M_{N+1}^{\mathrm{gp}})^{\otimes q} \to \mathcal{H}^q(\mathcal{S}'_N(q))$ and Proposition 6.1.4, we obtain the factorization in the last claim and the middle homomorphism becomes compatible with the symbol maps. For the last claim, we are reduced easily to the case $n = 1$, in which case, it follows from Theorem 6.2.1 and Proposition 5.4.1. □

7. Proof of C_{st}

We will first explain the idea of Fontaine, Messing and Kato to prove C_{crys} for a proper smooth scheme X over O_K. In the case $\dim X_K \leqslant p-1$ and $K = K_0$, using the theory of Fontaine and Laffaille on p-torsion crystalline representations, Fontaine and Messing proved that the de Rham cohomology $H^m_{\mathrm{dR}}(X_K/K)$ with its filtered φ-module structure is admissible ([**FM87**] II 2.8 Remark), that is, associated to a crystalline p-adic representation $\widetilde{V}^m$ and that there exist isomorphisms ([**FM87**] III 1.6 Corollary, 2.4 Proposition)

(7.0.1)

$$H^m(\overline{X}, S^r_{\mathbb{Q}_p}) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Fil}^r(B^+_{\mathrm{crys}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \xrightarrow{p^r-\varphi} B^+_{\mathrm{crys}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)) \xleftarrow{\sim} \widetilde{V}^m(r)$$

for $0 \leqslant m \leqslant r$ (cf. the beginning of § 5). Combining this with

$$H^m(\overline{X}, S^r_{\mathbb{Q}_p}) \longrightarrow H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \tag{7.0.2}$$

induced by (5.0.2) and using Poincaré duality, they proved $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \cong \widetilde{V}^m$. In the ramified case $K \neq K_0$, we don't have a good integral theory of p-torsion crystalline representations unless $[K : K_0] \times$ (length of filtration) $\leqslant p-2$. Kato and Messing constructed only a homomorphism ([**KM92**]):

$$H^m(\overline{X}, S^r_{\mathbb{Q}_p}) \longrightarrow (B^+_{\mathrm{crys}} \otimes_{K_0} H^m_{\mathrm{crys}}(X))^{\varphi=p^r} \cap \mathrm{Fil}^r(B^+_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K)). \tag{7.0.3}$$

To prove C_{crys} for $\dim X_K \leqslant (p-2)/2$, they needed the strong result of Kato and Kurihara for the étale cohomology side: that (7.0.2) is an isomorphism for $0 \leqslant m \leqslant r \leqslant p-2$ ([**Kat87**],[**Kur87**]).

In [**Kat94a**], K. Kato generalized the latter argument to the semi-stable case, which we will survey below. Now we have the isomorphisms without the restriction $r \leqslant p-2$ (Corollary 6.0.2) and hence we can remove the restriction $\dim X_K \leqslant (p-2)/2$ in [**Kat94a**].

7.1. Syntomic cohomology and étale cohomology. — Let (X, M) be a smooth fine log scheme over (S, N) such that X is proper over S and the special fiber (Y, M_Y) is of Cartier type over (s, N_s). We further assume that M_K is saturated. We construct a canonical G_K-equivariant homomorphisms functorial on X and compatible with the product structures (the semi-stable version of (7.0.3)):

(7.1.1) $$H^m((\overline{X},\overline{M}),S^r_{\mathbb{Q}_p})$$
$$\longrightarrow (B^+_{\mathrm{st}}\otimes_{K_0} H^m_{\mathrm{crys}}((X,M)))^{N=0,\varphi=p^r}\cap \mathrm{Fil}^r(B^+_{\mathrm{dR}}\otimes_K H^m_{\mathrm{dR}}((X_K,M_K)/K))$$

for integers $r,m\geqslant 0$ as follows.

First recall that we have the following commutative diagram (§ 4.3):

$$\begin{array}{ccccc} (S_n,N_n) & \xrightarrow{i_{E_n,\pi}} & (E_n,M_{E_n}) & \xleftarrow{i_{E_n,0}} & (\mathrm{Spec}(W_n),N^0_n) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Spec}(W_n) & & \end{array}$$

We define $H^m((\overline{X},\overline{M})/W,J^{[r]})$ to be

$$\varprojlim_n\left(\varinjlim_{K'} H^m_{\mathrm{crys}}(((X'_n,M'_n)/(W_n,pW_n,\gamma))_{\mathrm{crys}},J^{[r]}_{(X'_n,M'_n)/W_n})\right),$$

where K' ranges over all finite extensions of K contained in $\overline{K}$, (S',N') denotes $\mathrm{Spec}(O_{K'})$ with the log structure defined by the closed point and $(X',M')=(X,M)\times_{(S,N)}(S',N')$. $H^m((\overline{X},\overline{M})/W)$ is naturally endowed with a Frobenius endomorphism φ and there is a natural map:

(7.1.2)
$$H^m((\overline{X},\overline{M}),S^r_{\mathbb{Q}_p})\longrightarrow \mathrm{Ker}(H^m((\overline{X},\overline{M})/W,J^{[r]})_{\mathbb{Q}_p}\xrightarrow{p^r-\varphi} H^m((\overline{X},\overline{M})/W)_{\mathbb{Q}_p})$$

for $r,m\geqslant 0$. Here and hereafter, we denote the operation $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}$ simply by the subscript $\mathbb{Q}_p$. We define $H^m((\overline{X},\overline{M})/(S,N),J^{[r]}/J^{[s]})$ and $H^m((\overline{X},\overline{M})/(E,M_E))$ similarly using the base (S_n,N_n) and (E_n,M_{E_n}) respectively. The latter cohomology naturally endowed with φ and N satisfying $N\varphi=p\varphi N$ (cf. § 4.3, § 4.4). We have the following Künneth formulas:

Proposition 7.1.3

(1) ([**Tsu99**] Proposition 4.5.4, cf. [**Kat94a**] the proof of Lemma (4.2)). *The natural homomorphism:*

$$H^0((\overline{S}_n,\overline{N}_n)/(E_n,M_{E_n}))\otimes_{R_{E_n}} H^m((X_n,M_n)/(E_n,M_{E_n}))$$
$$\longrightarrow H^m((\overline{X}_n,\overline{M}_n)/(E_n,M_{E_n}))$$

is an isomorphism for $m\geqslant 0$.

(2) ([**Tsu99**] § 4.7, cf. [**KM92**] Proposition (1.3)). *The natural homomorphism obtained from Proposition* 4.4.1*:*

$$B^+_{\mathrm{dR}}\otimes_K H^m_{\mathrm{dR}}((X_K,M_K)/K)\longrightarrow \varprojlim_s H^m((\overline{X},\overline{M})/(S,N),\mathcal{O}/J^{[s]})_{\mathbb{Q}_p}$$

is an isomorphism for $m \geqslant 0$ and it induces an isomorphism for $r \geqslant 0$:

$$\mathrm{Fil}^r(B_{\mathrm{dR}}^+ \otimes_K H_{\mathrm{dR}}^m((X_K, M_K)/K)) \xrightarrow{\sim} \varprojlim_s H^m((\overline{X}, \overline{M})/(S, N), J^{[r]}/J^{[s]})_{\mathbb{Q}_p}.$$

To prove (2), we need the degeneration of the Hodge spectral sequence for $(X_K, M_K)/K$, where we use the assumption that M_K is saturated. I don't know whether the degeneration holds without this assumption.

To simplify the notation, we set

$$\begin{aligned} D^m &:= H_{\mathrm{crys}}^m((X, M)), \quad D_{\mathrm{dR}}^m := H_{\mathrm{dR}}^m((X_K, M_K)/K) \\ \mathcal{D}^m &:= H^m((X, M)/(E, M_E))_{\mathbb{Q}_p} \\ \widehat{B_{\mathrm{st}}^+} &:= H^0((\overline{S}, \overline{N})/(E, M_E))_{\mathbb{Q}_p} \quad \text{(the notation of C. Breuil)} \\ \overline{\mathcal{D}}^m &:= H^m((\overline{X}, \overline{M})/(E, M_E))_{\mathbb{Q}_p} \\ \overline{D}_{\mathrm{dR}}^m &:= \varprojlim_s H^m((\overline{X}, \overline{M})/(S, N), \mathcal{O}/J^{[s]})_{\mathbb{Q}_p} \\ \mathrm{Fil}^r\, \overline{D}_{\mathrm{dR}}^m &:= \varprojlim_s H^m((\overline{X}, \overline{M})/(S, N), J^{[r]}/J^{[s]})_{\mathbb{Q}_p} \end{aligned}$$

Then, from Proposition 4.3.2, Proposition 4.4.1 (1) and Proposition 7.1.3, we obtain the following commutative diagram:

$$\begin{array}{ccccc}
H^m((\overline{X}, \overline{M}), S_{\mathbb{Q}_p}^r) & & & & \\
\downarrow{\scriptstyle (7.1.2)} & & & & \\
H^m((\overline{X}, \overline{M})/W, J^{[r]})_{\mathbb{Q}_p}^{\varphi=p^r} & \longrightarrow & \mathrm{Fil}^r\, \overline{D}_{\mathrm{dR}}^m & \xleftarrow[7.1.3(2)]{\sim} & \mathrm{Fil}^r(B_{\mathrm{dR}}^+ \otimes_K D_{\mathrm{dR}}^m) \\
\downarrow & & \downarrow & & \downarrow \\
(\overline{\mathcal{D}}^m)^{\varphi=p^r, N=0} & \longrightarrow & \overline{D}_{\mathrm{dR}}^m & \xleftarrow[7.1.3(2)]{\sim} & B_{\mathrm{dR}}^+ \otimes_K D_{\mathrm{dR}}^m \\
\wr\uparrow{\scriptstyle 7.1.3(1)} & & & & \uparrow{\scriptstyle \iota_\pi \otimes \rho_\pi} \\
(\widehat{B_{\mathrm{st}}^+} \otimes_{R_E} \mathcal{D}^m)^{\varphi=p^r, N=0} & & \xleftarrow{\quad\sim\quad} & & (B_{\mathrm{st}}^+ \otimes_{K_0} D^m)^{\varphi=p^r, N=0}
\end{array}$$

Here the bottom left arrow is obtained from Proposition 4.4.1 (1) and Proposition 4.3.2. Thus we obtain the required homomorphism (7.1.1).

For a line bundle $\mathcal{L}$ on X, we define the syntomic first Chern class $c_{\mathrm{syn}}^1(\mathcal{L})$ to be the image of the class of $\mathcal{L}$ in $\mathrm{Pic}(X) = H_{\text{ét}}^1(X, \mathcal{O}_X^*)$ under the homomorphism $H_{\text{ét}}^1(X, \mathcal{O}_X^*) \to H_{\text{ét}}^1(X, M^{\mathrm{gp}}) \to H^2((\overline{X}, \overline{M}), S_{\mathbb{Q}_p}^1)$ induced by the symbol map. For a line bundle $\mathcal{L}$ on X_K, we define the de Rham first Chern class $c_{\mathrm{dR}}^1(\mathcal{L})$ similarly using $H_{\text{ét}}^1(X_K, M_K^{\mathrm{gp}}) \to H_{\mathrm{dR}}^1((X_K, M_K)/K)$ induced by $d\log\colon M_K^{\mathrm{gp}} \to \Omega^{\cdot}_{X_K/K}(\log(M_K))[1]$.

Proposition 7.1.4 **([Tsu99] Lemma 4.8.9).** — *For any line bundle $\mathcal{L}$ on X, the homomorphism* (7.1.1) *with $m = 2$, $r = 1$ sends $c^1_{\mathrm{syn}}(\mathcal{L})$ to $1 \otimes c^1_{\mathrm{dR}}(\mathcal{L}|_{X_K})$.*

Proof. — We are easily reduced to proving the diagram:

$$\begin{array}{ccc} H^1_{\text{ét}}(X_n, M_n^{\mathrm{gp}}) & \longrightarrow & H^1_{\text{ét}}(X_n, M_n^{\mathrm{gp}}) \\ \downarrow & & \downarrow \\ H^2_{\text{ét}}(Y, \mathcal{S}_n^{\sim}(1)_{(X,M)}) & \longrightarrow & H^2_{\mathrm{dR}}((X_n, M_n)/(S_n, N_n)) \end{array}$$

is commutative. This follows from its local analogue:

$$\begin{array}{ccc} M_n^{\mathrm{gp}} & \longrightarrow & M_n^{\mathrm{gp}} \\ \downarrow{\scriptstyle \text{symbol}} & & \downarrow \\ \mathcal{S}_n^{\sim}(1)_{(X,M),(Z,M_Z)} & \longrightarrow & \Omega^{\cdot}_{X_n/S_n}(\log(M_n/N_n))[1] \end{array}$$

trivial by definition. □

7.2. Proof of C_{st}. — Let (X, M) be as in the beginning of § 6. Then, from Corollary 6.0.2 and (7.1.1), we obtain a G_K-equivariant homomorphism:

(7.2.1)

$$H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \longrightarrow (B^+_{\mathrm{st}} \otimes_{K_0} H^m_{\mathrm{crys}}((X, M)))^{N=0,\varphi=p^r} \cap \mathrm{Fil}^r(B^+_{\mathrm{dR}} \otimes_K H^m_{\mathrm{dR}}(X_K/K))$$

for $0 \leqslant m \leqslant r$ compatible with the product structures and functorial on X. By tensoring $\mathbb{Q}_p(-r) = B_{\mathrm{st}}^{\varphi=p^{-r},N=0} \cap \mathrm{Fil}^{-r} B_{\mathrm{dR}}$, we obtain a G_K-equivariant homomorphism:

$$B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \longrightarrow B_{\mathrm{st}} \otimes_{K_0} H^m_{\mathrm{crys}}((X, M)) \qquad (7.2.2)$$

preserving φ, N and the filtrations after $B_{\mathrm{dR}} \otimes_{B_{\mathrm{st}}}$. We can verify that (7.2.1) for $m = 0$ is induced by $\mathbb{Q}_p(r) = \mathrm{Fil}^r B^+_{\mathrm{dR}} \cap (B^+_{\mathrm{st}})^{\varphi=p^r,N=0}$ and it implies that (7.2.2) is independent of the choice of $r(\geqslant m)$. Combining with Proposition 7.1.4 and Proposition 6.1.4, it also implies:

Proposition 7.2.3. — *For any line bundle $\mathcal{L}$ on X, the homomorphism* (7.2.2) *with $m = 1$ sends $t \otimes (c^1_{\text{ét}}(\mathcal{L}|_{X_K}) \otimes t^{-1})$ to $1 \otimes c^1_{\mathrm{dR}}(\mathcal{L}|_{X_K})$, where t denotes a non-zero element of $\mathbb{Q}_p(1)$.*

Now we will prove that (7.2.2) is a filtered isomorphism, which implies Theorem 1.1 by Corollary 2.2.10. Since the special fiber Y is reduced and X is smooth in a neighborhood of a codimension 0 point of the special fiber, by replacing K with a suitable finite unramified extension, we may assume that X_K is geometrically connected (SGA1 X Proposition 1.2) and has a section $s : S \to X$ whose image is contained in a smooth locus. Set $d := \dim X_K$. We have $\dim_{\mathbb{Q}_p} H^{2d}_{\text{ét}} = \dim_K H^{2d}_{\mathrm{dR}} = \dim_{K_0} H^{2d}_{\mathrm{crys}} = 1$.

Proposition 7.2.4 **([Tsu99] Lemma 4.10.3).** — *The homomorphism* (7.2.2) *for $m = 2d$ is a filtered isomorphism.*

Proof. — (The argument of Fontaine-Messing [**FM87**] III 6.3.) We take the blow up $\widetilde{X}$ of X along s and prove the proposition for $\widetilde{X}$ instead of X. Let P be the exceptional divisor, which is isomorphic to $\mathbb{P}_S^{d-1}$. Then, for a hyperplane $H \subset P$, we have $j^*([P_K]) = -[H_K]$ in $CH^1(P_K)$ where j denotes $P_K \hookrightarrow X_K$ and hence $[P_K]^d = (-1)^{d-1} j_*([H_K]^{d-1})$ in $CH^d(\widetilde{X}_K)$. This implies that the class of a rational point is $(-1)^{d-1} c^1(\mathcal{O}_{\widetilde{X}}(P)|_{\widetilde{X}_K})^d$ in $H^{2d}_{\text{ét}}$ and H^{2d}_{dR}. Hence the proposition for $\widetilde{X}$ follows from Proposition 7.2.3. □

By Proposition 7.2.4 and Poincaré duality, we see that the image of (7.2.2) is a direct factor of the RHS as B_{st}-modules and since $\dim_{\mathbb{Q}_p} H^m_{\text{ét}} = \dim_K H^m_{\text{dR}} (= \dim_{K_0} H^m_{\text{crys}})$ (by the Lefschetz principle and the equality over $\mathbb{C}$), it implies the bijectivity. For the isomorphism of the filtrations, we take gr of $B_{\text{dR}} \otimes_{B_{\text{st}}} (7.2.2)$ and prove that it is injective using Poincaré duality for étale cohomology and Serre duality.

Appendix. C_{st} implies C_{dR}

In this appendix, we will give an argument to derive C_{dR}: the theorem of G. Faltings ([**Fal89**] VIII) from C_{st} by using the alteration of de Jong ([**dJ96**]). As in the Notation in § 1, let K be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field and let $\overline{K}$ be an algebraic closure of K. We will prove the following theorem.

Theorem A1 (C_{dR}). — *For each finite extension L of K contained in $\overline{K}$ and each proper smooth scheme X over L, there exist* $\text{Gal}(\overline{K}/L)$*-equivariant* B_{dR}*-linear canonical isomorphisms:*

$$c_X : B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\cong} B_{\text{dR}} \otimes_L H^m_{\text{dR}}(X/L) \quad (m \in \mathbb{Z})$$

preserving the filtrations and satisfying the properties below. Here $X_{\overline{K}} := X \otimes_L \overline{K}$, the action of $g \in \text{Gal}(\overline{K}/L)$ on the LHS (resp. RHS) is $g \otimes g$ (resp. $g \otimes 1$) and the filtration on the LHS (resp. RHS) is $\text{Fil}^{\cdot} B_{\text{dR}} \otimes H^m_{\text{ét}}$ (resp. the tensor product of the filtrations on B_{dR} and H^m_{dR}). Let t denote any generator of $\mathbb{Z}_p(1) (\subset \text{Fil}^1 B_{\text{dR}})$.

(A1.1) *Functoriality : For other L' and X' such that $L \subset L'$ and a morphism $f : X' \to X$ compatible with $\text{Spec}(L') \to \text{Spec}(L)$, the following diagram is commutative:*

$$\begin{array}{ccc} B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X} & B_{\text{dR}} \otimes_L H^m_{\text{dR}}(X/L) \\ {\scriptstyle 1\otimes f^*}\downarrow & & {\scriptstyle 1\otimes f^*}\downarrow \\ B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X'_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_{X'}} & B_{\text{dR}} \otimes_{L'} H^m_{\text{dR}}(X'/L'). \end{array}$$

(A1.2) *Compatibility with cup products.*

(A1.3) *Compatibility with cycle classes: For any algebraic cycle Y on X of codimension r,*

$$c_X(1 \otimes (\text{cl}^{\text{ét}}_{X_{\overline{K}}}(Y_{\overline{K}}) \otimes t^{-r})) = t^{-r} \otimes \text{cl}^{\text{dR}}_X(Y).$$

(A1.4) *Compatibility with Chern classes: For any vector bundle E on X,*

$$c_X(1 \otimes (c_r^{\text{ét}}(E) \otimes t^{-r})) = t^{-r} \otimes c_r^{\text{dR}}(E).$$

(A1.5) *Compatibility with trace maps: If X is of equidimension d, then the following diagram commutes:*

$$\begin{array}{ccc} B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^{2d}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X} & B_{\text{dR}} \otimes_L H^{2d}_{\text{dR}}(X/L) \\ {\scriptstyle 1\otimes(t^d\cdot \text{Tr})}\downarrow & & \downarrow{\scriptstyle t^d \otimes \text{Tr}} \\ B_{\text{dR}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p & \xrightarrow{\cong} & B_{\text{dR}} \otimes_L L. \end{array}$$

(A1.6) *Compatibility with direct images: Under the same assumption as* (A1.1), *if $L' = L$, X is of equidimension d and X' is of equidimension e, then the following diagram commutes:*

$$\begin{array}{ccc} B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^{m}_{\text{ét}}(X'_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_{X'}} & B_{\text{dR}} \otimes_L H^{m}_{\text{dR}}(X'/L) \\ {\scriptstyle 1\otimes(t^{e-d}\cdot f_*)}\downarrow & & \downarrow{\scriptstyle t^{e-d} \otimes f_*} \\ B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^{m+2(d-e)}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X} & B_{\text{dR}} \otimes_L H^{m+2(d-e)}_{\text{dR}}(X/L). \end{array}$$

First one can derive the following weaker theorem easily from the results of [**Tsu99**].

Theorem A2. — *For each finite extension L of K contained in $\overline{K}$ and each proper smooth scheme X over L with semi-stable reduction, associated to each semi-stable model $\mathfrak{X}$, there exist* $\text{Gal}(\overline{K}/L)$*-equivariant B_{dR}-linear isomorphisms:*

$$c_{\mathfrak{X}} \colon B_{\text{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\cong} B_{\text{dR}} \otimes_L H^m_{\text{dR}}(X/L)$$

preserving the filtrations and satisfying the following properties, where t denotes a generator of $\mathbb{Z}_p(1) (\subset \text{Fil}^1 B_{\text{dR}})$.

(A2.1) *Functoriality I: For other L', X' and $\mathfrak{X}'$ such that $L \subset L'$ and a morphism $f \colon \mathfrak{X}' \to \mathfrak{X}$ compatible with* $\text{Spec}(O_{L'}) \to \text{Spec}(O_L)$, *the same diagram as in* (A1.1) *with c_X and $c_{X'}$ replaced by $c_{\mathfrak{X}}$ and $c_{\mathfrak{X}'}$ is commutative* ([**Tsu99**] *Proposition* 4.10.4).

(A2.2) *Compatibility with cup products.*

(A2.3) *Compatibility with cycle classes: For any algebraic cycle Y on X of codimension r,*

$$c_{\mathfrak{X}}(1 \otimes (\text{cl}^{\text{ét}}_{X_{\overline{K}}}(Y_{\overline{K}}) \otimes t^{-r})) = t^{-r} \otimes \text{cl}^{\text{dR}}_X(Y).$$

(A2.4) *Compatibility with Chern classes: For any vector bundle E on X,*

$$c_{\mathfrak{X}}(1 \otimes (c_r^{\text{ét}}(E) \otimes t^{-r})) = t^{-r} \otimes c_r^{\text{dR}}(E).$$

(A2.5) *Compatibility with trace maps: If X is of equidimension d, then the same diagram as in* (A1.5) *with c_X replaced by $c_{\mathfrak{X}}$ is commutative.*

(A2.6) *Compatibility with direct images: Under the same assumption as* (A2.1), *if $L' = L$, X is of equidimension d and X' is of equidimension e, then the same diagram as in* (A1.6) *with c_X and $c_{X'}$ replaced by $c_{\mathfrak{X}}$ and $c_{\mathfrak{X}'}$ is commutative.*

(A2.7) *Functoriality II: For any* $\sigma \in \mathrm{Gal}(\overline{K}/K)$*, if we denote by* X^σ*,* $\mathfrak{X}^\sigma$ *the base change of* X*,* $\mathfrak{X}$ *by* $\mathrm{Spec}(\sigma)\colon \mathrm{Spec}(O_{\sigma(L)}) \to \mathrm{Spec}(O_L)$*, then the following diagram commutes:*

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_{\mathfrak{X}}} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L) \\ {\scriptstyle \sigma\otimes\sigma^*}\downarrow \wr & & {\scriptstyle \sigma\otimes\sigma^*}\downarrow \wr \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X^\sigma_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_{\mathfrak{X}^\sigma}} & B_{\mathrm{dR}} \otimes_{\sigma(L)} H^m_{\mathrm{dR}}(X^\sigma/\sigma(L)). \end{array}$$

Here σ^* *denote the isomorphisms induced by the following cartesian diagrams:*

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\sim} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\sigma(L)) & \xrightarrow[\mathrm{Spec}(\sigma)]{\sim} & \mathrm{Spec}(L), \end{array} \qquad \begin{array}{ccc} X^\sigma_{\overline{K}} & \xrightarrow{\sim} & X_{\overline{K}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{K}) & \xrightarrow[\mathrm{Spec}(\sigma)]{\sim} & \mathrm{Spec}(\overline{K}). \end{array}$$

Proof. — The isomorphism $c_{\mathfrak{X}}$ compatible with the cup products (A2.2) is constructed in [**Tsu99**] § 4.10, (A2.1) is proven in [**Tsu99**] Proposition 4.10.4 and (A2.7) is trivial by the construction of $c_{\mathfrak{X}}$. We will prove the remaining properties.

(A2.3) (I learned the following argument from W. Messing.) Since $c_{\mathfrak{X}}$ is compatible with the Chern classes of a line bundle on $\mathfrak{X}$ (not on X!) ([**Tsu99**] Proposition 4.10.1) and $c_{\mathfrak{X}}$ is functorial on $\mathfrak{X}$ (A2.1), we see that $c_{\mathfrak{X}}$ is compatible with the Chern classes of a vector bundle on $\mathfrak{X}$ by the splitting principle. Here note that the flag variety associated to a vector bundle on $\mathfrak{X}$ is proper smooth over $\mathfrak{X}$. For any integral closed subscheme Y of X, if we denote by $\mathcal{Y}$ the closure of Y in $\mathfrak{X}$, then $\mathcal{O}_{\mathcal{Y}}$ has a resolution of finite length by locally free sheaves of finite rank (because $\mathfrak{X}$ is regular) and the cycle classes of Y in $H^*_{\text{ét}}$ and H^*_{dR} can be described in the same way in terms of the Chern classes of the locally free sheaves appearing in the resolution. Hence $c_{\mathfrak{X}}$ is also compatible with cycle classes.

(A2.4) Choose a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{E}$ such that $\mathcal{E}|_X \cong E$ (EGA I (9.4.8)). Then $\mathcal{E}$ has a resolution of finite length by locally free sheaves of finite rank. The rest is the same as the proof of (A2.3) above.

(A2.5) By (A2.1), $c_{\mathfrak{X}}$ decomposes into the sum of $c_{\mathfrak{X}'}$ for each irreducible component $\mathfrak{X}'$ of $\mathfrak{X}$. Hence, by (A2.1) again, we can replace L by a suitable finite unramified extension contained in $\overline{K}$ and assume that X is geometrically irreducible and has an L-rational point. In this case, H^{2d} are both one dimensional and (A2.5) follows from the compatibility with cycle classes of a point (A2.3).

(A2.6) follows from (A2.1), (A2.2) and (A2.5). □

In the rest of this appendix, we will derive Theorem A1 from Theorem A2 using the alteration of de Jong [**dJ96**]. First let us recall a result of de Jong. In this appendix, we say that a morphism $f\colon X \to Y$ between reduced noetherian schemes is an *étale alteration* if it is proper surjective and, for each $x \in X$ of codimension 0, f is étale in a neighborhood of x. If f is proper and surjective, the latter condition is equivalent to the following: For each $y \in Y$ of codimension 0, there exists an open neighborhood $V \subset Y$ of y such that $f^{-1}(V) \to V$ is étale and, for each $x \in X$ of codimension 0, $f(x)$ is also of codimension 0 in Y. Let L be a finite extension of K. For a scheme $\mathcal{X}$ of finite type over O_L, we say that $\mathcal{X}$ is *strictly semi-stable* over O_L if $\mathcal{X}$ is regular, the special fiber of $\mathcal{X}$ is a reduced divisor with normal crossings on $\mathcal{X}$, and the irreducible components of the special fiber and their intersections are smooth over the residue field of L.

***Theorem A3* (de Jong [dJ96]).** — *For a proper flat reduced scheme $\mathcal{X}$ over O_L, there exist a finite extension M of L, a proper strictly semi-stable scheme $\mathcal{Y}$ over O_M and a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ compatible with* $\mathrm{Spec}(O_M) \to \mathrm{Spec}(O_L)$ *such that the morphism $\mathcal{Y} \to \mathcal{X} \otimes_{O_L} O_M$ induced by f is an étale alteration.*

We will also need the following fact.

Proposition A4. — *For a proper strictly semi-stable scheme $\mathcal{X}$ over O_L, there exist a proper strictly semi-stable scheme $\mathcal{Z}$ over O_L and a proper surjective morphism $\mathcal{Z} \to \mathcal{X} \times_{\mathrm{Spec}(O_L)} \mathcal{X}$ over O_L which is an isomorphism on the generic fiber.*

Now let us construct the isomorphism c_X.

Proposition A5. — *Let L be a finite extension of K contained in $\overline{K}$, let X be a proper smooth scheme over L and let $\mathcal{X}$ be a proper flat model of X. (Such a model always exists by the compactification theorem of Nagata). Suppose that we are given a proper strictly semi-table scheme $\mathcal{Y}$ over O_L and an étale alteration $f\colon \mathcal{Y} \to \mathcal{X}$ over O_L. Then the homomorphism*

$$f^* : H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \longrightarrow H^m_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_p)$$

is injective, the homomorphism

$$f^* : H^m_{\text{dR}}(X/L) \longrightarrow H^m_{\text{dR}}(Y/L)$$

is injective and strictly compatible with the Hodge filtrations and $c_{\mathcal{Y}}$ in Theorem A2 *induces a* $\mathrm{Gal}(\overline{K}/L)$*-equivariant B_{dR}-linear isomorphism*

$$B_{\text{dR}} \otimes_{\mathbb{Q}_p} f^*(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong B_{\text{dR}} \otimes_L f^*(H^m_{\text{dR}}(X/L))$$

preserving the filtrations.

Proof. — (I learned this argument from T. Saito). By (A2.1), we may assume that X and $Y := \mathcal{Y} \otimes_{O_L} L$ is irreducible. (Note that $\mathcal{Y}$ and X are disjoint union of irreducible components but $\mathcal{X}$ is not in general. We replace $\mathcal{X}$ by the disjoint union of the irreducible components of $\mathcal{X}$ with the reduced induced closed subscheme structures.) Let g be the correspondence defined by the transpose $\Gamma^t_f = (f, \mathrm{id}_Y)\colon Y \hookrightarrow X \times Y$ of the graph $\Gamma_f := (\mathrm{id}_Y, f)\colon Y \hookrightarrow Y \times X$ of f. Then the composite $f \circ g$ is $n \cdot \mathrm{id}_X$. Here

n denotes the degree of $Y \to X$ at the generic point of X. Indeed, we see easily that the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{(\mathrm{id}_Y, f)} & Y \times X \\ {\scriptstyle (f,\mathrm{id}_Y)}\downarrow & & \downarrow{\scriptstyle \Gamma_f^t \times \mathrm{id}_X} \\ X \times Y & \xrightarrow{\mathrm{id}_X \times \Gamma_f} & X \times Y \times X \end{array}$$

is cartesian, and the direct image of the cycle $(f, \mathrm{id}_Y, f) : Y \subset X \times Y \times X$ in $X \times X$ is $n \cdot \Delta_X$. Here Δ_X denotes the diagonal of $X \times X$. Hence for the two cohomology groups in question, we have $g^* \circ f^* = (f \circ g)^* = n$ and hence f^* are injective. By applying the same argument to the Hodge cohomology $\oplus_i H^{m-i}(Z, \Omega^i_{Z/L}) \cong \oplus_i \mathrm{gr}^i H^m_{\mathrm{dR}}(Z/L)$, we see that the gr of f^* for de Rham cohomology is injective and hence f^* is strictly compatible with the Hodge filtrations. From the above argument, it also follows that g^* is surjective and hence the image of f^* coincides with the image of $f^* \circ g^* = (g \circ f)^*$. Set $H^m(Z)(r) = H^m_{\text{ét}}(Z_{\overline{K}}, \mathbb{Q}_p)(r)$ or $H^m_{\mathrm{dR}}(Z/L)$ (here we ignore the Hodge filtration) and denote by c the class in $H^{2d}(Y \times Y)(d)$ ($d = \dim X = \dim Y$) defined by the correspondence $g \circ f$. Then the composite $f^* \circ g^*$ is given by

$$H^m(Y) \xrightarrow{p_1^*} H^m(Y \times Y) \xrightarrow{-\cup c} H^{m+2d}(Y \times Y)(d) \xrightarrow{p_{2*}(d)} H^m(Y)$$

Now from Proposition A4, (A2.3), (A2.1), (A2.2) and (A2.6), we obtain the isomorphism in the proposition. The compatibility with the filtrations follows from the strict compatibility of f^* with the Hodge filtrations. □

Let L be a finite extension of K contained in $\overline{K}$ and let X be a proper smooth scheme over L. Choose a proper flat model $\mathcal{X}$ of X. Then by Theorem A3, there exist a finite extension M of L contained in $\overline{K}$, a proper strictly semi-stable scheme $\mathcal{Y}$ over O_M and a morphism $f \colon \mathcal{Y} \to \mathcal{X}$ compatible with $\mathrm{Spec}(O_M) \to \mathrm{Spec}(O_L)$ such that the induced morphism $f' \colon \mathcal{Y} \to \mathcal{X} \otimes_{O_L} O_M$ is an étale alteration. Choose such M, $\mathcal{Y}$ and f. Applying Proposition A5 to f', we obtain an isomorphism

$$c_{\mathcal{X},\mathcal{Y},f} \colon B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\cong} B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L)$$

which makes the following diagram commutative:

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \underset{\cong}{\xrightarrow{c_{\mathcal{X},\mathcal{Y},f}}} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L) \\ {\scriptstyle 1 \otimes f^*}\downarrow & & \downarrow{\scriptstyle 1 \otimes f^*} \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_p) & \underset{\cong}{\xrightarrow{c_{\mathcal{Y}}}} & B_{\mathrm{dR}} \otimes_M H^m_{\mathrm{dR}}(Y/M). \end{array}$$

Here the two vertical homomorphisms are injective and the homomorphism $c_{\mathcal{X},\mathcal{Y},f}$ is compatible with the actions of $\mathrm{Gal}(\overline{K}/M)$ and with the filtrations.

Proposition A6. — *Under the notations and assumptions as above, the homomorphism $c_{\mathcal{X},\mathcal{Y},f}$ is independent of the choice of $\mathcal{X}$, M, $\mathcal{Y}$ and f.*

Proof. — Choose other $\mathfrak{X}_1$, M_1, $\mathcal{Y}_1$ and f_1. Let $\mathfrak{X}_2$ be the scheme theoretic closure of the diagonal Δ_X of $X \times X$ in $\mathfrak{X} \times \mathfrak{X}_1$. Then $\mathfrak{X}_2$ is a proper flat model of X from which there are maps to $\mathfrak{X}$ and $\mathfrak{X}_1$. Let $M_2' := M \cdot M_1$, Let $\mathfrak{X}_2'$ be the base change of $\mathfrak{X}_2$ by $O_L \subset O_{M_2'}$ and let $\mathcal{Y}'$ (resp. $\mathcal{Y}_1'$) be the base change of the closed subscheme of $\mathcal{Y} \times_{\mathfrak{X}} \mathfrak{X}_2$ (resp. $\mathcal{Y}_1 \times_{\mathfrak{X}_1} \mathfrak{X}_2$) defined by the ideal consisting of all torsion elements by $M \subset M_2'$ (resp. $M_1 \subset M_2'$). Then we have natural étale alterations over $O_{M_2'}$ $\mathcal{Y}' \to \mathfrak{X}_2'$ and $\mathcal{Y}_1' \to \mathfrak{X}_2'$. Let $\mathcal{Y}_2'$ be the closure in $\mathcal{Y}' \times_{\mathfrak{X}_2'} \mathcal{Y}_1'$ of the inverse images of all points of codimension 0 on $\mathfrak{X}_2'$ (or equivalently X_2') endowed with the reduced closed subscheme structure. Note that, in general, the generic fiber of $\mathcal{Y}_2'$ is smooth over M_2' only in a neighborhood of the points of codimension 0. Then, by Theorem A3, there exist a finite extension M_2 of M_2' contained in $\overline{K}$, a proper strictly semi-stable scheme $\mathcal{Y}_2$ over O_{M_2}, and a morphism $\mathcal{Y}_2 \to \mathcal{Y}_2'$ compatible with $\operatorname{Spec}(O_{M_2}) \to \operatorname{Spec}(O_{M_2'})$ such that the induced morphism $\mathcal{Y}_2 \to \mathcal{Y}_2' \otimes_{O_{M_2'}} O_{M_2}$ is an étale alteration. Thus we obtain a commutative diagram

$$\begin{array}{ccccc} \mathfrak{X} & \longleftarrow & \mathfrak{X}_2 & \longrightarrow & \mathfrak{X}_1 \\ {\scriptstyle f}\uparrow & & {\scriptstyle f_2}\uparrow & & {\scriptstyle f_1}\uparrow \\ \mathcal{Y} & \longleftarrow & \mathcal{Y}_2 & \longrightarrow & \mathcal{Y}_1 \end{array}$$

over the commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec}(O_L) & = & \operatorname{Spec}(O_L) & = & \operatorname{Spec}(O_L) \\ \uparrow & & \uparrow & & \uparrow \\ \operatorname{Spec}(O_M) & \longleftarrow & \operatorname{Spec}(O_{M_2}) & \longrightarrow & \operatorname{Spec}(O_{M_1}). \end{array}$$

Here the middle vertical morphism induces an étale alteration $\mathcal{Y}_2 \to \mathfrak{X}_2 \otimes_{O_L} O_{M_2}$. Now, from the functoriality (A2.1), we obtain $c_{\mathfrak{X},\mathcal{Y},f} = c_{\mathfrak{X}_2,\mathcal{Y}_2,f_2} = c_{\mathfrak{X}_1,\mathcal{Y}_1,f_1}$. □

We set $c_X = c_{\mathfrak{X},\mathcal{Y},f}$.

Proposition A7. — *Under the notations and assumptions above, c_X is compatible with the actions of* $\operatorname{Gal}(\overline{K}/L)$.

Proof. — Choose $\mathfrak{X}$, M, $\mathcal{Y}$ and f as above. Then c_X is compatible with the actions of $\operatorname{Gal}(\overline{K}/M)$. Let σ be an arbitrary element of $\operatorname{Gal}(\overline{K}/L)$, let Y^σ, $\mathcal{Y}^\sigma$ be the base change of Y, $\mathcal{Y}$ by $\operatorname{Spec}(\sigma)\colon \operatorname{Spec}(\sigma(O_M)) \xrightarrow{\sim} \operatorname{Spec}(O_M)$, and let f^σ be the composite $\mathcal{Y}^\sigma \xrightarrow{\sim} \mathcal{Y} \xrightarrow{f} \mathfrak{X}$. Since the action of σ on L is trivial, f^σ is compatible with the embedding $L \hookrightarrow \sigma(M)$. By the definition of $c_{\mathfrak{X},\mathcal{Y},f}$ and the functoriality (A2.7), we obtain the following commutative diagram:

$$\begin{array}{ccc}
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X = c_{X,Y,f}} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L) \\
{\scriptstyle 1\otimes f^*}\big\downarrow \cap & & {\scriptstyle 1\otimes f^*}\big\downarrow \cap \\
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(Y_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_Y} & B_{\mathrm{dR}} \otimes_M H^m_{\mathrm{dR}}(Y/M) \\
{\scriptstyle \sigma\otimes\sigma^*}\big\downarrow \wr & & {\scriptstyle \sigma\otimes\sigma^*}\big\downarrow \wr \\
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(Y^{\sigma}_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_{Y^\sigma}} & B_{\mathrm{dR}} \otimes_{\sigma(M)} H^m_{\mathrm{dR}}(Y^\sigma/\sigma(M)) \\
{\scriptstyle 1\otimes(f^\sigma)^*}\big\uparrow \cup & & {\scriptstyle 1\otimes(f^\sigma)^*}\big\uparrow \cup \\
B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X = c_{X,Y^\sigma,f^\sigma}} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L).
\end{array}$$

Here the morphisms f^*, σ^* and $(f^\sigma)^*$ between étale and de Rham cohomology groups are induced by the following commutative diagrams respectively:

$$\begin{array}{ccc}
X & \xleftarrow{f} & Y \\
\big\downarrow & & \big\downarrow \\
\mathrm{Spec}(L) & \longleftarrow & \mathrm{Spec}(M) \\
\big\uparrow & & \big\uparrow \\
\mathrm{Spec}(\overline{K}) & \xleftarrow[\sim]{\mathrm{id}} & \mathrm{Spec}(\overline{K}),
\end{array}
\qquad
\begin{array}{ccc}
Y & \xleftarrow{\sim} & Y^\sigma \\
\big\downarrow & & \big\downarrow \\
\mathrm{Spec}(M) & \xleftarrow[\sim]{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\sigma(M)) \\
\big\uparrow & & \big\uparrow \\
\mathrm{Spec}(\overline{K}) & \xleftarrow[\sim]{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\overline{K})
\end{array}$$

$$\begin{array}{ccc}
X & \xleftarrow{f^\sigma} & Y^\sigma \\
\big\downarrow & & \big\downarrow \\
\mathrm{Spec}(L) & \longleftarrow & \mathrm{Spec}(\sigma(M)) \\
\big\uparrow & & \big\uparrow \\
\mathrm{Spec}(\overline{K}) & \xleftarrow[\sim]{\mathrm{id}} & \mathrm{Spec}(\overline{K})
\end{array}$$

If we denote by φ_σ the morphism between the two cohomology groups induced by the diagram

$$\begin{array}{ccc}
X & \xleftarrow[\sim]{\mathrm{id}} & X \\
\big\downarrow & & \big\downarrow \\
\mathrm{Spec}(L) & \xleftarrow[\sim]{\mathrm{id}} & \mathrm{Spec}(L) \\
\big\uparrow & & \big\uparrow \\
\mathrm{Spec}(\overline{K}) & \xleftarrow[\sim]{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\overline{K}),
\end{array}$$

then we have $\sigma^* \circ f^* = (f^\sigma)^* \circ \varphi_\sigma$. On the other hand, φ_σ is nothing but the action of σ for the étale cohomology and the identity for the de Rham cohomology. Hence the following diagram is commutative:

$$\begin{array}{ccc} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L) \\ {\scriptstyle \sigma\otimes\sigma}\downarrow\wr & & {\scriptstyle \sigma\otimes\mathrm{id}}\downarrow\wr \\ B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^m_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p) & \xrightarrow[\cong]{c_X} & B_{\mathrm{dR}} \otimes_L H^m_{\mathrm{dR}}(X/L). \end{array}$$

□

Finally, we will prove that c_X satisfies the properties (A1.1)-(A1.6). First let us prove the functoriality (A1.1). We can verify that c_X and c_{X_1} are compatible for any finite extension L_1 of L contained in $\overline{K}$ and the base change X_1 of X to L_1. (Choose a proper flat model $\mathfrak{X}$ of X, choose M, $\mathcal{Y}$ and f for the base change $\mathfrak{X}_1$ of $\mathfrak{X}$ to L_1, and use the same M and $\mathcal{Y}$ to define c_X and c_{X_1}.) If we denote by X_i, $i \in I$ the irreducible components of X, then we see easily $c_X = \oplus_{i\in I} c_{X_i}$. Hence, we may assume $L = L'$ and that X and X' are geometrically irreducible. We may further assume that there are a proper flat model $\mathfrak{X}$ of X, a proper strictly semi-stable scheme $\mathcal{Y}$ over O_L and an étale alteration $\mathcal{Y} \to \mathfrak{X}$ over O_L. Choose a proper flat model $\mathfrak{X}'_1$ of X and let $\mathfrak{X}'$ be the scheme theoretic image of $(\mathrm{id}_{X'}, f) : X' \hookrightarrow X' \times X$ in $\mathfrak{X}'_1 \times \mathfrak{X}$. Then $\mathfrak{X}'$ is a proper flat model of X' and the morphism f extends to a morphism $\mathfrak{X}' \to \mathfrak{X}$. Choose a closed point of the fiber of $\mathcal{Y} \times_{\mathfrak{X}} \mathfrak{X}' \to \mathfrak{X}'$ over the unique generic point of $\mathfrak{X}'$ and let $\widetilde{\mathfrak{X}}'$ be its closure in $\mathcal{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ endowed with the reduced induced closed subscheme structure. Then $\widetilde{\mathfrak{X}}' \to \mathfrak{X}'$ is an étale alteration. By Theorem A3, there is a finite extension M of L contained in $\overline{K}$, a proper strictly semi-stable scheme $\mathcal{Y}'$ over O_M and a morphism $\mathcal{Y}' \to \widetilde{\mathfrak{X}}'$ compatible with $\mathrm{Spec}(O_M) \to \mathrm{Spec}(O_L)$ such that the induced morphism $\mathcal{Y}' \to \widetilde{\mathfrak{X}}' \otimes_{O_L} O_M$ is an étale alteration. Define c_X and c_Y using $\mathcal{Y} \to \mathfrak{X}$ and $\mathcal{Y}' \to \widetilde{\mathfrak{X}}' \to \mathfrak{X}'$. Then (A1.1) follows from (A2.1).

The compatibility with the cup products (A1.2) follows easily from (A2.2). The compatibility with cycle classes (A1.3) follows from (A2.3) and the compatibility of the pull-back maps with cycle classes for étale and de Rham cohomologies. Similar for the compatibility with Chern classes (A1.4). For the compatibility with trace maps (A1.5), by replacing L by a finite extension of L contained in $\overline{K}$, we are reduced to the case that X is geometrically irreducible and has an L-rational point. Here we use the compatibility of c_X with base changes and with the decomposition of X into its irreducible components. Then (A1.5) follows from the compatibility of c_X with the cycle classes of a point (A1.3). Finally the compatibility with direct images (A1.6) follows from (A1.1), (A1.2) and (A1.5).

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