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ALMOST ÉTALE EXTENSIONS

by

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Abstract. — The theory of almost étale coverings allows to compare crystalline and p-adic étale cohomology, for schemes over a p-adic discrete valuation ring. Using Frobenius the main technical result (a purity theorem) is reproved and extended to all toroidal singularities. As a consequence one obtains Tsuji's comparison theorem for schemes with such type of singularities, even for cohomology with coefficients in suitable local systems. On the way we have to establish some basic results on finiteness of crystalline cohomology with such coefficients.

0. Introduction

The theory of almost étale coverings goes back to Tate for the case of discrete valuation rings. The higher dimensional theory has been invented by the author, and used to prove conjectures about Hodge-Tate and crystalline structures on the étale cohomology of varieties over p-adic fields. The main purpose of these notes is to extend the theory to cover toroidal singularities, especially all schemes with semistable reduction. This is possible because we have found a new method of proof for the main technical result, the purity theorem. This new method makes heavy use of Frobenius and of toroidal geometry. After starting and proving the new purity theorem we study duality, first local and then global. For the globalization we need a topos $\mathfrak{X}$ which (X a scheme or an algebraic space over a p-adic discrete valuation ring V, X_K its generic fibre) is an étale localization (in X) of the topos of locally constant sheaves on X_K. Thus its cohomology (or better the direct images to the étale topos of X) is Galois-cohomology. Our previous local theory applies to "coherent sheaves" on this topos, and they satisfy finiteness, Künneth and Poincaré-duality like any decent theory. However for this we have to work in the almost category, that is we divide by the subcategory of sheaves annihilated by any positive (fractional) power of p. A

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more systematic study of almost mathematics can be found in the recent preprint [GR] by O. Gabber and L. Ramero.

It is quite remarkable that one can still apply the Artin-Schreier exact sequence to derive from the results for "discrete" coefficients, like $\mathbb{Z}/p^s\mathbb{Z}$. Namely for these coefficients one cannot use the almost category. However at one key turn almost invariants under Frobenius coincides with real invariants. Let us also mention that we make again heavy use of Frobenius (now on the coefficients) and Fontaine's construction of rings $A_{\mathrm{inf}}(R)$. Finally we show that the theory with discrete coefficients gives étale cohomology of the generic fibre X_K. For this we have to compare characteristic classes or better Gysin maps. For regular embeddings one reduces to codimension one and an explicit calculation of local direct images. The latter replaces a more global (and less transparent) argument in [Fa 3]. In general one uses toroidal modifications to make embeddings regular. That is possible for the diagonal was a great surprise for the author.

At the end we explain how to derive comparison maps to crystalline cohomology. However this is only sketched as all these ideas have already been presented in a number of papers ([Fa 3], [Fa 5], [Fa 6], [Fa 7], so that everybody interested in the matter can learn it there. We also add some (a little bit) incomplete results about cohomology of isocrystals, which we could not find in the literature.

All in all this suffices to prove the C_{st}-conjecture of Fontaine-Jansen. This had been done previously by T. Tsuji ([Ts]). However we can also treat the case of non-constant coefficients where some new phenomena occur. Anyway the author profited from his participation in the p-adic year at Institut Poincaré in Paris, first by giving lectures on a preliminary version and then by learning about T. Tsuji's technique. I have to thank the referee who found many mistakes and made an enormous number of suggestions for improvements.

1. Almost-Mathematics

Let $\overline{V}$ denote a (commutative with unit) ring together with a sequence of principal ideals $\mathfrak{m}_\alpha \subseteq \overline{V}$ parametrized by positive elements $\alpha \in \Lambda^+$, where $\Lambda \subseteq \mathbb{Q}$ denotes a subgroup dense in $\mathbb{R}$. Denote by π a generator of $\mathfrak{m}_1$, and π^α a generator of $\mathfrak{m}_\alpha$. Assume furthermore that $\pi^\alpha\pi^\beta =$ unit $\cdot\pi^{\alpha+\beta}$, and that π^α is not a zero-divisor.

Examples

a) $\overline{V} = \mathbb{Z}_p =$ integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$, $\mathfrak{m}_\alpha =$ elements of valuation $\geqslant \alpha$, $\pi = p$ $(\Lambda = \mathbb{Q})$.

b) Consider $R = \varprojlim(\overline{\mathbb{Z}}_p/p\overline{\mathbb{Z}}_p, \text{Frobenius})$, $\overline{V} = W(R)$,

$$\pi = [\underline{p}, 0, \ldots, 0] \in W(R),$$

$\underline{p} = \varprojlim p^{1/p^n} \in R$, $\Lambda = \mathbb{Z}[1/p]$, $\pi^\alpha =$ the obvious element.

If $\mathfrak{m} = \bigcup_{\alpha>0} \mathfrak{m}_\alpha$, then a $\overline{V}$-module is almost zero if it is annihilated by $\mathfrak{m}$. Denote this by $M \approx 0$. Assume R is a $\overline{V}$-algebra and M an R-module.

1. Definition. — M is almost projective if $\mathrm{Ext}^i_R(M,N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)
M is almost flat if $\mathrm{Tor}_i^R(M,N) \approx 0$, all R-modules N and all $i > 0$ (or only $i = 1$)
M is almost finitely generated if for each $\alpha > 0, \alpha \in \Lambda$, there exists a finitely generated R-module N and a π^α-isomorphism $\psi_\alpha : N \to M$ (i.e., there exists $\phi_\alpha : M \to N$ with $\phi_\alpha \circ \psi_\alpha = \pi^\alpha \cdot \mathrm{id}_N$, $\psi_\alpha \circ \phi_\alpha = \pi^{\alpha_0} \circ \mathrm{id}_M$).
M is almost finitely presented: dito with N finitely presented.

2. Remarks

a) P almost projective
$\Leftrightarrow$ for all surjections $f : M \to N$ and maps $g : P \to N$, $\pi^\alpha \cdot g$ factors through f ($\alpha > 0$, $\alpha \in \Lambda$)
$\Leftrightarrow$ all $\alpha > 0$, $\alpha \in \Lambda$, there exists a free R-module L and maps $L \xrightarrow{g} P \xrightarrow{f} L$ with $g \circ f = \pi^\alpha \cdot \mathrm{id}_p$.

b) If P is almost projective and almost finitely generated, then P is almost finitely presented. Then also $P^* = \mathrm{Hom}_R(P,R), P \otimes_R P, \Lambda^i P$ etc. are almost finitely generated projective. Furthermore there is an almost isomorphism $P \otimes_R P^* \approx \mathrm{End}_R(P)$, and thus a trace-map $\mathrm{tr} : \mathfrak{m} \otimes_R \mathrm{End}_R(P) \to R$, or equivalently $\mathrm{End}_R(P) \to \mathrm{Hom}(\mathfrak{m}, R)$. Thus for $f \in \mathrm{End}_R(P)$ we can define a power-series

$$\det(1 + Tf) = \sum_{i=0}^{\infty} \mathrm{tr}(\Lambda^i f) \cdot T^i \in \mathrm{Hom}(\mathfrak{m}, R)[[T]].$$

One checks that $\mathrm{Hom}(\mathfrak{m}, R)$ is a ring

$$f \circ g(\pi^{\alpha+\beta}) = f(\pi^\alpha) \cdot g(\pi^\beta))$$

and that

$$\det(1 + Tf)\det(1 + Tg) = \det(1 + T(f + g + Tfg))(= \sum_{i=0}^{\infty} \mathrm{tr}(\Lambda^i(f + g + Tfg)) \cdot T^i$$

We say that P has rank $\leqslant r$ if $\Lambda^{r+1} P \approx 0$. Then letting $e \in \mathrm{Hom}(\mathfrak{m}, R)$ denote the coefficient of T^r in $\det(1 + T \cdot \mathrm{id}_p)$, one checks that $e = e^2$ is an idempotent, and thus $R \approx R_1 \times R_2$ factors (up to almost isomorphism). Also $P \approx (P \otimes_R R_1) \times (P \otimes_R R_2)$.

Over one of the factors (say R_1) $e = 1$. One then checks that after base-change to R_1, $L = \Lambda^r P$ satisfies $L \otimes_R L^* \approx R$, i.e. L behaves like a line-bundle: There are the maps

$$\mathrm{tr} : \mathrm{End}_R(L) \approx L \otimes_R L^* \longrightarrow R$$

and the identity: $R \to \mathrm{End}_R(L)$. We have $\mathrm{tr}(\mathrm{id}_p) = e = 1$. In fact for two $f, g \in \mathrm{End}_R(P)$ we have

$$\mathrm{tr}(\Lambda^r f) \cdot \mathrm{tr}(\Lambda^r g) = \mathrm{tr}(\Lambda^r(fg)) = \text{coefficient of } T^{2r} \text{ in } \det((1+Tf)(1+Tg)).$$

Apply this to f and g of the form

$$f(x) = \sum_{i=1}^{r} \lambda_i(x)\mu_i, \quad \lambda_i \in P^*,\ \mu_i \in P$$
$$g(x) = \sum_{i=1}^{r} \nu_i(x)\phi_i$$

to get $\mathrm{tr}(\Lambda^r f) \cdot \det(\nu_i(\phi_j)) = \det(\nu_i(f(\phi_j)))$, i.e.

$$\mathrm{tr}(\lambda^r f)(\nu_1 \wedge \cdots \wedge \nu_r \mid \phi_1 \wedge \cdots \wedge \phi_r) = (\nu_1 \wedge \cdots \wedge \nu_r \mid \Lambda^r f(\phi_1 \wedge \cdots \wedge \phi_r)).$$

Varying ν's and ϕ's gives $\Lambda^r f = \mathrm{tr}(\Lambda^r f)$ is a scalar. As the $\Lambda^r f$ (given by $x \mapsto \langle \lambda_1 \wedge \cdots \wedge \lambda_r \mid x(\mu_1 \wedge \cdots \wedge \mu_r)\rangle$ generate $\mathrm{End}_R(L)$, everything follows. The same argument gives:
If $e = 0$, then $\mathrm{tr}(\Lambda^r f) = 0$ and $\Lambda^r f = 0$, and then $\Lambda^r P = 0$. Thus by induction we get an almost isomorphism

$$R \approx R_0 \times \cdots \times R_r,$$

such that over R_i $\Lambda^i P$ is invertible, $\Lambda^{i+1} P = 0$. Also over R_i we have a determinant-function

$$\det(f) = \mathrm{tr}(\Lambda^i f) : \mathrm{End}_R(P) \longrightarrow \mathrm{Hom}(\mathfrak{m}, R_i)$$

homogeneous of degree i with $\det(fg) = \det(f)\det(g), \det(1) = 1$.

3. Remark. — I do not know whether for arbitrary finitely generated projective P's we have a decomposition $R \approx \prod_{i=0}^{\infty} R_i$ with $P \otimes_R R_i$ of rank i.

2. Almost étale coverings

1. Definition. — A ring homomorphism $A \to B$ is called an almost étale covering if
i) B is almost finitely generated projective as an A-module, of finite rank
ii) B is almost finitely generated projective as a $B \otimes_A B$-module

2. Remarks

a) There exists an idempotent $e_{B/A} \in \mathrm{Hom}(\mathfrak{m}, B \otimes_A B)$ measuring where B has rank 1 as $B \otimes_A B$-module. $e_{B/A}$ is annihilated by $(b \otimes_A 1 - 1 \otimes_A b)$ and maps to $1 \in \mathrm{Hom}(\mathfrak{m}, B)$ under multiplication.

b) Replacing A by $\mathrm{Hom}(\mathfrak{m}, A)$ we may reduce (in many cases) to the case where $\mathrm{rank}_A B = r$ is constant. Then there exists an almost faithfully flat base-change $A \to A'$ (A' is almost flat, and $M \otimes_A A' \approx 0$ implies $M \approx 0$) with $B' = B \otimes_A A' \approx A'^r$.

Proof. — If $r \geqslant 1, A \to B$ is almost faithfully flat: If $\mathfrak{a} \subseteq A$ is an ideal with $\mathfrak{a}B \approx B$, then $\mathfrak{m}A \subseteq \mathrm{Norm}_{B/A}(\mathfrak{a}B) \subseteq \mathfrak{a}\,\mathrm{Hom}(\mathfrak{m}, A)$. Furthermore after base-change to B we may split of one factor B from $B \otimes_A B$, thus reducing r to $r-1$.

c) $A \to B$ is called a G-covering if a finite group G acts on B such that after almost faithfully flat base-change $A \to A'$ we have $B' = B \otimes_A A' \approx A'^G$. If B has constant rank r, there exists $A \to B \to C$ with C/A a G-covering with $G = S_r$ (symm. group in r letters) C/B is an H-covering with $H = S_{r-1} \subseteq G$ ($C \subseteq B^{\otimes r}$ is an almost direct factor given by the idempotent $\prod_{i<j} \lambda_{ij}(1 - e_{B/A}), \lambda_{ij} : B^{\otimes 2} \to B^{\otimes r}$ the canonical maps).

2a. Hochschild-Cohomology and Lifting. — Recall that for an A-algebra B (with unit) we can form a complex of B-bimodules ($B \otimes B^{\mathrm{op}}$-modules)

$$C(B/A) : B \otimes_A B \leftarrow B \otimes_A B \otimes_A B \leftarrow B \otimes_A B \otimes_A B \otimes_A B \leftarrow,$$

with differential

$$d_n : C_n(B/A) = B^{\otimes n+2} \longrightarrow C_{n-1}(B/A) = B^{\otimes n+1}$$

$$d_n(b_0 \otimes b_1 \otimes \cdots \otimes b_{n+1}) = \sum_{i=0}^{n} (-1)^i b_0 \otimes \cdots \otimes b_i \cdot b_{i+1} \otimes \cdots \otimes b_{n+1}$$

This complex has an augmentation $C_0(B/A) \to B$ and is (with B in degree -1) nullhomotopic as complex of left respectively right B-modules. For a B-bimodule M define

$$H^*(B/A, M) = (\mathrm{Hom}_{B \otimes B^{\mathrm{op}}}(C_*(B/A), M))$$

Thus $H^0(B/A, M) = M^B = \{m \in M \mid bm = mb\,(\mathrm{all}\, b \in B)\}$
$H^1(B/A, M) = \{\text{biderivations } B \to M\}/\{\text{inner derivations } b \to [b, m] = bm - mb\}$ etc.

For any B-module N the bimodule $M = \mathrm{Hom}_A(B, N)$ has

$$H^i(B/A, M) = \begin{cases} 0, \; i > 0 \\ N, \; i = 0 \end{cases}$$

For example if B is a projective $B \otimes_A B$-module then for any bimodule M we have that $M = \mathrm{Hom}_B(B, M)$ (using say left B-module structures) is a direct summand in

$$\mathrm{Hom}_B(B \otimes_A B, M) = \mathrm{Hom}_A(B, M),$$

and thus its Hochschild-cohomology vanishes in positive degrees. By the same argument B almost $B \otimes_A B$-projective gives $H^i(B/A, M) \approx 0$ for $i > 0$.

Now suppose that $I \subseteq A$ is an ideal with square zero: $I^2 = 0$.

If B is an A-algebra which is almost unramified (B is an almost projective $B \otimes_A B$-module), C an A-algebra, $\overline{f} : B \to \overline{C} = C/IC$ an A-algebra morphism (or almost morphism), then there is at most one way to lift $\overline{f}$ to $f : B \to C$ (up to almost equality):

For two lifts f_1, f_2 the difference is an almost derivation $\overline{B} \to I \cdot C$, thus almost zero as $H^1(\overline{B}/\overline{A}, I \cdot C) \approx 0$.

If B is also an almost projective A-module, then in fact an almost lifting exists: For each $\alpha > 0$, $\pi^\alpha \overline{f}$ lifts to an A-linear map $f_\alpha : B \to C$. Consider

$$g_\alpha(x, y) = \pi^{2\alpha} f_\alpha(xy) - \pi^\alpha f_\alpha(x) f_\alpha(y) \in I \cdot C.$$

Then g_α defines a cochain in

$$\mathrm{Hom}_{B \otimes_A B}(C_2(\overline{B}/\overline{A}, I \cdot C)).$$

We claim that it is a cocycle. To check that compute the boundary

$$dg_\alpha(x, y, z) = x g_\alpha(y, z) - g_\alpha(xy, z) + g_\alpha(x, yz) - g_\alpha(x, y) z$$

Using that multiplication by $f_\alpha(u)$ coincides with multiplication by $\pi^\alpha u$ in IC, this is:

$$\begin{aligned} &f_\alpha(x)(\pi^\alpha f_\alpha(yz) - f_\alpha(y) f_\alpha(z)) - \pi^{2\alpha} f_\alpha(xyz) + \pi^\alpha f_\alpha(xy) f_\alpha(z) \\ &\qquad + \pi^{2\alpha} f_\alpha(xyz) - \pi^\alpha f_\alpha(x) f_\alpha(yz) - (\pi^\alpha f_\alpha(xy) - f_\alpha(x) f_\alpha(y)) f_\alpha(z) = 0. \end{aligned}$$

Thus $\pi^\alpha g_\alpha$ is a coboundary,

$$\pi^\alpha g_\alpha(x, y) = x h_\alpha(y) - h_\alpha(xy) + h_\alpha(x) y,$$

and $\widetilde{f_\alpha} = \pi^{3\alpha} f_\alpha + h_\alpha$ lifts $\pi^{4\alpha} \cdot \overline{f}$ and satisfies

$$\pi^{4\alpha} \cdot \widetilde{f_\alpha}(xy) = \widetilde{f_\alpha}(x) \widetilde{f_\alpha}(y)$$

Furthermore for $0 < \alpha \leqslant \beta/2 < \beta$

$$g_{\beta,\alpha} = \pi^{4\beta} f_\beta - \pi^{(\beta - 4\alpha)} f_\alpha$$

satisfies

$$\begin{aligned} g_{\beta,\alpha}(xy) &= f_\beta(x) f_\beta(y) - \pi^{8\beta - 8\alpha} f_\alpha(x) f_\alpha(y) \\ &= f_\beta(x)(f_\beta(y) - \pi^{4\beta - 4\alpha} f_\alpha(y)) + (f_\beta(x) - \pi^{4\beta - 4\alpha} f_\alpha(x)) \pi^{4\beta - 4\alpha} f_\alpha(y) \end{aligned}$$

(using the terms in brackets are in IC)

$$\begin{aligned} &= \pi^{4\beta} x (f_\beta(g) - \pi^{4\beta - 4\alpha} f_\alpha(y)) + (f_\beta(x) - \pi^{4\beta - 4\alpha} f_\alpha(x)) \pi^{4\beta} y \\ &= x\, g_{\beta,\alpha}(y) + g_{\beta,\alpha}(x) y. \end{aligned}$$

Thus $g_{\beta,\alpha}$ is an almost derivation, hence almost zero. Multiplying by π^β gives

$$\pi^{5\beta} \cdot f_\beta = \pi^{9(\beta - \alpha)} \pi^{5\alpha} \cdot f_\alpha$$

Hence considering the $\pi^{5\alpha} \cdot f_\alpha$ as multiplicative maps from $\pi^{9\alpha} \cdot V \otimes_V B$ to C (lifting $\overline{f}$), they glue to a map

$$f : \mathfrak{m} \otimes_V B = \varinjlim (\pi^{9\alpha} \cdot V) \otimes_V B \longrightarrow C$$

Also $f(1) \in \mathrm{Hom}(\mathfrak{m}, C)$ is an idempotent lifting 1, thus $f(1) = 1$.

Finally we lift almost étale coverings. First we treat the underlying projective module: Let $\overline{P}$ be an almost projective $\overline{A} = A/I$-module. For any $\alpha > 0$ choose a free $\overline{A}$-module $\overline{L}$ and maps $\overline{f}_\alpha : \overline{L}_\alpha \to P$, $\overline{g}_\alpha : P \to \overline{L}_\alpha$ with $\overline{f}_\alpha \circ \overline{g}_\alpha = \pi^\alpha \cdot \mathrm{id}$. Let $\overline{e}_\alpha \in \mathrm{End}(\overline{L}_\alpha)$ denote $\overline{g}_\alpha \circ \overline{f}_\alpha$, so $\overline{e}_\alpha^2 = \pi^\alpha \cdot \overline{e}_\alpha$.

Lift $\overline{e}_\alpha$ to an endomorphism f_α of L. Thus $f_\alpha^2 - \pi^\alpha f_\alpha$ takes values in $I \cdot L$, hence $(f_\alpha^2 - \pi^\alpha f_\alpha)^2 = 0$. If $e_\alpha = 3\pi^\alpha f_\alpha^2 - 2f_\alpha^3$, then e_α lifts $\pi^{2\alpha} e_\alpha$ and satisfies $e_\alpha^2 = \pi^{3\alpha} e_\alpha$. Hence $P_\alpha = L_\alpha/(\pi^{3\alpha} - e_\alpha)L_\alpha$ is "projective up to $\pi^{3\alpha}$", i.e. $\pi^{3\alpha}$ annihilates all higher $\mathrm{Ext}^i_A(P_\alpha, ?)$. Also $\overline{f}_\alpha$ and $\overline{g}_\alpha$ induce maps

$$\overline{P}_\alpha \xrightarrow{\overline{f}'_\alpha} \overline{P} \xrightarrow{\overline{g}'_\alpha} \overline{P}_\alpha$$

with

$$\overline{f}'_\alpha \circ \overline{g}'_\alpha = \pi^\alpha \cdot \mathrm{id}_{\overline{P}}, \pi^{2\alpha} \circ \overline{g}'_\alpha = \pi^{3\alpha} \cdot \mathrm{id}_{\overline{P}_\alpha}$$

For $\alpha \leqslant \beta/2 < \beta$,

$$\pi^{3\alpha+4\alpha-8\beta} \circ \overline{g}'_\beta \circ \overline{f}'_\alpha : P_\alpha \longrightarrow \overline{P}_\alpha \longrightarrow \overline{P} \longrightarrow \overline{P}_\beta$$

lifts to a map $g_{\alpha,\beta} : P_\alpha \to P_\beta$ with $\overline{f}'_\beta \circ \overline{g_{\alpha,\beta}} = \pi^{7\alpha-7\beta} \circ \overline{f}'_\alpha$. Considering $g_{\alpha,\beta}$ as a map from $\pi^{7\alpha} V \otimes_V P_\alpha$ to $\pi^{7\beta} V \otimes_V P_\beta$, we form $P = \varinjlim_\alpha \pi^{7\alpha} V \otimes_V P_\alpha$, for a suitable sequence $\alpha = \alpha_n$, $\alpha_{n+1} \leqslant \alpha_n/2$. Then the $\overline{f}_\alpha$ define an almost isomorphism $P \otimes_A A/I \approx \overline{P}$. As $\operatorname{coker} \overline{g}_{\alpha,\beta}$ is annihilated by $\pi^{7\alpha-7\beta+\alpha+3\beta}$ and hence by $\pi^{8\alpha}$, $\operatorname{coker} g_{\alpha,\beta}$ is annihilated by $\pi^{16\alpha}$, and $\pi^{7\alpha} V \otimes_V P_\alpha \to P$ is an isomorphism up to $\pi^{16\alpha}$. Thus P is almost projective and lifts $\overline{P}$.

Remark. — If $\overline{P}$ has rank $\leqslant r$, so has P.

Now assume $\overline{B}/\overline{A}$ is an almost étale covering, that $\overline{B}$ is almost projective over $\overline{A}$ as well as over $\overline{B} \otimes_{\overline{A}} \overline{B}$. We claim that there is an almost étale covering B/A with $B \otimes_A A/I \approx \overline{B}$:

We give a sketch, as the method follows the previous but the constants become more complicated. First lift $\overline{B}$ to an almost projective A-module B. Then for all $\alpha > 0$ the multiplication

$$\overline{m} : \overline{B} \otimes_A \overline{B} \longrightarrow \overline{B}$$

lifts after multiplication by π^α to

$$m_\alpha^0 : B \otimes_A B \longrightarrow B.$$

Then $\pi^{c\alpha}(m_\alpha^0(m_0^\alpha(a,b),c) = m_\alpha^0(a, m_0^\alpha(b,c)))$ (c = suitable constant) defines a class in $H^3(\overline{B}/\overline{A}, I \cdot B) \approx 0$. After modifying m_α we can make it associative. Then the commutator with a fixed element defines a derivation, thus $\pi^{c\alpha} \cdot m_\alpha$ (c suitable) is commutative, and similarly $1 \in \overline{B}$ lifts to an idempotent which is a unit. For $\alpha \leqslant \beta/2 < \beta$ a suitable multiple $\pi^{c\alpha} \cdot \mathrm{id}_{\overline{B}}$ lifts to a multiplicative map

$$(\pi^{c\alpha} V \otimes_V B, m_\alpha) \longrightarrow (\pi^{c\beta} V \otimes_V B, m_\beta).$$

Then $B = \varinjlim \pi^{c\alpha} V \otimes_V B$ with $\varinjlim m_\alpha$ does the job.

We have shown

3. Theorem. — *The functor sending B to $\overline{B} = B \otimes_A \overline{A}$ is an equivalence of categories (almost étale coverings of A) $\xrightarrow{\sim}$ (almost étale coverings of $\overline{A}$).*

2b. The Purity Theorem. — The previous theory is applied to the study of universal étale coverings. This was first done by Tate ([Ta]): Assume V is a p-adic discrete valuation-ring, with residue field $k = V/\pi V$ perfect of characteristic $p > 0$ (π a uniformiser) and fraction field K of characteristic zero. Assume furthermore that K_∞ is a totally ramified $\mathbb{Z}_p$-extension of $K, V_\infty \subseteq K_\infty$ the normalization of V. Then for any finite extension L_∞ of K_∞, the normal W_∞ of V_∞ in L_∞ is an almost étale covering of V_∞. This was generalised to non-perfect residue fields as follows: (see [F2], Theorem 1.2) Assume $[k : k^p] = p^d < \infty$. Assume given a tower $K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_\infty = \bigcup_{n\geqslant 0} K_n$. Let V_n = normalization of V in $K_n, \Omega_{V_n/V}$ the universal differential module. It is known that $\Omega_{V_n/V}$ can be generated by $\leqslant d+1$ elements, and that its length is the different of V_n/V. Now assume that for any $m \geqslant 0$ there exists an n such that $\Omega_{V_n/V}$ has as quotient $(V_n/\pi^m V_n)^{d+1}$. Then again for any finite $L_\infty \supseteq K_\infty$, W_∞ = normalization of V_∞ is an almost étale covering of V_∞.

Now suppose that over V we have an affine normal torus-embedding $T \cong \mathbb{G}_m^{d+1} \subseteq \overline{T}$ ([KKMS], § 1) If $L = \operatorname{Hom}(\mathbb{G}_m, T)$ denotes the characters of T, we furthermore assume given a rational convex polyhedron $\sigma \subseteq L_\mathbb{R}$ not containing any line, such that $\overline{T} = \operatorname{Spec}(V[L^* \cap \sigma^\vee])$. Finally we assume given an element $\lambda \in L^* \cap \sigma^\vee$ such that $\sigma \cap \ker(\lambda)$ is a simplex, i.e. consists of the positive linear combinations of elements $\rho_1, \ldots, \rho_r \in L \cap \sigma$ which are linearly independant over $\mathbb{Z}$. This hypotheses will later assure that the divisors at infinity in the generic fibres have normal crossings.

Let $\overline{T}_\lambda \subseteq \overline{T}$ denote the closed subscheme defined by the vanishing of $(\pi - \lambda)$. Assume given a noetherian ring R and a map $f : \operatorname{Spec}(R) \to \overline{T}_\lambda$ which is flat, has geometrically regular fibres and is formally étale in charateristic p, i.e. for a prime $\mathfrak{p} \subseteq R (p \in \mathfrak{p})$ with image $x \in \overline{T}$ is $R_\mathfrak{p}$ formally étale over $O_{\overline{T}_{\lambda,x}}$. This is equivalent to the fact that Frobenius induces an isomorphism $(\widehat{R}_\mathfrak{p}/p\widehat{R}_\mathfrak{p}) \otimes O_{\overline{T}_{\lambda,x}} \xrightarrow{\sim} \widehat{R}_\mathfrak{p}/p\widehat{R}_\mathfrak{p}$. ($\widehat{\ }$= completion in $\mathfrak{p}$-adic topology). Also for each residue-field κ of $\overline{O}_{T_\lambda}$ the fibre $R \otimes \kappa$ has dimension $\leqslant \log_p[\kappa : \kappa^p] \leqslant d$. Denote by $\overline{T}_n \to \overline{T}$ the covering induced by multiplication by $n!$, $\operatorname{Spec}(R_n) \to \operatorname{Spec}(R)$ its pullback via $\operatorname{Spec}(R) \to \overline{T}_\lambda \subseteq \overline{T}$ (then R_n is a module over $V_n = V[\pi^{1/n!}]$ because of the relation $\lambda = \pi$). If $R_\infty = \bigcup_{n>0} R_n$, and if $R_\infty \to S_\infty$ is the normalization of R_∞ in a finite étale covering of the preimage of $T \subseteq \overline{T}$ in $R_\infty[1/\pi]$, we claim that S_∞ is an almost étale covering of R_∞.

4. Theorem. — *S_∞/R_∞ is almost étale.*

Proof. Reduction-steps

i) Instead of the $\overline{T}_n$ we may choose $\operatorname{Spec} V[L_n^* \cap \sigma^\vee]$, where the $L_n \subseteq L$ form a decreasing sequence of sublattices such that for any integer $m, L_n \subseteq m \cdot L$ for $n \gg 0$.

ii) We may start with a fixed L_n (instead of L) right away. Especially we may assume that S_∞ is defined by an R-algebra S such that $S[1/\pi]$ is finite étale over $R[1/\pi]$ over the preimage of $T \subseteq \overline{T}$.

iii) For a suitable choice of L_n we can achieve that

$$\overline{T}_{\lambda,n} = \mathrm{Spec}(V[L_n^* \cap \sigma^\vee]/(\lambda - \pi)) \supseteq T_{\lambda,n} = \mathrm{Spec}(V[L_n^*]/(\lambda - \pi))$$

is over $\mathrm{Spec}(K)$ an embedding of the complement of a simple normal crossings divisor (use the properties of λ), and especially $\overline{T}_{\lambda,n} \otimes_V K$ is regular. By Abhyankhar's lemma we then may assume that $S[1/\pi]$ is finite étale over $R[1/\pi]$.

iv) From now on it will be sufficient to work with a sequence of lattices L_n cofinal with $p^n L$. The more general result then follows by base-extension (once we have that $S_\infty[1/\pi]$ extends to an almost étale covering, this covering must be the normalization of R_∞, and this remains true after base-change).

v) Next we consider what happens in the generic points in characteristic p. Suppose Z is an irreducible component $\overline{T}_\lambda \otimes_V k = \mathrm{Spec}(k[L \cap \sigma^\vee]/(\lambda))$. Then σ corresponds to an extremal ray (halfline) in σ. If it is generated by an indivisible element $\rho \in L$, and if $\lambda(\rho) = e$, then in a neighbourhood of the generic point z of Z the torus-embedding $\overline{T}$ is isomorphic to $T' \times \mathbb{A}^1, T' \cong \mathbb{G}_m^d$, and $\overline{T}_\lambda$ has as local ring there

$$\overline{T}_{\lambda,z} \cong V'[t]/(t^e - u\pi) \cong V'[\sqrt[e]{u\pi}]$$

where V' is the lcoal ring corresponding to $T', u \in V'$ is a unit. As $u\pi$ is a uniformiser of V' it follows that $\overline{T}_{\lambda,z}$ is a discrete valuation-ring. It follows that $\overline{T}_\lambda$ satisfies R_1. As it is Cohen-Maccaulay ($\overline{T}$ is as all torus-embeddings are [KKMS], § 3), $\overline{T}_\lambda$ is even normal. Also R and R_n are normal and Cohen-Macaulay.

After that interlude let us pass to the sublattices L_n, and choose a projective system of points $z_n \in \overline{T}_{\lambda,n}$ lifting z. We claim that the local rings $\mathcal{O}_{\overline{T}_{\lambda,n,z_n}}$ satisfy the ramification-conditions of [Fa2], Th. 1.2:

That is for given r there exists n such that the relative differentials $\Omega_{\overline{T}_{\lambda,n,z}/\overline{T}_\lambda}$ have a quotient $(\mathcal{O}_{\overline{T}_{\lambda,n,z_n}}/p^r)^{d+1}$ (the integer d in [Fa 2] coincides with our d here). If we compute relative logarithmic differentials we obtain $L_n^*/L^* \otimes \mathcal{O}_{\overline{T}_{\lambda,n,z_n}}$. The usual differentials map into these, with cokernel annihilated by $\prod_{i=0}^{d+1} \mu_i$, if the $\mu_i \in \sigma^\vee \cap L_n^*$ form a basis of L_n^*. Now for $L_n = p^n L$ the valuation of this product decreases like $c \cdot p^{-n}$, and the claim follows for those L_n and thus also for any sequence.

By formal étaleness the same applies to the sequence $(R_n, \mathfrak{p}_n)$, where $\mathfrak{p}_n \subseteq R_n$ is a projective system of primes above a fixed $\mathfrak{p} \subseteq R$ which is a minimal prime divisor of pR or πR. It follows that $S_{\infty,\mathfrak{p}}$ is almost étale over $R_{\infty,\mathfrak{p}}$.

vi) It now suffices to prove that S_∞ is almost flat over R_∞: There exists an idempotent $e_{S_\infty/R_\infty} \in S_\infty \otimes_{R_\infty} S_\infty[1/\pi]$ defining the diagonal (the direct summand $S_\infty[1/\pi]$). For each $\alpha > 0$ its product with π^α then lies in

$$\bigcap_{\mathfrak{p}} (S_\infty \otimes_{R_\infty} S_\infty)_{\mathfrak{p}} \cap S_\infty \otimes_{R_\infty} S_\infty[1/\pi] \approx S_\infty \otimes_{R_\infty} S_\infty$$

($\mathfrak{p}$ runs over all minimal prime-divisors of πR). Thus S_∞ is an almost direct summand in $S_\infty \otimes_{R_\infty} S_\infty$. Furthermore if

$$\pi^\alpha e_{S_\infty/R_\infty} = \sum_{i=1}^{r} x_i \otimes y_i \quad (x_i, y_i \in S_\infty),$$

then

$$\pi^\alpha x = \sum_{i=1}^{r} x_i \cdot Tr_{S_\infty/R_\infty}(y_i x)$$

for all $x \in S_\infty$, which exhibits S_∞ as an up to π^α direct summand in R_∞^r. Then S_∞ is almost projective over R_∞.

By localization we may thus assume that R is local with maximal ideal $\mathfrak{m}$, and that S_∞ is almost flat over R_∞ on the punctured spectrum $\mathrm{Spec}(R) - \{\mathfrak{m}\}$. It also follows that $S_\infty \otimes_R \widehat{R}\,(\widehat{R} = \mathfrak{m})$-adic completion) is almost equal to the normalization of $R_\infty \otimes_R \widehat{R}$, using that $\widehat{R}$ is normal (f is formally smooth) and that the two can only differ almost by $\mathfrak{m}$-torsion.

vii) Assume now that $\dim(R) = 2$: Let $z \in \overline{T}_\lambda$ denote the image of $\{\mathfrak{m}\}$, $z_n \in \overline{T}_{\lambda,n}$ a projective system of liftings. We claim that for a suitable choice of the lattices L_n the local rings of $\overline{T}_{\lambda,n}$ in z_n are regular. The same is then true for R_n. As S_n is normal of dimension two it is Cohen-Macaulay, and thus flat over R_n (depth + projective dimension $= 2$). Thus S_∞ is flat over R_∞.

To show the claim we may assume that $\pi \in \mathfrak{m}$. Thus $z \in \overline{T}_\lambda \otimes_N k \subseteq \overline{T} \otimes_V k$ corresponds two a codimension-two-point in $\overline{T} \otimes_V k$. If it is not the generic point of a codimension-two stratum each $\overline{T}_{\lambda,n,z_n}$ is already regular by the previous. If it is such a generic point then near z $\overline{T}_n$ looks like the product of a $\mathbb{G}_m^{d-1}$ with a torus-embedding of rank 2. This torus-embedding is given by a 2-dimensional convex polyhedral cone which must be a simplex. For suitable choice of the L_n this simplex is spanned by two basis-vectors of L_n, and $\overline{T}_n$ is smooth over V in z_n. It then follows easily that $\overline{T}_{\lambda,n}$ is still regular in z_n. (The singularity looks like $V_n[T_1, T_2]/(T_1^e T_2^f - \pi_n)$).

So finally now $\dim(R) \geqslant 3$. The Frobenius-mapping is surjective on R_∞/pR_∞ (or $R_\infty/\pi R_\infty$):

This holds for $\mathcal{O}_{\overline{T}_\infty}$, and S_∞ is generated by $\mathcal{O}_{\overline{T}_\infty}$ and R^p.

Following Fontaine we form the projective limit $\mathcal{R} = \varprojlim(R_\infty/pR_\infty)$ (transition-maps are Frobenius) and $A_{\inf}(R) = W(\mathcal{R})$. Then $A_{\inf}(R)/p \cdot A_{\inf}(R) \xrightarrow{\sim} \mathcal{R}$, $A_{\inf}(R)$ admits a Frobenius-automorphism ϕ, and there is a surjection $A_{\inf}(R) \twoheadrightarrow \widehat{R}_\infty$ (p-adic completion) with kernel generated by a single element $\xi = p - [\underline{p}]$. Here $[\underline{p}] = [\underline{p}, 0, \ldots, 0]$ with $\underline{p} = \varprojlim(p^{1/p^n}) \in \mathcal{R}$ a compatible system of p-power roots of p. Note that

$$\phi(\xi) = p - [\underline{p}]^p = p - (p - \xi)^p = \pm\xi^p + p \cdot (\text{unit}).$$

The elements p and ξ form a regular sequence in $A_{\inf}(R)$, and $A_{\inf}(R)$ is complete in the ξ-adic topology. However this topology is not ϕ-invariant. We thus consider the finer topology where a fundamental system of neighbourhoods of zero is given by the ideals generated by finite products ($m_n \geqslant 0$, almost all zero) $\prod_{n\in\mathbb{Z}}(\phi^n(\xi))^{m_n}$. Note that any product $\prod_{n>0}\phi^n(\xi)^{m_n}$ with positive ϕ-powers forms together with ξ a regular sequence in $A_{\mathrm{inf}}(R)$, and that the ideal generated by those two elements defines the same topology as (p,ξ). Thus $A_{\mathrm{inf}}(R)$ with the topology defined by $\prod_{n=0}^{N}\phi^n(\xi)$ is the fibred product of (A, ξ-adic topology) with (A, $\prod_{n=1}^{N}\phi^n(\xi)$-adic topology), over ($A$, (p,ξ)-adic topology).

Now S_∞ defines an almost étale covering V_∞ of $U_\infty = \mathrm{Spec}(R_\infty) - \mathrm{Spec}(R_\infty/\mathfrak{m}R_\infty)$. It obviously has finite rank $[L_\infty : K_\infty]$. Furthermore if we reduce modulo p it is almost invariant under Frobenius, i.e. Frobenius on $\mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty}$ induces an almost isomorphism

$$\mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty}\otimes_{\mathcal{O}_{U_\infty}/p\mathcal{O}_{U_\infty}}\mathcal{O}_{U_\infty}/p\mathcal{O}_{U_\infty}\approx\mathcal{O}_{V_\infty}/p\mathcal{O}_{V_\infty}:$$

Reduce to the split case.

Let $\mathfrak{m}_0 \subseteq A_{\mathrm{inf}}(R)$ denote the ideal generated by p and suitable lifts of generators of $\mathfrak{m}$. Then we can lift V_∞ first to an almost étale covering of $\mathrm{Spec}(A_{\mathrm{inf}}(R)/\xi^m) - \{\mathfrak{m}_0\}$, all m, which by transport of structure induces almost étale coverings on $\mathrm{Spec}(A_{\mathrm{inf}}(R)/\phi^n(\xi)^m) - \{\mathfrak{m}_0\}(n\in\mathbb{Z})$. These are all isomorphic modulo p because of Frobenius-invariance, and thus by induction (and gluing, that is taking fibred products with almost surjective maps) we obtain almost étale coverings of

$$\mathrm{Spec}\left(A_{\mathrm{inf}}(R)\Big/\Big(\prod_{n=0}^{N}\phi^n(\xi)\Big)^m\right)-\{\mathfrak{m}_0\}$$

and by applying ϕ of

$$\mathrm{Spec}\left(A_{\mathrm{inf}}(R)/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}\right)-\{\mathfrak{m}_0\}.$$

That is if we consider the ind-scheme defined by quotients $A_{\mathrm{inf}}(R)/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}$, then we have defined a ϕ-invariant almost étale covering of its punctured spectrum.

Notations

$$X_{\mathrm{inf}} = \text{“}\varinjlim\text{”}\,\mathrm{Spec}\left(A_{\mathrm{inf}}(R)\Big/\prod_{n\in\mathbb{Z}}\phi^n(\xi)^{m_n}\right)$$

$$U_{\mathrm{inf}} = X_{\mathrm{inf}} - \{\mathfrak{m}_0\}$$

$V_{\mathrm{inf}} \longrightarrow U_{\mathrm{inf}}$: the almost étale covering derived from S_∞.

Also “almost” should now be defined with respect to the sequence of ideals generated by $\underline{p}^{1/p^n} = \varphi^{-n}(\underline{p})$. Define the cohomology $H^*(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}})$ as Čech-cohomology of the affine covering defined by a system of generators of $\mathfrak{m}$. This is easily seen to be independant of the choice of the covering, and one has long exact cohomology-sequences from short (topologically) exact sequences. I conjecture but have not checked that it coincides with the cohomology of the ind-scheme V_{inf}. It certainly admits a Frobenius-automorphism (almost) ϕ.

For $i > 0$ the cohomology

$$H^i(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}}/\xi \cdot \mathcal{O}_{V_{\mathrm{inf}}}) \cong H^{i+1}_{\mathfrak{m}}(\widehat{S_\infty}),$$

is equal to the local cohomology. It satisfies a certain finiteness condition (SGA2, Exp. VIII, Th. 2.1):

Firstly for any n the cohomology $H^2_{\mathfrak{m}}(S_n)$ is a finitely generated S_n-module and thus of finite length, as for any prime $\mathfrak{p} \neq \mathfrak{m}$ $\mathrm{depth}(S_{n,\mathfrak{p}}) + \dim(S_{n/\mathfrak{p}}) \geqslant 3$. Furthermore there are maps

$$H^2_{\mathfrak{m}}(S_n) \otimes_{R_n} R_\infty = H^2_{\mathfrak{m}}(S_n) \otimes_{R_n} \widehat{R_\infty} \longrightarrow H^2_{\mathfrak{m}}(S_n \otimes_{R_n} \widehat{R_\infty}) \longrightarrow H^2_{\mathfrak{m}}(\widehat{S_\infty}).$$

For any positive $\alpha > 0$ the kernels and cokernels of both maps are annihilated by p^α, for n big enough:

If R_∞ where flat over R_n the first map would be an isomorphism. This is not true in general, but in the appendix 2 it is shown that for a sequence α_n with $\alpha_n \to 0$, p^{α_n} annihilates all $\mathrm{Tor}_i^{R_n}(R_m, M), (i > 0)$ $(m \geqslant n$ and M any R_n-module). The same then holds with R_n replaced by R_∞ or its p-adic completion. This implies the assertion for the first map.

For the second one studies $S_n \otimes_{R_n} R_\infty \to S_\infty$. Its kernel consists of p-torsion and is annihilated by the p^{α_n} from above. For the cokernel use that for each height-one prime $\mathfrak{p}$ the trace-form on $S_{n,\mathfrak{p}}$ has discriminant p^{β_n} with $\beta_n \to 0$. Thus

$$p^{\beta_n} \cdot S_{\infty,\mathfrak{p}} \subseteq \bigcap (S_n \otimes_{R_n} R_\infty)_{\mathfrak{p}}$$

and (using annihilation of Tor_1' s)

$$p^{\alpha_n+\beta_n} \cdot S_\infty \subseteq (\textstyle\bigcap_{\mathfrak{p}} S_{n,\mathfrak{p}}) \otimes_{R_n} R_\infty = S_n \otimes_{R_n} R_\infty.$$

Again we may pass to p-adic completions. It follows that for each $\alpha > 0$, $p^\alpha \cdot H^2_{\mathfrak{m}}(\widehat{S_\infty})$ is contained in a finitely generated $\mathfrak{m}$-torsion S_∞-module. In the appendix 2 it is also shown how to define an invariant (the normalized length)

$$\lambda(p^\alpha \cdot H^2_{\mathfrak{m}}(\widehat{S_\infty})) \in \mathbb{R}.$$

λ is $\geqslant 0$, is additive in short exact sequences, invariant under almost isomorphisms, and for an $\mathfrak{m}$-torsion module which is a submodule of a finitely presented $\mathfrak{m}$-torsion S_∞-module, λ vanishes only if the module is zero. Also if M is annihilated by $p^{1/p}$, then

$$\lambda(M \otimes_{R_\varphi} R/pR) = p^{d+1}\lambda(M).$$

Now consider

$$M = H^1(V_{\mathrm{inf}}, \mathcal{O}_{\mathrm{inf}}).$$

Then $M/\xi M$ injects into $H^1(V_\infty, \mathcal{O}_{V_\infty}) = H^2_{\mathfrak{m}}(\widehat{S_\infty})$. Thus for any $\alpha > 0$, $p^\alpha(M/\xi M)$ as well as $p^\alpha(M/(\xi, p)M)$ have finite λ-invariant. Furthermore Frobenius induces an almost isomorphism

$$M/(\phi^{-1}(\xi), p^{1/p})M \otimes_{R_\varphi} R/pR \approx M/(\xi, p)M$$

Thus

$$p^{d+1} \cdot \lambda(p^\alpha M/(\phi^{-1}(\xi), p^{1/p})M) \leqslant \lambda(p^{p\alpha} M/(\xi, p)M)$$

Choose $0 < \alpha < 1/p^2$. Then $p^{p\alpha-\beta} M/(\xi, p)M$ has a filtration of length $p+1$ by submodules

$$p^{\alpha_i} \cdot M/(\phi^{-1}(\xi), p)M, \quad \alpha_i = \alpha + i\frac{1-\alpha}{p+1} \quad (0 \leqslant i \leqslant p),$$

and each consecutive subquotient is a quotient of $p^\alpha M/(\phi^{-1}(\xi), p^{1/p})M$. Furthermore $p^{p\alpha} M/(\xi, p)M$ is contained in

$$\sum_{j=0}^{p-1} p^{(p-j)\alpha} \phi^{-1}(\xi)^j M/(\xi, p)M \quad (\xi \equiv \phi^{-1}(\xi)^p \bmod p),$$

which has a filtration of length p with consecutive subquotients of $p^\alpha M/(\phi^{-1}(\xi), p)M$. Thus

$$\lambda(p^{p\alpha} M/(\xi, p)M) \leqslant p(p+1) \cdot \lambda(p^\alpha M/(\phi^{-1}(\xi), p^{1/p})M).$$

As $p^{d+1} \geqslant p^3 > p(p+1)$, we see that both sides vanish, so $\lambda(p^\alpha \cdot M/(\xi, p)M) = 0$ (all $\alpha > 0$) and $\lambda(p^\alpha \cdot M/\xi M) = 0$ (p is nilpotent in $M/\xi M$). Now for $n \gg 0$ there is a map

$$M/\xi M \longrightarrow H^2_{\mathfrak{m}}(S_n) \otimes_{R_n} R_\infty$$

with kernel annihilated by p^α. As $\lambda(p^\alpha \cdot \text{image}) = 0$ it follows that $p^\alpha \cdot \mathit{image} = 0$, thus $p^{2\alpha}$ annihilates $M/\xi M$. As α was arbitrary, $M/\xi M \approx 0$, and by induction also any quotient $M/\prod_{n \in \mathbb{Z}} \phi^n(\xi)^{m_n} \cdot M \approx 0$ vanishes almost.

It follows that the map $\Gamma(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}}) \to \widehat{S_\infty} = \Gamma(V_\infty, \mathcal{O}_{V_\infty})$ is almost surjective: Order the finite products $\prod_{n \in \mathbb{Z}} \phi^n(\xi)^{m_n}$ in a countable sequence

$$g_1 = \xi, g_2, g_3, \ldots,$$

with

$$g_{n+1}/g_n = \text{ a } \phi\text{-power of } \xi.$$

Then

$$H^1(V_{\mathrm{inf}}, g_n \cdot \mathcal{O}_{V_{\mathrm{inf}}}/g_{n+1} \cdot \mathcal{O}_{V_{\mathrm{inf}}}) \approx 0,$$

as this ϕ-power induces an almost isomorphism with M. For $\alpha > 0$ choose a strictly increasing sequence

$$\alpha_1 = 0 < \alpha_n < \alpha_{n+1} < \cdots < \alpha.$$

For

$$f_1 \in \widehat{S_\infty} = \Gamma(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}}/g_1 \cdot \mathcal{O}_{V_{\mathrm{inf}}})$$

construct inductively $f_n \in \Gamma(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}}/g_n \cdot \mathcal{O}_{V_{\mathrm{inf}}})$ with f_{n+1} lifting $\underline{p}^{\alpha_{n+1}-\alpha_n} \cdot f_n$. This is possible because $\underline{p}^{\alpha_{n+1}-\alpha_n}$ annihilates the obstruction to lift f_n. Then the $p^{\alpha-\alpha_n} \cdot f_n$ form a compatible system of lifts of $\underline{p}^\alpha \cdot f_1 = p^\alpha \cdot f_1$. Thus finally $S_{\mathrm{inf}} = \Gamma(V_{\mathrm{inf}}, \mathcal{O}_{V_{\mathrm{inf}}})$

is an $A_{\inf}(R)$-algebra with $S_{\inf}/\xi \cdot S_{\inf} \approx \widehat{S_\infty}$. Furthermore Frobenius induces an almost automorphism of $S_{\inf}$.

Next consider the cokernel M of the map

$$S_{\inf} \otimes_{A_{\inf}(R)} S_{\inf} \longrightarrow \Gamma(V_{\inf}, \mathcal{O}_{V_{\inf}} \otimes_{A_{\inf}(R)} \mathcal{O}_{V_{\inf}}) :$$

On it Frobenius is an almost isomorphism. On the other hand $M/\xi \cdot M$ (almost) injects into the cokernel of

$$S_\infty \widehat{\otimes_{R_\infty}} S_\infty \longrightarrow \Gamma(V_\infty, \mathcal{O}_{V_\infty} \otimes_{R_\infty} \mathcal{O}_{V_\infty}),$$

that is into $H^1_{\mathfrak{m}}(S_\infty \widehat{\otimes_{R_\infty}} S_\infty)$.

By the same reasoning as before $H^1_{\mathfrak{m}}(S_\infty \widehat{\otimes_{R_\infty}} S_\infty)$ is for each $\alpha > 0$ p^α-isomorphic to a finitely presented torsion-module, thus $\lambda(p^\alpha \cdot M/\xi M) < \infty$ (all $\alpha > 0$) and $M/\xi M \approx 0$. Finally the canonical idempotent defining the diagonal defines an almost section of $\mathcal{O}_{V_{\inf}} \otimes_{A_{\inf(R)}} \mathcal{O}_{V_{\inf}}$ lifting the corresponding idempotent e_{S_∞/R_∞}. It is thus at least modulo ξ in the image of $S_{\inf} \otimes_{A_{\inf(R)}} S_{\inf}$, and e_{S_∞/R_∞} lies almost in the image of $S_\infty \otimes_{R_\infty} S_\infty$.

Appendix 1. Some toroidal geometry. — Assume $\mathrm{Spec}(R) \to \overline{T}_\lambda$ is formally étale in characteristic p. We assume that R is local with maximal ideal containing π. We want to study how much of this parametrization is canonical.

Obviously we can replace σ by its face parametrizing the stratum of $\overline{T}$ which contains the image of the closed point of $\mathrm{Spec}(R)$. Let $L' \subseteq L$ denote the lattice spanned by σ. Then we have a formally smooth map

$$\mathrm{Spec}(R) \longrightarrow \overline{T}'_\lambda \subseteq \overline{T}',$$

into the torus-embedding defined by (L', σ). Thus change notation and assume $L = L'$, but the map only formally smooth (i.e. flat with geometrically regular fibres).

4. Proposition. — *R is normal. The complement of the open stratum in* $\mathrm{Spec}(R)$ *is the union of irreducible divisors E. They correspond one to one to the extremal rays of σ. If $\rho \in L^\vee \cap \sigma^\vee$ is the indivisible generator of such a ray, then for any $\mu \in L$ the function $m \in R$ has valuation $\mu(\rho)$ in the generic point of the correspodning divisor.*

Furthermore $L^\vee \cap \sigma^\vee$ = invertible ideals $I \subset R$ with support in the complement of the open stratum, via $\mu \mapsto R \cdot \mu$.

Proof. — Choose a $\rho \in L$ which lies in the interior of σ. Then define a decreasing filtration on $V[L^\vee \cap \sigma^\vee]$ by $F^b(V[L^\vee \cap \sigma^\vee]) = \oplus_{\mu(\varphi) \geqslant n} V \cdot \mu$. It induces a quotient filtration on $V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)$ with

$$\mathrm{gr}^\circ_{F_\rho}(V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)) \cong k[L^\vee \cap \sigma^\vee],$$

and by pushout a filtration F°_ρ on R with $\mathrm{gr}_{F_\rho}(R) \cong \kappa[L^\vee \cap \sigma^\vee]$, κ = residue-field of R. Furthermore this filtration defines the $\mathfrak{m}$-adic topology on R. As the associated graded is normal, so is R.

For each extremal ρ we obtain a divisor in $\overline{T}$ isomorphic to the torus-embedding defined by $(L/\langle\rho\rangle$, projection of $\sigma)$, and the valuation of $\mu \in L^\vee$ along this divisor is $\mu(\rho)$. The intersection with $\overline{T}_\lambda$ is as follows: If $\lambda(\rho) = 0$, it is $\overline{T}'_\lambda$ with T' corresponding to $L/\langle\rho\rangle$. Its preimage in $\mathrm{Spec}(R)$ is normal and thus irreducible. If $\lambda(\rho) = n > 0$, then the intersection does not meet the generic fibre of $\overline{T}_\lambda$. In the special fibre

$$\overline{T}_\lambda \otimes_V k = \mathrm{Spec}(k[L^\vee \cap \sigma^\vee](\lambda))$$

the intersection is the boundary component of $\mathrm{Spec}(k[L^\vee \cap \sigma^\vee])$ defined by ρ. Its preimage in $\mathrm{Spec}(R)$ is irreducible for the same reasons as before, and also the valuation of μ is as required (the intersection with $\overline{T}_\lambda$ is transverse). As the complement of the open stratum in $\mathrm{Spec}(R)$ is the union of the divisors, we have shown the second assertion.

Finally assume $I = R \cdot f \subseteq R$ is an invertible ideal. For any $\rho \in L^\vee \cap \sigma^\vee$ we can as before define filtrations F_ρ on

$$V[L^\vee \cap \sigma^\vee], \quad V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)$$

and R. Now $\mathrm{gr}^0_{F_\rho}(R)$ is formally smooth over

$$\mathrm{gr}^0_{F_\rho}(V[L^\vee \cap \sigma^\vee]/(\pi - \lambda)) = k[L^\vee \cap \sigma^\vee \cap \rho^\perp],$$

and $\mathrm{gr}_{F_\rho}(R)$ is an integral domain (as this holds for $\overline{T}_\lambda$).

Now the conditions on I mean that f divides some $\mu_0 \in L^\vee \cap \sigma_j^\vee$, $\mu_0 = f \cdot g$. First choose $\rho \in \mathring{\sigma}$. Then if a and b are maximal with $f \in F^a_\rho(R)$, $g \in F^b_\rho(R)$, it follows by compution in $\mathrm{gr}_{F_\rho}(R) = \kappa[L^\vee \cap \sigma^\vee]$ that $a + b = \mu_0(\rho)$, and after multiplication by units that there is a decomposition $\mu_0 = \mu - \nu$ with $f - \mu \in F^{a+1}_\rho(R)$, $g - \nu \in F^{b+1}_\rho(R)$. μ and ν do not change if we vary slightly the ray $\mathbb{R}_+ \cdot \rho$. As σ^0 is convex they are thus independant of ρ.

Next choose ρ generating an extremal ray of σ. It corresponds to an irreducible divisor E. We show that for the corresponding valuation $v_E(f) \leqslant v_E(\mu) = \mu(\rho)$. If this holds for all E then f divides μ (by normality) and g divides ν (by symmetry). But then $I = R \cdot \mu$. Next assume that $v_E(f) > \mu(\rho) = n$. Thus $f \in F^{n+1}_\rho(R)$, which is generated by finitely many $\chi \in L^\vee \cap \sigma^\vee$ with $\chi(\rho) \geqslant n + 1$. Choose a ray in the interior $\mathring{\sigma}$ sufficiently close to ρ. It is generated by a $\widetilde{\rho}$ such that $\frac{n \cdot \widetilde{\rho}}{\mu(\widetilde{\rho})}$ is very close to ρ. Especially we can achieve that each of the κ's has values $> n$ on it. Thus if $\widetilde{n} = \mu(\widetilde{\rho})$, then $f \in F^{\widetilde{n}+1}_{\widetilde{\rho}}$, which cannot happen. Hence we have shown surjectivity of the map.

Injectivity follows because we already know how to recover μ from I_μ. Conversely once we have identified $\sigma^\vee$ with the semigroup of effective Cartier-divisors supported in the boundary, we obtain $L^\vee$ as the group of all such divisors, and then also L and σ. Also λ corresponds to π. Now extend λ to a basis of $L^\vee$, and choose equations for the remaining basis-elements. This defines a map $\mathrm{Spec}(R) \to \overline{R}_\lambda$ which differs from the given one by the action of T via a point z of $T(R)$. Passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ the two maps differ by an automorphism of $\widehat{R}$, which is the identity on $\kappa = R/\mathfrak{m}$ and multiplies $\mu \in L^\vee \cap \sigma^\vee$ by $\mu(z)$.

All in all (L,σ) is essentially unique up to automorphism once given $\mathrm{Spec}(R)$ and its open stratum.

5. Remark. — One might ask whether the fibre $R \otimes_V k$ is reduced. This happens if and only if this holds for $\overline{T}_\lambda$, i.e. if $k[L^\vee \cap \sigma^\vee]/(\lambda)$ is reduced, that is if $\lambda(\rho) \in \{0,1\}$ for all extremal rays ρ of σ. We can always achieve this by replacing L by $\widetilde{L} = \lambda^{-1}(e\mathbb{Z})$ with e sufficiently divisible, and λ by λ/e. Choose e such that $\lambda(\rho)$ divides e if $\lambda(\rho) \neq 0$. Then ρ changes to $\widetilde{\rho} = \frac{e}{\lambda(\rho)} \cdot \rho$, and $\lambda/e(\widetilde{\rho}) = 1$. This operation amounts to adjoining $\sqrt[e]{\pi}$ to V. We shall usually assume that this has been done. Then $R \otimes_V A$ is normal for any flat normal V-algebra A. Especially $R \otimes_V \overline{V}$ is normal.

There is also an intimate relation to log-structures as defined by Fontaine-Illusie-Kato: Namely on $X = \mathrm{Spec}(R)^{\text{ét}}$ consider the sheaf of monoids $P = j_*(\mathcal{O}^*_{X^0}) \cap \mathcal{O}_X$ ($j : X^0 \hookrightarrow X$ inclusion). Its fibre at $x \in \tau-$ stratum is an extension $0 \to \mathcal{O}^*_{x,x} \to P_x \to \tau^\vee \to 0$, with $\mathcal{O}^*_{x,x}$ acting freely on P_x. Thus P defines a fine log-structure, log-smooth over $\mathrm{Spec}(V)$ with the log-structure defined by $V - \{0\}$. Conversely if $X^0 \subseteq X$ is such that $P = j_*(\mathcal{O}^*_{X^0}) \cap \mathcal{O}_X$ defines a fine log-structure log-smooth over $\mathrm{Spec}(V)$ with $V - \{0\}$, then for each $x \in X$, $P_x(\mathcal{O}^{\mathrm{sh}}_{X,x})^*$ injects into the group of Weil-divisors supported on the complement of the open stratum in $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x})$. If $L^\vee_x = \langle P_x/(\mathcal{O}^{\mathrm{sh}}_{x,x})^*\rangle^{\mathrm{group}}$, $L = (L^\vee)^\vee$, then the valuations at irreducible components of the boundary induce maps $L^\vee \to \mathbb{Z}$ and thus elements $\rho \in L$, which span a polyhedral cone $\sigma \subseteq L_\mathbb{R}$ with $P_x/(\mathcal{O}^{\mathrm{sh}}_{x,x})^* = L^\vee \cap \sigma^\vee$. Furthermore elements of P_x lifting a basis of L contained in $\sigma^\vee$ define a formally smooth map $\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{x,x}) \to \overline{T}_\lambda$, where $\lambda \in L^\vee$ corresponds to π. $\lambda \in L^\vee$ may not be indivisible (it is only indivisible by p), but assume that it is. Hence our toroidal description can be translated into log-structures. However as all the important tools we need from the theory have been formulated and proved in toroidal geometry long before the advent of log-structures, we use the toroidal language.

Appendix 2: Length computations. — Let us start with the torus-embedding

$$T = \mathbb{G}_m^{d+1} \subseteq X = \mathrm{Spec}(V[L^\vee \cap \sigma^\vee]) = \mathrm{Spec}(R).$$

For a sublattice $L' \subseteq L$ let $X' = \mathrm{Spec}(R') = \mathrm{Spec}(V[L'^\vee \cap \sigma^\vee])$. As an R-module

$$R' = \bigoplus_{\mu \in L'^\vee/L^\vee} V[(\mu + L^\vee) \cap \sigma^\vee] = \bigoplus_\mu R'_\mu,$$

where the isomorphism-class of R'_μ as R-module depends only on the integral parts $[\mu(l)], l \in L$ the generator of an extremal ray of σ. Namely for $\kappa \in L^\vee$ $(\mu + \kappa)$ lies in $\sigma^\vee$ if any only if $\kappa(l) \geqslant -[\mu(l)]$ for all such l. As μ can be chosen in a compact set (representatives for $L^\vee_\mathbb{R}/L^\vee$) the numbers $[\mu(l)]$ are bounded. One derives that there are onyl finitely many isomorphism classes of R'_μ's, independant of the choice of L'. Furthermore, if for some integer N, $L' \subseteq N \cdot L$, then the multiplicity of a given isomorphism-class of R_μ's is $[L : L'] \cdot (\text{constant} + O(1/N))$. Also the constant is strictly positive for the multiplicity of the trivial R-module $R_\mu \cong R$.

Now firstly if $\lambda \in L^\vee$ is such that $\sigma \cap \ker(\lambda)$ is a simplex spanned by a partial basis of L, then R_λ is regular and R'_λ is flat over R_λ (being finite and Cohen-Macaulay). Thus if $P_\bullet \to P'$ is a resolution of R by finitely generated projective R-modules there exists on N such that λ^N is nullhomotopic on P_1, i.e. $\lambda^N \cdot \mathrm{id} = d \circ s + s \circ d$ for suitable $s : P_1 \to P_2, P_0 \to P_1$.

Thus λ^N annihilates $\mathrm{Tor}_1^R(R', M)$ for all R-modules M, and also all higher Tor's. It now follows that N can be chosen independantly from L'. The same holds after flat pushout. This already shows one of our claims:

If we let $L' = L_n$ (with L_n as before) and if $\sigma \cap \ker(\lambda)$ is a simplex, then for a suitable choice we obtain first on the level of torus-embeddings and then for $R \subseteq R_n \subseteq R_\infty$, that

$$p^\alpha \cdot \mathrm{Tor}_1^{R_n}(R_\infty, \text{any } R_n\text{-module}) = 0 \text{ for } n \gg 0.$$

Now for the definition of $\lambda()$: We assume that for a cofinal sequence of L_n's $L_n = N \cdot L$ with $N \to \infty$. Let $d_n = [L : L_n]$ denote its index. Then after pushout we obtain that for all n (in our cofinal sequence)

$$R_{n+m} = \bigoplus_{\mu \in L^\vee_{n+m}/L^\vee_n} R_{n+m,\mu}$$

where only finitely many isomorphism types of direct summands occur, and the relative frequency of each summand converges (for $m \to \infty$) to a constant, which is non-zero for the type $R_{n+m,\mu} \cong R$. Moreover this summand occurs with multiplicity $\geqslant c \cdot (d_{n+m}/d_m)$, $c > 0$ a constant independant of n and m.

Now assume M_∞ is a finitely presented R_∞-module which is $\mathfrak{m}$-torsion. ($\mathfrak{m}$ = maximal ideal of R). Then $M_\infty = M_n \otimes_{R_n} R_\infty$ for a finitely presented R_n-module M_n which is also $\mathfrak{m}$-torsion and thus of finite length. Define

$$\lambda(M_\infty) = \lim_{m \to \infty} \frac{d_n}{d_{n+m}} \mathrm{length}_R(M_n \otimes_{R_n} R_{n+m})$$

The limit exists because the relative multiplicities of the $R_{n+m,\mu}$ converge. Furthermore

$$\mathrm{length}_R(M_n \otimes_{R_n} R_{n+m}) \geqslant c \cdot d_{n+m} \cdot \lambda(M_\infty) \text{ for all } m.$$

Especially M_∞ vanishes if $\lambda(M_\infty) = 0$. In more generality we have:

Let $M_n \twoheadrightarrow \widetilde{M}_n$ denote a surjection, inducing $M_\infty \twoheadrightarrow \widetilde{M}_\infty$. Then for all m

$$\mathrm{length}(\mathrm{kernel}(M_n \otimes_{R_n} R_{n+m} \to \widetilde{M}_n \otimes_{R_n} R_{n+m}) \geqslant c \cdot d_{n+m} \cdot (\lambda(M_\infty) - \lambda(\widetilde{M}_\infty))$$

Next assume M_∞ is only finitely generated but still $\mathfrak{m}$-torsion. Choose a finite set of generators and denote by M_n the R_n-submodule generated by them. Then M_{n+m} is a quotient of $M_n \otimes_{R_n} R_{n+m}$, and the sequence $\lambda(M_n \otimes_{R_n} R_\infty)$ is decreasing.

6. Claim

$$\lim_{n\to\infty} \frac{1}{d_n} \mathrm{length}_R(M_n) = \lim_{n\to\infty} \lambda(M_n \otimes_{R_n} R_\infty)$$

Define this limit as $\lambda(M_\infty)$.

Proof of claim. — For $\varepsilon > 0$ choose n such that $\lambda(M_n \otimes_{R_n} R_\infty) \leqslant \lambda(M_\infty) + \varepsilon$. Then if $N_{n+m} = \ker(M_n \otimes_{R_n} R_{n+m} \to M_{n+m})$, we have that for $l \geqslant 0$

$$\begin{aligned}\mathrm{length}_R \ker(M_n \otimes_{R_n} R_{n+m+l} \to M_{n+m} \otimes_{R_{n+m}} R_{n+m+l})\\ \geqslant c \cdot (d_{n+m+l}/d_{n+m}) \cdot \mathrm{length}_R(N_{n+m})\end{aligned}$$

Passing to the limit $l \to \infty$ we obtain

$$\begin{aligned}\mathrm{length}_R(N_{n+m}) &\leqslant \frac{d_{n+m}}{c}(\lambda(M_n \otimes_{R_n} R_\infty) - \lambda(M_{n+m} \otimes_{R_{n+m}} R_\infty))\\ &\leqslant \frac{d_{n+m} \cdot \varepsilon}{c}\end{aligned}$$

Thus $0 \leqslant \frac{1}{d_{n+m}}(\mathrm{length}_R(M_n \otimes_{R_n} R_{n+m}) - \mathrm{length}_R(M_{n+m})) \leqslant \varepsilon/c$, which implies the claim.

Next for an arbitrary $\mathfrak{m}$-torsion R_∞-module M define

$$\lambda(M) = \sup\{\lambda(M') \mid M' \subseteq M \text{ finitely generated submodule}\},$$

so $0 \leqslant \lambda(M) \leqslant \infty$.

7. Lemma (additivity). — Suppose $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $\mathfrak{m}$-torsion R_∞-modules. Then

$$\lambda(M) = \lambda(M') + \lambda(M'')$$

Proof. — We easily reduce to M and M'' finitely generated. For a finite system of generators of M let M_n denote the R_n-module generated by them, M''_n its image in M'', and M'_n its intersection with M'. Then we have

$$\lim_{n\to\infty} \frac{1}{d_n} \mathrm{length}_R(M'_n) = \lambda(M) - \lambda(M'').$$

For any finite set of elements of M' the R_n-module $\widetilde{M'_n}$ generated by them is for big n contained in M'_n, and has length $\leqslant \mathrm{length}_R(M'_n)$. It follows that

$$\lambda(M') \leqslant \lambda(M) - \lambda(M'').$$

On the other hand for $\varepsilon > 0$ choose n such that

$$\lambda(M'') \geqslant \lambda(M''_n \otimes_{R_n} R_\infty) - \varepsilon.$$

Then for $m \geqslant 0$ the kernel of $M''_n \otimes_{R_n} R_{n+m} \to M''_{n+m}$ has length bounded by $\frac{\varepsilon}{c} \cdot d_{n+m}$, by the previous arguments. From the commutative diagram with exact rows

$$\begin{tikzcd} & M'_n \otimes_{R_n} R_{n+m} \arrow[r] \arrow[d] & M_n \otimes_{R_n} R_{n+m} \arrow[r] \arrow[d] & M''_n \otimes_{R_n} R_{n+m} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M'_{n+m} \arrow[r] & M_{n+m} \arrow[r] & M''_{n+m} \arrow[r] & 0 \end{tikzcd}$$

we obtain a surjection

$$\ker(M''_n \otimes_{R_n} R_{n+m} \to M''_{n+m}) \twoheadrightarrow M'_{n+m}/R_{n+m} \cdot M'_n$$

Thus

$$\mathrm{length}_R(M'_{n+m}) \leqslant \mathrm{length}_R(R_{n+m} \cdot M'_n) + \frac{\varepsilon}{c} \cdot d_{n+m}$$

and in the limit $m \to \infty$

$$\lambda(M) - \lambda(M'') \leqslant \lambda(R_\infty \cdot M'_n) + \frac{\varepsilon}{c} \leqslant \lambda(M') + \frac{\varepsilon}{c} \leqslant \lambda(M') + \frac{\varepsilon}{c}.$$

As ε was arbitrary we derive the lemma.

Finally if M is finitely presented and annihilated by $\pi^{1/p}$, then $\lambda(M \otimes_{R_\infty \varphi} R_\infty) = p^{d+1} \cdot \lambda(M)$:

This follows because first on the level of torus-embeddings pushout be Frobenius induces an isomorphism

$$k[L_n^\vee \cap \sigma^\vee]/(\lambda^{1/p}) \xrightarrow{\sim} k[pL_n^\vee \cap \sigma^\vee]/(\lambda)$$

Thus for modules annihilated by $\pi^{1/p}$ or $\lambda^{1/p}$ pushout by Frobenius replaces L_n by $\frac{1}{p}L_n$, hence d_n by $p^{-(d+1)}d_n$, and the assertion follows. It then also holds for finitely generated M by passage to the limit from $M_n \otimes_{R_n} R_\infty$.

Finally if M' is submodule of a finitely presented M_1 and $\lambda(M') = 0$, then $M'' = M/M'$ is also finitely presented, with $\lambda(M) = \lambda(M'')$. Then

$$\mathrm{length}_R(\ker(M_n \to M''_n)) \leqslant \frac{1}{c}(\lambda(M) - \lambda(M'')) = 0,$$

hence $M' = (0)$.

Remark. — There exist finitely generated $M \neq (0)$ with $\lambda(M) = 0$, for example the residue field of R_∞.

In fact $\lambda(M)$ vanishes if M is almost zero, that is annihilated by any positive p-power p^α :

We may assume that M is finitely generated. It suffices to show that for any finite $\mathfrak{m}$-torsion R_0-module M_0 and any sequence $\alpha_n \to 0$ the limit

$$\lim_{n\to\infty} \frac{1}{d_n} \mathrm{length}_R(M \otimes_R R_n/\pi^{\alpha_n} \cdot R_n) = 0.$$

This follows from toroidal geometry. Namely in the decomposition

$$R' = \bigoplus_{\mu \in L'^{\vee}/L^{\vee}} V[(\mu + L^{\vee}) \cap \sigma^{\vee}] = \bigoplus_{\mu} R'_{\mu}$$

the summand R'_{μ} lies in the ideal generated by λ^{α} (if $\alpha \cdot \lambda \in L'^{\vee}$) if $\mu(\ell)$ and $(\mu + \alpha\lambda)(\ell)$ have the same integral parts, $\ell \in L$ running though generators of extremal rays of σ. For $\alpha \to 0$ the fraction of μ's for which this does not happen is bounded asymptotically (as $L' \subseteq p^n \cdot L$ with $n \to \infty$) by constant $\cdot\, \alpha$. As the others disappear modulo π^{α_n} (after basechange) the assertion follows. By additivity we derive that λ is invariant under almost isomorphisms.

2c. Galois cohomology. — Suppose G is a finite group, A a commutative ring (with unit), B an A-algebra with G-action. Assume furthermore that for $b \in B$ $\mathrm{tr}(b) = \sum_{g \in G} g(b)$ lies in the image of A in B. If M is a B-module with semilinear G-action, we may compute the cohomology $H^*(G, M)$ using a G-acyclic resolution by such $B-G$-modules. If $b \in B$ has image $a = \mathrm{tr}(b) \in A$, then a annihilates all higher cohomology:

For $m \in M$ we have $\mathrm{tr}(b \cdot m) = \sum_{g \in G} g(b \cdot m) \in M^G$, and if $m \in M^G$ then $\mathrm{tr}(b \cdot m) = a \cdot m$. Applying this to a resolution of M the claim follows.

If we also assume that there exist elements $b_i, c_i \in B$ $(1 \leqslant i \leqslant r)$ such that $b \in B$

$$a = \sum_{i=1}^{r} b_i \cdot c_i$$

$$0 = \sum_{i=1}^{r} b_i \cdot g(c_i)$$

for $g \in G$, $g \neq 1$, we can define a map

$$M \longrightarrow B \otimes_A M^G$$

$$m \longmapsto \sum_{i=1}^{r} b_i \otimes_A \mathrm{tr}(c_i \cdot m)$$

The composition either way of this map with the natural

$$B \otimes_A M^G \longrightarrow M$$

will be a. For example if $A \subseteq B$ is an almost étale Galois-covering with group G, then we apply this to $a = \pi^{\alpha}$ (any $\alpha > 0$), $\sum_{i=1}^{r} b_i \otimes c_i = \pi^{\alpha} \cdot e_{B/A}$, and obtain that

$$H^i(G, M) \approx 0 \quad (i > 0)$$

$$M \approx B \otimes_A M^G$$

Passing to the limit this will also hold if B is an inductive limit of almost étale coverings and G the corresponding profinite group, i.e. for the extensions $R_{\infty} \subseteq \overline{R}$. Here $\overline{R}$ denotes the normalization of R in the maximal étale covering of the preimage of $T \subseteq \overline{T}$ in $\mathrm{Spec}(R)$, that is the union of all finite extensions $R \subseteq S$ with S normal and $S[1/\pi]/R[1/\pi]$ ramified only along the divisor "at infinity". For simplicity we assume that R is an integral domain (otherwise we would take everywhere products of integral domains). If $\overline{V}$ denotes the integral closure of V in $\overline{K}$, $\overline{R}$ contains as subring the normalization of the image of $R\otimes_V \overline{V} \to \overline{R}$. If $\lambda = n\cdot\lambda_0$ with λ_0 indivisible in L then firstly R contains the subring $V' = V[\sqrt[n]{\pi}]$. Replacing V by V' we then may assume that $\lambda = \lambda_0$. In this case the normalization of V in R is unramified over V. Thus if the residue field k is algebraically closed it coincides with V_j and $R\otimes_V \overline{V}$ is an integral domain. In any case we denote by Δ the Galois-group of $\overline{R}$ over the normalization of the image of $R\otimes_V\overline{V}$. If $R\otimes_V\overline{V}$ is an integral domain this is $\Delta = \pi_1(\operatorname{Spec}(R\otimes_V\overline{K})^0)$. The coverings defined by the R_n give a map

$$\Delta \longrightarrow L\otimes\widehat{\mathbb{Z}}(1) = \widehat{L}(1) \text{ (profinite completion).}$$

Let Δ_∞ denote its image and $\Delta_0\subseteq\Delta$ its kernel. As R is formally étale over $\overline{T}_\lambda$ and the R_n are totally ramified over R at all generic characteristic p-points if $[L:L_n]$ is a p-power, there is a surjection $\Delta_\infty \twoheadrightarrow L\otimes\mathbb{Z}_p(1)$, with kernel of order prime to p. For most arguments we can in fact replace Δ_∞ by $L\otimes\mathbb{Z}_p(1)$.

Now suppose $\overline{M}$ is a p-torsion $\overline{R}$-module with a semilinear continuous Δ-action. To study the cohomology $H^*(\Delta, M)$ we use the spectral sequence

$$E_2^{a,b} = H^a(\Delta_\infty, H^b(\Delta_0, M)) \Longrightarrow H^{a+b}(\Delta, M).$$

The extension governed by Δ_0 is a composition of almost étale extensions. Thus all higher Δ_0-cohomology almost vanishes,

$$M \approx (M^{\Delta_0})\otimes\overline{R},$$

and $H^*(\Delta, M)\approx H^*(\Delta_\infty, M^{\Delta_0})$. As Δ_∞ has cohomological dimension d this vanishes in degree $> d$. Also after choice of topological generators $\delta_1,\dots,\delta_d$ of Δ_∞ the cohomology is represented by the Koszul-complex of $\delta_1-1,\dots,\delta_d-1$ acting on M^{Δ_0}. This commutes with formally étale base change $R\to R'$.

For example we can study the cohomology of quotients $\overline{R}/p^s\cdot\overline{R}$. By the above we may (up to almost isomorphism) replace R by the affine ring of the torus-embedding $\overline{T}_\lambda$, and have to study the cohomology of $\mathbb{Z}_p(1)^d$ acting on

$$V[\sigma^\vee\cap L^\vee[1/p]]/(\lambda-\pi)\otimes_V \overline{V}/p^s\overline{V}$$

This module is free over $\overline{V}/p^s\overline{V}$, where a basis can be obtained as follows: λ acts freely on the monoid $\sigma^\vee\cap L^\vee[1/p]$. For each orbit $\{\mu, \mu+\lambda, \mu+2\lambda\dots\}$ choose the minimal representative μ. Then these μ form a basis.

All the basis-elements μ are eigenvectors for $\mathbb{Z}_p(1)^d$, with character $e^\mu : \mathbb{Z}_p(1)^d \to \mu_{p^\infty}$. If this character has order p^t the corresponding cohomology is annihilated by $\zeta_{p^t}-1$, ζ_{p^t} a primitive p^t-th root of unity. Thus up to terms annihilated by ζ_p-1 only the eigenspaces with trivial e^μ contribute to cohomology. e^μ is trivial if and only if $\mu\mid\ker(\lambda)$ is in $\ker(\lambda)^\vee$. Then there exists a unique $\alpha\in\mathbb{Z}[1/p]$, $0\leqslant\alpha<1$, with $\mu-\alpha\lambda\in L^\vee$ integral. Let $\mathcal{R} = \bigoplus_{\alpha,\mu} V\pi^\alpha\cdot\mu$ (sum over such α,μ). Then

$\mathcal{R} = R_\infty \cap R \otimes_V K_\infty$ is the normalization of $R \otimes_V V_\infty$, and $\pi\mathcal{R} \subseteq R \otimes_V \overline{V}$. Furthermore (as $H^*(\mathbb{Z}_p(1)^d, V) = \Lambda^\bullet(V(-1)^d)$)

$$H^*(\Delta, \overline{R}/p^s\overline{R}) \approx \mathcal{R} \otimes \Lambda^\bullet(\mathrm{Hom}(\Delta_\infty, \overline{V}/p^s\overline{V})) \oplus (\text{terms annihilated by } \zeta_p - 1).$$

Especially we get compatible almost maps

$$\mathrm{tr} : H^d(\Delta, \overline{R}/p^s\overline{R}) \longrightarrow \mathcal{R} \otimes_{V_\infty} \overline{V}/p^s\overline{V}(-d).$$

(Compatibility forces them to vanish on the second direct summand). The target can be more canonically identified with $\Omega^{d,\log}_{R/V} \otimes_R \mathcal{R} \otimes_{V_\infty} \overline{V}/p^s\overline{V}(-d)$. Here $\Omega^{\log}_{R/V}$ denotes the logarithmic module of differentials, for the log-structures on $\overline{T}_\lambda$ and $\mathrm{Spec}(V)$ given by $\sigma^\vee \cap L^\vee$ and $\mathbb{N}$ ($\mu \in \sigma^\vee \cap L^\vee$ maps to μ, and $1 \in \mathbb{N}$ to π). By étaleness

$$\Omega^{\log}_{R/V} \otimes V/p^sV \cong \ker(\lambda)^\vee \otimes R$$

Of course $\Omega^{d,\log}_{R/V}$ is the d'-th exterior power. At least modulo p^s it is free of rank one. Checking all the identification we get almost maps

$$\mathrm{tr} : H^d(\Delta, \overline{R}/p^s\overline{R}) \longrightarrow \Omega^{d,\log}_{R/V} \otimes_R \mathcal{R} \otimes_{V_\infty} \overline{V}/p^s\overline{V}(-d)$$

which are independant of the choice of generators $\delta_1, \ldots, \delta_d$. In fact they only depend on R (with its log-structure) and not on the formally étale map $\mathrm{Spec}(R) \to \overline{T}_\lambda$. This follows (as in [F2], Sect. I, 4) from the following facts, which can be checked by direct calculation for $\overline{T}_\lambda$ and then also hold for R by formal étaleness and almost étaleness:

i) The sequence

$$0 \longrightarrow \Omega^{\log}_{\overline{V}/V} \otimes_{\overline{V}} \overline{R} \longrightarrow \Omega^{\log}_{\overline{R}/R} \longrightarrow \Omega^{\log}_{\overline{R}/R\otimes\overline{V}} \longrightarrow 0$$

is almost exact

ii)

$$\Omega^{\log}_{\overline{V}/V} \cong \overline{V}[1/p]/\rho^{-1}\overline{V}(1)$$

for an element $\rho \in \overline{V}, \rho \neq 0$. The isomorphism is induced from

$$d\log : \mu_{p^\infty} \longrightarrow \Omega^{\log}_{\overline{V}/V}.$$

iii) Consider the exact sequence

$$\Omega^{\log}_{R/V} \otimes_R \overline{R} \longrightarrow \Omega^{\log}_{\overline{R}/\overline{V}} \longrightarrow \Omega^{\log}_{\overline{R}/R\otimes_V\overline{V}} \longrightarrow 0$$

Then the kernel of the first map is almost contained in $\bigcap_{s\geqslant 0} p^s \cdot (\Omega^{\log}_{R/V} \otimes_R \overline{R})$. Furthermore the induced map

$$\mathrm{Hom}(\tfrac{1}{p^s}\mathbb{Z}/\mathbb{Z}, \Omega^{\log}_{\overline{R}/R\otimes_V\overline{V}}) \longrightarrow \Omega^{\log}_{R/V} \otimes_R \overline{R}/p^s\overline{R}$$

is an almost isomorphism.

iv) Applying $\mathrm{Hom}(\frac{1}{p^s}\mathbb{Z}/\mathbb{Z},)$ to the exact sequence in i) gives an almost exact sequence

$$0 \longrightarrow \rho^{-1}(\overline{R}/p^s\overline{R}(1)) \longrightarrow \text{middle term} \longrightarrow \Omega^{\log}_{R/V} \otimes_R \overline{R}/p^s\overline{R} \longrightarrow 0,$$

thus a map

$$\Omega^{\log}_{R/V} \longrightarrow H^1(\Delta, (\overline{R}/p^s\overline{R})(1))$$

By skew-commutativity of the cup-product we obtain

$$\Omega^{d,\log}_{R/V} \otimes_V \overline{V}/p^s\overline{V}(-d) \longrightarrow H^d(\Delta, \overline{R}/p^s\overline{R})$$

which is $\rho^d \cdot d!$-times the map constructed before.

v) As $\rho^d \cdot d!$-times our maps are canonical, compatibility for varying s implies the assertion.

Remark. — So from now on we assume that $\mathcal{R} = R \otimes_V \overline{V}$, equivalent to the cone-condition above $((\mathbb{Q}^+ \cdot \lambda + L^\vee) \cap \sigma^\vee = \mathbb{Q}^+ \cdot \lambda + (L^\vee \cap \sigma^\vee))$. Other equivalences: if $\mu + \alpha\lambda(\mu \in L^\vee, \alpha \in \mathbb{Q}, 0 \leqslant \alpha < 1)$ lies in $\sigma^\vee$, then so does μ.
Also: The special fibre $\overline{T}_\lambda \otimes_V k$ is reduced.

Before we start to treat duality we have to introduce coefficients. Suppose $\mathbb{L}$ is a finite $\mathbb{Z}_p$-module with a continuous action of Δ. We denote by $\mathbb{L}^t = \mathrm{Hom}(\mathbb{L}, \mathbb{Q}_p/\mathbb{Z}_p)$ its dual. Next recall that $\sigma \cap \ker(\lambda)$ was supposed to be a simplex spanned by a partial basis $\rho_1, \ldots, \rho_r$ of L. On the generic fibre of $\overline{T}_\lambda$ these define normal crossings divisors $D_1, \ldots, D_r$, which are cut out by a (partial) dual basis of $L^\vee$. Choose a subset I of $\{1, \ldots, r\}$ and denote by D_I the union of the $\{D_i \mid i \in I\}$. Then for each n the preimage of D_I in $\overline{T}_{\lambda,n} \otimes_V K$ is a divisor. Let $J_{I,n} \subseteq V[L_n^\vee \cap \sigma^\vee]/(\lambda - \pi)$ denote the ideal of its reduced closure. That is $J_{I,n}$ is spanned by all $\mu \in L_n^\vee \cap \sigma^\vee$ with $\mu(\rho_i) > 0$ for $i \in I$. Finally $J_{I,\infty} \subseteq \Gamma(\overline{T}_{\lambda,\infty}, \mathcal{O})$ denotes the union of the $J_{I,n}$. One checks that for any $\mu \in L^\vee \cap \sigma^\vee$ with $\mu(\rho_i) > 0$ for $i \in I$, $\mu(\rho_i) = 0$ for $i \notin I$, $J_{I,\infty}$ is almost generated by the fractional powers μ^{1/p^n}. (Use that any μ' vanishing on $\sigma \cap \ker(\lambda)$ divides a power of π). Thus $J_{I,\infty}$ is almost a union of principal ideals and almost flat.

Finally we apply pushout to R. If we assume π is contained in the Jacobsen radical of R (for example if R is π-adically complete) then R is normal, and $R[1/\pi]$ is regular with the pullback of the D_i defining a normal crossings divisor. Then one checks that the ideal $J_{I,\infty} \otimes \overline{R}[1/\pi]$ defines the reduced pullback-divisor and thus only depends on the pullback of D_I to $\mathrm{Spec}(R[1/\pi])$, not on the particular representation $\mathrm{Spec}(R) \to \overline{T}_\lambda$. Finally be almost étaleness of $\overline{R}$ over $\Gamma(\overline{T}_{\lambda,\infty}, \mathcal{O})$ the ideal $J_{I,\infty} \cdot \overline{R}$ is almost equal to the ideal defining the closure, thus also as canonical as the divisor D_I. Call it $\overline{J}_I$.

Now our aim is to study the cohomology of Δ with coefficients $\mathbb{L} \otimes \overline{J}_I$. This depends only on the image of $\overline{J}_I$ in the π-adic completion of R, so that everything is canonical even if π should not lie in the Jacobsen radical of R.

Next comes finiteness. As $R\otimes_V\overline{V}$ is the inductive limit of noetherian rings ($R\otimes_V V'$, V' a finite extension of V) with flat transition-maps, it is coherent. That is any finitely presented $R\otimes_V\overline{V}$-module has a projective resolution by finitely generated free $R\otimes_V\overline{V}$-modules, and the finitely presented $R\otimes_V\overline{V}$-modules form an abelian subcategory of all $R\otimes_V\overline{V}$-modules.

We say that an $R\otimes_V\overline{V}$-module M is almost finitely presented if for each $\alpha>0$ there exists a finitely presented M_α and a π^α-isomorphism $f: M_\alpha\to M$ i.e. kernel and cokernel of f are annihilated by π^α. The almost finitely presented $R\otimes_V\overline{V}$-modules also form an abelian subcategory. Now suppose $\mathbb{L}$ is a finite $\mathbb{Z}_p$-module with a continuous Δ-action.

8. Proposition. — *The cohomology-groups $H^i(\Delta,\mathbb{L}\otimes\overline{J}_I)$ are almost finitely presented over $R\otimes_V\overline{V}$.*

Proof. — By devissage we may assume that $p\cdot\mathbb{L}=(0)$. We also may replace R by its p-adic completion, so $\mathrm{Spec}(R)$ is formally étale over $\overline{T}_\lambda$. Especially we have that $R\otimes_V K$ is regular, and the complement of $T\subseteq\overline{T}$ defines a normal crossing divisor. Replacing L by a sublattice $L'\subseteq L$ we may assume that $\mathbb{L}$ is unramified over $\mathrm{Spec}(R\otimes_V\overline{K})$. (If Δ' is the group corresponding to L', use the Hochschild spectral sequence for $0\to\Delta'\to\Delta\to\Delta/\Delta'\to 0$). Next by almost faithfully flat descent (for $\overline{R}/R_\infty$) $(\mathbb{L}\otimes\overline{R})^{\Delta_\infty}$ is almost finitely presented. Thus for each $\alpha>0$ there exists an integer n, a finitely presented R_n-module M_α action, and a π^α-isomorphism

$$M_\alpha\otimes_{R_n}R_\infty\longrightarrow(\mathbb{L}\otimes\overline{R})^{\mathrm{Gal}(\overline{R}/R_\infty\otimes_V\overline{K})}$$

By increasing n we make this $\mathrm{Gal}(R_\infty/R_n\otimes_V\overline{V})$-linear (trivial operation on M_α). It suffices if the images of generators of M_α are fixed.

Again by the Hochschild spectral sequence we may replace R by R_n, i.e. study $H^*(\Delta_\infty, M\otimes_R J_{\infty,I})$. Now for each n, $J_{\infty,I}$ is the direct sum of a finitely presented $R_n\otimes_V\overline{V}$-module and modules on which a generator δ of Δ_∞ acts like ζ_{p^n}, ζ_{p^n} a primitive p^n-th root of unity. (There are generated by μ's, μ such that $\ker(\lambda)$ not integral on $p^{n-1}L$.) For these the cohomology is annihilated by $\zeta_{p^n}-1$, while the first direct summand has finitely presented cohomology. (The p-adic valuation of $\zeta_{p^n}-1$ is $1/(p-1)p^{n-1}$ and converges to zero.)

Finally we come to duality. Recall ([H], Ch. V, § 2) that a dualizing complex D^* over a noetherian ring A is a complex with coherent cohomology which has an injective resolution in which each indecomposable injective appears precisely once. Equivalently for each prime $\mathfrak{p}\in\mathrm{Spec}(A)$ there exists an integer $d(\mathfrak{p})$ (related to the codimension) with $\mathrm{Ext}^i_{A_\mathfrak{p}}(K(\mathfrak{p}),D^*_\mathfrak{p})=\kappa(\mathfrak{p})$ for $i=d(\mathfrak{p})$, and 0 else. Then $K^*\to\mathbb{R}\mathrm{Hom}(K^*,D^*)$ defines a perfect duality on the derived category of finite complexes with finitely generated cohomology. If A is Cohen-Macaulay D^* has only one non-trivial cohomology group, the dualizing module ω_A. For example for the torus-embedding $\overline{T}$ a dualizing module is given by $\omega_{\overline{T}} = J \otimes \Omega^{d+1,\log}$, where J is the ideal generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. This follows because on the smooth locus the dualizing module coincides with the differentials, and in general it is the reflexive extension of its restriction to the smooth locus. Also for $\overline{T}_\lambda$ the dualizing module is $\mathrm{Ext}^1_{O_{\overline{T}}}(O_{\overline{T}_\lambda}, \omega_{\overline{T}}) \cong J \otimes O_{\overline{T}_\lambda}$.

It is the ideal in $\mathcal{O}_{\overline{T}_\lambda}$ (or better in $\Omega^{d,\log}_{\overline{T}_\lambda}$) generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. Finally if π is contained in the Jacobson-radical of R the map $\mathcal{O}_{\overline{T}_\lambda} \to R$ is flat with Gorenstein-fibres. Thus $\omega_{\overline{T}_\lambda} \otimes_{\overline{T}_\lambda} R = \omega_R$ is a dualizing module on R. It is isomorphic to J. Similarly the dualizing modules on R_n are isomorphic to J_n. By the general theory $\omega_{R_n} \cong \mathrm{Hom}_R(R_n, \omega_R)$. This isomorphism is determined by the evaluation at $1 \in R_n$, which defines a trace-map

$$\mathrm{tr} : \omega_{R_n} \longrightarrow \omega_R.$$

It is obtained by pushout from the trace-map on $O_{\overline{T}}$ or $O_{\overline{T}_\lambda}$. The latter sends to character $\mu \in L_n^\vee$ to itself if $\mu \in L^\vee$, and to zero else. Similarly we can define a dualizing module $\omega_{R\otimes_V \overline{V}} = \omega_R \otimes_V \overline{V}$.

By passing to the limit over finite extensions of V,

$$K^* \longrightarrow D(K^*) = \mathbb{R}\ \mathrm{Hom}_{R\otimes_V\overline{V}}(K^*, \omega_R \otimes_V \overline{V})$$

still defines a perfect duality for finite complexes with finitely presented (= coherent) cohomology, or an almost perfect duality for finite complexes with almost finitely presented cohomology. Again $\omega_{R\otimes_V\overline{V}} \subseteq R \otimes_V \overline{V}$ is the ideal J generated by all $\mu \in L^\vee$ which are strictly positive on $\sigma - \{0\}$. Similarly $\omega_{R_n\otimes_{V_n}\overline{V}} \cong J_n \subseteq R_n \otimes_{V_n} \overline{V}$ is the ideal generated by $\mu \in L_n^\vee$ strictly positive on $\sigma - \{0\}$, and we have the union $J_\infty \subseteq R_\infty \otimes_{V_\infty} \overline{V}$. It coincides almost with the ideal generated by $\mu \in \sigma^\vee$ which are strictly positive on $\sigma \cap \ker(\lambda)) - \{0\}$:

For such a μ $\mu + \alpha \cdot \lambda$ is strictly positive on $\sigma - \{0\}$, for each $\alpha > 0$.

That is J_∞ almost coincides with the ideal previously named by the same symbol. Finally the trace-map becomes more complicated:

As $V_n = V[\sqrt[e]{\pi}] = V[\pi_n]$ (for some e)

$$\frac{1}{e} \cdot \mathrm{tr}_{V_n/V} : \tfrac{\pi_n}{\pi} \cdot V_n \longrightarrow V$$

defines an isomorphism

$$\tfrac{\pi_n}{\pi} V_n \cong \mathrm{Hom}_V(V_n, V).$$

Thus our old trace defines a new trace from J_n to $J \otimes_V (\frac{\pi_n}{\pi} V_n)$ and by $\otimes_{V_n}\overline{V}$ a trace-map

$$\mathrm{tr} : J_n \otimes_{V_n} \overline{V} \longrightarrow J \otimes_V (\tfrac{\pi_n}{\pi} \overline{V}) = \tfrac{\pi_n}{\pi} \otimes (J \otimes_V \overline{V})$$

It sends a $\mu \in L_n^\vee$ strictly positive on $\sigma - \{0\}$ to itself or zero, depending on whether $\mu \in \ker(\lambda)^\vee + \mathbb{Q} \cdot \lambda$ or not. Obviously these combine to define

$$\mathrm{tr} : R_\infty \otimes_{V_\infty} \overline{V} \longrightarrow \tfrac{1}{\pi}(J \otimes_V \overline{V})$$

As on finite levels we obtain isomorphisms

$$J_n \otimes_V \overline{V} \cong \mathrm{Hom}_{R\otimes_V \overline{V}}(R_n \otimes_V \overline{V}, \tfrac{\pi_n}{\pi} \cdot (J \otimes_V \overline{V})),$$

we have in the limit for each $\mu \in \ker(\lambda)^\vee \otimes \mathbb{Q}/\mathbb{Z}$ an almost isomorphism of μ-components

$$(J_\infty \otimes_V \overline{V})_\mu \approx \mathrm{Hom}_{R\otimes_V \overline{V}}(R_\infty \otimes_V \overline{V}, \tfrac{1}{\pi}(J \otimes_V \overline{V}))_\mu$$

Of course the map

$$J_\infty \otimes_V \overline{V} \longrightarrow \mathrm{Hom}_{R\otimes_V \overline{V}}(R_\infty \otimes_V \overline{V}, \tfrac{1}{\pi} J \otimes_V \overline{V})$$

cannot be an almost-isomorphism because of the difference between direct sum and product (over all μ-components).

However this difficulty disappears if we take Δ_∞-cohomology: If μ has denominator p^n the cohomology of the μ-component is annihilated by $\zeta_{p^n} - 1$. It thus follows that we obtain almost-isomorphisms on cohomology. The same holds after tensoring with $(\mathbb{L} \otimes \overline{R})^{\mathrm{Gal}(\overline{R}/R_\infty \otimes_V \overline{V})}$ (π^α-isomorphic to $M_\alpha \otimes_{R_n} R_\infty$ as before). Thus (using almost étaleness).

9. Proposition. — tr *induces almost isomorphisms*

$$H^i(\Delta, \mathbb{L} \otimes \overline{R}) \approx H^i(\Delta, \mathbb{L} \otimes \mathrm{Hom}_{\overline{R}\otimes_V \overline{V}}(\overline{R}, \tfrac{1}{\pi} J \otimes_V \overline{V}))$$

Finally cohomology can be (almost) computed by taking invariants under

$$\mathrm{Gal}(\overline{R}/R_\infty \otimes_V \overline{V})$$

(an almost exact operation) and then applying the Koszul-complex with

$$(\delta_1 - 1), \ldots, (\delta_d - 1),$$

δ_i topological generators of $\Delta_\infty \cong \mathbb{Z}_p(1)^d$. However the dual of the Koszul-complex is again a Koszul-complex shifted by d. We thus obtain:

10. Theorem. — *Assume $\overline{T}_\lambda \otimes_V k$ reduced and that $\mathbb{L}$ is annihilated by p^s, and let $\mathbb{L}^t = \mathrm{Hom}(\mathbb{L}, \mathbb{Z}/p^s\mathbb{Z})$ denote the dual. Then*

i) *$\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \overline{R})$ is almost isomorphic to a finite complex of $R \otimes_V \overline{V}$-modules concentrated in degrees $[0, d]$ with almost finitely presented cohomology ($\mathbb{R}\Gamma(\Delta, \cdot)$ denotes one of the canonical complexes computing group-cohomology)*

ii) *The previously defined trace-maps* tr : $H^d(\Delta, \overline{R}/p^s\overline{R}) \to \mathcal{R}/p^s\mathcal{R}$ *induce maps* tr : $H^d(\Delta, \overline{J}/p^s\overline{J}) \to \frac{1}{\pi}\omega_R \otimes \overline{V}/p^s\overline{V}$ *($\mathcal{R}$ is contained in $\frac{1}{\pi} R \otimes_V \overline{V}$)*

iii) tr *defines an almost isomorphism*

$$\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \overline{J})(d)[d] \approx \mathbb{R}\,\mathrm{Hom}_{R\otimes_V \overline{V}/p^s\overline{V}}(\mathbb{R}\Gamma(\Delta, \mathbb{L}^\vee \otimes \overline{R}), \tfrac{1}{\pi}\omega_R \otimes \overline{V}/p^s\overline{V})$$

Proof. — Only some minor details need to be checked: Firstly our trace map (from duality) $J_\infty/(\delta_i - 1) \to J \otimes_V \frac{1}{\pi}\overline{V}$ induces on cohomology the previously defined trace, (obviously direct checking), and secondly we may replace R by its π-adic completion so that the duality-theory works.

Variant. — We may also use coefficients $\overline{J}_I; I \subseteq \{1, \ldots, r\}$ a subset. The ideals $J_I \subseteq R$ and $J_{I,n} \subseteq R_n$ are Cohen-Macaulay:

Reduce to the torus-embedding $\overline{T}$. If (say) $\delta_1 \in I$, let $\widetilde{L}$ denote the torus-embedding defined by $\widetilde{L} = L/\mathbb{Z} \cdot \delta_1$ and the image $\widetilde{\sigma}$ of σ in $\widetilde{L}_{\mathbb{R}}$. Then $\widetilde{\sigma}$ has extremal rays generated by $\delta_2, \ldots, \delta_s$. Let $\widetilde{I} = I \cap \{\delta_2, \ldots, \delta_r\}$. Then there is an exact sequence

$$0 \longrightarrow J_I \longrightarrow J_{I-\{1\}} \longrightarrow \widetilde{J}_{\widetilde{I}} \longrightarrow 0.$$

If the other two terms are Cohen-Macaulay (of relative dimension $d+1$ respectively d), then J_I also is.

Now duality holds between $J_{I,n}$ and $\mathrm{Hom}_R(J_{I,n}, \omega_R) = \mathrm{Hom}_{R_n}(J_{I,n}, \omega_{R_n})$, and similarly between $J_{I,n} \otimes_{V_n} \overline{V}$ and $\mathrm{Hom}_{R \otimes_V \overline{V}}(J_{I,n} \otimes_V \overline{V}, \omega_R \otimes_V \overline{V})$. However one checks that the union

$$\bigcup_n \mathrm{Hom}_{R \otimes_V \overline{V}}(J_{I,n} \otimes_V \overline{V}, \omega_R \otimes_V \overline{V}) \approx J_{I^c,\infty}$$

is almost $J^c_{I,\infty}$, $I^c \subseteq \{1, \ldots, r\}$ the complement of I:

The product $J_{I,\infty} \times J_{I^c,\infty} \to J_\infty$ shows that the union contains $J_{I^c,\infty}$. On the other hand it is contained in R_∞, and any character μ in it must be strictly positive on $\delta_i \in I^c$ (choose $\nu \in L^\vee \cap \sigma^\vee$ which is strictly positive on I but vanishes on δ_i. Then $\nu \in J_I$, and $\mu + \nu$ must be strictly positive on δ_i too). However these μ's are almost in $J_{I^c,\infty}$. With the same proofs as before we then have

10'. Theorem

$$\mathbb{R}\Gamma(\Delta, \mathbb{L} \otimes \overline{J}_{I^c})(d)[d] \approx \mathbb{R}\,\mathrm{Hom}_{R \otimes_V \overline{V}/p^s\overline{V}}(\mathbb{R}\Gamma(\Delta, \mathbb{L}^\vee \otimes \overline{R}), \tfrac{1}{\pi}\omega_R \otimes \overline{V}/p^s\overline{V}).$$

Last we need a local Künneth-formula. Suppose we have given two lattices L', L'' with cones $\sigma' \subseteq L'_{\mathbb{R}}$, $\sigma'' \subseteq L''_{\mathbb{R}}$ and elements $\lambda' \in L'^\vee$, $\lambda'' \in L''^\vee$. Then in general the fibered product $\overline{T}_{\lambda'} \otimes_V \overline{T}_{\lambda''}$ will not be normal. However we can achieve this by adjoining roots of π. Namely pass to $V[\sqrt[e]{\pi}]$ (replacing L' by $\lambda'^{-1}(e\mathbb{Z})$, L'' by $\lambda''^{-1}(e\mathbb{Z})$) such that $\mathcal{R}' = R' \otimes_V \overline{V}$, $\mathcal{R}'' = R'' \otimes_V \overline{V}$. That means for any linear combination $\mu' + \alpha\lambda' \in \sigma'^\vee$ ($\mu' \in L'$, $0 \leqslant \alpha < 1$) we have $\mu' \in \sigma'^\vee$, and similarly for $\mu'' + \alpha\lambda''$.

11. Lemma. — *Under the conditions $\overline{T}_{\lambda'} \otimes_V \overline{T}''_{\lambda''}$ is normal, and equal to $\overline{T}_\lambda$ where the torus-embedding $\overline{T}$ is defined by $L = \ker(\lambda', -\lambda'') \subseteq L' \times L''$, $\sigma = L_{\mathbb{R}} \cap \sigma' \times \sigma''$, $\lambda = (\lambda', 0)|L = (0, \lambda'')|L$. Also the special fibre is reduced.*

Proof. — There is a map

$$\overline{T}_\lambda \longrightarrow \overline{T}'_{\lambda'} \otimes_V \overline{T}''_{\lambda''}$$

induced from the tensor-product

$$V[\sigma' \cap L'^\vee] \otimes_V V[\sigma'' \cap L''^\vee] \longrightarrow V[\sigma \cap L^\vee],$$

by dividing by $(\pi - \lambda', \pi - \lambda'')$. The rest follows from remark 5.

Now assume that this condition holds. Suppose R is formally étale over $R_1' \otimes_V R''$. Furthermore assume given representations $\mathbb{L}'$ of Δ', L'' of Δ'', and let $\mathbb{L} = \mathbb{L}' \otimes \mathbb{L}''$. We assume that $\mathbb{L}'$ and $\mathbb{L}''$ are free $\mathbb{Z}/p^s\mathbb{Z}$-modules (to avoid using derived tensor-products). Furthermore choose finite subsets $I' \subseteq \{1, \ldots, r\}$, $I'' \subseteq \{1, \ldots, r''\}$. As $\sigma \cap \ker(\lambda) = (\sigma' \cap \ker(\lambda') \times (\sigma'' \cap \ker(\lambda''))$, $I = I' \times I''$ identifies with a subset of the vertices of $\sigma \cap \ker(\lambda)$. Then pullback and cup-product define a map

$$\mathbb{R}\Gamma(\Delta', \overline{J}_{I'} \otimes \mathbb{L}') \otimes^{\mathbb{L}}_{\overline{V}/p^s\overline{V}} \mathbb{R}\Gamma(\Delta'', \overline{J}_{I''} \otimes \mathbb{L}'') \otimes^{\mathbb{L}}_{R_1 \otimes_V R_2} R \longrightarrow \mathbb{R}\Gamma(\Delta, \overline{J}_I \otimes \mathbb{L})$$

12. Theorem. — *This induces almost isomorphisms on cohomology.*

Proof. — Reduce to $\Delta'_\infty, \Delta''_\infty, \Delta_\infty = \Delta'_\infty \times \Delta''_\infty$ acting on $R'_\infty, R''_\infty, R_\infty$, and apply the obvious isomorphism on the level of Koszul-complexes.

3. Global Cohomology

Assume $X \to \mathrm{Spec}(V)$ is a proper and flat algebraic space. For most purposes it suffices to consider schemes, but at one point (modification to regularise the diagonal embedding) we shall need algebraic spaces. Unfortunately the references on finiteness, traces, and duality only mention schemes and not algebraic spaces, so we have to indicate how to extend them. However if we start with a scheme the auxiliary algebraic spaces will be modifications of schemes and will have the same cohomology, so for them the scheme-theory suffices.

There exists a dualizing complex D_X on X, such that $D_X(K) = \mathbb{R}\operatorname{Hom}_{\mathcal{O}_X}(M, D_X)$ defines a perfect duality on bounded complexes with coherent cohomology. Furthermore (if X has say pure relative dimension d)

$$\mathbb{R}\Gamma(X, D_X(M)) \cong \mathbb{R}\operatorname{Hom}_V(\mathbb{R}\Gamma(M, R), V)[d],$$

induced from a trace-map

$$\mathrm{tr} : \mathbb{R}\Gamma(X, D_X) \longrightarrow V[-d]$$

(see [H], Ch. VII, Th. 3.3).

The theory of residual complexes and trace maps extends to algebraic spaces $f : X \to S$ over a scheme S. Namely if we choose an étale covering $Y \twoheadrightarrow X$ with Y a scheme, and I^* denotes a residual complex on S. Then the residual complex $f_Y^{\triangle}(I^*)$ (The Cousin-complex to $f_Y^!(I^*)$) descends to X, as for étale maps $f^{\triangle} = f^! = f^*$. To define a trace map

$$\mathrm{tr}_{X/S} : f_*(f^{\triangle}(I^*) \longrightarrow I^*$$

form $Y_2 = Y \times_X Y$ and use the exact sequence (the maps are not maps of complexes)

$$f_{Y_2,*}(f_{Y_2}^{\triangle}(I^*)) \longrightarrow f_{Y,*}(f_Y^{\triangle}(I^*)) \longrightarrow f_*(f^{\triangle}(I^*)) \longrightarrow 0.$$

That it is exact and that $\mathrm{tr}_{Y/S}$ factors over the quotient can be shown by étale descent on X. That $\mathrm{tr}_{X/S}$ is a map of complexes for proper maps follows from [H], as curves over Artinian rings are schemes. Finally to show that $\mathrm{tr}_{X/S}$ defines a perfect duality use devissage and Chow's lemma.

By base-extension $\otimes_V \overline{V}$ this extends to $X \otimes_V \overline{V}$, $D_X \otimes_V \overline{V}$, and bounded complexes with coherent cohomology. ($\mathcal{O}_{X\otimes_V \overline{V}}$ is still coherent). Only slightly less trivial is the extension to complexes $M^\bullet$ such that for any $\alpha > 0$ there exists a bounded complex $M^\bullet_\alpha$ with coherent cohomology, and maps $M^\bullet_\alpha \xrightarrow{\beta_\alpha} M^\bullet \xrightarrow{\lambda_\alpha} M^\bullet_\alpha$ with $\beta_\alpha \circ \lambda_\alpha = \pi^\alpha \cdot \mathrm{id}$, $\lambda_\alpha \circ \beta_\alpha = \pi^\alpha \cdot \mathrm{id}$. There duality holds up to almost quasi-isomorphisms (the induced maps on cohomology are almost isomorphic).

1. Lemma. — *A complex $M^\bullet$ with bounded cohomology satisfies these conditions (i.e. $\underline{M}^\bullet$ is π^α-isomorphic to a coherent $M^\bullet_\alpha$, for each $\alpha > 0$) if and only if the direct sum $\oplus \underline{H}^i(M^\bullet)$ is almost quasicoherent and locally (over each affine* $\mathrm{Spec}(R)$) *almost finitely presented.*

Proof. — The non-trivial direction is local $\Rightarrow$ global. Assume first that $M^\bullet$ is given by an almost quasicoherent sheaf M. Replacing M by $\mathfrak{m}M/\mathfrak{m}$-torsion we can assume that M is an honest quasicoherent sheaf. Then M is the filtering inductive limit of coherent sheaves $M_i, M = \varinjlim M_i$. For given $\alpha > 0$ and i big enough the cokernel of $M_i \to M$ is annihilated by π^α. We thus may replace M by the image of M_i, as this also satisfies the local condition (the coherent $R \otimes_V \overline{V}$-modules form an abelian category, and so do the almost finitely presented modules). Also the kernel N of $M_i \to M$ satisfies the local condition. Applying the same argument we may assume that it is also annihilated by π^α. Then $M_i \to M$ is an $\pi^{2\alpha}$-isomorphism.

For general M choose i minimal with $H^i(M) \neq (0)$. Then M is an extension (i.e. one has an exact triangle)

$$0 \longrightarrow H^i(M)[-i] \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

which up to isomorphism is determined by a class $c \in H^1(\mathbb{R}\,\mathrm{Hom}(M', H^i(M))[-i])$. Choose a π^α-approximation $M'_\alpha \xrightarrow{\beta'_\alpha} M' \xrightarrow{\lambda'_\alpha} M'_\alpha$ (by induction). By pullback with β'_α we get a class

$$c_\alpha \in H^1(\mathbb{R}\,\mathrm{Hom}(M'_\alpha, N))(N = H^i(M)[-i]),$$

defining

$$0 \longrightarrow N \longrightarrow M_\alpha \longrightarrow M'_\alpha \longrightarrow 0$$

and a map $M_\alpha \to M$. Further pullback (via λ'_α) to $H^1(\mathbb{R}\,\mathrm{Hom}(M', N))$ gives $\pi^\alpha \cdot c$, which also can be obtained from c via pushforward on N by $\pi^\alpha \cdot \mathrm{id}$. We thus obtain a map $M \to M_\alpha$ inducing λ'_α on M' and $\pi^\alpha \cdot \mathrm{id}$ on N. The two compositions of

$$M \longrightarrow M_\alpha \longrightarrow M$$

differ from $\pi^\alpha \cdot \mathrm{id}$ by maps $M' \to N$ or $M'_\alpha \to N$ (respectively), which are nullhomotopic for degree-reasons. Thus M'_α approximates M up to π^α. With the same type of argument we can replace N by an approximation N_α.

We want to apply this theory to complexes made up from $\mathbb{R}\Gamma(\Delta, \mathbb{L}\otimes\overline{R})'s$. For this we first have to define an appropriate topos.

For a scheme or algebraic space X we denote by $X^{\text{ét}}$ the étale topos. Furthermore $X^{\text{ét,lc}}$ denotes the topos defined by finite étale covers of X. If X is connected and x a geometric point, $X^{\text{ét,lc}}$ is equivalent to the topos of sets with continuous action of $\pi_1(X, x)$ (SGAI). In any case for noetherian X, $X^{\text{ét,lc}}$ is isomorphic to the topos of étale sheaves which are inductive limits of locally constant sheaves. Also there is a continuous $\rho_X : X^{\text{ét}} \to X^{\text{ét,lc}}$ with $\rho_{X,*}(\mathcal{F})$ = restriction of $\mathcal{F}$ to finite étale covers, $\rho_X^*(\mathbb{L}) = \mathbb{L}$, for $\mathbb{L}$ locally constant on X.

Next assume given $X \to \operatorname{Spec}(V)$, and an open subset $X^0 \subseteq X \otimes_V K$. Define a site whose objects consists of pairs (U, V), where $U \to X$ is étale and

$$V \longrightarrow (U \cap X^0) \otimes_K \overline{K}$$

is a finite étale covering. As maps we use compatible pairs of maps, and coverings are just pairs of surjective maps. Let $\mathfrak{X}^0_{\overline{K}}$ denote the corresponding topos. Its sheaves consist of ind-locally constant sheaves $\mathbb{L}_U$ on $(U \cap X^0) \otimes_K \overline{K}$, for each U as above, together with transition maps (for $U' \to U$) satisfying the usual cocycle condition. Furthermore for surjective $f : U' \twoheadrightarrow U$ $\mathbb{L}_U$ has to be the universal ind-locally constant sheaf mapping to the equaliser of

$$f_*\mathbb{L}_U \rightrightarrows f_*(\mathbb{L}_{U' \times_U U'}).$$

Equivalently sections $\mathbb{L}_U(V)$ can be glued for étale coverings of V induced from U. There exist a continuous

$$u_X : (X^0 \otimes_V \overline{K})^{\text{ét}} \longrightarrow \mathfrak{X}^0_{\overline{K}},$$

with

$$u_{X,*}(\mathcal{F})_U = \rho_{U,*}(\mathcal{F} \mid (U \cap X^0) \otimes_V \overline{K}).$$

The topos $\mathfrak{X}^0_{\overline{K}}$ has enough points. Namely for any geometric point x of X evaluation on the universal cover of an irreducible component of $\operatorname{Spec}(\mathcal{O}^{\text{sh}}_{X,x} \otimes_V \overline{K})$ is the fibre functor for such a point, and all of them are of this type. This is easily checked, starting with the fact that each point maps to a point in $X^{\text{ét}}$. However we only need that the points above form a faithful family of fibre functors. Also as affines are quasicompact cohomology will commute with filtering inductive limits.

Finally assume that $X \otimes_V K = \coprod X_\alpha$ is stratified, étale locally like by a simple normal crossings-divisor. The open strata X_α are smooth, and we define $c(\alpha)$ as the codimension of X_α. Also the normalization $\overline{X}_\alpha^{\text{norm}}$ of the the closure $\overline{X}_\alpha$ is smooth over K.

On this normalization there exists a locally constant étale (orientation) sheaf $\mathbb{O}_\alpha$, with stalks isomorphic to $\mathbb{Z}$, and structure-group $\{\pm 1\}$. If locally étale X_α is the intersection of $c(\alpha)$ divisors, each total ordering of these divisors defines a generator of $\mathbb{O}_\alpha$. Two such orderings differ by a permutation, and the two corresponding local generators of $\mathbb{O}_\alpha$ by the sign of this permutation. The sheaf $\mathbb{O}_\alpha$ is canonically trivial if $c(\alpha) = 1$, or if the closure $\overline{X}_\alpha$ is globally the transverse intersection of smooth divisors (which are strata). For example this holds if our stratification has simple normal crossings.

Define an order on indices by

$$\overline{X}_\alpha - X_\alpha = \coprod_{\beta > \alpha} X_\beta.$$

If $j_\alpha : X_\alpha \longrightarrow X$ denotes the inclusion, it is known that for $\mathbb{L}_\alpha$ locally constant on X_α and $\beta > \alpha$, the restrictions $j_\beta^* \mathbb{R}^\nu j_{\alpha,*}\mathbb{L}_\alpha$ are locally constant on X_β. This follows from Abhyankhar's lemma (and characteristic zero). Furthermore, if $\beta > \alpha$ and $c(\beta) - c(\alpha) = 1$, on the preimage of $\overline{X}_\beta$ the sheaves $\mathbb{O}_\alpha$ and the pullback of $\mathbb{O}_\beta$ are canonically isomorphic. Namely locally one adds an extra equation, which can be put in front of all others in an étale local ordering. It follows that in this case

$$j_\beta^* j_{\alpha,*}(\mathbb{O}_\alpha \otimes \mathbb{L}_\alpha) = \mathbb{O}_\beta \otimes j_\beta^* j_{\alpha,*}(\mathbb{L}_\alpha),$$

for any locally constant $\mathbb{L}_\alpha$ on X_α.

Now define a topos $\mathfrak{X}_{\overline{K}}$ as follows:

Sheaves in $\mathfrak{X}_{\overline{K}}$ associate to each U locally constant sheaves $\mathbb{L}_{U,\alpha}$ on $U_\alpha \otimes_K \overline{K}$, with pullback-maps (for $U' \to U$) as before and maps $\mathbb{L}_{U,\beta} \to j_\beta^* j_{\alpha,*}\mathbb{L}_{U,\alpha}$ for $\beta > \alpha$ satisfying transitivity for $\gamma > \beta > \alpha$ ($j_\gamma^* \circ j_{\alpha,*} = j_\gamma^* \circ j_{\beta,*} \circ j_\beta^* \circ j_{\alpha,*}$, by local calculation) and compatibility with pullbacks. Finally for fixed α the $\mathbb{L}_{U,\alpha}$ have to satisfy the previous gluing-condition for coverings $U' \twoheadrightarrow U$.

There are continuous maps of topoi

$$J_\alpha : \mathfrak{X}^0_{\alpha,\overline{K}} \longrightarrow \mathfrak{X}_{\overline{K}},$$

with inverse image J_α^* defined by chosing the α-component. The direct image $J_{\alpha,*}\mathbb{L}_\alpha$ has β-component $j_\beta^* j_{\alpha,*}(\mathbb{L}_\alpha)$.

Also we have a continuous $u_X : (X \otimes_V \overline{K})^{\text{ét}} \to \mathfrak{X}_{\overline{K}}$, with direct image $u_{X,*}(\mathcal{F})$ defined as follows: For minimal $\alpha, u_{X,*}(\mathcal{F})_{U,\alpha}$ is the restriction of $\mathcal{F}$ to finite étale covers of $U_\alpha \otimes_K \overline{K}$. If $u_{X,*}(\mathcal{F})_{U,\alpha}$ is already defined for all $\alpha < \beta$, then $u_{X,*}(\mathcal{F})_{U,\beta}$ is the universal (inductive limit of) locally constant sheaves fitting into a commutative diagram

$$\begin{tikzcd}
u_{X,*}(\mathcal{F})_{U,\beta} \arrow[r] \arrow[d] & \mathcal{F} \mid U_\beta \otimes_V \overline{K} \arrow[d] \\
\prod'_{\alpha<\beta} j_\beta^* j_{\alpha,*}(u_{X,*}(\mathcal{F})_{U,\alpha}) \arrow[r] & \prod_{\alpha<\beta} j_\beta^* j_{\alpha,*}(\mathcal{F} \mid U_\alpha \otimes_V \overline{K})
\end{tikzcd}$$

(That is form the fibred product as sheaves and apply $\rho_{\mathcal{U}_\beta,*}$) Here in the lower-keft corner $\prod'_{\alpha<\beta}$ denotes the subset of the product consisting of compatible families (for

$\widetilde{\alpha} < \alpha < \beta$ the α-components maps to the $\widetilde{\alpha}$-component via

$$j_\beta^* j_{\alpha,*}(u_{X,*}(\mathcal{F})_{U,\alpha}) \longrightarrow j_\beta^* j_{\alpha,*} j_\alpha^* j_{\widetilde{\alpha},*}(u_{X,*}(\mathcal{F})_{U,\widetilde{\alpha}}) = j_\beta^* j_{\widetilde{\alpha},*}(u_{X,*}(\mathcal{F})_{U,\widetilde{\alpha}})$$

One checks without much difficulty that this defines indeed a map of topoi. Also if $\mathcal{F}$ is such that all restrictions $\mathcal{F}_\alpha = \mathcal{F} \mid X_\alpha \otimes_K \overline{K}$ are locally constant, then $u_{X,*}(\mathcal{F})_{U,\alpha} = \mathcal{F} \mid U_\alpha \otimes_K \overline{K}$. Finally there is an obvious commutative diagram of maps

$$\begin{tikzcd} (X \otimes_V \overline{K})^{\text{ét}} \arrow[r] \arrow[dr] & \mathfrak{X}_{\overline{K}} \arrow[d, "v_X"] \\ & X^{\text{ét}} \end{tikzcd}$$

The direct image for the vertical map sends a collection $(\mathbb{L}_{U,\alpha})$ to the sheaf which associates to an étale $U \to X$ compatible families of sections in $\prod_\alpha \Gamma(U_\alpha, \mathbb{L}_{U,\alpha})$. The idea behind this definition of $\mathfrak{X}_{\overline{K}}$ is to consider the topos of étale sheaves on $X_{\overline{K}}$ which are (ind-) locally constant on each stratum, and add étale localization in X (to be able to reduce to étale local calculations). In principle we need such a construction to define cohomology with compact support, because it uses the extension by zero. However purists will note that this topos could be avoided by slight modification of the next construction.

We shall need another model for extensions by zero. For $\alpha < \beta$ define a functor $\Psi_{\alpha,\beta}$ on sheaves on $\mathfrak{X}^0_{\alpha,\overline{K}}$ (and taking values in them) by

$$\Psi_{\alpha,\beta}(\mathbb{L})_U(V) = \Gamma(\overline{V}_\beta, \mathbb{O}_\beta \otimes j_\beta^* j_{\alpha,*}(\mathbb{O}_\alpha^\vee \otimes \mathbb{L}_U \mid V)$$

Here $\overline{V}$ denotes the normalization of $U \otimes_V \overline{K}$ in $V \to U_\alpha \otimes_V \overline{K}$, $\overline{V}_\beta$ its β-stratum (the preimage of U_β) and $\mathbb{L}_U \mid V$ the pullback of $\mathbb{L}_U$ to V). One checks without difficulty that this satisfies the sheaf property with respect to étale coverings of U. Also for $\alpha > \beta > \gamma$ there are transition maps

$$\Psi_{\alpha,\beta}(\mathbb{L}) \longrightarrow \Psi_{\alpha,\gamma}(\mathbb{L})$$

satisfying the obvious commutativity for $\alpha < \beta < \gamma_1, \gamma_2 < \delta$.

To proceed further we need a number of local assumptions.

Local Conditions LC. — Let $R = \mathcal{O}^{\text{sh}}_{X,x}$ denote the strict henselization of X in a point x. We assume:

a) The closures of the strata on $\operatorname{Spec}(R \otimes_V \overline{K})$ are normal subvarieties of $\operatorname{Spec}(R \otimes_V \overline{V})$, defining a stratification (again indexed by $\alpha, \beta, \dots$)

b) For each α, the inclusion of the α-stratum admits a retraction

$$r_\alpha : \operatorname{Spec}(R \otimes_V \overline{V}) \longrightarrow \overline{\operatorname{Spec}(R \otimes_V \overline{V})_\alpha},$$

sending any β-stratum with $\beta < \alpha$ into the (open) α-stratum

c) There exists $l_1, \dots, l_r \in R$ such that the stratification on $\operatorname{Spec}(R \otimes_V \overline{K})$ is defined by the ideals generated by subsets of $\{l_1, \dots, l_r\}$

2. Lemma. — *Suppose LC holds. Then*

i) *The $\Psi_{\alpha,\beta}$ are exact*

ii) *If $\mathbb{L}$ is an injective sheaf on $\mathfrak{X}^0_{\alpha,\overline{K}}$, then all $\Psi_{\alpha,\beta}(\mathbb{L})$ are acyclic for the direct image $v_{X^0,*}$ (to $X^{\text{ét}}$). Also all $j^*_\beta j_{\alpha,*}(\mathbb{L})$ are acyclic for $v_{X_\beta} : \mathfrak{X}^0_\beta \otimes_V \overline{K} \to X^{\text{ét}}$, and the direct images $R^\nu J_{\alpha,*}\mathbb{L}$ are defined by the functors $j^*_\beta R^\nu j_{\alpha,*}$.*

Proof. — The assertions are local in $X^{\text{ét}}$. By an inductive limit argument we may replace $\mathfrak{X}$ by $\operatorname{Spec}(R), R = O^{\text{sh}}_{X,x}$ as before. Then $R \otimes_V \overline{V}$ is a finite product of strictly henselian local rings. We thus may assume that it is local (eventually replace K by a finite extension). Then by a) $\operatorname{Spec}(R \otimes_V \overline{V})$ is stratified with all closed strata normal and thus irreducible. Let Δ_α denote the fundamental group of the α-stratum. For each $\beta > \alpha$ we obtain subgroups $I_\beta \subseteq D_\beta \subseteq \Delta_\alpha$, well-determined up to conjugation, namely the inertia and decomposition-group of the β-stratum. If this is defined by (say) $l_1 = \cdots = l_t = 0$, then $I_\beta \cong \widehat{\mathbb{Z}}(1)^t$, via the coverings obtained by adjoining roots of l_i. Furthermore there is a canonical map $\Delta_\beta \to D_\beta/I_\beta$ (Δ_β- fundamental group of β-stratum) which is an isomorphism by b). Now any locally constant sheaf $\mathbb{L}$ on the generic stratum corresponds to a continuous representation of Δ. Then one checks that $\Psi_{0,\beta}(\mathbb{L})$ is defined by

$$\operatorname{Ind}^{\Delta}_{D_\beta}(\mathbb{L}) = \text{continuous } D_\beta\text{-linear maps } \Delta \to \mathbb{L}.$$

This is an exact functor and implies the first assertion (over proper subsets $U \subsetneqq \operatorname{Spec}(R)$ one has to localize further).

For ii) we need to compute direct images for $v_{X_\alpha} : \mathfrak{X}_\alpha \otimes_V \overline{K} \to X^{\text{ét}}$. Again the fibre in $x \in X$ is the cohomology of $\operatorname{Spec}(R \otimes_V \overline{V})$, that is the cohomology of the topos of locally constant sheaves on the α-stratum. (This topos is a retract of the full topos and has the same cohomology). This coincides with the group-cohomology $H^*(\Delta, \dots)$. However (using inclusions $\Delta_\alpha \supseteq D_{\alpha,\beta} \supseteq I_{\alpha,\beta} \cong \widehat{\mathbb{Z}}(1)^t$)

$$H^*(\Delta_\alpha, \operatorname{Ind}^{\Delta_\alpha}_{D_{\alpha,\beta}} \mathbb{L}) = H^*(D_{\alpha,\beta}, \mathbb{L})$$

vanishes in positive degrees for injective $\mathbb{L}'s$. Similar for the cohomology of $j^*_\beta j_{\alpha,*}\mathbb{L}$, which corresponds to $\mathbb{L}^{I_\beta}$.

Now we come to one of the main results. For a sheaf $\mathbb{L}$ on $\mathfrak{X}^0_{\overline{K}}$ define a complex $\Psi(\mathbb{L})$ (of sheaves on $\mathfrak{X}^0_{\overline{K}}$)

$$\Psi(\mathbb{L}) : 0 \longrightarrow \mathbb{L} \longrightarrow \bigoplus_{c(\beta)=1} \Psi_{0,\beta}(\mathbb{L}) \longrightarrow \bigoplus_{c(\beta)=2} \Psi_{0,\beta}(\mathbb{L}) \longrightarrow \dots$$

the maps being induced from local isomorphisms on $\mathbb{O}'_\beta s$. Étale locally all $\mathbb{O}_\beta$ can be trivialised, and they become sums of restrictions with $\pm$ signs.

On the other hand we have a sheaf $j_{0,!}\mathbb{L}$ on $\mathfrak{X}_{\overline{K}}$, which is $\mathbb{L}$ on $\mathfrak{X}^0_{\overline{K}}$ and trivial on all other strata.

3. Proposition. — $\mathbb{R}v_{X,*}(\Psi(\mathbb{L})) \cong \mathbb{R}v_{X,*}(j_{0,!}\mathbb{L})$ *canonically in the category of étale sheaves on X.*

Proof. — There exists a canonical map, thus the assertion becomes (étale) local, and we may assume that all $\mathbb{O}_\beta$ are trivial. Choose an injective resolution $\mathbb{L} \to I^\bullet$ in $\mathfrak{X}^0_{\overline{K}}$. Then $\Psi(I^\bullet)$ is $v_{X^0,*}$-acyclic, thus $\mathbb{R}v_{X^0,*}(\Psi(\mathbb{L}))$ is represented by $v_{X^0,*}(\Psi(I^\bullet))$. This associated to $U \to X$ étale the complex of global sections

$$\Gamma((U \cap X_0) \otimes_V \overline{K}, \Psi(I^\bullet)),$$

i.e. the double complex made up from

$$\bigoplus_{c(\beta)=a} \Gamma((U \cap X_0) \otimes_V \overline{K}, \Psi_{0,\beta}(I^b)) = \bigoplus_{c(\beta)=a} \Gamma((U \cap X_\beta) \otimes_V \overline{K}, \mathbb{O}_\beta \otimes j^*_\beta j_{0,*}(I^b))$$

This is the direct image of the total complex associated to a double complex on $\mathfrak{X} \otimes_V \overline{K}$ with components

$$\bigoplus_{c(\beta)=a} J_{\beta,*}(\mathbb{O}_\beta \otimes j^*_\beta j_{0,*}(I^b)).$$

The summands represent the derived direct image and are acyclic for $v_{X,*}$, as the sheaves $j^*_\beta j_{0,*}(I^b)$ are acyclic for J_β and v_{X_β}.

On the 0-stratum it is quasiisomorphic to $\mathbb{L}$. We claim that it is acyclic on all other strata:

This is a local assertion, thus we can trivialise all $\mathbb{O}'_\beta s$. Then on the α-stratum its components are

$$\bigoplus_{\substack{\beta \leqslant \alpha \\ c(\beta)=a}} j^*_\alpha j_{\beta,*} j^*_\beta j_0(I^b) = \bigoplus_{\substack{\beta \leqslant \alpha \\ c(\beta)=a}} j^*_\alpha j_{0,*}(I^b)$$

By one of the standard calculations in homological algebra it then follows that for $\operatorname{codim}(\alpha) > 0$ this forms an acyclic complex. Thus our complex represents $j_{0,!}(\mathbb{L})$ and its direct image represents $v_{X,*}(j_{0,!}(\mathbb{L}))$. As the resolution $I^\bullet$ is unique up to homotopy, this construction is also canonical (in the derived category) and functorial.

We leave it to the reader to formulate a variant where one extends $\mathbb{L}$ by zero along some of the codimension-one strata and by $\mathbb{R}_{j_0,*}(\mathbb{L})$ along the others. Of course one then uses only $\Psi'_{0,\beta}s$ with the β-stratum an intersection of the codimension-one strata of the first type.

Finally we apply our theory to the previous case of schemes formally étale over the torus-embedding $\overline{T}_\lambda \subseteq \overline{T}$. We have to check the local conditions (LC): By assumption $\sigma \cap \ker(\lambda)$ is spanned by a partial basis $\{\rho_1, \ldots, \rho_t\}$ of the lattice L. The strata are indexed by subsets $I \subseteq \{1, \ldots, t\}$, and the ideal of the I-stratum is spanned by those $\mu \in L^\vee \cap \sigma^\vee$ which are strictly positive on some $\rho_i, i \in I$. If L_I denotes the quotient-lattice $L_I = l/\langle \rho_i \mid i \in I \rangle$ and $\sigma_I \subseteq L_{I,\mathbb{R}}$ the image of σ, then σ_I defines a torus-embedding $T_I \subseteq \overline{T}_I$. Furthermore there is a projection $\overline{T} \to \overline{T}_I$ which has partial inverse given by an isomorphism $\overline{T}_I \xrightarrow{\sim} I$-stratum $\subseteq \overline{T}$. The latter map is given on affine rings by sending $\mu \in L^\vee \cap \sigma^\vee$ to either $\mu \in L^\vee_I \cap \sigma^\vee$ or to zero (if $\mu \notin L^\vee_I$, that is $\mu(\rho_i) > 0$ for some $i \in I$). This picture persists after base change to $\overline{T}_\lambda$ (dividing rings by the ideal $(\pi - \lambda)$), and also to the strictly henselian local ring R. Furthermore by ajoining a suitable root $\pi^{1/e}$ we can achieve that all strata in $X \otimes_V \overline{V}$ are normal. Thus we have conditions a) and b).

Next we define a sheaf $\overline{\mathcal{O}}$ on $\mathfrak{X}^0 \otimes_V \overline{K}$ as follows: To U and $V \to U_{\overline{K}}$ finite étale associate the global sections of the normalization $\mathcal{O}_V^{\mathrm{norm}}$ of $\mathcal{O}_U$ in V. We also want to define a sheaf $A_{\mathrm{inf}}(\overline{\mathcal{O}}_V)$, but for this we need that the Frobenius is surjective on $\overline{\mathcal{O}}/p \cdot \overline{\mathcal{O}}$. In fact we have a slightly stronger statement.

4. Lemma. — *Suppose $U = \mathrm{Spec}(R) \to \overline{T}_\lambda$ is étale over $\overline{T}_\lambda$. Then Frobenius is surjective on $\overline{R}/p \cdot \overline{R}$.*

Proof. — Frobenius is surjective on the affine ring of the curve $\overline{T}_{\lambda,\infty}$, and thus also on $R_\infty/p \cdot R_\infty$ by étaleness. Then for any almost étale cover S of R_∞ it is almost surjective on S/pS (by Frobenius-invariance of almost étale coverings) and thus also almost surjective on $\overline{R}/p \cdot \overline{R}$. Thus if $x \in \overline{R}$ there is a y with $p^{\frac{1}{2}} \cdot x \equiv y^p \mod p$. Then y is divisible by $p^{\frac{1}{2p}}$ and x is a p-th power modn $p^{1/2}$. Repeating we see that it is a p-th power mod p.

A little bit more involved is

5. Lemma. — *Suppose $a \in \overline{V}$ divides p^α for $\alpha < 1$. Then the map $z \mapsto z^p - az$ is surjective on $\overline{\mathcal{O}}/p \cdot \overline{\mathcal{O}}$.*

Proof. — It suffices to check local rings. So let $R = \mathcal{O}_{X,x}^{\mathrm{sh}} \subseteq \overline{R}$. Obviously we may assume that p lies in the maximal ideal of R and thus also in the Jacobson-radical of $\overline{R}$. For $y \in \overline{R}$ consider

$$S = \overline{R}[z]/(z^p - az - y).$$

It suffices to show that $S[1/p]$ is étale over $\overline{R}[1/p]$, or that $z^p - az - y$ and $pz^{p-1} - a$ have no common zero in any extension-field L of $\overline{R}[1/p]$. But if not, then

$$z = \sqrt[p-1]{a/p}, y = \sqrt[p-1]{a/p}\,\Big(\frac{a}{p} - a\Big).$$

Thus

$$y^{p-1}(p/a)^p = (1 - pa)^p$$
$$0 = (1 - pa)^p - y^{p-1}(p/a)^p$$

However the right hand side lies in $\overline{R}$ and is $\equiv 1 \mod p^\alpha$ (some $\alpha > 0$), thus is a unit.

We also need to consider the kernel of the endomorphism $z^p - az$ of $\overline{R}/p\overline{R}$. Assume now that $a = b^{p-1}$ with b^p dividing p^α, $\alpha < 1$. Then

$$z^p - b^{p-1}z \in p\overline{R}$$

if and only if

$$\Big(\frac{z}{b}\Big)^p - \Big(\frac{z}{b}\Big) \in \frac{p}{b^p} \cdot \overline{R}$$

This happens if and only if locally in the Zariski — or étale topology

$$z \in \mathbb{Z}_p \cdot b + b \cdot \frac{p}{b^p} \cdot \overline{R}$$

Thus

$$\ker(z^p - b^{p-1}z) = \mathbb{F}_p \cdot b \oplus \big(\frac{p}{b^{p-1}}\big)\overline{R}/p\overline{R} \text{ (as sheaves).}$$

We also need cohomology with values in the projective limit $\mathcal{R}(\overline{\mathcal{O}}) = \varprojlim \overline{\mathcal{O}}/p\overline{\mathcal{O}}$ (transition maps are Frobenius). This is a "topological" sheaf. However we do not need continuous cohomology. Projective system $\mathcal{F} = \{\mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \ldots\}$ indexed by integers form a new topos. The functor of global sections maps this to a projective system of abelian groups, whose projective limit is $H^0(\mathcal{F})$. Deriving this functor gives higher cohomology. The derived functor of (projective systems $\mathcal{F}_n$) $\to$ (projective systems of abelian groups) can be computed termwise. Finally the derived functors of $\varprojlim_{\mathbb{N}}$ are wellknown: There is only a non-zero $\varprojlim_{\mathbb{N}}^{(1)}$, which vanishes if the projective system satisfies a Mittag-Leffler-condition. By Leray we have short exact sequences

$$0 \longrightarrow \varprojlim{}^{(1)} H^{i-1}(\mathcal{F}_n) \longrightarrow H^i(\mathcal{F}) \longrightarrow \varprojlim H^i(\mathcal{F}_n) \longrightarrow 0.$$

Thus we have defined the cohomology of $\mathcal{R}(\overline{\mathcal{O}}) = \varprojlim \overline{\mathcal{O}}/p \cdot \overline{\mathcal{O}}$ as well as direct images $R^a v_{X,*}(\mathcal{R}(\overline{\mathcal{O}}))$ (which are projective systems on $X^{\text{ét}}$). Also choose any $a \in \mathcal{R}(\overline{V})$, $a \neq 0$. Then a can be represented by a system of elements $a_n \in \widehat{\overline{V}}$, $a_n = a_{n+1}^p$. Also $a = b^{p-1}$ for some b. Then for n big enough the $a_n' s$ and $b_n' s$ satisfy the condition in Lemma. Thus the sequence

$$0 \longrightarrow (\mathbb{F}_p)_s \oplus \big(\frac{p}{b_n^{p-1}}\overline{\mathcal{O}}\big)/p\overline{\mathcal{O}} \longrightarrow \overline{\mathcal{O}}/p\overline{\mathcal{O}} \xrightarrow{\Phi - a} \overline{\mathcal{O}}/p\overline{\mathcal{O}} \longrightarrow 0$$

(Φ = Frobenius) is exact.

Passing to the limit we obtain a sequence

$$0 \longrightarrow (\mathbb{F}_p)_s \longrightarrow \mathcal{R}(\overline{\mathcal{O}}) \xrightarrow{\Phi - a} \mathcal{R}(\overline{\mathcal{O}}) \longrightarrow 0$$

whose cohomology has trivial transition-maps for $n \gg 0$, and thus will disappear if we take cohomology. Here $(\mathbb{F}_p)_s$ stands for the restriction of the sheaf $\mathbb{F}_p$ to the special fibre. Namely its fibre at a geometric point is trivial in characteristic zero, and coincides with $\mathbb{F}_p$ in characteristic p. Here recall that geometric points are given by choice of a geometric point x of X, and then evaluation on the maximal étale cover of $\operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}} \otimes_V K)^0$. Equivalently the "generic fibre" defines an open subtopos of $\mathfrak{X}_{\overline{K}}^0$ (the corresponding site consists of objects with trivial special fibre), and $(\mathbb{F}_p)_s$ is the restriction of $\mathbb{F}_p$ to the complement. Also for any sheaf $\mathbb{L}$ on $\mathfrak{X}_{\overline{K}}^0$ the direct images under the map v_X to the étale topos of X commute with restriction to the special fibre:

This follows by computing stalks in geometric points of X. Namely if Δ_x denotes the fundamental group of an irreducible component of $\operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}} \otimes_V \overline{K})^0$, the stalks in

x of these direct images are (inductive limits of direct sums of copies of) the profinite group-cohomology of Δ_x, with values in the stalk upstairs.

As by the proper base-change theorem the cohomology of any torsion-sheaf on $X^{\text{ét}}$ is equal to that of its special fibre, it follows that also the cohomology of $\mathbb{L}$ and $(\mathbb{L})_s$ coincide.

In the same vein we can pass to $A_{\text{inf}}(\overline{\mathcal{O}}) = W(\mathcal{R}(\overline{\mathcal{O}}))$, which a priori is a double projective limit but can be written as a single projective limit. However we really only need the $A_{\text{inf}}(\overline{\mathcal{O}})/p^s A_{\text{inf}}(\overline{\mathcal{O}}) = W_s(\mathcal{R}(\overline{\mathcal{O}}))$. They fit into a sequence

$$0 \longrightarrow (\mathbb{Z}/p^s\mathbb{Z})_s \longrightarrow W_s(\mathcal{R}(\overline{\mathcal{O}})) \xrightarrow{\Phi-[a]} W_s(\mathcal{R}(\overline{\mathcal{O}})) \longrightarrow 0$$

with again essentially trivial cohomology. Finally we tensor with $u_{X,*}(\mathbb{L})$, $\mathbb{L}$ a locally constant $\mathbb{Z}/p^s\mathbb{Z}$-sheaf on $(X^0 \otimes_V \overline{K})^{\text{ét}}$, to obtain

$$0 \longrightarrow u_{X,*}(\mathbb{L})_s \longrightarrow u_{X,*}(\mathbb{L}) \otimes A_{\text{inf}}(\overline{\mathcal{O}}) \xrightarrow{\Phi-[a]} u_{X,*}(\mathbb{L}) \otimes A_{\text{inf}}(\overline{\mathcal{O}}) \longrightarrow 0.$$

As a consequence we obtain exact sequences

$$\begin{aligned} 0 \longrightarrow \operatorname{coker}(\Phi - [a] \mid H^{i-1}(\mathfrak{X}^0 \otimes_V \overline{K}, u_X^*(\mathbb{L}) \otimes A_{\text{inf}}(\overline{\mathcal{O}})) &\longrightarrow H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \\ \longrightarrow \ker(\Phi - [a] \mid H^i(\mathfrak{X}^0 \otimes_V \overline{K}, u_{X,*}(\mathbb{L}) \otimes A_{\text{inf}}(\overline{\mathcal{O}}))) &\longrightarrow 0 \end{aligned}$$

We also need a variant for cohomology with compact support. For each stratum α we have sheaves $\overline{\mathcal{O}}_\alpha$ on $\mathfrak{X}^0_\alpha \otimes_V \overline{K}$ as before. For $\alpha < \beta$ we define a sheaf $\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)$ on $\mathfrak{X}^0_\alpha \otimes_V \overline{K}$ as follows:

Suppose we have $U \to X$ étale and $V \to U_\alpha \otimes_V \overline{K}$ finite étale, with normalization $V^{\text{norm}} \to \overline{U}_\alpha \otimes_V \overline{K}$. It is stratified with reduced strata obtained by taking closures of generic fibres. We define $\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)$ as the normalization of $\mathcal{O}_U$ in V^{norm}_β, except that we have to twist as before with $\mathbb{O}_{alpha}$ and $\mathbb{O}_\beta$. This twisting goes as follows:

First $\mathbb{O}_\alpha$ defines a covering (of degree two) of $U_\alpha \otimes_V K$) and by pullback of V, which in turn extends to a ramified cover of V^{norm}. On its β-stratum we repeat this construction with $\mathbb{O}_\beta$ and obtain a new covering, with action by two copies of the group $\{\pm 1\}$. We now perform the previous normalization, and our global sections are the (-,-)-eigenspace under the group-action.

If $U = \operatorname{Spec}(R)$ is affine, all $\mathbb{O}_\alpha$ are trivial, and $S =$ normalization of R in V, then there are finitely many primes $\mathfrak{p}_\nu \subseteq S$ corresponding to the irreducible components of V^{norm}_β. $\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)$ is then the direct sum of the normalizations of $(S/\mathfrak{p}_\nu)$. $(S/\mathfrak{p}_\nu[1/\pi]$ is already normal).

Also for $\beta < \gamma$ there are morphisms

$$\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha) \longrightarrow \widetilde{\psi}_{\alpha,\gamma}(\overline{\mathcal{O}}_\alpha)$$

satisfying some transitivity for $\alpha < \beta < \gamma_1, \gamma_2 < \delta$.

6. Lemma

i) *Frobenius Φ is surjective on $\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)/p \cdot \widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)$*

ii) *For $b \in \overline{V}$ with b^p dividing p^α, $\alpha < 1$, $\Phi - b$ is surjective on*

$$\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha)/p \cdot \widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha).$$

Proof. — It suffices to check these over $\mathrm{Spec}(R)$, $R = \mathcal{O}^{\mathrm{sh}}_{X,x}$. We may assume $\alpha = 0$. Then $\widetilde{\psi}_{\alpha,\beta}(\overline{R})$ the subset of the product $\prod(\overline{R}/\mathfrak{p}_\nu)^{\mathrm{norm}}$ where

$$\Delta = \pi_1(\mathrm{Spec}(R \otimes_V \overline{K}))$$

acts continuously. The product is over all minimal primes of the β-stratum. Equivalently it is the induced module (via $D_\beta \subseteq \Delta$) from one $(\overline{R}/\mathfrak{p}_\nu)^{\mathrm{norm}}$, which by LC coincides with the version of $\overline{R}$ for the β-stratum. There assertions i) and ii) are already known. In fact for i) we can prove surjectivity of Frobenius on global section over sufficiently small affines, reducing to R_∞ and checking there directly. We now have the obvious complexes

$$\widetilde{\psi}(\overline{\mathcal{O}}_\alpha) : \overline{\mathcal{O}}_\alpha \longrightarrow \oplus_{c(\beta)=1}\widetilde{\psi}_{\alpha,\beta}(\overline{\mathcal{O}}_\alpha) \longrightarrow \oplus_{c(\beta)=2}\widetilde{\psi}_{\alpha,\beta}\ (\overline{\mathcal{O}}_\alpha) \longrightarrow \cdots$$

$\mathcal{R}(\widetilde{\psi}(\overline{\mathcal{O}}_\alpha))$ and $A_{\mathrm{inf}}(\widetilde{\psi}(\overline{\mathcal{O}}_\alpha))$. For any $\mathbb{L}$ locally constant $\mathbb{L}$ on $(X^0 \otimes_V \overline{K})^{\text{ét}}$, annihilated by p^s, we obtain a complex of projective systems

$$0 \longrightarrow \psi(u_{X,*}(\mathbb{L}))_s \longrightarrow u_{X,*}(\mathbb{L}) \otimes A_{\mathrm{inf}}(\widetilde{\psi}(\overline{\mathcal{O}})) \xrightarrow{\Phi-[a]} u_{X,*}(\mathbb{L}) \otimes A_{\mathrm{inf}}(\widetilde{\psi}(\overline{\mathcal{O}}))) \longrightarrow 0$$

with irrelevant cohomology, and thus short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{coker}(\Phi - [a] \mid H^{i-1}(\mathfrak{X}^0 \otimes_V \overline{K}, u_{X,*}(\mathbb{L}) \otimes A_{\mathrm{inf}}(\widetilde{\psi}(\overline{\mathcal{O}}))) &\longrightarrow H^i_!(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \\ \longrightarrow \ker(\Phi - [a] \mid H^i(\mathfrak{X}^0 \otimes_V \overline{K}, u_{X,*}(\mathbb{L}) \otimes A_{\mathrm{inf}}(\widetilde{\psi}(\overline{\mathcal{O}}))) &\longrightarrow 0 \end{aligned}$$

There are variants with partial compact support at infinity, with obvious formulations and proofs.

Finally we get back to "almost mathematics". There is also a subsheaf $\overline{\mathcal{J}} \subseteq \overline{\mathcal{O}}$, $\overline{\mathcal{J}} = \ker(\overline{\mathcal{O}} \to \bigoplus_{\mathrm{codim}\,\beta=1} \widetilde{\psi}_{0,\beta}\ (\overline{\mathcal{O}}))$.

7. Lemma. — *$\overline{\mathcal{J}} \approx \widetilde{\psi}(\overline{\mathcal{O}})$ is an almost isomorphism.*

Proof. — Again we may work with $R = \mathcal{O}^{\mathrm{sh}}_{X,x}, \mathrm{Spec}(R) \to \overline{T}_\lambda$. The assertion holds for $J_\infty \subseteq R_\infty$:

Here each stratum is irreducible, with ideal generated by the $\mu \in \bigcup_n L_n^\vee$ with $\mu(\rho_i) > 0$ for some $i \in I =$ index-set of the stratum. For a given μ the μ-eigenspace in the complex $\widetilde{\psi}(R_\infty)$ is a Čech-complex over the free module generated by those ρ_i with $\lambda(\rho_i) = \mu(\rho_i) = 0$. By direct calculation we have an isomorphism.

After that $\overline{R} \supseteq R_\infty$ is the limit of almost étale coverings. Assume we have such a covering $S \supseteq R_\infty$ with group G. Then if $\mathfrak{q}_\alpha \subseteq S$ is the prime ideal giving an irreducible component of the α-stratum, $D_\alpha \subseteq G$ its decomposition-group, and $\mathfrak{p}_\alpha = \mathfrak{q}_\alpha \cap R_\infty \subseteq R_\infty$, we have a map

$$R_\infty/\mathfrak{p}_\alpha \otimes_{R_\infty} S \longrightarrow \mathrm{Ind}_D^G((S/\mathfrak{q}_\alpha)^{\mathrm{norm}}),$$

which is an isomorphism after inverting π. As the left side is almost étale over $R_\infty/\mathfrak{p}_\alpha$ and thus has non-degenerate trace-form, the map must be an almost isomorphism. Passing to the limit over all S we obtain that $\widetilde{\psi}(\overline{R})$ is almost isomorphic to $\overline{J}_\infty \otimes_{R_\infty} \overline{R} \approx \overline{J}$.

If we define

$$\mathcal{R}(\overline{\mathcal{J}}) = \varprojlim \overline{\mathcal{J}}/p\overline{\mathcal{J}} = \ker(\mathcal{R}(\overline{\mathcal{O}}) \longrightarrow \mathcal{R}(\overline{\mathcal{O}}/\overline{\mathcal{J}}))$$

and similarly $A_{\inf}(\overline{\mathcal{J}}) = W(\mathcal{R}(\overline{\mathcal{J}}))$, we get almost isomorphisms

$$H^*(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\inf}(\overline{\mathcal{J}})) \approx H^*(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\inf}(\widetilde{\psi}(\overline{\mathcal{O}}))).$$

Next consider the direct images $R^\nu q_{X,*}(\overline{\mathcal{J}}/p^s\overline{\mathcal{J}})$ which are sheaves on $X^{\text{ét}}$. By Th. 2.10.i) there are almost zero unless $0 \leqslant \nu \leqslant d$, are almost quasi-isomorphic to bounded complexes of quasicoherent sheaves, and there is a canonical almost map

$$R^d q_{X,*}(\overline{\mathcal{J}}/p^s\overline{\mathcal{J}}) \longrightarrow \omega_X \otimes \overline{V}/p^s\overline{V},$$

inducing almost quasi-isomorphisms

$$\mathbb{R}q_{X,*}(\mathbb{L} \otimes \overline{\mathcal{J}}) \approx \mathbb{R}\operatorname{Hom}_{\mathcal{O}_X \otimes \overline{V}/p^s\overline{V}}(\mathbb{R}q_{X,*}(\mathbb{L}^\vee \otimes \overline{\mathcal{O}}), \omega_X \otimes \overline{V}/p^s\overline{V})[-d]$$

for $\mathbb{L}$ locally constant annihilated by p^s. Furthermore there is a canonical trace-map

$$H^d(X^{\text{ét}}, \omega_X) = H^d(X, \omega_X) \longrightarrow V$$

inducing global duality for coherent sheaves on X. As the direct images $\mathbb{R}q_{X,*}(\mathbb{L} \otimes \overline{\mathcal{J}})$ etc. are almost coherent these combine to produce almost quasi-isomorphisms

$$\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{\mathcal{J}}) \approx \mathbb{R}\operatorname{Hom}_{\overline{V}/p^s\overline{V}}(\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{\mathcal{O}}), \overline{V}/p^s\overline{V})[-2d]$$

Remark. — Previously we always worked with $\frac{1}{\pi}\omega_X$ instead of ω_X. This is because in the torus-embedding we identified the dualizing complex for the quotient $V[L^\vee \cap \sigma^\vee]/(\lambda - \pi)$ as the tensor-product of that for $V[L^\vee \cap \sigma^\vee]$ with the quotient, instead of the Ext^1. Then we used the trace-map for $V[L^\vee \cap \sigma^\vee]$. This differs from the trace-map used in relative duality by a factor π. That this is so can be seen easily for the affine torus-embedding $V[\lambda]$, and follows in the general case by functoriality $V[\lambda] \subseteq V[L^\vee \cap \sigma^\vee]$. It is also related to the fact that we have tacitly identified $\Lambda^{d+1}L^\vee$ and $\Lambda^d \ker(\lambda)^\vee$.

The structure of $H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\inf}(\overline{\mathcal{O}}))$ is surprisingly simple. Assume for the moment that $\mathbb{L}$ is annihilated by p. Then this cohomology is a module over $\mathcal{R}(\overline{V})$.

8. Theorem. — *For each i, $H^i(\mathfrak{X}^0 \otimes_V, \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{J}_I))$ is almost isomorphic to a finitely generated free $\mathcal{R}(\overline{V})$-module, namely to $H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \otimes \mathcal{R}(\overline{V})$ (support-condition given by I).*

Proof. — Let $M = H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{J}_I))$. Then for $\xi = \underline{p} \in \mathcal{R}(\overline{V})$,

$$M/\xi M \subseteq H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{J}_I),$$

which is almost finitely presented over $\mathcal{R}(\overline{V})/\xi \cdot \mathcal{R}(\overline{V}) = \overline{V}/p\overline{V}$. Also Frobenius Φ induces an isomorphism on M. Now for torsion $\overline{V}$-modules N we can define a normalised length $\lambda(N)$ similarly as before: Write $V = \cup V_\alpha$ with V_α finite over V. Then for $N = \overline{V} \otimes_{V_\alpha} N_\alpha$

$$\lambda(N) = \frac{1}{[V_\alpha : V]} \text{length}_V(N_\alpha),$$

which defines $\lambda(N)$ for N finitely presented. For N only finitely generated, $N = \overline{V} \cdot N_\alpha$, we have

$$\lambda(N) = \lim_{\beta \to \infty} \frac{1}{[V_\beta : V]} \text{length}_V(V_\beta \cdot N_\alpha),$$

and for arbitrary N

$$\lambda(N) = \sup\{\lambda(N') \mid N' \subseteq N \text{ finitely generated}\}.$$

Also $\lambda(\overline{V}/\mathfrak{a}) = \inf\{v(x) \mid x \in \mathfrak{a}\}$, where v is the valuation normalised such that $v(\pi) = 1$. Finally for a submodule N of a finitely presented $\overline{V}$-module, $\lambda(N) = 0$ implies $N = (0)$.

By additivity we extend λ to ξ-torsion $\mathcal{R}(\overline{V})$-modules. Pushout by Frobenius multiplies λ by p. Next for each $\alpha > 0$

$$\xi^\alpha \cdot H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{\mathcal{J}}_I)$$

has finite λ-invariant, and by devissage this holds for $\xi^\alpha \cdot H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^n)$, all n. Applying Frobenius-pushout to this module multiplies λ by p, and produces

$$\xi^{p\alpha} H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^{np}).$$

On the other hand filter $\mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^{np}$ by submodules $\xi^{in}(\mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I))/\xi^{np}$, apply the exact cohomology-sequences and use that for an exact sequence

$$N_1 \longrightarrow N_2 \longrightarrow N_3$$

we have

$$\lambda(\xi^{\alpha+\beta} N_2) \leqslant \lambda(\xi^\alpha N_1) + \lambda(\xi^\beta N_3),$$

with equality for all $\alpha, \beta > 0$ only possible if

$$\lambda(\xi^\alpha \cdot \ker(N_1 \to N_2)) = \lambda(\xi^\beta \cdot \text{coker}(N_2 \to N_3)) = 0, \quad \text{all} \quad \alpha, \beta > 0.$$

It then follows that for $m \geqslant 0$ the exact sequences

$$H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi) \longrightarrow H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^{m+1}) \\ \longrightarrow H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^m)$$

satisfy this condition. If (say) N denotes the kernel of the first map, for each $\alpha, \beta > 0$ multiplication by ξ^β on $\xi^\alpha N$ factors as $\xi^\alpha N \twoheadrightarrow N_1 \to \xi^\alpha N$ with N_1 submodule of a finitely presented $\overline{V}/p\overline{V}$-module. As

$$\lambda(N_1) \leqslant \lambda(\xi^\alpha \cdot N) = 0,$$

we have $N_1 = (0)$, and $N \approx 0$. Quite similarly (or from the long exact sequence in cohomology) the cokernel is almost zero. Especially the projective system of $H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^m)$ has almost surjective transition-maps, and $\lim^{(1)}$ of it is almost zero. Also if

$$M^i = H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)),$$

then

$$H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^m) \approx M^i/\xi^m \cdot M^i,$$

and $M = \varprojlim_m M/\xi^m M$:

If

$$z \in H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)/\xi^m),$$

we show that for each $\alpha > 0$ $\xi^\alpha \cdot z$ lifts to M. For this choose a strictly decreasing sequence

$$\{\alpha_n, n \geqslant m\}, \quad 0 < \alpha_{n+1} < \alpha_n < \alpha = \alpha_m.$$

Then choose

$$z_n \in H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{J}}_I)(\xi^n)$$

with $z_m = z$, z_{n+1} lifting $\xi^{\alpha_n - \alpha_{n+1}} \cdot z_n$. Then $\xi^\alpha \cdot z$ is lifted by $\varprojlim(\xi^{\alpha_n} \cdot z_n)$.

Also it follows that multiplication by ξ defines an almost isomorphism

$$\xi^n M^i/\xi^{n+1} M^i \approx \xi^{n+1} M^i/\xi^{n+2} M^i,$$

so ξ is almost injective on M^i. Next if $M^i/\xi M^i$ is not almost zero, choose an element $m \in M^i$. m is not almost zero in $M^i/\xi M^i$. Multiplication by m defines a map

$$\xi \cdot \mathcal{R}(\overline{V}) \cong \mathcal{R}(\overline{V}) \longrightarrow M^i,$$

injective as m is not ξ^l-torsion for any l. Consider α, $0 \leqslant \alpha \leqslant 1$, such that this map extends to $\xi^\alpha \cdot \mathcal{R}(\overline{V})$ (this is not possible for $\alpha < 0$, as $m \notin \xi \cdot M^i$). Such an extension is almost unique. Thus if $\underline{\alpha}$ is the infimum of these α's, the map extends uniquely to the union

$$\mathfrak{a}(\underline{\alpha}) = \bigcup_{\alpha < \underline{\alpha}} \xi^\alpha \cdot \mathcal{R}(\overline{V}).$$

Hence we get an injection $\mathfrak{a}(\underline{\alpha}) \subseteq M^i$, with quotient $\widetilde{M^i} = M^i/\mathfrak{a}(\underline{\alpha})$ again almost without ξ-torsion. If $\widetilde{M^i}$ is not almost zero we can repeat the construction to obtain an increasing sequences of submodules of M^i, with subquotients $\mathfrak{a}(\underline{\alpha}_\nu)$, and almost ξ-torsion-free quotients. Especially

$$\lambda(\xi \cdot M/\xi^2 \cdot M) \geqslant \sum_\nu \lambda(\xi \cdot \mathfrak{a}(\underline{\alpha}_\nu)/\xi^2 \cdot \mathfrak{a}(\underline{\alpha}_\nu)) = \text{number of } \underline{\alpha}_\nu\text{'s},$$

so the process must stop. Hence we have found a finite filtration

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_r = M = M^i,$$

with $M_n/M_{n-1} \approx \mathfrak{a}(\underline{\alpha}_n)$.

It then follows easily that for each $\alpha > 0$ there exists a free submodule $\mathcal{R}(\overline{V})^r \subseteq M$ with quotient annihilated by ξ^α. Especially M is almost projective of finite rank r over $\mathcal{R}(\overline{V})$. Next define an $\mathcal{R}(\overline{V})$-algebra

$$\mathcal{A} = S[M]/(\Phi(m) - m^p)$$

(symmetric algebra divided by the relations displayed). Obviously $\mathcal{A}[\xi^{-1}]$ is finite flat over $\mathcal{R}(\overline{V})[\xi^{-1}]$ of rank p^r, generated by monomials with exponents $< p$ in a basis of $M[\xi^{-1}] = \mathcal{R}(\overline{V})[\xi^{-1}]^r$. Also as the free submodules $N \cong \mathcal{R}(\overline{V})^r \subseteq M$ form a filtering inductive system with union almost M, we obtain (taking such monomials in a basis of N) a filtering inductive system of free submodules of $\mathcal{A}$ (of rank p^r) with union almost $\mathcal{A}$. Thus $\mathcal{A}$ is almost projective of finite rank p^r over $\mathcal{R}(\overline{V})$, and almost finitely generated. Also assume given such a free submodule with basis $\{m_1, \ldots, m_r\}$, quotient annihilated by ξ^α, and let

$$\xi^\alpha \cdot \Phi(m_i) = \sum_j a_{ij} \cdot m_j$$

with $a_{ij} \in \mathcal{R}(\overline{V})$. Then the matrix (a_{ij}) has an inverse up to ξ^β for any $\beta > (p+1)\alpha$, as $\xi^\beta \cdot m_i \in \xi^\alpha \cdot \Phi(\xi^\alpha \cdot M)$. Also all monomials with exponents $< p$ in the $\xi^{\alpha/p-1} \cdot m_i = T_i$ form an subalgebra $\mathcal{A}_\alpha$ isomorphic to

$$\mathcal{R}(\overline{V})[T_1, \ldots, T_r]/(T_i^p - \sum_j a_{ij} \cdot T_j).$$

From ramification-theory it follows that the trace-form identifies the dual of $\mathcal{A}_\alpha$ with $\det(a_{ij})^{-1} \cdot \mathcal{A}_\alpha$. Especially $\mathcal{A}[1/\xi] = \mathcal{A}_\alpha[1/\xi]$ is étale over $\mathcal{R}(\overline{V})[1/\xi]$, and the corresponding canonical idempotent $e \in \mathcal{A}_\alpha \otimes_{\mathcal{R}(\overline{V})} \mathcal{A}_\alpha[1/\xi]$ has denominator $\det(a_{ij})$ which divides $\xi^{r\beta}$ for any $\beta > (r+1)\alpha$. Letting $\alpha \to 0$ we see that the canonical idempotent in $\mathcal{A} \otimes_{\mathcal{R}(\overline{V})} \mathcal{A}[1/\xi]$ is almost integral, thus $\mathcal{A}$ is almost étale over $\mathcal{R}(\overline{V})$.

Hence we can almost lift $\mathcal{A}$ to an almost étale cover of $\mathcal{A}_{\inf}(\overline{V})$ and its quotient $\widehat{\overline{V}}$. However $\widehat{\overline{V}}[1/p]$ is algebraically closed, so this cover is almost trivial. It follows that $\mathcal{A} \approx \mathcal{R}(\overline{V})^{p^r}$. We thus obtain p^r different (almost) homomorphisms of $\mathcal{A}$ into $\mathcal{R}(\overline{V})$, corresponding to (almost) maps $M \to \mathcal{R}(\overline{V})$ which are Frobenius linear. These maps almost span the (module) dual of $\mathcal{A}$ and thus also of M (check with free submodules). Now the $\mathbb{F}_p$-vectorspace $\mathrm{Hom}^{\mathrm{almost}}_{\Phi,\mathcal{R}(\overline{V})}(M, \mathcal{R}(\overline{V}))$ has p^r elements and thus dimension r. A basis of it defines an almost map $M \to \mathcal{R}(\overline{V})^r$ inducing an almost surjection on duals, and thus an almost isomorphism. Also the canonical $\mathbb{F}_p$-structure on $\mathcal{R}(\overline{V})^r$ can be recovered as the space of Φ-invariants.

Next replace M by

$$H^i(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \widetilde{\psi}(\mathcal{R}(\overline{\mathcal{O}}))) = \widetilde{M}^i,$$

which is almost the same. Then we have the exact sequences ($b \in \mathcal{R}(\overline{V})$ as before)

$$0 \longrightarrow \mathrm{coker}(\Phi - b^{p-1} \mid \widetilde{M}^{i-1}) \longrightarrow H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow \ker(\Phi - b^{p-1} \mid \widetilde{M}_i) \longrightarrow 0.$$

But $\widetilde{M}^i \approx \mathcal{R}(\overline{V})^{r_i}$ (Φ-linear). Thus $\Phi - b^{p-1}$ is almost surjective on $\widetilde{M}^{i-1}$. Now the above sequence comes (with tensor-products etc.) from the first line in

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{F}_p \arrow[r, "b\cdot"] \arrow[d, equal] & \mathcal{R}(\overline{\mathcal{O}}) \arrow[r, "\Phi - b^{p-1}"] \arrow[d, "b'"] & \mathcal{R}(\overline{\mathcal{O}}) \arrow[r] \arrow[d, "b'^p"] & 0 \\
0 \arrow[r] & \mathbb{F}_p \arrow[r, "b\cdot b'"] & \mathcal{R}(\overline{\mathcal{O}}) \arrow[r, "\Phi - (bb')^{p-1}"] & \mathcal{R}(\overline{\mathcal{O}}) \arrow[r] & 0
\end{tikzcd}$$

Extend this as shown with $b' \in \mathcal{R}(\overline{V}), b' \neq 0$, unit, to get

$$\begin{tikzcd}
\operatorname{coker}(\Phi - b^{p-1})/\widetilde{M}^{i-1}) \arrow[r] \arrow[d, "b'^p\cdot"] & H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \arrow[r] \arrow[d, equal] & \ker(\Phi - b^{p-1}/\widetilde{M}^i) \arrow[r] \arrow[d, "b'"] & 0 \\
\operatorname{coker}(\Phi - (bb')^{p-1}/\widetilde{M}^{i-1}) \arrow[r] & H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \arrow[r] & \ker(\phi - (bb')^{p-1}/\widetilde{M}^i) \arrow[r] & 0
\end{tikzcd}$$

The first vertical arrow is zero, thus $H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L})$ injects into $\widetilde{M}_i$ for all b (we may vary b, b'). Also multiplication by b respectively b' applied to $\widetilde{M} \approx \mathcal{R}(\overline{V})^{r_i}$ induces maps

$$\begin{aligned}
\mathbb{F}_p^{r_i} = \Phi\text{-invariants in } \mathcal{R}(\overline{V})^{r_i} & \\
&\longrightarrow \ker(\Phi - b^{p-1}/\widetilde{M}^i) \\
&\longrightarrow \ker(\Phi - (bb')^{p-1}/\mathcal{R}(\overline{V})^{r_i}) = b\cdot \mathbb{F}_p^{r_i},
\end{aligned}$$

and one derives that indeed

$$\widetilde{M}^i \approx H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \otimes \mathcal{R}(\overline{V}).$$

Corollary 1. — *For any $\mathbb{L}$ (only annihilated by p^s)*

$$\begin{aligned}
&\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\text{inf}}(\overline{\mathcal{J}}_I)) \approx \mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \otimes A_{\text{inf}}(\overline{V}) \\
&\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes \overline{K}, \mathbb{L} \otimes \overline{\mathcal{J}}_I) \approx \mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, L) \otimes \overline{V}
\end{aligned}$$

Proof. — Dévissage in $\mathbb{L}$.

Corollary 2. — $\mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L})$ *satisfies Künneth.*

Proof. — Descente from $\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{\mathcal{J}}_I)$ (for $\overline{V}\otimes$ (a $\mathbb{Z}/p^s\mathbb{Z}$-module), "almost zero" implies zero).

Also the map $u_X : (X^0 \otimes_V \overline{K})^{\text{ét}} \to \mathfrak{X}^0 \otimes_V \overline{K}$ and its variants induce a map

$$\mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}).$$

Especially the trace-form on $H^{2d}_!((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z}(d))$ (if X has pure relative dimension d) induces a trace-form $\text{tr}^{\text{ét}}$ on $H^{2d}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}(d))$. We shall show later that it coincides with the restriction of the trace-form on $H^{2d}(\mathfrak{X}^0 \otimes_V \overline{K}, \overline{\mathcal{J}}/p^s\overline{\mathcal{J}}(d))$. Hence it induces by descente a perfect duality.

Corollary 3. — *The étale trace induces a perfect duality*

$$\mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma_{(!^c)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}^\vee)(d)[2d] \longrightarrow \mathbb{Z}/p^s\mathbb{Z}.$$

4. Chern-classes and Gysin-maps

Suppose $\mathcal{L}$ is a line-bundle on X. We define its first Chern-Class $c_1(\mathcal{L})$ in $H^2(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}(1))$ as follows: $\mathcal{L}$ is given by a class in

$$H^1(X, O_X^*) = H^1(X^{\text{ét}}, O_X^*).$$

Pullback via $v_X^* \mathcal{O}_X^* \to \overline{\mathcal{O}}^*$ gives a class in $H^1(\mathfrak{X} \otimes_V \overline{K}, \overline{\mathcal{O}}^*)$, and via the Kummer-sequence

$$0 \longrightarrow \mu_{p^s} \longrightarrow \overline{\mathcal{O}}^* \longrightarrow \overline{\mathcal{O}}^* \longrightarrow 0$$

we obtain a class in $H^2(\mathfrak{X} \otimes_V \overline{K}, \mu_{p^s})$. It is clear that pullback to $H^2((X \otimes_V \overline{K})^{\text{ét}}, \mu_{p^s})$ maps it to the étale first Chern-class of $\mathcal{L} \mid X \otimes_V \overline{K}$.

Next we consider a closed immersion $i : Z \to X$ such that locally in the étale topology

$$X \cong X_1 \times \mathbb{A}^1, Z \cong X_1 \times \{0\},$$

with X_1 having toroidal singularities, stratifications etc. as before. We intend to give a geometric description of the Gysin-map i_*. We define a new stratification on X by adding Z as well as its intersections with X_α's to the strata, and call the new topos $\widetilde{X}$. Thus $\widetilde{X}^0 = X^0 - Z^0$. Next there are maps of topoi

$$j : (\widetilde{\mathfrak{X}})^0 \otimes_V \overline{K} \longrightarrow \mathfrak{X}^0 \otimes_V \overline{K}$$
$$w : \widetilde{\mathfrak{X}} \otimes_V \overline{K} \longrightarrow \mathfrak{X} \otimes_V \overline{K}$$
$$i : \mathcal{Z} \otimes_V \overline{K} \longrightarrow \widetilde{\mathfrak{X}} \otimes_V \overline{K}$$

Also there are variants $i^{\text{ét}}, j^{\text{ét}}$ for the étale topos on the generic fibre. Now a sheaf $\mathcal{F}$ on $(X \otimes_V \overline{K})^{\text{ét}}$ which is locally constant on all strata induces $\mathbb{L} = u_{X,*}(\mathcal{F})$ on $\mathfrak{X} \otimes_V \overline{K}$ and $\widetilde{\mathbb{L}} = w^*(\mathbb{L})$ on $\widetilde{\mathfrak{X}} \otimes_V \overline{K}$. Furthermore on $(X \otimes_V \overline{K})^{\text{ét}}$ there is an exact sequence (or better an exact triangle)

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathbb{R}j_*^{\text{ét}} j^{\text{ét},*}\mathcal{F} \longrightarrow i_*^{\text{ét}}(\mathcal{F}/Z)(-1)[-1] \longrightarrow 0,$$

whose $\mathbb{R}u_{X,*}$ receives a map from

$$0 \longrightarrow \widetilde{\mathbb{L}} \longrightarrow \mathbb{R}j_*j^*\widetilde{\mathbb{L}} \longrightarrow \tau_{\geqslant 1}\mathbb{R}j_*j^*\widetilde{\mathbb{L}} \longrightarrow 0.$$

Furthermore there exists a canonical map (inertia at Z)

$$\mathbb{R}^1 j_* j^* \widetilde{\mathbb{L}} \longrightarrow i_*(\widetilde{\mathbb{L}}|Z)(-1)$$

which we shall show to be an isomorphism. Finally the higher direct images $\mathbb{R}^n j_* j^* \widetilde{\mathbb{L}}$ vanish for $n \geqslant 2$. This also applies to higher ($n \geqslant 1$) direct images $\mathbb{R}^n i_*(\widetilde{\mathbb{L}}|Z)$: The stalks are given by cohomology of profinite groups, and i induces an injection on local fundamental groups because it has étale local sections.

Hence by pullback we obtain an exact triangle

$$0 \longrightarrow \widetilde{\mathbb{L}} \longrightarrow \mathbb{R}j_* j^* \widetilde{\mathbb{L}} \longrightarrow i_*(\widetilde{\mathbb{L}}|Z)(-1)[-1] \longrightarrow 0$$

mapping to the two previous ones.

This is useful because it is wellknown that the Gysin-homomorphism in étale topology is defined by the connecting map

$$\mathbb{R}\Gamma((Z^0 \otimes_V \overline{K})^{\text{ét}}, \mathcal{F}(1))[-2] \longrightarrow \mathbb{R}\Gamma((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathcal{F})$$

from the first triangle, and similar for variants with (partial) compact support.

Sketch. — Taking cup-product with classes in $\mathbb{R}\Gamma_!((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathcal{F}^\vee)$ we reduce to constant coefficients and cohomology in highest degrees. Then the assertion follows from the explicit description of the cohomology-class of a point in Z^0.

Hence the second triangle defines the Gysin-homomorphism for $i : \mathcal{Z} \otimes_V \overline{K} \to \widetilde{\mathfrak{X}} \otimes_V \overline{K}$, and allows us to compare it to the first.

Finally we come to the analogous results with coefficients $\mathbb{L} \otimes \overline{\mathcal{O}}$. In fact we prove them first, and derive from them the assertion about discrete coefficients.

1. Proposition. — *Suppose I denotes support-conditions (at divisors transversal to the embedding i). Then on $\mathfrak{X}^0_{\overline{K}}$ there is an almost exact sequence (better triangle)*

$$0 \longrightarrow \mathbb{L} \otimes \overline{\mathcal{J}}_{I,X} \longrightarrow \mathbb{R}j_*(\mathbb{L} \otimes \overline{\mathcal{J}}_{I,\widetilde{X}}) \longrightarrow i_*(\mathbb{L} \otimes \overline{\mathcal{J}}_{I,Z})(-1)[-1] \longrightarrow 0.$$

Explanation of maps, and beginning of proof. — The maps can be described as follows:

i) The first is induced from

$$\overline{\mathcal{J}}_{I,X}/p^s \cdot \overline{\mathcal{J}}_{I,X} \longrightarrow j_* \overline{\mathcal{J}}_{I,\widetilde{X}}/p^s \overline{\mathcal{J}}_{I,\widetilde{X}}.$$

We claim that this is an almost isomorphism. Tensoring with $\mathbb{L}$ then gives an almost isomorphism of H^0's.

ii) Let ι denote the stratum Z in $\widetilde{\mathfrak{X}}$. Then there is a map

$$\mathbb{R}j_*(\mathbb{L} \otimes \overline{\mathcal{J}}_{I,\widetilde{X}})[1] \longrightarrow \mathbb{R}^1 j_*(\mathbb{L} \otimes \widetilde{\psi}_{0\iota}(\overline{\mathcal{J}}_{I,\widetilde{X}}))$$

Furthermore we claim that

$$j_* \widetilde{\psi}_{0,\iota}(\overline{\mathcal{J}}_{I,\widetilde{X}}) = i_*(\overline{\mathcal{J}}_{I,Z})$$

$$\mathbb{R}^1 j_* \mathbb{Z}/p^s\mathbb{Z}(1) = i_*(\widetilde{\mathbb{L}}/Z),$$

and that the cup-product with negative of the special class in $\mathbb{R}^1 j_* \mathbb{Z}/p^s\mathbb{Z}(1)$ defined by p-power roots of a local equation for Z gives an almost isomorphism

$$j_* \widetilde{\psi}_{0,\iota}(\overline{\mathcal{J}}_{I,\widetilde{X}})(-1) \longrightarrow \mathbb{R}^1 j_*(\overline{\mathcal{J}}_{I,Y}/p^s \overline{\mathcal{J}}_{I,Y})$$

Again this persists after tensoring with $\mathbb{L}$. The negative sign is related to the fact that the characteristic class of Z is the first Chern class of the line bundle $\mathcal{O}(Z)$ which is locally generated by the inverse of a local equation for Z.

iii) The higher direct images

$$\mathbb{R}^{\nu} j_* \overline{\mathcal{J}}_{I,\widetilde{X}} / p^s \overline{\mathcal{J}}_{I,\widetilde{X}} \approx 0 \text{ for } \nu \geqslant 2,$$

and also after tensoring with $\mathbb{L}$.

The assertions in i), ii), iii) imply almost exactness of the sequence in the proposition. From that we shall derive

iv) In addition

$$\mathbb{R}^1 j_* \mathbb{Z}/p^s\mathbb{Z}(1) = i_*(\widetilde{\mathbb{L}}/Z)(-1),$$

and all higher cohomology vanishes.

Thus we can construct the exact sequences from the beginning and obtain compatability of Gysin-maps.

Now to the required assertions. They are easily reduced to constant coefficients, as $\mathbb{L}$ becomes constant over $\overline{\mathcal{O}}$. That $j_*(\widetilde{\psi}_{0,\iota}(\overline{\mathcal{J}}_{I,\widetilde{X}})) = i_*(\mathcal{J}_{I,Z})$ follows from the definitions:

To evaluate the left hand side on a finite étale covering

$$V \longrightarrow \mathcal{U}^0 \otimes_V \overline{K} \quad (\mathcal{U} \subseteq X$$

étale) we first restrict to the preimage of $X \smallsetminus Z$, then extend by normalization to recover V, next restrict to the preimage of $Z \otimes_V \overline{K}$, and then form global sections of the normalization. But this gives the right hand side.

For the remaining assertions it suffices to check them on stalks

$$\operatorname{Spec}(R), \ R = \mathcal{O}_{X,x}^{\mathrm{sh}}.$$

Let $T \in R$ define the stratum Y. Extending V we may assume that $R \otimes_V \overline{V}$ is local and normal. Consider the normalizations $\overline{R}$ and $\overline{\overline{R}}$ of R in the maximal étale cover of $\operatorname{Spec}(R)^0 \otimes_V \overline{K}$ respectively of the complement of Z in it. By the theory of almost étale extension $\overline{R}$ and $\overline{\overline{R}}$ are limits of almost étale covers of R_∞ respectively $\widetilde{R}_\infty$, which in turn are induced by toroidal coverings. As locally in the étale topology $X \cong X_1 \times \mathbb{A}^1$ we may use for $\widetilde{R}_\infty$ the toroidal covering induced from toroidal coverings of X_1 and from adjoining roots of T, and for R_∞ we use the toroidal covering induced from X_1 as well as from p-power roots of a suitable unit, say of $1+T$. Especially $\overline{\overline{R}}$ is almost étale over the subextension obtained by adjoining to $\overline{R}$ all roots of T (and normalizing the result). Now it Δ and $\widetilde{\Delta}$ denote the relevant fundamental groups, and $\underset{\sim}{\Delta} \subseteq \widetilde{\Delta}$ the kernel of the surjection to Δ, we have (say without support conditions) to compute the cohomology $H^*(\underset{\sim}{\Delta}, \overline{\overline{R}}/p^s\overline{\overline{R}})$: For the first three assertions we need that

i) $H^0(\underset{\sim}{\Delta}, \overline{\overline{R}}/p^s\overline{\overline{R}}) \approx \overline{R}/p^s\overline{R}$

ii) $H^1(\underset{\sim}{\Delta}, \overline{\overline{R}}/p^s\overline{\overline{R}}) \approx (\overline{R}/T\overline{R})^{\mathrm{norm}} \otimes \mathbb{Z}/p^s\mathbb{Z}(-1)$

iii) $H^\nu(\underset{\sim}{\Delta}, \overline{\overline{R}}/p^s\overline{\overline{R}}) \approx (0)$ for $\nu \geqslant 2$.

By the theory of almost étale extensions we may first replace $\overline{R}$ by R_∞ and $\overline{\overline{R}}$ by the subextension corresponding to the quotient $\underset{\sim}{\Delta} \to \widehat{\mathbb{Z}}(1)$, i.e. where we adjoin roots of T. By Künneth it suffices to treat $d = 1$ (no factor X_1). At this stage also the support-condition disappears. Furthermore as groups of order prime to p have trivial cohomology, we reduce to the quotient $\underset{\sim}{\Delta} \to \mathbb{Z}_p(1)$. Let $S_\infty \subseteq \overline{\overline{R}}$ denote the corresponding normalization,

$$S_\infty/\overline{\overline{J}}_\infty \approx (R_\infty/T \cdot R_\infty)^{\mathrm{norm}} \cong \mathrm{Hom}_{cont}(\mathbb{Z}_p(1), \overline{V})$$

The assertions we have to show are

i) $H^0(\widehat{\mathbb{Z}}(1), S_\infty/p^s \cdot S_\infty) \approx R_\infty/p^s \cdot R_\infty$

ii) $H^1(\widehat{\mathbb{Z}}(1), S_\infty/p^s \cdot S_\infty) \approx (R_\infty/TR_\infty)^{\mathrm{norm}} \otimes \mathbb{Z}/p^s\mathbb{Z}(-1)$

iii) $H^\nu(\widehat{\mathbb{Z}}(1), S_\infty/p^s \cdot S_\infty) = 0$ for $\nu > 1$.

Here iii) is clear because if cohomological dimension one. Also we may replace $\widehat{\mathbb{Z}}(1)$ by the quotient $\mathbb{Z}_p(1)$, as the kernel $\prod_{l \neq p} \mathbb{Z}_l(1)$ has trivial cohomology. This amounts to changing S_∞ and $\widetilde{R}_\infty$ by adjoining only p-power roots of T. We do this, but retain the notation for simplicity. Now write $S_\infty = \bigcup_n S_n$, with

$$S_n = R_\infty[T^{1/p^n}]^{\mathrm{norm}}$$

Note that $S_n[1/T]$ is almost étale over $R_\infty[1/T]$, by the purity theorem. Also R_∞ is the inductive limit of regular rings

$$R_m = V_m[(1+T)^{p^{-n}}]^{\mathrm{sh}},$$

and thus $S_n = \bigcup_m S_{n,m}$ with

$$S_{n,m} = R_m[T^{p^{-n}}]^{\mathrm{norm}}$$

a free R_n-module of rank $q = p^n$. The group $\Delta_n = \mathbb{Z}/p^n\mathbb{Z}(1)$ operates on $S_{n,m}$. We have $H^0(\Delta_n, S_{n,m}) = R_m$. For the higher cohomology consider the infinite complex

$$K^\bullet_{n,m} : \cdots \longrightarrow S_{n,m} \xrightarrow{(\sigma-1)} S_{n,m} \xrightarrow{\mathrm{tr}} S_{n,m} \xrightarrow{(\sigma-1)} S_{n,m} \xrightarrow{\mathrm{tr}} S_{n,m} \longrightarrow \cdots$$

where $\mathrm{tr} = \sum_{i=0}^{q-1} \sigma^i$.

Its cohomology $H^i(K_{n,m}) = \widetilde{H}^i(\Delta_n, S_{n,m})$ is called the Tate-cohomology ($i \in \mathbb{Z}$). It coincides for $i > 0$ with $H^i(\Delta_n, S_{n,m})$, while

$$\widetilde{H}^0(\Delta_n, S_{n,m}) = R_m/\,\mathrm{tr}(S_{n,m}).$$

Obviously $\widetilde{H}^i$ is periodic with period two. If $S^\vee_{n,m} = \mathrm{Hom}_{R_m}(S_{n,m}, R_m)$, we can similarly compute $\widetilde{H}^i(\Delta_n, S^\vee_{n,m})$ from the version of $K_{n,m}$ where $S_{n,m}$ is replaced by duals. However this is isomorphic to $(K^\bullet_{n,m})^\vee[1]$. As R_m has finite cohomological dimension two we thus obtain a spectral sequence

$$E_2^{a,b} = \mathrm{Ext}^a_{R_m}(\widetilde{H}^{-b}(\Delta_n, S^\vee_{n,m}), R_m) \Rightarrow \widetilde{H}^{a+b+1}(\Delta_n, S_{n,m})$$

with $E_2^{a,b} = 0$ unless $0 \leqslant a \leqslant 2$.

Now we pass to the limit $m \to \infty$. The trace-form defines embeddings $S_{n,m} \hookrightarrow S_{n,m}^{\vee}$. If we localise at height one primes $\mathfrak{p}$ of R_m these are determined by the different of the extension. It is a unit except if $\mathfrak{p}$ divides p or T. However at the prime-divisors of P the p-valuation of the different converges to zero because $S_n[1/T]$ is almost étale over $R_\infty[1/T]$. For $\mathfrak{p}$ dividing T the different is generated by $T^{(q-1)/q}$. It thus follows that

$$T^{(q-1)/q} \cdot S_{n,m} \subset S_{n,m}^{\vee},$$

with the quotient annihilated by p^{α_m}, $\lim\limits_{m \to \infty} \alpha_m = 0$. It also follows that p^{α_m} annihilates $S_n / S_{n,m} \otimes_{R_m} R_\infty$. Hence we obtain:

a) S_n is almost projective finitely generated $/R_\infty$

b) The trace map factors

$$\mathrm{tr} : J_n = T^{1/q} \cdot S_n \longrightarrow T \cdot R_\infty,$$

and $\frac{1}{T} \cdot \mathrm{tr}$ induces an almost isomorphism

$$J_n \approx \mathrm{Hom}_{R_\infty}(S_n, R_\infty)$$

Especially $\mathrm{tr}(J_n) \approx T \cdot R_\infty$.

c) The $\widetilde{H}^i(\Delta_n, S_n)$ are almost finitely presented. There is a spectral sequence (up to almost zero modules)

$$E_2^{a,b} = \mathrm{Ext}^a_{R_\infty}(\widetilde{H}^{-b}(\Delta_n, J_n), R_\infty) \Rightarrow \widetilde{H}^{a+b+1}(\Delta_n, S_n)$$

with $E_2^{a.b} \approx 0$ unless $0 \leqslant a \leqslant 2$.

((c) follows because $S_{n,m} \otimes_{R_m} R_\infty \to S_n$ is a p^{α_n}-isomorphismus).
Now $\widetilde{H}^*(\Delta_n, J_n)$ is annihilated by p^n as well as by $\mathrm{tr}(S_n)$ which almost contains T. As (p^n, T) form a regular sequence in R_∞, $\mathrm{Ext}^a_{R_\infty}(M, R_\infty) = (0)$ for $a \leqslant 1$ if M is annihilated by p^n and T. Thus the spectral sequence almost degenerates to give (using period two)

$$\widetilde{H}^{a+1}(\Delta_n, S_n) \approx \mathrm{Ext}^2_{R_\infty}(\widetilde{H}^a(\Delta_n, J_n), R_\infty)$$

However

$$\widetilde{H}^0(\Delta_n, J_n) = (J_n \cap R_\infty)/\,\mathrm{tr}(J_n) \approx T \cdot R_\infty / T \cdot R_\infty \approx 0,$$

and thus

$$\widetilde{H}^{\mathrm{even}}(\Delta_n, J_n) \approx \widetilde{H}^{\mathrm{odd}}(\Delta_n, S_n) \approx (0).$$

Passing to the limit $n \to \infty$ implies

$$\begin{aligned} H^i(\Delta_\infty, J_\infty) &= (0), \quad i > 0 \text{ even} \\ H^i(\Delta_\infty, S_\infty) &= (0), \quad i > 0 \text{ odd} \\ H^0(\Delta_\infty, J_\infty) &\approx T \cdot R_\infty \\ H^0(\Delta_\infty, S_\infty) &\approx R_\infty \end{aligned}$$

Also all cohomology vanishes in degree > 1 for p-torsion modules).

Finally we compute

$$H^*(\Delta_\infty, S_\infty/J_\infty) \approx H^*(\mathbb{Z}_p(1), (R_\infty/TR_\infty)^{\mathrm{norm}}).$$

Here the action of $\mathbb{Z}_p(1)$ on $(R_\infty/TR_\infty)^{\mathrm{norm}}$ is trivial. Thus the module coincides with H^0. Also H^1 are the continuous homomorphisms $\mathbb{Z}_p(1) \to (R_\infty/T\cdot R_\infty)^{\mathrm{norm}}$. As the target has no torsion these vanish. Finally H^2 can be computed by tensoring with $0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$. The result:

$$\begin{aligned}
&H^0(\Delta_\infty, (R_\infty/T\cdot R_\infty)^{\mathrm{norm}}) = (R_\infty/T\cdot R_\infty)^{\mathrm{norm}}\\
&H^1(\Delta_\infty, (R_\infty/T\cdot R_\infty)^{\mathrm{norm}}) = (0)\\
&H^2(\Delta_\infty, (R_\infty/T\cdot R_\infty)^{\mathrm{norm}}) = (R_\infty/T\cdot R_\infty)^{\mathrm{norm}} \otimes \mathbb{Q}_p/\mathbb{Z}_p(-1)
\end{aligned}$$

Next using $0 \to J_\infty \to S_\infty \to S_\infty/J_\infty \to 0$ we compute

$$\begin{aligned}
&H^0(\Delta_\infty, S_\infty) = R_\infty\\
&H^1(\Delta_\infty, S_\infty) \approx (0)\\
&H^2(\Delta_\infty, S_\infty) \approx (R_\infty/T\cdot R_\infty)^{\mathrm{norm}} \otimes \mathbb{Q}_p/\mathbb{Z}_p(-1)
\end{aligned}$$

Finally $0 \to p^s\cdot S_\infty \to S_\infty \to S_\infty/p^s\cdot S_\infty \to 0$ gives

$$\begin{aligned}
&H^0(\Delta_\infty, S_\infty/p^s\cdot S_\infty) \approx R_\infty/p^s\cdot R_\infty\\
&H^1(\Delta_\infty, S_\infty/p^s\cdot S_\infty) \approx (R_\infty/T\cdot R_\infty)^{\mathrm{norm}} \otimes \mathbb{Z}/p^s\mathbb{Z}(-1)
\end{aligned}$$

Thus we have shown the first three claims (the maps are easily seen to be a described there). For the last assertion iv) it is easily checked if the residue field of x has characteristic zero. Thus assume that its characteristic is p. What we have to do is compute the first cohomology of the complement of the preimage of Y in $\mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{sh}} \otimes_V \overline{K})^0$, with coefficients $\mathbb{Z}/p^s\mathbb{Z}(1)$. Its irreducible components correspond to classes of Δ_x modulo the decomposition-group D of Y, and inertia at these components defines an injective map from our H^1 to the induced module $\mathrm{Ind}_D^\Delta(\mathbb{Z}/p^s\mathbb{Z})$, which defines $i_*(\mathbb{Z}/p^s\mathbb{Z})$. It remains to show that the map is also surjective. This uses the sheaf $\mathcal{R}(\overline{\mathcal{O}})$ on the topos $(\widetilde{X}^0 \otimes_V \overline{K})$. The assertions i),ii),iii), and projective limits, imply that the zero'th cohomology of $\mathcal{R}(\overline{\mathcal{O}}_{\widetilde{X}})$ is almost equal to $\mathcal{R}(\overline{\mathcal{O}}_X)$, and its first cohomology to $\mathrm{Ind}_D^\Delta(\mathcal{R}(\overline{\mathcal{O}}_Y))(-1)$. Higher cohomology is almost zero. Computing almost invariants under Frobenius we get the assertion with coefficients $\mathbb{Z}/p\mathbb{Z}$, and extending modulo p^s follows by by devissage.

After this hard work apply $\mathbb{R}q_{X,*}$ to the almost exact sequence

$$0 \longrightarrow \overline{\mathcal{O}}_X/p^s\cdot\overline{\mathcal{O}}_X \longrightarrow \mathbb{R}j_*\overline{\mathcal{O}}_{\widetilde{X}}/p^s\cdot\overline{\mathcal{O}}_{\widetilde{X}} \longrightarrow i_*(\overline{\mathcal{O}}_Y/p^s\cdot\overline{\mathcal{O}}_Y)[-1](-1) \longrightarrow 0$$

to obtain in highest degree

$$\mathbb{R}^d q_{X,*}(\overline{\mathcal{O}}_X/p^s\overline{\mathcal{O}}_X) \longrightarrow \mathbb{R}^d q_{\widetilde{X},*}(\overline{\mathcal{O}}_{\widetilde{X}}/p^s\overline{\mathcal{O}}_{\widetilde{X}}) \longrightarrow \mathbb{R}^{d-1}q_{Y,*}(\overline{\mathcal{O}}_Y/p^s\cdot\overline{\mathcal{O}}_Y)(-1) \longrightarrow 0$$

on $X^{\text{ét}}$. These map to the exact sequence

$$0 \longrightarrow \omega_X(-d) \longrightarrow \omega_X(Y)(-d) = \omega_{\widetilde{X}}(-d) \longrightarrow i_*\omega_Y(-d) \longrightarrow 0$$

with the resulting two squares commutative:

This is an easy local calculation. The first square commutes on $X \smallsetminus Z$ which is dense. For the second choose a local equation T for Z and extend to local parameter $X_1 = T, X_2, \dots, X_d$. Then we obtain a $\mathbb{Z}_p(1)^d = \Delta$-extension of $\mathcal{O}_{\widetilde{X}}$ by adjoining p-power roots of the X_i. Then the trace-map sends the generator of $H^d(\Delta, \mathbb{Z}_p(d)) \cong \mathbb{Z}_p$ to $d\log X_1 \wedge \cdots \wedge d\log X_d$. When mapping to the "$Z$-terms" the first goes to the canonical generator of $H^{d-1}(\Delta_Y, \mathbb{Z}_p(d-1)) = \mathbb{Z}_p$, where $\Delta_Y \cong \mathbb{Z}_p(1)^{d-1}$ corresponds to adjoining p-power roots of $X_2, \dots, X_d$. In turn this goes to $d\log X_2 \wedge \cdots \wedge d\log X_d$, which is the residue of $d\log X_1 \wedge \cdots \wedge d\log X_d$. However the direct image in coherent cohomology is defined by the exact sequence of ω's.

By taking cup-products with classes of complementary degree it follows that the Gysin-map

$$i_* : \mathbb{R}\Gamma_{(!)}(\mathcal{Z}, \mathbb{L} \otimes \overline{\mathcal{O}}_Y)(-1)[-2] \longrightarrow \mathbb{R}\Gamma_{(!)}(\mathfrak{X}, \mathbb{L} \otimes \overline{\mathcal{O}}_X)$$

is the connecting map from the exact sequence (triangle)

$$0 \longrightarrow \mathbb{L} \otimes \overline{\mathcal{O}}_X \longrightarrow \mathbb{R}j_*(\mathbb{L} \otimes \overline{\mathcal{O}}_{\widetilde{X}}) \longrightarrow i_*(\mathbb{L} \otimes \overline{\mathcal{O}}_Z)(-1)[-1] \longrightarrow 0$$

and its variants with support-conditions.

Also we have constructed an exact triangle

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{R}j_*j^*\mathbb{L} \longrightarrow i_*(\mathbb{L})(-1)[-1] \longrightarrow 0$$

mapping to the previous one (onto the almost invariants of Frobenius). For support-conditions one applies the functors $\Psi(\mathbb{L})$ and $\mathbb{R}v_{X,*}$, and obtains such a triangle on $X^{\text{ét}}$, with $\mathbb{R}v_{X,*}$ applied to all terms. It then induces an "adjoint"

$$\mathbb{R}\Gamma_{(!)}(\mathcal{Z}, \mathbb{L})(-1)[-2] \longrightarrow \mathbb{R}\Gamma_{(!)}(\mathfrak{X}, \mathbb{L})$$

for the trace-maps induced from cohomology with values in $\overline{\mathcal{O}}$. If we know that these trace-maps take values in $\mathbb{Z}/p^s\mathbb{Z}$ we could use them to define a Poincaré-duality and i_* would be the adjoint to i^*. In any case u_X^* maps our exact sequence to its pendant in $(X \otimes_V \overline{K})^{\text{ét}}$, which defines the direct image in étale cohomology. Thus one checks that our triangle defines an adjoint i_* for the inner product defined by $\mathrm{tr}^{\text{étale}}$.

The upshot of this discussion:

There exists an adjoint i_* for the $\mathrm{tr}^{\text{étale}}$-theory inducing the adjoint in $\overline{\mathcal{O}}$-cohomology.

Next let us compute in some examples that the comparison-maps

$$H^i(\mathfrak{X} \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow H^i((X \otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z})$$

tend to be isomorphisms:

Example 1. — Let

$$X = \mathbb{P}^n, \quad X^0 = \mathbb{G}_m^d = \mathbb{P}^d \smallsetminus \cup_{i=0}^d H_i,$$

H_i the standard-hyperplanes $\{T_i = 0\}$. Then X can be covered by standard affines $\mathbb{A}^n$ with fundamental group $\pi_1(\mathbb{G}_m^d \otimes_V \overline{K}) = \widehat{\mathbb{Z}}^d$ corresponding to n-power coverings

$\mathbb{P}^d \to \mathbb{P}^d$. Hence

$$\begin{aligned} H^*(\mathfrak{X}^0 \otimes_V \overline{K}, \overline{\mathcal{O}}/p^s\overline{\mathcal{O}}) &= \mathbb{R}\Gamma(\widehat{\mathbb{Z}}(+1)^d, \varinjlim \mathbb{R}\Gamma(\mathbb{P}^d, \mathcal{O}/p^s\mathcal{O}) \otimes_V \overline{V}) \\ &= (\text{exterior algebra in } \mathbb{Z}_p(-1)^d) \otimes \overline{V}/p^s\overline{V}. \end{aligned}$$

Taking Frobenius-invariants

$$\begin{aligned} H^*(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) &= \text{exterior algebra in} \quad \mathbb{Z}/p^s\mathbb{Z}(-1)^d \\ &= H^*((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z}) \end{aligned}$$

By using the fundamental triangle for the embeddings $H_i \hookrightarrow \mathbb{P}^d \hookleftarrow \mathbb{P}^d - H_i$ one obtains by induction that

$$H^*(\mathfrak{X} \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) = H^*\Big((X \otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z}) = \bigoplus_{i=0}^{d} \mathbb{Z}/p^s\mathbb{Z} \cdot \xi^i, \xi = c_1(\mathcal{O}(1))\Big).$$

Example 1*. — Let $\mathcal{E}$ denote a vectorbundle on X of rank $r+1$, $Y = \mathbb{P}_X(\mathcal{E}) \xrightarrow{\text{pr}} X$. Then

$$H^*(\mathcal{Y}^0 \otimes_V \overline{K}, \text{pr}^*(\mathbb{L}) \otimes \overline{\mathcal{O}}_Y) = \bigoplus_{i=0}^{r} H^{*-2i}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{\mathcal{O}}_X) \cdot \xi^i$$

with

$$\xi = c_1(O(1)), \quad \text{pr}^*(\mathcal{E}) \longrightarrow O(1)$$

the universal quotient. Also

$$H^*(\mathcal{Y}^0 \otimes_V \overline{K}, \text{pr}^* \mathbb{L}) = \bigoplus_{i=0}^{r} H^{*-2i}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \circ \xi^i.$$

Proof. — Both assertions are equivalent. We prove the corresponding equality for the derived image $\mathbb{R}\,\text{pr}_* \circ \mathbb{R}q_{X,*}$, that is étale locally in $X^{\text{ét}}$. But over some $\text{Spec}(O^{\text{sh}}_{X,x})$, Y is a product $X \times \mathbb{P}^r$, and the result follows from Künneth.

In example 1 the

$$\text{tr} : H^{2d}(\mathbb{P}^d, \overline{\mathcal{O}}/p^s\overline{\mathcal{O}}(d)) \longrightarrow \overline{V}/p^s\overline{V}$$

sends $c_1(\mathcal{O}(1))^d$ to 1: This follows by induction from (i = inclusion $H_0 \hookrightarrow \mathbb{P}^d$)

$$i_*c_1(\mathcal{O}(1))^{d-1} = c_1(\mathcal{O}(1))^d.$$

The formula holds in étale cohomology, and we already know that both versions of i_* are compatible.

By checking fibrewise we derive that in example 1* $\text{pr}_*(\xi^r) = 1$. Also one defines the Chern-classes

$$c_i(\mathcal{E}) \in H^{2i}(\mathfrak{X} \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}(i))$$

by the usual formula

$$\sum_{i=0}^{r+1} (-1)^i c_i(\mathcal{E}) \cdot \xi^{r-i} = 0.$$

They map to the corresponding Chern-classes in étale cohomology.

Example 2. — Let $\widetilde{X} \to \mathbb{P}^d = X$ denote the blow-up in the point $Y = (1:0\ldots)$. Then $\widetilde{X}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{d-1}$, with exceptional divisor $E \cong \mathbb{P}^{d-1}$. Using example 1^* one checks that for each i the sequence

$$0 \longrightarrow H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow H^i_{(!)}(\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) \oplus H^i_{()}(\mathcal{Y}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow H^i_{(!)}(\mathcal{E}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}) \longrightarrow 0$$

is exact.

Example 2*. — Let $Z \subseteq X$ denote a closed subvariety such that locally in the étale topology

$$(X, Z) \cong X_1 \times (\mathbb{A}^t, 0),$$

with X_1 toroidal etc. Let $\widetilde{X}$ denote the blow-up of X in $Z, E \subseteq \widetilde{X}$ the exceptional divisor, $E \cong \mathbb{P}_Z(J_Z/J_Z^2)$. Then for each i the sequence

$$0 \longrightarrow H^i_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow H^i_{(!)}(\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K}, \mathbb{L}) \oplus H^i_{(!)}(\mathcal{Z}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow H^i_{(!)}(\mathcal{E}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow 0$$

is exact.

Proof. — As in example 1^* one checks this by a localization in $X^{\text{ét}}$. This would give a priori a long exact sequence in cohomology, but for example the pullback

$$p_X^* : H^i_{(!)}(X^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow H^i_{(!)}(\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K}, \mathbb{L})$$

is injective, with left-inverse $p_{X,*}$.

Thus consider a product $X_1 \times \mathbb{A}^t = X$ and the blow-up $\widetilde{X}$ in $X_1 \times \{0\}$. $\mathbb{A}^t$ as well as its blow-up in zero obtain the structure of a torus-embedding from $\mathbb{G}_m^t \subseteq \mathbb{A}^t$. On $\mathbb{A}^t$ the codimension one strata are the coordinate hyperplanes, while on the blow-up we have their proper transforms as well as the exceptional divisor ($\cong \mathbb{P}^{t-1}$).

Now the blow-up induces an almost isomorphism on the cohomology of the stratum $X_1 \times \mathbb{G}_m^t$, using that we retain the same fundamental group and the theory of toroidal embeddings. Also the other strata can be treated by induction.

Remark. — For constant coefficients we also could have used local Künneth. The above procedure also works if $\mathbb{L}$ is not induced from the factor X_1 in $X_1 \times \mathbb{A}^t$.

Finally we need the self-intersection formula:

2. Lemma. — *For the inclusion $i : Z \hookrightarrow X$ of a divisor $i^* i_*(z) = -z \cup c_1(J_Z/J_Z^2) = z \cup c_1(\mathcal{N}_Z)$.*

Proof. — This amounts to computing the extension class of the pullback by i^* of the exact sequence

$$0 \longrightarrow \mathbb{Z}/p^s\mathbb{Z} \longrightarrow \mathbb{R}j_*\mathbb{Z}/p^s\mathbb{Z} \longrightarrow i_*\mathbb{Z}/p^s\mathbb{Z}(-1)[-1] \longrightarrow 0$$

Cover X by affines U_ν on which Z has a local equation f_ν. Then the p^s-th roots of f_ν form a μ_{p^s}-torsor on $\mathcal{U}_\nu \cap (X \smallsetminus Z)$, hence give a class in $\Gamma(\mathcal{U}_\nu \otimes_V \overline{K}, \mathbb{R}^1 j_* \mu_{p^s})$ which maps to $-1 \in i_*(\mathbb{Z}/p^s\mathbb{Z})$. Thus the restriction to $\mathcal{U}_\nu \cap Z$ trivializes the extension there. The trivialization is not global. Namely one intersections $\mathcal{U}_\nu \cap \mathcal{U}_\mu$ we get a difference given by the μ_{p^s}-torsor of roots of $u_{\nu\mu} = (f_\nu/f_\mu) \mid Y$. However this defines $c_1(J_Z/J_Z^2)$.

Finally we treat immersions of higher codimensions. Let $Z \subseteq X$ be a closed immersion such that locally in the étale topology $(X, Z) \cong X_1 \times (\mathbb{A}^t, 0)$. Let $\widetilde{X} =$ blow-up of X in $Z, E =$ exceptional divisor $= \mathbb{P}_Z(J_Z/J_Z^2)$.

$$\begin{tikzcd} E \arrow[r, hook, "\widetilde{i}"] \arrow[d, "P_Z"'] & \widetilde{X} \arrow[d, "P_X"] \\ Z \arrow[r, hook, "i"] & X \end{tikzcd}$$

On E we have the vectorbundle $\mathcal{F} = \mathrm{pr}_Y^*(J_Z/J_Z^2)^*/\mathcal{O}(-1)$. $\mathcal{F}$ has rank $t-1$, and from the usual formulas one computes that

$$c_{t-1}(\mathcal{F}) = \sum_{i=0}^{t-1} c_{r-1}(\mathrm{pr}_Z^*(J_Z/J_Z^2)^*) \cup \xi^i.$$

Especially $p_{Z,*}(c_{t-1}(\mathcal{F})) = 1$ (in all cohomology-theories).

3. Proposition. — *$p_X^* \circ i_*(z) = \widetilde{i}_*(c_{t-1}(\mathcal{F}) \cup p_Z^*(z))$ for $z \in H^*(\mathcal{Z}^0 \otimes_V \overline{K}, \mathbb{L})$.*

Proof. — The right-hand side lies in the image of pr_X^*: To check this it suffices to check the restriction to E. By the self-intersection formula this is

$$c_1(\mathcal{O}(-1)) \cup c_{t-1}(\mathcal{F}) \cup p_Z^*(z) = \mathrm{pr}_Z^*(c_t(J_Z/J_Z^2) \cup z)$$

and indeed induced from Z. It now suffices to apply $p_{X,*}$:

$$\begin{aligned} p_{X,*}(\widetilde{i}_*(c_{t-1}(\mathcal{F}) \cup p_Z^*(z)) &= i_* \circ p_{Z,*}(c_{t-1}(\mathcal{F}) \cup p_Z^*(z)) \\ &= i_*(z). \end{aligned}$$

Remark. — The proof actually shows the formula in the derived category.

4. Corollary. — *The transformation*

$$\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) \longrightarrow \mathbb{R}\Gamma((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L})$$

commutes with i_*.

Proof. — It commutes with everything else in the formula.

5. Corollary. — *The two trace-maps (induced from $\overline{\mathcal{O}}/p^s \cdot \overline{\mathcal{O}}$ respectively from étale cohomology of the generic fibre) on $H_!^{2d}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{Z}/p^s\mathbb{Z}(d))$ coincide.*

Proof. — We may assume that the generic fibre $X \otimes_V \overline{K}$ is irreducible (extend K if necessary). As the special fibre $X \otimes_V k$ is generically reduced, there exists (after perhaps again extending K) a section $i : Z = \mathrm{Spec}(V) \hookrightarrow X$ with image in the smooth locus. Then both traces are characterised by the fact that they take the value 1 on $i_*(1)$.

We intend to apply this theory to the diagonal embedding

$$\delta : X \hookrightarrow X \times_V X.$$

As this does not satisfy the conditions on the nose, we shall modify it to

$$\widetilde{\delta} : \widetilde{X} \longrightarrow (X \times_V X)^{\sim}.$$

This modification is done using toroidal geometry. We assume given a finite number of convex polyhedral cones $\{\sigma_\nu\}$, $\sigma_\nu \subseteq L_{\mathbb{R}}$, $L = \mathbb{Z}^{d+1}$, and also an indivisible $\lambda \in L^\vee$. We assume that each face of a σ_ν also appears among them, and that $\sigma_\nu \cap \ker(\lambda)$ is a simplex spanned by a partial basis of L.

Furthermore we assume given an étale covering of X by $X(\sigma_\nu)$'s, and étale maps $X(\sigma_\nu) \to \overline{T}_{\lambda,\sigma_\nu}$ (the torus-embedding defined by σ_ν). Also for τ a face of $\sigma, X(\tau)$ should be equal to the preimage of the open $\overline{T}_{\lambda,\{\tau\}} \subseteq \overline{T}_{\lambda,\{\sigma_\nu\}}$, with the induced map. Each $X(\sigma)$ is stratified, with strata indexed by faces τ of σ. If $x_1 \in X(\sigma_1)$ and $x_2 \in X(\sigma_2)$ lie over the same point $x \in X$, and in the closed σ_1-stratum respectively σ_2-stratum, then there exists an automorphism ($\langle\sigma_i\rangle \subseteq L$, the sublattice spanned by σ_i) $(\langle\sigma_1\rangle, \sigma_1, \lambda) \cong (\langle\sigma_2\rangle, \sigma_2, \lambda)$, such that the two induced isomorphisms between $\mathcal{O}^{\mathrm{sh}}_{X,x} = \mathcal{O}^{\mathrm{sh}}_{X(\sigma_1),x_1} = \mathcal{O}^{\mathrm{sh}}_{X(\sigma_2),x_2}$ and the strict henselizations of $\overline{T}_{\lambda,\{\sigma_1\}}$ respectively $\overline{T}_{\lambda,\{\sigma_2\}}$ differ by the induced automorphism and the action of an element of T.

We construct for each σ_ν a finite collection $M(\sigma_\nu)$ of T-invariant ideals I of $V[L^\vee \cap \sigma_\nu^\vee]$, as follows: First, for any integer n, consider the set of elements $\mu \in L^\vee \cap \sigma_\nu^\vee$ such that

i) $\mu(\rho) \leqslant n$ for each generator ρ of an extremal ray of σ_ν

ii) If $\lambda(\rho) = 0$ for such a ρ, then $\mu(\rho) = 0$ except for at most one such ρ, for which $\mu(\rho) = 1$.

Choose n so big that the elements generate the dual of the subgroup $\langle\sigma_\nu\rangle \subseteq L$ generated by σ_ν, for all ν. Then define $M_0(\sigma_\nu)$ as the (finite) set of principal ideals $V[L^\vee \cap \sigma_\nu^\vee] \cdot \mu$ generated by such μ. If τ is a face of σ then each ideal in $M_0(\sigma)$ generates an ideal in $M_0(\tau)$ in $V[L^\vee \cap \tau^\vee] \supseteq V[L^\vee \cap \sigma^\vee]$. Also the set of ideals $M_0(\sigma)$ is stable under isomorphisms $(\langle\sigma_1\rangle, \sigma_1, \lambda) \cong (\langle\sigma_2\rangle, \sigma_2, \lambda)$. We define $M(\sigma)$ as the set of all ideals of the form

$$\ker : V[L^\vee \cap \sigma^\vee] \longrightarrow V[L^\vee \cap \tau^\vee]/I,$$

with $I \in M_0(\tau), \tau$ a face of σ. Then:

i) $M(\sigma)$ is stable under isomorphisms of $(\langle\sigma\rangle, \sigma, \lambda)$.

ii) For τ a face of σ, the map $I \mapsto I \cdot V[L^\vee \cap \tau^\vee]$ induces a surjection $M(\sigma) \twoheadrightarrow M(\tau)$.

iii) If $\sigma \subseteq \ker(\lambda)$, then $M(\sigma)$ consists of the unit ideals and the ideals defining the irreducible components of the boundary-divisor

iv) $M(\sigma)$ contains principal ideals generated by μ, for μ a system of generators of $\langle\sigma\rangle^\vee$.

Now define new torus-embeddings $T \subseteq \tilde{\tilde{\overline{T}}}, \tilde{\tilde{\overline{T}}}_\lambda$ by normalised blow-up of all ideals $I_1 + I_2$, for $I_1, I_2 \in M(\sigma)$. By ii) this commutes with open immersions defined by faces $\tau \subseteq \sigma$, and by i) the induced modifications of $X(\sigma)$ descend to a proper map of algebraic spaces $f : \widetilde{X} \to X$ inducing $\widetilde{X}^0 \xrightarrow{\sim} X^0$. We call such an f a toroidal modification. By iii) on the generic fibre

$$\widetilde{X} \otimes_V K \longrightarrow X \otimes_V K$$

is common blow-up of all intersections of pairs of boundary-divisors. Locally in the étale topology of X f is a normalised blowup, thus quasiprojective, but I do not know whether this remains true globally, or whether $\widetilde{X}$ is always a scheme.

Similarly on $X \times_V X$ we blow-up (and normalise) all ideals

$$\mathrm{pr}_1^*(I_1) + \mathrm{pr}_1^*(I_2), \mathrm{pr}_1^*(I_1) + \mathrm{pr}_2^*(I_2), \mathrm{pr}_2^*(I_1) + \mathrm{pr}_2^*(I_2)$$

with $I_1 \in M(\sigma_1), I_2 \in M(\sigma_2)$ (over $X(\sigma_1) \times X(\sigma_2)$). Then the diagonal extends to $\widetilde{\Delta} : \widetilde{X} \to \widetilde{X \times_V X}$. Note that normalised blow-up has the following toroidal description: Each $I \in M(\sigma)$ defines a convex piecewise linear function ψ_I on σ by

$$\psi_I(\rho) = \min\{\mu(\rho) \mid \mu \in I \cap \sigma^\vee\}.$$

Then the normalised blow-up amounts to subdividing σ (in a minimal way) such that ψ_I becomes linear on each stratum.

Especially on $X \times_V X$ (defined locally by

$$L^{(2)} = \ker(\lambda, -\lambda) \subseteq L \times L, \sigma^{(2)} = \sigma \times \sigma \cap L_{\mathbb{R}}^{(2)}),$$

the functions $\min(\mu(x), \mu(y))$ are linear on the strata of the subdivision, for μ in a system of generators of $\langle\sigma\rangle^\vee$ (condition iv)). It follows that on each open stratum meeting the diagonal we have $\mu(x) = \mu(y)$, and thus this stratum is totally contained in the diagonal. Also on the diagonal we obtain the subdivision defining $\widetilde{X}$. Thus the diagonal embedding $\overline{T} \hookrightarrow \overline{T} \times_V \overline{T}$ extends to an open immersion

$$(T \times 1) \times \tilde{\tilde{\overline{T}}} \subseteq (\overline{T} \times_V \overline{T})^\sim.$$

Hence locally in the étale cohomology

$$(\widetilde{X \times_V X}, \widetilde{X}) \cong X_1 \times (\mathbb{A}^d, 0).$$

Next consider support-conditions at infinity for a toroidally defined $\widetilde{X} \to X$. If $J = J_I \subseteq \mathcal{O}_X$ partially defines the boundary, there are two canonical choices for an ideal $\widetilde{J} \subseteq \mathcal{O}_{\widetilde{X}}, \widetilde{J} \supseteq f^*(J)$. Either $\widetilde{J}$ defines the total preimage of $V(J)$ (the minimal choice), or the union of boundary divisors $\widetilde{D}$ not projecting onto irreducible components outside $V(J)$.

6. Lemma

i) $\widetilde{X} \otimes_V K$ *has normal crossings singularities*

ii) *For either choice of* $\widetilde{J}$ $\mathbb{R}f_*\widetilde{J} = J$.

Proof

i) On the generic fibre we have to treat the torus-embedding defined by $\sigma \cap \ker(\lambda)$, which up to a torus-factor is $\mathbb{G}_m^d \subseteq \mathbb{A}^d$. Then $\widetilde{X}$ is the union of $d!$ embeddings $\mathbb{G}_m^d \subseteq \mathbb{A}^d$. One of them maps to X via $(X_1, X_1 \cdot X_2, \dots, X_1 \cdot \cdots \cdot X_d)$, the others by permuting variables.

ii) If $\rho_1, \dots, \rho_r$ are the extremal rays for σ, those for the cones of $\widetilde{\sigma}$ are sums $\rho_S = \sum_{i\in S} \rho_i$ for non-empty subsets $S \subseteq \{1, \dots, d\}$. These lie in a common simplex if and only if the S's form a chain. Assume J is defined by $\{1, \dots, a\}$. Then the bigger transform $\widetilde{J}$ corresponds to $S \subseteq \{1, \dots, a\}$, while the total transform to $S \cap \{1, \dots a\} \neq \phi$.

By [KKMS], § 3, Th. 12 for any $\mu \in L^\vee$ the μ-component in $H^i(\widetilde{\widetilde{T}}; \widetilde{J})$ is $H^i_A(\sigma, k)$ for a certain closed subset $A \subseteq\mid \tau \mid$, namely the union of all faces τ with $\mu \in \Gamma$ (τ-open, $\widetilde{J}$). This is also equal to the relative cohomology $H^i(\sigma, \sigma - A; k)$. First choose $\widetilde{J}$ minimal. Now $\mu \notin \Gamma(\tau\text{-open}, \widetilde{J})$ if either not $\mu \in \tau^\vee$, or $\mu \in \tau^\vee$ and $\mu(\rho_S) = 0$.

For some ρ_S belonging to $\widetilde{J}$ with $\rho_S \in \tau$, i.e. $S \cap \{1, \dots, a\} \neq \phi$. Choose a $\delta\mu \in L^\vee$ with $\delta\mu(\rho_i) < 0$ for $i = 1, \dots, a, \delta_\mu(\rho_j) = 0$ for $j = a+1, \dots, r, \delta\mu(\rho) \geqslant 0$ for any extremal ρ of a face τ with $\lambda(\rho) > 0$. Then for $N \gg 0$

$$\mu \in \Gamma(\tau\text{-open}, \widetilde{J}) \Longleftrightarrow N\mu + \delta\mu \in \tau^\vee,$$

thus the μ-component becomes the $(N\mu+\delta\mu)$ component in $H^i(\widetilde{\widetilde{T}}, \mathcal{O})$ which is known to be 0 unless $i = 0$ and $N\mu + \delta\mu \in \sigma^\vee$, which means $\mu \in J$. For the maximal $\widetilde{J}$ choose $\delta\mu$ as before except that

$$\delta\mu(\rho_j) > -a \cdot \min\{\partial\mu(\rho_i) \mid 1 \leqslant i \leqslant a\}$$

for $a+1 \leqslant j \leqslant r$. This proves the assertion ii) for $\widetilde{\widetilde{T}} \to \overline{T}$, and by basechange also for $\widetilde{X} \to X$.

As an application let as before R_∞ denote the normalization of $R \otimes_V \overline{V}$ in the extension defined by adjoining all p-power roots of $\mu \in L^\vee, \widetilde{X}_\infty \to \widetilde{X}$ the corresponding normalised pullback to $f : \widetilde{X} \to X = \mathrm{Spec}(R)$, with Galois-group $\Delta_\infty = \mathbb{Z}_p(1)^d$. If M_∞ denotes an R_∞-module with continuous semilinear Δ_∞-action, and $J_\infty \subset R_\infty$ a support-ideal extending (in two possible ways) to $\widetilde{J}_\infty \subset \mathcal{O}_{\widetilde{X}_\infty}$, then

$$\mathbb{R}f_{\infty,*}(M_\infty \otimes^{\mathbb{L}} \widetilde{J}_\infty) = M_\infty \otimes^{\mathbb{L}} J_\infty$$

(by basechange), and this equality persists after applying $\mathbb{R}\Gamma(\Delta_\infty, \quad)$. However by the almost-étale theory these are cohomologies of $\mathbb{L} \otimes \overline{\mathcal{O}}$, if $M = (\overline{R} \otimes \mathbb{L})^{\mathrm{Gal}(\overline{R}/R_\infty)}$ for a locally constant p-torsion sheaf $\mathbb{L}$ on $\mathrm{Spec}(R)^0 \otimes_V \overline{K}$. That is we know that for the map $q_X : \mathfrak{X}^0 \otimes_V \overline{K} \to X^{\text{ét}}$ $Rq_{X,*}(\mathbb{L} \otimes \overline{J})$ is almost isomorphic to a complex of quasicoherent sheaves, and the above assertion means that the map

$$\mathbb{R}q_{X,*}(\mathbb{L} \otimes \overline{J}) \longrightarrow \mathbb{R}f_* \circ \mathbb{R}q_{\widetilde{X},*}(\mathbb{L} \otimes \widetilde{\overline{J}})$$

is an almost isomorphism. Thus:

7. Corollary. — $\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \overline{J}) \approx \mathbb{R}\Gamma(\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K}, \mathbb{L} \otimes \widetilde{\overline{J}})$

$$\mathbb{R}\Gamma_{(!)}(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L}) = \mathbb{R}\Gamma_{(!)}(\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K}, \mathbb{L})$$

In the second equation (!) *always denotes the relevant support-condition at infinity. The second assertion follows from the first by descente. Also the corresponding étale analogue holds:*

8. Lemma. — $\mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}) = \mathbb{R}\Gamma_{(!)}((\widetilde{X}^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L})$

Proof. — Let $X - X^0 = A \cup B$ denote the decomposition of the boundary in union of irreducible components, such that we have compact support along B. Similarly $\widetilde{X} - \widetilde{X}^0 = \widetilde{A} \cup \widetilde{B}$, with either $\widetilde{A} = f^{-1}(A)$ or $\widetilde{B} = f^{-1}(B)$. As Poincaré-duality exchanges A and B (and $\widetilde{A}, \widetilde{B}$) we may assume that $\widetilde{A} = f^{-1}(A)$. Now with $j_A : X - A \hookrightarrow X$ denoting the inclusion, we have $\mathbb{R}f_* \circ \mathbb{R}j_{\widetilde{A}} = \mathbb{R}j_A \circ \mathbb{R}f_*$. Thus it suffices if

$$\mathbb{R}f_* \circ \mathbb{R}j_{\widetilde{B},!}(\mathbb{L}) = \mathbb{R}j_{B,!} \circ \mathbb{R}f_*(\mathbb{L}) \quad \text{on} \quad X - A.$$

This assertion is local in $X - A$. We thus reduce to $X^0 = \mathbb{G}_m^d \subset X = \mathbb{A}^d$, $A = \phi$, $B = X - X^0$ and thus $\widetilde{B} = \widetilde{X} - \widetilde{X}^0$. As f is proper $\mathbb{R}f_*$ commutes with duality which exchanges $j_{B,!}$ and $\mathbb{R}j_{B,*}$, and similar for $\widetilde{B}$. This reduces us again to $A = X - X^0$, $B = \phi$, and then to $A = B = \phi$.

Now the main application: The diagonal-embedding

$$\delta : X \hookrightarrow X \times X$$

induces

$$\widetilde{\delta} : \widetilde{X} \hookrightarrow (\widetilde{X \times X}).$$

This embedding is such that locally in the étale topology

$$((\widetilde{X \times_V X}), \widetilde{X}) \cong X_1 \times (\mathbb{A}^d, 0).$$

Now choose a decomposition of the boundary

$$X - X^0 = A \cup B,$$

and extend to $\widetilde{X} - \widetilde{X}^0 = \widetilde{A} \cup \widetilde{B}$ with $\widetilde{A}$ = full preimage of A. Then define $\mathbb{R}\Gamma_{A,B}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L})$ as cohomology with compact support along B, and similarly for the other cohomology-theories or for $\widetilde{X}$. On $X \times_V X$ we consider the pair

$$(A^{(2)}, B^{(2)}) = (A \times_V X \cup X \times_V B, B \times_V X \cup X \times_V A),$$

and similarly $\widetilde{A}^{(2)}$ = full preimage of $A^{(2)}$ in $(\widetilde{X\times_V X})$, etc. Assume first that $\mathbb{L}$ is annihilated by p. We then have a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma((\widetilde{X}^0\otimes_V\overline{K})^{\text{ét}},\mathrm{End}(\mathbb{L})) & \xrightarrow{\widetilde{\delta}_*} & \mathbb{R}\Gamma_{\widetilde{A}^{(2)},\widetilde{B}^{(2)}}(((\widetilde{X\times_V X})^0\otimes_V\overline{K})^{\text{ét}},\mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d] \\ \uparrow & & \uparrow \\ \mathbb{R}\Gamma(\widetilde{\mathfrak{X}}^0\otimes_V\overline{K},\mathrm{End}(\mathbb{L})) & \xrightarrow{\widetilde{\delta}_*} & \mathbb{R}\Gamma_{\widetilde{A}^{(2)},\widetilde{B}^{(2)}}((\widetilde{\mathfrak{X}\times_V\mathfrak{X}})^0\otimes_V\overline{K},\mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d] \end{array}$$

which can be identified with

$$\begin{array}{ccc} \mathbb{R}\Gamma((X^0\otimes_V\overline{K})^{\text{ét}},\mathrm{End}(\mathbb{L})) & \xrightarrow{\delta_*} & \mathbb{R}\Gamma_{A^{(2)},B^{(2)}}((X\times_V X)^0\otimes_V\overline{K})^{\text{ét}},\mathbb{L}\otimes\mathbb{L}^V)(-d)[-2d] \\ \uparrow & & \uparrow \\ \mathbb{R}\Gamma(\mathfrak{X}^0\otimes_V\overline{K},\mathrm{End}(\mathbb{L})) & \xrightarrow{\delta_*} & \mathbb{R}\Gamma_{A^{(2)},B^{(2)}}((\mathfrak{X}\times_V\mathfrak{X}_0)^0\otimes_V\overline{K},\mathbb{L}\otimes\mathbb{L}^V)(-d)[-2d] \end{array}$$

Apply this to the canonical class $1\in H^0(X^0\otimes_V\overline{K})^{\text{ét}},\mathrm{End}(\mathbb{L}))$ or its variants. Its image in

$$\begin{aligned} &\mathbb{R}\Gamma_{A^{(2)},B^{(2)}}(((X\times_V X)^0\otimes_V\overline{K})^{\text{ét}},\mathbb{L}\otimes\mathbb{L}^\vee)(-d)[-2d] \\ &\qquad\cong\mathbb{R}\Gamma_{A,B}((X\otimes_V\overline{K})^{\text{ét}},\mathbb{L})\otimes\mathbb{R}\Gamma_{B,A}((X\otimes_V\overline{K})^{\text{ét}},\mathbb{L}^\vee)(-d)[-2d] \end{aligned}$$

corresponds via Poincaré-duality to the identity, and similarly for $\mathbb{R}\Gamma(\mathfrak{X}^0\otimes_V\overline{K},\dots)$. As the comparison-map

$$\mathbb{R}\Gamma_{A,B}(X^0\otimes_V\overline{K},\mathbb{L})\longrightarrow\mathbb{R}\Gamma_{A,B}((X^0\otimes_V\overline{K})^{\text{ét}},\mathbb{L})$$

respects the trace and cup-product it induces an isometry on cohomology, and thus an injection. That it also preserves $\delta_*(1)$ means that its adjoint also preserves products, thus is also an injection. Hence the comparison-map is a quasi-isomorphism. By devissage this also holds for general p-torsion $\mathbb{L}$'s.

9. Theorem. — $H^i_{A,B}(\mathfrak{X}^0\otimes_V,\mathbb{L})=H^i_{A,B}((X^0\otimes_V\overline{K})^{\text{ét}},\mathbb{L})$.

Remark. — We have given a "global" proof. In many cases one can also proceed "locally": Namely if for each $x\in X$ all strata in $\mathrm{Spec}(\mathcal{O}^{\text{sh}}_{X,x}\otimes_V\overline{K})$ are $K(\pi,1)$'s in the étale topology, then the stalks of $\mathbb{R}q_{X,*}\mathbb{L}$ coincide for

$$q_X:\mathfrak{X}^0\otimes_V\overline{K}\longrightarrow X^{\text{ét}}$$

respectively

$$(X^0\otimes_V\overline{K})^{\text{ét}}\longrightarrow X^{\text{ét}}.$$

5. Crystalline Cohomology

Let $X \to \mathrm{Spec}(V)$ be as before. That is X is a proper scheme or algebraic space over V with a log structure defined by an étale covering $\coprod X(\sigma) \twoheadrightarrow X$, étale $X(\sigma) \to \overline{T}_{\sigma,\lambda}$, gluing conditions, $L^\vee/\mathbb{Z}\lambda$ torsion free, etc. We also assume that the special fiber of X is reduced, which is equivalent to saying that $\rho(\lambda) = 0$ or 1 for the generator ρ of each extremal ray of σ. We can always achieve this by adjoining a root $\pi^{1/e}$. For most results it suffices that X is a fine and saturated log scheme smooth over V, proper and with reduced special fibre. The last condition is also equivalent to saying that the special fibre of X is of Cartier type, and it is used when we prove that the Frobenius endomorphism of crystalline cohomology is an isogeny and crystalline cohomology satisfies Poincaré duality. However I do not know how to regularise the diagonal embedding in this more general setting. Still I shall use the language of logarithmic structures as it now has become standard, restricting ourselves to such structures which are "toroidal" as above.

Choose a uniformiser π of V. Let $V_0 = W(k) \subseteq V$ denote the maximal unramified subring of Witt-vectors over k. Then

$$V = V_0[\pi] = V_0[t]/(f(t)),$$

where $f(t) = t^e + a_{e-1} \cdot t^{e-1} + \cdots + a_0$ is an Eisenstein polynomial: All a_i are divisible by p, and a_0/p is a unit in V_0. $D_{(f(t))}(V_0[t])$ denotes the divided power envelope of $(f(t))$, obtained by adjoining divided powers $f(t)^n/n!$ or $t^{en}/n!$, and R_V denotes its p-adic completion. $R_V \subset K_0[[t]]$ consists of power series $\sum_{n=0}^{\infty} a_n \cdot t^n$ with ([ ] denotes Gauss-brackets)

$$[n/e]! \cdot a_n \in V_0 \quad (\text{all } n), \quad \lim_{n\to\infty} ([n/e]! \cdot a_n) \longrightarrow 0$$

It has the PD-filtration $F^n(R_V)$ = p-adic closure of the ideal generated by divided powers $f^m/m!$, $m \geqslant n$. $R_V/F^1(R_V) = V$. We give R_V the logarithmic structure defined by the prelog structure $\mathbb{N} \to R_V$ which sends 1 to t. We want to study the logarithmic crystalline topos of X/R_V. For lack of suitable reference we first tensor with a fixed $\mathbb{Z}/p^s\mathbb{Z}$. So let $X_s = X \otimes_V V/p^sV$. Then the logarithmic crystalline site ([B], [K]) $(X_s/R_{V,s})_{\mathrm{crys}}$ consists of PD-embeddings (PD-structure compatible with that on $pR_{V,s} + F^1(R_{V,s})$)

$$U \hookrightarrow \mathcal{U}$$

where $U, \mathcal{U}$ are schemes/ $R_{V,s}$, $U \to X_s$ is étale and $\mathcal{U}$ has a logarithmic structure such that our embedding becomes fine. This means that $\mathcal{U}$ obtains a "toroidal structure" as follows:

The covering by $X(\sigma) \to X$ induces an étale covering by $U(\sigma) \to U$, with $U(\sigma) \to \overline{T}_{\lambda,\sigma} \subseteq \overline{T}_\sigma$. $\overline{T}_\sigma$ is a scheme over $V_0[t]$ with t corresponding to λ. As the ideal of U in $\mathcal{U}$ is nilpotent $U(\sigma)$ extends canonically to $\mathcal{U}(\sigma) \to \mathcal{U}$ étale, with $U(\sigma) \hookrightarrow \mathcal{U}(\sigma)$. Then the "toroidal structure" on $\mathcal{U}$ is an étale local extension of $U(\sigma) \to \overline{T}_\sigma$ to $\mathcal{U}(\sigma) \to \overline{T}_\sigma$

(as schemes over $V_0[t]$), unique up to homomorphisms

$$\mathcal{U}(\sigma) \longrightarrow T_\lambda = \ker(\lambda : T \to \mathbb{G}_m)$$

which is trivial on $U(\sigma)$. That is the extensions exist on an étale cover $\mathcal{U}'(\sigma) \twoheadrightarrow \mathcal{U}(\sigma)$ and the two pullbacks to $\mathcal{U}''(\sigma) = \mathcal{U}'(\sigma) \times_{\mathcal{U}(\sigma)} \mathcal{U}'(\sigma)$ differ by a map to $\mathcal{U}''(\sigma) \to T_\lambda$ satisfying the cocycle condition. In fact to construct $\mathcal{U}'(\sigma)$ it suffices to trivialise a class in the first cohomology of $1 + PD$-ideal, so any affine covering suffices. Also there are compatabilities for faces $\tau \subset \sigma$.

Maps

$$f : (U_1 \hookrightarrow \mathcal{U}_1) \longrightarrow (U_2 \to \mathcal{U}_2)$$

are pairs of log-maps $f_U : U_1 \to U_2, f_{\mathcal{U}} : \mathcal{U}_1 \to \mathcal{U}_2$ with the usual commutation rule. Being log-maps implies that locally in $\mathcal{U}(\sigma)$ we have $\rho : \mathcal{U}_1(\sigma) \to T_\lambda$, trivial on $U_1(\sigma)$, such that

$$\begin{tikzcd} \mathcal{U}_1(\sigma) \arrow[r] \arrow[dr] & \mathcal{U}_2(\sigma) \arrow[d] \\ & \overline{T}_\sigma \end{tikzcd}$$

commutes up to the action of ρ. (Composition is obvious) Coverings in the site are induced by étale covers of U. A crystal (of quasicoherent sheaves) on $(X_s/R_{V,s})_{\text{crys}}$ is a sheaf $\mathcal{E}$ such that for $U \hookrightarrow \mathcal{U}, \mathcal{E}_{\mathcal{U}}$ is a quasicoherent sheaf on $\mathcal{U}$. Also for any map $(U_1 \hookrightarrow \mathcal{U}_1) \to (U_2 \hookrightarrow \mathcal{U}_2), \mathcal{E}_{\mathcal{U}_1}$ should be isomorphic to the pullback of $\mathcal{E}_{\mathcal{U}_2}$, via the pullback map. A filtered crystal is an $\mathcal{E}$ as above together with a decreasing sequence of subsheaves $F^n(\mathcal{E}) \subseteq \mathcal{E}$ (indexed by $n \in \mathbb{Z}$) such that on each $U \hookrightarrow \mathcal{U}$, the $F^n(\mathcal{E})_{\mathcal{U}}$ are quasicoherent subsheaves with $F^a(\mathcal{O}_{\mathcal{U}}) \cdot F^b(\mathcal{E})_{\mathcal{U}} \subseteq F^{a+b}(\mathcal{E})_{\mathcal{U}}$ ($F^a(\mathcal{O}_{\mathcal{U}}) : PD$-filtration defined by the ideal of $U \hookrightarrow \mathcal{U}$), and for

$$f : (U_1, \mathcal{U}_1) \longrightarrow (U_2, \mathcal{U}_2)$$

$F^n(\mathcal{E})_{\mathcal{U}_1} \subseteq \mathcal{E}_{\mathcal{U}_1}$ is the subsheaf generated

$$\sum_{a+b \geqslant n} F^a(\mathcal{O}_{\mathcal{U}_1}) \cdot f_V^*(F^b(\mathcal{E})_{\mathcal{U}_2}).$$

Obviously any (unfiltered) $\mathcal{E}$ can be filtered by the $F^n(O_{\mathcal{U}}) \cdot \mathcal{E}_{\mathcal{U}}$. An example of a filtered crystal is $\mathcal{O}_X^{\text{crys}}\{a\}$, with underlying sheaf $\mathcal{E}_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}}$ and $F^n(\mathcal{E})_{\mathcal{U}} = F^{a+m}(\mathcal{O}_{\mathcal{U}})$. A filtered crystal of vectorbundles $(\mathcal{E}, F^\bullet)$ is called locally filtered free if for any $U \hookrightarrow \mathcal{U}$ $(\mathcal{E}_{\mathcal{U}}, F^\bullet(\mathcal{E}_{\mathcal{U}}))$ is locally isomorphic to a direct sum of $\mathcal{O}_{\text{crys}}\{a\}$'s (no compatibility with pullbacks is required).

Now choose an étale covering of X by $X(\sigma)$'s, with étale maps $X(\sigma) \to \overline{T}_{\sigma,\lambda}$. These lift to étale maps of formal schemes (with adic topology defined by $\pi - \lambda$) $\widehat{X(\sigma)} \to \widehat{\overline{T}_\sigma}$ = formal completion of $\overline{T}_\sigma$ along $\overline{T}_{\sigma,\lambda}$. Thus we can find an étale covering $Y \twoheadrightarrow X$ and a lift to an étale scheme $\widehat{Y}$ locally étale (and with the induced log structure) over $\widehat{\overline{T}_\sigma}$, namely for example $Y = \amalg X(\sigma)$. We assume that Y is affine. We need the exactified logarithmic products (better inductive limits of log-n-th infinitesimal neighbourhoods) $\widehat{Y}^{n+1,\log}$. These will be formal schemes with underlying topological space

$$Y \times_X Y \times \cdots \times_X Y \quad (n+1\text{-times}),$$

defined by étale glueing from the following: For $\overline{T}_\sigma$ the $(n+1)$-fold log-product is the formal completion of $\overline{T}_\sigma \times_{V_0} T_\lambda^n$ along $(\overline{T}_\sigma \times \{1,\ldots,1\})$. Its projections to $\overline{T}_\sigma$ are given by $\mathrm{pr}_0(z,g_1,\ldots,g_n) = z$, $\mathrm{pr}_i(z,g_1,\ldots,g_n) = g_i \cdot z$. But locally in the étale topology

$$Y \cong X \cong \overline{T}_{\sigma,\lambda}, \widehat{Y} \cong \widehat{\overline{T}_\sigma},$$

and we use pullback. It follows that via any of the projections pr_i $\widehat{Y}^{n+1,\log}$ is étale locally isomorphic to a product of formal schemes

$$\widehat{Y}^{n+1,\log} \cong \widehat{Y} \times \widehat{T}^n_{\sigma,\lambda}$$

($\widehat{T}_{\sigma,\lambda}$: formal completion along origine). The $\widehat{Y}^{n+1,\log}$ form a simplicial formal scheme.

Next if we reduce modulo some fixed p^s we can form the PD-hulls of the embeddings

$$Y \times_X Y \cdots \times_X Y \hookrightarrow \widehat{Y}^{n+1,\log}$$

and obtain schemes (not just formal) schemes over R_V/p^sR_V, or (again) formal simplicial schemes over R_V, with the p-adic topology:

$$Y_X^\bullet \hookrightarrow D(\widehat{Y}^{\bullet,\log}).$$

These are the log-PD-envelopes of [K]. Any crystal $\mathcal{E}$ on $(X_s/R_{V,s})_{\mathrm{crys}}$ can be evaluated on

$$D(\widehat{Y}^{\bullet;\log}) \otimes_{R_V} R_V/p^sR_V,$$

so gives $\mathcal{E}_n$ on each

$$D(\widehat{Y}^{\bullet;\log})_n \otimes_{R_V} R_V/p^sR_V$$

with pullback-maps which are all isomorphisms. As usual giving a crystal $\mathcal{E}$ is equivalent to giving $\mathcal{E}_0$ on

$$D(\widehat{Y}^{\bullet;\log})_0 \otimes_{R_V} R_V/p^sR_V$$

together with an isomorphism of the two pullbacks to

$$D(\widehat{Y}^{\bullet;\log})_1 \otimes_{R_V} R_V/p^sR_V,$$

satisfying a cocycle condition on

$$D(\widehat{Y}^{\bullet;\log})_2 \otimes_{R_V} R_V/p^sR_V,$$

or giving $\mathcal{E}_0$ with an integrable quasi-nilpotent log-connection

$$\nabla : \mathcal{E}_0 \longrightarrow \mathcal{E}_0 \otimes \Omega^1_{\widehat{Y},\log},$$

where $\Omega^1_{\widehat{Y},\log}$ is the sheaf of the logarithmic differential forms, pullback of dual to the Lie-algebra of T_λ (any character χ of T defines a closed differential form $d\chi/\chi$ on T_σ).

The relation is given by Taylor's formula. Its logarithmic version is mentioned in [K], 6.7.1.

With this connection we define the de Rham complex

$$\mathrm{DR}(\mathcal{E}) = (\mathcal{E} \otimes \Omega^\bullet_{\widehat{Y}^{\bullet;\log},\log}, \nabla + d)$$

on $D(\widehat{Y}^{\bullet,\log})_\bullet$. On $D(\widehat{Y}^{\bullet,\log})_n$ its underlying sheaves are $\mathrm{pr}_0^*(\mathcal{E})\otimes\wedge^\bullet(\bigoplus_{i=0}^{n} \mathrm{pr}_i^*((\Omega^1_{\widehat{Y},\log}))$, with

$$(\nabla + d)(\mathrm{pr}_0^*(e) \otimes \alpha) = \mathrm{pr}_0^*(\nabla e) \cdot \alpha + e \otimes d\alpha.$$

For any PD-thickening $U \hookrightarrow \mathcal{U}$ of $U \to X$ étale we find étale locally a map $(U,\mathcal{U}) \xrightarrow{f} (Y,\widehat{Y})$. Thus $\mathcal{E}_\mathcal{U}$ is determined by pullback. In fact one can form the log-product $(\mathcal{U} \times \widehat{Y}_\bullet)^{\log}$, a formal simplicial scheme with same underlying topological space as $U \times_X Y^\bullet_X$. Etale locally it is isomorphic to $\mathcal{U} \times \overline{T}_{\sigma,\lambda}$. As usual modulo p^s the divided power hull of this is a scheme or algebraic space, and we obtain a diagram

$$(U,\mathcal{U}) \xleftarrow{\mathrm{pr}_1} D((\mathcal{U} \times \widehat{Y}^\bullet)^{\log}) \xrightarrow{\mathrm{pr}_2} D(\widehat{Y}^\bullet)$$

Then it is known that

$$\mathbb{R}\,\mathrm{pr}_{1,*}\,\mathrm{pr}_2^*(\mathrm{DR}(\mathcal{E}))$$

is a resolution of $\mathcal{E}_\mathcal{U}$, and globalises to an u_X-acyclic resolution of $\mathcal{E}$ on the crystalline topos (One can drop the $\mathbb{R}$ because Y is affine). That it is a resolution follows as in [B] from the Poincaré-lemma, as locally $\mathcal{U} \times^{\log} \widehat{Y}^{n+1,\log}$ is isomorphic to the (usual) product $\mathcal{U} \times T_\lambda^{n+1}$. Also one uses cohomological descent (Y covers X).

That the resolution is acyclic follows from general category theory. Namely crystalline cohomology is the derived projective limit of étale cohomology of small thickenings. Here a thickening $U \hookrightarrow \mathcal{U}$ is called small if it is affine and maps to $D(\widehat{Y})$, and the derived projective limit is over the category of small thickenings. As this category has products and every object maps to $D(\widehat{Y})$, and the derived projective limit is the cohomology of the simplicial scheme $D(\widehat{Y}^{\bullet,\log})$. Also étale cohomology of quasicoherent sheaves on affines vanishes in higher degrees. As Y is affine we can represent the derived direct image by the usual direct image of quasicoherent sheaves. Thus $\mathbb{R}u_{X,*}(\mathcal{E})$ on $X^{\text{ét}}$ is representable by the direct image of the de Rham complex $\mathrm{DR}(\mathcal{E})$ on $D(\widehat{Y}^\bullet)$. One can check directly that the result is independant of the choice of a covering Y and its lift $\widehat{Y}$, up to canonical quasi-isomorphism in the derived category. Namely for two such choices Y_1, Y_2 and $\widehat{Y}_1, \widehat{Y}_2$ there are such comparison maps from the de Rham complex formed from $\widehat{Y}_1 \amalg \widehat{Y}_2$.

The same holds in the filtered context, that is for $\mathcal{E}$ filtered the de Rham complexes give filtered resolutions. In local coordinates $D((Y \times \widehat{Y}^\cdot)^{\log})_n$ corresponds to a free PD-algebra $\mathcal{O}_V\{t_\nu\}$ in $(n+1) \cdot d$ variables t_ν (= local coordinates near 1 in T_λ^{n+1}), and the de Rham complex is the tensor product of $\mathcal{E}_V$ and the de Rham complex in the $\{t_\nu\}$, with its product filtration.

Basically we are interested in the cohomology of crystals of vectorbundles. However we also need boundary conditions. Recall that $X \otimes_V k$ is reduced, or equivalently that $X \otimes_V \overline{V}$ is normal. Then étale locally on X there exists an étale morphism $X \to \overline{T}_{\sigma,\lambda}$, for some $\sigma \subset L_{\mathbb{R}}$ such that the extremal rays of σ are generated by ρ_i with $\lambda(\rho_i) \in \{0,1\}$. Any subset I of them defines a reduced divisor with support in the boundary (which includes the special fibre), and thus an ideal $J_I \subseteq \mathcal{O}_X$. For $I = \{\rho \mid \lambda(\rho) = 1\}$ we have $J_I = \pi \cdot \mathcal{O}_X$. Now restrict to subsets I with $\lambda(\rho) = 0$ for $\rho \in I$, and let I^c denote the complement of I in the set of all such ρ's, which defines the ideal $J \subseteq \mathcal{O}_X$. Then $J \cdot \Omega^{d,\log}_{X/V}$ is the relative dualizing complex (as in chapter 2 explanations proceeding proposition 9), $J_{I_c} = \mathrm{Hom}_{\mathcal{O}_X}(J_I, J)$. This is also equal to the derived Hom as J_I is Cohen-Macaulay (again chapter 2, explanations proceeding theorem 10'). Globally we assume given an ideal J_I which locally is of the type above. For example this is always possible for $\mathcal{O}_X$ or J. Also if J_I is defined so is J_{I^c}.

The ideals J_I extend naturally to $\overline{T}$ and thus to any object (U, V) of the log-crystalline topos, and form crystals of ideals. We are interested in the cohomologies $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I), \mathcal{E}$ a (filtered) crystal of vectorbundles on X^{crys}. As it stands this is defined for crystals modulo a fixed power p^s, and then can be represented (in the filtered derived category) by the hypercohomology of $\mathrm{DR}(\mathcal{E} \otimes J_I)$ on $D(\widehat{Y}^{\bullet,\log})$. Strictly speaking we still have to prove that because J_I is not itself a crystal of vectorbundles. However the same proof as before applies. As Y is affine so is each

$$(Y^{\bullet}_X)_n = \mathrm{Spec}(S_n), D((\widehat{Y}^{\bullet,\log})_n) = \mathrm{Spec}(D(S_n)),$$

and the de Rham complexes

$$(\mathcal{E} \otimes J_I)(D((\widehat{Y}^{\bullet,\log})_n) \otimes \Omega^{\cdot}_{S_n,\log}$$

form (modulo p^s) a double complex whose associated total complex represents the crystalline cohomology. It is in fact easily seen (comparing two coverings) that this is (up to canonical isomorphism in the filtered derived category) independent of choices. Also if we have a compatible system $\mathcal{E}$ of filtered crystals of vectorbundles $\mathcal{E}_s$ modulo each p^s, we may form the projective limit (for $s \to \infty$) of the corresponding complexes to get the cohomology of $\mathcal{E}$ (in the same spirit as earlier the cohomology of $A_{\inf}(\overline{\mathcal{O}})$). As the sheaves J_I are modulo each p^s flat over $R_{V,s} = R_V/p^s R_V$, the proof in [Fa7] can be modified to show that $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I)$ can be represented by a finite complex of filtered free R_V-modules if $\mathcal{E}$ is locally filtered free:

In [Fa7] it was shown that it suffices to verify the assertion modulo the PD-ideal and modulp p. Modulo the PD-ideal we have the cohomology of the relative de Rham complex, and everything is fine. Modulo p [Fa7] assumed that X is smooth, to guarantee that the relative Frobenius is flat. However this is not necessary. Namely in general the crystalline direct image under Frobenius is representable by a finite complex of quasicoherent sheaves on $X \otimes_V (R_V/p \cdot R_V)$, where the tensor product involves Frobenius. This complex is locally (in X) representable by a complex (bounded above) of finitely generated free modules, that is it is strictly pseudocoherent ([SGA6]). Namely one chooses a local smooth lift of X and its de Rham complex, and uses that relative Frobenius is strictly pseudocoherent because it is basechange from a noetherian situation. Thus it suffices to use the following result in the style of [SGA6]:

Lemma.1. — *Suppose $f : X \to S$ is a proper flat map of algebraic spaces, of finite presentation. Assume $\mathcal{K}^\bullet$ is a pseudocoherent complex on X. Then $\mathbb{R}f_*(\mathcal{K}^\bullet)$ is also strictly pseudocherent. If in addition $\mathcal{K}^\bullet$ has finite Tor-dimension over S, the derived direct $\mathbb{R}f_*(\mathcal{K}^\bullet)$ is perfect.*

Proof. — The assertion is local in S, so we may assume that $S = \operatorname{Spec}(R)$ is affine. Writing R is filtering union of noetherian subrings R_α we know that X is is already defined over some R_α, that is is basechange from a proper flat X_α/R_α. Now for a bounded above quasicoherent complex $\mathcal{K}_\alpha^\bullet$ on X_α we shall show below that it is quasi-isomorphic to such a complex of flat quasicoherent sheaves. Thus we can define the derived tensor product $\mathcal{K}_\alpha^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\alpha}} \mathcal{O}_X$. As every quasicoherent sheaf $\mathcal{F}$ on X is the filtering inductive limit of sheaves $\mathcal{F}_\alpha \otimes_{\mathcal{O}_{X_\alpha}} \mathcal{O}_X$ with $\mathcal{F}_\alpha$ coherent on X_α, it follows easily that for any integer N

$$\tau_{\geqslant -N}\mathcal{K}^\bullet = \tau_{\geqslant -N}(\mathcal{K}_\alpha^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_{X_\alpha}} \mathcal{O}_X)$$

for a coherent complex $\mathcal{K}_\alpha^\bullet$ on X_α, if α is big enough. So finally (d = cohomological dimension of $\mathbb{R}f_*$)

$$\tau_{\geqslant d-N}\mathbb{R}f_*(\mathcal{K}^\bullet) = \tau_{\geqslant d-N}\mathbb{R}f_{\alpha,*}(\mathcal{K}_\alpha^\bullet) \otimes^{\mathbb{L}}_{R_\alpha} R$$

is pseudocoherent in degrees $\geqslant d - N$, and we are done (N being arbitrary). If in addition $\mathcal{K}^\bullet$ has finite Tor-dimension over the base, the same holds for $\mathbb{R}f_*\mathcal{K}^\bullet$, and it is perfect.

It remains to show the assertion about flat resolutions. Choose an affine étale cover $Y \to X$, and denote by Y_n the open and closed subscheme of the n-fold product (over X) Y^n where all projections are different. Then Y_n is affine, and empty for big n. Also it admits a free action of the symmetric group S_n. Now over a ring any bounded above complex of modules has a functorial (even for change of rings) flat resolution. For example for a single module M write M as quotient of the free module with basis M, and continue with the kernel. Thus for a bounded above quasicoherent complex $\mathcal{K}^\bullet$ on X we choose such resolutions on each Y_n, form the anti-invariants under S_n, and their direct images form a double complex whose associated total complex is the desired resolution. The new maps are alternating sums of pullbacks via projections.

It is also shown in [Fa7] (the proof extends) that Berthelot's construction from [B] gives a trace-form

$$H^{2d}(X^{\mathrm{crys}}/R_V, J) \longrightarrow R_V\{-d\},$$

which induces a perfect duality (of complexes of filtered free R_V-modules)

$$\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E}^\vee \otimes J_{I^c}) \xrightarrow{\approx} \mathbb{R}\operatorname{Hom}_{R_V}(\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I), R_V\{-d\}[-2d])$$

Also there is a Künneth-formula. In short all the usual formalism is again available.

For the reader's convenience we recall the main steps in Berthelot's proof, and why our more exotic base, as well as the fact that X may be only an algebraic space, does not cause problems:

First of all it suffices to construct traces modulo any positive p-power p^s, that is work over quotients $R_{V,s}$. The filtration by dimension of support (a closed subset of the special fibre of X) defines a spectral sequence converging to cohomology. Its $E_1^{a,b}$-terms is a filtering union indexed by closed reduced subspaces Z of the special fibre, of codimension a, of the degree-$(a+b)$ crystalline cohomology of J with support in the generic points Z° of Z. This local cohomology is computed by the hypercohomology (with support in Z°) of the de Rham complex of a (flat) local lift of X near Z°. If $J \otimes \Omega_X^{\mathrm{lift},\cdot}$ denotes this complex, we need for each Z of dimension 0 a local trace

$$\mathrm{tr}_x : H^d_{\{Z\}}(J \otimes \Omega_X^{\mathrm{lift},d}) \longrightarrow R_{V,s}.$$

Furthermore these local traces must vanish in the image of $H^d_{\{Z^\circ\}}(J \otimes \Omega_X^{\mathrm{lift},d-1})$, and for Z of dimension 1 we need a reciprocity law, stating that for elements of $H^{d-1}_{\{Z^\circ\}}(J \otimes \Omega_X^{\mathrm{lift},d})$ the sum of traces of boundaries vanishes. Note that all these cohomologies live in the highest possible degree, so that they form a right exact functor. Now the local lifting can be already done over $V_0[[t]]$, or better its quotient under the ideal (p^s, t^N), with N so big that this ring still maps to $R_{V,s}$. This quotient is artinian and Gorenstein, and Berthelot's method gives local trace-maps first over $V_0[[t]]/(p^s, t^N)$ and then by tensor product over $R_{V,s}$. In fact they are defined by local duality theory. That they vanish on the image of d and that the reciprocity law holds then can be checked over $V_0[[t]]/(p^s, t^N)$. For the first we can compute directly as in [B], or choose a local projection to affine space. For the latter one glues as in [B] infinitesimal neighbourhoods of the closure Z, Z a closed subspace of dimension 1. Note that these are schemes.

For a closed immersion of a divisor $i : Z \hookrightarrow X$ transversal to all strata (i.e. étale locally $(X, Z) = X_1 \times (\mathbb{A}^1, 0)$) there is an exact sequence (triangle) with $\widetilde{X}$ corresponding to the new toroidal structure with open stratum $X^0 - Z^0$

$$\begin{aligned} 0 \longrightarrow \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) &\longrightarrow \mathbb{R}\Gamma(\widetilde{X}^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) \\ &\longrightarrow \mathbb{R}\Gamma(Z^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I)\{-1\}[-1] \longrightarrow 0 \end{aligned}$$

(see [Fa3]) and the connecting map represents i_*:

The exact sequence is obtained from de Rham complexes on $D(\widehat{Y}^\bullet)$. The usual de Rham complex maps to the one with logarithmic poles along the preimage of Z, and that in turn via the residue map to the de Rham complex of this preimage. To see that it represents i_* it suffices (as our exact sequences are compatible with cup-products) that the trace on $H^{2d-2}(Z^{\mathrm{crys}}/R_V, J)$ is equal to the composite of the connecting map with the trace on $H^{2d}(X^{\mathrm{crys}}/R_V, J)$. But the trace is characterised by its values on local cohomology

$$H^{2d-2}_{\{z\}}(Z^{\mathrm{crys}}/R_V, J) \quad (z \in Z \otimes_V k \quad \text{a closed point}).$$

This can be computed by locally lifting to

$$\mathcal{Z} = \mathfrak{X}^* \times \{0\} \hookrightarrow \mathfrak{X} = \mathfrak{X}^* \times \mathbb{A}^1,$$

with $\mathfrak{X}^*$ étale over some torus-embedding $\overline{T}^*$. The exact sequence reduces to the tensor-product of $J \circ \Omega^{\bullet,\log}_{\mathfrak{X}^*}$ with

$$0 \longrightarrow \Omega^\bullet_{\mathbb{A}^1} \longrightarrow \Omega^\bullet_{\mathbb{A}^1}(d\log 0) \xrightarrow{\text{Residue}} \mathcal{O}_{\{0\}}\{-1\}[-1] \longrightarrow 0,$$

and the assertion is wellknown.

Equivalently we may project locally to affine space, which reduces us to smooth schemes. It then suffices to verify the claim in one global example, as the inclusion of a hyperplane in projective space.

Finally for a line-bundle $\mathcal{L}$ on X the first Chern-class $c_1(\mathcal{L}) \in H^2(X^{\mathrm{crys}}/R_V, F^1(\mathcal{O}))$ is defined via the connecting map

$$\begin{aligned}\mathrm{Pic}(X) = H^1(X^{\mathrm{crys}}/R_V, (\mathcal{O}/F^1(\mathcal{O}))^*) \longrightarrow \\ H^2(X^{\mathrm{crys}}/R_V, (1+F^1(\mathcal{O}))) \xrightarrow{\log} H^2(X^{\mathrm{crys}}/R_V, F^1(\mathcal{O})).\end{aligned}$$

As usual this defines higher Chern classes, the selfintersection formula holds, and also (for immersions $Z \hookrightarrow X$ with locally $(X,Z) \cong X^* \times (\mathbb{A}^t, 0)$) the formula ($\widetilde{X}$ = blow-up of Z, E = exceptional divisor, $\mathcal{F}$ = normal bundle to Z)

$$\mathrm{pr}_X^* \circ i_* = \widetilde{i}_*(c_{t-1}(\mathcal{F}) \cup \mathrm{pr}_Z^*(\,))$$

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{i}} & \widetilde{X} \\ {\scriptstyle \mathrm{pr}_Z}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_X} \\ Z & \xrightarrow{i} & X \end{array}$$

So far the review.

Next relations to étale cohomology. For a ring R with $\mathrm{Spec}(R)$ étale over $\overline{T}_\lambda$ we have defined $A_{\mathrm{inf}}(\overline{R})$, with an isomorphism $A_{\mathrm{inf}}(\overline{R})/\xi \cdot A_{\mathrm{inf}}(\overline{R}) \xrightarrow{\sim} \widehat{\overline{R}}$ (p-adic completion). We define $A_{\mathrm{crys}}(\overline{R})$ as the p-adically completed divided power hull of $\xi \cdot A_{\mathrm{inf}}(\overline{R})$. It is filtered by the PD-filtration. For varying R the $A_{\mathrm{crys}}(\overline{R})/p^s \cdot A_{\mathrm{crys}}(\overline{R})$ form an almost sheaf of rings on $\mathfrak{X}^0 \otimes_V \overline{K}$, not just a prosheaf (like $A_{\mathrm{inf}}(\overline{R})$). This is so because the action of $\mathrm{Gal}(\overline{R}/R)$ on $A_{\mathrm{crys}}(\overline{R})/p^s \cdot A_{\mathrm{crys}}(\overline{R})$ (with discrete topology) is continuous. We denote these almost sheaves by $A_{\mathrm{crys}}(\overline{\mathcal{O}})/p^s \cdot A_{\mathrm{crys}}(\overline{\mathcal{O}})$, and by $A_{\mathrm{crys}}(\overline{\mathcal{O}})$ the prosheaf they define. Thus we may form the cohomology $\mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{\mathcal{O}}))$, which however turns out to be almost equal to $\mathbb{R}\Gamma((X^0 \otimes_V \overline{K})^{\mathrm{\acute{e}t}}, \mathbb{L}) \otimes^{\mathbb{L}} A_{\mathrm{crys}}(\overline{V})$, as

$$A_{\mathrm{crys}}(\overline{R})/p^s \cdot A_{\mathrm{crys}}(\overline{R}) \cong A_{\mathrm{inf}}(\overline{R})/p^s \cdot A_{\mathrm{inf}}(\overline{R}) \otimes_{A_{\mathrm{inf}}(\overline{V})} A_{\mathrm{crys}}(\overline{V}).$$

Namely use theorem 8 in chapter 3 and theorem 9 in chapter 4, as well as the fact that $A_{\mathrm{crys}}(\overline{R})/p^s A_{\mathrm{crys}}(\overline{R})$ is the quotient of the free "PD-polynomial ring" $(A_{\mathrm{inf}}(\overline{R})/p^s A_{\mathrm{inf}}(\overline{R}))\{u\}$ under the ideal generated by one regular element $u - \xi$. The same argument also applies to $F^i(A_{\mathrm{inf}}(\overline{\mathcal{O}})$. Similarly let $(A_{\mathrm{crys}}(\overline{R}/\overline{J}_I)$ is defined as is $A_{\mathrm{crys}}(\overline{R})$, replacing $\overline{R}$ by $\overline{R}/\overline{J}_I)$

$$A_{\mathrm{crys}}(\overline{J}_I) = \ker(A_{\mathrm{crys}}(\overline{R}) \longrightarrow A_{\mathrm{crys}}(\overline{R}/\overline{J}_I))$$

$=$ p-adic completion of $A_{\mathrm{inf}}(\overline{J}_I) \otimes_{A_{\mathrm{inf}}(\overline{V})} A_{\mathrm{crys}}(\overline{V})$, and then

$$\mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}) \otimes^{\mathbb{L}} A_{\mathrm{crys}}(\overline{V}) \approx \mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{J}_I)).$$

Also $A_{\mathrm{inf}}(\overline{R})$ and $A_{\mathrm{crys}}(\overline{R})$ get toroidal structures by choosing for each μ in a basis of $L^\vee$ a compatible system of p-power roots of μ, defining $\underline{\mu} = [\mu] \in A_{\mathrm{inf}}(\overline{R})$. Finally $A_{\mathrm{crys}}(\overline{V})$ becomes an R_V-algebra by mapping t to $\underline{\pi}$ which is defined by chosing a compatible system of p-power roots of the uniformiser π of V. This also makes each $A_{\mathrm{crys}}(\overline{R})$ into a logarithmic R_V-algebra.

Next the preimage $A^0_{\mathrm{crys}}(\overline{R})$ of R in $A_{\mathrm{crys}}(\overline{R})$ defines a PD-thickening. Thus if $\mathcal{E}$ is a crystal modulo p^s on X^{crys}/R_V, we can evaluate it on $A^0_{\mathrm{crys}}(\overline{R})$ to get a module with an action of $\mathrm{Gal}(\overline{R}/R \otimes_V \overline{V})$. This is functorial in R, thus defines a presheaf (in fact almost a sheaf for small enough $\mathrm{Spec}(R)$) on $\mathfrak{X}^0 \otimes_V \overline{K}$.

Now assume given a locally constant sheaf $\mathbb{L}$ (annihilated by p^s) on $(X^0 \otimes_V \overline{K})^{\text{ét}}$, and a functorial map

$$\mathcal{E}\,(A^0_{\mathrm{crys}}(\overline{R})) \longrightarrow \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{R}),$$

linear over $A^0_{\mathrm{crys}}(\overline{R})$ and commuting with the Galois-action. An example is $\mathcal{E} = \mathcal{O}_X$ and $\mathbb{L} = \mathbb{Z}/p^s\mathbb{Z}$. Then the de Rham resolution of $\mathcal{E}$ on the crystalline topos defines by evaluation a resolution of the presheaf $\mathcal{E}$ on $\mathfrak{X}^0 \otimes_V \overline{K}$, which maps to an injective resolution of $\mathbb{L} \otimes A_{\mathrm{crys}}(\overline{R})$. Hence we obtain a transformation

$$\begin{aligned} v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E}) &\longrightarrow \mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{\mathcal{O}})) \\ &\approx \mathbb{R}\Gamma((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}) \otimes A_{\mathrm{crys}}(\overline{V}). \end{aligned}$$

If $\mathcal{E}$ is filtered and the basic maps to $\mathbb{L} \otimes A_{\mathrm{crys}}(\overline{R})$ preserve filtrations, we obtain a transformation in the filtered derived category. Recall how to construct a good filtered injective resolution of a filtered object $\mathcal{E}, F^p(\mathcal{E})$ in a topos: We first inject each $\mathcal{E}/F^p(\mathcal{E})$ into an injective I^p, as well as $\mathcal{E}$ into I, and inject $\mathcal{E} \to I \times \prod I^p$, filtered by

$$F^p(I \times \textstyle\prod I^q) = I \times \prod_{q>p} I^q.$$

Finally continue with the quotient.

Also there is the obvious variant

$$\begin{aligned} v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) &\longrightarrow \mathbb{R}\Gamma(\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{J}_I)) \\ &\approx \mathbb{R}\Gamma_{(!)}((X^0) \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}) \otimes^{\mathbb{L}} A_{\mathrm{crys}}(\overline{V}). \end{aligned}$$

Next we check how much the map commutes with Chern-classes and direct images. Recall that there is a canonical map

$$\beta : \mathbb{Z}_p(1) \longrightarrow F^1(A_{\mathrm{crys}}(\overline{V})).$$

If $t \in \mathbb{Z}_p(1)$ is a generator, corresponding to a compatible sequence of primitive roots $\zeta_n \in \boldsymbol{\mu}_{p^n}(\overline{V}), \underline{1} = \varprojlim \zeta_n$ defines $[\underline{1}] \in A_{\mathrm{inf}}(\overline{V})$. Then $\beta(t) = \log \underline{1}$.

2. Theorem

i) *For a vectorbundle $\mathcal{E}$ on X, $c_i^{\mathrm{cr}}(\mathcal{E}) \in F^i H^{2i}(X^{\mathrm{crys}}/R_V, \mathcal{O}_X)$ maps to*

$$v_X(c_i^{\mathrm{cr}}(\mathcal{E})) = \beta^{\otimes i} \cdot c_i^{\text{ét}}(\mathcal{E}) \in H^{2i}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z}) \otimes F^i A_{\mathrm{crys}}(\overline{V}).$$

ii) *Assume $i : Z \hookrightarrow X$ is a closed immersion, such that locally in the étale topology $(X, Z) = X^* \times (\mathbb{A}^t, O)$. Then $v_X \circ i_*^{\mathrm{cr}} = \beta^{\otimes t} \cdot i_*^{\text{ét}} \circ v_Z$, as transformations (in the almost category)*

$$\mathbb{R}\Gamma(Z^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) \longrightarrow \mathbb{R}\Gamma((\mathfrak{X}^0 \otimes_V \overline{K}, \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{\mathcal{J}_I}))\{t\}[2t]$$

Proof. — For i) using example 1* of section 4 it suffices to treat $c_1(\mathcal{L})$ for line-bundles $\mathcal{L}$. Both Chern-classes are boundaries of the class of $\mathcal{L}$ in $H^1(X, \mathcal{O}_X^*)$, with respect to either

$$0 \longrightarrow 1 + F^1(\mathcal{O}_X^{\mathrm{crys}}) \longrightarrow \mathcal{O}_X^{\mathrm{crys},*} \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

(and composing with $1 + F^1(\mathcal{O}_X^{\mathrm{crys}}) \xrightarrow{\log} F^1(\mathcal{O}_X^{\mathrm{crys}})$), respectively

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathcal{R}(\overline{\mathcal{O}})^* \longrightarrow \widehat{\overline{\mathcal{O}}}^* \longrightarrow 0$$

However evaluated on $A_{\mathrm{crys}}(\overline{\mathcal{O}})$ the second sequence maps to the first via the composition (the first map is the Teichmüller-representative)

$$\mathcal{R}(\overline{\mathcal{O}})^* \longrightarrow A_{\mathrm{inf}}(\overline{\mathcal{O}})^* \longrightarrow A_{\mathrm{crys}}(\overline{\mathcal{O}})^*,$$

hence the assertion.

Similarly by blow-up and the fundamental formula

$$\mathrm{pr}_X^* \circ i_* = \widetilde{i}_*(c_{t-1}(\mathcal{F}) \cup \mathrm{pr}_Z^*())$$

ii) is reduced to the case of a divisor, that is $t = 1$. As before we denote by $\widetilde{X}$ X with the modified toroidal structure where Z is added to the boundary. There exists an étale covering of X by affines $\mathrm{Spec}(R)$ such that either $\mathrm{Spec}(R)$ lies over $X \setminus Z$ and has an étale toric parametrization $\mathrm{Spec}(R) \to \overline{T}_{\lambda,\sigma}$, or $\mathrm{Spec}(R) \to \overline{T}_{1,\lambda,\sigma} \times \mathbb{A}^1$ is an étale parametrization for the two possible toric structures (depending on whether $\{0\} \subseteq \mathbb{A}^1$ is added to the boundary). In the second case (that is Z not part of the boundary) we also assume that R has enough units to define a good R_∞. For example $\mathrm{Spec}(R)$ could lie above $\overline{T}_{1,\lambda} \times (\mathbb{A}^1 - \{1\})$. Let

$$Y_0 = \widetilde{Y}_0 = \amalg\, \mathrm{Spec}(R) = \mathrm{Spec}(S_0)$$

denote the corresponding affine cover of X,

$$\widetilde{Y}_n = \mathrm{Spec}(\widetilde{S}_n) \longrightarrow Y_n = \mathrm{Spec}(S_n)$$

the $(n+1)$-fold logarithmic fibre-products (in the two different log-structures),

$$\widehat{\widetilde{Y}}_n = Spf(\widetilde{S}'_n), \widehat{Y}_n = Spf(S'_n)$$

affine schemes locally étale over $\widehat{\overline{T}}^{n+1,\log}$ (respectively $(\overline{T}_1 \times \mathbb{A}^1)^{\wedge,n+1,\log}$, and denote by $Spf(D(\widetilde{S}'_n)), Spf(D(S'_n))$ the divided power envelopes. The crystalline cohomology $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I)$ can be represented by the total complex associated to the de Rham complexes $(\mathcal{E} \otimes J_I)(D(S'_n)) \otimes \Omega^{\bullet,\log}_{S'_n/R_V}$, and similarly for $\widetilde{X}$. However the $\widehat{\widetilde{Y}}'_n$ are also log-smooth in the log-structure of X, i.e. without adding Z to the boundary (the logarithmic product $(\mathbb{A}^1, \{0\}^{n+1,\log} = \mathbb{A}^1 \times \mathbb{G}^n_m$ is smooth). Thus they may also be used to compute the crystalline cohomology of X. That is there are the subcomplexes

$$(\mathcal{E} \otimes J_I)(D(\widetilde{S}'_n)) \otimes \Omega^{\bullet,\log}_{\sim\widetilde{S}'_n/R_V} \subseteq (\mathcal{E} \otimes J_I)(D(\widetilde{S}'_n)) \otimes_{\widetilde{S}'_n} \Omega^{\bullet,\log}_{\widetilde{S}'_n/R_V}$$

whose associated total complex represents

$$\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I).$$

The quotient represents $\mathbb{R}\Gamma(Z^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_{Z,I})\{-1\}[-1]$, with connecting map i_*^{crys}.

Also the local parametrizations give an equation $f_0 \in S'_0$ for the Z-stratum. By definition of the logarithmic product $f_1 = \mathrm{pr}_0^*(f_0)/\mathrm{pr}_1^*(f_0) \in \widetilde{S}_1^{\prime,*}$ is a unit. $d\log f_0$ is a 1-cochain with residue 1 in the de Rham complex $D(\widetilde{S}'_0) \otimes \Omega^{\bullet,\log}_{\widetilde{S}'_0/R_V}$. Also the de Rham complex for Z can be identified with the quotient of the

$$(\mathcal{E} \otimes J_I)(D(\widetilde{S}'_n)) \otimes \Omega^{\bullet,\log}_{\sim\widetilde{S}'_n/R_V}$$

obtained by setting

$$\mathrm{pr}_0^*(f_0) = d\log(\mathrm{pr}_0^*(f_0)) = 0.$$

Thus to compute $i_*^{\mathrm{crys}}(z)$ for a class $z \in H^m(Z^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_{Z,I})$, represent z by an m-cocycle $\overline{\lambda}$ in this quotient complex, lift to an m-cochain λ in the complex for X, form the product (in the de Rham complex for $\widetilde{X}$)

$$(d\log \mathrm{pr}_0^*(f_0), 0) \cup \lambda$$

and apply to it the differential to get a class in $H^{m+2}(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I)$ representing $i_*^{\mathrm{crys}}(z)$. For example to lift 1 we can use the cochain ψ represented by $(df_0/f_0, 0)$.

Now let us map to étale cohomology. There is a diagram of topoi

$$\widetilde{\mathfrak{X}}^0 \otimes_V \overline{K} \xhookrightarrow{j} \mathfrak{X}^0 \otimes_V \overline{K} \xleftarrow{i} \mathfrak{Z}^0 \otimes_V \overline{K}.$$

We know that

$$j_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I}))\approx \mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_I))$$
$$\mathbb{R}^1 j_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I}))\approx i_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{Z,I}))(-1),$$

and higher direct images

$$\mathbb{R}^\nu j_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I}))\quad(\nu\geqslant 2)\quad\text{or}\quad \mathbb{R}^\nu i_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{Z,I}))\quad(\nu\geqslant 1)$$

are almost zero. For $\overline{J}_{\widetilde{X},I}$ this has been shown in proposition 1 of chapter 4, and this implies the result for $\mathcal{R}(\overline{J}_{\widetilde{X},I}), A_{\mathrm{inf}}(\overline{J}_{\widetilde{X},I}), A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I})$.

Next our de Rham resolutions define complexes $L_X(\mathcal{E}\otimes J_I)$ or shorter L_X on $\mathfrak{X}^0\otimes_V\overline{K}, L_{\widetilde{X}}$ on $\widetilde{\mathfrak{X}}^0\otimes_V\overline{K}$ and L_Z on $\mathcal{Z}^0\otimes_V\overline{K}$, acyclic in degrees $\neq 0$, with an injection $L_X\hookrightarrow j_*L_{\widetilde{X}}$.

For general $\mathcal{E}$ and J_I we need cup-products. The de Rham and Čech complexes admit strictly associative differential products (in the Čech-setting, $(f\cup g)=\mathrm{pr}^*_{0,\ldots,a}(f)\cup\mathrm{pr}^*_{a,\ldots,a+b}(g)$ for f in degree a,g in degree b). This defines such products on $L_X(\mathcal{O}_X), L_{\widetilde{X}}(\mathcal{O}_{\widetilde{X}}), L_Z(\mathcal{O}_Z)$, and makes $L_X(\mathcal{E}\otimes J_I)$ etc. to differentially graded modules over them. Furthermore the elements $\mathrm{pr}_0^* f_0\in\widetilde{S}'_n$ generate an invertible ideal $\langle f_0\rangle$ invariant under all simplicial maps, and thus a differential ideal $\langle f_0, df_0\rangle\subseteq L_X(\mathcal{O})$. The quotient $L_X(\mathcal{O}_X)/\langle f_0,df_0\rangle$ naturally maps to $i_*L_Z(\mathcal{O}_X)$ induced from $A_{\mathrm{crys}}(\overline{\mathcal{O}}_X)\to i_*A_{\mathrm{crys}}(\overline{\mathcal{O}}_Z)$. Next we have defined a class

$$\psi\in\Gamma(\mathfrak{X}^0\otimes_V\overline{K}, j_*L^1_{\widetilde{X}}(\mathcal{O}_{\widetilde{X}})/L^1_X(\mathcal{O}_X)),$$

the image of $(\frac{df_0}{f_0},0)$. Under the multiplication of $L_X(\mathcal{O}_X)$ on $j_*L_{\widetilde{X}}(\mathcal{O}_{\widetilde{X}})/L_X(\mathcal{O}_X)$ this class is annihilated by $\langle f_0, df_0\rangle$. Thus cup-product defines a map

$$\cup\psi: L_X(\mathcal{E}\otimes J_I)/\langle f_0,df_0\rangle\longrightarrow j_*L_{\widetilde{X}}(\mathcal{E}\otimes J_I)/L_X(\mathcal{E}\otimes J_I)[+1]$$

We claim that

$$\begin{tikzcd}
L_X(\mathcal{E}\otimes J_I)/\langle f_0,df_0\rangle \arrow[r, "\cup\psi"] \arrow[d] & j_*L_{\widetilde{X}}(\mathcal{E}\otimes J_I)/L_X(\mathcal{E}\otimes J_I)[+1] \arrow[d] \\
i_*L_Z(\mathcal{E}\otimes J_I) \arrow[d] & \mathbb{R}j_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I}))/\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{X,I})[+1] \arrow[d] \\
i_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{Z,I})) \arrow[r, "\circ\beta"] & i_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{Z,I})(-1)
\end{tikzcd}$$

commutes in the derived category:

We will check that the domain (the upper left corner) has trivial higher cohomology. It then suffices to verify the assertion for the induced map on H^0, and the assertion becomes local in $\mathfrak{X}\otimes\overline{K}$. Hence we can work over a strictly henselian ring $R=\mathcal{O}^{\mathrm{sh}}_{X,x}$. Also the assertion does not depend on the choice of the covering Y, so we may assume that $S_0=R$. Then $L_X(\mathcal{E}\otimes J_I)$ becomes the tensor product of $\mathcal{E}\otimes J_I(A_{\mathrm{crys}})$, a free divided power algebra in certain variables t_μ, and the de Rham complex in the derivatives dt_μ. One of these say t_1, can be chosen as $t_1 = f_0 - h_0$, h_0 an element of $A_{\mathrm{crys}}(R)$ lifting f_0. It is (as it should be) a resolution of $\mathcal{E} \otimes J_I(A_{\mathrm{crys}})$. If we divide by f_0 and df_0 the cohomology is still concentrated in degree 0, but now is quotient of $\mathcal{E} \otimes J_I(A_{\mathrm{crys}})\langle t_1\rangle$ (tensor product with free DP-algebra in t_1) under the regular element $t_1 + h_0$ (multiplication by it is regular because this holds for multiplication by h_0 on $A_{\mathrm{crys}}(R)/p^s$). By an obvious compatability with cup-products (by $\mathcal{E}\otimes J_I(A_{\mathrm{crys}})$) we may reduce to constant coefficients, that is $\mathcal{E} = \mathcal{O}/p^s$, $\mathbb{L} = \mathbb{Z}/p^s$, $J_I = \mathcal{O}$. Furthermore it suffices to check commutativity for the image of the unit 1:

Namely from the description above for any class in $H^0(L_X(\mathcal{O}/p^s)/\langle f_0, df_0\rangle)$ we can find a p-power p^t such that first of all our class lifts to $H^0(L_X(\mathcal{O}/p^{s+t})/\langle f_0, df_0\rangle)$, and secondly becomes a multiple of 1 after multiplying it by p^t. If the required equality holds for 1 it holds in $A_{\mathrm{crys}}(R_Z)/p^{s+t}$ (the H^0 of the target) up to p^t-torsion, and everything follows.

Thus we check what happens to 1. As usual we have fundamental-groups $\Delta, \widetilde{\Delta}$, and $\underset{\sim}{\Delta} = \ker(\widetilde{\Delta} \to \Delta)$. Let $D \subseteq \Delta$ and $\widetilde{D} \subseteq \widetilde{\Delta}$ denote the decomposition groups of Z, that is for compatible extensions of the prime ideal defining Z. Then $\widetilde{D}$ projects onto D, with kernel $\widehat{\mathbb{Z}}(1)$.

We have to compare two elements in

$$H^1(\underset{\sim}{\Delta}, A_{\mathrm{crys}}(\widetilde{\overline{R}})/p^s) \approx \mathrm{Ind}_{\widetilde{D}}^{\widetilde{\Delta}}(H^1(\widehat{\mathbb{Z}}(1), A_{\mathrm{crys}}(\overline{R}_Z)/p^s).$$

It suffices to compare evaluations on compatible lifts of the prime dividing Z. Also we assumed $S_0 = R$. Especially f_0 now restricts to an element of R. Choosing compatible p-power roots of this restriction of f_0 defines an element $\underline{f}_0 \in \mathcal{R}(\widetilde{\overline{R}})$ and its Teichmüller lift $[\underline{f}_0]$ in $A_{\mathrm{inf}}(\widetilde{\overline{R}})$ and $A_{\mathrm{crys}}(\widetilde{\overline{R}})$. The quotient $g_0 = 1 \otimes f_0/[\underline{f}_0] \otimes 1$ is a 1-unit in the log-product used for $L_{\widetilde{X}}$, that is it is equal to one modulo the PD-ideal. Thus we may form its logarithm $\log(g_0)$, and the coboundary of the product $e \cup \log(g_0)$ is $e \cup (df_0/f_0, \log(f_1)$ (f_1 has turned into a 1-unit as well). Hence modulo this coboundary our chain $e \cup \psi$ lies in L_X. However this does not mean that we obtain the trivial class because g_0 and $\log(g_0)$ are not invariant under $\widetilde{\Delta}$, so they define only local and not global sections of $A_{\mathrm{crys}}(\overline{\mathcal{O}})/p^s$ over $\widetilde{X}^0 \otimes \overline{K}$. Namely the action of $\widetilde{\Delta}$ is described by the character

$$\chi : \widetilde{\Delta} \longrightarrow \mathbb{Z}_p(1)$$

defined by adjoining p-power roots of f_0, and

$$\delta(\log(g_0)) = \log(g_0) - \beta(\chi(\delta)).$$

Especially the obstruction to $\widehat{\mathbb{Z}}(1)$-invariance is given by β ($-\chi$ corresponds to 1).

Now i_*^{crys} is defined by the connecting map for

$$0 \longrightarrow \mathrm{DR}_X(\mathcal{E} \otimes J_I) \longrightarrow \mathrm{DR}_{\widetilde{X}}(\mathcal{E} \otimes J_I) \longrightarrow \mathrm{DR}_Y(\mathcal{E} \otimes J_I)[-1] \longrightarrow 0.$$

This maps to global sections in the exact sequence

$$0 \longrightarrow L_X(\mathcal{E}\otimes J_I) \longrightarrow L_{\widetilde{X}}(\mathcal{E}\otimes J_I) \longrightarrow j_*L_{\widetilde{X}}(\mathcal{E}\otimes J_I)/L_X(\mathcal{E}\otimes J_I) \longrightarrow 0$$

or also

$$0 \longrightarrow L_X(\mathcal{E}\otimes J_I) \longrightarrow L^*_{\widetilde{X}}(\mathcal{E}\otimes J_I) \longrightarrow L_X(\mathcal{E}\otimes J_I)/\langle f_0, df_0\rangle[-1] \longrightarrow 0,$$

the pullback via

$$\cup\psi : L_X(\mathcal{E}\otimes J_I)/\langle f_0, df_0\rangle[-1] \longrightarrow j_*L_{\widetilde{X}}(\mathcal{E}\otimes J_I)/L_X(\mathcal{E}\otimes J_I),$$

which in turn maps to

$$0 \longrightarrow \mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{X,I}) \longrightarrow \mathbb{R}j_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{\widetilde{X},I})) \longrightarrow i_*(\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{J}_{Z,I}))[-1](-1) \longrightarrow 0,$$

where the maps on the first two terms are the canonical comparison maps, while on the third we have the comparison-map multiplied by β. However as this last sequence defines $i_*^{\text{ét}}$, we get indeed $v_Y \circ i_*^{\mathrm{crys}} = \beta \cdot i_*^{\text{ét}} \circ v_X$. Thus the theorem is shown.

3. Corollary. — *For the trace-forms on $H^{2d}(X/R_V, J_X)$ respectively $H_!^{2d}((X^0\otimes_V \overline{K})^{\text{ét}}, \mathbb{Z}_p)$ we have*

$$\mathrm{tr}^{\text{ét}} \circ v_X = \beta^d \cdot \mathrm{tr}^{\mathrm{crys}}$$

Proof. — Check on the class of a smooth point.

Also toric blow-ups $\widetilde{X} \to X$ induce isomorphisms on crystalline cohomology. This is true on de Rham cohomology, thus modulo $F^1(R_V) \subseteq R_V$, and follows in general because $F^1(R_V)$ consists of topologically nilpotent elements. Hence:

4. Corollary. — *Assume we are given functorial transformations*

$$v_{\mathcal{E}} : \mathcal{E}(A^0_{\mathrm{crys}}(\overline{R})) \longrightarrow \mathbb{L}\otimes A_{\mathrm{crys}}(\overline{R})$$
$$v_{\mathcal{E}^\vee} : \mathcal{E}^\vee(A^0_{\mathrm{crys}}(\overline{R})) \longrightarrow \mathbb{L}^\vee\otimes A_{\mathrm{crys}}(\overline{R})$$
$$(\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X/p^s\mathcal{O}_X), \mathbb{L}^\vee = \mathrm{Hom}(\mathbb{L}, \mathbb{Z}/p^s\mathbb{Z})),$$

with $\langle v_{\mathcal{E}}(e), v_{\mathcal{E}^\vee}(e^\vee)\rangle = \beta^{\otimes m}\cdot v\langle e, e^\vee\rangle$ for some fixed m.

Then

$$v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E}\otimes J)\otimes_{R_V} A_{\mathrm{crys}}(\overline{V}) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0\otimes_V\overline{K})^{\text{ét}}, \mathbb{L})\otimes A_{\mathrm{crys}}(\overline{V})$$

has an inverse up to $\beta^{\otimes(d+m)}$, that is composition either way is multiplication by $\beta^{\otimes(d+m)}$

Proof. — The "inverse" is the adjoint of the map v_X for $\mathcal{E}^\vee$. One has to compare the classes of the diagonals, passing to $\widetilde{\Delta} : \widetilde{X} \to (\widetilde{X\times X})$. (The class of the diagonal $c_\delta \in H^{2d}_{(!)}(\widetilde{X\times X}, \mathbb{R}\underline{Hom}(\mathrm{pr}_2^*\mathcal{E}, \mathrm{pr}_1^*\mathcal{E}))$ is the image of a class in $H^{2d}_{(!)}(\widetilde{X\times X}, \underline{Hom}(\mathrm{pr}_2^*\mathcal{E}, \mathrm{pr}_1^*\mathcal{E}))$, the direct image of the identity map on $\mathcal{E}$).

Remarks

1) If $\mathcal{E}$ is in fact a crystal in the crystalline topos of X relative to V_0 (that is one has integrable connections

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^{1,\log}_{R'/V_0} = \mathcal{E} \otimes \mathrm{Lie}(T)^\vee),$$

the crystalline cohomology $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E})$ inherits a logarithmic connection (as a perfect complex). Especially all cohomology-groups have such a connection. Also the Galois-group $\mathrm{Gal}(\overline{R}/R)$ acts on

$$\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) \otimes_{R_V} A_{\mathrm{crys}}(\overline{V}),$$

although the map $R_V \to A_{\mathrm{crys}}(\overline{V})$ is not Galois-linear. It suffices that this holds modulo $F^1(A_{\mathrm{crys}}(\overline{V}))$. Finally if $\mathbb{L}$ is defined over $K, \mathrm{Gal}(\overline{R}/R)$ also operates on $\mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L})$. And if the local comparison-maps

$$\mathcal{E}(A^0_{\mathrm{crys}}(\overline{R})) \longrightarrow \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{R})$$

also respect the action of $\mathrm{Gal}(\overline{R}/R)$ the comparison-map v_X is $\mathrm{Gal}(\overline{K}/K)$-linear.

2) Everything works in the filtered context: If $\mathcal{E}$ is a filtered crystal, and the comparison

$$\mathcal{E}(A^0_{\mathrm{cr}}(\overline{R})) \longrightarrow \mathbb{L} \otimes A_{\mathrm{crys}}(\overline{R})$$

preserves filtrations, that is it sends $F^i(\mathcal{E}(A^0_{\mathrm{crys}}(\overline{R})))$ to $\mathbb{L} \otimes F^i(A_{\mathrm{crys}}(\overline{R}))$, the comparison-map

$$\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0 \otimes_K \overline{V})^{\text{ét}}, \mathbb{L}) \otimes A_{\mathrm{crys}}(\overline{V})$$

exists in the filtered derived catgeory, and similarly for inverese "up to $\beta^{\otimes(d+m)}$".

3) If $\mathcal{E}$ is a Frobenius-crystal (in the log-sense) then $\mathcal{E}(A^0_{\mathrm{crys}}(\overline{R})) \otimes_{A^0_{\mathrm{crys}}(\overline{R})} A_{\mathrm{crys}}(\overline{R})$ has a Frobenius-action ϕ. Namely $A_{\mathrm{inf}}(\overline{R})$ has a canonical Frobenius-lift (which is logarithmic: $\phi([\underline{\mu}]) = [\underline{\mu}]^p$, for $\mu \in \sigma^\vee \cap L^\vee$), which induces a Frobenius-lift on the preimage in $A_{\mathrm{crys}}(\overline{R})$ of $R/pR \subset \overline{R}/p\overline{R}$. Also this preimage is a PD-thickening of R/pR, so $\mathcal{E}$ can be evaluated on it, and this evaluation admits a semilinear Frobenius. Finally because $\mathcal{E}$ is a crystal in the tensor product above we may replace $A^0_{\mathrm{crys}}(\overline{R})$ by this preimage. It then makes sense to require that the comparison-map to $\mathbb{L}\otimes A_{\mathrm{crys}}(\overline{R})$ is Frobenius-linear. In this case

$$v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_V, \mathcal{E} \otimes J_I) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0 \otimes_K \overline{V})^{\text{ét}}, \mathbb{L}) \otimes A_{\mathrm{crys}}(\overline{V})$$

will also preserve Frobenius (note that $\mathrm{Frob}^*(J_I) \subseteq J_I$, as J_I is locally generated by elements $\mu \in \sigma^\vee \cap L^\vee$ with $\mathrm{Frob}(\mu) = \mu^p\cdot$ unit). Here the "Frobenius" on R_V is Frobenius on V_0, and sends t to t^p.

4) Combining 2) and 3) one can often recover the étale cohomology of $\mathbb{L}$ from the crystalline cohomology of $\mathcal{E}$. This uses Fontaine's functors which involve invariants under Frobenius. At this stage we descend from almost maps to "real" maps, just as we did earlier (in chapter 3) when forming almost invariants under Frobenius. Another possibility is to use Frobenius invariance for the comparison maps (this has

been proposed (in a 1999 preprint) by M. Kisin, who had overlooked this and thought it necessary to fill a "gap"). At this point we note for later use that for the ring $B^+_{\mathrm{DR}}(\overline{R})$ almost elements are "real" elements (reduce to $\widehat{\overline{R}}$).

5) In many cases (for example constant coefficients) one can pass to $s \to \infty$ to obtain a $\mathbb{Z}_p$-theory and then also a $\mathbb{Q}_p$-theory. In more generality this is done in the context of convergent isocrystals (see [BO], [O]):

Our base will be the "closed unit disk": That is consider the algebra $R_{V,0} = (V_0[t])^\wedge$, where $\wedge$ = p-adic completion. It has a toroidal or log structure generated by t. An affine logarithmic enlargement (we do not need others) is given by an $R_{V,0}$-algebra A with a log-structure (étale locally $\mathrm{Spec}(A) \to \overline{T}_\sigma$, $R_{V,0}$-linear), a p-adically closed ideal $I \subset A$ such that A is p-adically complete and such that some power of I is contained in pA, and an $R_{V,0}$-linear log-map $\mathrm{Spec}(A/I) \to X$. The definition of maps is the obvious one. A convergent (logarithmic) isocrystal associates to any enlargement A a finitely generated projective $A[1/p]$-module $\mathcal{E}(A)$, with $\mathcal{E}(B) = B \otimes_A \mathcal{E}(A)$ for maps $A \to B$. Similarly filtered convergent isocrystals associate filtered modules, $A[1/p]$ itself being filtered by powers of I.
For example for each $\varepsilon = 1/n$ consider the algebra $R_{V,\varepsilon} = V_0[t, t^{ne}/p]^\wedge$. For $n \geqslant e$, $R_{V,\varepsilon}$ with the ideal $I = (f(t))$ defines an enlargement of $\mathrm{Spec}(V)$. We define enlargements over $R_{V,\varepsilon}$ as enlargements (A, I) together with A an algebra over $R_{V,\varepsilon}$, and the obvious commutative diagram of log-maps. In the usual way (compare [Fa3], IV.e.) one defines the cohomology $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\varepsilon}, \mathcal{E} \otimes J)$. I do not know whether this is in general representable by a perfect complex over $R_{V,\varepsilon}[1/p]$, satisfying Poincaré-duality. However for Frobenius-crystals ($\Phi^*\mathcal{E} \cong \mathcal{E}$) one can at least show that the special fibre at $t = 0$ is representable by a perfect complex $\mathbb{K}_0$ (i.e. has finite-dimensional cohomology) on which Frobenius acts by isomorphisms, and that there are Φ-linear and horizontal (in case of an absolute crstal $\mathcal{E}$) comparison-maps

$$(H^i(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J) \otimes_{K_0} R_\varepsilon[1/p], d + N) \longrightarrow (H^i(X^{\mathrm{crys}}/R_{V,\varepsilon}, \mathcal{E} \otimes J), \nabla),$$

where $N = \mathrm{Res}(t\frac{d}{dt})$ is the endomorphism defined by the logarithmic connection, satisfying $\Phi \circ N = pN \circ \Phi$ (and thus N is nilpotent):

For the finite-dimensionality one constructs integral lattices and proceeds as in [Fa7]. The comparison-map is constructed as in [Fa4]: Start with any lift of the identity mod t. Then transform it by powers of Frobenius and pass to the limit. Also modulo t we have Poincaré-duality, characteristic classes, Gysin-maps etc.

I expect the comparison-map to be isomorphisms, but do not know how to prove this.

Define $A_{\mathrm{crys},\varepsilon}(\overline{R}) = p$-adic completion of $A_{\mathrm{inf}}(\overline{R})[\xi^n/p]$ (as before, $\varepsilon = 1/n$). This is an enlargement of R over $R_{V,\varepsilon}$, which maps to it sending t to $[\underline{\pi}]$. Thus $\mathcal{E}(A_{\mathrm{crys},\varepsilon}(\overline{R}))$ is defined. If we are given a smooth $\mathbb{Q}_p$-adic sheaf $\mathbb{L}$ on $(X^0 \otimes_V \overline{K})^{\text{ét}}$ and functorial (almost) maps

$$\mathcal{E}(A_{\mathrm{crys},\varepsilon}(\overline{R})) \longrightarrow \mathbb{L} \otimes A_{\mathrm{crys},\varepsilon}(\overline{R})$$

respecting Galois-actions and compatible for varying ε's) we obtain a map

$$v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\varepsilon}, \mathcal{E} \otimes J) \otimes_{R_{V,\varepsilon}} A_{\mathrm{crys},\varepsilon}(\overline{V}) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}; \mathbb{L}) \otimes A_{\mathrm{crys},\varepsilon}(\overline{V}),$$

with the usual properties concerning duality, characteristic classes, and action of $\mathrm{Gal}(\overline{K}/K)$. This also holds for filtrations if we assume that $\mathcal{E}$ is filtered and the maps $\mathcal{E}(A_{\mathrm{crys},\varepsilon}) \to \mathbb{L} \otimes A_{\mathrm{crys},\varepsilon}$ respect filtrations.

In this case we may also form the ring $B_{\mathrm{DR}}^+(\overline{V})$ as the completion in the filtration topology of $A_{\mathrm{crys},\varepsilon}(\overline{V})[1/p]$. For each quotient $B_{\mathrm{DR}}^+(\overline{V})/F^n$ one can find p-adically complete subrings S_n which are enlargements of $V/R_{V,\varepsilon}$. Also we may assume that S_n is big enough so that the natural map $V \to B_{\mathrm{DR}}^+(\overline{V})/F^n$ factors through it as log map (that is $([\underline{\pi}]/\pi)^{\pm 1} \in S_n$). Note that we have two different morphisms (of enlargements) from $R_{V,\varepsilon}$ to S_n, namely once mapping through $A_{\mathrm{crys},\varepsilon}(\overline{V})$ and once through evaluation at V. If we could show a reasonable base-change theorem it would follows that the pushforwards to S_n of $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\varepsilon}, \mathcal{E} \otimes J)$ will coincide with the cohomology relative S_n. In fact to get a canonical filtered isomorphism between the two pushforwards to B_{DR}^+/F^n and (in the limit) B_{DR}^+ it suffices to have an integrable connection satisfying Griffiths-duality on either $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\varepsilon}, \mathcal{E} \otimes J)$ or whatever elso one wishes to push forward. This fact is necessary because in Fontaine's comparison machinery one uses the second push forward, while our maps naturally respect filtrations if one uses the first.

Also for a Frobenius-crystal we can compose with the comparison-maps to obtain (with $B_{\mathrm{crys}}^+ = A_{\mathrm{crys}} \otimes_{\mathbb{Z}} \mathbb{Q}$)

$$v_X : \mathbb{R}\Gamma(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J) \otimes_{K_0} B_{\mathrm{crys},\varepsilon}^+(\overline{V}) \longrightarrow \mathbb{R}\Gamma_{(!)}((X^0 \otimes_V \overline{K})^{\text{ét}}, \mathbb{L}) \otimes B_{\mathrm{crys},\varepsilon}^+(\overline{V})$$

This will respect Galois-actions as the other maps do, where however on the left hand side the Galois-action is not just via the second factor but has to be twisted, because of the connection $d + N$. There is a 1-cocycle $\mathrm{Gal}(\overline{K}/K) \to \mathbb{Z}_p(1)$

$$\sigma \longmapsto \overline{\sigma}$$

describing the action on all p-power roots of π, or equivalently on $[\underline{\pi}] \in A_{\mathrm{inf}}(\overline{V})$. Then the Galois-action of σ on

$$H^i(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J_I) \otimes A_{\mathrm{crys},\varepsilon}(\overline{V})$$

is defined as

$$\sigma \longmapsto \exp(\beta(\overline{\sigma})N) \otimes \sigma$$

This corresponds to Fontaine's description: If ϵ is small enough $A_{\mathrm{crys},\epsilon}$ maps to the usual A_{crys} defined using the PD-hull, which in turn maps to B_{crys}^+ and B_{crys}. If $B_{\mathrm{st}} = B_{\mathrm{cr}}[u]$ with $Nu = 1(u = \text{“}\log([\underline{\pi}]/\pi)\text{”})$ then $(E_0 = H^i(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J))$

$$E_0 \otimes B_{\mathrm{cr}} \cong \ker(N \otimes 1 + 1 \otimes N \mid E_0 \otimes B_{\mathrm{st}})$$

via $\exp(-N \otimes u)$, and as $\sigma(u) = u + \beta(\overline{\sigma})$ this intertwines Galois-actions.

In the appendix it is shown how to endow $\mathbb{R}\Gamma(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J_I) \otimes^{\mathbb{L}}_{V_0} R_{V,\epsilon}$ with a filtration such that it can take the place of $\mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I)$. With it all the theory works, and we get comparison maps as needed in Fontaine's theory of crystalline representations.

5*. Appendix: Cohomology of Frobenius-isocrystals

As before X is étale locally étale over some $\overline{T}_\lambda$. The cohomology of convergent isocrystals can be defined via de Rham complexes on smooth local lifts, as it is done in crystalline cohomology. However one has to replace PD-envelopes by open "$> \varepsilon$"-neighbourhoods, as follows:

Denote for $\epsilon = a/b$ by $R_{V,\epsilon}$ the p-adic completion of the normalization of $V_0[[t]][t^b/p^a]$. Then

$$R_{K,\varepsilon} = R_{V,\varepsilon} \otimes_{V_0} K_0$$

is the rings of formal powerseries convergent in the closed disk of radius p^ε, that is formal powerseries $f = \sum_{n\geqslant 0} a_n t^n$ with ($v_p = p$-adic valuation).

$$a_n \in K_0, \lim_{n\to\infty} (v_p(a_n) + n\epsilon) = \infty.$$

We need the following variant of the Poincaré-lemma:

Consider the de Rham complex

$$R_{K,\varepsilon} \xrightarrow{d} R_{K,\varepsilon} \cdot dt.$$

Obviously the constants K_0 lie in the kernel of d. If we divide by them and map to the corresponding complex with ϵ replaced by $\delta > \epsilon$, the induced map (from "ϵ" to "δ") on cohomology vanishes.

Thus the projective system of de Rham complexes, for all ϵ bigger than a fixed $\delta \geqslant 0$, forms an acyclic pro-object resolving the constants. It is called the "de Rham complex of the open δ-tube".

Next to define and compute cohomology we consider elements δ, ε smaller than the valuation of π. If X embeds into a p-adic affine formal scheme Y smooth over $\overline{T}_\sigma \otimes_{V_0[t]} V_0[[t]]$, respecting toroidal structures, let $J_X \subseteq \mathcal{O}_Y$ denote the ideal of $X \times_V k$. For $\delta = c/d \leqslant \epsilon$ define the closed (δ, ϵ)-tube of X in Y as the affine Y-scheme with algebra

$$((\mathcal{O}_Y[J_X^d/p^c]) \otimes_{V_0[[t]][t^d/p^c]} R_{V,\epsilon})^\wedge.$$

Call it $D_{X,\delta,\epsilon}(Y)$. Define (for $\delta < \epsilon$) the open (δ, ϵ)-tube as the *ind*-scheme

$$\text{“}\varinjlim_{\delta<\delta'\leqslant\epsilon}\text{”} D_{X,\delta',\epsilon}(Y) = D_{X,>\delta,\epsilon}(Y).$$

As each $D_{X,\delta',\epsilon}(Y)$ is an enlargement of X over $R_{V,0}$ we can evaluate $\mathcal{E}$ on it, and form the de Rham complex $\mathcal{E}(D_{X,\delta',\epsilon}(Y)) \otimes \Omega^{\bullet,\log}_{Y/V_0[[t]]}$. Then define $\mathbb{R}\Gamma_{>\delta}(X/R_{V,\varepsilon}, \mathcal{E})$ as represented by the derived projective limit ($\delta' \to \delta+$) of the de Rham complexes. Up to quasi-isomorphism it is independant of the choice of δ. Similarly we define

$\mathbb{R}\Gamma(X/R_{V,\varepsilon}, \mathcal{E} \otimes J)$, and if X cannot be embedded globally we use local embeddings and hypercoverings. One checks that the result is independant of all choices, the key being the Poincaré-lemma which allows to compare two local smooth embeddings (and there we unfortunately need open tubes).

In general we know little about finiteness or base change for these cohomologies. However for Frobenius-isocrystals the situation is much better. We define a semilinear "Frobenius" ϕ on $R_{V,\varepsilon}$ by $\phi(t) = t^p$. Then Frobenius operates on enlargements of $X \otimes_V k/V_0[[t]]$, and a Frobenius-isocrystal is an isocrystal $\mathcal{E}$ together with an isomorphism $\mathrm{Frob}^*(\mathcal{E}) \cong \mathcal{E}$. We first claim that for a Frobenius-isocrystal the cohomology modulo t is wellbehaved.

For this choose an affine étale covering $X' \to X \otimes_V k$ and lift to a toroidal or log-smooth $\widehat{X'}$ over $V_0 = V_0[[t]]/(t)$. Also choose a Frobenius-lift Φ on $\widehat{X'}$. This induces a Frobenius Φ on all fibered products $\widehat{X'}^{\log,n+1}$ and $D(\widehat{X'}^{\log,n+1})$, where $D()$ stands for the p-adic completion of the (log) divided power envelopes of $(X'/X \otimes_V k)^{n+1}$. We denote its affine ring by $\widehat{R}'_n$. These are affine enlargements of $X \otimes_V k$, and thus by pullback we can evaluate $\mathcal{E}$ on them. We claim that there exists a crystal $\mathcal{E}^0$ of coherent sheaves on $(X \otimes_V k)^{\mathrm{crys}}/V_0$, flat over V_0, with

$$\mathcal{E}(D(\widehat{X'}^{\log,n+1})) = \mathcal{E}^0(D(\widehat{X'}^{\log,n+1})) \otimes_{V_0} K_0,$$

functorially:

Choose a finitely generated $\widehat{R}'$-submodule $M_0 \subset \mathcal{E}(\widehat{R}'_0)$ spanning $\mathcal{E}(\widehat{R}'_0)$, and stable under $p^N \cdot \Phi$ (some fixed N) and $\bigtriangledown$ (replace M_0 by its transform under some big Frobenius-power, if necessary). Then $\mathrm{pr}_1^*(M_0) \subseteq p^{-N} \cdot \mathrm{pr}_0^*(M_0)$ in $\mathcal{E}(\widehat{R}'_1)$, after (if necessary) enlarging N. If we define

$$\mathcal{E}^0(\widehat{R}'_0) = \{e \in M_0 : \mathrm{pr}_1^*(e) \in \mathrm{pr}_0^*(M_0)\}$$

then $p^N \cdot M_0 \subseteq \mathcal{E}^0(\widehat{R}'_0) \subseteq M_0$, $\mathcal{E}^0(\widehat{R}'_0)$ is stable under $p^N \cdot \Phi$ and $\bigtriangledown$, and

$$\mathrm{pr}_1^*(\mathcal{E}^0(\widehat{R}')) = \mathrm{pr}_0^*(\mathcal{E}^0(\widehat{R}')) \subseteq \mathcal{E}(\widehat{R}'_1).$$

This last identity is checked locally in $\widehat{X'}_1$. We thus may assume that

$$\widehat{X'}_n = \widehat{X'} \times D(\widehat{\widehat{T}}_\lambda)^n.$$

Note that $\mathrm{pr}_0^*(M_0)$ is always a submodule of $\mathcal{E}(\widehat{R}'_n)$ as $\widehat{R}'_0$ is noetherian pr_0 is flat. Now start with the exact sequence

$$0 \longrightarrow \mathcal{E}_0(\widehat{R}')/p^N \cdot M_0 \longrightarrow M_0/p^N \cdot M_0 \xrightarrow{\mathrm{pr}_1^*} p^{-N} \cdot \mathrm{pr}_0^*(M_0)/\mathrm{pr}_0^*(M_0),$$

tensor with $\widehat{R}'_1/p^N$ which is flat over $\widehat{R}'_0/p^N$ via pr_0^*, and use simplicial identities computing in the subquotient $p^{-N} \cdot \mathrm{pr}_0^*(M_0)/\mathrm{pr}_0^*(M_0)$ of $\mathcal{E}(\widehat{R}'_2)$). It then follows that

$$\mathrm{pr}_1^*(\mathcal{E}^0(\widehat{R}') \subseteq \mathrm{pr}_0^*(\mathcal{E}^0(\widehat{R}')$$

and vice-versa. Thus $\mathcal{E}^0(\widehat{R}')$ defines a crystal.

By the methods of [Fa7] (most difficulties disappear because V_0 is noetherian) it follows that for each $s > 0$

$$\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0/p^s)$$

is representable by a perfect complex over V_0/p^s, and we may form the projective limit to represent

$$\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0)$$

by a perfect complex over V_0. Also the perfect K_0-complex

$$\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes_{V_0} K_0$$

is independant of all choices, and Φ^* induces an isomorphism on cohomology: It suffices to show that Frobenius has a left-inverse (the cohomology has finite dimension). To construct this use that local Frobenius-lifts are finite, and that duality theory defines a trace-map from the de Rham complex of $\Phi^*(\mathcal{E})$ to that of $\mathcal{E}$. Namely the trace without coefficients satisfies $\mathrm{tr}(\Phi^*(\alpha) \wedge \beta) = \alpha \wedge \mathrm{tr}(\beta)$, and it extends to coefficients in $\mathcal{E}$ by the rule $\mathrm{tr}(\Phi^*(e) \otimes \alpha) = e \otimes \alpha$. This applies to all the open tubes in the definition of cohomology, except that the radius of convergence decreases: The ϵ-tube is replaced by the $p\epsilon$-tube (for Frobenius it is the converse). Also the trace respects coefficients J_I, by a local calculation. If we divide it by the relative rank of Frobenius (that is $p^{\dim}$) it even commutes with all simplicial pullbacks. The induced map on cohomology is our left-inverse.

Next we study relations between $\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes_{V_0} K_0$ and

$$\mathbb{R}\Gamma(X^{\text{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I).$$

For this use the morphism $R_{V,\epsilon} \to V_0$ which sends t to 0. It induces an isomorphism $R_{K,\epsilon}/tR_{K,\epsilon} \cong K_0$. By an easy variant of base-change the derived product $\mathbb{R}\Gamma(X^{\text{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I) \otimes^{\mathbb{L}}_{R_{K,\epsilon}} K_0$ is equal the cohomology of the corresponding isocrystal over V_0, which is defined by de Rham complexes on open δ-tubes, for any $\delta > 0$. The δ-tubes involve the cosimplicial ring $\widehat{R}'_\bullet$. Now if $\delta \geqslant \frac{1}{p-1}$ the PD-envelope maps to the projective system of affine rings which define the open tube, and the induced map on de Rham complexes gives a comparison

$$\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes_{V_0} K_0 \longrightarrow \mathbb{R}\Gamma(X^{\text{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I) \otimes^{\mathbb{L}}_{R_{V,\epsilon}} K_0.$$

We claim that it is a quasi-isomorphism.

This can be checked on the level of de Rham complexes on each $\widehat{R}'_n$. However as the special fibre lifts to a toroidal or log-smooth scheme over V_0, the de Rham complexes is quasi-isomorphic to that of such a lift, which involves neither PD-hulls nor δ-tubes.

Next we try to lift to a Frobenius-linear

$$\mathbb{R}\Gamma((X \otimes_V k)^{\text{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes_{V_0} K_0 \longrightarrow \mathbb{R}\Gamma(X^{\text{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I).$$

Namely let L^* denote the de Rham complex representing the cohomology of the closed tube $D_{X,\delta,\epsilon}$ for some $\delta < \epsilon$. It is a complex of p-adic Banach-modules over

$R_{V,\epsilon}$, Frobenius on it is a bounded operator. Also for $\delta_1 < \delta$ we may map

$$\text{open } \delta\text{-tube} \subset \text{closed } \delta\text{-tube} \subset \text{open } \delta_1\text{-tube},$$

thus factoring the identity on the cohomology of the isocrstal over $R_{V,\epsilon}$. Choose a split complex M_0^* of finite dimensional K_0-vectorspaces representing

$$\mathbb{R}\Gamma(X^{\mathrm{crys}} \otimes_V k/V_0, \mathcal{E} \otimes J_I).$$

Then Frobenius acts by automorphisms on the M_0^i, and thus $N^* = \mathrm{Hom}^*_{K_0}(M_0^*, L^*)$ is also a complex of $R_{V,\epsilon}$-Banach-modules with bounded Frobenius. Mapping via an open δ_1-tube we obtain a Frobenius-invariant element of $H^0(N^*/tN^*)$ which we want to lift. Thus chose some lift $\widetilde{\alpha} \in N^0$. Then

$$\Phi(\widetilde{\alpha}) - \widetilde{\alpha} = t \cdot \beta + d(\gamma)$$

with

$$\beta \in N^0, \gamma \in N^{-1}.$$

Define $\alpha = \widetilde{\alpha} + \sum_{m=0}^{\infty} \Phi^m(t \cdot \beta)$. This sum converges in N^0 as

$$\Phi^m(t \cdot \beta) = t^{p^m} \cdot \Phi^m(\beta),$$

and the t-power in front has norm p to the power $-p^m\epsilon$, while all other norms grow at most exponentially in m. Then

$$\begin{aligned}
\Phi(\alpha) - \alpha &= d(\gamma), \\
d(\alpha) &= d(\widetilde{\alpha}) + \sum_{m=0}^{\infty} \Phi^m \circ d(t \cdot \beta) \\
&= d(\widetilde{\alpha}) + \sum_{m=0}^{\infty} \Phi^m \circ d \circ \Phi(\widetilde{\alpha}) - d \circ \Phi^m(\widetilde{\alpha}) \\
&= d(\widetilde{\alpha}) + \lim_{m\to\infty} \Phi^{m+1} \circ d(\widetilde{\alpha}) - d(\widetilde{\alpha}) \\
&= \lim_{m\to\infty} \Phi^{m+1} \circ d(\widetilde{\alpha}) = 0,
\end{aligned}$$

by the same reasoning, as $d(\widetilde{\alpha}) \in t \cdot N^{i+1}$. Hence

5. Proposition. — *There exists a lift*

$$\alpha : \mathbb{R}\Gamma((X \otimes_V k)^{\mathrm{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes^{\mathbb{L}}_{V_0} R_{V,\epsilon} \longrightarrow \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J),$$

Frobenius-linear up to homotopy. It is called the Hyodo-Kato map.

How unique is that lift? We claim that for any finite complex M_0^* of finite dimensional K_0-vectorspaces with Frobenius automorphism, a Frobenius invariant cohomology class in $\mathrm{Hom}^*(M_0^*, \Gamma(X^{\mathrm{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J_I)$ vanishes if it vanishes modulo t:

Namely using the Hyodo-Kato map our class comes from a Frobenius-invariant class in the homomorphisms of M_0^* into $t \cdot \Gamma(X^{\mathrm{crys}}/R_{V,\epsilon}, \mathcal{E} \otimes J)$. Mapping again via closed tubes we obtain in some $N^i = \mathrm{Hom}^i(M_0^*, L^*)$ a class $t \cdot \alpha$ with

$$t \cdot \alpha - \Phi(t \cdot \alpha) = d(t \cdot \gamma).$$

But then

$$t \cdot \alpha = d\Big(\sum_{m=0}^{\infty} \Phi^m(t \cdot \gamma)\Big)$$

is exact.

Also it may happen that $\mathcal{E}$ is actually a crystal over V_0, that is in the absolute sense. We show that in this case the Hyodo-Kato map is horizontal for the Gauss-Manin connection. Namely in this case we may repeat all our constructions using de Rham complexes relative V_0, with trivial log structure. These are extensions of the previous de Rham complexes, say

$$0 \longrightarrow \mathrm{DR}(\mathcal{E}/R_{V,\epsilon})\{-1\}[-1] \longrightarrow \mathrm{DR}(\mathcal{E}/V_0) \longrightarrow \mathrm{DR}(\mathcal{E}/R_{V,\epsilon}) \longrightarrow 0.$$

The shifts come from multiplication by dt/t.

If we reduce modulo t we get complexes over K_0 with finite dimensional cohomology. As before denote by M_0^* a split complex representing the relative cohomology of $\mathcal{E} \otimes J_I$. Then the connecting map of the above sequence defines a Frobenius invariant map $N : M_0^* \to M_0^*\{-1\}$. By linear algebra one checks that the extension is then quasi-isomorphic to that given by the mapping cone $C^*(N)$:

$$0 \longrightarrow M_0^*\{-1\}[-1] \longrightarrow C^*(N) \longrightarrow M_0^* \longrightarrow 0.$$

The quasi-isomorphism between the two extensions is not unique up to homotopy, but two choices differ by an automorphism of the extension above, that is by an element of $H^{-1}(\mathrm{End}^*(M_0)\{-1\})$. Also the natural Frobenius on the cone has to be modified by such an element to make the quasi-isomorphism Frobenius-linear, up to homotopy. (This modification defines an invariant of cohomology which I have not found in the literature.)

Next our lifting works as before and defines a Hyodo-Kato map

$$C^*(N) \otimes_{K_0} R_{K,\epsilon} \longrightarrow \mathbb{R}\Gamma(X^{\mathrm{crys}}/R_{V,\epsilon}/V_0, \mathcal{E} \otimes J_I).$$

It is compatible with the previous ones, and from its existence it follows that these respect Gauss-Manin connections.

Finally mapping $R_{\geqslant \varepsilon} \to K$ (or $R_V \to V$) we know that the pushout is de Rham cohomology of $\mathcal{E}(\widehat{X})$ over V. This also satisfies Poincaré-duality. By the usual calculation with characteristic of diagonals the induced maps

$$H^i(X \otimes_V k/V_0, \mathcal{E} \otimes J_I) \otimes_{K_0} K \longrightarrow H^i_{\mathrm{DR}}(X, \mathcal{E} \otimes J_I)$$

are isomorphisms (the Hyodo-Kato isomorphisms). To check that characteristic classes etc. are respected represent them by short exact sequences. It then follows that the Hyodo-Kato map modulo t as well as the pushout induced from $R_{V,\epsilon}$ respect them. But then this also follows for the Hyodo-Kato map itself, because of uniqueness of lifts for Frobenius-invariant maps.

Finally if $\mathcal{E}$ is filtered we can define a pullback-filtration on

$$\mathbb{R}\Gamma((X \otimes_V k)^{\mathrm{crys}}/V_0, J_I \cdot \mathcal{E}^0) \otimes^{\mathbb{L}}_{V_0} R_{V,\epsilon}$$

by the following general construction:

Construction. — Suppose $\alpha : A^* \to B^*$ is a morphism of complexes, with B^* filtered by subcomplexes $F^n(B^*)$. Then identify A^* with the mapping cone of $(\alpha, -\mathrm{id})$: $A^* \oplus B^* \to B^*$ and filter this cone by the subcomplexes which are mapping cones of $A^* \oplus F^n(B^*) \to B^*$.

This makes (trivially) the Hyodo-Kato map filtered. Also if $\mathcal{E}$ is absolute filtered this construction gives a filtration satisfying Griffith's transversality. That is $\nabla(t \cdot \partial/\partial t)$ lifts naturally to a (nonlinear) derived endomorphism of filtration-degree -1. Finally the pushout to B^+_{DR} defines an isomorphism on cohomology of gr_F, thus becomes a filtered quasi-isomorphism.

Remarks. — Pragmatically one could replace $H^i(X/R_{\geqslant\varepsilon}, \mathcal{E} \otimes J_I)$ by

$$H^i(X \otimes_V k/V_0, \mathcal{E} \otimes J_I) \otimes_{K_0} R_{\geqslant\varepsilon}$$

and then have all the desired properties. However philosophically this looks a little bit like cheating.

6. Complements: Relative case, Fontaine-Lafaille theory

As in [Fa3] one can generalise to the case of proper maps $f : X \to Y$ with $f(X^0) \subseteq Y^0$. For a reasonable theory one needs some local conditions on f. We assume that étale locally f is isomorphic to a map of torus-embeddings $\overline{S}_\lambda \to \overline{T}_\lambda$, where $\overline{S}_\lambda \subseteq \overline{S}$, $\overline{T}_\lambda \subseteq \overline{T}$, $\overline{S}$ corresponds to a lattice L and a cone $\sigma \subseteq L_{\mathbb{R}}$, $\overline{T}$ to a lattice M and a cone $\tau \subseteq M_{\mathbb{R}}$, and the map f to a homomorphism $L \to M$ which sends σ into τ such that each face of σ maps onto a face of τ. We assume that the cokernel of $L \to M$ is torsion free. Also $\lambda \in M^\vee$ maps to the corresponding element $\lambda \in L^\vee$. Finally σ and τ should satisfy our usual assumptions, namely that $\sigma \cap \ker(\lambda)$ and $\tau \cap \ker(\lambda)$ are spanned by partial basis of L, respectively M. This implies that $f_K : X \otimes_V K \to Y \otimes_V K$ sends each open stratum of X smoothly onto an open stratum of Y. Also for each locally constant torsion-sheaf $\mathbb{L}$ on $X^0 \otimes_V \overline{K}$, the direct images $\mathbb{R}^\nu f_* \mathbb{L}$ on $Y^0 \otimes_V \overline{K}$ are again locally constant. Furthermore f induces a map of topoi $f : \mathfrak{X}^0 \otimes_V \overline{K} \to \mathfrak{Y}^0 \otimes_V \overline{K}$:

For $U \to Y$ étale, $V \to U^0 \otimes_V \overline{K}$ an étale covering, the pullbacks to X are in the situs defining $\mathfrak{X}^0 \otimes_V \overline{K}$. This defines f^* on sites, etc. We intend to sketch the proof of the following

6. Theorem. — *Suppose $\mathbb{L}$ is locally constant torsion on $(X^0 \otimes_V \overline{K})^{\text{ét}}$, defining such a sheaf $\mathbb{L}$ on $\mathfrak{X}^0 \otimes_V \overline{K}$. Then the direct images $\mathbb{R}^\nu f_* \mathbb{L}$ under*

$$f : (X^0 \otimes \overline{K})^{\text{ét}} \longrightarrow (Y^0 \otimes_V \overline{K})^{\text{ét}}$$

and

$$f : \mathfrak{X}^0 \otimes_V \overline{K} \longrightarrow \mathcal{Y}^0 \otimes_V \overline{K}$$

correspond. Furthermore

$$R^\nu f_*(\mathbb{L} \otimes \overline{\mathcal{O}}_X) \approx R^\nu f_* \mathbb{L} \otimes \overline{\mathcal{O}}_Y,$$

and similarly for $A_{\text{inf}}(\overline{\mathcal{O}})$, or direct images with compact support (along boundary strata of X mapping to Y^0).

Proof. — This is done as in the absolute case, replacing $\overline{V}$ by $\overline{R}$, for $\operatorname{Spec}(R) \to Y$ étale and also $\operatorname{Spec}(R)$ étale over $\overline{S}_\lambda$, R strictly henselian. As usual we can for many purposes replace $\overline{R}$ by R_∞. First we have to study Poincaré-duality for the direct images $\mathbb{R}f_*(\mathbb{L} \otimes \overline{\mathcal{O}}_X)$. There is an (almost defined) trace-map

$$\mathbb{R}^{2\delta} f_*(\overline{\mathcal{J}}_X / p^s \overline{\mathcal{J}}_X)(\delta) \longrightarrow \overline{\mathcal{J}}_Y / p^s \overline{\mathcal{J}}_Y$$

(δ = relative dimension) inducing an almost quasi-isomorphism

$$\mathbb{R}f_*(\mathbb{L} \otimes \overline{\mathcal{O}}_X)[2\delta](\delta) \approx \mathbb{R}\operatorname{Hom}_{\overline{R}}(\mathbb{R}f_*(\mathbb{L}^\vee \otimes \overline{\mathcal{J}}_X), \overline{\mathcal{J}}_Y p^s \overline{\mathcal{J}}_Y)$$

(and similarly for cohomology with compact support at infinity). The proofs are easy generalizations of the previous ones.

Secondly $\mathbb{R}f_*(\mathbb{L})$ is given by the "almost Frobenius-invariants" on $\mathbb{R}f_*(\mathbb{L} \otimes A_{\text{inf}}(\overline{\mathcal{O}}_X))$. We claim that (if R is strictly henselian)

$$R^\nu f_*(\mathbb{L} \otimes \overline{\mathcal{O}}_X) \approx R^\nu f_*(\mathbb{L}) \otimes \overline{R}$$

(and similarly with coefficients $A_{\text{inf}}(\overline{R})$):

Reduce to $\mathbb{L}$ annihilated by $p, s = 1$. Then choose the highest index ν for which $\mathcal{M} = R^\nu f_*(\mathbb{L} \otimes \mathcal{R}(\overline{\mathcal{O}}_X))$ is not almost isomorphic to (almost Φ-invariants in M) $\otimes$ $\mathcal{R}(\overline{R})$. Then $M = \mathcal{M}/\xi\mathcal{M}$ is almost finitely presented over $\overline{R}/p\overline{R}$, and Frobenius is an almost isomorphism on $\mathcal{M}$. Equivalently $\mathcal{M} = \varprojlim(M, \Phi)$, and Frobenius induces an isomorphism $M/p^{1/p} \cdot M \approx M$. Thus everything can be formulated in terms of M. Furthermore we can replace $\overline{R}$ by R_∞ and M by $M_\infty = \operatorname{Gal}(\overline{R}/R_\infty)$-invariants, as $M \approx M_\infty \otimes_{R_\infty} \overline{R}$. M_∞ is an almost finitely presented R_∞/pR_∞-module with Frobenius $\Phi : M_\infty/p^{1/p} M_\infty \approx M_\infty$. In the appendix it is shown that for some almost étale covering $\overline{A}$ of R_∞/pR_∞

$$M_\infty \otimes_{R_\infty} \overline{A} \approx (\text{almost } \Phi\text{-invariants}) \otimes A.$$

Thus be descending induction over ν we can find such an A almost étale over $\overline{R}/p\overline{R}$ with

$$\mathbb{R}^\nu f_*(\mathbb{L} \otimes \overline{\mathcal{O}}_X) \otimes_{\overline{R}} \overline{A} \approx M_0^\nu \otimes_{\mathbb{F}_p} \overline{A}$$

for certain $\mathbb{F}_p$-vectorspaces M_0^ν. $\overline{A}$ lifts to an almost étale covering $\widehat{A}$ of $\widehat{\overline{R}}$, the p-adic completion of $\overline{R}$. Temporarily working over $\widehat{A}$ we see that the M_0^ν are the direct images of $\mathbb{L}$ on the topos $\mathfrak{X}^0 \otimes_{\mathcal{Y}} \widehat{A}$, the analogue of the topos $\mathfrak{X}^0$ with the base $\overline{V}$ replaced by $\widehat{A}$. These map to the étale images, and these maps are isomorphisms (by duality, or basechange to discrete valuation rings). If we replace $\widehat{A}$ by a Galois-covering the Galois-group $\mathrm{Gal}(\widehat{A}/\widehat{\overline{R}})$ then operates trivially on the M_0^ν, as it operates trivially on étale direct images. Hence by taking invariants

$$R^\nu f_*(\mathbb{L} \otimes \overline{\mathcal{O}}) \cong M_0^\nu \otimes \overline{R},$$

and comparison is shown.

Obviously there are consequences for comparison to crystalline direct images. Finally the Fontaine-Lafaille theory from [Fa3] carries over without much change: Consider a filtered Frobenius-crystal $(\mathcal{E}, F^i(\mathcal{E}))$ which locally is the direct sum of shifted filtered modules $\mathcal{O}_X/p^s\mathcal{O}_X\{e_i\}$, with shifts $0 \leqslant e_i \leqslant p-2$. Assume that Frobenius Φ on F^i is divisible by p^i, that is there exist (assuming we evaluate on a PD-thickening with Frobenius-lift) $\Phi_i : F^i(\mathcal{E}) \otimes_\rho \mathcal{O}_X \to \mathcal{E}$ with $\Phi_{i-1} \mid F^i = p \cdot \Phi^i$. Furthermore assume that for a local filtered basis m_i, of degree e_i, the $\Phi^{e_i}(m_i)$ also form a basis. Then

$$\mathbb{L} = \mathrm{Hom}_{F,\Phi}(\mathcal{E} \otimes A_{\mathrm{crys}}(\overline{R})), A_{\mathrm{crys}}(\overline{R}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

defines a locally constant étale sheaf on $X \otimes_V \overline{K}$. This is shown as in [Fa3] (and uses in fact little of the structure of $\overline{R}$). Finally sometimes such crystals can be constructed as crystalline direct images under $f : X \to Y$. However for this one at least needs more stringent restrictions on f ("of relative Cartier-type" in Kato's terminology, see also [Fa3]), and also criteria under which the derived direct image splits into the direct sum of its cohomology groups.

6*. Appendix: Some more almost mathematics

The ring $R_\infty = \lim\limits_{n\to\infty} R_n$ is the increasing union of noetherian rings R_n, such that for a sequence $\varepsilon_n \to 0$ we have

$$p^{\varepsilon_n} \cdot \mathrm{Tor}^{R_n}(R_{n+m},\ \text{any } R_n\text{-module}) \quad = 0.$$

If M is any almost finitely presented R_∞-module, then for each $\alpha > 0, M$ is p^α-isomorphic to $M_n \otimes_{R_n} R_\infty$ for some finitely presented R_n-module. Resolve M_n by finitely generated free R_n-module, to get by base extension a complex of finitely generated free R_∞-modules which up to $p^{\alpha+\varepsilon_n}$ resolves M. Similarly for a map $M \to N$ of such modules. With these methods one sees easily that the almost finitely presented R_∞-modules form an abelian subcategory of all R_∞-modules, which is closed under forming $\mathrm{Tor}_i^{R_\infty}(M, N)$ or $\mathrm{Ext}_i^{R_\infty}(M, N)$. The same holds for R_∞/pR_∞

(and by the way all this remains true for $\overline{R}$). From now on assume that R is strictly henselian.

We now show that for an almost finitely presented R_∞/pR_∞-module M with a Frobenius-linear almost isomorphism $\Phi : M/p^{1/p}M \approx M$, for an almost faithfully flat almost étale covering A of R_∞/pR_∞,

$$M \otimes_{R_\infty} A \approx M_0 \otimes_{\mathbb{F}_p} A$$

with

$$M_0 = \mathrm{Hom}(\mathfrak{m}; M)^\Phi$$

the almost invariants under Frobenius.

We proceed by induction over $\dim(R)$ and may assume that the assertion already holds for the strict localization in any prime $\mathfrak{p} \subseteq R, \mathfrak{p} \neq \mathcal{M}$. By descent it follows that the localizations $M_\mathfrak{p}$ are almost projective on $(R_\infty/pR_\infty)_\mathfrak{p}$. Also the assertion has already been shown for $d = 0$(R a discrete valuation ring), so we may assume that $\dim(R) \geqslant 2$. Replace M by $\mathrm{Hom}(\mathcal{M}, M)$. Then $M_0 = M^\Phi$, Frobenius is surjective on M, and M has no $\mathcal{M}$-torsion. If $\mathcal{M} = \varprojlim(M, \Phi)$, then $\mathcal{M}$ has no ξ-torsion, $\mathcal{M}/\xi\mathcal{M} \cong M$, and Frobenius is bijective in the $\mathcal{R}_\infty$-module $\mathcal{M}$. Next consider $\mathcal{N} = \mathrm{Ext}^1_{R_\infty}(\mathcal{M}, R_\infty)$. Then Frobenius is also bijective on $\mathcal{N}$, and $\mathcal{N}/\xi\mathcal{N}$ injects into

$$\mathrm{Ext}^1_{R_\infty}(\mathcal{M}, R_\infty/pR_\infty) = \mathrm{Ext}^1_{R_\infty}/pR_\infty(M, R_\infty/pR_\infty),$$

which is almost finitely presented and almost $\mathcal{M}$-torsion. By a previously used argument (using $d > 0$ and $\lambda(\xi^\alpha\mathcal{N}/\xi\mathcal{N})$), $\mathcal{N}/\xi\mathcal{N} \approx 0$, and it then follows that the maps any map $\mathcal{M} \to \mathcal{R}_\infty/\xi^n \cdot \mathcal{R}_\infty$ almost lifts to $\mathcal{R}_\infty/\xi^{n+1} \cdot \mathcal{R}_\infty$, and thus in the limit to $\mathcal{R}_\infty$.

Next consider the cokernel $\mathcal{N}$ of

$$\mathcal{M} \otimes_{\mathcal{R}_\infty} \mathrm{Hom}_{\mathcal{R}_\infty}(\mathcal{M}, \mathcal{R}_\infty) \longrightarrow \mathrm{End}_{R_\infty}(\mathcal{M}).$$

Then Frobenius is almost isomorphic on $\mathcal{N}$. Also $\mathcal{N}/\xi \cdot \mathcal{N}$ almost injects into the cokernel of

$$M \otimes_{R_\infty} \mathrm{Hom}_{R_\infty}(M, R_\infty/pR_\infty) \longrightarrow \mathrm{End}_{R_\infty}(M).$$

This cokernel is almost finitely presented and almost $\mathcal{M}$-torsion. By the usual argument $\mathcal{N}/\xi \cdot \mathcal{N}$ almost vanishes, and so does $\mathcal{N}$ (lifting modulo ξ^n step by step). Hence for each $\alpha > 0$, $\xi^\alpha \cdot \mathrm{id}_\mathcal{N}$ lies in the image, thus is of the form $\sum x_i \otimes y_i$ with $x_i \in \mathcal{M}, y_i \in \mathrm{Hom}_{\mathcal{R}_\infty}(\mathcal{M}, \mathcal{R}_\infty)$. Thus $\xi^d \cdot z = \sum y_i(z) \cdot x_i$ for each $z \in \mathcal{M}$, and $\mathcal{M}$ is almost projective.

Each localization $M_\mathfrak{p}$ has finite rank, for $\mathfrak{p} \neq \mathfrak{m}$. If r denotes the maximum of these ranks for $\mathfrak{p}$ a minimal prime divisor of pR, then M and $\mathfrak{m}$ have rank $\leqslant r$. For $n > r(p-1)$ consider the cokernel $\mathcal{N}$ of the map

$$\begin{aligned} \mathfrak{m} \otimes S^{n-p}(\mathfrak{n}) &\longrightarrow S^n(\mathfrak{n}) \\ m \otimes m' &\longmapsto m^p \cdot m! \end{aligned}$$

Then $\mathcal{N}/\xi\mathcal{N}$ injects into the corresponding cokernel for M, which is $\mathcal{M}$-torsion and almost finitely presented. Also Frobenius is an isomorphism on $\mathcal{N}$, hence as usual $\mathcal{N}/\xi N \approx 0$, and the map above is almost surjective ($\mathcal{N} \approx 0$). Thus

$$\mathcal{A} = S^{\bullet}(\mathcal{N})/\langle m^p - \Phi(m)\rangle$$

is almost generated by symmetric powers $S^n(\mathcal{N})$ ($n \leqslant r(p-1)$), hence almost finitely presented. Also Frobenius is almost isomorphic on $\mathcal{A}$, thus $\mathcal{A}$ itself is almost projective (same argument as for $\mathcal{N}$). Finally $\mathcal{A}$ defines an almost étale covering of $A_{\text{inf}}(R_\infty)$ which is almost faithfully flat ($A_{\text{inf}}(R_\infty)$ is a direct summand). The universal map $\mathcal{N} \to \mathcal{A}$ is Φ invariant. Passing to a Galois-hull of $\mathcal{A}$ we obtain the desired A.

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