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Kyo Nishiyama
Hiroyuki Ochiai
Kenji Taniguchi
Hiroshi Yamashita
Shohei Kato

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Kyo Nishiyama
Hiroyuki Ochiai
Kenji Taniguchi
Hiroshi Yamashita
Shohei Kato

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K. Nishiyama
Faculty of Integrated Human Studies, Kyoto University, Kyoto 606-8501, Japan.
E-mail : `kyo@math.h.kyoto-u.ac.jp`

H. Ochiai
Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro, Tokyo 152-8551, Japan.
E-mail : `ochiai@math.titech.ac.jp`

K. Taniguchi
Department of Mathematics, Aoyama Gakuin University, 6-16-1, Chitose-dai, Tokyo 157-8572, Japan.
E-mail : `taniken@gem.aoyama.ac.jp`

H. Yamashita
Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan.
E-mail : `yamasita@math.sci.hokudai.ac.jp`

S. Kato
6-7-8-205, Nakakasai, Edogawa-ku, Tokyo 134-0083, Japan.
CTC, 1-11-5, Fujimi, Chiyoda-ku, Tokyo 102-8166, Japan.
E-mail : `cato@ctc-g.co.jp`

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NILPOTENT ORBITS, ASSOCIATED CYCLES AND WHITTAKER MODELS FOR HIGHEST WEIGHT REPRESENTATIONS

Kyo Nishiyama, Hiroyuki Ochiai, Kenji Taniguchi, Hiroshi Yamashita, Shohei Kato

Abstract. — Let G be a reductive Lie group of Hermitian type. We investigate irreducible (unitary) highest weight representations of G which are not necessarily in the holomorphic discrete series. The results of three articles of this volume include the determination of the associated cycles, the Bernstein degrees, and the generalized Whittaker models for such representations. We give a convenient description of K-types by branching rules of representations of classical groups. An integral formula of the degrees of small nilpotent orbits is established for arbitrary Hermitian Lie algebras. The generalized Whittaker models for each unitary highest weight module are specified by means of the principal symbol of a gradient type differential operator, and also in relation to the multiplicity in the associated cycle. In the text, we also present some expository introductions of the key notions treated in this volume, such as associated cycles, Howe correspondence for dual pairs where one member of the pair is compact, and the realization of highest weight representations on the kernels of the differential operators of gradient type.

***Résumé* (Orbites nilpotentes, cycles associés et modèles de Whittaker pour les représentations de plus haut poids)**

Soit G un groupe de Lie réductif de type hermitien. Nous étudions les représentations irréductibles (unitaires) de G de plus haut poids, qui ne sont pas nécessairement dans la série discrète holomorphe. Les résultats obtenus dans les trois articles de ce volume comprennent la détermination des cycles associés, des degrés de Bernstein et des modèles de Whittaker généralisés de ces représentations. Nous donnons une description commode des K-types par les règles de branchement des représentations des groupes classiques. Une formule intégrale pour les degrés des petites orbites nilpotentes est établie pour les algèbres de Lie hermitiennes quelconques. Les modèles de Whittaker généralisés pour chaque module unitaire de plus haut poids sont spécifiés au moyen du symbole principal d'un opérateur différentiel de type gradient, et également en relation avec la multiplicité dans le cycle associé. Le texte comporte aussi des exposés introductifs concernant les principales notions considérées : cycles associés, correspondance de Howe dans le cas où la paire duale contient un membre compact et réalisation des représentations de plus haut poids dans les noyaux d'opérateurs différentiels de type gradient.

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INTRODUCTION TO THIS VOLUME

1. Associated cycle

Let G be a reductive group over $\mathbb{R}$, and consider an irreducible admissible representation π of G. There are many kinds of invariants attached to π in order to study, even classify, such representations. Among these invariants, one of the most important ones is the global character Θ_π of π. However, since the global character determines π completely, it is hard to compute Θ_π explicitly. Besides it, there are several invariants which are easier to handle; such as the infinitesimal character λ_π, the Gelfand-Kirillov dimension $\operatorname{Dim}\pi$, the Bernstein degree $\operatorname{Deg}\pi$, minimal K-types, etc., where K is a maximal compact subgroup of G. These invariants are "coarse" in the sense that a single invariant cannot specify π by itself. However, they are strong enough when you use them together to analyze the properties of π.

It is Vogan who was first aware of the importance of using the *associated variety* $\mathcal{AV}_\pi$ of π to study admissible representations of a real reductive group G (**[18, 19]**). Let $\mathfrak{g}_\mathbb{R}$ be the Lie algebra of G, and let $U(\mathfrak{g})$ be the universal enveloping algebra of the complexification $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ of $\mathfrak{g}_\mathbb{R}$. The associated variety $\mathcal{AV}_\pi$ is defined to be the support of graded $S(\mathfrak{g})$-module $\operatorname{gr} X_\pi$ corresponding to the Harish-Chandra $(U(\mathfrak{g}), K)$-module X_π of π, where $\operatorname{gr} X_\pi$ is defined through a good filtration of X_π compatible with the natural filtration of $U(\mathfrak{g})$, and $S(\mathfrak{g}) = \operatorname{gr} U(\mathfrak{g})$ denotes the symmetric algebra of $\mathfrak{g}$ (see **[19]** for precise definition). The associated variety is a kind of geometric counterpart of the purely algebraic notion of primitive ideals. It is not so hard to compute, but, as an invariant of π, it contains rich information on π. Later, Vogan refined the notion of associated variety to define the *associated cycle*. Let us see what is the associated cycle of π briefly (for precise definition, see **[19]**, and also **[NOT]** in this volume). Before that, we need some notation.

Fix a maximal compact subgroup K of G. The choice of K determines a complexified Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We denote by $K_\mathbb{C}$ the complexification of K, which has the Lie algebra $\mathfrak{k}$. Let $\mathcal{N}(\mathfrak{p})$ be the nilpotent variety in $\mathfrak{p}$. By the adjoint representation, the group $K_\mathbb{C}$ acts on $\mathcal{N}(\mathfrak{p})$ with finitely many orbits.

The associated variety of an irreducible admissible representation π is a union of the closure of equi-dimensional nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$:

$$\mathcal{AV}_\pi = \bigcup_{i=1}^{l} \overline{\mathcal{O}_i},$$

where $\{\mathcal{O}_i\}_{i=1}^{l} \subset \mathcal{N}(\mathfrak{p})/K_{\mathbb{C}}$ is a family of $K_{\mathbb{C}}$-orbits which generate the same nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_\pi^{\mathbb{C}}$ in $\mathfrak{g}$ ($G_{\mathbb{C}}$ is a connected Lie group with Lie algebra $\mathfrak{g}$). Then, the associated cycle of π is a linear combination of the closure of $\mathcal{O}_i$:

$$\mathcal{AC}_\pi = \sum_{i=1}^{l} m_i[\overline{\mathcal{O}_i}],$$

where m_i is a positive integer called the *multiplicity* of π at $\mathcal{O}_i$. Roughly speaking, the orbits $\mathcal{O}_i$ describe the "directions" in which π spreads most rapidly (cf. [**5**]). The multiplicity m_i gives the "rank" of $U(\mathfrak{g})$-module X_π localized at $\overline{\mathcal{O}_i}$. Take $\lambda \in \mathcal{O}_i$ and let $K_{\mathbb{C}}(\lambda)$ be the fixed subgroup of $K_{\mathbb{C}}$ at λ. Then $K_{\mathbb{C}}(\lambda)$ acts on the space of multiplicities, and therefore m_i can be interpreted as the dimension of the representation of $K_{\mathbb{C}}(\lambda)$ (see [**19**, Definition 2.12]).

The cycle $\mathcal{AC}_\pi$ behaves very well as an invariant of π. For example, the orbits $\mathcal{O}_i$ are equi-dimensional, and their complex dimension is equal to the Gelfand-Kirillov dimension $\operatorname{Dim}\pi$. Also the Bernstein degree is expressed as

$$\operatorname{Deg}\pi = \sum_{i=1}^{l} m_i \deg \overline{\mathcal{O}_i},$$

where $\deg \overline{\mathcal{O}_i}$ denotes the degree of the nilpotent cone $\overline{\mathcal{O}_i}$, and it should be understood as that of the corresponding projectivised variety in $\mathbb{P}(\mathfrak{p})$ (cf. [**NOT**]; see [**4, 6**] for the definition of the degree of a projective variety).

The authors of the first article in this volume, namely, Nishiyama, Ochiai and Taniguchi (abbreviated as NOT in the following), were interested in the computation of the Bernstein degree $\operatorname{Deg}\pi$. It seemed rather hard to compute $\operatorname{Deg}\pi$ for a particular instance of π. It is directly related to the associated cycle, but only few (non-trivial) examples were known at that time.

Assume that G/K is an irreducible Hermitian symmetric space, and take an irreducible unitary *highest weight* representation π of G. As a representation of $K_{\mathbb{C}}$, the space $\mathfrak{p}$ decomposes into two irreducible components:

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

Then it is well known that $\mathcal{AV}_\pi$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}_\pi \subset \mathfrak{p}^+$. Hence the associated cycle can be written as

$$\mathcal{AC}_\pi = m_\pi[\overline{\mathcal{O}_\pi}] \quad (m_\pi \in \mathbb{Z}_{>0}),$$

where m_π is the multiplicity of π at $\mathcal{O}_\pi$.

In [**NOT**], NOT derive an explicit K-type formula of π via the method of reductive dual pairs in the stable range, where G is assumed to be classical. The asymptotics of the K-types implies a formula for $\operatorname{Deg}\pi = m_\pi \deg \overline{\mathcal{O}_\pi}$. Further, the multiplicity m_π is interpreted as the dimension of an irreducible finite dimensional representation σ of a compact Lie group G' which forms a reductive dual pair (G, G') with G. The representation σ naturally determines π through the Weil (or oscillator) representation, and vice versa. The correspondence between σ and π is called the *theta correspondence* (see below for the precise formulation which requires metaplectic covers). As a byproduct, NOT also get an integral formula of $\deg \overline{\mathcal{O}_\pi}$, which can be calculated explicitly.

For any real reductive group G, the author of the second article, Yamashita, has been interested in the embeddings of irreducible admissible representations π into a series of representations induced from certain nilpotent subgroups of G. Such an induced module is called a *generalized Gelfand-Graev representation* of G (cf. [**9**]). By construction, it is attached to each nilpotent G-orbit $\mathcal{O}^{\mathbb{R}}$ in the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ through the Dynkin-Kostant theory. We say that π has a *generalized Whittaker model* of type $\mathcal{O}^{\mathbb{R}}$ if there exists an embedding of π into the generalized Gelfand-Graev representation attached to $\mathcal{O}^{\mathbb{R}}$.

The existence of generalized Whittaker models (or such vectors) reflects some regularity of the irreducible representation π of G in question. For example, as is shown by Kostant (for quasi-split groups) and Matumoto (for any real reductive groups), π has the largest possible Gelfand-Kirillov dimension if and only if the algebraic dual of the Harish-Chandra module of π has nonzero Whittaker vectors attached to the principal nilpotent orbits (see [**NOT**, Th.2.4]). Further, Matumoto ([**11**], [**12**]) established some results of this nature on generalized Whittaker vectors in connection with the associated variety $\mathcal{AV}(\operatorname{Ann}_{U(\mathfrak{g})} X_\pi)$ of the primitive ideal $\operatorname{Ann}_{U(\mathfrak{g})} X_\pi$, or the wave front set $\mathrm{WF}(\pi)$ of π. For details, we refer to [**Y**, Introduction]. It is well-known that $\mathcal{AV}(\operatorname{Ann}_{U(\mathfrak{g})} X_\pi)$ is the closure of a single nilpotent orbit in $\mathfrak{g}$ which contains $\mathcal{AV}_\pi$. The wave front set $\mathrm{WF}(\pi)$ describes the singularity of the distribution character Θ_π of π, and it is a union of some nilpotent G-orbits in $\mathfrak{g}_{\mathbb{R}}$. By Rossmann [**14**], it is shown that $\mathrm{WF}(\pi)$ coincides with the asymptotic support of π introduced in [**1**]. Recently, Schmid and Vilonen proved that the wave front "cycle", which is a refinement of $\mathrm{WF}(\pi)$, corresponds to the associated cycle via Kostant-Sekiguchi correspondence ([**16**]).

Then, it is natural to ask whether the associated cycle characterizes the generalized Whittaker models of interest. At first glance, this problem may seem to be more difficult to handle directly, since the associated cycle lives in $\mathcal{N}(\mathfrak{p})$, contrary to the above two invariants $\mathcal{AV}(\operatorname{Ann}_{U(\mathfrak{g})} X_\pi)$ and $\mathrm{WF}(\pi)$ of π. But, in [**5**], Gyoja and Yamashita found evidence for a strong relationship between the associated variety $\mathcal{AV}_\pi$ and the embeddings in question. Moreover, for any unitary highest weight representation π

of the indefinite unitary group $G = U(p,q)$, Tagawa calculated in his master thesis **[17]** the dimension of the space of generalized Whittaker models attached to the Cayley transform of $\mathcal{O}_\pi$, by using the realization of π in the oscillator representation of reductive dual pair (G, G') with $G' = U(m)$. It is equal to the dimension of the corresponding irreducible representation σ of G', if the dual pair (G, G') is in the stable range, i.e., $m \leqslant \min(p,q)$.

Concerning the irreducible admissible (unitary) highest weight representations π in Hermitian symmetric case, Yamashita gives in **[Y]** some structure theorems for the space $\mathcal{Y}_\pi$ of all $(\mathfrak{g}, K)$-homomorphisms from the Harish-Chandra module X_π into the generalized Gelfand-Graev representation $\Gamma(\mathcal{O}_\pi^{\mathbb{R}})$. Here, $\mathcal{O}_\pi \in \mathcal{N}(\mathfrak{p})/K_{\mathbb{C}}$ and $\mathcal{O}_\pi^{\mathbb{R}} \in \mathcal{N}(\mathfrak{g}_{\mathbb{R}})/G$ are related by the Kostant-Sekiguchi correspondence, and the representation $\Gamma(\mathcal{O}_\pi^{\mathbb{R}})$ is attached to the nilpotent G-orbit $\mathcal{O}_\pi^{\mathbb{R}}$. It is proved that, if the representation π is unitary, the associated cycle $\mathcal{AC}_\pi$ completely characterizes the generalized Whittaker models of type $\mathcal{O}_\pi^{\mathbb{R}}$.

In 1998 and 1999, we, NOT and Yamashita, had several occasions to discuss what is going on, and we gradually had been understanding that some of our results are very close in terms of associated cycles. For example, it turns out that the dimension of the vector space $\mathcal{Y}_\pi$ and the multiplicity m_π in the associated cycle of π coincide each other; they are both equal to $\dim \sigma$ in the stable range case, where σ is the irreducible representation of G' associated with π via theta correspondence.

Both methods have their own advantages.

In **[Y]**, it is shown that the space $\mathcal{Y}_\pi$, which characterizes the generalized Whittaker models, carries a natural $K_{\mathbb{C}}(X)$-action, where $X \in \mathcal{O}_\pi$, and $K_{\mathbb{C}}(X)$ denotes the fixed subgroup of X in $K_{\mathbb{C}}$. As mentioned before, the multiplicity in the associated cycle is naturally interpreted as the dimension of a certain representation of $K_{\mathbb{C}}(X)$. An application of such interpretation is given in **[Y]** by showing that $\mathcal{Y}_\pi$ is contragredient to the $K_{\mathbb{C}}(X)$-module attached to π by Vogan **[19]**. Moreover, in view of the work **[16]** of Schmid and Vilonen, the result of **[Y]** naturally gives an interpretation of the multiplicity in the wave front cycle for unitary highest weight modules. In fact, Yamashita's proof uses the Cayley transform (Kostant-Sekiguchi correspondence) essentially. It should be noticed that the work **[Y]** does not deal with the degree of the nilpotent orbit, which is the other important quantity in the associated cycle.

On the other hand, in **[NOT]**, their computation also gives the degree $\deg \overline{\mathcal{O}_\pi}$ itself for the representation π in the stable range of reductive dual pairs by some definite integral. Although, the formula of $\deg \overline{\mathcal{O}_\pi}$ for such representations π is already known (Giambelli-Thom-Porteous formula, see **[NOT]**), this seems to be a new proof which is purely representation theoretic. Also, we can read off the strong relationship between K-type decomposition $\pi\big|_K$ and $K_{\mathbb{C}}$-module structure of the regular function ring $\mathbb{C}[\overline{\mathcal{O}_\pi}]$, which is predicted also by Vogan (cf. **[20]**).

Later, Kato and Ochiai **[KO]** obtained a formula for the degree $\deg \overline{\mathcal{O}_\pi}$ for irreducible unitary highest weight modules π of an arbitrary simple Lie group of Hermitian type, including exceptional groups. Here, we can not use the theory of reductive dual pairs or classical invariant theory. Instead, we use the structure of root systems corresponding to the orbits. The same method is applicable for orbits of irreducible representations with multiplicity-free action. The results in **[KO]** cover all the irreducible multiplicity-free representation. As an application, we determine the explicit value of the Bernstein degree and the associated cycle of irreducible unitary highest weight representations with scalar extreme K-type. This is a generalization of **[NOT]**.

After discussing, we, all of five authors of these three articles, finally were led to an idea that we should publish our results in a unified volume, and here it is.

2. Main results

Up to now, we are concentrated only on the associated cycles. Although each article contains its own introduction, let us briefly take a look at the other aspects of the three articles.

2.1. Description of the generalized Whittaker models via gradient type differential operators. — Let us first explain the results of [**Y**].

Let π be an irreducible admissible representation of G. We denote by π^* the representation of G contragredient to π. Suppose that the Harish-Chandra module $X_{\pi^*} = (X_\pi)^*$ of π^* is realized as the K-finite kernel of a certain invariant differential operator $\mathcal{D}$ of gradient type acting on the C^∞-sections of a G-homogeneous vector bundle over G/K. For example, discrete series representations or derived functor modules satisfy this assumption, if the infinitesimal character is sufficiently regular.

The main object of the article is a description of generalized Whittaker models for each representation π with highest weight, by using the principal symbol of the differential operator $\mathcal{D}$.

To be more precise, let G be a connected simple Lie group of Hermitian type. As a representation, we take an irreducible admissible highest weight representation $\pi = \pi(\tau)$ with extreme K-type $\tau \in \operatorname{Irr}(K)$. Note that π is *not* necessarily unitary for a while. We write $L(\tau) = X_{\pi(\tau)}$ for the Harish-Chandra module of π. It is known that the dual lowest weight module $L(\tau)^*$ can be realized as the K-finite kernel of a G-invariant differential operator $\mathcal{D}_{\tau^*}$ of gradient type (this fact is due to Davidson, Enright and Stanke; see [**Y**, Section 2.3] for the definition of $\mathcal{D}_{\tau^*}$).

Take an arbitrary $K_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{p}^+$. Let $\mathcal{O}^{\mathbb{R}}$ denote the nilpotent G-orbit in $\mathfrak{g}_{\mathbb{R}}$ attached to $\mathcal{O}$ by the Kostant-Sekiguchi correspondence. Then, a standard argument in the Dynkin-Kostant theory on the nilpotent orbit $\mathcal{O}^{\mathbb{R}}$ allows us to define a nilpotent Lie subalgebra $\mathfrak{n}(\mathcal{O}^{\mathbb{R}})$ of $\mathfrak{g}$ and its character $\eta(\mathcal{O}^{\mathbb{R}})$. An infinitesimally induced representation

$$\Gamma(\mathcal{O}^{\mathbb{R}}) = \operatorname{Ind}_{\mathfrak{n}(\mathcal{O}^{\mathbb{R}})}^{G} \eta(\mathcal{O}^{\mathbb{R}})$$

is called a generalized Gelfand-Graev representation of G associated to $\mathcal{O}^{\mathbb{R}}$. For each $L(\tau)$ and each $\Gamma(\mathcal{O}^{\mathbb{R}})$, we are concerned with the space

$$\mathcal{Y}(\tau, \mathcal{O}) := \operatorname{Hom}_{\mathfrak{g},K}(L(\tau), \Gamma(\mathcal{O}^{\mathbb{R}})),$$

which describes the generalized Whittaker models for $L(\tau)$ of type $\mathcal{O}^{\mathbb{R}}$.

Let $\mathcal{AV}_{\pi} = \overline{\mathcal{O}_{\pi}} \subset \mathfrak{p}^{+}$ be the associated variety of π, and $\mathcal{O}_{\pi}^{\mathbb{R}} \subset \mathcal{N}(\mathfrak{g}_{\mathbb{R}})$ the Cayley transform of $\mathcal{O}_{\pi}$.

The following theorem is a consequence of Theorems 4.7 and 4.9 in [**Y**].

Theorem A

(1) *The dimension of the vector space $\mathcal{Y}(\tau, \mathcal{O})$ is given by*

$$\dim \mathcal{Y}(\tau, \mathcal{O}) = \begin{cases} 0 & \text{if } \dim \mathcal{O} > \dim \mathcal{O}_{\pi}, \\ \text{finite } (\neq 0) & \text{if } \mathcal{O} = \mathcal{O}_{\pi}, \\ \infty & \text{if } \dim \mathcal{O} < \dim \mathcal{O}_{\pi}. \end{cases}$$

(2) *Let $\sigma(\mathcal{D}_{\tau^*})$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ at the origin (see* [**Y**, Section 3.3] *for the precise definition). Then, the kernel of the linear map $\sigma(\mathcal{D}_{\tau^*})(X, \cdot\,)$ does not vanish if the element X lies in $\mathcal{O}_{\pi}$. For such an X, there exists a canonical linear embedding of this kernel space into $\mathcal{Y}_{\pi} := \mathcal{Y}(\tau, \mathcal{O}_{\pi})$.*

This theorem tells us that the embeddings impose a strict restriction on the associated variety (cf. [**11**]). Note that the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^{+}$ are distinguished by their dimension. In particular, we have $\mathcal{O} \subset \mathcal{AV}_{\pi} = \overline{\mathcal{O}_{\pi}}$ if and only if $\dim \mathcal{O} \leqslant \operatorname{Dim} \pi = \dim \mathcal{O}_{\pi}$.

As for the unitary highest weight modules, we get the following neat description [**Y**, Theorem 4.8] of generalized Whittaker models.

Theorem B. — *Suppose that the representation $\pi = \pi(\tau)$ of G is unitary. Then, the linear embedding of* $\ker \sigma(\mathcal{D}_{\tau^*})(X, \cdot\,)$ *into $\mathcal{Y}_{\pi}$ given in Theorem A is surjective, where $X \in \mathcal{O}_{\pi}$. Moreover, the common dimension of these two spaces coincides with the multiplicity m_{π} in the associated cycle, i.e.,*

$$\mathcal{AC}_{\pi} = (\dim \mathcal{Y}_{\pi}) \cdot [\overline{\mathcal{O}_{\pi}}] = (\dim \ker \sigma(\mathcal{D}_{\tau^*})(X, \cdot\,)) \cdot [\overline{\mathcal{O}_{\pi}}].$$

This theorem tells us that the associated cycle gives some control even on the embeddings of π into some kind of representations. We note that the kernel $\ker \sigma(\mathcal{D}_{\tau^*})(X, \cdot\,)$ has a structure of $K_{\mathbb{C}}(X)$-module in a natural way, where $K_{\mathbb{C}}(X)$ is the fixed subgroup of X in $K_{\mathbb{C}}$ as before.

As an application of Theorem B, one can compute the multiplicity $m_{\pi} = \dim \mathcal{Y}_{\pi}$ explicitly, if π is the theta lift of a finite dimensional representation of a compact group G'. To be more precise, assume that G is a classical group of type AIII, CI or DIII. Let G' be a compact group dual to G in the sense of Howe's theory on reductive dual pairs in the large symplectic group $Sp(2N, \mathbb{R})$. Let $Mp(2N, \mathbb{R})$ be the metaplectic group which is the unique non-trivial double cover of $Sp(2N,\mathbb{R})$. We denote by $\widetilde{G}$, $\widetilde{K}$ and $\widetilde{G'}$ the inverse images of G, K and G' by the covering map $Mp(2N,\mathbb{R}) \to Sp(2N,\mathbb{R})$ respectively. Then, the Weil representation ω of $Mp(2N,\mathbb{R})$ restricted to the pair $(\widetilde{G},\widetilde{G'})$ decomposes into a direct sum of irreducible representations of $\widetilde{G}\times\widetilde{G'}$ without multiplicity:

$$\omega \simeq \bigoplus_{\sigma} \theta(\sigma)\,\hat{\otimes}\,\sigma,$$

and the assignment $\sigma \mapsto \theta(\sigma)$ gives a one-one correspondence between a set of (equivalence classes of) irreducible unitary representations of $\widetilde{G'}$ and a set of such representations of $\widetilde{G}$. Note that σ is necessarily finite dimensional since $\widetilde{G'}$ is compact. The representation $\theta(\sigma)$ of $\widetilde{G}$ is called the *theta lift* of σ. It is well known that $\theta(\sigma)$ is a unitary highest weight representation $\pi = \pi(\tau)$ for some $\tau \in \mathrm{Irr}(\widetilde{K})$ (see **[10]** and **[3]**). Although $\theta(\sigma)$ is a genuine representation of the double covering $\widetilde{G}$, after the twist by an appropriate character, or just taking the connected component, we also get almost all the unitary highest weight representations of G itself in this way.

By using the above realization of the irreducible representation $\pi = \theta(\sigma)$ in ω, the $K_{\mathbb{C}}(X)$-module $\ker\sigma(\mathcal{D}_{\tau^*})(X,\,\cdot\,)$, which is isomorphic to the space $\mathcal{Y}_\pi$ of all $(\mathfrak{g},K)$-homomorphisms from π into $\Gamma(\mathcal{O}_\pi^{\mathbb{R}})$, can be described through some algebraic and geometric techniques. See Theorems 5.14 and 5.15 (together with the isomorphism (4.15)) in **[Y]** for the precise statements. We deduce in particular the following

Theorem C. — *Assume that the pair (G,G') is in the stable range with G' the smaller member. Let $\pi = \theta(\sigma)$ be the theta lift of an irreducible finite dimensional representation $\sigma \in \mathrm{Irr}(\widetilde{G'})$. Then we have*

$$\mathcal{AC}_\pi = \dim\sigma\cdot[\overline{\mathcal{O}_\pi}],$$

where $\mathcal{O}_\pi$ does not depend on the individual $\pi = \theta(\sigma)$, but it depends only on the group G'.

NOT proved the same statement in a completely different way.

2.2. Asymptotics of K-types and the stable branching coefficients.— Now let us turn to **[NOT]**.

Let (G,G') be a reductive dual pair of type I which is irreducible. We consider the case where G' is a *compact* group. Then G is necessarily of Hermitian type. Namely, the pair is one of the following.

$$(G,G') = \begin{cases} (Sp(2n,\mathbb{R}),O(m)), \\ (U(p,q),U(m)), \\ (O^*(2p),Sp(2m)). \end{cases}$$

We further assume that the pair is in the stable range with G' the smaller member. For $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G'})$, there corresponds an irreducible unitary representation $\theta(\widetilde{\sigma}) \in \mathrm{Irr}(\widetilde{G})$

called the *theta lift* of $\widetilde{\sigma}$. Note that $\theta(\widetilde{\sigma})$ is possibly zero. We assume that $\theta(\widetilde{\sigma})$ does not vanish in the following. If we twist $\widetilde{\sigma}$ by a certain unitary character χ of $\widetilde{G'}$, then $\sigma = \widetilde{\sigma} \otimes \chi^{-1}$ factors through to the representation of G'. We denote $L(\sigma) = \theta(\sigma \otimes \chi)$ for $\sigma \in \mathrm{Irr}(G')$ (see **[NOT**, § 5] for more details). In this case, $L(\sigma)$ is an irreducible unitary highest weight representation which is singular with few exceptions.

In **[NOT]**, we first study the K-type decomposition $L(\sigma)|_{\widetilde{K}}$ via the branching coefficients of finite dimensional representations of classical groups. It has a nice K-type formula. Put $\pi = L(\sigma)$. Let $\mathcal{O}_\pi$ be the open $K_{\mathbb{C}}$-orbit of the associated variety of $\pi = L(\sigma)$ which actually depends only on G'. It is well known among experts that $\overline{\mathcal{O}_\pi}$ is a geometric quotient of a vector space by a linear action of $G'_{\mathbb{C}}$. From the description, we can compute the regular function ring $\mathbb{C}[\overline{\mathcal{O}_\pi}]$ explicitly. These considerations are more or less folklore in the representation theory.

The first main result of **[NOT]** is an integral expression of the degree of nilpotent orbits. If $\sigma = \mathbf{1}_{G'}$ is the trivial representation of G', then its theta lift $\pi = L(\mathbf{1}_{G'})$ has almost the same K-type structure as $\mathbb{C}[\mathcal{AV}_\pi]$. The difference between these two K-type structures are only a small constant shift in the highest weights. This is essentially due to Davidson, Enright and Stanke **[2]**. The coincidence of K-types imply that the multiplicity of the associated cycle $\mathcal{AC}_\pi$ is one, and we have $\operatorname{Deg}\pi = \deg\overline{\mathcal{O}_\pi}$. Thus the calculation of $\deg\overline{\mathcal{O}_\pi}$ reduces to that of the Bernstein degree of the theta lift of the trivial representation. This is a purely representation theoretic problem.

Theorem D. — *For the $K_{\mathbb{C}}$-orbit $\mathcal{O}_\pi$ in $\mathfrak{p}^+$, there exist integers F, m, n and $1 \leqslant \alpha \leqslant 4$ such that*

$$\deg\overline{\mathcal{O}_\pi} = \frac{1}{F}\frac{(\dim\mathcal{O}_\pi)!}{m!}\times \int_{\substack{x_i \geqslant 0 \\ x_1+\cdots+x_m \leqslant 1}} \Big|\prod_{1\leqslant i<j\leqslant m}(x_i - x_j)\Big|^\alpha (x_1\cdots x_m)^{n-\alpha m}dx_1\cdots dx_m,$$

where F, n, m, α are explicitly given in **[NOT**, § 7.6].

The above definite integral can be expressed explicitly in terms of Gamma functions, and it gives Giambelli's formula.

The second result is a comparison theorem of K-types of $\pi = L(\sigma)$ and those of $\mathbb{C}[\mathcal{AV}_\pi]$ for general σ. If σ is not the trivial representation, the description of K-types $L(\sigma)|_{\widetilde{K}}$ in terms of $\mathbb{C}[\mathcal{AV}_\pi]$ requires the notion of *stable branching coefficients*, which is first introduced by Sato **[15]**. Let (L, H) be a spherical pair of reductive algebraic groups. Let Φ^+ be the lattice semigroup consisting of the dominant integral weights for L. We denote by Φ the integral weight lattice generated by Φ^+. Put

$$\Phi^+(H) = \{\eta \in \Phi^+ \mid (\tau_\eta)^H \neq 0\},$$

where τ_η denotes an irreducible finite dimensional representation of L with highest weight $\eta \in \Phi^+$ and $(\tau_\eta)^H$ is the subspace of H-invariants (or H-spherical vectors). Let

$\Phi(H)$ be the sublattice in Φ generated by $\Phi^+(H)$. Sato's observation is as follows. Take an irreducible finite dimensional representation $\sigma \in \mathrm{Irr}(H)$ of H. Then the branching coefficient $m(\lambda, \sigma)$ $(\lambda \in \Phi^+)$ defined by

$$\tau_\lambda\big|_H \simeq \sum_{\sigma \in \mathrm{Irr}(H)}^{\oplus} m(\lambda, \sigma)\, \sigma$$

does not depend on a particular λ, but only depends on the coset $[\lambda] \in \Phi^+/\Phi(H)$ for sufficiently "large" λ. We call the value $m(\lambda, \sigma)$ for a sufficiently large λ the stable branching coefficient and denote it as $m([\lambda], \sigma)$.

Now let us take $H = G'_{\mathbb{C}}$ and L as follows according as the pair (G, G').

(G, G')	(L, H)	l
$(Sp(2n, \mathbb{R}), O(m))$	(GL_m, O_m)	n
$(U(p, q), U(m))$	$(GL_m \times GL_m, GL_m)$	$\min(p, q)$
$(O^*(2p), Sp(2m))$	(GL_{2m}, Sp_{2m})	p

Note that L is the complexification of a member of "seesaw" dual pairs.

We do *not* assume that the pair (G, G') is in the stable range in the following theorem.

Theorem E. — *Fix l in the above table, and take $\sigma \in \mathrm{Irr}(G') = \mathrm{Irr}(H)$. Then $L(\sigma)\big|_{\widetilde{K}}$ is asymptotically a multiple of $\mathbb{C}[\mathcal{AV}_\pi]$ $(\pi = L(\sigma) = \theta(\sigma \otimes \chi))$. The multiplicity, denoted by m_π, is given as follows. There are a sublattice $\Lambda_l^+ \subset \Phi^+$ and a dimension function $r([\lambda])$ on $\Lambda_l^+/\Phi(H)$ such that*

$$m_\pi = \sum_{[\lambda] \in \Lambda_l^+/\Phi(H)} m([\lambda], \sigma) r([\lambda]).$$

Moreover, m_π coincides with the multiplicity of $\overline{\mathcal{O}_\pi}$ in the associated cycle $\mathcal{AC}_\pi$.

If the dual pair (G, G') is in the stable range, then $m \leqslant l$ holds and the above Λ_l^+ coincides with the whole Φ^+. In that case, we can use *Sato's summation formula* to deduce that $m_\pi = \dim \sigma$, which will imply Theorem C.

2.3. Degree of nilpotent orbits. — We now explain the paper **[KO]**.

Let G be a simple Lie group of Hermitian type, and consider K-invariant decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ as above. Then there are precisely $r + 1$ $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \leqslant m \leqslant r)$ in $\mathfrak{p}^+$, where r is the real rank of $\mathfrak{g}_{\mathbb{R}}$.

For three types of groups G in Section 2.2, the degree of the closure of the nilpotent orbit $\overline{\mathcal{O}_m}$ is given by Giambelli formula. In **[NOT]**, the degree in this case is computed using the explicit $K_{\mathbb{C}}$-structure of the regular function ring $\mathbb{C}[\overline{\mathcal{O}_m}]$ and Weyl's dimension formula. The result is expressed by some definite integral over some simplex. The integral can be evaluated explicitly by using the gamma function associated to the Hermitian symmetric cone. This gives an alternative, and possibly new, proof of Giambelli's formula.

In [**KO**], we consider the degree of the closure of the nilpotent orbit $\overline{\mathcal{O}_m}$ for a simple Lie group of Hermitian type. It is significant that the representation of $K_{\mathbb{C}}$ on $\mathbb{C}[\mathfrak{p}^+]$ is multiplicity-free. In general, let us consider an irreducible representation V of a complex reductive algebraic group $\mathbf{K}$ whose action on $\mathbb{C}[V]$ is multiplicity-free. We obtain the integral formula of the degree of the closure of an arbitrary $\mathbf{K}$-orbit in V.

Let Δ be the root system of $\mathbf{K}$ and define $\rho_{\mathbf{K}} = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, the half sum of all the positive roots. We may assume that there exists $Z \in \mathrm{Lie}(\mathbf{K})$ which represents the degree operator for the polynomial ring $\mathbb{C}[V]$.

Theorem F. — *Let Y be an irreducible closed $\mathbf{K}$-stable subset of V. Then there exist a set of dominant integral weights $\{\varphi_1, \ldots, \varphi_m\}$ and a subset $\Delta_Y \subset \Delta$ such that*

$$\deg Y = \frac{(m + |\Delta_Y^+|)!}{\prod_{\alpha\in\Delta_Y^+}\langle\alpha, \rho_{\mathbf{K}}\rangle} \int_D \prod_{\alpha\in\Delta_Y^+} \langle\alpha, \textstyle\sum_{i=1}^m x_i\varphi_i\rangle \, dx_1\cdots dx_m,$$

where the domain D of the integration is given by

$$D = \Big\{(x_1, \ldots, x_m) \mid x_1, \ldots, x_m \geqslant 0,\ \textstyle\sum_{i=1}^m x_i\varphi_i(Z) \leqslant 1\Big\}.$$

In the case of a simple Lie group of Hermitian type, the set of linear functions $\{\langle\alpha, \sum_{i=1}^m x_i\varphi_i\rangle \mid \alpha \in \Delta_Y^+\}$ can be described in terms of the restricted root systems. Then, we have a unified formula to express the degree of the orbits $\overline{\mathcal{O}_m}$, which covers the Giambelli formula in [**NOT**] as well as that for two exceptional groups of Hermitian type. We also apply this method to obtain the associated cycles and the Bernstein degrees of certain unitary highest weight representations of exceptional G, which are in the Wallach set.

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BERNSTEIN DEGREE AND ASSOCIATED CYCLES OF HARISH-CHANDRA MODULES - HERMITIAN SYMMETRIC CASE -

by

Kyo Nishiyama, Hiroyuki Ochiai & Kenji Taniguchi

Dedicated to Professor Ryoshi Hotta on his 60th anniversary

Abstract. — Let $\tilde{G}$ be the metaplectic double cover of $Sp(2n, \mathbb{R})$, $U(p, q)$ or $O^*(2p)$. we study the Bernstein degrees and the associated cycles of the irreducible unitary highest weight representations of $\tilde{G}$, by using the theta correspondence of dual pairs. The first part of this article is a summary of fundamental properties and known results of the Bernstein degrees and the associated cycles. Our first result is a comparison theorem between the K-module structures of the following two spaces; one is the theta lift of the trivial representation and the other is the ring of regular functions on its associated variety. Secondarily, we obtain the explicit values of the degrees of some small nilpotent $K_{\mathbb{C}}$-orbits by means of representation theory. The main result of this article is the determination of the associated cycles of singular unitary highest weight representations, which are the theta lifts of irreducible representations of certain compact groups. In the proofs of these results, the multiplicity free property of spherical subgroups and the stability of the branching coefficients play important roles.

***Résumé* (Le degré de Bernstein et le cycle associé des modules de Harish-Chandra — le cas hermitien symétrique)**

Soit $\tilde{G}$ le revêtement double métaplectique de $Sp(2n, \mathbb{R})$, $U(p, q)$ ou $O^*(2p)$. Nous étudions les degrés de Bernstein et les cycles associés des représentations irréductibles unitaires de $\tilde{G}$ de plus haut poids, en utilisant la correspondance thêta par paires duales. La première partie de cet article est un résumé des propriétés fondamentales et des résultats connus concernant les degrés de Bernstein et les cycles associés. Notre premier résultat est un théorème de comparaison entre les structures en tant que K-modules des deux espaces suivants : l'un est le relèvement thêta de la représentation évidente, l'autre est l'anneau des fonctions régulières sur la variété associée. Deuxièmement, nous obtenons de manière explicite les valeurs des degrés de quelques petites $K_{\mathbb{C}}$-orbites nilpotentes au moyen de la théorie des représentations. Le résultat principal de cet article est la détermination des cyles associés aux représentations singulières unitaires de plus haut poids, qui sont les relèvements thêta des représentations irréductibles de certains groupes compacts. Dans les démonstrations de ces résultats, la non-multiplicité des sous-groupes sphériques et la stabilité des coefficients de branchement jouent des rôles importants.

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Introduction

Let G be a semisimple (or more generally, reductive) Lie group. For an irreducible admissible representation π of G, there exist several important invariants such as irreducible characters, primitive ideals, associated varieties, asymptotic supports, Bernstein degrees, Gelfand-Kirillov dimensions, etc. They are interrelated with each other, and intimately related to the geometry of coadjoint orbits.

For example, at least if G is compact and π is finite dimensional, the character of π is the Fourier transform of an orbital integral on a semisimple coadjoint orbit (**[29]**). This is also the case for a general semisimple G and fairly large family of the representations (see **[41]**). This intimate relation between coadjoint orbits and irreducible representations invokes the philosophy of so-called *orbit method*, which is exploited by pioneer works of Kirillov and Kostant, and is now being developed by many contributors. However, for a general semisimple Lie group G, it seems that the orbit method still requires much to do. In particular, we should understand some small representations corresponding to nilpotent coadjoint orbits, which are called *unipotent*.

On the other hand, by definition, most of invariants are directly related to *nilpotent* coadjoint orbits. In a sense, the corresponding nilpotent orbits represent the leading term of irreducible characters (**[1]**, **[44]**). The invariants of large representations correspond to the largest nilpotent coadjoint orbit, namely, the principal nilpotent orbit. For large representations, the orbit method seems to behave considerably well. Therefore we are now interested in 'small' representations whose invariants are related to smaller nilpotent coadjoint orbits.

One extreme case is the case of finite dimensional representations. In this case, however, the corresponding orbit is zero, and there is not a so much interesting phenomenon. The next to the extreme case is the case of minimal representations, which corresponds to the minimal nilpotent orbit. The minimal nilpotent orbit is unique in the sense that it is the only orbit among non-zero nilpotent ones with the smallest possible dimension. These representations have a simple structure. For example, their K-type structure is in a ladder form and is multiplicity free (**[50]**). Against its simple structure, though, systematic and thorough study of the minimal representations is still progressing now through the works of Kostant-Brylinski and many other mathematicians. If we turn our attention to the small representations other than minimal ones, it seems that there is relatively less knowledge on them up to now. In this paper, we study small representations which are unitary lowest (or highest) weight representations of G. Such representations exist if and only if G/K enjoys a structure of Hermitian symmetric space, where K denotes a maximal compact subgroup of G.

To be more specific, let us introduce notations. We assume that the symmetric space G/K is irreducible and Hermitian. Moreover, we assume that G is classical other than $SO(n, 2)$, i.e., $G = Sp(2n, \mathbb{R}), U(p, q)$ or $O^*(2p)$. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the Cartan decomposition with respect to K. We denote the complexified decomposition by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since G/K is an irreducible Hermitian symmetric space, the induced adjoint representation of K on $\mathfrak{p}$ breaks up into precisely two irreducible components $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Note that, as a representation of K, $\mathfrak{p}^-$ is contragredient to $\mathfrak{p}^+$ via the Killing form. We extend this representation of K to the representation of the complexification $K_{\mathbb{C}}$ of K holomorphically.

Let L be an irreducible unitary lowest weight module of G. Then it is well-known that the associated variety of L, denoted by $\mathcal{AV}(L)$, is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit contained in $\mathfrak{p}^-$ (we choose an appropriate positive system which is compatible with $\mathfrak{p}^+$).

Put $r = \mathbb{R}\text{-rank}\, G$, the real rank of G. Then there exist exactly $(r+1)$ nilpotent $K_{\mathbb{C}}$-orbits $\{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_r\}$ in $\mathfrak{p}^-$. We choose an indexing of the orbits so that $\dim \mathcal{O}_{i-1} < \dim \mathcal{O}_i$ holds for $1 \leqslant i \leqslant r$; in particular, $\mathcal{O}_0 = \{0\}$ is the trivial one, and $\mathcal{O}_r$ is the open dense orbit. Most of lowest weight representations L correspond to the largest orbit $\mathcal{O}_r$. For example, the associated variety of a holomorphic discrete series (or its limit) is $\overline{\mathcal{O}_r} = \mathfrak{p}^-$. The invariants of the holomorphic discrete series representations are completely understood (see **[14]**, **[43]**, **[7]**; also see § 2.4 below). However, for each orbit $\mathcal{O}_m$ $(0 < m < r)$, there exists a relatively small family of lowest weight representations whose associated variety is indeed the closure of the orbit $\mathcal{O}_m$. Thanks to the theory of reductive dual pairs via the Weil representation of metaplectic groups, we have a complete knowledge of such a family of lowest weight representations (at least for classical groups listed above).

Although we can define a specific 'small' representations even for the largest orbit $\mathcal{O}_r$, we restrict ourselves to the case $\mathcal{O}_m$ $(m < r)$ in this introduction. Then there exists a compact group G_2 corresponding to each m (cf. § 3, Table 2) such that (G_1, G_2) forms a dual pair in a large symplectic group $Sp(2N, \mathbb{R})$. Let $Mp(2N, \mathbb{R})$ be the metaplectic double cover of $Sp(2N, \mathbb{R})$. We denote by $\widetilde{H} \subset Mp(2N, \mathbb{R})$ the inverse image of a subgroup $H \subset Sp(2N, \mathbb{R})$ of the covering map.

The family of unitary irreducible lowest weight representations of $\widetilde{G}$ whose associated variety is $\overline{\mathcal{O}_m}$ is parametrized by $\mathrm{Irr}(G_2)$, the set of the irreducible finite dimensional representations of G_2. We denote the lowest weight representation of $\widetilde{G}$ corresponding to $\sigma \in \mathrm{Irr}(G_2)$ by $L(\sigma)$ (see § 5 for precise description). Roughly, the correspondence $\sigma \mapsto L(\sigma)$ is the theta lift after twisted by a certain unitary character of $\widetilde{G_2}$.

Our first observation is the following.

Theorem A. — *Let $\mathbf{1}_{G_2}$ be the trivial representation of G_2 and $L(\mathbf{1}_{G_2})$ the unitary lowest weight representation of $\widetilde{G}$ corresponding to $\mathbf{1}_{G_2}$. The Bernstein degree of $L(\mathbf{1}_{G_2})$ coincides with the degree of the closure of the nilpotent orbit $\overline{\mathcal{O}_m}$ (defined in the sense of algebraic geometry) ;*

$$\operatorname{Deg} L(\mathbf{1}_{G_2}) = \deg \overline{\mathcal{O}_m}.$$

We also get an explicit and computable formula for $\operatorname{Deg} L(\mathbf{1}_{G_2})$.

Note that the varieties $\overline{\mathcal{O}_m}$ are determinantal varieties of various type and an explicit formula of their degree is known as *Giambelli-Thom-Porteous formula.* Our representation theoretic proof of the formula seems new, and gives an alternative proof.

To prove Theorem A, we construct a $K_{\mathbb{C}}$-equivariant map $\psi : V \to \overline{\mathcal{O}_m}$, where V is a certain $K_{\mathbb{C}} \times (G_2)_{\mathbb{C}}$-module. This map induces an algebra isomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \xrightarrow{\sim} \mathbb{C}[V^*]^{(G_2)_{\mathbb{C}}},$$

which means that $\overline{\mathcal{O}_m} = V//(G_2)_{\mathbb{C}}$. The map ψ is closely related to the dual pair (G, G_2), and we call it *unfolding* of $\overline{\mathcal{O}_m}$. By this, the proof of Theorem A reduces to a problem of classical invariant theory.

The 'smallest' unipotent representation attached to the orbit $\mathcal{O}_m$ should be realized on the section of a certain line bundle on $\mathcal{O}_m$ called half-form bundle ([**5**], [**6**], [**52**]). We investigate such half-form bundles, and get an evidence of strong relationship between the space of global sections of the half-form bundles and $L(\sigma)$, where σ is a special one-dimensional character of G_2.

Next, let us consider a general unitary lowest weight module $L(\sigma)$ ($\sigma \in \mathrm{Irr}(G_2)$). We describe its K-type decomposition and the Poincaré series in terms of certain branching coefficient of finite dimensional representations of general linear groups and G_2. Such descriptions are well-known among experts. However, references to them are scattered in many places, and sometimes their treatments are ad hoc. Since we need an explicit and unified picture for the K-types of $L(\sigma)$, we reproduce the decompositions in the sequel.

Now our main theorem says

Theorem B. — *Let $L(\sigma)$ be an irreducible unitary lowest weight module of $\widetilde{G}$ corresponding to $\sigma \in \mathrm{Irr}(G_2)$. Then its Bernstein degree is given by*

$$\operatorname{Deg} L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}_m}.$$

There is a notion of *associated cycle* which is a refinement of the notion of associated variety. Roughly speaking, it expresses associated variety with multiplicity. For a precise definition, see §§ 1.1 and 1.3. Then the following is an immediate corollary to Theorem B.

Theorem C. — *The associated cycle of $L(\sigma)$ is given by $\mathcal{AC}\,(L(\sigma)) = \dim \sigma \cdot [\overline{\mathcal{O}_m}]$.*

The proof of Theorem B is based on the theory of multiplicity free action of algebraic groups, which is a subject of § 8. The key ingredients of the proof are multiplicity free property of spherical subgroups and Sato's summation formula of the stable branching coefficients.

Lastly, we would like to comment on several aspects of our results.

First, the Bernstein degree of an irreducible representation π is closely related to the dimension of its "Whittaker vectors". In fact, for large representations, Matumoto proved that the Bernstein degree and the dimension of algebraic Whittaker vectors coincide ([**36**]). For 'small' representations, we cannot hope the same story, because they do not have any Whittaker vector in a naive sense. However, for complex semisimple Lie groups, Matumoto observed that the finite-dimensionality and non-vanishing of the space of certain degenerate Whittaker vectors determines the wave front set of π ([**34**], [**35**]). Recently, Yamashita has found a strong relation between the multiplicity of associated cycles and the dimension of generalized Whittaker vectors in the case of unitary highest weight module ([**54**]).

Second, let us consider the (twisted) theta correspondence (or Howe correspondence, dual pair correspondence, ...) between $L(\sigma) \in \operatorname{Irr}(\widetilde{G})$ and $\sigma \in \operatorname{Irr}(G_2)$. Since G_2 is compact and σ is finite dimensional, its associated cycle is simply given by $\mathcal{AC}(\sigma) = \dim \sigma \cdot [\{0\}]$. Recall $\mathcal{AC}(L(\sigma)) = \dim \sigma \cdot [\overline{\mathcal{O}_m}]$ from Theorem C. These formulas strongly indicate the following; there should be a correspondence between nilpotent orbits of the dual pairs, and it induces certain relation between associated cycles of representations in theta correspondence. An optimistic reflection suggests that, if $L(\sigma)$ is a theta lift of σ, then their associated varieties are related as

$$\mathcal{AC}(L(\sigma)) = \sum_i m_i[\overline{\mathcal{O}_i}] \quad \longleftrightarrow \quad \mathcal{AC}(\sigma) = \sum_i m_i[\overline{\mathcal{O}_i'}],$$

with the same multiplicity, where $\mathcal{O}_i \leftrightarrow \mathcal{O}_i'$ indicates the orbit correspondence. However we do not have an intuitive evidence of such a kind of correspondence other than the cases treated here.

Third, Theorem A (or K-type decompositions) suggests that we should "quantize" the orbit $\mathcal{O}_m$ to get an irreducible unitary representation $L(\mathbf{1}_{G_2})$, which certainly should be a unipotent representation. For this, it will be helpful to try the similar method exploited by Kostant-Brylinski in the case of the minimal orbit. However, this will require much more than what we have presented in this note.

Now let us explain each section briefly.

In § 1, we define the associated cycles and other important invariants of representations in a general setting. After that, we collect their basic properties which will be needed later. In particular, in Lemma 1.1 and Theorem 1.4, we clarify the relationship between the associated cycles and the Bernstein degree (or the degree of the projectivised nilpotent cone); also, we recall the fact that the associated variety is the projection of the characteristic variety under the moment map (Lemma 1.6).

In § 2, we briefly summarize known facts and examples of associated cycles of various types of representations. To see what is going on in this paper, §§ 1.3 and 2.4 will be extremely useful.

In § 3, we review the properties of a reductive dual pair which we will need later. After an explicit description of the Fock realization of the Weil representation in

§ 4, § 5 is devoted to giving the complete description of the unitary lowest weight representations of G via theta correspondence.

In § 6, we give a formula of K-type decomposition of the unitary lowest weight representations, using the branching coefficient of finite dimensional representations of compact groups. These formulas are well-known among experts, however, we need full detailed formulas in the following sections.

In § 7, we study the geometry of nilpotent orbits in the case of Hermitian symmetric pair. Take a $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}^-$ and the unfolding $\psi : V \to \overline{\mathcal{O}_m}$ as above. We use ψ to study the ring of regular functions $\mathbb{C}[\overline{\mathcal{O}_m}]$ on $\overline{\mathcal{O}_m}$, and clarify its $K_{\mathbb{C}}$-module structure. This leads us an identification of $\deg \overline{\mathcal{O}_m}$ and the Bernstein degree of one of the smallest unitary lowest weight module attached to $\mathcal{O}_m$. As a result, Theorem A is proved. We also study the global sections of the half-form bundle over $\mathcal{O}_m$ in the tame cases.

In § 8, we study a general theory of multiplicity free actions of a pair of reductive algebraic groups. We define the notions of degree and dimension of the space of covariants. The main result in this section is the formula of the degree and the dimension of covariants (Theorem 8.6).

In § 9, we treat general unitary lowest weight representations of G, which are singular. By the results of § 8, we prove Theorems B and C in this section.

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Notation. — We denote the field of real (respectively, complex, quaternionic) numbers by $\mathbb{R}$ (respectively, $\mathbb{C}, \mathbb{H}$). If $\mathbb{K}$ is one of these fields, we use the following notation for subsets of matrices:

$M(n, m, \mathbb{K})$	the set of all $n \times m$ matrices,
$\mathrm{Sym}\,(n, \mathbb{K})$	the set of all symmetric matrices of size n,
$\mathrm{Alt}\,(n, \mathbb{K})$	the set of all alternating matrices of size n.

These subsets are abbreviated as $M_{n,m}, \mathrm{Sym}_n, \mathrm{Alt}_n$ respectively, if there is no confusion on the base fields. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, we also denote by skew-Her$\,(n, \mathbb{K})$ the set of all skew Hermitian matrices of size n. If τ_λ is an irreducible finite dimensional representation of $GL(m, \mathbb{C})$ (or $U(m)$) with highest weight λ, we often write it as $\tau_\lambda^{(m)}$, denoting the rank m of the group explicitly by the superscript.

1. Invariants of representations

1.1. A review on the commutative ring theory. — First of all, we review well-known results in the commutative ring theory, which we need in the subsequent sections. For more details of what are discussed here, we refer the readers to textbooks on the commutative ring theory, for example, **[10]**, **[21]**, **[33]**.

Let V be an n-dimensional vector space over the field $\mathbb{C}$ and let $A := \mathbb{C}[V]$ be the ring of polynomials on V. For a finitely generated A-module M, the *support* $\operatorname{Supp} M$ of M is defined to be the set of prime ideals $\mathfrak{p}$ with $M_{\mathfrak{p}} \neq 0$. Since M is finitely generated, $\operatorname{Supp} M$ coincides with the Zariski closure of $\operatorname{Ann} M := \{a \in A \mid aM = 0\}$, which is denoted by $\mathcal{V}(M)$. We often identify $\mathcal{V}(M)$ with the affine variety

$$\mathcal{V}(M) \cap \text{m-Spec}\, A = \{x \in V \mid p(x) = 0 \ \ (\forall p \in \operatorname{Ann} M)\}.$$

Let A_n be the set of homogeneous polynomials of degree n. By the natural grading $A = \oplus_{n=0}^{\infty} A_n$, A is a graded $\mathbb{C}$-algebra. Let $M = \oplus_{n=0}^{\infty} M_n$ be a finitely generated graded A-module. As usual, we denote the Poincaré series by $P(M;t)$. It is well-known that there exists a unique polynomial $Q(t)$ and a non-negative integer d such that

$$P(M;t) = \sum_{n=0}^{\infty} (\dim M_n) t^n = \frac{Q(t)}{(1-t)^d}, \quad Q(1) \neq 0. \tag{1.1}$$

It turns out that $Q(1)$ is a positive integer. By the expression (1.1), we know that $\dim M_n$ is a polynomial in n for sufficiently large n, and it is written as

$$\dim M_n = \frac{Q(1)}{(d-1)!} n^{d-1} + (\text{lower order terms of } n).$$

Note that the integer d is the dimension of $\mathcal{V}(M)$. The integer $Q(1)$ is called the *multiplicity of* M, and we denote it by $\mathbf{m}(M)$.

A prime ideal $\mathfrak{P} \in \operatorname{Spec} A$ is called an associated prime of M if $\mathfrak{P}$ is an annihilator of some non-zero element of M. The set of associated primes is denoted by $\operatorname{Ass} M$. It is easy to see that $\operatorname{Ass} M \subset \operatorname{Supp} M$. The set of minimal elements of $\operatorname{Ass} M$ and that of $\operatorname{Supp} M$ coincide, and they form a finite set. Let $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$ be the set of minimal primes in $\operatorname{Supp} M$, and let $\mathcal{V}(M) = \cup_{i=1}^{r} C_i$ be the corresponding irreducible decomposition of the variety $\mathcal{V}(M)$.

Choose $\mathfrak{Q}^1 \in \operatorname{Ass} M$. Then there exists a submodule $M^1 \subset M$ such that $M^1 \simeq A/\mathfrak{Q}^1$. By induction, there exists a finite sequence $0 = M^0 \subset M^1 \subset \cdots \subset M^l = M$ such that $M^k/M^{k-1} \simeq A/\mathfrak{Q}^k$ for some $\mathfrak{Q}^k \in \operatorname{Spec} A$ $(k = 1, 2, \ldots, l)$. It is not hard to check that the integer

$$\operatorname{mult}_{\mathfrak{P}}(M) := \#\{\mathfrak{Q}^k \mid \mathfrak{Q}^k = \mathfrak{P}\}, \quad \mathfrak{P} : \text{minimal prime}$$

is independent of the choice of the sequence $\{M^k\}_k$. This integer is called the *multiplicity of* M *at* $\mathfrak{P}$. Note that $\operatorname{mult}_{\mathfrak{P}}(M)$ is reinterpreted as the length of Artinian $A_{\mathfrak{P}}$-module $M_{\mathfrak{P}}$. By the correspondence of minimal $\mathfrak{P} \in \operatorname{Ass} M$ and the irreducible component C of $\mathcal{V}(M)$, we also denote the multiplicity by $\operatorname{mult}_C(M)$.

As a refinement of $\operatorname{Supp} M$, we consider the formal linear combination of the minimal primes $\mathfrak{P}_i$ (or irreducible components C_i) with coefficients $\operatorname{mult}_{\mathfrak{P}_i}(M) = \operatorname{mult}_{C_i}(M)$,

$$\underline{\operatorname{Supp}} M := \sum_i \operatorname{mult}_{\mathfrak{P}_i}(M)\ [\mathfrak{P}_i] = \sum_i \operatorname{mult}_{C_i}(M)\ [C_i].$$

More generally, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on an algebraic variety X. We can refine $\operatorname{Supp} \mathcal{F}$ analogously. For any irreducible component C of the support of $\mathcal{F}$, the rank of the module $\mathcal{F}$ at a generic point of C is a well-defined positive integer $\operatorname{mult}_C(\mathcal{F})$. This is called the multiplicity of C in the support of $\mathcal{F}$. Then we consider the formal linear combination of the components C of $\operatorname{Supp}(\mathcal{F})$ with coefficients $\operatorname{mult}_C(\mathcal{F})$,

$$\underline{\operatorname{Supp}}(\mathcal{F}) = \sum_C \operatorname{mult}_C(\mathcal{F})\ [C].$$

The multiplicity of M can be obtained from $\underline{\operatorname{Supp}} M$. Let $\deg C$ be the degree of the variety C, i.e. $\deg C = \mathbf{m}(A/\mathfrak{P})$ (see, e.g., **[16]**). Since the Poincaré series is additive, $\mathbf{m}(M)$ is the sum of $\mathbf{m}(M^k/M^{k-1})$'s with $\dim \mathcal{V}(M^k/M^{k-1}) = \dim \mathcal{V}(M)$. By the definition of the sequence $\{M^k\}_k$ and the multiplicity $\operatorname{mult}_{\mathfrak{P}}(M)$, we have

Lemma 1.1. — $\mathbf{m}(M) = \displaystyle\sum_{\substack{i \\ \dim C_i = \dim \mathcal{V}(M)}} \operatorname{mult}_{C_i}(M) \deg C_i.$

Remark 1.2. — The notion of degree is usually defined for projective varieties. In our case, we can projectivise $\mathcal{V}(M)$ and its irreducible components since $\operatorname{Ann} M$ is graded. Then $\deg C_i$ should be interpreted as the degree of the projectivised variety.

1.2. Invariants of $U(\mathfrak{g})$-modules. — In this subsection, we introduce invariants of representations of Lie algebras after **[49]**, **[51]**. These invariants are main objects of this paper.

Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. We denote by $U_n(\mathfrak{g})$ the finite dimensional subspace of $U(\mathfrak{g})$, spanned by products of at most n-elements of $\mathfrak{g}$. Then $\{U_n(\mathfrak{g})\}_{n=0}^{\infty}$ is a filtration of $U(\mathfrak{g})$, called the *standard filtration*. By the Poincaré-Birkhoff-Witt (PBW) theorem, the associated graded algebra $\operatorname{gr} U(\mathfrak{g}) = \oplus_{n=0}^{\infty} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$.

Let V be a $U(\mathfrak{g})$-module. A chain $0 = V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V$, where the V_n's are subspaces of V, is called a *filtration of* V if it satisfies the following conditions:

$$\bigcup_{n=0}^{\infty} V_n = V, \qquad U_n(\mathfrak{g}) V_m \subset V_{n+m}, \qquad \dim V_n < \infty.$$

By the second condition, the graded object

$$\operatorname{gr} V = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n V, \quad \operatorname{gr}_n V := V_n / V_{n-1}$$

has the structure of a graded $S(\mathfrak{g})$-module. A filtration is called *good* if it also satisfies

$$U_n(\mathfrak{g})V_m = V_{n+m} \quad \text{(for all } m \text{ sufficiently large, all } n \geqslant 0). \tag{1.2}$$

In this case, V is a finitely generated $U(\mathfrak{g})$-module and $\operatorname{gr} V$ is a finitely generated $S(\mathfrak{g})$-module. Conversely, if V is finitely generated, we can construct a good filtration by choosing a finite dimensional generating subspace V_0 and by putting $V_n = U_n(\mathfrak{g})V_0$.

Regarding the symmetric algebra $A = S(\mathfrak{g})$ as the polynomial ring on the dual space $\mathfrak{g}^*$, we define several invariants of V using those defined via commutative ring theory.

Definition 1.3. — For a finitely generated $U(\mathfrak{g})$-module V, we define the *associated variety* $\mathcal{AV}(V)$, the *associated cycle* $\mathcal{AC}(V)$, the *Gelfand-Kirillov dimension* $\operatorname{Dim} V$, and the *Bernstein degree* $\operatorname{Deg} V$ by

$$\mathcal{AV}(V) = \mathcal{V}(\operatorname{gr} V), \qquad \mathcal{AC}(V) = \underline{\operatorname{Supp}}(\operatorname{gr} V),$$
$$\operatorname{Dim} V = \dim \mathcal{AV}(V), \qquad \operatorname{Deg} V = \mathbf{m}(\operatorname{gr} V)$$

respectively. They are independent of the choice of good filtrations of V, and therefore well-defined for V.

For an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

of finitely generated $U(\mathfrak{g})$-modules, we have $\operatorname{Dim} V_2 = \max\{\operatorname{Dim} V_1, \operatorname{Dim} V_3\}$, and

$$\mathcal{AV}(V_2) = \mathcal{AV}(V_1) \cup \mathcal{AV}(V_3). \tag{1.3}$$

Note that the associated cycle is not additive in general, i.e., $\mathcal{AC}(V_2) \neq \mathcal{AC}(V_1) + \mathcal{AC}(V_3)$. If we write

$$c_d(V) = \begin{cases} \operatorname{Deg} V & \text{if } d = \operatorname{Dim} V, \\ 0 & \text{if } d > \operatorname{Dim} V, \end{cases}$$

then the Bernstein degree becomes additive in the sense that

$$\operatorname{Deg} V_2 = c_{\operatorname{Dim} V_2}(V_2) = c_{\operatorname{Dim} V_2}(V_1) + c_{\operatorname{Dim} V_2}(V_3).$$

The right hand side is equal to $\operatorname{Deg} V_1 + \operatorname{Deg} V_3$ if $\operatorname{Dim} V_1 = \operatorname{Dim} V_3$.

1.3. The structure of invariants of Harish-Chandra modules. — The associated variety of a module (with some assumption, of course) over a reductive Lie algebra $\mathfrak{g}$ is contained in the nilpotent cone in $\mathfrak{g}^*$. Moreover, if it is a Harish-Chandra $(\mathfrak{g}, K)$-module, the associated variety has a $K_{\mathbb{C}}$-orbit structure. In this subsection, we shall review these well-known results.

Let G be a connected reductive group over $\mathbb{R}$ and $\mathfrak{g}_0$ its Lie algebra. Take a maximal compact subgroup $K \subset G$ and let $K_{\mathbb{C}}$ be its complexification. Denote by $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition associated to K and by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its complexification. For a Harish-Chandra $(\mathfrak{g}, K)$-module $\mathcal{H}$, we choose a finite dimensional K-invariant generating subspace $\mathcal{H}_0$ and define a filtration by $\mathcal{H}_n = U_n(\mathfrak{g})\mathcal{H}_0$. Then the graded object $\operatorname{gr}\mathcal{H}$ has compatible $S(\mathfrak{g})$- and $K_{\mathbb{C}}$-actions.

By the compatibility of $\mathfrak{g}$- and $K_{\mathbb{C}}$-actions, $\mathcal{AV}(\mathcal{H})$ is invariant under the action of $K_{\mathbb{C}}$ and $\mathfrak{k}$ acts on $\operatorname{gr}\mathcal{H}$ trivially. It follows that $\mathcal{AV}(\mathcal{H})$ is a $K_{\mathbb{C}}$-invariant subvariety in $(\mathfrak{g}/\mathfrak{k})^* \simeq \mathfrak{p}$.

Fix a connected algebraic group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. The algebra $U(\mathfrak{g})^{G_{\mathbb{C}}}$ of $\operatorname{Ad}(G_{\mathbb{C}})$-invariants in $U(\mathfrak{g})$ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, since $G_{\mathbb{C}}$ is connected. Filter the algebra $U(\mathfrak{g})^{G_{\mathbb{C}}}$ by the standard filtration of $U(\mathfrak{g})$, then $\operatorname{gr} U(\mathfrak{g})^{G_{\mathbb{C}}}$ is isomorphic to $S(\mathfrak{g})^{G_{\mathbb{C}}}$, the algebra of $\operatorname{Ad}(G_{\mathbb{C}})$-invariants in $S(\mathfrak{g})$. Since any irreducible $U(\mathfrak{g})$-module is annihilated by a maximal ideal in $U(\mathfrak{g})^{G_{\mathbb{C}}}$, any $U(\mathfrak{g})$-module of finite length is annihilated by the product of a finite number of maximal ideals in $U(\mathfrak{g})^{G_{\mathbb{C}}}$. Such a product is of finite codimension in $U(\mathfrak{g})^{G_{\mathbb{C}}}$. Therefore, the radical of the graded object of this product is the ideal $S^+(\mathfrak{g})^{G_{\mathbb{C}}}$, the set of invariant polynomials without constant term. This argument implies that the associated variety $\mathcal{AV}(\mathcal{H})$ is contained in the zero set $\mathcal{V}(S^+(\mathfrak{g})^{G_{\mathbb{C}}})$ of $S^+(\mathfrak{g})^{G_{\mathbb{C}}}$. Note that $\mathcal{V}(S^+(\mathfrak{g})^{G_{\mathbb{C}}})$ coincides with the set $\mathcal{N}^*$ of nilpotent elements in $\mathfrak{g}^*$, since $G_{\mathbb{C}}$ is connected.

Consequently, $\mathcal{AV}(\mathcal{H})$ is a union of $K_{\mathbb{C}}$-orbits in $\mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^* \simeq \mathcal{N}_{\mathfrak{p}}$, the set of nilpotent elements in $\mathfrak{p}$. By a theorem of Kostant-Rallis, $\mathcal{N}_{\mathfrak{p}}$ is a finite union of $K_{\mathbb{C}}$-orbits. Summarizing the above discussion and the results of many contributors, we have the following well-known theorem.

Theorem 1.4. — *If $\mathcal{H}$ is a Harish-Chandra $(\mathfrak{g}, K)$-module, then the associated variety $\mathcal{AV}(\mathcal{H})$ is a finite union of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$. Moreover, if $\mathcal{H}$ is irreducible, we have the following.*

(1) *There exist nilpotent $K_{\mathbb{C}}$-orbits $\{C_i\} \subset \mathcal{N}_{\mathfrak{p}}$ with dimension equal to* $\operatorname{Dim}\mathcal{H}$ *such that*

$$\mathcal{AV}(\mathcal{H}) = \bigcup_{i=1}^{l} \overline{C_i}. \tag{1.4}$$

(2) *Denote the associated cycle as $\mathcal{AC}(\mathcal{H}) = \sum_i m_i[\overline{C_i}]$. Then the Bernstein degree is given by*

$$\operatorname{Deg}\mathcal{H} = \sum_{i=1}^{l} m_i \deg \overline{C_i}. \tag{1.5}$$

(3) *Let $I = I_{\mathcal{H}} \subset U(\mathfrak{g})$ be the associated primitive ideal. Then $\mathcal{AV}(U(\mathfrak{g})/I)$ is the closure of a single nilpotent $G_{\mathbb{C}}$-orbit $C_{\mathcal{H}}$, and for any i, the $G_{\mathbb{C}}$-orbit through C_i coincides with $C_{\mathcal{H}}$. In fact, $C_{\mathcal{H}} \cap \mathfrak{p}$ decomposes into a finite union of equidimensional nilpotent $K_{\mathbb{C}}$-orbits, and $\{C_i\}$ is a subset of its irreducible components:*

$$C_{\mathcal{H}} \cap \mathfrak{p} \supset C_1, \dots, C_l. \tag{1.6}$$

Remark 1.5. — Take $\lambda \in C_i$ and denote by $K_{\mathbb{C}}(\lambda)$ the isotropy subgroup of $K_{\mathbb{C}}$ at λ. The multiplicity m_i in (2) can be interpreted as the dimension of a certain representation of $K_{\mathbb{C}}(\lambda)$. For this, we refer to [**51**, Definition 2.12].

1.4. Invariants of $\mathcal{D}_X$-modules. — The relation between the associated varieties (associated cycles) and the characteristic varieties (characteristic cycles) is discussed in [**3**]. First, we recall the definition of the characteristic varieties and the characteristic cycles, which is analogous to that of the associated variety and its cycle for $\mathfrak{g}$-module given in § 1.2.

Let X be a smooth algebraic variety over an algebraically closed field $\mathbb{C}$. We denote by $\mathcal{D}_X$ the sheaf of (algebraic) differential operators on X. On $\mathcal{D}_X$, we have a natural increasing filtration by the $\mathcal{O}_X$-submodules $\mathcal{D}_X(n)$, the subsheaf of all differential operators of order $\leqslant n$;

$$0 = \mathcal{D}_X(-1) \subset \mathcal{D}_X(0) = \mathcal{O}_X \subset \mathcal{D}_X(1) \subset \cdots \subset \mathcal{D}_X.$$

The associated graded sheaf

$$\operatorname{gr} \mathcal{D}_X = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n \mathcal{D}_X, \qquad \operatorname{gr}_n \mathcal{D}_X = \mathcal{D}_X(n)/\mathcal{D}_X(n-1)$$

is naturally identified with the direct image sheaf $\pi_*(\mathcal{O}_{T^*X})$, where $\pi : T^*X \to X$ is the cotangent bundle of X, and $\mathcal{O}_{T^*X}$ is the structure sheaf of T^*X.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then there is a good filtration

$$0 = \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}.$$

The corresponding graded module is defined by

$$\operatorname{gr} \mathcal{M} = \bigoplus_{n=0} \operatorname{gr}_n \mathcal{M}, \qquad \operatorname{gr}_n \mathcal{M} = \mathcal{M}_n/\mathcal{M}_{n-1}.$$

Then $\operatorname{gr} \mathcal{M}$ is coherent over $\pi_*(\mathcal{O}_{T^*X})$. The support of $\operatorname{gr} \mathcal{M}$ as a module on T^*X, more precisely, the support of $\mathcal{O}_{T^*X} \otimes_{\operatorname{gr} \mathcal{D}_X} \operatorname{gr} \mathcal{M}$, is called the *characteristic variety* of $\mathcal{M}$. This is a closed conic algebraic subvariety of the cotangent bundle T^*X, which is usually denoted by

$$\operatorname{Ch}(\mathcal{M}) = \operatorname{Supp}(\operatorname{gr} \mathcal{M}).$$

The variety does not depend on the choice of a good filtration. As a refinement of $\operatorname{Ch}(\mathcal{M})$, we define the *characteristic cycle* of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ by

$$\underline{\operatorname{Ch}}(\mathcal{M}) = \underline{\operatorname{Supp}}(\operatorname{gr} \mathcal{M}).$$

The characteristic cycle is also independent of the choice of a good filtration.

From now on in this subsection, let G be a reductive algebraic group over $\mathbb{C}$ and let X be the set of Borel subgroups of G. Then it is known that X is a complete G-homogeneous variety, and the Lie algebra $\mathfrak{g}$ of G acts on X by vector fields on X. This gives a Lie algebra homomorphism

$$\mathfrak{g} \to D(X),$$

where $D(X) = \Gamma(X, \mathcal{D}_X)$ denotes the set of all global sections of the sheaf $\mathcal{D}_X$ on X. This map extends to an algebra homomorphism

$$\psi : U(\mathfrak{g}) \to D(X),$$

which is known to be surjective. With the natural filtrations, $\operatorname{gr} U(\mathfrak{g})$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$, while $\operatorname{gr} D(X)$ to the set of global sections $\mathbb{C}[T^*X]$ of (algebraic) holomorphic functions on the cotangent bundle T^*X. Since the map ψ is compatible with the natural filtrations, we have the associated graded ring homomorphism

$$\phi = \operatorname{gr} \psi : S(\mathfrak{g}) \to \mathbb{C}[T^*X].$$

The map ϕ gives rise to the moment map

$$\mu : T^*X \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$. It is known that the image of μ is normal and that the map μ is birational onto its image. The moment map is the key to give a relation between the characteristic variety and the associated variety.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then the set of all global sections $M = \Gamma(X, \mathcal{M})$ is a module over $D(X) = \Gamma(X, \mathcal{D}_X)$. Using the algebra homomorphism ψ, a $D(X)$-module is considered as a $\mathfrak{g}$-module. Then M is a finitely generated $\mathfrak{g}$-module with the trivial central character. Conversely, any finitely generated $\mathfrak{g}$-module M with the trivial central character can be obtained in this manner from a coherent $\mathcal{D}_X$-module $\mathcal{M}$. Indeed, $\mathcal{M}$ is obtained by, so called, the localization such as $\mathcal{M} = \mathcal{D}_X \otimes_{U(\mathfrak{g})} M$ using the homomorphism ψ.

Lemma 1.6. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Consider $M = \Gamma(X, \mathcal{M})$ as a $\mathfrak{g}$-module.*

(1) *The associated variety of M is the image of the characteristic variety of $\mathcal{M}$ under the moment map:*

$$\mathcal{AV}(M) = \mu(\operatorname{Ch}(\mathcal{M})).$$

(2) *Suppose, moreover, that $\mathcal{M}$ has a good filtration $\{\mathcal{M}_j\}_j$ such that $H^1(X, \mathcal{M}_j) = 0$. We denote the direct image under the moment map of the $\mathcal{O}_{T^*X}$-module $\operatorname{gr} \mathcal{M}$ by $\mu_*(\operatorname{gr} \mathcal{M})$, which is a coherent $\mathcal{O}_{\mathfrak{g}^*}$-module. Then the associated cycle is described by the cycle of this module*

$$\mathcal{AC}(M) = \underline{\operatorname{Supp}}(\mu_*(\operatorname{gr} \mathcal{M})),$$

where the definition of the cycle of $\mathcal{O}_{\mathfrak{g}^*}$-module is given in § 1.1.

More general statement would be found in Theorem 1.9 and Remark to Lemma 1.6 in [**3**]. The condition of the vanishing of the first cohomology appearing in (2) of the lemma holds for sufficiently regular infinitesimal characters, due to a result of Serre. See Appendix A of [**3**], for details.

2. Known results and examples

In this section, we summarize known results and examples of the invariants defined in § 1. Some of them are immediately obtained from the definition, others are non-trivial.

2.1. Finite dimensional representation. — For a finite dimensional representation V of a complex Lie algebra $\mathfrak{g}$, we may take $V_0 = V$ and consequently $V_n = V$ for all $n \geqslant 0$. Then the Poincaré series is a constant $\dim V$, and we have

$$\operatorname{Dim} V = 0 \quad \text{and} \quad \operatorname{Deg} V = \dim V.$$

From this, we conclude that the associated variety of V is $\{0\}$, and the associated cycle equals $\mathcal{AC}\,(V) = (\dim V) \cdot [\{0\}]$.

2.2. Generalized Verma module. — Let $\mathfrak{q} = \mathfrak{l} + \bar{\mathfrak{u}}$ be the Levi decomposition of a parabolic subalgebra $\mathfrak{q}$ of a complex reductive Lie algebra $\mathfrak{g}$, where $\mathfrak{l}$ is a Levi subalgebra and $\bar{\mathfrak{u}}$ is the nilpotent radical of $\mathfrak{q}$. Denote by $\mathfrak{u}$ the opposite nilpotent Lie algebra to $\bar{\mathfrak{u}}$. Take an irreducible finite dimensional representation τ_λ of $\mathfrak{l}$ with the highest weight λ and extend it to a representation of $\mathfrak{q}$ trivially. The generalized Verma module $M(\lambda)$ is defined by $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} \tau_\lambda$.

Proposition 2.1. — *The invariants for the generalized Verma module $M(\lambda)$ are*

$$\operatorname{Dim} M(\lambda) = \dim \mathfrak{u} = \dim \bar{\mathfrak{u}}, \qquad \operatorname{Deg} M(\lambda) = \dim \tau_\lambda, \tag{2.1}$$

$$\mathcal{AV}\,(M(\lambda)) = \bar{\mathfrak{u}} \quad \textit{and} \qquad \mathcal{AC}\,(M(\lambda)) = (\dim \tau_\lambda)[\bar{\mathfrak{u}}]. \tag{2.2}$$

Here, we identified $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form.

Proof. — By the PBW theorem, $M(\lambda) = U(\mathfrak{u}) \otimes_{\mathbb{C}} \tau_\lambda$ as a vector space and $M(\lambda)_n := U_n(\mathfrak{u}) \otimes_{\mathbb{C}} \tau_\lambda$ $(n = 0, 1, 2, \dots)$ defines a good filtration of $M(\lambda)$. We denote the associated graded module by $\operatorname{gr} M(\lambda)$. Since

$$\dim \operatorname{gr}_n M(\lambda) = (\dim \tau_\lambda) \times \binom{n + \dim \mathfrak{u} - 1}{\dim \mathfrak{u} - 1},$$

we immediately conclude that $\operatorname{Dim} M(\lambda) = \dim \mathfrak{u}$ and $\operatorname{Deg} M(\lambda) = \dim \tau_\lambda$.

Next, we shall calculate $\mathcal{AC}\,(M(\lambda))$. Since $\mathfrak{q}$ is contained in $\operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ and the intersection $S(\mathfrak{u}) \cap \operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ is $\{0\}$, $\operatorname{Ann}_{S(\mathfrak{g})} \operatorname{gr} M(\lambda)$ coincides with $S(\mathfrak{g})\mathfrak{q}$. Then

$$\mathcal{AV}\,(M(\lambda)) = \{x \in \mathfrak{g}^* \mid \langle x, \mathfrak{q} \rangle = \{0\}\} \simeq \bar{\mathfrak{u}}.$$

Moreover, since $\bar{\mathfrak{u}} \simeq \mathbb{C}^{\dim \bar{\mathfrak{u}}}$ is irreducible and its degree is one, the multiplicity is $\dim \tau_\lambda$ by Lemma 1.1. $\square$

2.3. Lowest weight module. — We use the same notation as in the previous subsection.

Let V be a $\mathfrak{q}$-lowest weight $U(\mathfrak{g})$-module, i.e. there exists an irreducible finite dimensional $\mathfrak{l}$-submodule V_0 in V such that $\bar{\mathfrak{u}}$ acts trivially on it and V is generated by it. Let λ be the highest weight of V_0. By the universality of the generalized Verma module, there exists a unique surjective $U(\mathfrak{g})$-homomorphism

$$\Phi : M(\lambda) \twoheadrightarrow V.$$

By this homomorphism, a good filtration on V is induced from that of $M(\lambda)$. By (1.3) and (2.2), we have

$$\mathcal{AV}(V) \subset \bar{\mathfrak{u}}. \tag{2.3}$$

2.4. Hermitian symmetric case. — Let (G, K) be an irreducible Hermitian symmetric pair. We use the notation in § 1.3. The adjoint representation of K on $\mathfrak{p}$ decomposes into two irreducible components $\mathfrak{p}^{\pm}$. Since $\mathfrak{q} := \mathfrak{k} + \mathfrak{p}^-$ is a maximal parabolic subalgebra of $\mathfrak{g}$, we can apply the results in §§ 1.3 and 2.2 for a $\mathfrak{q}$-lowest weight module. By (2.3) and Theorem 1.4, the associated variety of a $\mathfrak{q}$-lowest weight $(\mathfrak{g}, K)$-module is a finite union of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$.

In particular, since the $(\mathfrak{g}, K)$-module of the holomorphic discrete series is a generalized Verma module, the invariants for it are given by (2.1) and (2.2), where τ_λ is the minimal K-type and $\bar{\mathfrak{u}} = \mathfrak{p}^-$. Namely we have

Proposition 2.2. — *Let π_λ be a holomorphic discrete series representation of G with the minimal K-type τ_λ. Then invariants of π_λ are given as*

$$\operatorname{Dim} \pi_\lambda = \dim \mathfrak{p}^- = \frac{1}{2} \dim G/K, \qquad \operatorname{Deg} \pi_\lambda = \dim \tau_\lambda, \tag{2.4}$$

$$\mathcal{AV}(\pi_\lambda) = \mathfrak{p}^- \quad \textit{and} \qquad \mathcal{AC}(\pi_\lambda) = (\dim \tau_\lambda)[\mathfrak{p}^-]. \tag{2.5}$$

Let us consider the Poincaré series of a $\mathfrak{q}$-lowest weight module V. Let Z be the center of K and let $\mathfrak{z}_0$ be its Lie algebra. Under our setting, every element of Z acts on $\mathfrak{p}^{\pm}$ by a non-trivial scalar and it acts on the minimal K-type of V also by a scalar.

Choose a base H of $\mathfrak{z}_0$ and denote by α the scalar $\operatorname{ad}(H)|_{\mathfrak{p}^+}$. Let $h(s) := \exp sH \in K$. The action of $h(s)$ on V gives the Poincaré series of V. More precisely,

Proposition 2.3. — *The Poincaré series of a $\mathfrak{q}$-lowest weight module V is*

$$P(\operatorname{gr} V; t) = t^{-n_0} \left(\operatorname{trace} h(s)|_V\right),$$

where $t = e^{\alpha s}$ and n_0 is the scalar by which H acts on the minimal K-type of V.

Proof. — First, we consider the generalized Verma module $M(\lambda)$. The action of $h(s)$ on $M(\lambda)_n$ is a scalar $e^{(n+n_0)\alpha s}$. By the definition of the Poincaré series, we have $P(\operatorname{gr} M(\lambda); t) = t^{-n_0} \left(\operatorname{trace} h(s)|_{M(\lambda)}\right)$. Using the universality of the Verma module, we obtain the Poincaré series of a lowest weight module V in the same way. □

2.5. Discrete series of real rank one groups. — For the discrete series representations of real rank one groups, the associated cycles are explicitly obtained by Chang [**8**].

By many contributors, the associated variety of a discrete series is well-known. Especially, it is a closure of a single $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$, and irreducible (see Theorem 1.4). Then the problem reduces to the determination of the multiplicity. Using the relation between the associated cycle and the characteristic cycle (Lemma 1.6), he calculated it by investigating the fiber of the moment map.

For the explicit value of the multiplicity, we refer to his paper.

2.6. Large representation. — Let G be a real reductive Lie group and let $G = KA_mN_m$, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_{m,0} + \mathfrak{n}_{m,0}$ be the Iwasawa decomposition of G and $\mathfrak{g}_0 := \operatorname{Lie} G$, respectively.

For a Harish-Chandra $(\mathfrak{g}, K)$-module V, it is known that the Gelfand-Kirillov dimension is at most $\dim \mathfrak{n}_{m,0}$ ([**49**]). We call V *large* if $\operatorname{Dim} V = \dim \mathfrak{n}_{m,0}$. In this case, V has Whittaker models and the dimension of models coincides with the Bernstein degree of V.

To state more precisely, we need some notation. Let $\psi : N_m \to \mathbb{C}^\times$ be a unitary character. We denote the differential character of $\mathfrak{n}_{m,0}$ by the same symbol ψ. Then ψ is identified with an element of $\sqrt{-1}(\mathfrak{n}_{m,0}/[\mathfrak{n}_{m,0}, \mathfrak{n}_{m,0}])^*$. We call ψ *admissible* if the coadjoint M_mA_m-orbit of ψ is open in $(\mathfrak{n}_{m,0}/[\mathfrak{n}_{m,0}, \mathfrak{n}_{m,0}])^*$. Here, M_m is the centralizer of A_m in K. For an admissible ψ, we define the space of dual Whittaker vectors $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ by

$$\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V) := \{v^* \in V^* \mid Xv^* = \psi(X)v^* \quad (\forall X \in \mathfrak{n}_{m,0})\},$$

where V^* is the dual space of V.

Theorem 2.4 ([**36**]). — *The space* $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ *is not zero if and only if* $\operatorname{Dim} V = \dim \mathfrak{n}_{m,0}$. *In this case, the dimension of* $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ *equals* $\operatorname{Deg} V$.

If V is a principal series representation, the dimension of $\mathrm{Wh}^*_{\mathfrak{n}_{m,0},\psi}(V)$ is obtained by Kostant (quasi-split case, [**30**]) and Lynch (non-quasi-split case, [**32**]). Thus by the above theorem, we know the Bernstein degree of V:

Theorem 2.5 ([**30**], [**32**]). — *The principal series representation* $\operatorname{Ind}^G_{M_mA_mN_m}(\sigma \otimes e^\nu \otimes 1)$ *is large, and the Bernstein degree is* $\#W(\mathfrak{g}_0, \mathfrak{a}_{m,0}) \cdot \dim \sigma$, *where* $W(\mathfrak{g}_0, \mathfrak{a}_{m,0})$ *is the little Weyl group.*

Remark 2.6. — The associated variety of a principal series representation is a finite union of the closure of regular nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$. Let $\{\mathcal{N}_1, \ldots, \mathcal{N}_l\} \subset \mathfrak{p}$ be the set of all regular nilpotent $K_{\mathbb{C}}$-orbits. Then we have

$$\sum_{i=1}^{l} \deg \overline{\mathcal{N}_i} = \#W(\mathfrak{g}_0, \mathfrak{a}_{m,0})$$

(see **[31]**). Since $\deg \overline{\mathcal{N}_i} = \deg \overline{\mathcal{N}_j}$, we see that $\deg \overline{\mathcal{N}_i} = \# W(\mathfrak{g}_0, \mathfrak{a}_{m,0})/l$.

There are explicit calculations of Whittaker models of some low rank groups. For the following representations, the Whittaker models are explicitly determined.
(1) Large discrete series representations of $Sp(2,\mathbb{R})$ (by Oda **[38]**).
(2) Large discrete series representations of $SU(2,2)$ (by Yamashita **[53]** and Hayata-Oda **[19]**).
(3) The generalized principal series representation $\mathrm{Ind}_{P_J}^G(\sigma \otimes e^{\nu+\rho_J} \otimes 1)$ of $G = Sp(2,\mathbb{R})$ (by Hayata **[18]**). Here, $P_J = M_J A_J N_J$ is the Jacobi parabolic subgroup of G and σ is a discrete series representation of $M_J \simeq \mathbb{C}^\times \times SU(1,1)$.
(4) Large discrete series representations of $SU(n,1)$ and $Spin(2n,1)$ (by Taniguchi **[46]**).

From these calculations, we know their Bernstein degrees. The Bernstein degrees of (1)–(3) are all four. Those of (4) are twice the multiplicities, which are obtained by Chang (see § 2.5). In other words, the degree of the associated varieties of large discrete series representations of $SU(n,1)$ and $Spin(2n,1)$ is two (cf. Lemma 1.1).

2.7. Minimal representation. — In this subsection, we will give Bernstein degrees of so-called minimal representations. Here we only consider *non-Hermitian* symmetric space G/K, though the arguments below equally works well for general situations.

If G/K is non-Hermitian, G has a minimal representation if and only if G/K is in the following list.
– Classical case : $SO(p,q)/SO(p) \times SO(q)$ where $p \geqslant q \geqslant 3, p+q \in 2\mathbb{Z}$ or $p \in 2\mathbb{Z}, q = 3$.
– Exceptional case : The following 8 cases.

$F_{4,4}/Sp(3) \times SU(2)$	$G_2/SO(4)$	$E_{6,4}/SU(2) \times SU(6)$	$E_{6,6}/Sp(4)$
$E_{7,4}/Spin(12) \times SU(2)$	$E_{7,7}/SU(8)$	$E_{8,4}/E_7 \times SU(2)$	$E_{8,8}/Spin(16)$

Take the minimal nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{\min} \subset \mathfrak{g}$. Then in this case $\mathcal{O}_{\min} \cap \mathfrak{p} =: Y$ is a single nilpotent $K_{\mathbb{C}}$-orbit, which is minimal among non-zero nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$ with respect to the closure relation.

***Theorem 2.7* (Vogan).** — *Let $\pi_{\min}$ be a minimal representation of G. Then there exists some weight ν such that*

$$\pi_{\min}|_K \simeq \bigoplus_{m \geqslant 0} \tau_{m\psi+\nu},$$

where ψ is the highest weight of $\mathfrak{p}$ (= the highest root), and $\tau_{m\psi+\nu}$ is the irreducible representation of K with highest weight $m\psi+\nu$.

Remark 2.8. — The weight ν is the highest weight of the minimal K-type of $\pi_{\min}$. For an explicit description of ν, we refer to Table 1 of **[5]** and the references cited there.

Put $\Delta_c^+(\psi) = \{\alpha \in \Delta_c^+ \mid \langle\psi,\alpha\rangle \neq 0\}$, where Δ_c^+ denotes the totality of positive compact roots. For $\alpha \in \Delta_c^+$, note that $\langle\psi,\alpha\rangle \neq 0$ if and only if $2\langle\psi,\alpha\rangle/\langle\psi,\psi\rangle = 1$ ([**27**, Lemma 2.2]).

Proposition 2.9. — *With the above notation, we have*

$$\operatorname{Dim}(\pi_{\min}) = \#\Delta_c^+(\psi) + 1 = \dim_{\mathbb{C}} \mathcal{O}_{\min}/2 = \dim_{\mathbb{C}} Y,$$

$$\operatorname{Deg}(\pi_{\min}) = \deg \overline{Y} = (\#\Delta_c^+(\psi))! \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\psi\rangle}{2\langle\rho_c,\alpha\rangle},$$

$$\mathcal{AC}(\pi_{\min}) = [\overline{Y}].$$

Proof. — From the explicit description of ν (cf. [**5**, Table 1]), we conclude that $\langle\nu,\alpha\rangle = 0$ holds for each positive compact root $\alpha \notin \Delta_c^+(\psi)$. Also we know a good filtration of $(\pi_{\min}, V)$ is given by

$$V_n = \bigoplus_{m\leqslant n} \tau_{m\psi+\nu}$$

(see [**50**]). Put $d = \#\Delta_c^+(\psi) + 1$. By Weyl's dimension formula, we calculate the dimension of $\operatorname{gr}_n V$ as

$$\begin{aligned}
\dim \operatorname{gr}_n V = \dim \tau_{n\psi+\nu} &= \prod_{\alpha\in\Delta_c^+} \frac{\langle n\psi+\nu+\rho_c,\alpha\rangle}{\langle\rho_c,\alpha\rangle} \\
&= n^{d-1} \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\alpha\rangle}{\langle\rho_c,\alpha\rangle} \prod_{\alpha\notin\Delta_c^+(\psi)} \frac{\langle\nu+\rho_c,\alpha\rangle}{\langle\rho_c,\alpha\rangle} + (\text{lower order terms of } n) \\
&= \frac{1}{(d-1)!}\left\{(d-1)! \prod_{\alpha\in\Delta_c^+(\psi)} \frac{\langle\psi,\psi\rangle}{2\langle\rho_c,\alpha\rangle}\right\} n^{d-1} + (\text{lower order terms of } n)
\end{aligned}$$

From the last formula, we can read off the desired formulas of dimension and degree.

On the other hand, since Y is a $K_{\mathbb{C}}$-orbit through a highest weight vector in $\mathfrak{p}$, $\overline{Y}$ is a highest weight variety (see [**48**]). Then the decomposition of the coordinate ring as a $K_{\mathbb{C}}$-module becomes

$$\mathbb{C}[\overline{Y}] \simeq \bigoplus_{m\geqslant 0} \tau_{m\psi},$$

with grading given by m. By the same method as above, we conclude that $\deg \overline{Y}$ is equal to $\operatorname{Deg} \pi_{\min}$ which proves that $\mathcal{AC}(\pi_{\min}) = [\overline{Y}]$. □

3. Reductive dual pair

Let W be a real symplectic space of dimension $2N$. We put $\mathcal{G} = Sp(W) = Sp(2N,\mathbb{R})$ and $\widetilde{\mathcal{G}} = Mp(2N,\mathbb{R})$, the metaplectic double cover of $\mathcal{G}$ (see [**47**, §I.2] for example). A pair of reductive subgroups (G_1, G_2) of $\mathcal{G}$ is called a *reductive dual pair* if they are mutually commutant to each other in $\mathcal{G}$ (see [**22**], for example). We denote by $(\widetilde{G_1}, \widetilde{G_2})$ the inverse image of these subgroups under the covering map $\widetilde{\mathcal{G}} \to \mathcal{G}$. Then they are also commutant to each other in $\widetilde{\mathcal{G}}$.

Let us assume that the pair (G_1, G_2) is irreducible (see [**25**, § 4] for definition). Then there are two possibilities.

(I) The pair (G_1, G_2) jointly acts on W. This action is irreducible.

(II) There exists a maximally totally isotropic space U of W, such that $W = U \oplus U^*$ gives the irreducible decomposition with respect to the joint action of the pair.

In the following, *we only treat the dual pair of type* (I), so that we assume that the joint action of $G_1 \times G_2$ on W is irreducible. Then, by the irreducibility, there exist a division algebra D over $\mathbb{R}$ and vector spaces V_1/D and $D\backslash V_2$ over D for which the following two properties hold. First, W is the tensor product of V_1 and V_2 over D :

$$W = V_1 \otimes_D V_2.$$

Second, G_i $(i = 1, 2)$ acts on V_i irreducibly as D-linear transformations. We put

$$2n = \dim_{\mathbb{R}} V_1, \qquad m = \dim_D V_2, \tag{3.1}$$

hence $\dim_{\mathbb{R}} W = 2N = 2nm$. Note that the division algebra is given by $D \simeq \operatorname{End}_{G_1}(V_1) \simeq \operatorname{End}_{G_2}(V_2)$.

Since W carries a symplectic structure (and (G_1, G_2) is a pair in the symplectic group $Sp(W)$), it produces some additional structure on the vector spaces V_1 and V_2. Namely, we have the following.

First, there exists an involution ι of D (possibly trivial). Second, V_i $(i = 1, 2)$ carries a sesqui-linear form $(\ ,\ )_i$ which is invariant under G_i. One of the forms, say $(\ ,\ )_1$, is skew-Hermitian with respect to the involution ι and the other $(\ ,\ )_2$ is Hermitian; and the original symplectic form $\langle\ ,\ \rangle_W$ on W is given by the product of these forms:

$$\langle\ ,\ \rangle_W = \operatorname{Re}(\ ,\ )_1 \otimes_D (\ ,\ )_2.$$

Moreover, the group G_i is the full isometry group with respect to $(\ ,\ )_i$. In the following, we always assume that $(\ ,\ )_1$ is skew-Hermitian, and $(\ ,\ )_2$ is Hermitian.

Here is a table (Table 1) of such pairs borrowed from [**25**, Table 4.1].

(D, ι)	$\mathcal{G}$	(G_1, G_2)	
$(\mathbb{R}, \mathbf{1})$	$Sp(2nm, \mathbb{R})$	$(Sp(2n, \mathbb{R}), O(p, q))$	$m = p + q$
$(\mathbb{C}, \mathbf{1})$	$Sp(4nm, \mathbb{R})$	$(Sp(2n, \mathbb{C}), O(m, \mathbb{C}))$	
$(\mathbb{C}, {}^-)$	$Sp(2nm, \mathbb{R})$	$(U(p, q), U(r, s))$	$n = p + q, m = r + s$
$(\mathbb{H}, {}^-)$	$Sp(2nm, \mathbb{R})$	$(O^*(2p), Sp(r, s))$	$n = 2p, m = r + s$

TABLE 1. Reductive dual pairs of type (I).

In this paper, we only treat the case where one of the pair, say G_2, is compact. In fact, we have the following explicit cases in Table 2 in mind. However, we try to keep general situation whenever possible. In any case, G_2 *is always assumed to be compact.* Let us specify an explicit embedding of (G_1, G_2) into $\mathcal{G} = Sp(2nm, \mathbb{R})$.

	(D, ι)	$\mathcal{G}$	(G_1, G_2)	
Case (Sp, O)	$(\mathbb{R}, \mathbf{1})$	$Sp(2nm, \mathbb{R})$	$(Sp(2n, \mathbb{R}), O(m))$	
Case (U, U)	$(\mathbb{C}, ^-)$	$Sp(2nm, \mathbb{R})$	$(U(p,q), U(m))$	$n = p + q$
Case (O^*, Sp)	$(\mathbb{H}, ^-)$	$Sp(2nm, \mathbb{R})$	$(O^*(2p), Sp(2m))$	$n = 2p$

TABLE 2. Reductive dual pairs (G_1, G_2) with G_2 being compact.

Although our arguments below are fairly general, sometimes it is convenient to use a concrete realization. In each of three cases, we will give a symplectic vector space $\mathbb{R}^{2nm}$ endowed with an explicit symplectic form in terms of invariant bilinear forms of V_1 and V_2. This will determine the group $\mathcal{G} = Sp(2nm, \mathbb{R})$.

Case (Sp, O). — Let $\mathbb{R}^{2n}$ be a symplectic vector space with a symplectic form

$$(u, v)_1 = {}^t u J_n v \quad (u, v \in \mathbb{R}^{2n}), \qquad J_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}, \tag{3.2}$$

and consider $G_1 = Sp(2n, \mathbb{R})$ as the isometry group of $(\mathbb{R}^{2n}, (\ ,\ )_1)$. For $G_2 = O(m)$, we take the standard Euclidean bilinear form $(u, v)_2 = {}^t uv \quad (u, v \in \mathbb{R}^m)$, and consider $O(m) = O(\mathbb{R}^m, (\ ,\ )_2)$. Then the tensor product $W = \mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{R}^m$ with a symplectic form

$$\langle\ ,\ \rangle_W = (\ ,\ )_1 \otimes_{\mathbb{R}} (\ ,\ )_2$$

gives the embedding $(G_1, G_2) \hookrightarrow \mathcal{G} = Sp(W, \langle\ ,\ \rangle_W)$.

Let us see this embedding infinitesimally. So, first consider $\mathfrak{sp}(2n, \mathbb{R})$:

$$\begin{aligned} \mathfrak{sp}(2n, \mathbb{R}) &= \{Z \in \mathfrak{gl}(2n, \mathbb{R}) \mid {}^t Z J_n + J_n Z = 0\} \\ &= \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^t X_{11} \end{pmatrix} \;\middle|\; \begin{array}{l} X_{11} \in \mathfrak{gl}(n, \mathbb{R}), \\ X_{12}, X_{21} \in \mathrm{Sym}\,(n, \mathbb{R}) \end{array} \right\}. \end{aligned} \tag{3.3}$$

Then it is embedded into larger $\mathfrak{sp}(2nm, \mathbb{R})$ as

$$\mathfrak{sp}(2n, \mathbb{R}) \ni \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -{}^t X_{11} \end{pmatrix} \longmapsto \begin{pmatrix} X_{11}^{\oplus m} & X_{12}^{\oplus m} \\ X_{21}^{\oplus m} & -{}^t X_{11}{}^{\oplus m} \end{pmatrix} \in \mathfrak{sp}(2nm, \mathbb{R}), \tag{3.4}$$

where

$$X^{\oplus m} = \mathrm{diag}\,(X, X, \ldots, X) \quad (m\text{-times}).$$

Similarly, $\mathfrak{o}(m,\mathbb{R}) = \mathrm{Alt}\,(m,\mathbb{R})$ is embedded into $\mathfrak{sp}(2nm,\mathbb{R})$ as

$$\mathfrak{o}(m,\mathbb{R}) \ni X \longmapsto \begin{pmatrix} X * 1_n & 0 \\ 0 & X * 1_n \end{pmatrix},$$

where

$$X * A = \begin{pmatrix} x_{11}A & x_{12}A & \cdots & x_{1m}A \\ x_{21}A & x_{22}A & \cdots & x_{2m}A \\ \vdots & \vdots & & \vdots \\ x_{m1}A & x_{m2}A & \cdots & x_{mm}A \end{pmatrix}. \tag{3.5}$$

Case (U,U). — Consider an indefinite Hermitian form $(\ ,\ )_1$ on $\mathbb{C}^n$ of signature (p,q) $(n = p+q)$:

$$(u,v)_1 = {}^t\overline{u} I_{p,q} v \quad (u,v \in \mathbb{C}^n), \qquad I_{p,q} = \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix}. \tag{3.6}$$

Then, $G_1 = U(p,q)$ is the full isometry group of $(\mathbb{C}^n, (\ ,\ )_1)$. Also we take a definite Hermitian form $(\ ,\ )_2$ on $\mathbb{C}^m$ as $(u,v)_2 = {}^t\overline{u}v$ $(u,v \in \mathbb{C}^m)$. This determines the unitary group $G_2 = U(m)$. Then the tensor product $W = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^m$ naturally inherits a Hermitian form $(\ ,\ )_1 \otimes_{\mathbb{C}} (\ ,\ )_2$. We make use of its imaginary part to define a symplectic form on $W \simeq \mathbb{R}^{2nm}$:

$$\langle\ ,\ \rangle_W = \mathrm{Re}\left(\sqrt{-1}\,(\ ,\ )_1 \otimes_{\mathbb{C}} (\ ,\ )_2\right).$$

The form $\langle\ ,\ \rangle_W$ is clearly non-degenerate and it defines the isometry group $\mathcal{G} = Sp(W, \langle\ ,\ \rangle_W) \simeq Sp(2nm,\mathbb{R})$.

Under our explicit realization of $U(p,q)$, its Lie algebra is given as

$$\begin{aligned} \mathfrak{u}(p,q) &= \left\{ Z \in \mathfrak{gl}(p+q,\mathbb{C}) \mid {}^t\overline{Z} I_{p,q} + I_{p,q} Z = 0 \right\} \\ &= \left\{ Z = \begin{pmatrix} Z_{11} & Z_{12} \\ {}^t\overline{Z_{12}} & Z_{22} \end{pmatrix} \;\middle|\; \begin{matrix} Z_{11} \in \text{skew-Her}\,(p,\mathbb{C}) \\ Z_{22} \in \text{skew-Her}\,(q,\mathbb{C}) \\ Z_{12} \in M(p,q,\mathbb{C}) \end{matrix} \right\}. \end{aligned} \tag{3.7}$$

Let us write $Z = X + \sqrt{-1}\,Y$ with $X, Y \in M(n,\mathbb{R})$. Then an explicit embedding into $\mathfrak{sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{u}(p,q) \ni X + \sqrt{-1}\,Y \mapsto \begin{pmatrix} X & -Y I_{p,q} \\ I_{p,q} Y & I_{p,q} X I_{p,q} \end{pmatrix} \in \mathfrak{sp}(2n,\mathbb{R}). \tag{3.8}$$

Now the above embedding composed by the embedding (3.4) will give the desired realization of $\mathfrak{u}(p,q)$ in $\mathfrak{sp}(2nm,\mathbb{R})$.

On the other hand, the compact companion $\mathfrak{u}(m)$ is embedded into $\mathfrak{sp}(2nm,\mathbb{R})$ as

$$\begin{aligned} \mathfrak{u}(m) &= \mathrm{Alt}\,(m,\mathbb{R}) + \sqrt{-1}\ \mathrm{Sym}\,(m,\mathbb{R}) \ni X + \sqrt{-1}\,Y \\ &\longmapsto \begin{pmatrix} X * 1_n & -Y * I_{p,q} \\ Y * I_{p,q} & X * 1_n \end{pmatrix} \in \mathfrak{sp}(2nm,\mathbb{R}). \end{aligned} \tag{3.9}$$

Case (O^*, Sp). — Let $O(2p,\mathbb{C})$ be the complex orthogonal group with respect to the following bilinear form

$$(u,v) = {}^t u S_p v \quad (u, v \in \mathbb{C}^{2p}), \qquad S_p = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}.$$

We realize $G_1 = O^*(2p)$ as a subgroup of $O(2p,\mathbb{C})$, namely,

$$O^*(2p) = O(2p,\mathbb{C}) \cap U(p,p) \subset M(2p,\mathbb{C}), \tag{3.10}$$

where $U(p,p)$ is realized in the same way as Case (U,U). Similarly, we realize $G_2 = Sp(2m)$ as a compact subgroup of $Sp(2m,\mathbb{C})$:

$$Sp(2m) = Sp(2m,\mathbb{C}) \cap U(2m) \subset M(2m,\mathbb{C}).$$

First we describe embedding of $\mathfrak{o}^*(2p)$ into $\mathfrak{sp}(2n,\mathbb{R})$ $(n = 2p)$. Our realization of $O^*(2p)$ gives its Lie algebra as

$$\begin{aligned} \mathfrak{o}^*(2p) &= \{Z \in \mathfrak{gl}(2p,\mathbb{C}) \mid {}^t\overline{Z} I_{p,p} + I_{p,p} Z = 0, \quad {}^t Z S_p + S_p Z = 0\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mid X \in \text{skew-Her}\,(p,\mathbb{C}),\ Y \in \text{Alt}\,(p,\mathbb{C}) \right\}, \end{aligned}$$

where $I_{p,p}$ is given by (3.6).

It is subtle to describe a symplectic form of the larger $Sp(2nm,\mathbb{R})$ $(n = 2p)$ in terms of the original (skew-)Hermitian forms over $\mathbb{H}$ which define $O^*(2p)$ and $Sp(2m)$ as the full isometry groups. Instead, we give here only an explicit embedding of $O^*(2p)$ infinitesimally. Let us write $X = X_1 + \sqrt{-1}\ X_2$ and $Y = Y_1 + \sqrt{-1}\ Y_2$ with real matrices X_i, Y_i $(i = 1, 2)$. Then, the infinitesimal embedding of $\mathfrak{o}^*(2p)$ into $\mathfrak{sp}(2n,\mathbb{R})$ is given by

$$\mathfrak{o}^*(2p) \ni \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} X_1 & -Y_1 & -X_2 & -Y_2 \\ Y_1 & X_1 & Y_2 & -X_2 \\ \hline X_2 & -Y_2 & X_1 & Y_1 \\ Y_2 & X_2 & -Y_1 & X_1 \end{array}\right) \in \mathfrak{sp}(2n,\mathbb{R}). \tag{3.11}$$

This embedding is compatible with the embedding given in Case (U,U), i.e., we have a sequence of subgroups

$$O^*(2p) \hookrightarrow U(p,p) \hookrightarrow Sp(2n,\mathbb{R}).$$

The embedding into the larger $\mathfrak{sp}(2nm,\mathbb{R})$ is given by the composition of (3.4) and (3.11).

Let us see the embedding of the compact companion $\mathfrak{sp}(2m)$. Its Lie algebra becomes

$$\begin{aligned} \mathfrak{sp}(2m) &= \{Z \in \mathfrak{gl}(2m,\mathbb{C}) \mid {}^t\overline{Z} + Z = 0, \quad {}^t Z J_m + J_m Z = 0\} \\ &= \left\{ Z = \begin{pmatrix} X & -Y \\ \overline{Y} & \overline{X} \end{pmatrix} \mid X \in \text{skew-Her}\,(m,\mathbb{C}),\ Y \in \text{Sym}\,(m,\mathbb{C}) \right\}. \end{aligned}$$

If we denote $Z = A + \sqrt{-1}\ B$ with real matrices A and B, then the embedding is given by

$$\mathfrak{sp}(2m) \ni Z = A + \sqrt{-1}\ B \longmapsto \begin{pmatrix} A * 1_p & -B * 1_p \\ B * 1_p & A * 1_p \end{pmatrix} \in \mathfrak{sp}(2nm, \mathbb{R}).$$

4. Fock realization of Weil representation

Let ω be the Weil representation of $Mp(2n, \mathbb{R})$, the metaplectic double cover of $Sp(2n, \mathbb{R})$. Weil representation plays central roles in many fields, and a large amount of results are known. For example, see **[24]**, **[25]**, **[28]**, **[47]**, **[40]**, etc. We introduce here, among all, explicit realization of Harish-Chandra module of ω on a polynomial ring (e.g., see **[25]** and **[9]**). It is called *Fock model.*

For the time being, we write $G = Sp(2n, \mathbb{R})$ and $\widetilde{G} = Mp(2n, \mathbb{R})$. Since we only consider Harish-Chandra modules, in fact we do not need entire $Mp(2n, \mathbb{R})$ but only its complexified Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and a maximal compact subgroup $\widetilde{K} = U(n)^{\sim}$. We fix a maximal compact subgroup $K \simeq U(n)$ in $Sp(2n, \mathbb{R})$ as follows. Put

$$Sp(2n, \mathbb{R}) = \left\{ g \in GL(2n, \mathbb{R}) \mid {}^t g J_n g = J_n \right\}, \qquad J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \tag{4.1}$$

Then K is given as

$$K = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in M(n, \mathbb{R}), \quad a + ib \in U(n) \right\}. \tag{4.2}$$

We identify K and $U(n)$ as above and sometimes we will write $a + ib \in K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the corresponding Cartan decomposition, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its complexification.

Let E_{ij} be the matrix unit, and put

$$F_{ij} := E_{ij} - E_{ji}, \qquad G_{ij} := E_{ij} + E_{ji}.$$

Then it is easy to see that a basis of $\mathfrak{k}$ is given by

(4.3)

$$A_{ij} := \begin{pmatrix} F_{ij} & O_n \\ O_n & F_{ij} \end{pmatrix} \quad (1 \leqslant i < j \leqslant n), \qquad B_{ij} := \begin{pmatrix} O_n & -G_{ij} \\ G_{ij} & O_n \end{pmatrix} \quad (1 \leqslant i \leqslant j \leqslant n),$$

and that of $\mathfrak{p}$ is given by

(4.4)

$$C_{ij} := \begin{pmatrix} G_{ij} & O_n \\ O_n & -G_{ij} \end{pmatrix} \quad (1 \leqslant i \leqslant j \leqslant n), \qquad D_{ij} := \begin{pmatrix} O_n & G_{ij} \\ G_{ij} & O_n \end{pmatrix} \quad (1 \leqslant i \leqslant j \leqslant n).$$

The representation space of ω in Fock model is a polynomial ring in n variables. Here we only give the explicit action of each basis element on the polynomial ring

$\mathbb{C}[x_1, x_2, \ldots, x_n]$:

$$\begin{aligned} &\omega(A_{ij}) = x_i\partial_{x_j} - x_j\partial_{x_i}, &&\omega(B_{ij}) = \sqrt{-1}\,(x_i\partial_{x_j} + \partial_{x_i}x_j), \\ &\omega(C_{ij}) = 2\partial_{x_i}\partial_{x_j} - \tfrac{1}{2}x_ix_j, &&\omega(D_{ij}) = -\sqrt{-1}\,(2\partial_{x_i}\partial_{x_j} + \tfrac{1}{2}x_ix_j). \end{aligned} \tag{4.5}$$

The action of $\widetilde{K}$ on $\mathbb{C}[x_1, \ldots, x_n]$ is the symmetric tensor product of the natural representation of $U(n)$ on $\mathbb{C}^n$ tensored by $\det^{1/2}$, which requires the double cover $\widetilde{K}$.

Put $H_i = -\sqrt{-1}\,B_{ii}/2$ and let

$$\mathfrak{t} = \left\{ \sum_{i=1}^{n} t_iH_i \mid t_i \in \mathbb{C} \right\}$$

be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. We define $\varepsilon_j \in \mathfrak{t}^*$ as $\varepsilon_j(H_i) = \delta_{ij}$. Then the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ is given by

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i \neq j \leqslant n\} \cup \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leqslant i \leqslant j \leqslant n\},$$

where $\varepsilon_i - \varepsilon_j$ is a compact root while $\pm(\varepsilon_i + \varepsilon_j)$ is non-compact. We take a positive system in the standard way :

$$\Delta^+(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i \leqslant j \leqslant n\}.$$

Then root vectors X_α ($\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$) and its action on Weil representation are given as

$$H_i = -\frac{\sqrt{-1}}{2}B_{ii}, \qquad \omega(H_i) = x_i\partial_{x_i} + \frac{1}{2}, \tag{4.6}$$

$$X_{\varepsilon_i - \varepsilon_j} = \frac{1}{2}(A_{ij} - \sqrt{-1}\,B_{ij}) \quad (i \neq j), \qquad \omega(X_{\varepsilon_i - \varepsilon_j}) = x_i\partial_{x_j}, \tag{4.7}$$

$$X_{\varepsilon_i + \varepsilon_j} = -\frac{1}{2}(C_{ij} - \sqrt{-1}\,D_{ij}), \qquad \omega(X_{\varepsilon_i + \varepsilon_j}) = \frac{1}{2}x_ix_j, \tag{4.8}$$

$$X_{-\varepsilon_i - \varepsilon_j} = \frac{1}{2}(C_{ij} + \sqrt{-1}\,D_{ij}), \qquad \omega(X_{-\varepsilon_i - \varepsilon_j}) = 2\partial_{x_i}\partial_{x_j}. \tag{4.9}$$

Note that

$$\mathfrak{k} \simeq \mathfrak{gl}(n, \mathbb{C}) \ni E_{ij} \leftrightarrow \begin{cases} X_{\varepsilon_i - \varepsilon_j} = x_i\partial_{x_j} & (i \neq j), \\ H_i = x_i\partial_{x_i} + \dfrac{1}{2} & (i = j). \end{cases} \tag{4.10}$$

We write

$$\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i, j \leqslant n\}, \qquad \Delta_n = \Delta_n^+ \sqcup (-\Delta_n^+),$$

the set of non-compact roots, and

$$\Delta_k^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}, \qquad \Delta_k = \Delta_k^+ \sqcup (-\Delta_k^+),$$

the set of compact roots. Then $\mathfrak{p}$ decomposes up into two K-irreducible components $\mathfrak{p}^\pm$ given by

$$\mathfrak{p}^\pm = \sum_{\pm\alpha \in \Delta_n^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the roots space corresponding to α. Note that $\omega(\mathfrak{p}^-)$ is realized as differential operators of degree two, and that $\omega(\mathfrak{p}^+)$ is the multiplication by homogeneous polynomials of degree two. So $\mathfrak{p}^\pm$ increases/decreases the degree of the representation space $\mathbb{C}[x_1, x_2, \ldots, x_n]$ by 2, while $\omega(\mathfrak{k})$ keeps the degree stable.

5. Unitary lowest weight representations

Let Ω be the Weil representation of $\widetilde{\mathcal{G}} = Mp(2N, \mathbb{R})$ $(N = nm)$ and consider reductive dual pair (G_1, G_2) of compact type in $\mathcal{G} = Sp(2N, \mathbb{R})$. In the following, we often write $G = G_1$ without the subscription 1. In fact, our main concern is on the irreducible infinite dimensional representations of $G = G_1$ which appear in the restriction of Weil representation Ω. Moreover, we assume that G_2 is contained in the specified maximal compact subgroup $\mathcal{K} \simeq U(N)$ of $\mathcal{G}$ given in the former section (cf. (4.2)). Each of our three cases (and their realization) clearly satisfies this condition.

For a subgroup $H \subset \mathcal{G}$, we denote by $\widetilde{H}$ the inverse image of H in $\widetilde{\mathcal{G}}$ of the covering map $\widetilde{\mathcal{G}} \to \mathcal{G}$, and call it the metaplectic cover of H by abuse of terminology. Since the metaplectic covers $\widetilde{G_1}$ and $\widetilde{G_2}$ commute with each other, we have a natural projection $\widetilde{G_1} \times \widetilde{G_2} \to \widetilde{G_1} \cdot \widetilde{G_2}$ (product in $\widetilde{\mathcal{G}}$). By this projection, we consider the restriction $\Omega|_{\widetilde{G_1}\cdot\widetilde{G_2}}$ as a representation of $\widetilde{G_1} \times \widetilde{G_2}$. Then we have a discrete and multiplicity free decomposition

$$\Omega \simeq \sum_{\widetilde{\sigma}\in\mathrm{Irr}(\widetilde{G_2})}^{\oplus} L(\widetilde{\sigma} \otimes \chi^{-1}) \boxtimes \widetilde{\sigma} \tag{5.1}$$

as a representation of $\widetilde{G_1} \times \widetilde{G_2}$. Here we denote an irreducible representation of $\widetilde{G} = \widetilde{G_1}$ corresponding to $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G_2})$ by $L(\widetilde{\sigma} \otimes \chi^{-1})$, where $\chi \in \mathrm{Irr}(\widetilde{G_2})$ is the unique one-dimensional character which appears in $\Omega|_{\widetilde{G_2}}$ (cf. Theorem 4.3 in **[25]**).

To be more specific, we argue like this. The representation space of Ω is realized on a polynomial ring of $N = nm$ variables. We consider it as the polynomial ring on the *dual* space of $n \times m$ matrices $M_{n,m}$ over $\mathbb{C}$. Since the one-dimensional space of constant polynomials in $\mathbb{C}[M_{n,m}^*]$ is preserved by the action of $\widetilde{\mathcal{K}}$, it is also preserved by $\widetilde{G_2}$ because of our assumption. Hence it gives the one-dimensional character and we denote it by $\chi \in \mathrm{Irr}(\widetilde{G_2})$.

The representation $L(\widetilde{\sigma} \otimes \chi^{-1})$ is possibly zero, and if it is not zero, then the representation $\sigma = \widetilde{\sigma} \otimes \chi^{-1}$ factors through to the representation of G_2. Therefore we write $L(\sigma) = L(\widetilde{\sigma} \otimes \chi^{-1})$ for $\sigma \in \mathrm{Irr}(G_2)$. The decomposition (5.1) can be rewritten as

$$\Omega \simeq \sum_{\sigma\in\mathrm{Irr}(G_2)}^{\oplus} L(\sigma) \boxtimes (\sigma \otimes \chi). \tag{5.2}$$

In the following, as explained above, we always twist the representation $\widetilde{\sigma} \in \mathrm{Irr}(\widetilde{G_2})$ by χ, and consider $\sigma = \widetilde{\sigma} \otimes \chi^{-1}$ as the representation of G_2. This twist might be sometimes misleading, but it reduces considerable amount of *untwisting*. For example, under this convention, we have $L(\mathbf{1}_{G_2}) \neq 0$, where $\mathbf{1}_{G_2}$ denotes the trivial representation of G_2. This representation turns out to be strongly related to geometric properties of nilpotent orbits.

It is known that $L(\sigma)$ ($\sigma \in \mathrm{Irr}(G_2)$) is an irreducible unitary lowest weight module of $\widetilde{G}$, if it is not zero (cf. Theorem 4.4 in **[25]**). Every irreducible unitary lowest weight module of $\widetilde{G}$ arises in this manner if $G = Sp(2n, \mathbb{R})$ or $U(p,q)$ and the compact companion G_2 moves all the possible rank. If G is $O^*(2p)$, there are other unitary lowest weight modules which can not be obtained in this manner (**[9]**, **[11]**).

In our cases, the compact subgroup G_2 naturally acts on its defining vector space V_2 keeping the non-degenerate Hermitian form $(\ ,\ )_2$ invariant (see § 3). Put $2n = \dim_{\mathbb{R}} V_1$ and $m = \dim_D V_2$ as in (3.1). Then we can realize G in a smaller symplectic group: $G \hookrightarrow Sp(2n, \mathbb{R})$, putting $m = 1$.

Let us denote the Weil representation of the smaller group $Mp(2n, \mathbb{R})$ by ω. Then, it is easy to see that $\Omega \simeq \omega^{\otimes m}$ as a representation of $Mp(2n, \mathbb{R})$, and the Harish-Chandra $(\mathfrak{g}, K)$-module of the Weil representation Ω (resp. ω) is realized on the polynomial ring $\mathbb{C}[M_{n,m}^*] \simeq \otimes^m \mathbb{C}[(\mathbb{C}^n)^*]$ over $n \times m$ matrices (resp. the polynomial ring $\mathbb{C}[(\mathbb{C}^n)^*]$). Note that we take a contragredient representation of $M_{n,m}$ rather than $M_{n,m}$ itself.

Let $K \subset G$ be a maximal compact subgroup of G which lives in $U(n) \subset Sp(2n, \mathbb{R})$, where $U(n)$ is a maximal compact subgroup of $Sp(2n, \mathbb{R})$ (cf. (4.2)). We will explain briefly how we get K-type decomposition of $L(\sigma)$ for each $\sigma \in \mathrm{Irr}(G_2)$. Note that the product $K \cdot G_2$ is compact and that it is contained in the maximal compact subgroup $\mathcal{K} \simeq U(nm)$ of $\mathcal{G} = Sp(2nm, \mathbb{R})$. It is well-known that $\mathcal{K}$-types of Ω can be described as

$$\Omega\big|_{\widetilde{\mathcal{K}}} = \sum_{k=0}^{\infty}{}^{\oplus} \tau(k\psi + 1/2\,\mathbb{I}), \quad \mathbb{I} = (1, 1, \ldots, 1), \tag{5.3}$$

where ψ is the highest weight of the natural (or defining) representation of $\mathcal{K} \simeq U(nm)$ on $\mathbb{C}^{nm}$ and $\tau(\lambda)$ is an irreducible finite-dimensional representation of $\mathcal{K}$ with highest weight λ. Note that the representation space of $\tau(k\psi + 1/2\,\mathbb{I})$ coincides with the space of homogeneous polynomials of degree k. Decompose $\tau(k\psi + 1/2\,\mathbb{I})$ by the joint action of $\widetilde{K} \times \widetilde{G_2}$:

$$\tau(k\psi + 1/2\,\mathbb{I})\big|_{\widetilde{K} \times \widetilde{G_2}} = \sum_{\tau_1 \in \mathrm{Irr}(\widetilde{K}),\ \sigma \in \mathrm{Irr}(G_2)}^{\oplus} m_k(\tau_1, \sigma)\, \tau_1 \boxtimes \widetilde{\sigma} \qquad (\widetilde{\sigma} = \sigma \otimes \chi).$$

Note that we again use the projection map $\widetilde{K} \times \widetilde{G_2} \to \widetilde{K} \cdot \widetilde{G_2} \subset \widetilde{\mathcal{K}}$ here. In particular, the one-dimensional space $\tau(1/2\,\mathbb{I})$ is decomposed as

$$\tau(1/2\,\mathbb{I})\big|_{\widetilde{K}\times\widetilde{G_2}} = \chi_1 \boxtimes \chi.$$

In other words, the multiplicity for $k = 0$ has the property

$$m_0(\tau_1, \sigma) = \begin{cases} 1 & \tau_1 = \chi_1, \sigma = \mathbf{1}_{G_2} \\ 0 & \text{otherwise.} \end{cases}$$

The explicit form of χ and χ_1 is given in Section 7 after we fix the embedding $K \subset \mathcal{K}$.

Since $L(\sigma)$ consists of the space of multiplicities of $\widetilde{\sigma}$ in Ω, we get K-type decomposition of $L(\sigma)$ as

$$L(\sigma)\big|_{\widetilde{K}} = \sum_{\tau_1 \in \mathrm{Irr}(\widetilde{K})}^{\oplus} \sum_{k=0}^{\infty} m_k(\tau_1, \sigma)\, \tau_1, \tag{5.4}$$

where the sum $\sum_{k=0}^{\infty} m_k(\tau_1, \sigma)$ is necessarily finite.

Let k_σ be the lowest possible degree of $\widetilde{\sigma}$-isotypic component of $\mathbb{C}[M_{n,m}^*]$. We define the Poincaré series of $L(\sigma)$ in terms of the multiplicity $m_k(\tau_1, \sigma)$ as

$$P(L(\sigma); t^2) = t^{-k_\sigma} \sum_{k=0}^{\infty} \sum_{\tau_1 \in \mathrm{Irr}(\widetilde{K})} m_k(\tau_1, \sigma)\, \dim \tau_1\, t^k. \tag{5.5}$$

Note that the action of $\mathfrak{p}^+$ increases the degree k by two (cf. (4.8)), so we write $P(L(\sigma); t^2)$ instead of $P(L(\sigma); t)$. We denote the center of $\widetilde{\mathcal{K}}$ by $Z(\widetilde{\mathcal{K}})$. We know that $Z(\widetilde{\mathcal{K}})$ is isomorphic to $U(1)$ and there exists an element H in the Lie algebra of $Z(\widetilde{\mathcal{K}})$ such that $\Omega(H)$ acts on the space of homogeneous polynomials of degree k by $k + nm/2$. Indeed, $H = \sum_{i=1}^{nm} H_i$ with the notation (4.10). The operator $\Omega(H)$ is semisimple, and the decomposition into the H-isotypic components is given by (5.3). Moreover, the natural embedding $\widetilde{K} \subset \widetilde{\mathcal{K}}$ induces an isomorphism between the Lie algebra of $Z(\widetilde{K})$ and that of $Z(\widetilde{\mathcal{K}})$. We denote the element in the Lie algebra of $Z(\widetilde{K})$ corresponding to H by H'. Then the formal character of $L(\sigma)$ on the compact Cartan subgroup restricted to the center of $\widetilde{K}$ can be expressed by the Poincaré series:

$$\operatorname{trace}_{L(\sigma)} t^{H'} = t^{k_\sigma + nm/2} P(L(\sigma); t^2). \tag{5.6}$$

To get explicit multiplicity formulas, we are involved in case-by-case analysis.

6. Description of K-types of the lowest weight modules

Assume that the pair (G_1, G_2) is in the *stable range* where G_2 is the smaller member. This means that $m \leqslant \mathbb{R}\text{-rank}\, G_1$, where $m = \dim_D V_2$ (cf. § 3). Take $\sigma \in \mathrm{Irr}(G_2)$ for which $L(\sigma)$ is not zero, and put $\widetilde{\sigma} = \sigma \otimes \chi$ as above. Let us describe K-type decomposition of $L(\sigma)$ in each explicit cases.

6.1. Case (Sp, O). — Assume that $m \leqslant n = \mathbb{R}\text{-rank}\, Sp(2n, \mathbb{R})$. This means the pair $(Sp(2n, \mathbb{R}), O(m))$ is in the stable range. As before, we shall write $G = G_1 = Sp(2n, \mathbb{R})$ and $\mathcal{G} = Sp(2nm, \mathbb{R})$.

Let $K = U(n)$ be a maximal compact subgroup of $G = Sp(2n, \mathbb{R})$ which is realized in the standard way (cf. (4.2)). Let $\mathcal{K} = U(n \times m) \subset \mathcal{G}$ act on $M_{n,m} = M(n, m, \mathbb{C})$ naturally as unitary transformation group. The product group $U(n) \times O(m)$ acts on $M_{n,m}$ naturally as

$$(6.1) \qquad (k, h)X = kX\,{}^t h \qquad ((k, h) \in U(n) \times O(m),\ X \in M_{n,m}).$$

Since the action is also unitary, it induces a map $U(n) \times O(m) \to U(nm) = \mathcal{K}$. The image of the above map coincides with $K \cdot G_2$. Note that the kernel of the map is $\{(\pm 1_n, \pm 1_m) \in U(n) \times O(m)\}$.

The metaplectic cover $\widetilde{\mathcal{K}}$ acts on $M_{n,m}$ as the composition of the projection $\widetilde{\mathcal{K}} \to \mathcal{K}$ and the natural action of the unitary group $\mathcal{K} = U(nm)$. This action induces the representation of $\widetilde{\mathcal{K}}$ on the polynomial ring $\mathbb{C}[M_{n,m}^*]$, which is isomorphic to the symmetric tensor of $M_{n,m}$. By the formula (4.10), we conclude that the action of $\widetilde{\mathcal{K}}$ on $\mathbb{C}[M_{n,m}^*]$ via Weil representation Ω is twisted by $\det^{1/2}$. We shall denote this representation by $\mathbb{C}[M_{n,m}^*] \otimes \det^{1/2}$. Therefore $\widetilde{K}$ acts on $\mathbb{C}[M_{n,m}^*]$ as $\mathbb{C}[M_{n,m}^*] \otimes \det^{m/2}$ and $\widetilde{G_2}$ acts as $\mathbb{C}[M_{n,m}^*] \otimes \det^{n/2}$. So the one-dimensional representation χ_1 of $\widetilde{K}$ coincides with $\det^{m/2}$, and the one-dimensional representation $\chi \in \operatorname{Irr}(\widetilde{G_2})$ coincides with $\det^{n/2}$. However, we should be more precise about χ because $G_2 = O(m)$ is not connected.

The metaplectic cover $\widetilde{\mathcal{K}}$ has a realization

$$\widetilde{\mathcal{K}} = \{(k, z) \in \mathcal{K} \times \mathbb{C}^\times \mid \det k = z^2\}$$

and the representation $\det^{1/2}$ of $\widetilde{\mathcal{K}}$ is given by the map $(k, z) \mapsto z$. Then the subgroup $\widetilde{G_2}$ is realized as

$$\widetilde{G_2} = \{(k, z) \in G_2 \times \mathbb{C}^\times \mid \det^n k = z^2\}$$

and its character $\chi = \det^{n/2}$ is given by $\chi(k, z) = z$. The identity component of $\widetilde{G_2}$ is

$$\left(\widetilde{G_2}\right)_0 = \{(k, z) \in G_2 \times \mathbb{C}^\times \mid \det k = 1, z = 1\} \simeq SO(m).$$

The map $(k, z) \mapsto (\det k, z)$ induces the isomorphism of the component group $\widetilde{G_2}/\left(\widetilde{G_2}\right)_0$ onto the group

$$A(G_2) = \{(t, z) \in \mathbb{Z}_2 \times \mathbb{C}^\times \mid t^n = z^2\}$$

of order four. Since the one-dimensional character χ is trivial on the identity component $\left(\widetilde{G_2}\right)_0$, it induces the character of the component group $A(G_2)$. We denote it by the same letter χ, then $\chi(t, z) = z$. First, we consider the case where n is odd. Then $A(G_2) = \{(\zeta^2, \zeta) \mid \zeta = \pm 1, \pm\sqrt{-1}\ \} \cong \mathbb{Z}_4$. If we define

$\varepsilon = ((\mathrm{diag}\,(1_{m-1}, -1), \sqrt{-1}\,) \in \widetilde{G_2}$, then $\varepsilon \in A(G_2)$ generates the component group $\mathbb{Z}_4$. Then we see that

$$\widetilde{G_2} \simeq SO(m) \rtimes \mathbb{Z}_4 \qquad \text{if } n \text{ is odd.}$$

The character χ satisfies $\chi(\varepsilon) = \sqrt{-1}$, which determines the character χ of $\mathbb{Z}_4$. Second, let us consider the case where n is even. In this case,

$$\widetilde{G_2} = G_2 \times \mathbb{Z}_2 \qquad \text{if } n \text{ is even.} \tag{6.2}$$

The character χ is trivial on $G_2 = O(m)$ and is non-trivial on $\mathbb{Z}_2$.

By the argument in § 5, we will get K-type decomposition of $L(\sigma)$ if we know the explicit decomposition of $\mathbb{C}[M_{n,m}^*] \otimes \det{}^{1/2}$ as $\widetilde{K} \times \widetilde{G_2}$-module. We first consider the space $\mathbb{C}[M_{n,m}^*]$ as the usual symmetric tensor of the natural representation of the unitary group $\mathcal{K} = U(n \times m)$, then afterwards we will twist it by $\det^{1/2}$ to fit it to the Weil representation Ω.

We extend the $U(n) \times O(m)$-action (6.1) on $M_{n,m}$ naturally to the $U(n) \times U(m)$-action. It is well known (cf. **[23]**) that, as $U(n) \times U(m)$-module, $\mathbb{C}[M_{n,m}^*]$ decomposes as

$$\mathbb{C}[M_{n,m}^*]\big|_{U(n)\times U(m)} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} \tau_\lambda^{(n)} \boxtimes \tau_\lambda^{(m)},$$

where $\mathcal{P}_m$ denotes the set of all partitions of length less than or equal to m. We make use of this decomposition. We identify K with $U(n)$ above, and consider $G_2 = O(m)$ in $U(m)$ in the standard manner, i.e., $O(m) = U(m) \cap GL(m, \mathbb{R})$. Decompose $\tau_\lambda^{(m)} \in \mathrm{Irr}(U(m))$ as $O(m)$-module:

$$\tau_\lambda^{(m)}\big|_{O(m)} \simeq \sum_{\sigma\in\mathrm{Irr}(O(m))}^{\oplus} m(\lambda, \sigma)\, \sigma. \tag{6.3}$$

Then we have a joint decomposition

$$\mathbb{C}[M_{n,m}^*]\big|_{U(n)\times O(m)} = \sum_{\sigma\in\mathrm{Irr}(O(m))}^{\oplus} \left\{ \sum_{\lambda\in\mathcal{P}_m}^{\oplus} m(\lambda, \sigma)\, \tau_\lambda^{(n)} \right\} \boxtimes \sigma.$$

So we completely know σ-isotypic component of $\mathbb{C}[M_{n,m}^*]$ in terms of the multiplicity $m(\lambda, \sigma)$. Twist of this representation by $\det^{1/2}$ causes the twist by $\det^{m/2} \boxtimes \chi$ as a representation of $\widetilde{K} \times \widetilde{G_2}$. Therefore $L(\sigma)|_{\widetilde{K}}$ decomposes as

$$L(\sigma)\big|_{U(n)^\sim} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} m(\lambda, \sigma)\, \tau_\lambda^{(n)} \otimes \det{}^{m/2}. \tag{6.4}$$

This formula describes the multiplicities of K-types of $L(\sigma)$ in the case of Case (Sp, O).

To describe the lowest weight and the lowest K-type of $L(\sigma)$, we give a classification of $\mathrm{Irr}(O(m))$ briefly. For more detailed discussion, see **[23**, § 3.6] for example. Let $\sigma(\mu)$ be an irreducible representation of $SO(m)$ with highest weight μ.

Lemma 6.1. — *Let σ be an irreducible representation of $O(m)$.*

(1) *If* $\sigma\big|_{SO(m)}$ *is irreducible, then* σ *and* $\sigma \otimes \det$ *are mutually inequivalent.*

(2) *If* $\sigma\big|_{SO(m)}$ *is reducible, then* σ *and* $\sigma \otimes \det$ *are equivalent. In this case,* m *is necessarily even. Moreover, there exist positive integers* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{m/2} > 0$ *such that*

$$\sigma\big|_{SO(m)} \simeq \sigma(\mu^+) \oplus \sigma(\mu^-),$$

where $\mu^+ = (\mu_1, \mu_2, \ldots, \mu_{m/2})$ *and* $\mu^- = (\mu_1, \mu_2, \ldots, -\mu_{m/2})$.

In case (1) in the above lemma, it is subtle to tell the difference between σ and $\sigma \otimes \det$. However, since the difference causes strong influence on our result, we discuss this point.

Take a Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{o}(m)$ as

$$\mathfrak{h}_0 = \{H = \operatorname{diag}(a(\theta_1), a(\theta_2), \ldots, a(\theta_{[m/2]}), 0) \mid \theta_i \in \mathbb{R}\}, \qquad a(\theta) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

where the last 0 in the expression of H appears if and only if m is odd. We define $\varepsilon_j \in \mathfrak{h}^*$ as $\varepsilon_j(H) = \sqrt{-1}\,\theta_j$ in the above expression. Then, positive roots are given by

$$\Delta^+ = \begin{cases} \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant m/2\} & \text{if } m \text{ is even,} \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \leqslant i < j \leqslant [m/2]\} \sqcup \{\varepsilon_j \mid 1 \leqslant j \leqslant [m/2]\} & \text{if } m \text{ is odd.} \end{cases}$$

Assume that $\sigma\big|_{SO(m)}$ be irreducible. Write $\sigma\big|_{SO(m)} = \sigma(\mu)$ for some highest weight $\mu = \sum_{j=1}^{[m/2]} \mu_j \varepsilon_j$. Let $\delta = \operatorname{diag}(1_{m-1}, -1) \in O(m) \backslash SO(m)$. Then, $\sigma\big|_{SO(m)}$ is irreducible if and only if the twisted representation $\sigma(\mu)^\delta$ is equivalent to $\sigma(\mu)$. Consequently, the highest weight space of $\sigma(\mu)$ is preserved by the action of δ. In particular, if m is even, we get $\mu_{m/2} = 0$.

Since $\delta^2 = 1_m$, its action on the highest weight space is the multiplication by ± 1. If it is 1, we will write $\sigma = \sigma(\mu)$ by abuse of notation; if it is -1, then we denote $\sigma = \sigma(\mu) \otimes \det$. Let $k = \ell(\mu)$ so that $\mu_k > \mu_{k+1} = 0$. We put $\mu^+ = \mu$ if $\sigma = \sigma(\mu)$. If $\sigma = \sigma(\mu) \otimes \det$, we add 1 to μ $(m-2k)$-times after μ_k, i.e.,

$$\mu^+ = (\mu_1, \mu_2, \ldots, \mu_k, 1, \ldots, 1) = (\mu, 1^{m-2k}).$$

The following theorem is due to Kashiwara-Vergne [**28**] (see also [**23**, § 3.6]).

Theorem 6.2. — *Assume that* $m \leqslant n = \mathbb{R}\text{-rank}\, Sp(2n, \mathbb{R})$. *Then* $L(\sigma)$ *is not zero for any* $\sigma \in \operatorname{Irr}(O(m))$ *and it gives an irreducible unitary lowest weight module of* $Sp(2n, \mathbb{R})^\sim$. *Let* μ^+ *be as above, and extend* μ^+ *to the weight of* $Sp(2n, \mathbb{R})$ *by adding zero. Then the lowest weight of* $L(\sigma)$ *is given by*

$$w_K\left(\mu^+ + \frac{m}{2}\mathbb{I}\right),$$

where w_K *is the longest element of the Weyl group of* $K = U(n)$ *and* $\mathbb{I} = (1, 1, \ldots, 1)$. *Consequently, the lowest* K*-type of* $L(\sigma)$ *is* $\tau(\mu^+) \otimes \det^{m/2}$.

From this theorem, we get the Poincaré series of $L(\sigma)$ as

$$P(L(\sigma);t^2) = t^{-|\mu^+|} \sum_{\lambda\in\mathcal{P}_m} m(\lambda,\sigma)\ \dim\tau_\lambda^{(n)}\ t^{|\lambda|}. \tag{6.5}$$

Consider the special case where $\sigma \in \operatorname{Irr}(O(m))$ is the trivial representation $\mathbf{1}_{O(m)}$.

Corollary 6.3. — *We have the K-type decomposition of $L(\mathbf{1}_{O(m)})$ as*

$$L(\mathbf{1}_{O(m)})\big|_{U(n)^\sim} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} \tau_{2\lambda}^{(n)} \otimes \det{}^{m/2},$$

where $\mathcal{P}_m$ is the set of all partitions such that $\ell(\lambda) \leqslant m$. The Poincaré series of $L(\mathbf{1}_{O(m)})$ is given by

$$P(L(\mathbf{1}_{O(m)});t) = \sum_{\lambda\in\mathcal{P}_m} \dim\tau_{2\lambda}^{(n)}\ t^{|\lambda|}. \tag{6.6}$$

Proof. — It is well-known that

$$m(\lambda,\mathbf{1}_{O(m)}) = \begin{cases} 1 & \text{if } \lambda \text{ is an even partition,} \\ 0 & \text{otherwise.} \end{cases}$$

Apply this formula to (6.4) and (6.5). □

6.2. Case (U,U). — We consider the pair $(G_1,G_2) = (U(p,q),U(m))$. We put $G = G_1 = U(p,q)$ in this subsection. Assume that $m \leqslant \min(p,q) = \mathbb{R}\text{-rank}\,U(p,q)$. This means that the pair $(U(p,q),U(m))$ is in the stable range.

A maximal compact subgroup of G is isomorphic to $U(p)\times U(q)$, and we realized it as

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A\in U(p), B\in U(q) \right\} \subset U(p,q). \tag{6.7}$$

Put $n = p+q$. In this case, $K\times G_2$ acts on $M_{n,m}$ in somewhat distorted manner. Let us identify $M_{n,m} = M_{p,m}\oplus M_{q,m}$. Then the action of $\operatorname{diag}(a,b)\times g \in (U(p)\times U(q))\times U(m)$ is given by

$$M_{p,m}\oplus M_{q,m} \ni X\oplus Y \longmapsto aX\,{}^t g \oplus \bar{b} Y\,{}^t\bar{g}. \tag{6.8}$$

This action gives the projection $K\times G_2 \to K\cdot G_2 \subset \mathcal{K} = U(nm)$, where $\mathcal{K}$ is the maximal compact subgroup of $\mathcal{G} = Sp(2nm,\mathbb{R})$ (cf. (3.8) and (3.9)). The kernel of the projection $K\times G_2 \to K\cdot G_2$ is given by $\{((\alpha 1_p,\alpha 1_q),\alpha^{-1}1_m) \mid \alpha\in\mathbb{C}^\times, |\alpha|=1\}$.

Let us consider the Weil representation Ω of $\widetilde{\mathcal{G}} = Mp(2nm,\mathbb{R})$ on $\mathbb{C}[M_{n,m}^*]$. Then the representation of $\Omega|_{\widetilde{\mathcal{K}}}$ is isomorphic to $\mathbb{C}[M_{n,m}^*]\otimes\det{}^{1/2}$. If we consider the underlying space $\mathbb{C}[M_{n,m}^*]$ as $\mathbb{C}[M_{n,m}^*] = \mathbb{C}[M_{p,m}^*]\otimes\mathbb{C}[M_{q,m}]$, the above embedding of $K\times G_2$ into $\mathcal{K}$ tells us that the representation of $U(p)\times U(q)$ is isomorphic to

$$\left(\mathbb{C}[M_{p,m}^*]\otimes\det{}^{m/2}\right) \boxtimes \left(\mathbb{C}[M_{q,m}]\otimes\det{}^{-m/2}\right); \tag{6.9}$$

while that of $U(m)$ is isomorphic to

(6.10)

$$\left(\mathbb{C}[M^*_{p,m}] \otimes \det{}^{p/2}\right) \otimes \left(\mathbb{C}[M_{q,m}] \otimes \det{}^{-q/2}\right) \simeq \left(\mathbb{C}[M^*_{p,m}] \otimes \mathbb{C}[M_{q,m}]\right) \otimes \det{}^{(p-q)/2}.$$

Therefore the one-dimensional character $\chi \in \mathrm{Irr}(\widetilde{G_2})$ is equal to $\det{}^{(p-q)/2}$, and the one-dimensional character χ_1 of $\widetilde{K}$ is $\det{}^{m/2} \boxtimes \det{}^{-m/2}$.

Let us first consider the untwisted representation $\mathbb{C}[M^*_{n,m}]$ of $\mathcal{K}$. To decompose the restriction to $K \times G_2$, we make use of $U(p) \times U(m)$ (or $U(q) \times U(m)$) duality. We have the decomposition as $(U(p) \times U(q)) \times U(m)$-module

$$\begin{aligned}
\mathbb{C}[M^*_{n,m}] &= \mathbb{C}[M^*_{p,m}] \otimes \mathbb{C}[M_{q,m}] \\
&= \left(\sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau^{(p)}_\lambda \boxtimes \tau^{(m)}_\lambda\right) \otimes \left(\sum_{\mu \in \mathcal{P}_m}^{\oplus} (\tau^{(q)}_\mu)^* \boxtimes (\tau^{(m)}_\mu)^*\right) \\
&= \sum_{\lambda,\mu \in \mathcal{P}_m}^{\oplus} \left(\tau^{(p)}_\lambda \boxtimes (\tau^{(q)}_\mu)^*\right) \boxtimes \left(\tau^{(m)}_\lambda \otimes (\tau^{(m)}_\mu)^*\right).
\end{aligned}$$

Therefore, if we define the branching coefficient $m(\lambda, \mu; \nu)$ by

$$\tau^{(m)}_\lambda \otimes (\tau^{(m)}_\mu)^* = \sum_\nu^{\oplus} m(\lambda, \mu; \nu)\, \tau^{(m)}_\nu, \tag{6.11}$$

we get

$$\mathbb{C}[M^*_{n,m}]|_{K \times U(m)} = \sum_\nu^{\oplus} \left\{ \sum_{\lambda,\mu \in \mathcal{P}_m}^{\oplus} m(\lambda, \mu; \nu)\, \tau^{(p)}_\lambda \boxtimes (\tau^{(q)}_\mu)^* \right\} \boxtimes \tau^{(m)}_\nu.$$

To get the representation $\Omega|_{\widetilde{K_1} \times \widetilde{G_2}}$, we should twist the above decomposition by $(\det^{m/2} \boxtimes \det^{-m/2}) \boxtimes \det^{(p-q)/2}$. After this twisting, for $\sigma = \tau^{(m)}_\nu \in \mathrm{Irr}(U(m))$, we get the K-type decomposition of $L(\sigma)$:

$$L(\tau^{(m)}_\nu)|_{\widetilde{K}} \simeq \sum_{\lambda,\mu \in \mathcal{P}_m}^{\oplus} m(\lambda, \mu; \nu)\, (\tau^{(p)}_\lambda \otimes \det{}^{m/2}) \boxtimes (\tau^{(q)}_\mu \otimes \det{}^{m/2})^*. \tag{6.12}$$

To determine the lowest weight of $L(\tau^{(m)}_\nu)$, we prove a lemma.

Lemma 6.4. — *Take an arbitrary dominant integral weight ν of $U(m)$, and write it as*

$$\nu = (a_1, a_2, \ldots, a_s, 0, \ldots, 0, -b_t, \ldots, -b_2, -b_1),$$

where

$$\begin{aligned}
&a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0, \quad b_1 \geqslant b_2 \geqslant \cdots \geqslant b_t > 0, \\
&a_i, b_j \in \mathbb{Z}; \quad s, t \geqslant 0 \text{ and } s + t \leqslant m.
\end{aligned}$$

Consider a set of pairs of partitions $\{(\lambda, \mu) \in \mathcal{P}_m \times \mathcal{P}_m \mid m(\lambda, \mu; \nu) \neq 0\}$. Then partitions

$$\begin{cases} \lambda = \alpha := (a_1, a_2, \ldots, a_s, 0, \ldots, 0) & \text{and} \\ \mu = \beta := (b_1, b_2, \ldots, b_t, 0, \ldots, 0) \end{cases}$$

minimize the degree $|\lambda| + |\mu|$ among such pairs. Moreover, (α, β) is a unique pair which attains the minimal degree. In this case, $m(\alpha, \beta; \nu) = 1$ holds.

Proof. — Take a sufficiently large $l \geqslant 0$ such that $\nu' = \nu + (l, \ldots, l)$ becomes a partition. We have

$$\begin{aligned}
&(6.13) \qquad & m(\lambda, \mu; \nu) &= \dim \left((\tau_\lambda \otimes \tau_\mu^*) \otimes \tau_\nu^* \right)^{U(m)} \\
&(6.14) \qquad & &= \dim \left((\tau_\lambda \otimes \det{}^l) \otimes (\tau_\mu \otimes \tau_{\nu'})^* \right)^{U(m)} \\
&(6.15) \qquad & &= c_{\mu,\nu'}^{\lambda + (l^m)} \neq 0,
\end{aligned}$$

where $c_{\gamma,\delta}^{\eta} = [\tau_\gamma \otimes \tau_\delta : \tau_\eta]$ denotes the Littlewood-Richardson coefficient. Since $c_{\gamma,\delta}^{\eta} \neq 0$ implies $|\eta| = |\gamma| + |\delta|$, we have $|\lambda| + ml = |\mu| + |\nu'|$, or equivalently $|\lambda| = |\mu| + |\nu|$. Therefore, in order to minimize $|\lambda| + |\mu|$, we only have to make $|\lambda|$ minimal. However, if ν' is not contained in $\lambda + (l^m)$, the coefficient $c_{\mu,\nu'}^{\lambda+(l^m)}$ vanishes. Therefore, $\lambda = \alpha$ is the smallest possible partition (e.g., see [**13**, § 5.2, Proposition 3]). If we take $\mu = \beta$, then it is easy to see that $m(\alpha, \beta; \nu) = 1$ (loc. cit.).

If we denote the highest weight of τ_ν^* by

$$\nu^* = (b_1, b_2, \ldots, b_t, 0, \ldots, 0, -a_s, \ldots, -a_2, -a_1),$$

it holds that $m(\lambda, \mu; \nu) = m(\mu, \lambda; \nu^*)$. By the same argument as above, we conclude that $\mu = \beta$ is the only possibility for $m(\alpha, \beta; \nu) \neq 0$. □

Remark 6.5. — If ν is also a partition, the above proof tells us that $m(\lambda, \mu; \nu) = c_{\mu,\nu}^{\lambda}$, where $c_{\mu,\nu}^{\lambda}$ is the Littlewood-Richardson coefficient.

Theorem 6.6. — *Assume that $m \leqslant \min(p, q) = \mathbb{R}\text{-rank}\, U(p, q)$. Then $L(\sigma)$ is not zero for any $\sigma = \tau_\nu^{(m)} \in \mathrm{Irr}(U(m))$ and it gives an irreducible unitary lowest weight module of $U(p,q)^{\sim}$. For ν, define α, β as in Lemma 6.4, and put $\beta^* = (0, \ldots, 0, -b_t, \ldots, -b_2, -b_1)$. Then the lowest weight of $L(\tau_\nu^{(m)})$ is given by*

$$w_K \left(\alpha + \frac{m}{2}\mathbb{I}_p, \beta^* - \frac{m}{2}\mathbb{I}_q \right),$$

where w_K is the longest element of the Weyl group of $K = U(p) \times U(q)$ and $\mathbb{I}_p = (1, \ldots, 1) = (1^p)$. Consequently, the lowest K-type of $L(\tau_\nu^{(m)})$ is $(\tau_\alpha^{(p)} \otimes \det^{m/2}) \boxtimes (\tau_\beta^{(q)} \otimes \det^{m/2})^*$.

Proof. — It is known that $L(\sigma)$ is an irreducible unitary lowest weight module of the metaplectic cover $U(p,q)^{\sim}$. So we simply have to determine its lowest weight. To do that, we only need to know the lowest K-type (or harmonic K-type) which is unique. By Lemma 6.4, we conclude that $\tau_\alpha^{(p)} \boxtimes \tau_\beta^{(q)*}$ gives such a K-type with a twist by $\chi_1 = \det^{m/2} \boxtimes \det^{-m/2}$. □

By this theorem, we obtain Poincaré series of $L(\tau_\nu^{(m)})$:

$$\begin{aligned} P(L(\tau_\nu^{(m)});t) &= t^{-|\beta|} \sum_{\lambda,\mu\in\mathcal{P}_m} m(\lambda,\mu;\nu) \dim\tau_\lambda^{(p)} \dim\tau_\mu^{(q)} \, t^{|\mu|} \\ &= t^{-|\alpha|} \sum_{\lambda,\mu\in\mathcal{P}_m} m(\lambda,\mu;\nu) \dim\tau_\lambda^{(p)} \dim\tau_\mu^{(q)} \, t^{|\lambda|}. \end{aligned}$$

This formula follows from (5.5) after a reflection on degrees. Note that the summation is taken over (λ,μ) satisfying $|\lambda| - |\mu| = |\nu|$ (see the proof of Lemma 6.4). Hence the total degree of $\tau_\lambda^{(p)} \boxtimes (\tau_\mu^{(q)})^*$ is given by $|\lambda| + |\mu| = 2|\mu| + |\nu| = 2|\lambda| - |\nu|$, while $|\nu| = |\alpha| - |\beta|$ and $k_\sigma = |\alpha| + |\beta|$ for $\sigma = \tau_\nu^{(m)}$.

Consider the special case where $\tau_\nu^{(m)}$ is trivial, i.e., $\nu = 0$. Then it is easy to see that

$$m(\lambda,\mu;0) = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get

Corollary 6.7. — *We have the K-type decomposition of $L(\mathbf{1}_{U(m)})$ as*

$$L(\mathbf{1}_{U(m)})|_{\widetilde{K}} \simeq \sum_{\lambda\in\mathcal{P}_m}^{\oplus} (\tau_\lambda^{(p)} \otimes \det{}^{m/2}) \boxtimes (\tau_\lambda^{(q)} \otimes \det{}^{m/2})^*. \tag{6.16}$$

Its Poincaré series becomes

$$P(L(\mathbf{1}_{U(m)});t) = \sum_{\lambda\in\mathcal{P}_m} \dim\tau_\lambda^{(p)} \dim\tau_\lambda^{(q)} \, t^{|\lambda|}. \tag{6.17}$$

6.3. Case (O^*,Sp). — We consider the pair $(G_1,G_2) = (O^*(2p), Sp(2m))$ in $\mathcal{G} = Sp(2nm,\mathbb{R})$ $(n = 2p)$, which is in the stable range, i.e., we assume that $m \leqslant [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$.

In this case, a maximal compact subgroup K of $O^*(2p)$ is isomorphic to $U(p)$. We realize the isomorphism as

$$O^*(2p) \supset K = \left\{ \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix} \mid X \in U(p) \right\} \leftrightarrow X \in U(p). \tag{6.18}$$

Then $K \times G_2$ is imbedded into $\mathcal{K} = U(2pm)$ canonically. To be more precise, this embedding of $K = U(p)$ and $G_2 = Sp(2m) = Sp(2m,\mathbb{C}) \cap U(2m)$ is given by the action on $M_{p,2m}$ as

$$M_{p,2m} \ni X \longmapsto gX\,{}^t h \qquad ((g,h) \in K \times G_2).$$

The action induces a projection $K \times G_2 \to K \cdot G_2 \subset \mathcal{K}$ with kernel $\{(\pm 1_p, \pm 1_{2m}) \in U(p) \times Sp(2m)\}$.

Let us consider the Weil representation Ω. As the representation space of Ω, we take the polynomial ring $\mathbb{C}[M_{p,2m}^*]$ as before. Then we know that $\Omega|_{\widetilde{K}}$ is isomorphic to

$$\mathbb{C}[M_{p,2m}^*] \otimes \det{}^m,$$

where $\mathbb{C}[M^*_{p,2m}]$ is considered as the symmetric tensor product of the representation $M_{p,2m}$ above. On the other hand, we have $\widetilde{G_2} \simeq Sp(2m) \times \mathbb{Z}_2$, and the one-dimensional character χ arises as the non-trivial character of $\mathbb{Z}_2$ as in (6.2). Therefore, we have $\Omega|_{\widetilde{G_2}} \simeq \mathbb{C}[M^*_{p,2m}] \otimes \chi$.

First, let us treat the untwisted symmetric tensor. So we decompose $\mathbb{C}[M^*_{p,2m}]$ by using $U(p) \times U(2m)$ duality :

$$\mathbb{C}[M^*_{p,2m}]\big|_{U(p)\times U(2m)} \simeq \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} \tau^{(p)}_\lambda \boxtimes \tau^{(2m)}_\lambda .$$

Take a highest weight λ for $U(2m)$ and μ for $Sp(2m)$. Let us define the branching coefficient $m(\lambda, \mu)$ by

$$\tau^{(2m)}_\lambda\big|_{Sp(2m)} \simeq \sum_{\mu}^{\oplus} m(\lambda,\mu)\sigma_\mu ,$$

where $\sigma_\mu \in \operatorname{Irr}(Sp(2m))$ is the irreducible representation of $Sp(2m)$ with highest weight μ. We also write $m(\lambda, \sigma_\mu)$ instead of $m(\lambda, \mu)$. With this notation, we can write down the decomposition :

$$\mathbb{C}[M^*_{p,2m}]\big|_{U(p)\times Sp(2m)} = \sum_{\sigma_\mu\in\operatorname{Irr}(Sp(2m))}^{\oplus} \left\{ \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} m(\lambda,\mu)\, \tau^{(p)}_\lambda \right\} \boxtimes \sigma_\mu .$$

To get the restricted representation $\Omega|_{\widetilde{K}\times\widetilde{G_2}}$, we must twist the above representation by $\det^m \boxtimes \chi$. Therefore $L(\sigma)|_{\widetilde{K}}$ decomposes as

$$L(\sigma_\mu)\big|_{U(p)^\sim} \simeq \sum_{\lambda\in\mathcal{P}_{2m}}^{\oplus} m(\lambda,\mu)\, \tau^{(p)}_\lambda \otimes \det{}^m . \tag{6.19}$$

This formula describes the multiplicities of K-types of $L(\sigma_\mu)$ in the case of Case (O^*, Sp).

Theorem 6.8. — *Assume that $m \leqslant [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$. Then $L(\sigma)$ is not zero for any $\sigma = \sigma_\mu \in \operatorname{Irr}(Sp(2m))$ and it gives an irreducible unitary lowest weight module of $O^*(2p)^\sim$. Extend μ to the weight of $O^*(2p)$ by adding zero. Then the lowest weight of $L(\sigma_\mu)$ is given by*

$$w_K\,(\mu + m\mathbb{I}_p)\,,$$

where w_K is the longest element of the Weyl group of $K = U(p)$ and $\mathbb{I}_p = (1, \ldots, 1) = (1^p)$. Consequently, the lowest K-type of $L(\sigma_\mu)$ is $\tau^{(p)}_\mu \otimes \det{}^m$.

Proof. — See [**23**, § 3.8.5]. □

From the above theorem, we obtain the Poincaré series of $L(\sigma_\mu)$:

$$P(L(\sigma_\mu); t^2) = t^{-|\mu|} \sum_{\lambda\in\mathcal{P}_{2m}} m(\lambda,\mu)\, \dim \tau^{(p)}_\lambda\, t^{|\lambda|}.$$

Consider the special case where $\sigma_\mu = \mathbf{1}_{Sp(2m)}$, i.e., $\mu = 0$. It is well-known that

$$m(\lambda, \mathbf{1}_{Sp(2m)}) = \begin{cases} 1 & \text{if } \lambda_{2i-1} = \lambda_{2i} \text{ for } 1 \leqslant i \leqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

So we get

Corollary 6.9. — *We have the K-type decomposition of $L(\mathbf{1}_{Sp(2m)})$ as*

$$L(\mathbf{1}_{Sp(2m)})\big|_{\widetilde{K}} \simeq \sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau_{\lambda^\#}^{(p)} \otimes \det{}^m, \tag{6.20}$$

where $\lambda^\# = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots)$ is a transposed even partition which is obtained by doubling each row of λ. Its Poincaré series is given by

$$P(L(\mathbf{1}_{Sp(2m)}); t) = \sum_{\lambda \in \mathcal{P}_m} \dim \tau_{\lambda^\#}^{(p)}\, t^{|\lambda|}. \tag{6.21}$$

7. Degree of nilpotent orbits

7.1. Automorphism groups of Hermitian symmetric spaces. — Let G be one of real reductive Lie groups $Sp(2n, \mathbb{R})$, $U(p,q)$, or $O^*(2p)$. These groups appear as the group G_1 in Table 2. The division algebra D is specified there. Let K be a maximal compact subgroup of G specified in § 6. In all cases, the corresponding Riemannian symmetric spaces G/K have G-invariant complex structure. In other words, the spaces G/K are Hermitian symmetric spaces. For $G = Sp(2n, \mathbb{R})$, $U(p,p)$, or $O^*(4k)$, the corresponding Hermitian symmetric space G/K is of *tube type.* For $G = U(p,q)$ with $p \neq q$, or $O^*(4k+2)$, G/K is not of tube type. For definitions and properties of symmetric spaces, see **[20]**.

We fix a complexification $G_\mathbb{C}$ of the real Lie group G. Let $K_\mathbb{C}$ be the minimal complex Lie subgroup of $G_\mathbb{C}$ containing K. We list up here $(G_\mathbb{C}, K_\mathbb{C})$ for the convenience of readers.

G	$G_\mathbb{C}$	$K_\mathbb{C}$
$Sp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$GL(n, \mathbb{C})$
$U(p,q)$	$GL(p+q, \mathbb{C})$	$GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$
$O^*(2p)$	$O(2p, \mathbb{C})$	$GL(p, \mathbb{C})$

TABLE 3. Complexifications of (G, K).

For real Lie groups such as G and K, we denote the corresponding Lie algebra by $\mathfrak{g}_0$, $\mathfrak{k}_0$, respectively. Its complexification is denoted by $\mathfrak{g}$ and $\mathfrak{k}$. The corresponding

Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is stable under the restriction of the adjoint action to $K_{\mathbb{C}}$. Moreover, in our cases, the subspace $\mathfrak{p}$ breaks up into the sum of two non-isomorphic irreducible representations of $K_{\mathbb{C}}$, say

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

The representation $\mathfrak{p}^-$ is the contragredient representation of $\mathfrak{p}^+$.

Let us describe the pair $(\mathfrak{p}^+, \mathfrak{p}^-)$ and the action of $K_{\mathbb{C}}$ on them for each case. Although the action itself is fairly well-known, we need more explicit features in the following.

For $G = Sp(2n, \mathbb{R})$, we realize it as in (4.1) and a maximal compact subgroup $K \simeq U(n)$ is also specified there (4.2). The complexification $G_{\mathbb{C}}$ is identified naturally with $Sp(2n, \mathbb{C})$ with respect to the same symplectic form as G (see (3.2) for the symplectic form). Then, the decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is given by

$$\mathfrak{p}^{\pm} = \left\{ \begin{pmatrix} \pm\sqrt{-1}\, A & A \\ A & \mp\sqrt{-1}\, A \end{pmatrix} \mid A \in \mathrm{Sym}\,(n, \mathbb{C}) \right\}.$$

Therefore, we can identify the both spaces with the space of symmetric matrices of size n. To see the action of $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$, it is more convenient to use the different realization of $Sp(2n, \mathbb{R})$. Let

$$\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_n & -\sqrt{-1}\, 1_n \\ -\sqrt{-1}\, 1_n & 1_n \end{pmatrix},$$

which is called the Cayley transform. The conjugation of $Sp(2n, \mathbb{R})$ by γ produces a different (but isomorphic) real form of $Sp(2n, \mathbb{C})$, and we denote it by $G^{\gamma} = Sp(2n, \mathbb{R})^{\gamma}$. In G^{γ}, the conjugated maximal compact subgroup K^{γ} has a simple diagonal form:

$$K^{\gamma} = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^t k^{-1} \end{pmatrix} \mid k \in U(n) \right\}.$$

The complexification $K^{\gamma}_{\mathbb{C}}$ is also expressed similarly as above, but k belonging to $GL(n, \mathbb{C})$. Then $\mathfrak{p}^{\gamma}$ is represented by off diagonal matrices

$$\mathfrak{p}^{\gamma} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathrm{Sym}\,(n, \mathbb{C}) \right\},$$

and

$$\mathfrak{p}^{\gamma+} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in \mathrm{Sym}\,(n, \mathbb{C}) \right\}, \quad \mathfrak{p}^{\gamma-} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in \mathrm{Sym}\,(n, \mathbb{C}) \right\}.$$

We denote the element $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ of $\mathfrak{p}^{\gamma}$ by (B, C). Then the adjoint action of an element $k \in K^{\gamma}_{\mathbb{C}}$ on $\mathfrak{p}^{\gamma}$ is given by

$$k(B, C) = (kB\,{}^t k,\ {}^t k^{-1} C k^{-1}).$$

We sometimes identify the $K_{\mathbb{C}}$-module $\mathfrak{p}^+$ with $\mathrm{Sym}\,(n, \mathbb{C})$.

For $G = U(p,q)$, we realized it as the full isometry group of the indefinite Hermitian form (3.6) (cf. (3.7)), and a maximal compact subgroup $K \simeq U(p) \times U(q)$ is given in (6.7). The complexification $G_{\mathbb{C}}$ is naturally identified with $GL(p+q,\mathbb{C})$, and $K_{\mathbb{C}}$ is given by

$$K_{\mathbb{C}} = \left\{ k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \mid k_1 \in GL(p,\mathbb{C}),\ k_2 \in GL(q,\mathbb{C}) \right\}.$$

The other member of the Cartan decomposition is expressed by off diagonal matrices

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B \in M(p,q,\mathbb{C}), C \in M(q,p,\mathbb{C}) \right\},$$

and such an element is denoted by (B,C). Irreducible subspaces $\mathfrak{p}^{\pm}$ are given as

$$\mathfrak{p}^{+} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in M(p,q,\mathbb{C}) \right\}, \quad \mathfrak{p}^{-} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in M(q,p,\mathbb{C}) \right\}.$$

The adjoint action of an element $k = (k_1,k_2) \in K_{\mathbb{C}}$ on $\mathfrak{p}$ is given by

$$(k_1,k_2)(B,C) = (k_1 B k_2^{-1}, k_2 C k_1^{-1}).$$

Therefore, the representation $\mathfrak{p}^{+}$ of $K_{\mathbb{C}}$ is identified with $M(p,q,\mathbb{C})$.

For $G = O^*(2p)$, we gave a realization in (3.10). A maximal compact subgroup $K \simeq U(p)$ is chosen again as diagonal matrices (6.18). The complexified Lie group $G_{\mathbb{C}}$ is identified with

$$O(2p,\mathbb{C}) = \left\{ Z \in GL(2p,\mathbb{C}) \mid {}^{t}Z S_p Z = S_p \right\}, \qquad S_p = \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}, \tag{7.1}$$

and

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} k & 0 \\ 0 & {}^{t}k^{-1} \end{pmatrix} \mid k \in GL(p,\mathbb{C}) \right\}.$$

We identify $K_{\mathbb{C}}$ and $GL(p,\mathbb{C})$ in the following, so $k \in K_{\mathbb{C}}$ denotes a matrix in $GL(p,\mathbb{C})$. Now $\mathfrak{p}$ becomes

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in \mathrm{Alt}\,(p,\mathbb{C}) \right\}.$$

As above, we denote the element $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ by (B,C). The $K_{\mathbb{C}}$ stable decomposition of $\mathfrak{p}$ is given by

$$\mathfrak{p}^{+} = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in \mathrm{Alt}\,(p,\mathbb{C}) \right\}, \quad \mathfrak{p}^{-} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in \mathrm{Alt}\,(p,\mathbb{C}) \right\}.$$

The adjoint action of an element $k \in K_{\mathbb{C}}$ on $\mathfrak{p}$ is

$$k(B,C) = (kB\,{}^{t}k, {}^{t}k^{-1} C k^{-1}).$$

We identify the $K_{\mathbb{C}}$-module $\mathfrak{p}^{+}$ with $\mathrm{Alt}\,(p,\mathbb{C})$.

7.2. Kostant-Rallis decomposition. — In this subsection, we summarize the $K_{\mathbb{C}}$-orbit decomposition of $\mathfrak{p}^-$. The orbit decomposition of $\mathfrak{p}^+$ is the same. We denote the real rank of Lie group G by r (cf. § 6). Then there are exactly $(r+1)$ $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$ in each of the above three cases. We give a parametrization of these orbits by $(r+1)$ integers, $0, 1, \dots, r$,

$$\mathfrak{p}^- = \coprod_{j=0}^{r} \mathcal{O}_j.$$

We know that the dimension of the orbits are distinct (see below). We arrange the numbering of orbits so that an orbit with the larger index has the larger dimension. With this indexing, the set $\mathcal{O}_r$ is an open dense subset of $\mathfrak{p}^-$ in the classical topology (or, also in Zariski topology). On the contrary, the orbit $\mathcal{O}_0 = \{0\}$. We also know that the closure in classical topology (or, also in Zariski topology),

$$\overline{\mathcal{O}_m} = \coprod_{j \leqslant m} \mathcal{O}_j.$$

In other words, the closure relation of the orbits is linear ordering. Each closure is a Zariski closed subset of the affine space $\mathfrak{p}^-$, then $\overline{\mathcal{O}_m}$ is an affine algebraic variety. We denote the defining ideal of these subset $\overline{\mathcal{O}_m}$ by

$$I_m := \{p \in \mathbb{C}[\mathfrak{p}^-] \mid p|_{\overline{\mathcal{O}_m}} = 0\}.$$

This is an ideal of the polynomial ring $\mathbb{C}[\mathfrak{p}^-]$ on $\mathfrak{p}^-$. Then, by definition, the coordinate ring $\mathbb{C}[\overline{\mathcal{O}_m}]$ is isomorphic to the residual ring $\mathbb{C}[\mathfrak{p}^-]/I_m$.

Note that we can identify $\mathfrak{p}^+$ with the dual vector space of $\mathfrak{p}^-$ via Killing form. Therefore, by the natural identification, $\mathbb{C}[\mathfrak{p}^-] = S(\mathfrak{p}^+)$, where $S(\mathfrak{p}^+)$ denotes the symmetric algebra. Since $\mathfrak{p}^+$ is an abelian subspace of $\mathfrak{g}$, we also identify $S(\mathfrak{p}^+)$ with the enveloping algebra $U(\mathfrak{p}^+)$. We use these identification freely in the following. In particular, as a $K_{\mathbb{C}}$-module, $\mathbb{C}[\overline{\mathcal{O}_m}]$ is isomorphic to a quotient module of $S(\mathfrak{p}^+)$.

Let $L = L(\sigma)$ be an irreducible unitary lowest weight module of G treated in § 5. We construct a good filtration of L by taking the lowest K-type as a generating subspace of L (cf. § 1.2). Let $M = \operatorname{gr} L$ be the associated graded $S(\mathfrak{g})$-module. Since the generating subspace is preserved by $\mathfrak{k}$ and $\mathfrak{p}^-$, the $S(\mathfrak{g})$-module M is annihilated by $\mathfrak{k}$ and $\mathfrak{p}^-$. Therefore, its associated variety $\mathcal{AV}(L)$ is contained in $\mathfrak{p}^-$, by the identification above, and is a $K_{\mathbb{C}}$ stable closed subset. Since the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$ has linear ordering with respect to the closure relation, we can conclude that $\mathcal{AV}(L) = \overline{\mathcal{O}_m}$ for some $0 \leqslant m \leqslant r$.

In the following subsections, we see that there is a strong relationship between $\mathbb{C}[\overline{\mathcal{O}_m}]$ and the K-type decomposition of $L(\mathbf{1}_{G_2})$. In fact, they are the same as $K_{\mathbb{C}}$-modules up to some character. This relationship is an example of general phenomenon and is well-known among experts. It is a part of Vogan's philosophy of orbit method [**52**].

Our main aim of the following subsections is calculation of the Bernstein degree of $L(\mathbf{1}_{G_2})$. Our approach is purely representation theoretic. It turns out that $\operatorname{Deg} L(\mathbf{1}_{G_2})$ coincides with the classical degree of the corresponding orbit $\overline{\mathcal{O}_m} = \mathcal{AV}(L(\mathbf{1}_{G_2}))$, which coincides with determinantal variety of various type (see, e.g, **[12]** or **[16**, Lecture 9]). Hence our calculation here will give a new proof of the formula of $\deg \overline{\mathcal{O}_m}$ called *Giambelli-Thom-Porteous formula* (**[15]**, **[17]**; also see **[12**, Chapter 14]).

7.3. The case $G = Sp(2n,\mathbb{R})$. — Consider $G = Sp(2n,\mathbb{R})$. In this case, as is given above, $K = U(n)$, $K_{\mathbb{C}} = GL(n,\mathbb{C})$, $\mathfrak{p}^- = \operatorname{Sym}(n,\mathbb{C})$. The action of $k \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by

$$k \cdot A = {}^t k^{-1} A k^{-1} \qquad (k \in GL(n,\mathbb{C}), A \in \operatorname{Sym}(n,\mathbb{C})). \tag{7.2}$$

We define a locally closed subset of $\operatorname{Sym}(n,\mathbb{C})$ by

$$\mathcal{O}_m = \{A \in \operatorname{Sym}(n,\mathbb{C}) \mid \operatorname{rank}(A) = m\}, \qquad (m = 0,1,\ldots,n).$$

By the definition of the action of $K_{\mathbb{C}}$, it is easy to see that $\mathcal{O}_m$ is stable under the action of $K_{\mathbb{C}}$. Moreover, they classify all the $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$. The matrix $\sum_{j=1}^m E_{jj}$ belongs to the orbit $\mathcal{O}_m$. Here, E_{ij} is the matrix unit, that is, (i,j)-entry of the matrix E_{ij} is one and all other entries are zero. The dimension of the orbit $\mathcal{O}_m$ is given by

$$\dim \mathcal{O}_m = rm - (m-1)m/2.$$

For subsets $I = \{i_1, i_2, \ldots, i_{m+1}\}$ and $J = \{j_1, j_2, \ldots, j_{m+1}\}$ of $\{1,2,\ldots,n\}$ with the same cardinality $(m+1)$, we define the minor

$$D_{IJ}(A) = \det(a_{i_p j_q})_{1 \leqslant p,q \leqslant m+1},$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in \operatorname{Sym}(n,\mathbb{C})$. Then the defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by these minors

$$\{D_{IJ} \mid I, J \subset \{1,2,\ldots,n\}, |I| = |J| = m+1\}.$$

Recall the dual pair $(Sp(2n,\mathbb{R}), O(m))$ in § 3. We define an *unfolding* of the orbit $\mathcal{O}_m$ by an extra action of $O(m)$, or more precisely, its complexification $O(m,\mathbb{C})$. Let us consider the space of $m \times n$ matrices $M_{m,n} = M(m,n,\mathbb{C})$ and define an action of $K_{\mathbb{C}} \times O(m,\mathbb{C}) = GL(n,\mathbb{C}) \times O(m,\mathbb{C}) \ni (k,h)$ on $M_{m,n}$ by

$$(k,h) \cdot X = hXk^{-1} \qquad (X \in M_{m,n}).$$

For $X \in M_{m,n}$, we define

$$\psi(X) = {}^t X X \in \operatorname{Sym}(n,\mathbb{C}).$$

This is a polynomial map of degree two. With the trivial action of $O(m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi : M_{m,n} \to \overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}} \times O(m,\mathbb{C})$-equivariant, that is, $\psi(hXk^{-1}) = {}^t k^{-1} \psi(X) k^{-1}$ for all $k \in K_{\mathbb{C}}$, $h \in O(m,\mathbb{C})$. We see that the image of ψ coincides with $\overline{\mathcal{O}_m}$.

Lemma 7.1. — *The map ψ above induces the $\mathbb{C}$-algebra isomorphism*

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \ni f \mapsto f \circ \psi \in \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})} = S(M_{n,m})^{O(m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ is the geometric quotient $M_{n,m}//O(m,\mathbb{C})$. In particular, $\overline{\mathcal{O}_m}$ is a normal variety. Here we consider $M_{m,n} = M_{n,m}^*$ as the algebraic dual of $M_{n,m}$.

Proof. — The induced map ψ^* is injective since ψ is surjective. The classical invariant theory, in the modern reformulation [**24**, § 3.4], says that every $O(m,\mathbb{C})$-invariants on $M_{m,n}$ is generated by typical invariants of degree two, which implies the map ψ^* is surjective. □

Now we come back to the dual pair $(Sp(2n,\mathbb{R}), O(m))$ in $\mathcal{G} = Sp(2nm,\mathbb{R})$ and the Weil representation Ω of $\widetilde{\mathcal{G}}$ (cf. § 3). Let $L(\mathbf{1}_{O(m)})$ be an irreducible unitary lowest weight module of $\widetilde{G}$ which corresponds to the trivial representation of $O(m)$. We should clarify the relationship between $\mathcal{O}_m$ and the representation $L(\mathbf{1}_{O(m)})$.

Since the associated variety of $L(\mathbf{1}_{O(m)})$ is contained in $\mathfrak{p}^-$, it is enough to see the annihilator of $\operatorname{gr} L(\mathbf{1}_{O(m)})$ in $U(\mathfrak{p}^+)$. Therefore, let us see the action of the non-compact root vector $X_{\varepsilon_a+\varepsilon_b} \in \mathfrak{p}^+$ via $\Omega : \mathfrak{sp}(2nm,\mathbb{R}) \to \operatorname{End}_{\mathbb{C}}(\mathbb{C}[M_{n,m}])$,

$$\Omega(X_{\varepsilon_a+\varepsilon_b}) = \frac{1}{2}\sum_{j=1}^{m} x_{aj}x_{bj},$$

(see (4.8)). By this formula, we see $\Omega(X_{\varepsilon_a+\varepsilon_b}) \in \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$. Moreover, we have

$$2\,\Omega(X_{\varepsilon_a+\varepsilon_b}) = \psi_{ab},$$

here $\psi_{ab} \in \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ is the ab-component of ψ. This means that the subspace spanned by typical invariants ψ_{ab} coincides with the image $\Omega(\mathfrak{p}^+)$. Thus, the subalgebra of $\Omega(U(\mathfrak{p}^+))$ generated by $\Omega(\mathfrak{p}^+)$ is isomorphic to $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$, which is generated by typical invariants as is explained above. Let us define the natural good filtration of $L = L(\mathbf{1}_{O(m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is the constant polynomial with value 1. Then we have an isomorphism

$$L(\mathbf{1}_{O(m)}) \cong \operatorname{gr} L(\mathbf{1}_{O(m)}) \cong U(\mathfrak{p}^+)/I_m$$

as $U(\mathfrak{p}^+)$-modules, $K_{\mathbb{C}}$-modules and filtered modules. The filtration induced by the degree of polynomials coincides with the natural filtration up to a shift. This implies

Lemma 7.2. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{O(m)}) = \operatorname{Ann} \operatorname{gr} L(\mathbf{1}_{O(m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — As is explained above, we have the desired isomorphisms. For the annihilator, note that the representation space $\mathbb{C}[M_{n,m}^*]^{O(m,\mathbb{C})}$ of $L(\mathbf{1}_{O(m)})$ has the natural grading as K-module and $\mathfrak{p}^+$ acts on $\mathbb{C}[M^*_{n,m}]^{O(m,\mathbb{C})} = \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ as a homogeneous operator of degree two. This means that the annihilator in $U(\mathfrak{p}^+)$ does not change after taking gradation as a filtered $U(\mathfrak{g})$-module. □

Corollary 7.3. — *The representation $L(\mathbf{1}_{O(m)})$ has the following properties.*
(1) *The associated variety of $L(\mathbf{1}_{O(m)})$ is $\overline{\mathcal{O}_m}$.*
(2) *As a $\widetilde{K}$-module, $L(\mathbf{1}_{O(m)})$ is isomorphic to $\mathbb{C}[\overline{\mathcal{O}_m}] \otimes \det^{m/2}$.*
(3) *The Bernstein degree of $L(\mathbf{1}_{O(m)})$ coincides with $\deg \overline{\mathcal{O}_m}$.*

Proof. — (1) is a direct consequence of the above lemma.

Let us consider (2). By definition, $L(\mathbf{1}_{O(m)})$ is realized on $\mathbb{C}[M^*_{n,m}]^{O(m,\mathbb{C})} = \mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ (see § 5). As is explained in § 5, to get $\widetilde{K}$-module structure of $L(\mathbf{1}_{O(m)})$, we must twist $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ by $\det^{m/2}$. Therefore, untwisting of $L(\mathbf{1}_{O(m)})$ produces $\mathbb{C}[M_{m,n}]^{O(m,\mathbb{C})}$ itself, and the module structure factors through to that of K.

Since the unfolding map ψ has degree two, it is easy to see the definition of $\operatorname{Deg} L(\mathbf{1}_{O(m)})$ and $\deg \overline{\mathcal{O}_m}$ coincides, which proves (3). □

Let us calculate $\operatorname{Deg} L(\mathbf{1}_{O(m)}) = \deg \overline{\mathcal{O}_m}$ explicitly. Recall the good filtration $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$. By (6.6) and the Weyl's dimension formula, we know

$$
\begin{aligned}
\dim L_k &= \sum_{\lambda \in \mathcal{P}_m, |\lambda| \leqslant k} \dim \tau^{(n)}_{2\lambda} \\
&= \sum_{\lambda \in \mathcal{P}_m, |\lambda| \leqslant k} \frac{\prod_{1 \leqslant i < j \leqslant n} (2\lambda_i - 2\lambda_j - i + j)}{\prod_{1 \leqslant i < j \leqslant n} (j - i)} \\
&= \frac{2^{m(m-1)/2 + m(n-m)} k^{m(m-1)/2 + m(n-m) + m}}{\prod_{i=1}^m (n-i)!} \times \\
&\quad \times \int\limits_{\substack{0 \leqslant x_m \leqslant x_{m-1} \leqslant \cdots \leqslant x_1, \\ x_1 + \cdots + x_m \leqslant 1}} \prod_{1 \leqslant i < j \leqslant m} (x_i - x_j) \prod_{i=1}^m x_i^{n-m} dx_1 \cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k) \\
&= \frac{2^{mn - m(m+1)/2} k^{mn - m(m-1)/2}}{m! \prod_{i=1}^m (n-i)!} \times \\
&\quad \times \int\limits_{\substack{x_i \geqslant 0 \\ x_1 + \cdots + x_m \leqslant 1}} \prod_{1 \leqslant i < j \leqslant m} |x_i - x_j| \prod_{i=1}^m x_i^{n-m} dx_1 \cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k)
\end{aligned}
$$

for sufficiently large k. Here, in the third equality, we devide the formula by a suitable power of k and interprete the leading term as a Riemann sum for the integral.

Let us generalize the integral above slightly, and denote it as

$$I^\alpha(s,m) = \int\limits_{\substack{x_i \geqslant 0,\\ x_1+\cdots+x_m\leqslant 1}} |\Delta|^\alpha \left(\textstyle\prod_{i=1}^m x_i\right)^s dx_1\cdots dx_m, \tag{7.3}$$

where $\Delta = \prod_{1\leqslant i<j\leqslant m}(x_i - x_j)$ is the difference product. An explicit formula of this integral is given by using Gamma function of Hermitian symmetric cone ([**37**]).

Theorem 7.4. — *Let $I^\alpha(s,m)$ be as in (7.3). For $\operatorname{Re} s > -1$ and $\alpha = 1,2,4$, we have*

$$I^\alpha(s,m) = \frac{\prod_{j=1}^m \Gamma(j\alpha/2+1)\Gamma(s+1+(j-1)\alpha/2)}{\Gamma(\alpha/2+1)^m\Gamma(sm+N+1)}, \tag{7.4}$$

where $N = m + \dfrac{\alpha}{2}m(m-1)$.

Summarizing above, we have the following theorem.

Theorem 7.5. — *Assume that $m \leqslant n = \mathbb{R}\text{-rank}\, Sp(2n,\mathbb{R})$, and consider the reductive dual pair $(Sp(2n,\mathbb{R}), O(m))$.*

(1) The unitarizable lowest weight module $L(\mathbf{1}_{O(m)})$ of $Sp(2n,\mathbb{R})^\sim$ has the lowest weight $\frac{m}{2}(1,1,\ldots,1) = \frac{m}{2}\sum_{i=1}^n \varepsilon_i$. Its associated cycle is multiplicity-free and given by $\mathcal{AC}\,(L(\mathbf{1}_{O(m)})) = [\overline{\mathcal{O}_m}]$.

(2) The Gelfand-Kirillov dimension and the Bernstein degree of $L(\mathbf{1}_{O(m)})$ are

$$\begin{aligned} \operatorname{Dim} L(\mathbf{1}_{O(m)}) &= \dim \overline{\mathcal{O}_m} = m\left(n - \frac{m-1}{2}\right), \\ \operatorname{Deg} L(\mathbf{1}_{O(m)}) &= \deg \overline{\mathcal{O}_m} = \prod_{l=0}^{m-1} \frac{l!}{l!!}\frac{(2n-2m+l)!!}{(n-m+l)!}, \end{aligned}$$

where $l!! = l(l-2)(l-4)\cdots 2$ for an even integer l, and $l!! = l(l-2)(l-4)\cdots 1$ for odd l.

Proof. — From the top degree term of $\dim L_k$ above, we get the Gelfand-Kirillov dimension

$$\operatorname{Dim} L(\mathbf{1}_{O(m)}) = mn - \frac{m(m-1)}{2} =: d,$$

and

$$\begin{aligned} \operatorname{Deg} L(\mathbf{1}_{O(m)}) &= \frac{2^{d-m}d!}{m!\prod_{i=1}^m (n-i)!} I^1(n-m,m) \\ &= \frac{2^d}{\pi^{m/2}m!}\prod_{j=1}^m \frac{\Gamma(j/2+1)\Gamma(d/m-(j-1)/2)}{\Gamma(n-j+1)} \\ &= \prod_{l=0}^{m-1} \frac{l!}{l!!}\frac{(2n-2m+l)!!}{(n-m+l)!}. \end{aligned}$$

$\square$

We close this subsection by giving the relation between the lowest weight module $L(\mathbf{1}_{O(m)})$ and the half-form bundle on the orbit $\mathcal{O}_m$. We choose a representative

$$\lambda = \sum_{j=1}^{m} E_{jj} \in \mathcal{O}_m \subset \mathrm{Sym}\,(n, \mathbb{C}) \cong \mathfrak{p}^-$$

of the orbit $\mathcal{O}_m$. The group $K_{\mathbb{C}} = GL(n, \mathbb{C})$ acts on $\mathcal{O}_m$ transitively by (7.2). The stabilizer $(K_{\mathbb{C}})_\lambda$ of λ in $K_{\mathbb{C}}$ is

$$(K_{\mathbb{C}})_\lambda = \left\{ k = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix} \mid g_1 \in O(m, \mathbb{C}), g_2 \in GL(n-m, \mathbb{C}) \right\}. \tag{7.5}$$

We denote the determinant of the isotropy representation by $\det(\mathrm{Ad}\,|_{T_\lambda \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \to \mathbb{C}^\times$, where $T_\lambda \mathcal{O}_m$ is the tangent space of $\mathcal{O}_m$ at λ. It is written by

$$\det(\mathrm{Ad}\,|_{T_\lambda \mathcal{O}_m}) = (\det g_1)^{n-m} (\det g_2)^{-m} = (\det g_1)^n (\det k)^{-m},$$

with the notation (7.5). The cotangent bundle $T^*\mathcal{O}_m$ is a $K_{\mathbb{C}}$-equivariant vector bundle. The line bundle $\Lambda^{\mathrm{top}} = \bigwedge^{\dim \mathcal{O}_m} T^*\mathcal{O}_m$ consisting of volume forms on the orbit $\mathcal{O}_m$ is a $K_{\mathbb{C}}$-equivariant line bundle. Then it corresponds to the one-dimensional representation of the isotropy subgroup $(K_{\mathbb{C}})_\lambda$. In this case it is given by the coisotropy representation

$$\det(\mathrm{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \ni k \mapsto (\det g_1)^{-n} (\det k)^m \in \mathbb{C}^\times,$$

with the notation (7.5). We introduce the square root of the line bundle Λ^{top}, denoted by ξ, and consider the set $\Gamma(\mathcal{O}_m, \xi)$ of its global sections. We will give the relation between this line bundle on the orbit $\mathcal{O}_m$ and the lowest weight representation under consideration.

In what follows, *we assume that n is even.* We define the one-dimensional representation

$$\xi : (\widetilde{K_{\mathbb{C}}})_\lambda \ni k \mapsto \det{}^{m/2} k \in \mathbb{C}^\times.$$

By the definition, the coisotropy representation is the square of ξ;

$$\det(\mathrm{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) = \det(\mathrm{Ad}\,|_{T_\lambda \mathcal{O}_m})^{-1} = \xi^2.$$

This means that ξ corresponds to the half-form bundle on the orbit $\mathcal{O}_m = K_{\mathbb{C}}/(K_{\mathbb{C}})_\lambda = \widetilde{K_{\mathbb{C}}}/(\widetilde{K_{\mathbb{C}}})_\lambda$. The set of global sections $\Gamma(\mathcal{O}_m, \xi)$ has a natural $\widetilde{K_{\mathbb{C}}}$-module structure.

Proposition 7.6. — *For $0 \leqslant m < n$ and $n \in 2\mathbb{Z}$, the lowest weight module $L(\mathbf{1}_{O(m)})$ is isomorphic to $\Gamma(\mathcal{O}_m, \xi)$ as $\widetilde{K}$-modules.*

Proof. — We denote the complexification of the character $\chi_1 : \widetilde{K} \to \mathbb{C}^\times$ introduced in Section 6.1 by the same character. To be more explicit, we define the character

$\chi_1 : \widetilde{K_{\mathbb{C}}} \to \mathbb{C}^\times$ by $\chi_1(k) = \det^{m/2} k$. The restriction of χ_1 to the isotropy subgroup coincides with ξ. Then, we see that

$$\Gamma(\mathcal{O}_m, \xi) = \operatorname{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi = \chi_1 \otimes \operatorname{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \mathbf{1}_{(\widetilde{K_{\mathbb{C}}})_\lambda} = \chi_1 \otimes \mathbb{C}[\mathcal{O}_m]$$

for $0 \leqslant m \leqslant n$ as $\widetilde{K_{\mathbb{C}}}$-modules. On the other hand, we have seen in Corollary 7.3(2) that

$$L(\mathbf{1}_{O(m)}) = \mathbb{C}[\overline{\mathcal{O}_m}] \otimes \det^{m/2}.$$

Since $\overline{\mathcal{O}_m}$ is normal (cf. Lemma 7.1), and, for $m \neq n$, $\operatorname{codim}_{\overline{\mathcal{O}_m}} \mathcal{O}_m \geqslant 2$ for $m \neq n$, the restriction map gives a natural isomorphism $\mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathcal{O}_m]$ (cf. [**10**, Chapter 11, § 11.2]). This shows the proposition. $\square$

7.4. The case $G = U(p,q)$. — Let $G = U(p,q)$. In this case, $K = U(p) \times U(q)$, $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$, $\mathfrak{p}^- = M(q,p,\mathbb{C})$. The action of $(k_1,k_2) \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by

$$k_2 A k_1^{-1}. \tag{7.6}$$

Put $r = \mathbb{R}\text{-rank}\, U(p,q) = \min(p,q)$. We define a subset of $M_{q,p} = M(q,p,\mathbb{C})$ by

$$\mathcal{O}_m = \{A \in M_{q,p} \mid \operatorname{rank}(A) = m\}, \qquad (m = 0,1,\dots,r).$$

By an argument similar to the case $Sp(2n,\mathbb{R})$, we know that $\mathcal{O}_m$ is a $K_{\mathbb{C}}$-orbit, and they give a complete classification of $K_{\mathbb{C}}$-orbits in $\mathfrak{p}^-$. Note that the matrix $\sum_{j \leqslant m} E_{jj}$ is contained in $\mathcal{O}_m$. It is easy to see that

$$\dim \mathcal{O}_m = (p+q)m - m^2,$$

hence all the orbits have different dimensions. The defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by the minors

$$\{D_{IJ} \mid I \subset \{1,2,\dots,q\}, J \subset \{1,2,\dots,p\}, |I| = |J| = m+1\}.$$

The affine algebraic variety $\overline{\mathcal{O}_m}$ is called the determinantal variety.

Now recall the dual pair $(U(p,q), U(m))$. Let $GL(m,\mathbb{C})$ be the complexification of $U(m)$. We consider the natural action of $K_{\mathbb{C}} \times GL(m,\mathbb{C}) = (GL(p,\mathbb{C}) \times GL(q,\mathbb{C})) \times GL(m,\mathbb{C}) \ni (k_1,k_2,h)$ on $(A,B) \in M_{m,p} \oplus M_{m,q} \simeq M_{m,p+q}$ by

$$({}^t h^{-1} A k_1^{-1}, hB\, {}^t k_2), \tag{7.7}$$

which comes from (6.8). For $(A,B) \in M_{m,p} \oplus M_{m,q}$, we define an unfolding map ψ by

$$\psi(A,B) = {}^t BA \in M_{q,p}.$$

This is a polynomial map of degree two. Note that $\psi(\sum_{l \leqslant j} E_{ll}, \sum_{l \leqslant j} E_{ll}) = \sum_{l \leqslant j} E_{ll} \in \mathcal{O}_m$. From this, we see that the image of ψ coincides with $\overline{\mathcal{O}_m}$. With the trivial action of $GL(m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi : M_{m,p+q} \to \overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}} \times GL(m,\mathbb{C})$-equivariant, that is, $\psi({}^t h^{-1} A k_1^{-1}, hB\,{}^t k_2) = k_2 \psi(A,B) k_1^{-1}$ for all $(k_1,k_2) \in K_{\mathbb{C}}$, $h \in GL(m,\mathbb{C})$. This map induces the $\mathbb{C}$-algebra homomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \ni f \mapsto f \circ \psi \in \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})}.$$

As a summary we have

Lemma 7.7. — *There exists a $\mathbb{C}$-algebra isomorphism*

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \to \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})} = S(M_{p+q,m})^{GL(m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ is the geometric quotient $M_{p+q,m}//GL(m,\mathbb{C})$. In particular, $\overline{\mathcal{O}_m}$ is a normal variety. Here we consider $M_{p+q,m}$ as the contragredient space to $M_{m,p+q}$.

Proof. — It is injective since ψ is surjective. The classical invariant theory also says that every $GL(m,\mathbb{C})$-invariants on $M_{m,p+q}$ is generated by typical invariants of degree two, that is, this map ψ^* is surjective. □

For the Weil representation of the dual pair $(U(p,q),U(m)) \in Sp(2nm,\mathbb{R})$ and the unitary lowest weight module $L(\mathbf{1}_{U(m)})$, we have expected the same story. Take a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ consisting of diagonal matrices

$$\mathfrak{t} = \{H = \operatorname{diag}(a_1,\ldots,a_p,b_1,\ldots,b_q) \mid a_i, b_j \in \mathbb{C}\}.$$

This is also a Cartan subalgebra of $\mathfrak{g}$. We define $\varepsilon_i, \delta_j \in \mathfrak{t}^*$ by $\varepsilon_i(H) = a_i, \delta_j(H) = b_j$ for above $H \in \mathfrak{t}$. Then the set of positive non-compact roots is

$$\Delta_n^+ = \{\varepsilon_i - \delta_j \mid 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}.$$

Put

$$X_{\varepsilon_a-\delta_b} = \left(\begin{array}{c|c} 0 & E_{ab} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{gl}(p+q,\mathbb{C}) = \mathfrak{g}.$$

Then $X_{\varepsilon_a-\delta_b}$ is a non-compact root vector in $\mathfrak{p}^+$. From the embedding (3.8) and the Fock realization (4.5) of the Weil representation Ω, we conclude that

$$\Omega(-2X_{\varepsilon_a-\delta_b}) = \psi_{ab} = \sum_{j=1}^{m} x_{aj} y_{bj} \qquad (1 \leqslant a \leqslant p, 1 \leqslant b \leqslant q), \tag{7.8}$$

where $(x_{aj})_{1\leqslant a\leqslant p, 1\leqslant j\leqslant m} \in M_{p,m}$ and $(y_{bj})_{1\leqslant b\leqslant q, 1\leqslant j\leqslant m} \in M_{q,m}$. Note that these quadratics (7.8) generate the full invariants $S(M_{p+q,m})^{GL(m,\mathbb{C})}$. From this, we get

Lemma 7.8. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{m,p+q}]^{GL(m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{U(m)}) = \operatorname{Ann} \operatorname{gr} L(\mathbf{1}_{U(m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — The proof is similar to that of Lemma 7.2. □

Corollary 7.9
(1) *The associated variety of* $L(\mathbf{1}_{U(m)})$ *is* $\overline{\mathcal{O}_m}$.
(2) *As a* $\widetilde{K}$*-module,* $L(\mathbf{1}_{U(m)})$ *is isomorphic to* $\mathbb{C}[\overline{\mathcal{O}_m}] \otimes (\det{}^{m/2} \boxtimes \det{}^{-m/2})$.
(3) *Bernstein degree of* $L(\mathbf{1}_{U(m)})$ *coincides with* $\deg \overline{\mathcal{O}_m}$.

Proof. — The proof is similar to that of Corollary 7.3. For the K-type decomposition of $L(\mathbf{1}_{U(m)})$, see (6.16). □

Let us define the natural filtration of $L = L(\mathbf{1}_{U(m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is a constant polynomial. By (6.17), we know

$$
\begin{aligned}
\dim L_k &= \sum_{\substack{\lambda \in \mathcal{P}_m \\ |\lambda| \leqslant k}} \dim \tau_\lambda^{(p)} \dim \tau_\lambda^{(q)} \\
&= \sum_{\substack{\lambda, l(\lambda) \leqslant m \\ |\lambda| \leqslant k}} \frac{\prod_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j - i + j) \prod_{\substack{1 \leqslant i \leqslant m, \\ m+1 \leqslant j \leqslant p}} (\lambda_i - i + j) \prod_{m+1 \leqslant i < j \leqslant p} (j - i)}{\prod_{1 \leqslant i < j \leqslant p} (j - i)} \\
&\quad \times \frac{\prod_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j - i + j) \prod_{\substack{1 \leqslant i \leqslant m, \\ m+1 \leqslant j \leqslant q}} (\lambda_i - i + j) \prod_{m+1 \leqslant i < j \leqslant q} (j - i)}{\prod_{1 \leqslant i < j \leqslant q} (j - i)} \\
&= \frac{k^{m(m-1)/2 \times 2 + m(p+q-2m) + m}}{\prod_{i=1}^m (p-i)!(q-i)!} \\
&\quad \times \int_{\substack{0 \leqslant x_m \leqslant x_{m-1} \leqslant \cdots \leqslant x_1, \\ x_1 + \cdots + x_m \leqslant 1}} \prod_{1 \leqslant i < j \leqslant m} (x_i - x_j)^2 \prod_{i=1}^m x_i^{p+q-2m} dx_1 \ldots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k) \\
&= \frac{k^{m(p+q-m)}}{m! \prod_{i=1}^m (p-i)!(q-i)!} \int_{\substack{0 \leqslant x_i \\ x_1 + \cdots + x_m \leqslant 1}} \prod_{1 \leqslant i < j \leqslant m} |x_i - x_j|^2 \prod_{i=1}^m x_i^{p+q-2m} dx_1 \ldots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k)
\end{aligned}
$$

for sufficiently large k.

Theorem 7.10. — *Assume that* $m \leqslant \min(p,q) = \mathbb{R}\text{-rank}\, U(p,q)$, *and consider the reductive dual pair* $(U(p,q), U(m))$.
(1) *The unitarizable lowest weight module* $L(\mathbf{1}_{U(m)})$ *of* $U(p,q)^\sim$ *has the lowest weight* $m/2\mathbb{I}_{p,q} = m/2 \left(\sum_{i=1}^p \varepsilon_i - \sum_{j=1}^q \delta_j\right)$, *where* $\mathbb{I}_{p,q} = (1, \ldots, 1, -1, \ldots, -1)$. *Its associated cycle is given by* $\mathcal{AC}\left(L(\mathbf{1}_{U(m)})\right) = [\overline{\mathcal{O}_m}]$.

(2) *The Gelfand-Kirillov dimension and the Bernstein degree of* $L(\mathbf{1}_{U(m)})$ *is given by*

$$\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{U(m)}) &= \dim \overline{\mathcal{O}_m} = m(p+q-m), \\
\operatorname{Deg} L(\mathbf{1}_{U(m)}) &= \deg \overline{\mathcal{O}_m} = \prod_{j=1}^{m} \frac{(j-1)!\,(p+q-m-j)!}{(p-j)!\,(q-j)!}
\end{aligned}$$

Proof. — By the formula of $\dim L_k$ above, we have

$$\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{U(m)}) &= m(p+q-m) =: d, \\
\operatorname{Deg} L(\mathbf{1}_{U(m)}) &= \frac{d!}{m! \prod_{i=1}^{m} (p-i)!(q-i)!} I^2(p+q-2m, m).
\end{aligned}$$

Now apply Theorem 7.4. □

We show that the half-form bundle on $\mathcal{O}_m$ is related to some lowest weight representation $L(\sigma)$. We put

$$\lambda = \sum_{j=1}^{m} E_{jj} \in \mathcal{O}_m \subset M_{q,p} \cong \mathfrak{p}^-.$$

The group $K_{\mathbb{C}} = GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ acts on $\mathcal{O}_m$ by (7.6). The stabilizer $(K_{\mathbb{C}})_\lambda$ of λ in $K_{\mathbb{C}}$ is

$$(7.9) \quad (K_{\mathbb{C}})_\lambda = \left\{ (k_1, k_2) = \left(\begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix}, \begin{pmatrix} g_1 & * \\ 0 & g_3 \end{pmatrix} \right) \in K_{\mathbb{C}} \mid g_1 \in GL(m,\mathbb{C}) \right\}.$$

The determinant of the isotropy representation is

$$\det(\operatorname{Ad}|_{T_\lambda \mathcal{O}_m}) = (\det g_1)^{p-q} (\det k_1)^{-m} (\det k_2)^{m},$$

and that of the coisotropy representation

$$\det(\operatorname{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) : (K_{\mathbb{C}})_\lambda \ni (k_1,k_2) \mapsto (\det g_1)^{-(p-q)} (\det k_1)^{m} (\det k_2)^{-m} \in \mathbb{C}^\times,$$

with the notation (7.9). We denote the line bundle consisting of volume forms on $\mathcal{O}_m$ by Λ^{top}, and its square root by ξ. Let us clarify the meaning of the square root ξ of Λ^{top}. We denote the inverse image of the subgroup $K_{\mathbb{C}} \subset \mathcal{K}_{\mathbb{C}}$ in $\widetilde{\mathcal{K}_{\mathbb{C}}}$ by $\widetilde{K_{\mathbb{C}}}$. This is a double covering group of $K_{\mathbb{C}}$, which is not necessarily connected, with the covering map $\widetilde{K_{\mathbb{C}}} \longrightarrow K_{\mathbb{C}}$. We have an realization

$$\widetilde{K_{\mathbb{C}}} = \{(k,z) \in K_{\mathbb{C}} \times \mathbb{C}^\times \mid k = (k_1,k_2), (\det k_1)^m (\det k_2)^{-m} = z^2\}.$$

Through the natural projection, $\widetilde{K_{\mathbb{C}}}$ also acts on $\mathcal{O}_m$. We denote the isotropy subgroup at $\lambda \in \mathcal{O}_m$ by $(\widetilde{K_{\mathbb{C}}})_\lambda$. This is the inverse image of $(K_{\mathbb{C}})_\lambda$, that is,
(7.10)

$$(\widetilde{K_{\mathbb{C}}})_\lambda = \left\{ ((k_1,k_2),z) \in \widetilde{K_{\mathbb{C}}} \mid k_1 = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix}, k_2 = \begin{pmatrix} g_1 & * \\ 0 & g_3 \end{pmatrix}, g_1 \in GL(m,\mathbb{C}) \right\}.$$

In what follows, *we assume that* $p-q$ *is even* and calculate the K-types of $\Gamma(\mathcal{O}_m,\xi)$. There exists a well-defined character

$$\xi : (\widetilde{K_{\mathbb{C}}})_\lambda \ni (k_1,k_2,z) \mapsto (\det g_1)^{-(p-q)/2} z \in \mathbb{C}^\times,$$

with the notation (7.10). By the construction of ξ, the coisotropy representation is the square of ξ:

$$\det(\operatorname{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) = \xi^2.$$

This means that ξ determines the half-form bundle $\sqrt{\Lambda^{\text{top}}}$ on the orbit $\mathcal{O}_m$. As a $\widetilde{K_{\mathbb{C}}}$-module, the set of global sections $\Gamma(\mathcal{O}_m,\xi)$ is isomorphic to the induced module

$$\Gamma(\mathcal{O}_m,\xi) = \operatorname{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi.$$

We define a character $\chi_1 : \widetilde{K_{\mathbb{C}}} \longrightarrow \mathbb{C}^\times$ by $\chi_1(k_1,k_2,z) = z$, and $\xi' : (K_{\mathbb{C}})_\lambda \to \mathbb{C}^\times$ by $\xi'(k_1,k_2) = (\det g_1)^{-(p-q)/2}$ in the notation above. The character ξ' lifts up to a character of $(\widetilde{K_{\mathbb{C}}})_\lambda$ via projection map, and we denote it by the same letter ξ' again. Roughly speaking, χ_1 equals "$\det^{m/2} k_1 \det^{-m/2} k_2$". Then, ξ is the tensor product of ξ' with the restriction of χ_1 to the subgroup $(\widetilde{K_{\mathbb{C}}})_\lambda$. By the reciprocity law,

$$\operatorname{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi = \chi_1 \otimes \operatorname{Ind}_{(\widetilde{K_{\mathbb{C}}})_\lambda}^{\widetilde{K_{\mathbb{C}}}} \xi' = \chi_1 \otimes \operatorname{Ind}_{(K_{\mathbb{C}})_\lambda}^{K_{\mathbb{C}}} \xi'.$$

Lemma 7.11. — *We assume that* $m \leqslant \min(p,q)$ *and* $p-q \in 2\mathbb{Z}$ *as before, and that* $\max(p,q) \neq m$. *Then, as a* $K_{\mathbb{C}}$*-module, we have an isomorphism*

$$\operatorname{Ind}_{(K_{\mathbb{C}})_\lambda}^{K_{\mathbb{C}}} \xi' = \sum_{\lambda\in\mathcal{P}_m}^{\oplus} \tau_{\lambda+l\mathbb{I}_m} \boxtimes \tau_\lambda^*$$

with $l = (q-p)/2$. *Here we denote* $\mathbb{I}_m = (1,\ldots,1,0,\ldots,0)$, *in which* 1 *appears* m*-times.*

This shows that

$$\Gamma(\mathcal{O}_m,\xi) = \sum_{\lambda\in\mathcal{P}_m}^{\oplus} (\tau_{\lambda+l\mathbb{I}_m} \otimes \det^{m/2}) \boxtimes (\tau_\lambda \otimes \det^{m/2})^*.$$

On the other hand, by (6.12), the lowest weight module $L(\chi^{-1})$ also has the same $\widetilde{K}$-types. Indeed, the character χ of $G_2 = U(m)$ is $\det^{(p-q)/2} = \det^{-l}$ as is shown in (6.10). For $\nu = l\mathbb{I}_m$, we see that the multiplicity $m(\lambda,\mu;\nu)$ defined by (6.11) is

$$m(\lambda,\mu;\nu) = \begin{cases} 1 & \text{if } \lambda = \mu+\nu \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing above, we have

Proposition 7.12. — *Suppose* $p-q \in 2\mathbb{Z}$ *and* $0 \leqslant m < \min(p,q)$. *Let* $\chi = \det^{(p-q)/2}$ *be the character of* $G_2 = U(m)$ *given in* (6.10). *Then the lowest weight module* $L(\chi^{-1})$ *is isomorphic to* $\Gamma(\mathcal{O}_m,\xi)$ *as a* $\widetilde{K_{\mathbb{C}}}$*-module.*

7.5. The case $G = O^*(2p)$. — Let us consider the case $G = O^*(2p)$. In this case $K = U(p)$, $K_{\mathbb{C}} = GL(p,\mathbb{C})$, $\mathfrak{p}^- = \mathrm{Alt}\,(p,\mathbb{C})$. The action of $k \in K_{\mathbb{C}}$ on $A \in \mathfrak{p}^-$ is given by ${}^t k^{-1} A k^{-1}$.

Put $r = \mathbb{R}\text{-rank}\, O^*(2p) = [p/2]$, where $[x]$ is the Gauss symbol. We define a subset of $\mathrm{Alt}\,(p,\mathbb{C})$ by

$$\mathcal{O}_m = \{A \in \mathrm{Alt}\,(p,\mathbb{C}) \mid \mathrm{rank}\,(A) = 2m\}, \qquad (m = 0, 1, \ldots, r).$$

Since the rank of alternative matrices is always even, these $\{\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r\}$ form the set of all $K_{\mathbb{C}}$-orbits on $\mathrm{Alt}\,(p,\mathbb{C})$. The matrix $\sum_{j=1}^m (E_{m+j,j} - E_{j,m+j})$ is contained in $\mathcal{O}_m$. The dimension of the orbit is given by

$$\dim \mathcal{O}_m = 2pm - m(2m+1),$$

and the defining ideal I_m of $\overline{\mathcal{O}_m}$ is generated by

$$\{D_{IJ} \mid I, J \subset \{1, 2, \ldots, 2p\}, |I| = |J| = 2m+1\}.$$

Recall the dual pair $(O^*(2p), Sp(2m))$. Let $Sp(2m,\mathbb{C})$ be the complexification of $Sp(2m)$. We define the action of $K_{\mathbb{C}} \times Sp(2m,\mathbb{C})$ on $A \in M_{2m,p}$ by

$$(k,h)\cdot A = hAk^{-1}, \qquad \text{for } k \in GL(p,\mathbb{C}) = K_{\mathbb{C}},\ h \in Sp(2m,\mathbb{C}).$$

We define an unfolding map ψ by

$$\psi(A) = {}^t A J_m A \qquad \text{for } A \in M_{2m,p},$$

where J_m is defined as in (3.2). This is a polynomial map of degree two. Since

$$\psi\Big(\sum_{j\leqslant 2m} E_{jj}\Big) = \sum_{j=1}^{m} (E_{m+j,j} - E_{j,m+j}) \in \mathcal{O}_m,$$

we see that the image of ψ coincides with $\overline{\mathcal{O}_m}$. With the trivial action of $Sp(2m,\mathbb{C})$ on $\overline{\mathcal{O}_m}$, the map

$$\psi : M_{2m,p} \to \overline{\mathcal{O}_m}$$

is $K_{\mathbb{C}} \times Sp(2m,\mathbb{C})$-equivariant, that is, $\psi(hAk^{-1}) = {}^t k^{-1}\psi(A)k^{-1}$ for all $k \in K_{\mathbb{C}}$ and $h \in Sp(2m,\mathbb{C})$. This map induces a $\mathbb{C}$-algebra homomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \ni f \mapsto f \circ \psi \in \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})}.$$

Lemma 7.13. — *We have a $\mathbb{C}$-algebra isomorphism*

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \to \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})},$$

which means that $\overline{\mathcal{O}_m}$ is the geometric quotient $M_{2m,p}//Sp(2m,\mathbb{C})$. In particular, $\overline{\mathcal{O}_m}$ is a normal variety.

Proof. — The proof is similar to that of Lemma 7.1. □

Let us consider the Weil representation of the dual pair $(O^*(2p), Sp(2m)) \in Sp(2nm, \mathbb{R})$ $(n = 2p)$ and the unitary lowest weight module $L(\mathbf{1}_{Sp(2m)})$. Take a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ consisting of diagonal matrices

$$\mathfrak{t} = \{H = \operatorname{diag}(a_1, \dots, a_p, -a_1, \dots, -a_p) \mid a_i \in \mathbb{C}\}.$$

This is also a Cartan subalgebra of $\mathfrak{g}$. We define $\varepsilon_i \in \mathfrak{t}^*$ by $\varepsilon_i(H) = a_i$ for above $H \in \mathfrak{t}$. Then the set of positive non-compact roots is

$$\Delta_n^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leqslant i < j \leqslant p\}.$$

Put

$$X_{\varepsilon_a+\varepsilon_b} = \left(\begin{array}{c|c} 0 & E_{ab} - E_{ba} \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{o}(2p, \mathbb{C}) = \mathfrak{g}.$$

Note that the complexification $\mathfrak{o}(2p, \mathbb{C})$ is given in (7.1), in which we adopt rather non-standard symmetric bilinear form S_p. Then $X_{\varepsilon_a+\varepsilon_b}$ is a non-compact root vector in $\mathfrak{p}^+$. From the embedding (3.11) and the Fock realization (4.5) of Ω, we get

$$\Omega(-2X_{\varepsilon_a+\varepsilon_b}) = \psi_{ab} = \sum_{j=1}^{m}(x_{aj}y_{bj} - x_{bj}y_{aj}) \qquad (1 \leqslant a < b \leqslant p), \tag{7.11}$$

where $\left((x_{aj})_{1\leqslant a\leqslant p, 1\leqslant j\leqslant m}, (y_{bj})_{1\leqslant b\leqslant p, 1\leqslant j\leqslant m}\right) \in M_{p,2m} = M^*_{2m,p}$. These quadratics (7.11) generate the invariants $S(M_{p,2m})^{Sp(2m,\mathbb{C})}$.

Lemma 7.14. — *There are algebra isomorphisms*

$$\Omega(U(\mathfrak{p}^+)) \simeq \mathbb{C}[M_{2m,p}]^{Sp(2m,\mathbb{C})} \simeq \mathbb{C}[\overline{\mathcal{O}_m}] = \mathbb{C}[\mathfrak{p}^-]/I_m.$$

We have $\operatorname{Ann} L(\mathbf{1}_{Sp(2m)}) = \operatorname{Ann} \operatorname{gr} L(\mathbf{1}_{Sp(2m)}) = I_m$ *in* $U(\mathfrak{p}^+)$.

Proof. — The proof is similar to that of Lemma 7.2. □

Corollary 7.15

(1) *The associated variety of* $L(\mathbf{1}_{Sp(2m)})$ *is* $\overline{\mathcal{O}_m}$.
(2) *As a* K*-module,* $\mathbb{C}[\overline{\mathcal{O}_m}]$ *is isomorphic to* $L(\mathbf{1}_{Sp(2m)})$.
(3) *The Bernstein degree of* $L(\mathbf{1}_{Sp(2m)})$ *coincides with* $\deg \overline{\mathcal{O}_m}$.

Proof. — The proof is similar to that of Corollary 7.3. For the K-type decomposition of $L(\mathbf{1}_{Sp(2m)})$, see (6.20). □

Let us define the natural filtration of $L = L(\mathbf{1}_{Sp(2m)})$ by $L_k = U_k(\mathfrak{p}^+)\mathbf{1}$, where $\mathbf{1}$ is the constant polynomial. By (6.21), we know

$$\dim L_k = \sum_{\substack{\lambda\in\mathcal{P}_m \\ |\lambda|\leqslant k}} \dim \tau^{(p)}_{\lambda^\#}$$

$$
\begin{aligned}
&= \sum_{\substack{|\lambda|\leqslant k\\ l(\lambda)\leqslant m}} \frac{\prod_{1\leqslant i<j\leqslant m} (\lambda_i-\lambda_j-2i+2j)^2((\lambda_i-\lambda_j-2i+2j)^2-1)}{\prod_{1\leqslant i<j\leqslant p}(j-i)} \\
&\quad \times \prod_{\substack{1\leqslant i\leqslant m\\ 2m+1\leqslant j\leqslant p}} (\lambda_i-2i+j)(\lambda_i-2i+1+j) \prod_{2m+1\leqslant i<j\leqslant p}(j-i) \\
&= \frac{k^{m(m-1)/2\times 4+2m(p-2m)+m}}{\prod_{i=1}^{2m}(p-i)!} \\
&\quad \times \int_{\substack{0\leqslant x_m\leqslant\cdots\leqslant x_1\\ x_1+\cdots+x_m\leqslant 1}} \prod_{1\leqslant i<j\leqslant m}(x_i-x_j)^4\prod_{i=1}^m x_i^{2(p-2m)}dx_1\cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k) \\
&= \frac{k^{m(2p-2m-1)}}{m!\prod_{i=1}^{2m}(p-i)!} \int_{\substack{0\leqslant x_i\\ x_1+\cdots+x_m\leqslant 1}} \prod_{1\leqslant i<j\leqslant m}|x_i-x_j|^4\prod_{i=1}^m x_i^{2(p-2m)}dx_1\cdots dx_m \\
&\qquad\qquad + (\text{lower order terms of } k)
\end{aligned}
$$

for sufficiently large k.

Theorem 7.16. — *Assume that $m \leqslant [p/2] = \mathbb{R}\text{-rank}\, O^*(2p)$, and consider the reductive dual pair $(O^*(2p), Sp(2m))$.*
(1) The unitarizable lowest weight module $L(\mathbf{1}_{Sp(2m)})$ of $O^*(2p)^\sim$ has the lowest weight $m(1,\ldots,1) = m\sum_{i=1}^p \varepsilon_i$, and its associated cycle is given by $\mathcal{AC}\,(L(\mathbf{1}_{Sp(2m)})) = [\overline{\mathcal{O}_m}]$.
(2) The Gelfand-Kirillov dimension and the Bernstein degree of $L(\mathbf{1}_{Sp(2m)})$ is given by

$$
\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{Sp(2m)}) &= \dim\overline{\mathcal{O}_m} = m(2p-2m-1), \\
\operatorname{Deg} L(\mathbf{1}_{Sp(2m)}) &= \deg\overline{\mathcal{O}_m} = (2m-1)!!\prod_{j=1}^m \frac{(2(j-1))!\,(2(p-m-j))!}{(p-j)!\,(p-m-j)!}.
\end{aligned}
$$

Proof. — By the formula of $\dim L_k$, we get

$$
\begin{aligned}
\operatorname{Dim} L(\mathbf{1}_{Sp(2m)}) &= m(2p-2m-1) =: d, \\
\operatorname{Deg} L(\mathbf{1}_{Sp(2m)}) &= \frac{d!}{m!\prod_{i=1}^{2m}(p-i)!} I^4(2p-4m, m).
\end{aligned}
$$

Apply Theorem 7.4 to get the desired formula. □

We have a relation between the half-form bundle and $L(\mathbf{1}_{Sp(2m)})$ similar to that in Proposition 7.6. We define $\lambda = \begin{pmatrix} J_m & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{Alt}(p,\mathbb{C})$ with $J_m = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}$.

The isotropy subgroup of λ in $K_{\mathbb{C}}$ is

$$(K_{\mathbb{C}})_\lambda = \left\{ k = \begin{pmatrix} g_1 & 0 \\ * & g_2 \end{pmatrix} \mid g_1 \in Sp(2m, \mathbb{C}), g_2 \in GL(p-2m, \mathbb{C}) \right\}.$$

Therefore the determinant of the coisotropy representation becomes

$$\det(\mathrm{Ad}^*|_{T^*_\lambda \mathcal{O}_m}) = (\det k)^{2m},$$

and we define its square root by

$$\xi : (K_{\mathbb{C}})_\lambda \ni k \mapsto (\det k)^m \in \mathbb{C}^\times.$$

Proposition 7.17. — *For $m < p/2$, the set of global section of the half-form bundle $\Gamma(\mathcal{O}_m, \xi)$ is isomorphic to $L(\mathbf{1}_{Sp(2m)})$ as a $\widetilde{K}$-module.*

7.6. A unified formula. — Consider the reductive dual pair $(G_1, G_2) \subset \mathcal{G} = Sp(2nm, \mathbb{R})$ of compact type. We put $G = G_1$, which is a non-compact companion. We use the notation in § 3 freely in this subsection. In particular, $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is a division algebra over $\mathbb{R}$, and $n = 1/2 \dim_{\mathbb{R}} V_1, m = \dim_D V_2$. Put $r = \mathbb{R}\text{-rank}\, G$, and $\alpha = \dim_{\mathbb{R}} D = 1, 2, 4$.

Summarizing the above three explicit calculations, we have a unified expression of the Gelfand-Kirillov dimension and the Bernstein degree of the unitary lowest weight module $L(\mathbf{1}_{G_2})$.

Theorem 7.18. — *Assume that the dual pair (G_1, G_2) is in the stable range, i.e., $m \leqslant r$. We denote by $L(\mathbf{1}_{G_2})$ the irreducible lowest weight module of $\widetilde{G_1}$ which is the (twisted) theta lift of the trivial representation of the compact companion G_2. Then the associated cycle $\mathcal{AC}\, L(\mathbf{1}_{G_2})$ is the closure of the m-th $K_{\mathbb{C}}$-orbit $\overline{\mathcal{O}_m}$ in $\mathfrak{p}^-$. Moreover, we have*

$$\operatorname{Dim} L(\mathbf{1}_{G_2}) = m\Big(n + 1 - \frac{\alpha}{2}(m+1)\Big) = \dim \overline{\mathcal{O}_m} =: d,$$

and

$$\operatorname{Deg} L(\mathbf{1}_{G_2}) = F^{-1} \frac{d!}{m!} I^\alpha(n - \alpha m, m) = \deg \overline{\mathcal{O}_m},$$

where $I^\alpha(s, m)$ is the integral (7.4), and the integer F is given by

$$F = \begin{cases} \prod_{j=1}^m (2(n-j))!! = 2^{m-d} \prod_{j=1}^m (n-j)! & \text{Case } (Sp, O), \\ \prod_{j=1}^m (p-j)!(q-j)! & \text{Case } (U, U), \\ \prod_{j=1}^{2m} (p-j)! & \text{Case } (O^*, Sp). \end{cases}$$

Remark 7.19. — If G/K is of tube type, we have

$$F = \begin{cases} \prod_{j=1}^m (2(n-j))!! = 2^{m-d} \prod_{j=1}^m (n-j)! & \text{Case } (Sp, O), \\ \prod_{j=1}^m \{(n/\alpha - j)!\}^2 & \text{Case } (U, U), \\ \prod_{j=1}^{2m} (n/\alpha - j)! & \text{Case } (O^*, Sp). \end{cases}$$

8. Multiplicity free action and Poincaré series

In this section, we develop a general theory on Poincaré series of graded modules which arise from multiplicity free action of reductive groups. All the groups in this section are complex algebraic groups and irreducible representations are finite dimensional ones.

8.1. Poincaré series of covariants. — Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be complex reductive groups and X a vector space on which $\mathcal{G}_1$ and $\mathcal{G}_2$ jointly act linearly. We assume that the action of $\mathcal{G}_1 \times \mathcal{G}_2$ is multiplicity free. This means that the polynomial ring $\mathbb{C}[X] = \Gamma(X)$ decomposes, as a $\mathcal{G}_1 \times \mathcal{G}_2$-module, into irreducible representations with multiplicity one. Namely, there exists a subset $R_X(\mathcal{G}_1 \times \mathcal{G}_2) \subset \mathrm{Irr}(\mathcal{G}_1 \times \mathcal{G}_2)$ such that

$$\Gamma(X) = \sum_{\pi_1 \boxtimes \pi_2 \in R_X(\mathcal{G}_1 \times \mathcal{G}_2)}^{\oplus} \pi_1 \boxtimes \pi_2.$$

We assume further, in the decomposition, the correspondence $\pi_1 \leftrightarrow \pi_2$ is one to one. Hence, π_1 determines π_2 and vice versa.

We choose suitable positive systems of roots for $\mathcal{G}_1$ and $\mathcal{G}_2$, and fix them in what follows. Let λ be the highest weight of $\pi_1 = \pi_1(\lambda)$ with respect to the positive system we chose. Then we will denote the corresponding highest weight of π_2 by $\varphi(\lambda)$ so that $\pi_2 = \pi_2(\varphi(\lambda))$. Let Λ^+ be a lattice semigroup of the highest weights of $\pi_1 \in \mathrm{Irr}(\mathcal{G}_1)$ which occur in $\Gamma(X)$. Then we can write the decomposition as

$$\Gamma(X) = \sum_{\lambda \in \Lambda^+}^{\oplus} \pi_1(\lambda) \boxtimes \pi_2(\varphi(\lambda)).$$

Note that the correspondence $\Lambda^+ \ni \lambda \mapsto \varphi(\lambda)$ is a semigroup morphism from Λ^+ into the dominant weight lattice of $\mathcal{G}_2$, i.e., $\varphi(\lambda + \eta) = \varphi(\lambda) + \varphi(\eta)$.

We consider a (reductive) spherical subgroup H of $\mathcal{G}_1$. Since H is spherical, for any irreducible representation (π_1, V) of $\mathcal{G}_1$, V has at most one-dimensional invariants under the action of H : $\dim V^H \leqslant 1$. We put

$$\Lambda^+(H) = \{\lambda \in \Lambda^+ \mid \dim V_\lambda^H = 1\} \subset \Lambda^+,$$

where V_λ is a representation space of $\pi_1(\lambda)$. Let Λ (respectively $\Lambda(H)$) be the lattice generated by Λ^+ (respectively $\Lambda^+(H)$). Note that it is not necessary to hold that $\Lambda^+(H) = \Lambda^+ \cap \Lambda(H)$. Since Λ^+ is a free abelian semigroup generated by finite elements (see the argument in [**26**, § 2]), we can extend the correspondence $\varphi(\cdot)$ to Λ as a group morphism.

The set of H-invariants of $\Gamma(X)$ is denoted by $\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H$. Then it decomposes multiplicity freely as a $\mathcal{G}_2$-module

$$\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H \simeq \sum_{\lambda \in \Lambda^+(H)}^{\oplus} \pi_2(\varphi(\lambda)).$$

Since $\Gamma(X; \mathbf{1}_H)$ is a finitely generated graded Noetherian algebra, it has a Poincaré series $P(\mathbf{1}_H; t)$, where t is an indeterminate. More precisely, we define $P(\mathbf{1}_H; t)$ in the following way. If the representation $\pi_1(\lambda) \boxtimes \pi_2(\varphi(\lambda))$ occurs in the k-th degree of the polynomial ring $\Gamma(X) = \mathbb{C}[X]$, we write $|\lambda| = k$. This degree map is obviously additive $|\lambda + \eta| = |\lambda| + |\eta|$. We put

$$P(\mathbf{1}_H; t) = \sum_{\lambda \in \Lambda^+(H)} \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|} = \operatorname{trace}_{\Gamma(X;\mathbf{1}_H)}(t^E), \tag{8.1}$$

where E denotes the degree operator. Let $\{a_1, \dots, a_d\} \subset \mathbb{C}[X]^H$ be a set of homogeneous and algebraically independent elements such that $\mathbb{C}[X]^H$ is integral over a subalgebra $\mathbb{C}[a_1, \dots, a_d]$ generated by $a_1, \dots a_d$. Put $h_i = \deg a_i$. Then there exists a polynomial $Q(t)$ such that

$$P(\mathbf{1}_H; t) = \frac{Q(t)}{\prod_{i=1}^{d}(1 - t^{h_i})}, \tag{8.2}$$

and $Q(1)$ gives a positive integer (see, e.g., [**45**, Theorem 2.5.6]). The integer $Q(1)$ is independent of the choice of $\{a_1, \dots, a_d\}$ above. We call it the *degree* of $\mathbb{C}[X]^H$ and denote $Q(1) = \operatorname{Deg}\Gamma(X; \mathbf{1}_H)$. The number d coincides with the transcendental degree of the quotient field of $\mathbb{C}[X]^H$, and we denote it by $d = \operatorname{Dim}\Gamma(X; \mathbf{1}_H)$, which is the dimension of the geometric quotient $X//H$.

More generally, for any $\sigma(\mu) \in \operatorname{Irr}(H)$ with highest weight μ, we denote $\sigma(\mu)$-covariants of $\Gamma(X)$ by $\Gamma(X; \sigma(\mu))$, i.e.,

$$\Gamma(X; \sigma(\mu)) := (\sigma(\mu)^* \otimes \mathbb{C}[X])^H.$$

The space of covariants $\Gamma(X; \sigma(\mu))$ is a finitely generated $\Gamma(X; \mathbf{1}_H) = \mathbb{C}[X]^H$-module by polynomial multiplication against the second factor (see, e.g., [**39**]). Note that it carries also a representation of $\mathcal{G}_2$ on the second factor.

If we decompose the restriction of $\pi_1(\lambda)$ to H as

$$\pi_1(\lambda)\big|_H \simeq \sum_{\mu}^{\oplus} m(\lambda, \mu)\, \sigma(\mu)$$

with multiplicity $m(\lambda, \mu)$, we have the decomposition

$$\Gamma(X; \sigma(\mu)) \simeq \sum_{\lambda \in \Lambda^+}^{\oplus} m(\lambda, \mu)\, \pi_2(\varphi(\lambda)),$$

as a $\mathcal{G}_2$-module. We define the Poincaré series $P(\sigma(\mu); t)$ of $\Gamma(X; \sigma(\mu))$ by

$$P(\sigma(\mu); t) = \sum_{\lambda \in \Lambda^+} m(\lambda, \mu) \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|}.$$

Since $\Gamma(X; \sigma(\mu))$ is a finitely generated graded module over $\Gamma(X; \mathbf{1}_H)$, its Poincaré series has rational expression as

$$P(\sigma(\mu); t) = \frac{Q(\sigma(\mu); t)}{\prod_{i=1}^{d}(1 - t^{h_i})}$$

with the same d and $h_1, \dots, h_d$ as in (8.2). Here, $Q(\sigma(\mu); t)$ is a polynomial in t and its value at $t = 1$ gives a non-negative integer, which is independent of the choice of $a_1, \dots, a_d$ again. We call it the *degree* of covariants $\Gamma(X; \sigma(\mu))$ and denote $\operatorname{Deg}\Gamma(X; \sigma(\mu)) = Q(\sigma(\mu); 1)$.

The purpose of this subsection is to relate the dimension $\operatorname{Dim}\Gamma(X; \sigma(\mu))$ and the degree $\operatorname{Deg}\Gamma(X; \sigma(\mu))$ to those of invariants.

For "sufficiently large" λ, the multiplicity $m(\lambda, \mu)$ depends only on the coset $[\lambda] = \lambda + \Lambda(H) \in \Lambda^+/\Lambda(H)$. Here $\Lambda^+/\Lambda(H)$ is an abbreviation for $(\Lambda^+ + \Lambda(H))/\Lambda(H)$. To be precise, we have

***Lemma 8.1* (Sato).** — *For any $\lambda \in \Lambda^+$ and $\sigma(\mu) \in \operatorname{Irr}(H)$, there exists $\eta^M \in \Lambda^+(H)$ which satisfies*

$$m(\lambda + \eta^M, \mu) = m(\lambda + \eta^M + \eta, \mu) \qquad (\forall \eta \in \Lambda^+(H)).$$

The integer $m(\lambda + \eta^M, \mu)$ does not depend on the choice of η^M. We denote this integer by $m([\lambda], \mu)$ and call it the stable branching coefficient *after F. Sato.*

Proof. — Our setting here fits into Sato's assumption **[42]**. □

Let Δ_2^+ be a positive root system of $\mathcal{G}_2$. We define a subset $\Delta_2^+(H) \subset \Delta_2^+$ by

$$\Delta_2^+(H) = \{\alpha \in \Delta_2^+ \mid \langle \varphi(\eta), \alpha \rangle = 0 \quad (\forall \eta \in \Lambda(H))\}, \tag{8.3}$$

where $\langle , \rangle$ denotes the inner product which is invariant under the Weyl group action. For $\lambda \in \Lambda^+$, we put

$$r(\lambda) = r([\lambda]) = \prod_{\alpha \in \Delta_2^+(H)} \frac{\langle \varphi(\lambda) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}, \tag{8.4}$$

where ρ is the half sum of positive roots in Δ_2^+. Note that the right hand side of (8.4) does not depend on individual λ, but depends only on the coset $[\lambda] \in \Lambda^+/\Lambda(H)$. By definition, $r([\lambda])$ is a positive quantity.

Proposition 8.2. — *We assume that, for any $\lambda \in \Lambda^+$, there exists $\lambda^\flat \in \Lambda^+$ such that*

$$(\lambda + \Lambda(H)) \cap \Lambda^+ = \lambda^\flat + \Lambda^+(H). \tag{8.5}$$

Then, for any $\mu \in \Lambda^+(H)$, we have

$$\lim_{t \uparrow 1} \frac{P(\sigma(\mu); t)}{P(\mathbf{1}_H; t)} = \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda]).$$

Remark 8.3. — Condition (8.5) determines $\lambda^\flat \in \Lambda^+$ uniquely if it exists. Hence, $\lambda^\flat$ depends only on the coset $[\lambda] = \lambda + \Lambda(H)$. If we set $S = \{\lambda^\flat \mid \lambda \in \Lambda^+\}$, this amounts to

$$\Lambda^+ = S \oplus \Lambda^+(H);$$

or, equivalently to say, Λ^+ is a free $\Lambda^+(H)$-module over the base set S. From this observation, the map

$$(\cdot)^\flat : \Lambda^+/\Lambda(H) \ni [\lambda] \mapsto [\lambda]^\flat := \lambda^\flat \in S \subset \Lambda^+$$

is a well-defined section of the projection map $\Lambda^+ \to \Lambda^+/\Lambda(H)$.

Corollary 8.4. — *Under the same assumption, we have*

$$\operatorname{Deg} \Gamma(X; \sigma(\mu)) \quad = \quad \operatorname{Deg} \Gamma(X; \mathbf{1}_H) \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda]).$$

We need a technical lemma to prove the proposition.

Lemma 8.5. — *Take arbitrary $\lambda \in \Lambda^+$.*
(1) *There exists $\eta_\lambda \in \Lambda^+(H)$ such that*

$$\dim \pi_2(\varphi(\lambda + \eta)) \leqslant r(\lambda) \cdot \dim \pi_2(\varphi(\eta_\lambda + \eta)) \qquad (\forall \eta \in \Lambda^+(H)).$$

(2) *We have*

$$\dim \pi_2(\varphi(\lambda + \eta)) \geqslant r(\lambda) \cdot \dim \pi_2(\varphi(\eta)) \qquad (\forall \eta \in \Lambda^+(H)).$$

Proof. — By Weyl's dimension formula, we have

$$\begin{aligned} \dim \pi_2(\varphi(\lambda + \eta)) &= \prod_{\alpha \in \Delta_2^+} \frac{\langle \varphi(\lambda + \eta) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\ &= r(\lambda) \prod_{\alpha \notin \Delta_2^+(H)} \frac{\langle \varphi(\lambda + \eta) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}. \end{aligned} \tag{8.6}$$

To prove (1), it is enough to take $\eta_\lambda \in \Lambda^+(H)$ so that $\langle \varphi(\lambda), \alpha \rangle \leqslant \langle \varphi(\eta_\lambda), \alpha \rangle$ holds for any $\alpha \notin \Delta_2^+(H)$. This is certainly possible. Since $\varphi(\cdot)$ is a group homomorphism, (8.6) becomes

$$r(\lambda) \prod_{\alpha \notin \Delta_2^+(H)} \frac{\langle \varphi(\lambda + \eta) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \leqslant r(\lambda) \prod_{\alpha \notin \Delta_2^+(H)} \frac{\langle \varphi(\eta_\lambda + \eta) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

Now we are to prove (2). Since $\langle \varphi(\lambda), \alpha \rangle \geqslant 0$, we get

$$\dim \pi_2(\varphi(\lambda + \eta)) \geqslant r(\lambda) \prod_{\alpha \notin \Delta_2^+(H)} \frac{\langle \varphi(\eta) + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = r(\lambda) \dim \pi_2(\varphi(\eta)).$$

This proves (2). □

Proof of Proposition 8.2. — Let us take arbitrary $0 < t < 1$.

First we note that $m(\lambda, \mu) \leqslant m([\lambda], \mu)$ for any λ (see [**42**, Corollary 1.2]). Therefore, we have

$$\begin{aligned}
P(\sigma(\mu); t) &= \sum_{\lambda \in \Lambda^+} m(\lambda, \mu) \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|} \\
&\leqslant \sum_{\lambda \in \Lambda^+} m([\lambda], \mu) \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|} \\
&= \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) \sum_{\eta \in \Lambda^+(H)} \dim \pi_2(\varphi([\lambda]^\flat + \eta))\, t^{|[\lambda]^\flat + \eta|}.
\end{aligned} \tag{8.7}$$

For $[\lambda]^\flat$, take $\eta_{[\lambda]^\flat} \in \Lambda^+(H)$ as in Lemma 8.5 (1), and recall the definition of $P(\mathbf{1}_H; t)$ from (8.1). Then we can calculate the above formula as

$$\begin{aligned}
(8.7) &\leqslant \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat| - |\eta_{[\lambda]^\flat}|} \times \sum_{\eta \in \Lambda^+(H)} \dim \pi_2(\varphi(\eta_{[\lambda]^\flat} + \eta))\, t^{|\eta_{[\lambda]^\flat} + \eta|} \\
&\leqslant \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat| - |\eta_{[\lambda]^\flat}|}\, P(\mathbf{1}_H; t).
\end{aligned}$$

Note that, for fixed μ, there are only a finite number of cosets $[\lambda]$ for which $m([\lambda], \mu)$ does not vanish ([**42**, Corollary 2.5 (iii)]).

On the other hand, if we choose $\eta^M \in \Lambda^+(H)$ large enough, we have that $m(\lambda + \eta^M, \mu) = m([\lambda], \mu)$ by the definition of the stable branching coefficient. We can take η^M uniformly for $\lambda \in \Lambda^+$, since there are only a finite number of $[\lambda]$'s which count. So, by Lemma 8.5 (2), we get the following inequality:

$$\begin{aligned}
P(\sigma(\mu); t) &= \sum_{\lambda \in \Lambda^+} m(\lambda, \mu) \dim \pi_2(\varphi(\lambda))\, t^{|\lambda|} \\
&\geqslant \sum_{\lambda \in \Lambda^+} m(\lambda + \eta^M, \mu) \dim \pi_2(\varphi(\lambda + \eta^M))\, t^{|\lambda + \eta^M|} \\
&= \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) \sum_{\eta \in \Lambda^+(H)} \dim \pi_2(\varphi([\lambda]^\flat + \eta^M + \eta))\, t^{|[\lambda]^\flat + \eta^M + \eta|} \\
&\geqslant \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat + \eta^M|} \sum_{\eta \in \Lambda^+(H)} \dim \pi_2(\varphi(\eta))\, t^{|\eta|} \\
&= \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat + \eta^M|}\, P(\mathbf{1}_H; t).
\end{aligned}$$

From these inequalities, we have

$$\begin{aligned}
&\sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat| - |\eta_{[\lambda]^\flat}|} \\
&\qquad \geqslant \frac{P(\sigma(\mu); t)}{P(\mathbf{1}_H; t)} \geqslant \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda])\, t^{|[\lambda]^\flat + \eta^M|}.
\end{aligned}$$

If we take the limit $t \uparrow 1$, we get

$$\lim_{t \uparrow 1} \frac{P(\sigma(\mu); t)}{P(\mathbf{1}_H; t)} = \sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda]). \qquad \square$$

8.2. Examples of multiplicity free actions and Poincaré series. — We keep the notation in the former subsection §8.1. So $\mathcal{G}_1 \times \mathcal{G}_2$ acts on X multiplicity freely, and H is a spherical subgroup of $\mathcal{G}_1$.

In many cases, we have an identity

$$\sum_{[\lambda] \in \Lambda^+/\Lambda(H)} m([\lambda], \mu) r([\lambda]) = \dim \sigma(\mu). \tag{8.8}$$

It will prove that

$$\operatorname{Deg} \Gamma(X; \sigma(\mu)) \;=\; \dim \sigma(\mu) \cdot \operatorname{Deg} \mathbb{C}[X]^H, \tag{8.9}$$

under the technical condition (8.5). However, at the same time, there also exist exceptions to (8.8). In this subsection, we will give three examples in which (8.8) and hence (8.9) hold. We need these examples later on.

Let B be a Borel subgroup of $\mathcal{G}_1$ such that $HB \subset \mathcal{G}_1$ is dense. Such a Borel subgroup exists since H is spherical. Define a parabolic subgroup $P \subset \mathcal{G}_1$ as

$$P = \{g \in \mathcal{G}_1 \mid HBg = HB\} \supset B.$$

Then $L = P \cap H$ is a reductive subgroup which contains the derived group of a Levi subgroup of P. The identity component of $B \cap H$ is a Borel subgroup of the identity component of L. Let $B = TU$ be a Levi decomposition with T being a Cartan subgroup of $\mathcal{G}_1$. We will denote by $\tau_L(\lambda)$ an irreducible representation of L with highest weight $e^\lambda|_{H \cap B}$.

Let Φ^+ be the semigroup lattice of dominant weights of $\mathcal{G}_1$ and Φ the weight lattice. We define

$$\Phi^+(H) = \{\lambda \in \Phi^+ \mid \dim V_\lambda^H = 1\},$$

and denote by $\Phi(H)$ a lattice generated by $\Phi^+(H)$ in Φ. It is known that

$$\Phi(H) = \{\lambda \in \Phi \mid e^\lambda|_{H \cap T} \equiv 1\}.$$

To get the identity (8.8), we use *Sato's formula* ([**42**, Corollary 2.5])

$$\sum_{[\lambda] \in \Phi^+/\Phi(H)} m([\lambda], \mu) \dim \tau_L(\lambda) = \dim \sigma(\mu). \tag{8.10}$$

However, there are two obstructions to get identity (8.8) by using Sato's formula (8.10).

One obstruction is in the range of the summation. The representatives $[\lambda]$ must move all the coset of dominant weight lattice in Sato's formula. However, in general, $\Lambda^+/\Lambda(H)$ is a strict subset of $\Phi^+/\Phi(H)$. This obstruction is serious.

The other obstruction is the difference between $r(\lambda)$ and $\dim \tau_L(\lambda)$. However, in most cases, they are identical. We do not know an exception up to now.

We summarize here desired conditions which enables us to use Sato's formula.

(S1) Coincidence of coset spaces: $\Lambda^+/\Lambda(H) = \Phi^+/\Phi(H)$.

(S2) Coincidence of dimension functions: $r(\lambda) = \dim \tau_L(\lambda) \quad (\forall\lambda \in \Lambda^+)$.

(S3) Existence of good representatives: for any $\lambda \in \Lambda^+$, there exists $\lambda^\flat \in \Lambda^+$ such that

$$(\lambda + \Lambda(H)) \cap \Lambda^+ = \lambda^\flat + \Lambda^+(H). \tag{8.11}$$

Condition (S3) is equivalent to the following condition (S3′) (see Remark 8.3).

(S3′) There is a subset $S \subset \Lambda^+$ which satisfies $\Lambda^+ = S \oplus \Lambda^+(H)$.

If once we check the above conditions, we conclude the formula (8.9).

Theorem 8.6. — *If the above three conditions* (S1)–(S3) *hold, we have*

$$\operatorname{Deg} \Gamma(X;\sigma(\mu)) \quad = \quad \dim \sigma(\mu) \cdot \operatorname{Deg} \mathbb{C}[X]^H, \tag{8.12}$$

for any $\sigma(\mu) \in \operatorname{Irr}(H)$.

In the following, we examine the above three conditions (S1)–(S3) in each case.

Example A. — Let $\mathcal{G}_1 = GL(m,\mathbb{C}), \mathcal{G}_2 = GL(n,\mathbb{C})$ and assume that $m \leqslant n$. This assumption is essential in the following. We take $H = SO(m,\mathbb{C})$. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. We put $X = M_{m,n}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^n)^*$ and let $\mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$M_{m,n}(\mathbb{C}) \ni A \to {}^t g_1^{-1} A g_2^{-1}, \qquad (g_i \in \mathcal{G}_i, i = 1,2).$$

The decomposition of $\mathbb{C}[X]$ is given by

$$\mathbb{C}[X] \simeq \sum_{\lambda \in \mathcal{P}_m}^{\oplus} \tau_{GL_m}(\lambda) \boxtimes \tau_{GL_n}(\lambda),$$

where $\mathcal{P}_m$ denotes the set of partitions with length at most m. Therefore the action of $\mathcal{G}_1 \times \mathcal{G}_2$ is multiplicity free, and we have

$$\Lambda^+ = \mathcal{P}_m, \qquad \Lambda = \Phi \simeq \mathbb{Z}^m.$$

Since we can naturally identify $\lambda \in \mathcal{P}_m$ with $\varphi(\lambda) \in \mathcal{P}_n$, we will denote $\varphi(\lambda)$ simply by the same letter λ. If we denote by $\mathcal{P}_m^{even}$ the set of even partitions, then it is well-known that

$$\Lambda^+(H) = \mathcal{P}_m^{even}, \qquad \Lambda(H) = \Phi(H) \simeq (2\mathbb{Z})^m.$$

In this case, the coset space $\Lambda^+/\Lambda(H) = \Lambda/\Lambda(H) \simeq (\mathbb{Z}_2)^m$ is a finite set, and it coincides with $\Phi^+/\Phi(H)$.

We have $\Delta_2^+(H) = \{\varepsilon_i - \varepsilon_j \mid m < i < j \leqslant n\}$ in the standard notation. Using this, one can conclude easily that $r(\lambda) = 1$ for any $\lambda \in \mathcal{P}_m$. On the other hand, let B be a Borel subgroup consisting of upper triangular matrices. Then $HB \subset \mathcal{G}_1$ is dense and $P = B$. Since $L = H \cap P = H \cap B \simeq (\mathbb{Z}_2)^{m-1}$, L is a finite abelian group. Hence we have $r(\lambda) = \dim \tau_L(\lambda) = 1$.

Next we verify the condition (S3), i.e., (8.11). Put

$$\varpi_i = \sum_{k=1}^{i} \varepsilon_k = (1, \dots, 1, 0, \dots, 0) \qquad (1 \leqslant i \leqslant m),$$

the fundamental weights for $GL(m, \mathbb{C})$. Note that Λ^+ has a basis $\{\varpi_k \mid 1 \leqslant k \leqslant m\}$ and $\Lambda^+(H)$ has a basis $\{2\varpi_k \mid 1 \leqslant k \leqslant m\}$. Any $\lambda \in \Lambda^+$ can be expressed as

$$\lambda = \sum_{k=1}^{m} n_k \varpi_k \qquad (n_k \in \mathbb{Z}_{\geqslant 0}).$$

We put

$${n_k}^\flat = \begin{cases} 0 & \text{if } n_k \in 2\mathbb{Z}, \\ 1 & \text{otherwise} \end{cases}$$

Under the notation above, we define

$$\lambda^\flat = \sum_{k=1}^{m} {n_k}^\flat \varpi_k.$$

It is a simple task to verify that $\lambda^\flat$ satisfies the condition (8.11), and we have

$$\Lambda^+ = \Lambda^+(H) \oplus \left\{ \sum_{k=1}^{m} n_k \varpi_k \mid n_k = 0, 1 \right\}.$$

From observations above, we conclude Theorem 8.6 holds in this case.

Example B. — Let $\mathcal{G}_1 = GL(m, \mathbb{C}) \times GL(m, \mathbb{C})$, $\mathcal{G}_2 = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ and assume that $m \leqslant \min(p, q)$. We take $H = \Delta GL(m, \mathbb{C}) \simeq GL(m, \mathbb{C})$ the diagonal subgroup. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. Let $n = p + q$. We put

$$X = M_{m,n}(\mathbb{C}) = M_{m,p}(\mathbb{C}) \oplus M_{m,q}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^p)^* \oplus (\mathbb{C}^m \otimes \mathbb{C}^q),$$

and let $((x_1, x_2), (y_1, y_2)) \in \mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$M_{m,n}(\mathbb{C}) \ni (A, B) \to ({}^t x_1^{-1} A y_1^{-1}, x_2 B \, {}^t y_2) \qquad (A \in M_{m,p}(\mathbb{C}), B \in M_{m,q}(\mathbb{C})).$$

Then the action is multiplicity free, and the decomposition of $\mathbb{C}[X]$ is given by

$$\mathbb{C}[X] \simeq \sum_{(\mu,\nu) \in \mathcal{P}_m \times \mathcal{P}_m}^{\oplus} (\tau_{GL_m}(\mu) \boxtimes \tau_{GL_m}(\nu)^*) \boxtimes \left(\tau_{GL_p}(\mu) \boxtimes \tau_{GL_q}(\nu)^*\right).$$

Therefore we have

$$\Lambda^+ = \mathcal{P}_m \times \mathcal{P}_m, \qquad \Lambda = \Phi \simeq \mathbb{Z}^m \times \mathbb{Z}^m.$$

Here, to avoid the confusion, we have twisted the second factor of Λ^+ by $-w_0$, where w_0 is the longest element in Weyl group. The correspondence between $\pi_1(\lambda)$ and $\pi_2(\varphi(\lambda))$ is given by

$$\lambda = (\mu, \nu) \in \mathcal{P}_m \times \mathcal{P}_m \leftrightarrow \varphi(\lambda) = (\mu, \nu) \in \mathcal{P}_p \times \mathcal{P}_q$$

simply extended by zero. Again, we shall identify $\varphi(\lambda)$ with λ.

Since $\tau_{GL_m}(\mu) \boxtimes \tau_{GL_m}(\nu)^*$ contains non-trivial H-fixed vector if and only if $\mu = \nu$, we get

$$\Lambda^+(H) = \Delta\mathcal{P}_m, \qquad \Lambda(H) = \Phi(H) \simeq \Delta\mathbb{Z}^m.$$

In this case, the coset space $\Lambda^+/\Lambda(H) = \Lambda/\Lambda(H) \simeq \mathbb{Z}^m$ is an infinite set, and it coincides with $\Phi^+/\Phi(H)$.

We have

$$\Delta_2^+(H) = \{\varepsilon_i - \varepsilon_j \mid m < i < j \leqslant p\} \sqcup \{\delta_i - \delta_j \mid m < i < j \leqslant q\},$$

in the standard notation, which concludes $r(\lambda) = 1$. We take a Borel subgroup $B = B_1 \times \overline{B_1} \subset \mathcal{G}_1$, where B_1 is the standard Borel subgroup of $GL(m, \mathbb{C})$ consisting of upper triangular matrices and $\overline{B_1}$ is its opposite. Then $HB \subset \mathcal{G}_1$ is dense. Again, the parabolic subgroup P coincides with B. Hence $L = H \cap P = H \cap B = \Delta T_1$ is isomorphic to a maximal torus T_1 in $GL(m, \mathbb{C})$. Therefore, we conclude that $r(\lambda) = 1 = \dim \tau_L(\lambda)$.

For $\lambda = (\mu, \nu) \in \Lambda^+$, let

$$\mu - \nu = \sum_{k=1}^{m} n_k \varpi_k \qquad (n_k \in \mathbb{Z}),$$

where $\{\varpi_k\}$ is the set of fundamental weights of $GL(m, \mathbb{C})$. Put

$$\mu^\flat = \sum_k \max(n_k, 0)\varpi_k, \qquad \nu^\flat = \sum_k \max(-n_k, 0)\varpi_k.$$

If we define $\lambda^\flat = (\mu^\flat, \nu^\flat)$, it satisfies the condition (8.11). In this case, we get

$$\Lambda^+ = \Lambda^+(H) \oplus \left\{ \left(\sum_{k=1}^{m} n_k \varpi_k, \sum_{k=1}^{m} n'_k \varpi_k \right) \mid n_k n'_k = 0, \ n_k, n'_k \in \mathbb{Z}_{\geqslant 0} \right\}.$$

Now we conclude that Theorem 8.6 also holds in this case.

Example C. — Let $\mathcal{G}_1 = GL(2m, \mathbb{C})$, $\mathcal{G}_2 = GL(p, \mathbb{C})$ and assume that $2m \leqslant p$. We take $H = Sp(2m, \mathbb{C})$. Therefore $(\mathcal{G}_1, H)$ is a symmetric pair. We realize $Sp(2m, \mathbb{C})$ as

$$Sp(2m, \mathbb{C}) = \left\{ g \in GL(2m, \mathbb{C}) \mid g \operatorname{diag}(J_2, \dots, J_2)\, {}^t g = \operatorname{diag}(J_2, \dots, J_2) \right\},$$

where

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We put $X = M_{2m,p}(\mathbb{C}) \simeq (\mathbb{C}^m \otimes \mathbb{C}^p)^*$ and let $\mathcal{G}_1 \times \mathcal{G}_2$ act naturally on X as

$$M_{2m,p}(\mathbb{C}) \ni A \to {}^t g_1^{-1} A g_2^{-1}, \qquad (g_i \in \mathcal{G}_i, i = 1, 2).$$

The action of $\mathcal{G}_1 \times \mathcal{G}_2$ is multiplicity free, and we have the decomposition of $\mathbb{C}[X]$ as

$$\mathbb{C}[X] \simeq \sum_{\lambda \in \mathcal{P}_{2m}}^{\oplus} \tau_{GL_{2m}}(\lambda) \boxtimes \tau_{GL_p}(\lambda).$$

Therefore

$$\Lambda^+ = \mathcal{P}_{2m}, \qquad \Lambda = \Phi \simeq \mathbb{Z}^{2m}.$$

We shall identify $\lambda \in \mathcal{P}_{2m}$ with $\varphi(\lambda) \in \mathcal{P}_p$. The irreducible representation $\pi_1(\lambda)$ has a non-trivial H-fixed vector if and only if $\lambda_{2i-1} = \lambda_{2i}$ ($1 \leqslant i \leqslant m$), i.e., $\lambda = \sum_{k=1}^m n_{2k}\varpi_{2k}$ ($n_{2k} \in \mathbb{Z}_{\geqslant 0}$). Therefore, we have

$$\Lambda^+(H) = \sum_{k=1}^m \mathbb{Z}_{\geqslant 0}\, \varpi_{2k}, \qquad \Lambda(H) = \Phi(H) = \sum_{k=1}^m \mathbb{Z}\, \varpi_{2k} \simeq \mathbb{Z}^m.$$

Then it is easy to see that

$$\Lambda^+/\Lambda(H) = \Phi^+/\Phi(H) \simeq \sum_{k=1}^m \mathbb{Z}_{\geqslant 0}\, \varpi_{2k-1}.$$

For $\lambda = \sum_{k=1}^{2m} n_k\varpi_k \in \Lambda^+$, we define

$$\lambda^\flat = \sum_{k=1}^m n_{2k-1}\varpi_{2k-1}.$$

Then it is clear that $\lambda^\flat$ satisfies the condition (8.11), hence we get

$$\Lambda^+ = \Lambda^+(H) \oplus \left\{ \sum_{k=1}^m n_k\varpi_{2k-1} \mid n_k \in \mathbb{Z}_{\geqslant 0} \right\}.$$

Take a Borel subgroup of $\mathcal{G}_1$ consisting of upper triangular matrices. Then $HB \subset \mathcal{G}_1$ is dense and the parabolic subgroup P is given by

$$P = \{\mathrm{diag}\,(p_1, p_2, \ldots, p_m) \mid p_k \in SL(2,\mathbb{C})\} \cdot B.$$

Then we have

$$L = H \cap P = \{\mathrm{diag}\,(p_1, p_2, \ldots, p_m) \mid p_k \in SL(2,\mathbb{C})\} \simeq SL(2,\mathbb{C})^m.$$

Therefore,

$$\tau_L(\lambda) = \tau_{SL_2}(\lambda_1 - \lambda_2) \boxtimes \tau_{SL_2}(\lambda_3 - \lambda_4) \boxtimes \cdots \boxtimes \tau_{SL_2}(\lambda_{2m-1} - \lambda_{2m}),$$

where $\tau_{SL_2}(\mu)$ is an irreducible representation of $SL(2,\mathbb{C})$ with highest weight μ. Since $\dim \tau_{SL_2}(\mu) = \mu + 1$, $\dim \tau_L(\lambda)$ is given by

$$\dim \tau_L(\lambda) = \prod_{k=1}^m (\lambda_{2k-1} - \lambda_{2k} + 1).$$

On the other hand, we have

$$\Delta_2^+(H) = \{\varepsilon_{2k-1} - \varepsilon_{2k} \mid 1 \leqslant k \leqslant m\} \sqcup \{\varepsilon_i - \varepsilon_j \mid 2m < i < j \leqslant p\}.$$

Hence we get

$$r(\lambda) = \prod_{k=1}^m (\lambda_{2k-1} - \lambda_{2k} + 1) = \dim \tau_L(\lambda).$$

Now we conclude that Theorem 8.6 is also valid in this case.

9. Associated cycle of unitary lowest weight modules

Let (G_1, G_2) be a reductive dual pair with G_2 being compact. We often write $G = G_1$ without subscription. We treat the three cases given in § 3 ; namely, $(G, G_2) = (Sp(2n, \mathbb{R}), O(m)), (U(p, q), U(m))$, or $(O^*(2p), Sp(2m))$.

In this section, we will prove the following theorem.

Theorem 9.1. — *We assume that the pair (G, G_2) is in the stable range where G_2 is a smaller member, i.e., $m \leqslant \mathbb{R}\text{-rank}\, G$.*

Take a finite dimensional irreducible representation $\sigma \in \mathrm{Irr}(G_2)$. Then the corresponding representation $L(\sigma) \in \mathrm{Irr}(\widetilde{G})$ is a unitary lowest weight module of the metaplectic cover $\widetilde{G}$ of G. The associated cycle of $L(\sigma)$ is given by

$$\mathcal{AC}\, L(\sigma) = \dim \sigma \cdot [\overline{\mathcal{O}_m}], \tag{9.1}$$

where $\mathcal{O}_m$ is a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}^-$ given in § 7.

Corollary 9.2. — *Let the notation be as above. Then, the Gelfand-Kirillov dimension and the Bernstein degree of $L(\sigma)$ are given by*

$$\mathrm{Dim}\, L(\sigma) = \dim \overline{\mathcal{O}_m}, \qquad \mathrm{Deg}\, L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}_m}.$$

Explicit formulas for $\dim \overline{\mathcal{O}_m}$ and $\deg \overline{\mathcal{O}_m}$ are given in Theorems 7.5, 7.10 and 7.16.

Let us prove Theorem 9.1 for the pair $(Sp(2n, \mathbb{R}), O(m))$. This pair is the most complicated one, because $O(m)$ is not connected. The other pairs can be treated similarly.

Take an irreducible representation $\sigma \in \mathrm{Irr}(O(m))$ and consider the lowest weight module $L(\sigma)$ of $\widetilde{G} = \widetilde{Sp(2n, \mathbb{R})}$. First, let us recall the Poincaré series (6.5) of $L(\sigma)$

$$P(L(\sigma); t^2) = t^{-|\mu^+|} \sum_{\lambda \in \mathcal{P}_m} m(\lambda, \sigma) \dim \tau_\lambda^{(n)}\, t^{|\lambda|},$$

where $\tau_\lambda^{(n)}$ is an irreducible finite dimensional representation of $K_{\mathbb{C}} \simeq GL(n, \mathbb{C})$ with highest weight $\lambda \in \mathcal{P}_m$ and $\mathcal{P}_m$ is the set of all partitions of length less than or equal to m.

We consider two cases, according to $\sigma\big|_{SO(m)}$ is irreducible or not (see Lemma 6.1).

1) Let us assume that $\sigma\big|_{SO(m)}$ is irreducible. We denote by $\sigma(\mu) \in \mathrm{Irr}(SO(m))$ the restriction, where μ is the highest weight. In this case, the branching coefficient $m(\lambda, \sigma)$ satisfies

$$m(\lambda, \sigma) + m(\lambda, \sigma \otimes \det) = m(\lambda, \mu),$$

where $m(\lambda, \mu)$ is the branching coefficient with respect to $SO(m)$, i.e.,

$$\tau_\lambda^{(m)}\big|_{SO(m)} = \sum_{\mu}^{\oplus} m(\lambda, \mu) \sigma(\mu).$$

This means that

$$t^{|\mu|}P(L(\sigma);t^2) + t^{|\mu|+m-2k}P(L(\sigma \otimes \det);t^2) = \sum_{\lambda \in \mathcal{P}_m} m(\lambda,\mu) \dim \tau_\lambda^{(n)} t^{|\lambda|}, \tag{9.2}$$

where $\sigma = \sigma(\mu)$ (with the convention after Lemma 6.1) and $k = \ell(\mu)$. The right hand side of (9.2) coincides with the Poincaré series $P(\sigma(\mu);t)$ of covariants of $\sigma(\mu)$ defined in § 8, if we take $\mathcal{G}_1 = GL(m,\mathbb{C}) \supset H = SO(m,\mathbb{C}), \mathcal{G}_2 = GL(n,\mathbb{C})$, and $X = M_{m,n} = M_{n,m}^*$ as in Example A there. To distinguish two types of Poincaré series, we shall write $P(\Gamma(X;\sigma(\mu));t)$ instead of $P(\sigma(\mu);t)$ in this section.

Let $d = \operatorname{Dim} L(\mathbf{1}_{O(m)})$. Note that the Gelfand-Kirillov dimension of $L(\sigma)$ and $L(\sigma \otimes \det)$ also coincides with d. Then we have

$$\begin{aligned} &\lim_{t\uparrow 1}(1-t^2)^d\Big\{t^{|\mu|}P(L(\sigma);t^2) + t^{|\mu|+m-2k}P(L(\sigma\otimes\det);t^2)\Big\} \\ &\qquad = \operatorname{Deg} L(\sigma) + \operatorname{Deg} L(\sigma\otimes\det). \end{aligned} \tag{9.3}$$

This implies that $d = \operatorname{Dim}\Gamma(X;\sigma(\mu))$ and

$$\lim_{t\uparrow 1}(1-t^2)^d P(\Gamma(X;\sigma(\mu));t) = \operatorname{Deg}\Gamma(X;\sigma(\mu)). \tag{9.4}$$

Lemma 9.3. — *For any $\sigma \in \operatorname{Irr}(O(m))$, we have*

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} L(\sigma \otimes \det).$$

Proof. — We denote a subspace of the symmetric algebra $S(M_{n,m}) = \mathbb{C}[M_{n,m}^*]$ on which $O(m)$ acts via σ by V_σ. Then the representation space of $L(\sigma)$ is identified with the σ-covariants $(V_\sigma \otimes \sigma^*)^{O(m)}$. In order to get the $\widetilde{K}$-action on it, we must twist it by $(\det k)^{m/2}$ $(k \in GL(n,\mathbb{C}))$, though it does not affect on the gradation itself. Since we only consider the Poincaré series, we simply ignore this twist.

Put

$$\delta = \det\big(E_{ij}\big)_{1\leqslant i,j\leqslant m} \in S(M_{n,m}),$$

where E_{ij} is the matrix unit. Then, clearly δ represents $\det \in \operatorname{Irr}(O(m))$. The multiplication by δ maps V_σ injectively to $V_{\sigma\otimes\det}$,

$$\delta : V_\sigma \longrightarrow V_{\sigma\otimes\det}.$$

This map increases the degree by $\deg\delta = m^2$, and we conclude that

$$t^{m^2}P(L(\sigma);t) \leqslant P(L(\sigma\otimes\det);t)$$

for $0 < t < 1$. Since $(\sigma\otimes\det)\otimes\det = \sigma$, we finally get

$$t^{2m^2}P(L(\sigma);t) \leqslant t^{m^2}P(L(\sigma\otimes\det);t) \leqslant P(L(\sigma);t).$$

If we multiply $(1-t)^d$ $(d = \operatorname{Dim} L(\sigma))$ and take limit $t \uparrow 1$, we get

$$\operatorname{Deg} L(\sigma) \leqslant \operatorname{Deg} L(\sigma\otimes\det) \leqslant \operatorname{Deg} L(\sigma).$$

□

By Lemma 9.3, formulas (9.3) and (9.4) imply

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} L(\sigma \otimes \det) = 2^{-1} \operatorname{Deg} \Gamma(X; \sigma(\mu)). \tag{9.5}$$

Consider a special case where $\sigma = \mathbf{1}_{O(m)}$, the trivial representation of $O(m)$. Then the above formula (9.5) becomes

$$\operatorname{Deg} L(\mathbf{1}_{O(m)}) = \operatorname{Deg} L(\det) = 2^{-1} \operatorname{Deg} \Gamma(X; \mathbf{1}_{SO(m)}). \tag{9.6}$$

Theorem 8.6 implies that

$$\begin{aligned} \operatorname{Deg} L(\sigma) &= 2^{-1} \operatorname{Deg} \Gamma(X; \sigma(\mu)) \\ &= 2^{-1} \dim \sigma(\mu) \ \operatorname{Deg} \Gamma(X; \mathbf{1}_{SO(m)}) = \dim \sigma(\mu) \ \operatorname{Deg} L(\mathbf{1}_{O(m)}). \end{aligned}$$

Since the associated cycle of $L(\sigma)$ is a multiple of $\overline{\mathcal{O}_m}$, the multiplicity is given by

$$\operatorname{Deg} L(\sigma) / \deg \overline{\mathcal{O}_m} = \operatorname{Deg} L(\sigma) / \operatorname{Deg} L(\mathbf{1}_{O(m)}) = \dim \sigma(\mu) = \dim \sigma$$

(cf. Theorems 1.4 and 7.5).

2) Assume that $\sigma|_{SO(m)} = \sigma(\mu^+) \oplus \sigma(\mu^-)$ as in Lemma 6.1 (2). Then it is easy to see that

$$m(\lambda, \sigma) = m(\lambda, \mu^+) = m(\lambda, \mu^-).$$

Therefore we have

$$t^{|\mu^+|} P(L(\sigma); t^2) = \sum_{\lambda \in \mathcal{P}_m} m(\lambda, \mu^+) \dim \tau_\lambda^{(n)} t^{|\lambda|} = P(\Gamma(X; \sigma(\mu^+)); t).$$

Multiply $(1-t^2)^d$ both hand sides, and take limit $t \uparrow 1$. Then we get

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} \Gamma(X; \sigma(\mu^+)) = \dim \sigma(\mu^+) \ \operatorname{Deg} \Gamma(X; \mathbf{1}_{SO(m)}).$$

By (9.6), we get

$$\operatorname{Deg} L(\sigma) = 2 \dim \sigma(\mu^+) \ \operatorname{Deg} L(\mathbf{1}_{O(m)}) = \dim \sigma \ \operatorname{Deg} L(\mathbf{1}_{O(m)}),$$

which proves (9.1) by the same reasoning as 1). This completes the proof of Theorem 9.1 for the pair $(Sp(2n, \mathbb{R}), O(m))$.

For the other pairs, we use Examples B and C in § 8 instead of Example A. In these cases, we have

$$\operatorname{Deg} L(\sigma) = \operatorname{Deg} \Gamma(X; \sigma)$$

for appropriate choice of X. This formula and Theorem 8.6 prove the theorem by almost the same arguments above.

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CAYLEY TRANSFORM AND GENERALIZED WHITTAKER MODELS FOR IRREDUCIBLE HIGHEST WEIGHT MODULES

by

Hiroshi Yamashita

Dedicated to Professor Ryoshi Hotta on his sixtieth birthday

Abstract. — We study the generalized Whittaker models for irreducible admissible highest weight modules $L(\tau)$ for a connected simple Lie group G of Hermitian type, by using certain invariant differential operators $\mathcal{D}_{\tau^*}$ of gradient type on the Hermitian symmetric space $K\backslash G$. It is shown that each $L(\tau)$ embeds, with nonzero and finite multiplicity, into the generalized Gelfand-Graev representation $\Gamma_{m(\tau)}$ attached to the unique open orbit $\mathcal{O}_{m(\tau)}$ (through the Kostant-Sekiguchi correspondence) in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. The embeddings can be intrinsically analyzed by means of the Cayley transform which carries the bounded realization of $K\backslash G$ to unbounded one. If $L(\tau)$ is unitarizable, the space $\mathcal{Y}(\tau)$ of infinitesimal homomorphisms from $L(\tau)$ into $\Gamma_{m(\tau)}$ can be described in terms of the principal symbol at the origin of the differential operator $\mathcal{D}_{\tau^*}$. For the classical groups $G = SU(p,q)$, $Sp(n,\mathbb{R})$ and $SO^*(2n)$, the space $\mathcal{Y}(\tau)$ is clearly understood through the oscillator representations of reductive dual pairs.

***Résumé* (Transformation de Cayley et modèles de Whittaker généralisés pour les modules irréductibles de plus haut poids)**

Soit G un groupe de Lie connexe simple de type hermitien. On considère les G-modules irréductibles admissibles $L(\tau)$ de plus haut poids. Dans cet article, nous étudions les modèles de Whittaker généralisés pour $L(\tau)$ en utilisant certains opérateurs différentiels de type gradient $\mathcal{D}_{\tau^*}$ sur l'espace hermitien symétrique $K\backslash G$. Il est montré que chaque $L(\tau)$ apparaît, avec une multiplicité finie et non nulle, dans la représentation de Gelfand-Graev généralisée $\Gamma_{m(\tau)}$ qui est attachée à l'unique orbite ouverte $\mathcal{O}_{m(\tau)}$ (par la correspondance de Kostant-Sekiguchi) dans la variété $\mathcal{V}(L(\tau))$ associée à $L(\tau)$. On peut analyser intrinsèquement les isomorphismes de $L(\tau)$ dans $\Gamma_{m(\tau)}$ au moyen de la transformation de Cayley qui donne un rapport entre la réalisation de $K\backslash G$ comme domaine borné et celle comme domaine non borné. Si $L(\tau)$ est unitarisable, l'espace $\mathcal{Y}(\tau)$ des homomorphismes infinitésimaux de $L(\tau)$ dans $\Gamma_{m(\tau)}$ s'exprime par le symbole principal à l'origine de l'opérateur différentiel $\mathcal{D}_{\tau^*}$. Pour les groupes classiques $G = SU(p,q)$, $Sp(n,\mathbb{R})$ et $SO^*(2n)$, on peut comprendre l'espace $\mathcal{Y}(\tau)$ en utilisant les représentations oscillateur pour les paires duales réductives.

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Introduction

For a semisimple algebraic group G, the generalized Gelfand-Graev representations introduced by Kawanaka **[14]** form a family of representations of G induced from certain one-dimensional characters of various unipotent subgroups. By construction, each of these induced G-modules is naturally attached to a nilpotent G-orbit $\mathcal{O}_G$ in the Lie algebra through the Dynkin-Kostant theory. The original (non generalized) Gelfand-Graev representations are induced from nondegenerate characters of a maximal unipotent subgroup, and they correspond to the principal nilpotent orbits. We say that an irreducible representation π of G has a generalized Whittaker model of type $\mathcal{O}_G$ if π admits an embedding into the generalized Gelfand-Graev representation attached to $\mathcal{O}_G$. The problem of describing the generalized Whittaker models is important not only in representation theory but also in connection with the theory of automorphic forms.

Generalized Whittaker models (or vectors) for irreducible representations of G have been studied by many authors (e.g., **[14]**, **[15]**, **[26]**, **[22]**, **[24]**, **[39]**, etc.). For real or complex groups, it is Kostant **[18]** who initiated a systematic study on the existence of nonzero Whittaker vectors attached to the principal nilpotent orbits of quasi-split groups, in connection with the primitive ideals of the irreducible representations in question. Later, some results of Kostant have been extended by Matumoto to those on generalized Whittaker vectors associated to arbitrary (not necessarily principal) nilpotent orbits $\mathcal{O}_G$. In fact, it is shown in **[22]** that the Harish-Chandra module of an irreducible admissible representation π has a nonzero generalized Whittaker vector of type $\mathcal{O}_G$ only if the nilpotent orbit $\mathcal{O}_G$ is contained in the associated variety of the primitive ideal $\operatorname{Ann}\pi$ in the universal enveloping algebra. For complex groups G, one of the main results in **[24]** tells us that, under certain assumptions on $\mathcal{O}_G$ and on π, the space of $C^{-\infty}$-generalized Whittaker vectors of type $\mathcal{O}_G$ is nonzero and finite-dimensional if and only if the closure of $\mathcal{O}_G$ coincides with the wave front set of π.

As to p-adic groups, Mœglin and Waldspurger have already established in 1987 a stronger result of this nature, by showing that the wave front cycle (asymptotic cycle) of an irreducible representation π of G completely controls the spaces of generalized Whittaker vectors of interest. Namely, it is proved in **[26]** that, if $\mathcal{O}_G$ is a nilpotent orbit which is maximal in the wave front set (asymptotic support) of π, the dimension of the space of generalized Whittaker vectors of type $\mathcal{O}_G$ is equal to the multiplicity of $\mathcal{O}_G$ in the wave front cycle. However, up to this time, the corresponding phenomenon is not yet fully understood in the case of real groups, except for the representations with the largest Gelfand-Kirillov dimension (see **[23]** and **[25]**).

In this article, we focus our attention on the irreducible admissible (unitary) highest weight representations of real simple Lie groups. These are representations with rather small Gelfand-Kirillov dimensions. We reveal a structure of the spaces of generalized Whittaker models in relation to the associated cycles of highest weight modules.

Now, let G be a connected simple Lie group with finite center, and let K be a maximal compact subgroup of G. Assume that $K\backslash G$ is Hermitian symmetric. The Lie algebras of G and K are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. We write $K_{\mathbb{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) for the complexifications of K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$, and let θ denote the corresponding Cartan involution of $\mathfrak{g}$. The G-invariant complex structure on $K\backslash G$ gives a triangular decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ of $\mathfrak{g}$. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$).

The group G of Hermitian type has a distinguished family of irreducible admissible Hilbert representations with highest weights. The Harish-Chandra module of such a G-representation is obtained as the unique simple quotient $L(\tau)$ of generalized Verma module induced from an irreducible representation (τ, V_τ) of K. Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{q} := \mathfrak{k} + \mathfrak{p}_+$ of $\mathfrak{g}$ by making $\mathfrak{p}_+$ act on V_τ trivially. We call τ the extreme K-type of $L(\tau)$.

The purpose of this paper is to describe the generalized Whittaker models for irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$. To be more precise, let $\{\mathcal{O}_m \mid m = 0, \dots, r\}$ be the totality of nilpotent $K_{\mathbb{C}}$-orbits in the nilradical $\mathfrak{p}_+$ of $\mathfrak{q}$, arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$. We write $\mathcal{O}'_m$ for the the nilpotent G-orbit in $\mathfrak{g}_0$ corresponding to $\mathcal{O}_m$ by the Kostant-Sekiguchi bijection. Following the recipe by Kawanaka **[14]** (see also **[40]**), we can construct a generalized Gelfand-Graev representation $\Gamma_m = \mathrm{Ind}_{\mathfrak{n}(m)}^{G}(\eta_m)$ (GGGR for short; see Definition 4.3) of G attached to $\mathcal{O}'_m$. On the other hand, it is well-known that the associated variety $\mathcal{V}(L(\tau))$ of a highest weight module $L(\tau)$ is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in $\mathfrak{p}_+$, where $m(\tau)$ depends on τ. Then our aim is to specify the $(\mathfrak{g}, K)$-embeddings of $L(\tau)$ into these GGGRs Γ_m ($m = 0, \dots, r$). This is a continuation of our earlier work **[41]** on Whittaker models for the holomorphic discrete series.

In order to specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 2.3). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke (**[2]**, **[3]**, **[4]**). The K-finite kernel of $\mathcal{D}_{\tau^*}$ realizes the dual lowest weight module $L(\tau)^*$. By virtue of the kernel theorem given as Corollary 1.8, we find that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions of the differential equation $\mathcal{D}_{\tau^*}F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question, where η_m is the character of nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$ that defines Γ_m.

The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by means of the unbounded realization of $K\backslash G$ via the Cayley transform (cf. **[32]**, **[9]**). Some remarkable results of

Enright and Joseph [5], Jakobsen [20] on the annihilator ideal of (unitarizable) highest weight modules are useful in the course of our study. Also, elementary properties (cf. Vogan [33, Section 2]) on the associated (characteristic) cycle of Harish-Chandra modules guarantee that the space $\mathcal{Y}(\tau, m)$ does not vanish for the most relevant $m = m(\tau)$. As a result, we get the following conclusions (see Theorems 4.7–4.9).

Theorem 1. — *$L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. In this case, the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ consists only of elementary functions on the unbounded domain $\mathcal{S}\,(\subset \mathfrak{p}_-)$ which realizes $K\backslash G$.*

Theorem 2. — *If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau)$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at a certain point $X \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$, that is, the multiplicity of embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$, coincides with the multiplicity of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$.*

For the classical groups $G = SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$, the theory of reductive dual pair gives explicit realizations of unitarizable highest weight modules $L(\tau)$ (cf. [12], [7], [3]). In this setting, it is not difficult to specify the generalized Whittaker models for such $L(\tau)$'s more explicitly by using the oscillator representation of a pair (G, G') with a compact group G' dual to G. In fact, this has been done by Tagawa [31] for the case $SU(p,q)$, motivated by author's observation in 1997 for the case $Sp(n,\mathbb{R})$. We include this observation as well as Tagawa's result at the end of this paper (see Theorems 5.14 and 5.15 together with the isomorphism (4.15)), handling all the groups $SU(p,q)$, $Sp(2n,\mathbb{R})$ and $SO^*(2n)$ in a unified manner.

The last statement in Theorem 2 clarifies the relationship between the generalized Whittaker models and the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of unitarizable highest weight module $L(\tau)$. In fact, $\mathcal{Y}(\tau)$ turns to be the dual of the isotropy representation of $K_{\mathbb{C}}(X)$ attached to $\mathcal{AC}(L(\tau))$ in the sense of Vogan [33]. We note that the associated cycle and the Bernstein degree of $L(\tau)$ have been specified by Nishiyama, Ochiai and Taniguchi [27] for the above classical groups G through detailed study of K-types of $L(\tau)$, where $L(\tau)$ is assumed to be an irreducible constituent of the oscillator representations of pairs (G, G') in the stable range (with smaller G'). Recently, Kato and Ochiai [13] have generalized the technique in [27] to a large extent. They established in particular a unified formula for the degrees of nilpotent orbits $\mathcal{O}_m$, which is valid for any simple Lie group of Hermitian type.

An η_m-equivariant linear form on $L(\tau)$ is called an (algebraic) generalized Whittaker vector of type η_m. Each $(\mathfrak{g}, K)$-embedding of $L(\tau)$ into the GGGR Γ_m, composed with the evaluation at the identity $e \in G$ of functions in Γ_m, naturally gives rise to a generalized Whittaker vector of type η_m on $L(\tau)$. We can show that the converse is also true for the most relevant case $m = m(\tau)$. Namely, it turns out that every generalized Whittaker vector of type η_m comes from a function in the space $\mathcal{Y}(\tau)$ for any $L(\tau)$ (see Proposition 4.19). This allows us to interpret the main results of this article in terms of algebraic generalized Whittaker vectors associated to irreducible highest weight $(\mathfrak{g}, K)$-modules (Theorem 4.22).

We organize this paper as follows.

Section 1 gives general theory on the embeddings of irreducible $(\mathfrak{g}, K)$-modules into induced G-representations. The kernel theorem (Corollary 1.8) is our main tool for studying generalized Whittaker models. We introduce in Section 2 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [4]. In addition, the solutions F of $\mathcal{D}_{\tau^*}F = 0$ of exponential type are specified in Proposition 2.8. Section 3 is devoted to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 3.11). In Section 4 we give our main results (Theorems 4.7–4.9) that describe the generalized Whittaker models for $L(\tau)$. Relation to algebraic generalized Whittaker vectors is also investigated. Last in Section 5, we discuss the case of classical groups $SU(p, q)$, $Sp(2n, \mathbb{R})$ and $SO^*(2n)$ more explicitly.

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1. Embeddings of Harish-Chandra modules

This section prepares some generalities about the embeddings of irreducible Harish-Chandra modules into C^∞-induced representations of a semisimple Lie group, by developing our earlier observation [**42**, I, §2] for the discrete series in full generality. The results stated in this section are more or less folklore for the experts, or they are consequences of some known facts concerning the maximal globalization of Harish-Chandra modules due to Schmid and Kashiwara (cf. [**29**], [**11**]). Nevertheless we include here the detail with direct proofs in order to keep this paper more accessible and self-contained. In fact, a kernel theorem, Corollary 1.8, will be essentially used in the succeeding sections to describe generalized Whittaker models for highest weight representations.

1.1. A duality of Peter-Weyl type. — Throughout this section, let G be any connected semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. We keep the same notation and convention employed at the beginning of Introduction.

A $U(\mathfrak{g})$-module $\boldsymbol{X}$ is called a $(\mathfrak{g}, K)$-*module* if the subalgebra $U(\mathfrak{k})$ acts on $\boldsymbol{X}$ locally finitely, and if the $\mathfrak{k}_0$-action gives rise to a representation of K on $\boldsymbol{X}$ through exponential map. By a *Harish-Chandra module*, we mean a $(\mathfrak{g}, K)$-module of finite length as a $U(\mathfrak{g})$-module. By basic results of Harish-Chandra (see e.g., [**35**, Chap.3]), any admissible (i.e., K-multiplicity finite) representation of G on a Hilbert space $\boldsymbol{H}$ yields, through differentiation, a $(\mathfrak{g}, K)$-module structure on the subspace $\boldsymbol{H}_K$ of all K-finite vectors in $\boldsymbol{H}$. The continuous G-module $\boldsymbol{H}$ is irreducible if and only if the corresponding $\boldsymbol{H}_K$ is irreducible as a $(\mathfrak{g}, K)$-module. Each irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ can be extended to an irreducible Hilbert G-module $\boldsymbol{H}$ with K-finite part $\boldsymbol{H}_K = \boldsymbol{X}$. Notice that the $(\mathfrak{g}, K)$-module corresponding to the irreducible G-module $\boldsymbol{H}^*$ contragredient to $\boldsymbol{H}$ is isomorphic to the K-finite part of the full dual space $\boldsymbol{X}' = \mathrm{Hom}_{\mathbb{C}}(\boldsymbol{X}, \mathbb{C})$. We denote this irreducible $(\mathfrak{g}, K)$-module by $\boldsymbol{X}^*$, and call it the dual Harish-Chandra module of $\boldsymbol{X}$.

We study in this paper the embeddings of irreducible $(\mathfrak{g}, K)$-modules $\boldsymbol{X}$ into certain smoothly induced Fréchet G-modules $\boldsymbol{F}$. Such an $\boldsymbol{F}$ has a compatible $\mathfrak{g}$ and K module structure through differentiation, and its K-finite part $\boldsymbol{F}_K$ is a $(\mathfrak{g}, K)$-module. We note that the image of $\boldsymbol{X}$ by any $\mathfrak{g}$ and K homomorphism into $\boldsymbol{F}$ is necessarily contained in $\boldsymbol{F}_K$, i.e., $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}) = \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, \boldsymbol{F}_K)$.

The group G acts on the space $C^\infty(G)$ of all smooth functions on G by left translation and by right translation as follows:

$$g^L f(x) := f(g^{-1}x), \quad g^R f(x) := f(xg) \quad (g \in G, x \in G, f \in C^\infty(G)).$$

These two actions L and R commute with each other. Through differentiation one gets two $U(\mathfrak{g})$-representations on $C^\infty(G)$ denoted again by L and R respectively. Let $C^\infty_K(G)$ be the space of functions $f \in C^\infty(G)$ which are left K-finite and also right K-finite. Then $C^\infty_K(G)$ becomes a $(\mathfrak{g}, K)$-module through L or R.

If the group G is compact, i.e., $G = K$, the regular representation $(L \hat{\otimes} R, C^\infty_G(G))$ of $G \times G$ decomposes into irreducibles as

$$C^\infty_G(G) \simeq \bigoplus_{\delta \in \hat{G}} V_\delta \otimes V_\delta^* \quad \text{as } G \times G\text{-modules}$$

by the Peter-Weyl theorem, where $\hat{G}$ denotes the set of all equivalence classes of irreducible finite-dimensional representations of G and we write V_δ for an irreducible G-module of class $\delta \in \hat{G}$. The following lemma says that we have a similar duality of Peter-Weyl type for irreducible Harish-Chandra modules of noncompact semisimple Lie groups.

Lemma 1.1. — *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let f be in $C^\infty_K(G)$. Then the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})^L f$ generated by f through the action L is isomorphic to $\boldsymbol{X}$ if and only if the corresponding $U(\mathfrak{g})^R f$ through the action R is isomorphic to $\boldsymbol{X}^*$.*

We give a proof below introducing some important notion which we use throughout this paper.

Proof of Lemma 1.1. — Let us prove the *if* part only since the converse can be proved in the same way. So, assume that $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$ as $(\mathfrak{g}, K)$-modules.

Take a finite-dimensional K-module (τ, V_τ) which is isomorphic to $U(\mathfrak{k})^L f$. Let $i : V_\tau \xrightarrow{\sim} U(\mathfrak{k})^L f$ denote a K-isomorphism. We define a V_τ^*-valued smooth function F on G by

$$\langle F(g), v\rangle = i(v)(g) \quad (v \in V_\tau,\ g \in G),$$

where $\langle\ \cdot, \cdot\ \rangle$ denotes the natural dual pairing on $V_\tau^* \times V_\tau$. Then it is immediate to verify that F lies in the following space:

$$C_{\tau^*}^\infty(G) := \{\Phi : G \xrightarrow{C^\infty} V_\tau^* \mid \Phi(kg) = \tau^*(k)\Phi(g) \quad (g \in G,\ k \in K)\}. \tag{1.1}$$

Here (τ^*, V_τ^*) denotes the representation of K contragredient to τ. The space $C_{\tau^*}^\infty(G)$ has G- and $U(\mathfrak{g})$-module structures through right translation R. The function F is in the K-finite part, say $C_{\tau^*}^\infty(G)_K$, of $C_{\tau^*}^\infty(G)$ since $U(\mathfrak{k})^L f \subset C_K^\infty(G)$. By definition we see

$$f(g) = \langle F(g), i^{-1}(f)\rangle. \tag{1.2}$$

Now the assignment $D^R F \mapsto D^R f = \langle D^R F(\cdot), i^{-1}(f)\rangle$ $(D \in U(\mathfrak{g}))$ gives a $(\mathfrak{g}, K)$-homomorphism from $U(\mathfrak{g})^R F$ onto $U(\mathfrak{g})^R f \simeq \boldsymbol{X}^*$. We see that this homomorphism is injective. In fact, suppose $D^R f = 0$ for some $D \in U(\mathfrak{g})$. It then follows that

$$\begin{aligned} 0 = D^R f(kg) &= \langle D^R F(kg), i^{-1}(f)\rangle \\ &= \langle \tau^*(k) D^R F(g), i^{-1}(f)\rangle = \langle D^R F(g), i^{-1}((k^{-1})^L f)\rangle \end{aligned} \tag{1.3}$$

for all $g \in G$ and all $k \in K$. This implies that $D^R F = 0$ since f is a K-cyclic vector for $U(\mathfrak{k})^L f \simeq V_\tau$. Thus we have found a $(\mathfrak{g}, K)$-module embedding, say A_0, from $\boldsymbol{X}^*$ into $C_{\tau^*}^\infty(G)_K$ whose image equals $U(\mathfrak{g})^R F$.

Let $(\pi, \boldsymbol{H})$ be an irreducible admissible G-representation with Harish-Chandra module $\boldsymbol{X}$, and let $(\pi^*, \boldsymbol{H}^*)$ be the representation of G contragredient to π. We have $\boldsymbol{H}_K^* = \boldsymbol{X}^*$ as remarked before. By virtue of the Frobenius reciprocity for smoothly induced representation $\mathrm{Ind}_K^G(\tau^*)$ of G acting on $C_{\tau^*}^\infty(G)$, one obtains a linear isomorphism

$$\mathrm{Hom}_K(\boldsymbol{X}^*,\ V_\tau^*) \simeq \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}^*,\ C_{\tau^*}^\infty(G)_K), \tag{1.4}$$

which is given as follows. Take a K-homomorphism $T : \boldsymbol{X}^* \to V_\tau^*$. Then we can define $A(\varphi) \in C_{\tau^*}^\infty(G)$ for every $\varphi \in \boldsymbol{X}^*$ by

$$A(\varphi)(g) = \tilde{T}(\pi^*(g)\varphi) \quad (g \in G). \tag{1.5}$$

Here $\tilde{T}$ denotes the unique continuous extension of $T : \boldsymbol{X}^* \to V_\tau^*$ to $\boldsymbol{H}^*$. Then, the assignment $T \mapsto A$ gives (1.4).

We now consider our specified embedding $A_0 : \boldsymbol{X}^* \simeq U(\mathfrak{g})^R F \hookrightarrow C^\infty_{\tau^*}(G)_K$. Let T_0 denote the element of $\mathrm{Hom}_K(\boldsymbol{X}^*, V_\tau^*)$ corresponding to A_0 by (1.5). Set $\varphi_0 := A_0^{-1}(F) \in \boldsymbol{X}^*$ and $\psi_0 := i^{-1}(f) \circ \tilde{T}_0 \in \boldsymbol{X}$. Here ψ_0 is regarded as an element of $\boldsymbol{X} = ((\boldsymbol{H}^*)^*)_K$ through

$$\psi_0 : \boldsymbol{H}^* \xrightarrow{\tilde{T}_0} V_\tau^* \xrightarrow{i^{-1}(f)} \mathbb{C}$$

with $i^{-1}(f) \in V_\tau = \mathrm{Hom}_{\mathbb{C}}(V_\tau^*, \mathbb{C})$. In view of (1.2) and (1.5) we find

$$f(g) = \langle \pi^*(g)\varphi_0, \psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} = \langle \varphi_0, \pi(g)^{-1}\psi_0 \rangle_{\boldsymbol{H}^* \times \boldsymbol{H}} \quad (g \in G). \tag{1.6}$$

Finally, (1.6) implies that the map

$$\boldsymbol{X} \ni D\psi_0 \mapsto D^L f = \langle \varphi_0, \pi(g)^{-1} D\psi_0 \rangle \in U(\mathfrak{g})^L f \quad (D \in U(\mathfrak{g}))$$

gives a $(\mathfrak{g}, K)$-isomorphism, i.e., $\boldsymbol{X} \simeq U(\mathfrak{g})^L f$ as desired. □

1.2. Maximal globalization. — Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module. We fix once and for all an irreducible finite-dimensional representation (τ, V_τ) of K which occurs in $\boldsymbol{X}$, and fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules. Then the adjoint operator i_τ^* of i_τ gives a surjective K-homomorphism from $\boldsymbol{X}^*$ to V_τ^*. We denote by A_{τ^*} the $(\mathfrak{g}, K)$-embedding from $\boldsymbol{X}^*$ into $C^\infty_{\tau^*}(G)$ (see (1.1)) corresponding to i_τ^* through (1.4) and (1.5).

Equip $C^\infty_{\tau^*}(G)$ with a Fréchet space topology of compact uniform convergence of functions on G and each of their derivatives. The following proposition characterizes the closure $A_{\tau^*}(\boldsymbol{X}^*)^-$ of $A_{\tau^*}(\boldsymbol{X}^*)$ in $C^\infty_{\tau^*}(G)$.

***Theorem 1.2* (cf. [29], [11]).** — *Under the above notation, $A_{\tau^*}(\boldsymbol{X}^*)^-$ is a G-submodule of $C^\infty_{\tau^*}(G)$, and one gets an isomorphism of G-modules*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \ni W \overset{\sim}{\longmapsto} F \in A_{\tau^*}(\boldsymbol{X}^*)^-$$

through

$$\langle F(g), v \rangle = ((W \circ i_\tau)(v))(g) \quad (g \in G,\ v \in V_\tau). \tag{1.7}$$

Here $C^\infty(G)$ is viewed as a smooth G-module by left translation L, and the right action R on $C^\infty(G)$ naturally gives a G-module structure on $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$.

It follows essentially from [**29**, page 316] that the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ gives a *maximal globalization* of the Harish-Chandra module $\boldsymbol{X}^*$. Namely, if a complete, locally convex Hausdorff topological vector space $\boldsymbol{F}$ admits a continuous G-action with underlying Harish-Chandra module $\boldsymbol{X}^*$, then the identity map on $\boldsymbol{X}^*$ extends uniquely to a continuous embedding $\boldsymbol{F} \hookrightarrow A_{\tau^*}(\boldsymbol{X}^*)^-$ as G-modules. One can get the above theorem from the first statement of Theorem 2.8, or equivalently (2.9), in [**11**].

In what follows, we give a direct proof of the above theorem to keep this article self-contained. This is done by generalizing our argument in [**42**, I, §2].

Proof of Theorem 1.2. — Let W be a $(\mathfrak{g}, K)$-embedding of $\boldsymbol{X}$ into $C^\infty(G)$. Since $W \circ i_\tau : V_\tau \xrightarrow{\sim} W(i_\tau(V_\tau)) \subset C^\infty(G)$ is a K-isomorphism, we can see just as in the beginning of the proof of Lemma 1.1 that there exists a unique $F \in C^\infty_{\tau^*}(G)$ satisfying (1.7). It is then easy to observe that the map $W \mapsto F$ sets up a G-homomorphism, say Υ, from $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$ to $C^\infty_{\tau^*}(G)$ and that Υ is injective because of the irreducibility of $\boldsymbol{X}$. Hence we will get the theorem if we can show

$$\mathrm{Im}\,\Upsilon = A_{\tau^*}(\boldsymbol{X}^*)^-, \tag{1.8}$$

where $\mathrm{Im}\,\Upsilon$ denotes the image of Υ.

To prove (1.8) we use the projection to K-isotypic component. Let $\boldsymbol{M}$ be any smooth Fréchet K-module. For each $\delta \in \hat{K}$, the unitary dual of K, the integral operator Q_δ defined by

$$Q_\delta(v) = (\dim \delta) \cdot \int_K \overline{\mathrm{tr}(\delta(k))} \cdot kv\, dk \quad (v \in \boldsymbol{M}),$$

gives a continuous K-equivariant projection of $\boldsymbol{M}$ onto its δ-isotypic component $\boldsymbol{M}_\delta$. Here dk denotes the normalized Haar measure on K. By Harish-Chandra, the Fourier series $\sum_{\delta \in \hat{K}} Q_\delta(v)$ converges absolutely to v. (cf. [**36**, Th.4.4.2.1]).

Now the right hand side of (1.8) is described in terms of the projections Q_δ as

$$A_{\tau^*}(\boldsymbol{X}^*)^- = \{F \in C^\infty_{\tau^*}(G) \mid F_\delta := Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*) \quad \text{for all } \delta \in \hat{K}\}. \tag{1.9}$$

In reality, the inclusion $\supset$ is evident since the sum $F = \sum_{\delta \in \hat{K}} F_\delta$ converges in $C^\infty_{\tau^*}(G)$. Conversely assume F be in the closure $A_{\tau^*}(\boldsymbol{X}^*)^-$. Then there exists a sequence $\{F_j\}_{j=1,2,..}$ in $A_{\tau^*}(\boldsymbol{X}^*)$ such that $F = \lim_{j\to\infty} F_j$. Since the projection Q_δ is continuous, one obtains $F_\delta = \lim_{j\to\infty}(F_j)_\delta$ for every $\delta \in \hat{K}$. Noting that $(F_j)_\delta$ lies in a finite-dimensional (and hence closed) subspace $A_{\tau^*}(\boldsymbol{X}^*)_\delta \simeq \boldsymbol{X}^*_\delta$, we find that $F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)_\delta$.

We are going to show just as in the proof of [**42**, I, Th.2.4] that $\mathrm{Im}\,\Upsilon$ coincides with the right hand side of (1.9) by using Lemma 1.1 instead of [**42**, I, Lemma 2.5].

Let F be a nonzero function in $C^\infty_{\tau^*}(G)$ such that $F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$. We write Ξ for the totality of finite subsets S of $\hat{K}$ consisting of elements δ such that $F_\delta \neq 0$. Define $F_S \in A_{\tau^*}(\boldsymbol{X}^*)$ and $f_{S,v} \in C^\infty(G)$ by

$$F_S = \sum_{\delta \in S} F_\delta, \quad f_{S,v} = \langle F_S(\cdot), v\rangle$$

for every $S \in \Xi$ and $v \in V_\tau \backslash \{0\}$. Then, $U(\mathfrak{g})^R F_S = A_{\tau^*}(\boldsymbol{X}^*) \simeq \boldsymbol{X}^*$ as $(\mathfrak{g}, K)$-modules. This implies that $U(\mathfrak{g})^R f_{S,v} \simeq \boldsymbol{X}^*$ for all S and v (cf. (1.3)). Set $Q_S := \sum_{\delta \in S} Q_\delta$ and $f_v := \langle F(\cdot), v\rangle$. We now use Lemma 1.1 to deduce

$$Q_S(U(\mathfrak{g})^L f_v) = U(\mathfrak{g})^L f_{S,v} \simeq \boldsymbol{X} \quad (S \in \Xi,\ v \in V_\tau \backslash \{0\}), \tag{1.10}$$

by noting that Q_S commutes with $U(\mathfrak{g})$-action L. It then follows from the irreducibility of $\boldsymbol{X}$ that the kernel of projection Q_S restricted to $U(\mathfrak{g})^L f_v$ is independent of a choice of $S \in \Xi$. Indeed, let S_1 and S_2 be in Ξ, and set $S' := S_1 \cup S_2$. Note that $Q_{S_i} = Q_{S_i}Q_{S'}$ $(i = 1, 2)$. One sees from (1.10) that Q_{S_i} on $U(\mathfrak{g})^L f_{S',v}$ gives a $U(\mathfrak{g})$-isomorphism from $U(\mathfrak{g})^L f_{S',v}$ onto $U(\mathfrak{g})^L f_{S_i,v}$ by Schur's lemma. This implies that

$$\operatorname{Ker}(Q_{S_1}|U(\mathfrak{g})^L f_v) = \operatorname{Ker}(Q_{S'}|U(\mathfrak{g})^L f_v) = \operatorname{Ker}(Q_{S_2}|U(\mathfrak{g})^L f_v).$$

The above kernel space must be $\{0\}$ since

$$\bigcap_{S \in \Xi} \operatorname{Ker}(Q_S|U(\mathfrak{g})^L f_v) \subset \bigcap_{S} \operatorname{Ker} Q_S = \{0\},$$

where S in the middle term runs over all finite subsets of $\hat{K}$. We thus find an embedding

$$\boldsymbol{X} \simeq U(\mathfrak{g})^L f_v \hookrightarrow C^\infty(G)$$

corresponding to F through Υ.

Conversely, let $W : \boldsymbol{X} \hookrightarrow C^\infty(G)$ be any $(\mathfrak{g}, K)$-embedding. Set $F := \Upsilon(W)$. We want to prove $F_\delta = Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$. To do this, define an element $\xi \in \boldsymbol{X}^*$ by

$$\langle \xi, a \rangle = ((Q_\delta \circ W)(a))(e) \quad (a \in \boldsymbol{X}),$$

where e denotes the identity element of G. It then follows for any $D \in U(\mathfrak{g})$ and $v \in V_\tau$ that

$$\begin{aligned} \langle D^L F_\delta(e), v \rangle &= D^L((Q_\delta \circ W \circ i_\tau)(v))(e) \\ &= ((Q_\delta \circ W)(D i_\tau(v)))(e) \\ &= \langle \xi, D i_\tau(v) \rangle = \langle i_\tau^* {}^T D \xi, v \rangle, \end{aligned}$$

since D^L commutes with Q_δ and with W. Here $U(\mathfrak{g}) \ni D \mapsto {}^T D \in U(\mathfrak{g})$ denotes the principal anti-automorphism of $U(\mathfrak{g})$ such that ${}^T X = -X$ if $X \in \mathfrak{g}$. We thus deduce

$$D^L F_\delta(e) = i_\tau^*({}^T D \xi) \quad \text{for all } D \in U(\mathfrak{g}). \tag{1.11}$$

This yields that

$$F_\delta(g) = \tilde{i}_\tau^*(\pi^*(g)\xi) = A_{\tau^*}(\xi)(g) \quad (g \in G)$$

as desired, because the both functions F_δ and $A_{\tau^*}(\xi)$ are real analytic on the connected Lie group G, and because they have the same Taylor series expansion at e by (1.11).

Thus the theorem has been proved completely. □

1.3. Kernel theorem. — To study the embeddings of $\boldsymbol{X}$ into various induced G-modules, it is useful to characterize the G-module $A_{\tau^*}(\boldsymbol{X}^*)^-$ as the full kernel space of a continuous G-homomorphism $\mathcal{D}$ defined on $C^\infty_{\tau^*}(G)$ in the following way.

Theorem 1.3. — *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let (τ, V_τ) be a K-type of $\boldsymbol{X}$. Fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules, and write A_{τ^*} for the $(\mathfrak{g}, K)$-embedding $\boldsymbol{X}^* \hookrightarrow C^\infty_{\tau^*}(G)$ associated with the adjoint operator i_τ^* by* (1.4) *and* (1.5).

If $\mathcal{D}$ is any continuous G-homomorphism from the $C^\infty_{\tau^*}(G)$ to a smooth Fréchet G-module $\boldsymbol{M}$ such that

$$(1.12)\qquad A_{\tau^*}(\boldsymbol{X}^*) = \{F \in C^\infty_{\tau^*}(G) \mid F \textit{ is right } K\textit{-finite and } \mathcal{D}F = 0\},$$

then the full kernel space $\operatorname{Ker}\mathcal{D}$ *of* $\mathcal{D}$ *in* $C^\infty_{\tau^*}(G)$ *equals the G-module* $A_{\tau^*}(\boldsymbol{X}^*)^-$, *the closure of* $A_{\tau^*}(\boldsymbol{X}^*)$ *in* $C^\infty_{\tau^*}(G)$. *Hence one gets from Theorem 1.2*

$$(1.13)\qquad \operatorname{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \simeq \operatorname{Ker}\mathcal{D} = A_{\tau^*}(\boldsymbol{X}^*)^- \quad \textit{as } G\textit{-modules.}$$

Proof. — We show that $\operatorname{Ker}\mathcal{D} = A_{\tau^*}(\boldsymbol{X}^*)^-$. The inclusion $\supset$ is obvious because $\operatorname{Ker}\mathcal{D}$ is a closed subspace of $C^\infty_{\tau^*}(G)$ by the continuity of $\mathcal{D}$ and because $A_{\tau^*}(\boldsymbol{X}^*) \subset \operatorname{Ker}\mathcal{D}$ by (1.12). Conversely if $F \in \operatorname{Ker}\mathcal{D}$, then it follows from (1.12) that $F_\delta = Q_\delta(F) \in A_{\tau^*}(\boldsymbol{X}^*)$ for every $\delta \in \hat{K}$, because $\mathcal{D}F_\delta = Q_\delta(\mathcal{D}F) = 0$. Hence we get $F = \sum_{\delta\in\hat{K}} F_\delta \in A_{\tau^*}(\boldsymbol{X}^*)^-$. Now the assertion follows from Theorem 1.2. $\square$

Remark 1.4. — The above proof tells us that the assumption on $\mathcal{D}$ can be weakened. Namely, the theorem is still true for any K-homomorphism $\mathcal{D}$ from $C^\infty_{\tau^*}(G)$ to a smooth K-module $\boldsymbol{M}$ satisfying (1.12).

Example 1.5. — We mention that an operator $\mathcal{D}$ satisfying the requirement in Theorem 1.3 has been constructed when $\boldsymbol{X}^*$ is the $(\mathfrak{g}, K)$-module associated with: (a) discrete series ([**28**], [**10**]) more generally Zuckerman cohomologically induced module ([**38**], [**1**]), with parameter "far from the walls", or (b) highest weight representation ([**2**], [**4**]; see also Theorem 2.6). In each of these cases, $\mathcal{D}$ is given as a *G-invariant differential operator of gradient type* acting on $C^\infty_{\tau^*}(G)$, where τ^* is the unique extreme K-type of $\boldsymbol{X}^*$ which occurs in $\boldsymbol{X}^*$ with multiplicity one.

We conclude this section by giving an application of Theorem 1.3. For this we need

Definition 1.6. — Let $\mathfrak{n}$ be a complex Lie subalgebra of $\mathfrak{g}$, and $(\eta, \boldsymbol{E})$ be a representation of $\mathfrak{n}$ on a Fréchet space $\boldsymbol{E}$ such that the linear endomorphism $\eta(Z)$ is continuous on $\boldsymbol{E}$ for every $Z \in \mathfrak{n}$. Then the space

$$C^\infty(G;\eta) := \{f : G \xrightarrow{C^\infty} \boldsymbol{E} \mid Z^R f = -\eta(Z) f \quad (Z \in \mathfrak{n})\},$$

endowed with the natural Fréchet space topology, has a structure of smooth G-module by L. We write Γ_η for the resulting G-representation on $C^\infty(G;\eta)$, and call it the *representation of G induced from η in C^∞-context.*

Remark 1.7. — If $\mathfrak{n}$ is the complexification of real Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}_0$ corresponding to a simply connected analytic subgroup N of G, then $C^\infty(G;\eta)$ coincides with the space of $\boldsymbol{E}$-valued smooth functions f on G such that

$$f(gn) = \eta(n)^{-1} f(g) \quad (g \in G,\ n \in N),$$

at least when $\boldsymbol{E}$ is finite-dimensional. Here η denotes the well-defined representation of the group N defined by $\eta : \mathfrak{n}_0 \to \boldsymbol{E}$ through exponential map.

Let the notation and assumption be as in Theorem 1.3 and in Definition 1.6. We write $C^\infty_{\tau^*}(G;\eta)$ for the space of C^∞-functions on G with values in $V_\tau^* \otimes \boldsymbol{E}$ satisfying the following conditions.

$$\begin{cases} Z^R F = -(\mathrm{id}_{V_\tau^*} \otimes \eta(Z))F & (Z \in \mathfrak{n}), \\ k^L F = (\tau^*(k^{-1}) \otimes \mathrm{id}_{\boldsymbol{E}})F & (k \in K), \end{cases}$$

where id_V denotes the identity map on a set V. Let $\boldsymbol{E}'$ be the space of continuous linear functionals on $\boldsymbol{E}$ equipped with dual $U(\mathfrak{n})$-action. We define a linear map

$$\mathcal{D}(\eta) : C^\infty_{\tau^*}(G;\eta) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\boldsymbol{E}', \boldsymbol{M})$$

through $\mathcal{D}$ by

$$(\mathcal{D}(\eta)F)(\zeta) = \mathcal{D}(\langle F(\cdot), \zeta\rangle) \quad (F \in C^\infty_{\tau^*}(G;\eta),\ \zeta \in \boldsymbol{E}').$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the canonical dual pairing on $(V_\tau^* \otimes \boldsymbol{E}) \times \boldsymbol{E}'$ with values in V_τ^*. If η is a one-dimensional $\mathfrak{n}$-representation, the above $\mathcal{D}(\eta)$ is naturally identified with the restriction of $\mathcal{D}$ to the subspace $C^\infty_{\tau^*}(G;\eta)$ of $C^\infty_{\tau^*}(G)$.

By using (1.13), we can now deduce the following

***Corollary 1.8* (Kernel Theorem).** — *Under the above notation, assume that the representation $(\eta, \boldsymbol{E})$ of $\mathfrak{n}$ is weakly cyclic in the following sense: there exists a $\zeta_0 \in \boldsymbol{E}'$ such that $U(\mathfrak{n})\zeta_0$ is dense in $\boldsymbol{E}'$ with respect to the weak $*$-topology. Then the embeddings of irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ into induced module $C^\infty(G;\eta)$ are characterized as*

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X}, C^\infty(G;\eta)) \simeq \mathrm{Ker}\,\mathcal{D}(\eta) \quad \textit{as vector spaces.}$$

Here the isomorphism is given as in (1.7).

Remark 1.9. — The above kernel thoerem has been proved in our earlier work [**42**, I, Th.2.4] in case that $\boldsymbol{X}$ is the $(\mathfrak{g}, K)$-module of discrete series and that $\mathcal{D}$ is a differential operator of gradient type (Schmid operator).

Proof of Corollary 1.8. — First, we observe just as in the proof of Theorem 1.2 that the map

$$\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G;\eta)) \ni W \overset{\Upsilon_\eta}{\longmapsto} F \in C^\infty_{\tau^*}(G;\eta)$$

defined as in (1.7) yields an injective linear map. For a nonzero element $F \in C^\infty_{\tau^*}(G;\eta)$ and a nonzero vector $v \in V_\tau$, we put $f_v := \langle F(\cdot), v\rangle_{(V_\tau^* \otimes \boldsymbol{E}) \times V_\tau} \in C^\infty(G;\eta)$. Then F lies in the image of Υ_η if and only if

$$U(\mathfrak{g})^L f_v \simeq \boldsymbol{X} \quad \text{as } (\mathfrak{g}, K)\text{-modules.} \tag{1.14}$$

It follows from the Hahn-Banach extension theorem that the G-homomorphism

$$C^\infty(G;\eta) \ni f \longmapsto \langle f(\cdot), \zeta_0\rangle_{\boldsymbol{E}\times\boldsymbol{E}'} \in C^\infty(G) \tag{1.15}$$

is injective because $U(\mathfrak{n})\zeta_0$ is weak $*$-dense in $\boldsymbol{E}'$. Then (1.14) and (1.15) together with (1.13) imply that $F \in \mathrm{Im}\,\Upsilon_\eta$ if and only if $(\mathcal{D}(\eta)F)(\zeta_0) = 0$. Since the function

F is η-covariant, the latter condition is equivalent to $(\mathcal{D}(\eta)F)(U(\mathfrak{n})\zeta_0) = 0$. This implies $\mathcal{D}(\eta)F = 0$ (and vice versa) by virtue of the Banach-Steinhaus theorem. □

We will apply the above kernel theorem later in this paper to describe generalized Whittaker models for irreducible highest weight representations.

2. Differential operators, and lowest or highest weight modules

From now on, we assume that $K\backslash G$ is an irreducible Hermitian symmetric space with G-invariant complex structure. We consider the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ. In this section we describe, following [**4**], the differential operators $\mathcal{D}_{\tau^*}$ of gradient type on $K\backslash G$ whose K-finite kernels realize the dual lowest weight $(\mathfrak{g}, K)$-modules $L(\tau)^*$ (Theorem 2.6). This combined with Theorem 1.3 enables us to identify the maximal globalization of $L(\tau)^*$ with the full (not necessarily K-finite) kernel space of $\mathcal{D}_{\tau^*}$ (Proposition 2.7). We also specify for later use the solutions of differential equation $\mathcal{D}_{\tau^*}F = 0$ of exponential type.

2.1. Simple Lie group of Hermitian type. — We begin with summarizing some basic facts on fine structure for simple Lie groups of Hermitian type, following the notation in [**41**, Part I, §5] and [**8**, 3.3]. It is known that there exists a unique (up to sign) central element Z_0 of $\mathfrak{k}_0$ such that $\operatorname{ad} Z_0$ restricted to $\mathfrak{p}_0$ gives an $\operatorname{Ad}(K)$-invariant complex structure on $\mathfrak{p}_0$. One gets a triangular decomposition of $\mathfrak{g}$ as follows:

$$\begin{aligned} &\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{such that} \\ &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \end{aligned} \tag{2.1}$$

where $\mathfrak{p}_\pm$ denotes the eigenspace of $\operatorname{ad} Z_0$ on $\mathfrak{g}$ with eigenvalue $\pm\sqrt{-1}$ respectively. We extend $\operatorname{ad} Z_0$ on $\mathfrak{p}_0$ to a G-invariant complex structure on the Hermitian symmetric space $K\backslash G$ canonically through the identification $\mathfrak{p}_0 = T(K\backslash G)_{Ke}$, the tangent space of $K\backslash G$ at the origin Ke.

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and for each $\gamma \in \Delta$ the corresponding root subspace of $\mathfrak{g}$ will be denoted by $\mathfrak{g}(\mathfrak{t}; \gamma)$:

$$\mathfrak{g}(\mathfrak{t}; \gamma) = \{X \in \mathfrak{g} \mid (\operatorname{ad} H)X = \gamma(H)X \ \text{ for all } H \in \mathfrak{t}\}.$$

We can choose root vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ $(\gamma \in \Delta)$ such that

$$X_\gamma - X_{-\gamma},\ \sqrt{-1}(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma, \tag{2.2}$$

where H_γ is the element of $\sqrt{-1}\mathfrak{t}_0$ corresponding the coroot $\gamma^\vee := 2\gamma/(\gamma,\gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B of $\mathfrak{g}$. Let Δ_c (resp. Δ_n) denote the subset of all compact (resp. noncompact) roots in Δ.

Take a positive system Δ^+ of Δ compatible with the decomposition (2.1):

$$\mathfrak{p}_\pm = \bigoplus_{\gamma\in\Delta_n^+} \mathfrak{g}(\mathfrak{t};\pm\gamma) \quad \text{with} \quad \Delta_n^+ := \Delta^+ \cap \Delta_n,$$

and fix a lexicographic order on $\sqrt{-1}\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \ldots, \gamma_r$.

Now, put $\mathfrak{t}^- := \sum_{k=1}^r \mathbb{C}H_{\gamma_k} \subset \mathfrak{t}$, and denote by $\gamma^- \in (\mathfrak{t}^-)^*$ the restriction to $\mathfrak{t}^-$ of a linear form $\gamma \in \mathfrak{t}^*$. For integers k, l with $1 \leqslant l < k \leqslant r$, we define subsets P_{kl}, P_k, P_0 of Δ_n^+ and subsets C_{kl}, C_k, C_0 of Δ_c^+ respectively by

$$P_{kl} := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k + \gamma_l}{2}\right)^- \right\}, \tag{2.3}$$

$$C_{kl} := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k - \gamma_l}{2}\right)^- \right\}, \tag{2.4}$$

$$P_k := \left\{ \gamma \in \Delta_n^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k}{2}\right)^- \right\}, \quad C_k := \left\{ \gamma \in \Delta_c^+ \,\middle|\, \gamma^- = \left(\frac{\gamma_k}{2}\right)^- \right\}, \tag{2.5}$$

$$P_0 := \{\gamma_1, \gamma_2, \ldots, \gamma_r\}, \quad C_0 := \{\gamma \in \Delta_c^+ \mid \gamma^- = 0\}. \tag{2.6}$$

By Harish-Chandra the subsets Δ_n^+ and Δ_c^+ are decomposed as

$$\Delta_n^+ = \Big(\bigcup_{1\leqslant k\leqslant r} P_k\Big) \bigcup P_0 \bigcup \Big(\bigcup_{1\leqslant l<k\leqslant r} P_{kl}\Big),$$
$$\Delta_c^+ = C_0 \bigcup \Big(\bigcup_{1\leqslant k\leqslant r} C_k\Big) \bigcup \Big(\bigcup_{1\leqslant l<k\leqslant r} C_{kl}\Big),$$

where the unions are disjoint. Moreover the maps

$$C_{kl} \ni \gamma \longmapsto \gamma + \gamma_l \in P_{kl} \quad \text{and} \quad C_k \ni \gamma \longmapsto -\gamma + \gamma_k \in P_k \tag{2.7}$$

give rise to bijections from C_{kl} to P_{kl} and from C_k to P_k respectively. Note that the subsets P_{kl} and C_{kl} are always non-empty, and that P_k and C_k $(1 \leqslant k \leqslant r)$ are empty if and only if the Hermitian symmetric space $K\backslash G$ is analytically equivalent to a tube domain.

We now introduce a *Cayley transform* $\boldsymbol{c} = \mathrm{Ad}(c)$ on $\mathfrak{g}$ defined by the following element of $G_{\mathbb{C}}^\circ$:

$$c = \exp\Big(\frac{\pi}{4}\sum_{k=1}^r (X_{\gamma_k} - X_{-\gamma_k})\Big), \tag{2.8}$$

where $G_{\mathbb{C}}^\circ$ denotes a connected Lie group with Lie algebra $\mathfrak{g}$. Note that $-\boldsymbol{c}^2$ gives the identity map on $\mathfrak{t}^-$. It follows that

$$\mathfrak{a}_{\mathfrak{p},0} := \boldsymbol{c}^{-1}(\mathfrak{t}^- \cap \sqrt{-1}\mathfrak{t}_0) = \boldsymbol{c}(\mathfrak{t}^- \cap \sqrt{-1}\mathfrak{t}_0) \tag{2.9}$$

is a maximal abelian subspace of $\mathfrak{p}_0$, and that the elements

$$H_k := c^{-1}(H_{\gamma_k}) = -c(H_{\gamma_k}) = X_{\gamma_k} + X_{-\gamma_k} \quad (k = 1, 2, \ldots, r) \tag{2.10}$$

form an orthogonal basis of vector space $\mathfrak{a}_{\mathfrak{p},0}$ with respect to the inner product defined by the Killing form B. This implies in particular that r equals the real rank of G. The restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}_{\mathfrak{p}}$ has been described by Moore in terms of linear forms $\psi_k := \gamma_k \circ (c|\mathfrak{a}_{\mathfrak{p}})$ on $\mathfrak{a}_{\mathfrak{p}}$ (see e.g., [**8**, Th.3.5] for the description).

2.2. Generalized Verma module and its maximal submodule. — Let (τ, V_τ) be any irreducible finite-dimensional representation of K with Δ_c^+-highest weight $\lambda = \lambda(\tau)$. We consider the *generalized Verma $U(\mathfrak{g})$-module* induced from τ:

$$M(\tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau.$$

Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{k} + \mathfrak{p}_+$ by letting $\mathfrak{p}_+$ act on V_τ trivially: $\mathfrak{p}_+ V_\tau = \{0\}$. $M(\tau)$ has a structure of $(\mathfrak{g}, K)$-module through

$$D' \cdot (D \otimes v) := D'D \otimes v, \quad k \cdot (D \otimes v) := \mathrm{Ad}(k)D \otimes \tau(k)v$$

for $D' \in U(\mathfrak{g})$, $k \in K$ and $D \otimes v \in M(\tau)$ with $D \in U(\mathfrak{g})$, $v \in V_\tau$. Let $N(\tau)$ be the unique maximal proper $(\mathfrak{g}, K)$-submodule of $M(\tau)$. Then the quotient $L(\tau) := M(\tau)/N(\tau)$ gives an irreducible $(\mathfrak{g}, K)$-module with Δ^+-highest weight λ.

We now summarize for later use some basic known results concerning the structure of $N(\tau)$. One finds from the decomposition (2.1) that $M(\tau) = U(\mathfrak{p}_-)V_\tau$ is canonically isomorphic to the tensor product $S(\mathfrak{p}_-) \otimes V_\tau = S(\mathfrak{p}_-) \otimes_{\mathbb{C}} V_\tau$ as a K-module, where $S(\mathfrak{p}_-)$ ($\simeq U(\mathfrak{p}_-)$) denotes the symmetric algebra of $\mathfrak{p}_-$ looked upon as a K-module by the adjoint action. This isomorphism yields a gradation of the K-module $M(\tau)$:

$$M(\tau) = \bigoplus_{j=0}^{\infty} M_j(\tau) \quad \text{with} \quad M_j(\tau) := S^j(\mathfrak{p}_-)V_\tau \simeq S^j(\mathfrak{p}_-) \otimes V_\tau. \tag{2.11}$$

Here we write $S^j(\mathfrak{p}_-)$ for the K-submodule of $S(\mathfrak{p}_-)$ consisting of all homogeneous elements of $S(\mathfrak{p}_-)$ of degree j. Note that the submodule $N(\tau)$ is graded:

$$N(\tau) = \bigoplus_{j=0}^{\infty} N_j(\tau) \quad \text{with} \quad N_j(\tau) := N(\tau) \cap M_j(\tau), \tag{2.12}$$

because $N(\tau)$ is stable under the action of the central element $\sqrt{-1}Z_0 \in \mathfrak{t}$ which gives the gradation $S(\mathfrak{p}_-) = \oplus_{j=0}^{\infty} S^j(\mathfrak{p}_-)$.

Since $M(\tau) = S(\mathfrak{p}_-)V_\tau$ is finitely generated over the Noetherian ring $S(\mathfrak{p}_-)$, so is the submodule $N(\tau)$, too. This implies that, if $N(\tau) \neq \{0\}$, there exist finitely many irreducible K-submodules $W_1, \ldots, W_q$ of $N(\tau)$ such that

$$N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.13}$$

for some positive integers i_u $(u = 1, \dots, q)$ arranged as

$$i(\tau) := i_1 = \min\{j \mid N_j(\tau) \neq \{0\}\}. \tag{2.14}$$

We call $i(\tau)$ *the level of reduction* of $M(\tau)$.

An irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$ is called *unitarizable* if $\boldsymbol{X}$ is isomorphic to the Harish-Chandra module $\boldsymbol{H}_K$ of an irreducible unitary representation of G on a Hilbert space $\boldsymbol{H}$. The unitarizable highest weight $(\mathfrak{g}, K)$-modules have been completely classified by Enright, Howe and Wallach [**7**]. Note that their work contains case-by-case analysis, and that it uses some results of former contributors such as [**12**], [**6**], etc. Later, Enright and Joseph [**5**], and also Jakobsen [**20**] gave a more intrinsic classification.

For unitarizable $L(\tau)$'s, [**5**] gives a simple description of the maximal submodule $N(\tau)$ as follows. Assume that $L(\tau)$ is unitarizable and that $N(\tau) \neq \{0\}$. Then the level $i(\tau)$ of reduction of $M(\tau)$ is an integer such that $1 \leqslant i(\tau) \leqslant r$, where r is the real rank of G as in 2.1. Let $Q_{i(\tau)}$ be the irreducible K-submodule of $S^{i(\tau)}(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \cdots - \gamma_{r-i(\tau)+1}$. Then the tensor product $Q_{i(\tau)} \otimes V_\tau$ has a unique irreducible K-submodule W_1, called the Parthasarathy, Rao and Varadarajan component (the PRV-component for short), with extreme weight $\lambda - \gamma_r - \cdots - \gamma_{r-i(\tau)+1}$. Noting that

$$Q_{i(\tau)} \otimes V_\tau \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau \simeq M_{i(\tau)}(\tau), \tag{2.15}$$

we regard W_1 as a K-submodule of $M_{i(\tau)}(\tau)$.

Theorem 2.1 **([5, 5.2, 6.5 and 8.3], see also [3, 3.1]).** — *Keep the above notation. If $L(\tau)$ is unitarizable and if the maximal submodule $N(\tau)$ of $M(\tau)$ does not vanish, $N(\tau)$ is a highest weight $(\mathfrak{g}, K)$-module generated over $S(\mathfrak{p}_-)$ by the PRV-component W_1:*

$$N(\tau) = S(\mathfrak{p}_-) W_1.$$

2.3. A realization of lowest weight module $L(\tau)^*$. — For each irreducible representation (τ, V_τ) of K, let $L(\tau)^*$ be the irreducible lowest weight $(\mathfrak{g}, K)$-module which is dual to $L(\tau)$. This subsection gives after [**4**] a realization of $L(\tau)^*$ as the K-finite kernel of a certain G-invariant differential operator of gradient type defined on the symmetric space $K\backslash G$. This together with the kernel theorem (Corollary 1.8) will tell us how to describe the $(\mathfrak{g}, K)$-embeddings of highest weight module $L(\tau)$ into various induced G-representations.

Now, let $O^*_{\tau^*}(G)$ denote the space of functions F in $C^\infty_{\tau^*}(G)$ (see (1.1)) satisfying

$$X^L F = 0 \quad \text{for all} \quad X \in \mathfrak{p}_+. \tag{2.16}$$

Then we see that $O^*_{\tau^*}(G)$ is a closed G-submodule of $C^\infty_{\tau^*}(G)$ through right translation R, and that it is canonically isomorphic to the space of anti-holomorphic sections of the G-homogeneous vector bundle on $K\backslash G$ associated to the K-module V^*_τ.

It is useful to employ another realization of the G-module $O^*_{\tau^*}(G)$ as a space of holomorphic V^*_τ-valued functions on a bounded domain $\mathcal{B}$ of $\mathfrak{p}_-$. To be more precise, we take a connected linear Lie group G° with a covering homomorphism

$$\varpi : G \to G^\circ.$$

Such a G° always exists (we can take $G^\circ = \mathrm{Ad}(G)$ for example). Let $G^\circ_{\mathbb{C}}$ denote the connected complexification of G°. We write K°, $K^\circ_{\mathbb{C}}$ and $P_\pm = \exp \mathfrak{p}_\pm$ for the connected Lie subgroups of $G^\circ_{\mathbb{C}}$ with Lie algebras $\mathfrak{k}_0$, $\mathfrak{k}$ and $\mathfrak{p}_\pm$, respectively. Note that the exponential map gives holomorphic diffeomorphisms from $\mathfrak{p}_\pm$ onto $P_\pm$. Consider an open dense subset $P_+K^\circ_{\mathbb{C}}P_-$ of $G^\circ_{\mathbb{C}}$, which is holomorphically diffeomorphic to the direct product $P_+ \times K^\circ_{\mathbb{C}} \times P_-$ through multiplication. For each $x \in P_+K^\circ_{\mathbb{C}}P_-$, let $p_+(x)$, $k_{\mathbb{C}}(x)$, and $p_-(x)$ denote respectively the elements of P_+, $K^\circ_{\mathbb{C}}$, and P_- such that $x = p_+(x)k_{\mathbb{C}}(x)p_-(x)$. We set $\xi(x) := \log p_-(x) \in \mathfrak{p}_-$. Note that $G^\circ \subset P_+K^\circ_{\mathbb{C}}P_-$. We extend the assignment $x \mapsto \xi(x) \quad (x \in G^\circ)$ to a map, denoted again by $\xi(x)$, from G to $\mathfrak{p}_-$ through ϖ. This (extended) ξ naturally induces an anti-holomorphic diffeomorphism, say $\tilde{\xi}$, from the symmetric space $K\backslash G$ onto a bounded domain

$$\mathcal{B} := \{\xi(x) \in \mathfrak{p}_- \mid x \in G\} \tag{2.17}$$

of $\mathfrak{p}_-$, where $\tilde{\xi}(Kx) := \xi(x)$. (See for example [**16**, 7.129].) Let $K_{\mathbb{C}}$ denote the complexification of K. Then, ϖ restricted to K yields a covering homomorphism from $K_{\mathbb{C}}$ to $K^\circ_{\mathbb{C}}$, and the map $x \mapsto k_{\mathbb{C}}(x)$ $(x \in G^\circ)$ lifts to a map from G to $K_{\mathbb{C}}$ which we denote again by $k_{\mathbb{C}}(x)$ $(x \in G)$.

Let $O(\mathcal{B}, V^*_\tau)$ be the space of all V^*_τ-valued holomorphic functions on $\mathcal{B}$. It is easily verified that the above $\tilde{\xi}$ gives rise to a linear isomorphism Θ from $O^*_{\tau^*}(G)$ onto $O(\mathcal{B}, V^*_\tau)$ by

$$(\Theta F)(\tilde{\xi}(Kx)) := \tau^*(k_{\mathbb{C}}(x))^{-1}F(x) \quad (x \in G) \tag{2.18}$$

for $F \in O^*_{\tau^*}(G)$. Then $O(\mathcal{B}, V^*_\tau)$ has a G-module structure inherited from $(R, O^*_{\tau^*}(G))$ through Θ:

$$(g \cdot f)(\xi(x)) = \tau^*(k_{\mathbb{C}}(\exp \xi(x)\, g))f(\xi(xg)) \quad (x \in G) \tag{2.19}$$

for $g \in G$ and $f \in O(\mathcal{B}, V^*_\tau)$. Here one should notice that

$$\exp \xi(x)\, g = (p_+(x)k_{\mathbb{C}}(x))^{-1}xg \in P_+K^\circ_{\mathbb{C}}G^\circ \subset P_+K^\circ_{\mathbb{C}}P_-$$

for $x, g \in G^\circ$, and that the map

$$\mathcal{B} \times G^\circ \ni (z, g) \longmapsto k_{\mathbb{C}}(\exp z\, g) \in K^\circ_{\mathbb{C}}$$

lifts to a map from $\mathcal{B} \times G$ to $K_{\mathbb{C}}$ in the canonical way (cf. [**4**, Prop.4.7]).

By differentiating the G-action (2.19) one obtains a $\mathfrak{g}$-module $O(\mathcal{B}, V^*_\tau)$. We remark that the action of each element Y in $\mathfrak{p}_-$ is described simply as

$$(Y \cdot f)(z) = \frac{d}{dt} f(z + tY)|_{t=0} \quad (z \in \mathcal{B}). \tag{2.20}$$

An $f \in O(\mathcal{B}, V_\tau^*)$ is K-finite if and only if f is a polynomial, because one sees from (2.19) that

$$(h_t \cdot f)(z) = \tau^*(h_t) f(e^{\sqrt{-1}t} z)$$

for $h_t := \exp tZ_0 \in K$ $(t \in \mathbb{R})$, where Z_0 is the central element of $\mathfrak{k}_0$ defined in 2.1. Hence the K-finite part $O^*_{\tau^*}(G)_K$ of $O^*_{\tau^*}(G)$ is isomorphic, through Θ, to the space $\mathcal{P}(\mathfrak{p}_-, V_\tau^*) = S(\mathfrak{p}_+) \otimes V_\tau^*$ of V_τ^*-valued polynomial functions on $\mathfrak{p}_-$. Here we identify the symmetric algebra $S(\mathfrak{p}_+)$ of $\mathfrak{p}_+$ with the ring of polynomial functions on $\mathfrak{p}_-$ through the Killing form B restricted to $\mathfrak{p}_+ \times \mathfrak{p}_-$.

We now define a bilinear form $\langle \cdot, \cdot \rangle_\tau$ on $O^*_{\tau^*}(G) \times (U(\mathfrak{g}) \otimes_{\mathbb{C}} V_\tau)$ by

$$\langle F, D \otimes v \rangle_\tau := \langle D^L F(e), v \rangle = \langle ({}^T D)^R F(e), v \rangle \tag{2.21}$$

for $F \in O^*_{\tau^*}(G), D \in U(\mathfrak{g})$, and $v \in V_\tau$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $V_\tau^* \times V_\tau$, and $D \mapsto {}^T D$ the principal anti-automorphism of $U(\mathfrak{g})$, respectively. Then it is a routine task to verify that $\langle \cdot, \cdot \rangle_\tau$ naturally gives rise to a $(\mathfrak{g}, K)$-invariant bilinear form on $O^*_{\tau^*}(G) \times M(\tau)$, which we denote again by $\langle \cdot, \cdot \rangle_\tau$. Note that this pairing is described through the above isomorphism Θ as

$$\langle F, D \otimes v \rangle_\tau = \langle ({}^T D \cdot f)(0), v \rangle \quad \text{with} \quad f := \Theta F \in O(\mathcal{B}, V_\tau^*),$$

where $D \in U(\mathfrak{p}_-) = S(\mathfrak{p}_-)$, $v \in V_\tau$, and ${}^T D \cdot f$ is defined through the directional derivative action (2.20). This implies the following

Lemma 2.2 **(cf. [3, §2])**

(1) *The $(\mathfrak{g}, K)$-invariant pairing $\langle \cdot, \cdot \rangle_\tau$ is nondegenerate on $O^*_{\tau^*}(G)_K \times M(\tau)$.*

(2) *Let $R(\tau^*)$ be the orthogonal of the maximal submodule $N(\tau)$ in $O^*_{\tau^*}(G)_K \simeq \mathcal{P}(\mathfrak{p}_-, V_\tau^*)$ with respect to $\langle \cdot, \cdot \rangle_\tau$. Then $R(\tau^*)$ is the unique, nonzero irreducible $(\mathfrak{g}, K)$-submodule of $O^*_{\tau^*}(G)_K$, and it is isomorphic to the lowest weight module $L(\tau)^*$ dual to $L(\tau) = M(\tau)/N(\tau)$. The $(\mathfrak{g}, K)$-isomorphism A_{τ^*} from $L(\tau)^*$ onto $R(\tau^*)$ is given by*

$$\langle A_{\tau^*}(\varphi), w \rangle_\tau = \langle \varphi, \ w + N(\tau) \rangle_{L(\tau)^* \times L(\tau)} \quad (w \in M(\tau))$$

for $\varphi \in L(\tau)^*$.

We are now going to introduce a differential operator of gradient type whose K-finite kernel characterizes the $(\mathfrak{g}, K)$-module $R(\tau^*) = A_{\tau^*}(L(\tau)^*)$. For this, we take a basis $X_1, \ldots, X_s$ of the $\mathbb{C}$-vector space $\mathfrak{p}_+$ such that $B(X_j, \overline{X}_k) = \delta_{jk}$ (Kronecker's δ), where $\overline{X}_i \in \mathfrak{p}_-$ denotes the complex conjugate of $X_i \in \mathfrak{p}_+$ with respect to the real form $\mathfrak{g}_0$. Set

$$X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s} \in U(\mathfrak{p}_+) \quad \text{and} \quad \overline{X}^\alpha := \overline{X}_1^{\alpha_1} \cdots \overline{X}_s^{\alpha_s} \in U(\mathfrak{p}_-)$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_s)$ of nonnegative integers $\alpha_1, \ldots, \alpha_s$. We denote by $|\alpha| := \alpha_1 + \cdots + \alpha_s$ the length of α. For each positive integer n we define the gradients ∇^n and $\overline{\nabla}^n$ of order n on $C^\infty_{\tau^*}(G)$ as follows.

$$\nabla^n F(x) := \sum_{|\alpha|=n} \frac{1}{\alpha!}\overline{X}^\alpha \otimes (X^\alpha)^L F(x),$$

$$\overline{\nabla}^n F(x) := \sum_{|\alpha|=n} \frac{1}{\alpha!}X^\alpha \otimes (\overline{X}^\alpha)^L F(x),$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$, where $\alpha! := \alpha_1!\cdots\alpha_n!$. It is then easy to see that $\nabla^n F$ and $\overline{\nabla}^n F$ are independent of the choice of a basis $X_1, \dots, X_s$, and that the operators ∇^n and $\overline{\nabla}^n$ give continuous G-homomorphisms

$$\nabla^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(-n)}(G), \quad \overline{\nabla}^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(+n)}(G).$$

Here $\tau^*(\pm n)$ denotes the K-representation on the tensor product $S^n(\mathfrak{p}_\pm) \otimes V^*_\tau$ respectively.

Let W_u $(u = 1, \dots, q)$ be, as in (2.13), the irreducible K-submodules of $S^{i_u}(\mathfrak{p}_-)V_\tau \subset N(\tau)$ which generate $N(\tau)$ over $S(\mathfrak{p}_-)$ when $N(\tau) \neq \{0\}$. For each u, the adjoint operator P_u of the embedding

$$W_u \hookrightarrow S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.22}$$

gives a surjective K-homomorphism:

$$P_u : S^{i_u}(\mathfrak{p}_+) \otimes V^*_\tau \simeq (S^{i_u}(\mathfrak{p}_-) \otimes V_\tau)^* \longrightarrow W^*_u. \tag{2.23}$$

Definition 2.3. — Under the above notation, let $\mathcal{D}_{\tau^*}$ be a G-invariant differential operator from $C^\infty_{\tau^*}(G)$ to $C^\infty_\rho(G)$ defined by

$$\mathcal{D}_{\tau^*}F(x) := \nabla^1 F(x) \oplus (\oplus_{u=1}^q P_u(\overline{\nabla}^{i_u}F(x)))$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. Here we write $\rho = \rho(\tau^*)$ for the representation of K on

$$(\mathfrak{p}_- \otimes V^*_\tau) \oplus (\oplus_{u=1}^q W^*_u),$$

and $\mathcal{D}_{\tau^*}$ should be understood as $\mathcal{D}_{\tau^*} = \nabla^1$ if $N(\tau) = \{0\}$, or equivalently $M(\tau) = L(\tau)$. We call $\mathcal{D}_{\tau^*}$ the *differential operator of gradient type* associated to τ^*.

Remark 2.4. — A function $F \in C^\infty_{\tau^*}(G)$ lies in the G-submodule $O^*_{\tau^*}(G)$ defined by (2.16) if and only if $\nabla^1 F = 0$. Hence we have $\operatorname{Ker}\mathcal{D}_{\tau^*} \subset O^*_{\tau^*}(G)$ for every τ^*, and the equality holds if and only if $N(\tau) = \{0\}$.

Remark 2.5. — If $L(\tau)$ is unitarizable, one sees from Theorem 2.1 that

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (P_1 \circ \overline{\nabla}^{i(\tau)}).$$

Here $i(\tau)$ is the level of reduction of $M(\tau)$, and the K-homomorphism P_1 is defined through the PRV-component $W_1 \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau$.

The following theorem, equivalent to [**4**, Prop.7.6] due to Davidson and Stanke, realizes the lowest weight module $L(\tau)^*$ by means of $\mathcal{D}_{\tau^*}$.

***Theorem 2.6* (cf. [4]).** — *The image $R(\tau^*)$ of the $(\mathfrak{g},K)$-embedding A_{τ^*} from $L(\tau)^*$ into $O^*_{\tau^*}(G)_K$ defined in Lemma 2.2 coincides with the K-finite kernel of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type:*

$$R(\tau^*) = \{F \in C^\infty_{\tau^*}(G) \mid F \text{ is right } K\text{-finite and } \mathcal{D}_{\tau^*}F = 0\}.$$

2.4. Maximal globalization of $L(\tau)^*$. — The above theorem together with Theorem 1.3 implies that the full kernel space $\operatorname{Ker}\mathcal{D}_{\tau^*}$ gives a maximal globalization of the lowest weight module $L(\tau)^*$, as follows.

Proposition 2.7. — (1) *The closure $R(\tau^*)^-$ of $R(\tau^*) = A_{\tau^*}(L(\tau)^*)$ in $C^\infty_{\tau^*}(G)$ coincides with* $\operatorname{Ker}\mathcal{D}_{\tau^*}$. *It coincides also with the orthogonal, say $R'(\tau^*)$, of $N(\tau)$ in the whole (not necessarily K-finite) space $O^*_{\tau^*}(G)$ with respect to the paring $\langle\ \cdot, \cdot\ \rangle_\tau$ in* (2.21).

(2) *One has an isomorphism of G-modules*

$$\operatorname{Hom}_{\mathfrak{g},K}(L(\tau),\ C^\infty(G)) \simeq \operatorname{Ker}\mathcal{D}_{\tau^*}(= R(\tau^*)^- = R'(\tau^*))$$

by the correspondence given in Theorem 1.2 through the canonical K-embedding i_τ of V_τ into $L(\tau)$.

Proof. — The statements except $R(\tau^*)^- = R'(\tau^*)$ follow immediately from Theorems 1.3 and 2.6. The equality $R(\tau^*)^- = R'(\tau^*)$ can be shown just as in the proof of Theorem 1.3, by bearing in mind that $R'(\tau^*)$ is K-stable. □

We end this section by specifying for later use the solutions $F \in O^*_{\tau^*}(G)$ of exponential type of the differential equation $\mathcal{D}_{\tau^*}F = 0$.

For each $X \in \mathfrak{p}_+$ and each $v^* \in V^*_\tau$, let $f_{X,v^*} = \exp X \otimes v^*$ denote the V^*_τ-valued holomorphic function on $\mathfrak{p}_-$ defined by

$$f_{X,v^*}(z) := \exp B(X,z)\cdot v^* \quad (z \in \mathfrak{p}_-).$$

We set $F_{X,v^*} := \Theta^{-1}f_{X,v^*} \in O^*_{\tau^*}(G)$. Then the function F_{X,v^*} is described as

$$F_{X,v^*}(x) = \exp B(X,\xi(x))\cdot\tau^*(k_{\mathbb{C}}(x))v^* \quad (x \in G) \tag{2.24}$$

by the definition of Θ (see (2.18)).

Proposition 2.8. — *The function F_{X,v^*} satisfies the differential equation $\mathcal{D}_{\tau^*}F = 0$ if and only if*

$$P_u(X^{i_u} \otimes v^*) = 0 \quad \text{for } u = 1,\ldots,q. \tag{2.25}$$

Here P_u is a K-homomorphism from $S^{i_u}(\mathfrak{p}_+)\otimes V^*_\tau$ onto W^*_u defined in (2.22) *and in* (2.23).

Proof. — Let $D \in S(\mathfrak{p}_-)$. In view of (2.20) we observe that

$$D^R F_{X,v^*} = \Theta^{-1}(D\cdot f_{X,v^*}) = D(X)F_{X,v^*},$$

because f_{X,v^*} is an exponential function defined by X, where $D \in S(\mathfrak{p}_-)$ in the right hand side is looked upon as a polynomial on $\mathfrak{p}_+ = \mathfrak{p}_-^*$. It then follows that F_{X,v^*} is orthogonal to $N(\tau) = \sum_{u=1}^q S(\mathfrak{p}_-)W_u$ with respect to the $\mathfrak{g}$-invariant pairing $\langle \cdot, \cdot \rangle_\tau$ in (2.21) if and only if

$$\langle F_{X,v^*}, w\rangle_\tau = 0 \quad \text{for all } w \in W_u \ (u = 1, \dots, q). \tag{2.26}$$

We now express $w \in W_u$ as $w = \sum_{j=1}^N D_j v_j$ with $D_j \in S^{i_u}(\mathfrak{p}_-)$ and $v_j \in V_\tau$. Then the left hand side of (2.26) is calculated as

$$\langle F_{X,v^*}, w\rangle_\tau = (-1)^{i_u} \sum_{j=1}^N D_j(X)\langle v^*, v_j\rangle = (-1)^{i_u}(X^{i_u} \otimes v^*, w),$$

where $(\cdot, \cdot)$ denotes the dual pairing between $S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^*$ and $S^{i_u}(\mathfrak{p}_-) \otimes V_\tau$. Hence, the element F_{X,v^*} is orthogonal to W_u with respect to $\langle \cdot, \cdot \rangle_\tau$ if and only if the linear form $X^{i_u} \otimes v^*$ on $S^{i_u}(\mathfrak{p}_-) \otimes V_\tau$ vanishes on the subspace W_u, or equivalently $P_u(X^{i_u} \otimes v^*) = 0$ by the definition of P_u. We thus conclude that (2.25) gives a necessary and sufficient condition for $\mathcal{D}_{\tau^*} F_{X,v^*} = 0$ by Proposition 2.7 (1). □

In the next section we will study the condition (2.25) in connection with the associated variety and the multiplicity of highest weight module $L(\tau)$.

3. Associated variety and multiplicity of highest weight modules

The purpose of this section is to understand the associated variety and multiplicity for each irreducible highest weight module $L(\tau)$ by means of the principal symbol $\boldsymbol{\sigma}$ of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type. The harvest of our discussion is summarized as Theorem 3.11. The symbol $\boldsymbol{\sigma}$ yields a $K_{\mathbb{C}}$-homogeneous vector bundle on the unique open orbit $\mathcal{O}_{m(\tau)}$ in $\mathcal{V}(L(\tau))$. The dimension of fibers can be understood as the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$ which defines the variety $\mathcal{V}(L(\tau))$, and a result of Joseph (cf. Theorem 3.7) tells us $I_{m(\tau)}L(\tau) = \{0\}$, i.e., $L(\tau)/I_{m(\tau)}L(\tau) = L(\tau)$, for unitarizable $L(\tau)$'s.

3.1. $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ in $\mathfrak{p}_+$. — We keep the notation in 2.1. Let us begin by describing the $K_{\mathbb{C}}$-orbit decomposition of the vector space $\mathfrak{p}_+$ under the adjoint action. For every integer m such that $0 \leqslant m \leqslant r = \mathbb{R}\text{-rank}\, G$, we set

$$\mathcal{O}_m := \mathrm{Ad}(K_{\mathbb{C}})X(m) \quad \text{with} \quad X(m) := \sum_{k=r-m+1}^{r} X_{\gamma_k}. \tag{3.1}$$

Here $X_{\gamma_k} \in \mathfrak{g}(\mathfrak{t}; \gamma_k)$ (see (2.2)) is a root vector for noncompact positive root γ_k, and $X(0)$ should be understood as 0. It then follows that every $X \in \mathfrak{p}_+$ is conjugate to some $X(m)$ under $K_{\mathbb{C}}$:

$$\mathfrak{p}_+ = \mathcal{O}_0 \cup \cdots \cup \mathcal{O}_r.$$

In reality, there exists an element $k \in K$ such that

$$\mathrm{Ad}(k)(X + \overline{X}) \in \mathfrak{a}_{\mathfrak{p},0} = \sum_{k=1}^{r} \mathbb{R}(X_{\gamma_k} + X_{-\gamma_k})$$

(see (2.9) and (2.10)), since $X + \overline{X} \in \mathfrak{p}_0 = \mathrm{Ad}(K)\mathfrak{a}_{\mathfrak{p},0}$. This shows that

$$\mathrm{Ad}(k)X = \sum_{k \in I} c_k X_{\gamma_k} \quad \text{with} \quad c_k \in \mathbb{R}\backslash\{0\},$$

for some subset I of $\{1, \ldots, r\}$. Note that the complex torus

$$\exp\Big(\sum_{k \in I} \mathbb{C} H_{\gamma_k}\Big) \subset K_{\mathbb{C}}$$

acts on the set $\sum_{k \in I} \mathbb{C}^{\times} X_{\gamma_k}$ transitively, and that $\sum_{k \in I} X_{\gamma_k}$ is conjugate to $X(|I|)$ under the action of the Weyl group $N_K(\mathfrak{a}_{\mathfrak{p},0})/Z_K(\mathfrak{a}_{\mathfrak{p},0})$ of the pair $(\mathfrak{g}_0, \mathfrak{a}_{\mathfrak{p},0})$ (see e.g., [**41**, Prop.5.1(3)]), where $|J|$ denotes the cardinal number of any set J. We thus find that $X \in \mathcal{O}_m$ with $m = |I|$, and that the elements $X(0), \ldots, X(m-1)$ are in the closure of the orbit $\mathcal{O}_m$ with respect to the usual topology (or the Zariski topology) on $\mathfrak{p}_+$.

One can compute the the dimension of each $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ as follows. In view of Harish-Chandra's result (2.3)–(2.7) on the restricted roots, we easily find that the tangent space $T_{X(m)}(\mathcal{O}_m) = [\mathfrak{k}, X(m)]$ of $\mathcal{O}_m$ at the point $X(m) \in \mathcal{O}_m$ is described as

$$[\mathfrak{k}, X(m)] = \bigoplus_{\gamma \in \Delta^+(m)} \mathfrak{g}(\mathfrak{t}; \gamma) \tag{3.2}$$

with

$$\Delta^+(m) := \{\gamma_r, \ldots, \gamma_{r-m+1}\} \bigcup \Big(\bigcup_{\substack{k>l \\ k>r-m}} P_{kl} \Big) \bigcup \Big(\bigcup_{k>r-m} P_k \Big). \tag{3.3}$$

Hence one obtains

$$\dim \mathcal{O}_m = m + \sum_{\substack{k>l \\ k>r-m}} |P_{kl}| + \sum_{k>r-m} |P_k|.$$

This implies in particular that

$$\dim \mathcal{O}_r > \dim \mathcal{O}_{r-1} > \cdots > \dim \mathcal{O}_0 = 0,$$

and that

$$\dim \mathcal{O}_m - \dim \mathcal{O}_{m-1} = 1 + |P_{r-m+1}| + \sum_{l<r-m+1} |P_{r-m+1,l}|.$$

Note that the right hand side of the above equality is at least two if either $P_{r-m+1} \neq \varnothing$ (namely, $K\backslash G$ is not of tube-type) or $m < r$.

Thus we have proved the following well-known result.

Proposition 3.1. — *The subspace $\mathfrak{p}_+$ splits into a disjoint union of $r+1$ number of $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \leqslant m \leqslant r)$: $\mathfrak{p}_+ = \coprod_{0 \leqslant m \leqslant r} \mathcal{O}_m$, and the closure $\overline{\mathcal{O}_m}$ of each orbit $\mathcal{O}_m$ is equal to $\cup_{k \leqslant m} \mathcal{O}_k$ for every m.*

3.2. Associated variety $\mathcal{V}(L(\tau))$. — Let $L(\tau) = M(\tau)/N(\tau)$ be, as in 2.2, the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type (τ, V_τ). Consider the annihilator ideal

$$\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau) := \{D \in S(\mathfrak{p}_-) \mid Dw = 0 \quad \text{for all } w \in L(\tau)\}.$$

of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$. It should be remarked that an element $D \in S(\mathfrak{p}_-)$ belongs to $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ if and only if $Dv = 0$ for all $v \in V_\tau$, since $L(\tau) = S(\mathfrak{p}_-) V_\tau$ with commutative algebra $S(\mathfrak{p}_-)$.

Definition 3.2. — The affine algebraic variety

$$\mathcal{V}(L(\tau)) := \{X \in \mathfrak{p}_+ \mid D(X) = 0 \quad \text{for all } D \in \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)\} \subset \mathfrak{p}_+$$

defined by the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is called the *associated variety* of the $(\mathfrak{g}, K)$-module $L(\tau)$. Here $S(\mathfrak{p}_-)$ is identified with the ring of polynomial functions on $\mathfrak{p}_+$ through the Killing form B of $\mathfrak{g}$.

Remark 3.3. — The notion of the associated variety has been introduced by Vogan **[33]** for arbitrary Harish-Chandra modules (see also **[44]**,**[8]**). As for the highest weight modules $L(\tau)$, the above definition of $\mathcal{V}(L(\tau))$ coincides with Vogan's original one. Indeed, let

$$\operatorname{gr} L(\tau) := \bigoplus_{n=0}^{\infty} U_n(\mathfrak{g}) V_\tau / U_{n-1}(\mathfrak{g}) V_\tau$$

be the graded $(S(\mathfrak{g}), K)$-module defined through the filtration

$$\{0\} := U_{-1}(\mathfrak{g}) V_\tau \subset V_\tau = U_0(\mathfrak{g}) V_\tau \subset \cdots \subset U_{n-1}(\mathfrak{g}) V_\tau \subset U_n(\mathfrak{g}) V_\tau \subset \dots,$$

of $L(\tau)$. Here $U_n(\mathfrak{g})$ $(n = 0, 1, \dots)$ denotes the natural increasing filtration of $U(\mathfrak{g})$, and $S(\mathfrak{g}) \simeq \oplus_{n=0}^{\infty} U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. Then one easily sees that $\mathfrak{k} + \mathfrak{p}_+$ annihilates $\operatorname{gr} L(\tau)$, and that

$$\operatorname{gr} L(\tau) \simeq L(\tau) \quad \text{as } (S(\mathfrak{p}_-), K)\text{-modules}$$

by (2.11) and (2.12). Hence the algebraic variety in $\mathfrak{g}^* = \mathfrak{g}$ (the identification through B) defined by the annihilator of $\operatorname{gr} L(\tau)$ in $S(\mathfrak{g})$, which is the associated variety by Vogan, is nothing but $\mathcal{V}(L(\tau))$.

Since the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is stable under $\mathrm{Ad}(K_{\mathbb{C}})$, so is the variety $\mathcal{V}(L(\tau))$. In view of Proposition 3.1, we see that there exists a unique integer $m = m(\tau)$ $(0 \leqslant m \leqslant r)$ such that

$$\mathcal{V}(L(\tau)) = \overline{\mathcal{O}_m} \quad \text{with} \quad \mathcal{O}_m = \mathrm{Ad}(K_{\mathbb{C}}) X(m) \quad \text{and} \quad m = m(\tau). \tag{3.4}$$

In particular, the variety $\mathcal{V}(L(\tau))$ is irreducible.

Now let I_m be the prime ideal of $S(\mathfrak{p}_-)$ associated to the irreducible variety $\overline{\mathcal{O}_m}$ $(m = 0, \ldots, r)$:

$$(3.5) \qquad I_m := \{D \in S(\mathfrak{p}_-) \mid D(X) = 0 \quad \text{for all } X \in \overline{\mathcal{O}_m}\}.$$

It holds that $I_r = \{0\}$ since $\overline{\mathcal{O}_r} = \mathfrak{p}_+$. If $m < r$, one knows that

$$(3.6) \qquad I_m = S(\mathfrak{p}_-)Q_{m+1}$$

by [**5**, 8.1] and [**21**, Prop.2.3], where Q_{m+1} denotes as in (2.15) the irreducible K-submodule of $S^{m+1}(\mathfrak{p}_-) \subset S(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \cdots - \gamma_{r-m}$.

By Hilbert's Nullstellensatz, $I_{m(\tau)}$ coincides with the radical of the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ for every τ. This allows us to deduce the following

Lemma 3.4. — *The annihilator in $S(\mathfrak{p}_-)$ of $(S(\mathfrak{p}_-), K)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ is equal to $I_{m(\tau)}$.*

Proof. — Since $\sqrt{\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)} = I_{m(\tau)}$, there exists an integer $n_0 > 0$ such that $B^{n_0} \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ for every $B \in Q_{m(\tau)+1}$, the finite-dimensional generating subspace of $I_{m(\tau)}$. This implies that

$$(3.7) \qquad (I_{m(\tau)})^{n_0} \subset \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau).$$

If $D \in \mathrm{Ann}_{S(\mathfrak{p}_-)}(L(\tau)/I_{m(\tau)}L(\tau))$, then $DL(\tau) \subset I_{m(\tau)}L(\tau)$. Inductively, one gets

$$(3.8) \qquad D^n L(\tau) \subset (I_{m(\tau)})^n L(\tau) \quad (n = 1, 2, \ldots).$$

We thus find from (3.7) and (3.8) that $D^{n_0} \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$, and so $D \in I_{m(\tau)}$. This proves the inclusion $\mathrm{Ann}_{S(\mathfrak{p}_-)}(L(\tau)/I_{m(\tau)}L(\tau)) \subset I_{m(\tau)}$. The converse inclusion is obvious. □

For each $X \in \mathfrak{p}_+$, let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ which defines the variety $\{X\}$ of one element X:

$$(3.9) \qquad \mathfrak{m}(X) := \sum_{Y \in \mathfrak{p}_-} (Y - B(X, Y))S(\mathfrak{p}_-).$$

We set

$$(3.10) \qquad \mathcal{W}(X, \tau) := L(\tau)/\mathfrak{m}(X)L(\tau).$$

Then we see that $\dim \mathcal{W}(X, \tau) < \infty$, and that the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at X acts on $\mathcal{W}(X, \tau)$ naturally. Note that, if J is an ideal of $S(\mathfrak{p}_-)$ that defines the variety $\mathcal{V}(L(\tau))$, then

$$(3.11) \qquad \mathfrak{m}(X) \supset J \iff X \in \mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}.$$

By applying [**33**, Cor.2.10 and Def.2.12] in view of Lemma 3.4, we immediately deduce

Proposition 3.5. — *Assume that $X \in \mathcal{O}_{m(\tau)}$. Then the dimension of $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ coincides with the multiplicity of the $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the unique minimal associated prime $I_{m(\tau)}$:*

$$\dim \mathcal{W}(X,\tau) = \operatorname{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau)) \quad (X \in \mathcal{O}_{m(\tau)}).$$

So in particular, one has $\mathcal{W}(X,\tau) \neq \{0\}$.

Remark 3.6. — See [**33**, Section 2] and also [**27**, 1.1] for the definition and elementary properties of the multiplicities of finitely generated modules over a commutative Noetherian ring (in connection with Harish-Chandra modules). The multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ of the whole $L(\tau)$ at $I_{m(\tau)}$ is described as

$$\sum_{j=0}^{n_0-1} \dim\{(I_{m(\tau)})^j L(\tau)/\mathfrak{m}(X)(I_{m(\tau)})^j L(\tau)\} \qquad (X \in \mathcal{O}_{m(\tau)}), \tag{3.12}$$

through the filtration

$$L(\tau) = (I_{m(\tau)})^0 L(\tau) \supset (I_{m(\tau)})^1 L(\tau) \supset \cdots \supset (I_{m(\tau)})^{n_0} L(\tau) = \{0\}$$

of the $(S(\mathfrak{p}_-), K)$-module $L(\tau)$. Here n_0 is as in (3.7), and the summand at $j = 0$ in (3.12) is equal to the above $\dim \mathcal{W}(X,\tau)$.

The above proposition will be used in the next subsection to study the associated variety $\mathcal{V}(L(\tau))$ in connection with the principal symbol of differential operator $\mathcal{D}_{\tau^*}$ of gradient type.

As for the unitarizable highest weight modules, the following remarkable result of Joseph (due to Davidson, Enright and Stanke [**3**] for $\mathfrak{g}$ classical) gives a clearer understanding of the above proposition.

Theorem 3.7 **([21, Lem.2.4 and Th.5.6]).** — *If $L(\tau)$ is unitarizable, the annihilator $\operatorname{Ann}_{S(\mathfrak{p}_-)} w$ in $S(\mathfrak{p}_-)$ of any nonzero vector $w \in L(\tau)$ coincides with the prime ideal $I_{m(\tau)}$. Especially, one has $\operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau) = I_{m(\tau)}$.*

Remark 3.8. — For unitarizable $L(\tau) = M(\tau)/N(\tau)$ with nonzero $N(\tau)$, the above theorem together with (3.6) implies the inequality:

$$i(\tau) \leqslant m(\tau) + 1,$$

where $i(\tau)$ is as in (2.14) the level of reduction of the generalized Verma module $M(\tau)$. A description of the number $m(\tau)$ in terms of $i(\tau)$ has been given in [**21**].

Corollary 3.9 **(to Prop.3.5 and Th.3.7).** — *One has*

$$\dim \mathcal{W}(X,\tau) = \operatorname{mult}_{I_{m(\tau)}}(L(\tau)) \quad (X \in \mathcal{O}_{m(\tau)})$$

for every irreducible unitarizable highest weight module $L(\tau)$.

For the classical groups $Sp(2n,\mathbb{R})$, $U(p,q)$ and $O^*(2p)$, Nishiyama, Ochiai and Taniguchi [**27**, Th.7.18 and Th.9.1] have described the associated cycle:

$$\mathcal{AC}(L(\tau)) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \cdot [\overline{\mathcal{O}_{m(\tau)}}], \tag{3.13}$$

and also the Bernstein degree

$$\mathrm{Deg}\, L(\tau) = \mathrm{mult}_{I_{m(\tau)}}(L(\tau)) \cdot \deg(\overline{\mathcal{O}_{m(\tau)}}) \tag{3.14}$$

of the unitarizable highest weight module $L(\tau)$ (our $\mathfrak{p}_+$ is replaced by $\mathfrak{p}_-$ in [**27**]) by using the theory of reductive dual pairs (G, G') with compact G'. They treat the case where the dual pair (G, G') is in the stable range with smaller G', and then the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$ is specified as the dimension of the corresponding irreducible representation of G', through detailed study of K-types of $L(\tau)$. On the other hand, the above corollary allows us to give another simple proof of this description of the multiplicity by investigating the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$, where the dual pairs (G, G') need not be in the stable range. We will do it later in Section 5 (see Theorems 5.14 and 5.15).

3.3. Principal symbol $\boldsymbol{\sigma}$ and associated cycle. — Let

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (\oplus_{u=1}^q P_u \circ \overline{\nabla}^{i_u})$$

be, as in Definition 2.3, the differential operator of gradient type whose kernel realizes the maximal globalization of dual lowest weight module $L(\tau)^*$ (see Proposition 2.7). We put

$$\boldsymbol{\sigma}(X, v^*) := \sum_{u=1}^{q} P_u(X^{i_u} \otimes v^*) \in W^* := \oplus_{u=1}^q W_u^* \tag{3.15}$$

for $X \in \mathfrak{p}_+$ and $v^* \in V_\tau^*$, where $P_u : S^{i_u}(\mathfrak{p}_+) \otimes V_\tau^* \longrightarrow W_u^*$ is the K-homomorphism in (2.23). Here $\boldsymbol{\sigma}$ should be understood as $\boldsymbol{\sigma}(X, v^*) = 0$ for every $X \in \mathfrak{p}_+$ and every $v^* \in V_\tau^*$, when $\mathcal{D}_{\tau^*} = \nabla^1$, or equivalently $N(\tau) = \{0\}$. Note that $\boldsymbol{\sigma}$ is naturally identified with the principal symbol at the origin Ke of differential operator $\mathcal{D}_{\tau^*}$, where the symbol is considered only on $\mathfrak{p}_+ \times V_\tau^*$ with the anti-holomorphic cotangent space $\mathfrak{p}_+ = \mathfrak{p}_-^*$ of $K\backslash G$ at Ke. By abuse of language, we call $\boldsymbol{\sigma}$ the principal symbol of $\mathcal{D}_{\tau^*}$ at the origin, since we are concerned mainly with the anti-holomorphic sections of G-homogeneous vector bundle $V_\tau^* {}_K\times G$.

We are now going to describe the associated variety $\mathcal{V}(L(\tau))$ by means of $\boldsymbol{\sigma}$. To do this, fix any $X \in \mathfrak{p}_+$ for a while. Then the map $v^* \mapsto \boldsymbol{\sigma}(X, v^*)$ gives a $K_{\mathbb{C}}(X)$-homomorphism $\boldsymbol{\sigma}(X, \cdot\,)$ from V_τ^* to W^*. Hence $\mathrm{Ker}\, \boldsymbol{\sigma}(X, \cdot\,)$ is a $K_{\mathbb{C}}(X)$-submodule of V_τ^*. By Proposition 2.8 we can describe $\mathrm{Ker}\, \boldsymbol{\sigma}(X, \cdot\,)$ as

$$\mathrm{Ker}\, \boldsymbol{\sigma}(X, \cdot\,) = \{v^* \in V_\tau^* \mid \mathcal{D}_{\tau^*} F_{X,v^*} = 0\},$$

where $F_{X,v^*} \in C_{\tau^*}^\infty(G)$ is the function of exponential type defined by (2.24).

The following lemma relates the above kernel with the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X,\tau)$ in (3.10).

Lemma 3.10. — *For each $X \in \mathfrak{p}_+$, the natural map*

$$V_\tau \hookrightarrow M(\tau) \to L(\tau) = M(\tau)/N(\tau) \to \mathcal{W}(X,\tau) = L(\tau)/\mathfrak{m}(X)L(\tau) \tag{3.16}$$

from V_τ onto $\mathcal{W}(X,\tau)$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}(X,\tau)^* \simeq \operatorname{Ker}\boldsymbol{\sigma}(X,\,\cdot\,) \subset V_\tau^* \tag{3.17}$$

through the contravariant functor $\operatorname{Hom}_{\mathbb{C}}(\,\cdot\,,\mathbb{C})$.

Proof. — First, the natural map from $M(\tau)$ to $\mathcal{W}(X,\tau)$ in (3.16) induces a linear isomorphism from $\mathcal{W}(X,\tau)^*$ onto the space $\mathcal{U}$ of all linear forms ψ on $M(\tau)$ satisfying

$$\psi \circ D = D(X)\psi \quad \text{for} \quad D \in S(\mathfrak{p}_-) \tag{3.18}$$

and

$$\psi|N(\tau) = 0 \quad \text{with} \quad N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{as in (2.13).} \tag{3.19}$$

In view of (3.18), one sees that the second condition (3.19) is equivalent to

$$\psi|W_u = 0 \quad \text{for } u = 1,\dots,q.$$

Second, pull back each $\psi \in \mathcal{U}$ to an element of V_τ^* through the embedding $V_\tau \hookrightarrow M(\tau)$:

$$\mathcal{U} \ni \psi \mapsto v^* := \psi|V_\tau \in V_\tau^*. \tag{3.20}$$

By (3.18), this map is injective. We can show just as in the proof of Proposition 2.8 that an element $v^* \in V_\tau^*$ lies in the image of the map (3.20) if and only if $P_u(X^{i_u} \otimes v^*) = 0$ for $u = 1,\dots,q$, or equivalently, $v^* \in \operatorname{Ker}\boldsymbol{\sigma}(X,\,\cdot\,)$. One thus gets the linear isomorphism (3.17), which is in fact a $K_{\mathbb{C}}(X)$-homomorphism since so is the map (3.16). $\square$

We are now in a position to give a characterization of the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$ and the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau)/I_{m(\tau)}L(\tau))$ in terms of the principal symbol $\boldsymbol{\sigma}$, as follows.

Theorem 3.11. — *Let $L(\tau)$ be any irreducible highest weight $(\mathfrak{g},K)$-module with extreme K-type τ, and let $\boldsymbol{\sigma} : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type associated to τ^*. Then it holds that*

$$\mathcal{V}(L(\tau)) = \{X \in \mathfrak{p}_+ \mid \operatorname{Ker}\boldsymbol{\sigma}(X,\,\cdot\,) \neq \{0\}\}. \tag{3.21}$$

Moreover, if X is an element of the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ of $\mathcal{V}(L(\tau))$, the dimension of vector space $\operatorname{Ker}\boldsymbol{\sigma}(X,\,\cdot\,)$ coincides with the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$ of $S(\mathfrak{p}_-)$ corresponding to the variety $\mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}$.

Remark 3.12. — We can get the same kind of characterization of the associated variety and the multiplicity also for irreducible $(\mathfrak{g}, K)$-modules of discrete series G-representations, by using the results of **[10]** and **[44]**. We will discuss it elsewhere.

Proof of Theorem 3.11. — We write $\mathcal{V}'$ for the set in the right hand side of (3.21). First, we immediately find that $\mathcal{V}'$ is an affine algebraic variety of $\mathfrak{p}_+$, by noting that

$$\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot) \neq \{0\} \iff \operatorname{rank}\boldsymbol{\sigma}(X, \cdot) \leqslant \dim V_\tau^* - 1.$$

Moreover $\mathcal{V}'$ is $K_\mathbb{C}$-stable, because one has

$$\operatorname{Ker}\boldsymbol{\sigma}(\operatorname{Ad}(k)X, \cdot) = \tau^*(k)\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot) \quad \text{for all } k \in K_\mathbb{C}$$

by the definition of $\boldsymbol{\sigma}$.

Second, the inclusion $\mathcal{O}_{m(\tau)} \subset \mathcal{V}'$ and the second assertion of the theorem are direct consequences of Proposition 3.5 and Lemma 3.10. If $X \notin \overline{\mathcal{O}_{m(\tau)}} = \mathcal{V}(L(\tau))$, we get $\mathfrak{m}(X) + \operatorname{Ann}_{S(\mathfrak{p}_-)}L(\tau) = S(\mathfrak{p}_-)$ by (3.11). This implies that

$$\mathfrak{m}(X)L(\tau) = (\mathfrak{m}(X) + \operatorname{Ann}_{S(\mathfrak{p}_-)}L(\tau))L(\tau) = L(\tau).$$

So one gets $\operatorname{Ker}\boldsymbol{\sigma}(X, \cdot) \simeq \mathcal{W}(X,\tau)^* = \{0\}$ again by Lemma 3.10. We thus find $\mathcal{O}_{m(\tau)} \subset \mathcal{V}' \subset \overline{\mathcal{O}_{m(\tau)}}$, and so $\mathcal{V}' = \mathcal{V}(L(\tau))$ as desired. □

4. Generalized Whittaker models for highest weight modules

In this section we describe the generalized Whittaker models for irreducible highest weight modules $L(\tau)$. The main results are summarized as Theorems 4.7–4.9. We find that each $L(\tau)$ embeds, with nonzero and finite multiplicity, into the generalized Gelfand-Graev representation $\Gamma_{m(\tau)}$ attached to the Cayley transform of the open $K_\mathbb{C}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. It is shown that, if $L(\tau)$ is unitarizable, the multiplicity of $(\mathfrak{g}, K)$-embeddings $L(\tau) \hookrightarrow \Gamma_{m(\tau)}$ coincides with the multiplicity of $L(\tau)$ at the defining prime ideal of $\mathcal{V}(L(\tau))$.

4.1. Generalized Gelfand-Graev representations. — We keep the notation in 2.1 and 3.1. We begin with introducing in this subsection the generalized Gelfand-Graev representations of G attached to the Cayley transforms of nilpotent $K_\mathbb{C}$-orbits $\mathcal{O}_m = \operatorname{Ad}(K_\mathbb{C})X(m)$ in $\mathfrak{p}_+$, where m ranges over the integers such that $0 \leqslant m \leqslant r = \mathbb{R}\text{-rank}\, G$.

For this, we consider an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$(4.1) \quad X(m) = \sum_{k=r-m+1}^{r} X_{\gamma_k}, \quad H(m) := \sum_{k=r-m+1}^{r} H_{\gamma_k}, \quad Y(m) := \sum_{k=r-m+1}^{r} X_{-\gamma_k},$$

with commutation relation

$$\begin{cases} [H(m), X(m)] = 2X(m), & [H(m), Y(m)] = -2Y(m), \\ [X(m), Y(m)] = H(m). \end{cases}$$

We put

$$
(4.2)\quad \begin{cases} X'(m) := -\sqrt{-1}\boldsymbol{c}^{-1}(X(m)) = \dfrac{\sqrt{-1}}{2}(H(m) - X(m) + Y(m)), \\ H'(m) := \boldsymbol{c}^{-1}(H(m)) = X(m) + Y(m) = \displaystyle\sum_{k=r-m+1}^{r} H_k \ \text{(cf. (2.10))}, \\ Y'(m) := \sqrt{-1}\boldsymbol{c}^{-1}(Y(m)) = -\dfrac{\sqrt{-1}}{2}(H(m) + X(m) - Y(m)), \end{cases}
$$

where $\boldsymbol{c} = \mathrm{Ad}(c)$, with c as in (2.8), is the Cayley transform on $\mathfrak{g}$. Then $(X'(m), H'(m), Y'(m))$ forms an $\mathfrak{sl}_2$-triple in the real form $\mathfrak{g}_0$ of $\mathfrak{g}$, since $\overline{H(m)} = -H(m)$, $\overline{X(m)} = Y(m)$ by (2.2). Set $\mathcal{O}'_m := \mathrm{Ad}(G)X'(m)$. We note that the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponds to the $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+ \subset \mathfrak{p}$ through the Kostant-Sekiguchi correspondence (cf. [**8**, Th.3.1]).

Lemma 4.1 ([**8**, Lemma 3.2])

(1) *The Lie algebra* $\mathfrak{g}$ *decomposes into a direct sum of the* j*-eigensubspaces* $\mathfrak{g}_j(m)$ *for* $\mathrm{ad}\,H'(m)$ *as*

$$\mathfrak{g} = \mathfrak{g}_{-2}(m) \oplus \mathfrak{g}_{-1}(m) \oplus \mathfrak{g}_0(m) \oplus \mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m).$$

(2) *Let* $\Delta(m,j)$ $(j = 0, \pm1, \pm2)$ *be the subsets of the root system* Δ *of* $(\mathfrak{g}, \mathfrak{t})$ *defined by*

$$\Delta(m,2) := \{\gamma_{r-m+1}, \ldots, \gamma_r\} \bigcup \Big(\bigcup_{r-m<l<k} P_{kl} \Big),$$

$$(4.3)\quad \Delta(m,1) := \Big(\bigcup_{l\leqslant r-m<k} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{r-m<k} (P_k \cup C_k) \Big),$$

$$(4.4)\quad \begin{aligned} \Delta^+(m,0) := C_0 \bigcup \{\gamma_1, \ldots, \gamma_{r-m}\} \bigcup \Big(\bigcup_{r-m<l<k} C_{kl} \Big) \\ \bigcup \Big(\bigcup_{l<k\leqslant r-m} (P_{kl} \cup C_{kl}) \Big) \bigcup \Big(\bigcup_{k\leqslant r-m} (P_k \cup C_k) \Big), \end{aligned}$$

$$(4.5)\quad \Delta(m,0) := \Delta^+(m,0) \cup (-\Delta^+(m,0)), \quad \Delta(m,-j) := -\Delta(m,j) \quad (j = 1,2).$$

Then each subspace $\boldsymbol{c}(\mathfrak{g}_j(m)) = \mathrm{Ad}(c)\mathfrak{g}_j(m)$ *is described in terms of the root subspaces* $\mathfrak{g}(\mathfrak{t};\gamma)$ *as*

$$\boldsymbol{c}(\mathfrak{g}_j(m)) = \begin{cases} \oplus_{\gamma\in\Delta(m,j)}\mathfrak{g}(\mathfrak{t};\gamma) & \text{if } j \neq 0, \\ \mathfrak{t} \oplus \left(\oplus_{\gamma\in\Delta(m,0)}\mathfrak{g}(\mathfrak{t};\gamma)\right) & \text{if } j = 0. \end{cases}$$

Now we set

$$\Delta^-(m) := (\Delta(m,-2) \cup \Delta(m,-1)) \cap \Delta_n = -\Delta^+(m) \ \text{(cf. (3.3))}.$$

Let $\mathfrak{p}_-(m)$ and $\mathfrak{n}(m)$ be nilpotent, abelian Lie subalgebras of $\mathfrak{g}$ defined respectively by

$$(4.6)\qquad \mathfrak{p}_-(m) := \bigoplus_{\gamma\in\Delta^-(m)} \mathfrak{g}(\mathfrak{t};\gamma) \quad \text{and} \quad \mathfrak{n}(m) := \boldsymbol{c}(\mathfrak{p}_-(m)).$$

Note that $\mathfrak{p}_-(r) = \mathfrak{p}_-$. If $K\backslash G$ is of tube type, the Lie subalgebra $\mathfrak{n}(m)$ is stable under the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

We get the following lemma on the structure of these subalgebras $\mathfrak{p}_-(m)$ and $\mathfrak{n}(m)$.

Lemma 4.2

(1) *One has the equality*

$$(4.7)\qquad \mathfrak{p}_-(m) = [\mathfrak{k}, Y(m)].$$

Namely, $\mathfrak{p}_-(m)$ *is canonically isomorphic to the tangent space at the point* $Y(m)$ *of* $K_{\mathbb{C}}$*-orbit* $\mathrm{Ad}(K_{\mathbb{C}})Y(m)$ *in* $\mathfrak{p}_-$*.*

(2) *Let* $\mathfrak{v}(m)$ *be the subspace of* $\mathfrak{g}_1(m)$ *such that*

$$\mathfrak{v}(m) := \boldsymbol{c}^{-1}(\oplus_{\gamma\in\Xi(m)}\mathfrak{g}(\mathfrak{t};\gamma))$$

with

$$(4.8)\qquad \Xi(m) := \Big(\bigcup_{l\leqslant r-m<k} P_{kl}\Big) \bigcup \Big(\bigcup_{k>r-m} C_k\Big) \subset \Delta(m,1).$$

Then it holds that

$$(4.9)\qquad \mathfrak{n}(m) = \mathfrak{v}(m)\oplus\mathfrak{g}_2(m) \quad \textit{and} \quad \dim\mathfrak{v}(m) = \frac{1}{2}\dim\mathfrak{g}_1(m).$$

Proof. — First, (4.7) is a direct consequence of (3.2). To prove (2), we note that

$$\boldsymbol{c}^2 = \mathrm{Ad}(c)^2 = \prod_{k=1}^{r} s_{\gamma_k} \quad \text{with} \quad s_{\gamma_k} := \mathrm{Ad}\big(\exp\frac{\pi}{2}(X_{\gamma_k} - X_{-\gamma_k})\big),$$

gives rise to an element of the Weyl group of $(\mathfrak{g},\mathfrak{t})$ such that

$$\boldsymbol{c}^2\gamma_k = -\gamma_k, \quad \boldsymbol{c}^2 C_k = -P_k, \quad \boldsymbol{c}^2 P_k = -C_k$$

for $k = 1,\ldots,r$. In fact, s_{γ_k} gives the orthogonal reflection with respect to γ_k, and (2.7) implies $\boldsymbol{c}^2 C_k = -P_k$ and so $\boldsymbol{c}^2 P_k = -C_k$. We thus find that

$$\boldsymbol{c}^2\Delta^-(m) = \Delta(m,2)\cup\Xi(m) \quad \text{(disjoint union)},$$

and correspondingly

$$\mathfrak{n}(m) = \mathfrak{v}(m)\oplus\mathfrak{g}_2(m)$$

by Lemma 4.1(2). In view of (4.3) and (4.8), one gets the second equality in (4.9). □

Let η_m be the one-dimensional representation (i.e., character) of abelian Lie subalgebra $\mathfrak{n}(m) = \mathfrak{v}(m)\oplus\mathfrak{g}_2(m)$ defined by

$$(4.10)\qquad \eta_m(U) := \sqrt{-1}B(U,\theta X'(m)) = -\sqrt{-1}B(U,Y'(m)) \quad \text{for} \quad U\in\mathfrak{n}(m).$$

Here θ denotes the complexified Cartan involution of $\mathfrak{g}$, and B the Killing form of $\mathfrak{g}$. Then, just as in Definition 1.6 we get a C^∞-induced G- and $(\mathfrak{g}, K)$-representation $\Gamma_m := \Gamma_{\eta_m}$ acting on the space

$$C^\infty(G; \eta_m) = \{f \in C^\infty(G) \mid U^R f = -\eta_m(U) f \quad (U \in \mathfrak{n}(m))\} \tag{4.11}$$

through left translation L. Note that

$$C^\infty(G; \eta_r) \subset C^\infty(G; \eta_{r-1}) \subset \cdots \subset C^\infty(G; \eta_0) = C^\infty(G), \tag{4.12}$$

since one sees $\mathfrak{n}(m) \subset \mathfrak{n}(m')$ and $\eta_{m'}|\mathfrak{n}(m) = \eta_m$ for $m \leqslant m'$.

Definition 4.3. — We call $(\Gamma_m, C^\infty(G; \eta_m))$ the *generalized Gelfand-Graev representation* (GGGR for short) of G attached to the nilpotent orbit $\mathcal{O}'_m = \mathrm{Ad}(G)X'(m)$ in $\mathfrak{g}_0$.

Remark 4.4. — The GGGRs attached to arbitrary nilpotent orbits have been constructed in full generality by Kawanaka [**14**] for reductive algebraic groups. See also [**40**] for the GGGRs of real semisimple Lie groups.

Remark 4.5. — It should be noticed that the above Γ_m's are slightly different from the C^∞-induced GGGRs discussed in [**40**]. In fact, we extend η_m to a linear form on the Lie subalgebra $\mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m)$ by (4.10). Let ζ_m be the irreducible unitary representation of the nilpotent Lie subgroup

$$\tilde{N}(m) := \exp((\mathfrak{g}_1(m) \oplus \mathfrak{g}_2(m)) \cap \mathfrak{g}_0)$$

of G which corresponds to the coadjoint orbit $\mathrm{Ad}^*(\tilde{N}(m))(-\sqrt{-1}\eta_m)$ by the Kirillov orbit method. In [**40**, Def.1.11], the C^∞-GGGR attached to $\mathcal{O}'_m$ is defined to be the representation $C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m)$ of G induced from ζ_m in C^∞-context.

Nevertheless, we can show just as in [**40**, Prop.4.10] that $\mathfrak{n}(m)$ is a totally complex, positive polarization of the linear form $-\sqrt{-1}\eta_m$ on the Lie algebra of $\tilde{N}(m)$. This implies that

$$C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m) \hookrightarrow \Gamma_m \quad \text{as } G\text{-modules},$$

and the image of this embedding is always dense in Γ_m. So we treat Γ_m in this paper instead of $C^\infty\text{-}\mathrm{Ind}_{\tilde{N}(m)}^G(\zeta_m)$.

4.2. Generalized Whittaker models. — For any irreducible finite-dimensional K-module (τ, V_τ), let $L(\tau) = M(\tau)/N(\tau)$ (see 2.2) be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Consider the GGGRs $(\Gamma_m, C^\infty(G; \eta_m))$ ($m = 0, \ldots, r$) induced from the characters $\eta_m : \mathfrak{n}(m) \to \mathbb{C}$. We say that $L(\tau)$ has a *generalized Whittaker model* of type η_m if $L(\tau)$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $C^\infty(G; \eta_m)$.

We are going to describe the generalized Whittaker models for $L(\tau)$ by specifying the vector space $\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G; \eta_m))$ of $(\mathfrak{g}, K)$-homomorphisms from $L(\tau)$ into $C^\infty(G;\, \eta_m)$. To do this, let $\mathcal{D}_{\tau^*} : C^\infty_{\tau^*}(G) \to C^\infty_\rho(G)$ be, as in Definition 2.3, the

G-invariant differential operator of gradient type whose kernel realizes the maximal globalization of lowest weight module $L(\tau)^*$ (see Proposition 2.7). We set

$$(4.13)\qquad \begin{aligned}\mathcal{Y}(\tau, m) &:= \operatorname{Ker} \mathcal{D}_{\tau^*}(\eta_m)\\ &= \{F \in C^\infty_{\tau^*}(G) \mid \mathcal{D}_{\tau^*}F = 0,\ \ U^R F = -\eta_m(U)F\ (U \in \mathfrak{n}(m))\}.\end{aligned}$$

Then the kernel theorem (Corollary 1.8) gives a linear isomorphism

$$(4.14)\qquad \operatorname{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau, m)$$

through the correspondence (1.7).

Now our aim is to describe the space $\mathcal{Y}(\tau, m)$ for each τ and m. For this purpose, we essentially utilize the following unbounded realization of Hermitian symmetric space $K\backslash G$.

***Proposition 4.6* (cf. [16, page 455], [9], [32]).** — *Retain the notation at the beginning of 2.3, and consider the open dense subset $P_+K^\circ_{\mathbb{C}}P_-$ of $G^\circ_{\mathbb{C}}$ with $P_\pm = \exp\mathfrak{p}_\pm$.*

(1) One has $G^\circ c \subset P_+K^\circ_{\mathbb{C}}P_-$, where c is the Cayley element of $G^\circ_{\mathbb{C}}$ in (2.8).

(2) Set $\xi'(x) := \log p_-(xc) \in \mathfrak{p}_-$ $(x \in G^\circ)$, where $xc = p_+(xc)k_{\mathbb{C}}(xc)p_-(xc)$ with $k_{\mathbb{C}}(xc) \in K^\circ_{\mathbb{C}}$ and $p_\pm(xc) \in P_\pm$. Extend the assignment $x \mapsto \xi'(x)$ $(x \in G^\circ)$ to a map from G to $\mathfrak{p}_-$ through the covering homomorphism $\varpi : G \to G^\circ$. Then, the extended $\xi'(x)$ $(x \in G)$ sets up an anti-holomorphic diffeomorphism, say $\tilde{\xi}'$, from $K\backslash G$ onto an unbounded domain

$$\mathcal{S} := \{\xi'(x) \mid x \in G\} \subset \mathfrak{p}_-.$$

Note that the map $x \mapsto k_{\mathbb{C}}(xc)$ $(x \in G^\circ)$ lifts to a map from G to $K_{\mathbb{C}}$ (cf. [32]). We write $k_{\mathbb{C}}(x\cdot c)$ $(x \in G)$ for this lift.

We are now in a position to state the principal results of this article. Let $\mathcal{O}_{m(\tau)}$ be, as in (3.4), the unique open $K_{\mathbb{C}}$-orbit in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. Among the generalized Whittaker models for $L(\tau)$, those of type $\eta_{m(\tau)}$ are most important. We obtain the following theorem on the corresponding linear space $\mathcal{Y}(\tau, m)$ with $m = m(\tau)$.

Theorem 4.7. — *Let (τ, V_τ) be an irreducible finite-dimensional representation of K. Set $m = m(\tau)$ and $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m)$ for short. Then,*

(1) $\mathcal{Y}(\tau)$ is a nonzero, finite-dimensional vector space.

(2) For any $F \in \mathcal{Y}(\tau)$, there exists a unique polynomial function φ on $\mathfrak{p}_-$ with values in V^*_τ such that

$$F(x) = \exp B(X(m), \xi'(x))\tau^*(k_{\mathbb{C}}(x\cdot c))\varphi(\xi'(x)) \quad (x \in G).$$

(3) Let $\boldsymbol{\sigma} : \mathfrak{p}_+ \times V^*_\tau \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type, defined by (3.15). Consider the functions $F_{X(m),v^*} \in C^\infty_{\tau^*}(G)$ of exponential type in Proposition 2.8. Then the assignment

$$v^* \longmapsto c^R F_{X(m),v^*} := F_{X(m),v^*}(\cdot\, c) \quad (v^* \in \operatorname{Ker}\boldsymbol{\sigma}(X(m),\ \cdot\ ))$$

yields an injective linear map

$$\chi_\tau : \operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,) \hookrightarrow \mathcal{Y}(\tau).$$

The linear map χ_τ is not surjective in general. In fact, if $L(\tau)$ is finite-dimensional, one has $\operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,) = V_\tau^*$ since $X(m) = X(0) = 0$ in this case. However Lemma 1.1 implies that $\mathcal{Y}(\tau) \simeq L(\tau)^*$.

Nevertheless, we can show the surjectivity of χ_τ for relevant $L(\tau)$'s.

Theorem 4.8. — *Assume that $L(\tau)$ is unitarizable. Then the linear embedding χ_τ in Theorem 4.7 is surjective. Hence one gets*

$$\operatorname{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau) \simeq \operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,) \simeq \mathcal{W}(X(m), \tau)^* \tag{4.15}$$

as vector spaces, where $m = m(\tau)$, and $\mathcal{W}(X(m),\tau) = L(\tau)/\mathfrak{m}(X(m))L(\tau)$ is as in (3.10). *Moreover, the dimension of the vector spaces in* (4.15) *equals the multiplicity* $\operatorname{mult}_{I_m}(L(\tau))$ *of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the unique associated prime $I_m \subset S(\mathfrak{p}_-)$ by Corollary 3.9.*

Theorem 4.7 for $m = m(\tau)$ allows us to deduce the following result on the structure of $\mathcal{Y}(\tau, m')$ for $m' \neq m(\tau)$.

Theorem 4.9. — *The linear space $\mathcal{Y}(\tau, m')$ vanishes (resp. is infinite-dimensional) if $m' > m(\tau)$ (resp. $m' < m(\tau)$).*

Remark 4.10. — Theorem 4.8 recovers, to a great extent, our earlier work [**41**, Part II] on the generalized Whittaker models for the holomorphic discrete series $L(\tau) = M(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+p_+)} V_\tau$:

$$\operatorname{Hom}_{\mathfrak{g},K}(M(\tau), C^\infty(G;\eta_r)) \simeq V_\tau^*.$$

Moreover, the above three theorems applied to the special case $m(\tau) = r$ gives an answer to Problem 12.7 (*for* $i = 0$) posed in [**41**]. But this answer does not seem to be new. In fact, D. H. Collingwood kindly informed me in 1992 that he had settled Problem 12.7.

Remark 4.11. — The vanishing of $\mathcal{Y}(\tau, m')$ ($m' > m(\tau)$) in Theorem 4.9 follows also from a general result of Matumoto [**22**, Th.2].

The following three subsections 4.3–4.5 will be devoted to proving the above three theorems.

4.3. Key lemmas. — In this subsection we prepare two lemmas which are crucially important to prove Theorems 4.7 and 4.8.

The first lemma is the following somewhat surprising result of Jakobsen.

Lemma 4.12 ([**20**, Prop.2.9]). — *Let g be any element of $G_{\mathbb{C}}^{\circ}$, and let $L(\tau)$ be an irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Then one has the equality*

$$\mathrm{Ann}_{U(\mathrm{Ad}(g)\mathfrak{p}_-)} L(\tau) = \mathrm{Ad}(g)(\mathrm{Ann}_{U(\mathfrak{p}_-)} L(\tau))$$

on the annihilator of $L(\tau)$ in $U(\mathrm{Ad}(g)\mathfrak{p}_-)$ and that in $U(\mathfrak{p}_-) = S(\mathfrak{p}_-)$.

Second, for each integer $m = 0, \ldots, r$, let J_m denote the ideal of $S(\mathfrak{p}_-)$ generated by the elements $Y - B(X(m), Y)$ $(Y \in \mathfrak{p}_-(m))$:

$$J_m = \sum_{Y \in \mathfrak{p}_-(m)} (Y - B(X(m), Y)) S(\mathfrak{p}_-). \tag{4.16}$$

A method of Joseph (cf. [**5**, 2.4]) for describing the lowest weight vector of irreducible K-module $Q_{m+1} = I_m \cap S^{m+1}(\mathfrak{p}_-)$ (see (3.6)) can be applied to deduce the following

Lemma 4.13. — *It holds that $I_m + J_m = \mathfrak{m}(X(m))$. Here I_m (see (3.5)) is the prime ideal of $S(\mathfrak{p}_-)$ corresponding to the irreducible algebraic variety $\overline{\mathcal{O}_m}$, and $\mathfrak{m}(X(m))$ (see (3.9)) is the maximal ideal of $S(\mathfrak{p}_-)$ corresponding to $X(m) \in \mathfrak{p}_+$.*

Proof. — The inclusion $I_m + J_m \subset \mathfrak{m}(X(m))$ is obvious since any polynomial in I_m or in J_m vanishes at $X(m)$ by definition. If $m = r$, the equality $I_r + J_r = \mathfrak{m}(X(r))$ holds since $I_r = \{0\}$ and $J_r = \mathfrak{m}(X(r))$.

Now we assume that $m < r$. In order to prove the sum $I_m + J_m$ exhausts the whole $\mathfrak{m}(X(m))$, we consider the subspace

$$\mathfrak{q}_m := \sum_{\gamma \in \Delta_n \cap \Delta^+(m,0)} \mathfrak{g}(\mathfrak{t}; -\gamma) \subset \mathfrak{p}_-.$$

(See (4.4) for the definition of $\Delta^+(m, 0)$.) Then one gets $\mathfrak{p}_- = \mathfrak{p}_-(m) \oplus \mathfrak{q}_m$ as vector spaces, and hence

$$\mathfrak{m}(X(m)) = J_m + \mathfrak{q}_m S(\mathfrak{p}_-) \tag{4.17}$$

by the definitions of $\mathfrak{m}(X(m))$ and J_m.

We set

$$\Omega(m) := (\Delta_c \cap \Delta(m,0)) \setminus \Big(\bigcup_{r-m<l<k} (C_{kl} \cup -C_{kl}) \Big) \quad \text{(cf. (4.5))},$$

and let $\mathfrak{k}_m$ be the Lie subalgebra of $\mathfrak{k}$ defined by

$$\mathfrak{k}_m := \mathfrak{t} \oplus (\oplus_{\gamma \in \Omega(m)} \mathfrak{g}(\mathfrak{t}; \gamma)).$$

We write $(K_{\mathbb{C}})_m$ for the analytic subgroup of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{k}_m$. Note that $\mathfrak{q}_m, J_m$ and I_m are all stable under the adjoint action of $(K_{\mathbb{C}})_m$. Further, by using (2.7) one easily checks that $\mathfrak{q}_m$ is an irreducible $(K_{\mathbb{C}})_m$-module with lowest weight vector $X_{-\gamma_{r-m}} \in \mathfrak{q}_m$. This together with (4.17) reduces our task to showing

$$X_{-\gamma_{r-m}} \in I_m + J_m, \tag{4.18}$$

which can be done as follows.

Let $Q_{m+1} = I_m \cap S^{m+1}(\mathfrak{p}_-)$ be the irreducible K-submodule of I_m with lowest weight $-\gamma_r - \cdots - \gamma_{r-m}$. Take a nonzero lowest weight vector $D_{m+1} \in Q_{m+1}$. By virtue of [**5**, 2.4(*)], we find that

$$D_{m+1} \equiv cX_{-\gamma_r} \cdots X_{-\gamma_{r-m}} \mod J_m$$

for some nonzero constant $c \in \mathbb{C}$. This implies that

$$D_{m+1} \equiv \{c \prod_{k>r-m} B(X_{\gamma_k}, X_{-\gamma_k})\} \cdot X_{-\gamma_{r-m}} \mod J_m.$$

Thus we have obtained (4.18) as desired. □

4.4. A role of the Cayley transform. — Keep the notation at the beginning of 2.3. We recall that the bounded realization $\mathcal{B} = \{\xi(x) \mid x \in G\} \subset \mathfrak{p}_-$ of $K\backslash G$ gives a linear isomorphism

$$\Theta : O^*_{\tau^*}(G) \xrightarrow{\sim} O(\mathcal{B}, V^*_\tau)$$

by (2.18). Let $O(\mathfrak{p}_-, V^*_\tau)$ be the space of all holomorphic functions on the whole $\mathfrak{p}_-$ with values in V^*_τ. Naturally, we regard $O(\mathfrak{p}_-, V^*_\tau)$ as a subspace of $O(\mathcal{B}, V^*_\tau)$. Set

$$O^*_{\tau^*}(G)_0 := \Theta^{-1} O(\mathfrak{p}_-, V^*_\tau).$$

Just in the same way, the unbounded realization

$$\mathcal{S} = \{\xi'(x) = \xi(\varpi(x)c) \mid x \in G\} \subset \mathfrak{p}_-$$

of $K\backslash G$ in Proposition 4.6 gives a linear isomorphism

$$\Theta^c : O^*_{\tau^*}(G) \xrightarrow{\sim} O(\mathcal{S}, V^*_\tau),$$

by

$$\Theta^c F(\xi'(x)) := \tau^*(k_{\mathbb{C}}(x \cdot c))^{-1} F(x) \quad (x \in G; F \in O^*_{\tau^*}(G)). \tag{4.19}$$

See also [**32**, 2.4]. Similarly we put

$$O^*_{\tau^*}(G)^c_0 := (\Theta^c)^{-1} O(\mathfrak{p}_-, V^*_\tau).$$

Then the composite $(\Theta^c)^{-1} \circ \Theta$ induces an isomorphism from $O^*_{\tau^*}(G)_0$ onto $O^*_{\tau^*}(G)^c_0$ as vector spaces. This is exactly the (well-defined) right translation of functions on G by the Cayley element $c \in G^\circ_{\mathbb{C}}$:

$$O^*_{\tau^*}(G)_0 \ni F \longmapsto c^R F \in O^*_{\tau^*}(G)^c_0, \tag{4.20}$$

where

$$c^R F(x) = \tau^*(k_{\mathbb{C}}(x \cdot c))(\Theta F)(\xi'(x)) \quad \text{with} \quad \xi'(x) = \log p_-(\varpi(x)c)$$

for $x \in G$.

The function $c^R F = ((\Theta^c)^{-1} \circ \Theta)(F)$ can be interpreted as follows. First, take an open neighbourhood U of e (the identity element) in G such that $\varpi : G \to G^\circ$

restricted to U gives a diffeomorphism from U onto $U^\circ := \varpi(U)$ ($\subset G^\circ$). Define a V_τ^*-valued function F° on U° by setting

$$F^\circ(\varpi(x)) = F(x) \quad (x \in U). \tag{4.21}$$

One sees that F° extends, in a unique way, to a (multi-valued) complex analytic function $\tilde{F}^\circ$ on the open dense subset $P_+K_{\mathbb{C}}^\circ P_-$ of $G_{\mathbb{C}}^\circ$, and that $\tilde{F}^\circ$ comes from the function

$$P_+ \times K_{\mathbb{C}} \times P_- \ni (p_+, k_{\mathbb{C}}, p_-) \longmapsto \tau^*(k_{\mathbb{C}})(\Theta F)(\log p_-) \in V_\tau^* \tag{4.22}$$

through the covering map from $K_{\mathbb{C}}$ to $K_{\mathbb{C}}^\circ$. Second, we consider the right translation $c^R\tilde{F}^\circ$ of $\tilde{F}^\circ$ by the Cayley element $c \in G_{\mathbb{C}}^\circ$. This is a complex analytic function defined on $P_+K_{\mathbb{C}}^\circ P_-c^{-1}$. Noting that $G^\circ \subset P_+K_{\mathbb{C}}^\circ P_-c^{-1}$ by Proposition 4.6 (1), we write c^RF° for the restriction to G° of $c^R\tilde{F}^\circ$. Then our $c^RF = ((\Theta^c)^{-1} \circ \Theta)(F)$ gives a (single-valued) lift of c^RF° to G.

The following proposition assures that the above right translation c^R preserves the kernel of differential operator $\mathcal{D}_{\tau^*}$.

Proposition 4.14. — *Let $\mathcal{D}_{\tau^*} : C_{\tau^*}^\infty(G) \to C_\rho^\infty(G)$ be the differential operator of gradient type associated to τ^*. Then* (4.20) *yields a linear isomorphism*

$$\operatorname{Ker} \mathcal{D}_{\tau^*} \cap O_{\tau^*}^*(G)_0 \simeq \operatorname{Ker} \mathcal{D}_{\tau^*} \cap O_{\tau^*}^*(G)_0^c.$$

Namely, a function F in $O_{\tau^*}^*(G)_0$ satisfies the differential equation $\mathcal{D}_{\tau^*}F = 0$ if and only if the corresponding c^RF in $O_{\tau^*}^*(G)_0^c$ satisfies the same equation.

As shown in the next subsection, this proposition together with two key lemmas in 4.3 allows us to describe the space $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, m(\tau))$ of generalized Whittaker functions on G associated to the highest weight module $L(\tau)$.

Proof of Proposition 4.14. — Let $F \in O_{\tau^*}^*(G)_0$. We employ the interpretation of c^RF and also the notation given just before the proposition. Note that $\mathcal{D}_{\tau^*}$ naturally gives rise to a right $G_{\mathbb{C}}^\circ$-invariant, holomorphic differential operator, say $\tilde{\mathcal{D}}_{\tau^*}^\circ$ defined on the complex group $G_{\mathbb{C}}^\circ$.

Now assume that $\mathcal{D}_{\tau^*}F = 0$. Then one finds that $\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ = 0$ on $P_+K_{\mathbb{C}}^\circ P_-$. In reality, $\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ$ is the complex analytic extension of $\tilde{\mathcal{D}}_{\tau^*}^\circ F^\circ$ on U°, and the latter $\tilde{\mathcal{D}}_{\tau^*}^\circ F^\circ$ equals zero by assumption (cf. (4.21)). We thus get

$$\tilde{\mathcal{D}}_{\tau^*}^\circ(c^R\tilde{F}^\circ) = c^R(\tilde{\mathcal{D}}_{\tau^*}^\circ\tilde{F}^\circ) = 0 \quad \text{on } P_+K_{\mathbb{C}}P_-c^{-1}.$$

This implies that $\mathcal{D}_{\tau^*}(c^RF) = 0$, because $\mathcal{D}_{\tau^*}(c^RF)$ is a lift to G of the restriction $(\tilde{\mathcal{D}}_{\tau^*}^\circ(c^R\tilde{F}^\circ))|G^\circ$.

The reverse implication can be proved in the same way by using the inverse Cayley transform. □

4.5. Proof of the main theorems. — We are now ready to prove our main theorems given in 4.2.

Proof of Theorem 4.7. — Let F be any function in $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, m)$ (see (4.13)) with $m = m(\tau)$. Set $f^c := \Theta^c F \in \mathcal{O}(\mathcal{S}, V_\tau^*)$.

STEP 1. We first see that the requirement $U^R F = -\eta_m(U)F$ ($U \in \mathfrak{n}(m) = \boldsymbol{c}(\mathfrak{p}_-(m))$) for F is equivalent to

$$D_1 \cdot f^c = 0 \quad \text{for} \quad D_1 \in J_m \quad \text{(cf. (4.16))}, \tag{4.23}$$

for the corresponding f^c, by noting that

$$\eta_m(\boldsymbol{c}Y) = -\sqrt{-1}B(Y, \boldsymbol{c}^{-1}(Y'(m))) = -B(Y, X(m)) \quad (Y \in \mathfrak{p}_-(m)).$$

Here the action of $S(\mathfrak{p}_-)$ on $\mathcal{O}(\mathcal{S}, V_\tau^*)$ is defined by the directional derivative (2.20).

STEP 2. Consider the point $Y_0 := \xi'(e) \in \mathcal{S}$ (e the identity element of G), which is expressed as

$$Y_0 = \log p_-(c) = -\sum_{k=1}^{r} X_{-\gamma_k}.$$

Let D_2 be any element of the annihilator ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$. Then it is standard to verify that

$$(D_2 \cdot f^c)(Y_0) = (\boldsymbol{c}(D_2))^R F(e) = (\boldsymbol{c}({}^T D_2))^L F(e).$$

Here $D \mapsto {}^T D$ denotes the principal anti-automorphism of $U(\mathfrak{g})$ as in 1.2.

Noting that the ideal $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)$ is homogeneous, we can apply Lemma 4.12 to deduce that $\boldsymbol{c}({}^T D_2)$ lies in the annihilator of $L(\tau)$ in $U(\boldsymbol{c}(\mathfrak{p}_-))$. This implies that

$$(\boldsymbol{c}({}^T D_2))^L F = 0,$$

because $U(\mathfrak{g})^L \langle v^*, F(\cdot) \rangle \simeq L(\tau)$ for every nonzero vector $v^* \in V_\tau^*$. We thus conclude

$$(D_2 \cdot f^c)(Y_0) = 0 \quad (D_2 \in \mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau)). \tag{4.24}$$

STEP 3. We are going to specify the function $f^c \in \mathcal{O}(\mathcal{S}, V_\tau^*)$. It follows from Hilbert's Nullstellensatz that

$$\sqrt{\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) + J_m} = \mathfrak{m}(X(m)),$$

since $\mathrm{Ann}_{S(\mathfrak{p}_-)}L(\tau) + J_m$ defines the variety $\{X(m)\}$ of one point $X(m)$ in $\mathfrak{p}_+$ by virtue of Lemma 4.13. Hence (4.23) and (4.24) imply that there exists a nonnegative integer N such that

$$((Y - B(X(m), Y))^{N+1} \cdot f^c)(Y_0) = 0 \quad \text{for all } Y \in \mathfrak{p}_-. \tag{4.25}$$

This means that the function f^c is of the form

$$f^c(Y) = \exp B(X(m), Y)\varphi(Y), \tag{4.26}$$

where φ is a V_τ^*-valued polynomial function on $\mathfrak{p}_-$ of degree at most N. In particular, F lies in $O_{\tau^*}^*(G)_0^c$ by (4.26), and so one finds that

$$\mathcal{Y}(\tau) \subset O_{\tau^*}^*(G)_0^c.$$

Thus we have proved the claim (2) of Theorem 4.7 as well as the finite-dimensionality of $\mathcal{Y}(\tau)$ in the assertion (1).

STEP 4. Let $v^* \in V_\tau^*$. By Proposition 2.8, the function $F_{X(m),v^*} \in O_{\tau^*}^*(G)_0$ of exponential type (see (2.24)) satisfies $\mathcal{D}_{\tau^*} F_{X(m),v^*} = 0$ if and only if the vector v^* lies in $\operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,)$. Proposition 4.14 says that the former condition is equivalent to $\mathcal{D}_{\tau^*}(c^R F_{X(m),v^*}) = 0$. Noting that $\Theta^c(c^R F_{X(m),v^*})$ is of the form (4.26) with constant function $\varphi(Y) = v^*$ ($Y \in \mathfrak{p}_-$), we deduce that $c^R F_{X(m),v^*} \in \mathcal{Y}(\tau)$ for every $v^* \in \operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,)$. This proves the assertion (3). Finally, the vector space $\mathcal{Y}(\tau)$ does not vanish because

$$\{0\} \neq \mathcal{W}(X(m), \tau)^* \simeq \operatorname{Ker} \boldsymbol{\sigma}(X(m), \,\cdot\,) \hookrightarrow \mathcal{Y}(\tau),$$

thanks to Proposition 3.5 and Lemma 3.10. □

Proof of Theorem 4.8. — Suppose that $L(\tau)$ is unitarizable. We set $m = m(\tau)$. Then one knows that $I_m = \operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ by Theorem 3.7. This combined with Lemma 4.12 allows us to refine the discussion in Step 3 of the proof of Theorem 4.7. As a result, we find that, for any $F \in \mathcal{Y}(\tau)$, the corresponding function $f^c = \Theta^c F$ in (4.26) is necessarily of exponential type, i.e., $f^c(Y) = \exp B(X(m), Y) v^*$ ($Y \in \mathfrak{p}_-$) for some $v^* \in V_\tau^*$. This proves the surjectivity of χ_τ in Theorem 4.7. Now the remainder of the theorem is a consequence of Corollary 3.9 and Lemma 3.10. □

Proof of Theorem 4.9. — First, assume that $m' > m := m(\tau)$. Let F be any function in the space $\mathcal{Y}(\tau, m')$. By (4.12), F belongs to $\mathcal{Y}(\tau) = \mathcal{Y}(\tau, m(\tau))$ also. Hence the corresponding $f^c := \Theta^c F \in O(\mathfrak{p}_-, V_\tau^*)$ is of the form (4.26). It follows in particular that

$$(X_{-\gamma_{r-m'+1}})^n \cdot f^c = \exp B(X(m), \,\cdot\,)((X_{-\gamma_{r-m'+1}})^n \cdot \varphi) = 0$$

for sufficiently large integers n, because $B(X(m), X_{-\gamma_{r-m'+1}}) = 0$ and because φ in (4.26) is a polynomial on $\mathfrak{p}_-$. On the other hand, since F is in $C^\infty(G; \eta_{m'})$, we see just as in Step 1 of the proof of Theorem 4.7 that

$$(X_{-\gamma_{r-m'+1}}) \cdot f^c = B(X(m'), X_{-\gamma_{r-m'+1}}) f^c = B(X_{\gamma_{r-m'+1}}, X_{-\gamma_{r-m'+1}}) f^c.$$

Thus one gets $f^c = 0$ since $B(X_{\gamma_{r-m'+1}}, X_{-\gamma_{r-m'+1}}) \neq 0$. This shows that $\mathcal{Y}(\tau, m') = \{0\}$.

Second, assume that $m' < m = m(\tau)$. Take a nonzero function F in $\mathcal{Y}(\tau)$ by Theorem 4.7 (1). Note that $\mathcal{Y}(\tau) \subset \mathcal{Y}(\tau, m')$. For each $t \in \mathbb{R}$, we define an element $a_t \in G$ by

$$a_t := \exp\{-t(X_{\gamma_{r-m+1}} + X_{-\gamma_{r-m+1}})\} = \exp t\boldsymbol{c}(H_{\gamma_{r-m+1}}) \text{ (cf. (4.2))}.$$

Then it is easily checked that

$$\mathrm{Ad}(a_t)\mathfrak{n}(m') = \mathfrak{n}(m') \quad \text{and} \quad \eta_{m'} \circ \mathrm{Ad}(a_t) = \eta_{m'}.$$

This implies that the functions $(a_t)^R F$ still lie in $\mathcal{Y}(\tau, m')$ for all $t \in \mathbb{R}$, by noting that the differential operator $\mathcal{D}_{\tau^*}$ is right G-invariant. These vectors $(a_t)^R F$ $(t \in \mathbb{R})$ in $\mathcal{Y}(\tau, m')$ turn to be linearly independent, because one gets

$$(\boldsymbol{c}(X_{-\gamma_{r-m+1}}))^R((a_t)^R F) = e^{2t} \cdot B(X_{\gamma_{r-m+1}}, X_{-\gamma_{r-m+1}}) \cdot ((a_t)^R F).$$

by direct computation. Hence the vector space $\mathcal{Y}(\tau, m')$ in question is infinite-dimensional if $m' < m(\tau)$. □

Now we have completely proved the main theorems, Theorems 4.7–4.9.

4.6. Relation to generalized Whittaker vectors. — We end this section by interpreting our results (Theorems 4.7–4.9) in terms of generalized Whittaker vectors in the algebraic dual of an irreducible highest weight $(\mathfrak{g}, K)$-module. To do this we prepare the following lemma.

Lemma 4.15. — *Set* $\mathfrak{n} := \mathfrak{n}(r) = \boldsymbol{c}(\mathfrak{p}_-)$ *(cf.* (4.6)*). Let b be the linear map from $\mathfrak{n}$ to $\mathfrak{p}_-$ defined by*

$$b(Z) \equiv Z \qquad \mathrm{mod}\ \mathfrak{k} + \mathfrak{p}_+$$

for $Z \in \mathfrak{n}$. Then b is a surjective linear isomorphism.

Proof. — Write an element $Y \in \mathfrak{p}_-$ as a linear combination of root vectors:

$$Y = \sum_{\gamma \in \Delta_n^+} c_\gamma X_{-\gamma} \quad \text{with} \quad c_\gamma \in \mathbb{C}.$$

Then it is easy to compute the Cayley transform $\boldsymbol{c}(X_{-\gamma})$ of $X_{-\gamma}$ for each noncompact positive root γ (see [**32**, 2.1 and 2.2] and also [**41**, 9.1]). As a result, one finds that

$$\boldsymbol{c}(X_{-\gamma}) \equiv \kappa_\gamma X_{-\gamma} \qquad \mathrm{mod}\ \ \mathfrak{p}_+ + \mathfrak{k},$$

where $\kappa_\gamma = 1/2$ or $1/\sqrt{2}$ depending on $\gamma \in \Delta_n^+$. This implies that

$$b(\boldsymbol{c}(Y)) = \sum_{\gamma \in \Delta_n^+} \kappa_\gamma c_\gamma X_{-\gamma}.$$

We thus get the lemma. □

Let $L(\tau)$ be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Let us look upon $L(\tau)$, by restriction, as a module over $U(\mathfrak{n}) = S(\mathfrak{n})$. (Note that $\mathfrak{n}$ is an abelian subalgebra of $\mathfrak{g}$.) Then Lemma 4.15 immediately implies the following

Proposition 4.16. — *$L(\tau)$ is finitely generated as a $U(\mathfrak{n})$-module. Moreover, the Gelfand-Kirillov dimension* $\mathrm{Dim}(\mathfrak{n}; L(\tau))$ *and the Bernstein degree* $\mathrm{Deg}(\mathfrak{n}; L(\tau))$ *of $L(\tau)$ as a $U(\mathfrak{n})$-module coincide with those* $\mathrm{Dim}\, L(\tau) = \mathrm{Dim}(\mathfrak{g}; L(\tau))$ *and* $\mathrm{Deg}\, L(\tau) = \mathrm{Deg}(\mathfrak{g}; L(\tau))$ *as a $U(\mathfrak{g})$-module, respectively.*

Remark 4.17. — The argument in the proof of Lemma 4.15 allows us to show

$$\mathfrak{n}^{\perp} \cap \mathcal{V}(L(\tau)) \subset \mathfrak{n}^{\perp} \cap \mathfrak{p}_{+} = \{0\},$$

where $\mathfrak{n}^{\perp}$ denotes the orthogonal of $\mathfrak{n}$ in $\mathfrak{g}$ with respect to the Killing form. In view of this property, we can apply a criterion [**43**, Th.2.2] for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras. This gives another proof of the above proposition.

In view of Lemma 4.12, the annihilator ideal of $L(\tau)$ in $U(\mathfrak{n})$ turns to be

$$\operatorname{Ann}_{U(\mathfrak{n})} L(\tau) = \boldsymbol{c}(\operatorname{Ann}_{U(\mathfrak{p}_{-})} L(\tau)), \tag{4.27}$$

and it defines the associated variety

$$\mathcal{V}(\mathfrak{n}; L(\tau)) = \overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$$

of $U(\mathfrak{n})$-module $L(\tau)$, which is an irreducible affine algebraic variety in $\mathfrak{n}^* = \boldsymbol{c}(\mathfrak{p}_{+})$. Thus, the associated cycle $\mathcal{AC}(\mathfrak{n}; L(\tau))$ of $U(\mathfrak{n})$-module $L(\tau)$ is of the form

$$\mathcal{AC}(\mathfrak{n}; L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)) \cdot [\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}],$$

where $\operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau))$ denotes the multiplicity of $U(\mathfrak{n})$-module $L(\tau)$ at the unique associated prime $\boldsymbol{c}(I_{m(\tau)})$. Further, the Bernstein degree of $U(\mathfrak{n})$-module $L(\tau)$ is described as

$$\operatorname{Deg}(\mathfrak{n}; L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)) \cdot \deg(\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}), \tag{4.28}$$

where $\deg(\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})})$ denotes the degree of the nilpotent cone $\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$ (cf. [**27**, Lemma 1.1]).

The above discussion tells us the following coincidence of two types of multiplicities of $L(\tau)$.

Proposition 4.18. — *One has the equality*

$$\operatorname{mult}_{I_{m(\tau)}}(L(\tau)) = \operatorname{mult}_{\boldsymbol{c}(I_{m(\tau)})}(\mathfrak{n}; L(\tau)),$$

where $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ *is the multiplicity in the associated cycle of* $(\mathfrak{g}, K)$*-module* $L(\tau)$ (*cf.* (3.13)).

Proof. — The assertion follows from Proposition 4.16 together with the equalities (3.14) and (4.28), by noting that the degrees of orbits $\overline{\mathcal{O}_{m(\tau)}}$ and $\overline{\boldsymbol{c}(\mathcal{O}_{\mathrm{m}(\tau)})}$ coincide with each other. □

Now, for each $m = 0, \ldots, r = \mathbb{R}\text{-rank}(G)$, let η_m be the one-dimensional representation of $\mathfrak{n}(m) = \boldsymbol{c}(\mathfrak{p}_{-}(m))$ which induces the GGGR $(\Gamma_m, C^{\infty}(G; \eta_m))$ (cf. (4.10) and (4.11)). A linear form ψ on $L(\tau)$ is called an (algebraic) *generalized Whittaker vector* of type η_m if

$$\psi(Uw) = \eta_m(U)\psi(w) \quad \text{for all} \quad U \in \mathfrak{n}(m) \text{ and } w \in L(\tau).$$

We write $\mathrm{Wh}^*_{\eta_m}(L(\tau))$ for the space of such generalized Whittaker vectors. By definition, one observes that

$$\mathrm{Wh}^*_{\eta_m}(L(\tau)) \simeq \left(L(\tau)/\boldsymbol{c}({}^T\! J_m)L(\tau)\right)^* := \mathrm{Hom}_{\mathbb{C}}(L(\tau)/\boldsymbol{c}({}^T\! J_m)L(\tau),\ \mathbb{C}), \tag{4.29}$$

as vector spaces, where J_m is the ideal of $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$ defined by (4.16), and T denotes the automorphism of $S(\mathfrak{p}_-)$ such that ${}^T Y = -Y$ for $Y \in \mathfrak{p}_-$. Further, every $(\mathfrak{g}, K)$-embedding T from $L(\tau)$ into $C^\infty(G; \eta_m)$ yields a generalized Whittaker vector $\psi \in \mathrm{Wh}^*_{\eta_m}(L(\tau))$ by

$$\psi(w) = (Tw)(e) \quad (w \in L(\tau)).$$

This assignment $T \mapsto \psi$ sets up a linear embedding

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \hookrightarrow \mathrm{Wh}^*_{\eta_m}(L(\tau)). \tag{4.30}$$

We can show that this embedding is actually surjective for the most relevant case, as follows.

Proposition 4.19. — *If $m = m(\tau)$, the map (4.30) is surjective. Namely, every nonzero generalized Whittaker vector in $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ gives an embedding of $L(\tau)$ into the GGGR $C^\infty(G; \eta_{m(\tau)})$.*

Remark 4.20. — Let $L(\tau)^\infty$ denote the smooth G-module consisting of all C^∞-vectors for an irreducible admissible representation of G corresponding to $L(\tau)$. In view of the discussion in [**41**, 12.5], one finds that, if $L(\tau)$ is a member of holomorphic discrete series, any vector in $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ extends also to a continuous G-isomorphism from $L(\tau)^\infty$ into $C^\infty(G; \eta_{m(\tau)})$. This appears to be true for any $L(\tau)$ not necessarily in the discrete series, but we do not discuss it here.

Proof of Proposition 4.19. — First, we set $\tilde{\mathfrak{m}} := \mathrm{Ann}_{S(\mathfrak{p}_-)} L(\tau) + J_{m(\tau)}$. By virtue of Lemma 4.13, $\tilde{\mathfrak{m}}$ is an ideal of $S(\mathfrak{p}_-)$ that defines the one point variety $\{X(m(\tau))\}$, and in particular, the codimension of $\tilde{\mathfrak{m}}$ in $S(\mathfrak{p}_-)$ is finite. By (4.27), the isomorphism (4.29) turns out to be

$$\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau)) \simeq \left(L(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})L(\tau)\right)^*. \tag{4.31}$$

Second, we consider the generalized Verma module $M(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau$ and its unique maximal submodule $N(\tau)$. The natural quotient map $M(\tau) \to L(\tau) = M(\tau)/N(\tau)$ induces a linear isomorphism

$$L(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})L(\tau) \simeq M(\tau)/(N(\tau) + \boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)) \tag{4.32}$$

in the canonical way. Now let $\langle\ \cdot, \cdot\ \rangle_\tau$ be the $(\mathfrak{g}, K)$-invariant bilinear form on $O^*_{\tau^*}(G) \times M(\tau)$ constructed in 2.3. We write $\mathcal{E}$ for the orthogonal of $\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)$ in $O^*_{\tau^*}(G)$. Then, the bilinear form $\langle\ \cdot, \cdot\ \rangle_\tau$ naturally induces a linear embedding

$$\mathcal{E} \hookrightarrow (M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau))^*. \tag{4.33}$$

Third, just as in the proof of Theorem 4.7 (see 4.5), one finds that an element $F \in O^*_{\tau^*}(G)$ belongs to $\mathcal{E}$ if and only if

$$D \cdot f^c(Y_0) = 0 \quad \text{for all} \quad D \in \tilde{\mathfrak{m}}, \tag{4.34}$$

where $Y_0 = \xi'(e)$, and $f^c = (\Theta^c)^{-1}F \in O(\mathcal{S}, V^*_\tau)$ as in (4.19). It then follows from (4.34) that

$$\dim \mathcal{E} = \dim V_\tau \times \dim S(\mathfrak{p}_-)/\tilde{\mathfrak{m}}. \tag{4.35}$$

Since we have

$$M(\tau) = U(\mathfrak{n})V_\tau \simeq U(\mathfrak{n}) \otimes V_\tau$$

by Lemma 4.15, the dimension of the quotient space $M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)$ is equal to the right hand side of (4.35). This shows that the linear isomorphism (4.33) is surjective:

$$\mathcal{E} \simeq (M(\tau)/\boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau))^*. \tag{4.36}$$

In view of Proposition 2.7 (1), (4.31), (4.32) and (4.36) give rise to isomorphisms

$$\mathcal{E} \cap \mathrm{Ker}\mathcal{D}_{\tau^*} \simeq (M(\tau)/(N(\tau) + \boldsymbol{c}({}^T\tilde{\mathfrak{m}})M(\tau)))^* \simeq \mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$$

as vector spaces, where $\mathcal{D}_{\tau^*}$ is the differential operator of gradient-type associated to τ^*. This proves the proposition, because every function in $\mathcal{E} \cap \mathrm{Ker}\mathcal{D}_{\tau^*}$ gives a $(\mathfrak{g}, K)$-embedding of $L(\tau)$ into $C^\infty(G; \eta_{m(\tau)})$ by virtue of (4.14). □

Proposition 4.21. — *If $L(\tau)$ is unitarizable, one gets*

$$\dim \mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau)) = \dim L(\tau)/\mathfrak{m}(X(m(\tau)))L(\tau) = \dim L(\tau)/\boldsymbol{c}(\mathfrak{m}(-X(m(\tau))))L(\tau),$$

where $\mathfrak{m}(X)$ is the maximal ideal of $S(\mathfrak{p}_-)$ defining a point $X \in \mathcal{O}_{m(\tau)}$ (cf. (4.1)). Moreover, the above dimension is equal to the multiplicity in the associated cycle $\mathcal{AC}(L(\tau))$ of $L(\tau)$.

Proof. — The assertions follow from Theorem 4.8 and Proposition 4.19 by noting the isomorphism (4.31), where ${}^T\tilde{\mathfrak{m}} = \mathfrak{m}(-X(m(\tau)))$ in this case. □

Concerning the spaces of algebraic generalized Whittaker vectors, we are now in a position to give the following consequence of the main results of this article.

Theorem 4.22. — *The dimension of the vector space $\mathrm{Wh}^*_{\eta_m}(L(\tau))$ is given as*

$$\dim \mathrm{Wh}^*_{\eta_m}(L(\tau)) = \begin{cases} 0 & \text{if } m > m(\tau), \\ \text{finite } (\neq 0) & \text{if } m = m(\tau), \\ \infty & \text{if } m < m(\tau). \end{cases}$$

Here $\mathcal{O}_{m(\tau)}$ is the unique open $K_{\mathbb{C}}$-orbit in the associated variety of $L(\tau)$. Moreover, if $L(\tau)$ is unitarizable, the dimension of $\mathrm{Wh}^*_{\eta_{m(\tau)}}(L(\tau))$ coincides with the multiplicity $\mathrm{mult}_{I_{m(\tau)}}(L(\tau))$.

Proof. — The claims for the cases $m < m(\tau)$ and $m = m(\tau)$ follow from Theorems 4.7 and 4.8 coupled with (4.30) and Proposition 4.19. The property $\mathrm{Wh}^*_{\eta_m}(L(\tau)) = \{0\}$ for $m > m(\tau)$ can be proved by an argument similar to the one given in the proof of Theorem 4.9, or alternatively, one can apply a general result [**22**, Corollary 4] of Matumoto. □

5. Case of the classical groups

Hereafter, we assume that G is one of the classical groups $SU(p,q)$ $(p \geqslant q)$, $Sp(n,\mathbb{R})$ or $SO^*(2n)$. The theory of reductive dual pairs gives concrete realizations of unitarizable highest weight modules $L[\sigma] = L(\tau[\sigma])$ for these groups G (see Theorem 5.1), by decomposing the oscillator representation of the pair (G, G'), where $G' = U(k)$, $O(k)$, or $Sp(k)$ respectively, and $\sigma \in \hat{G}'$.

For such $L[\sigma]$'s, we specify in this section the $K_{\mathbb{C}}(X(m))$-modules $\mathcal{W}(X(m), \tau[\sigma]) = L[\sigma]/\mathfrak{m}(X(m))L[\sigma]$ (cf. (3.10)) with $m = m(\tau[\sigma])$ explicitly by using the Fock model of the oscillator module (see Theorems 5.14 and 5.15). In view of Theorem 4.8, this leads us to a clearer understanding of the generalized Whittaker models for $L[\sigma]$.

5.1. Oscillator representation. — We start with constructing the oscillator representation of the pair (G, G'), following [**3**, §7]. First, realize our classical groups G as

$$\begin{cases} SU(p,q) = \left\{ g \in SL(p+q, \mathbb{C}) \;\middle|\; g \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix} \right\} \quad (p \geqslant q), \\ Sp(n,\mathbb{R}) = \left\{ g \in SU(n,n) \;\middle|\; {}^t g J_n g = J_n \right\} \text{ with } J_n := \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}, \\ SO^*(2n) = \left\{ g \in SU(n,n) \;\middle|\; {}^t g \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} g = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} \right\}, \end{cases}$$

where I_n denotes the identity matrix of size n. The totality of unitary matrices in G forms a maximal compact subgroup K.

Let $M_{p,q}$ denote the space of all complex matrices of size $p \times q$. We write Sym_n (resp. Alt_n) for the set of all symmetric (resp. alternating) complex matrices of size n. Then, the real rank $r = \mathbb{R}\text{-rank}\, G$, the complexification $K_{\mathbb{C}}$ of K, and the irreducible $K_{\mathbb{C}}$-module $\mathfrak{p}_+$ under Ad, can be described for each G respectively as in the following table.

(5.1)

G	r	$K_{\mathbb{C}}$	$\mathfrak{p}_+$
$SU(p,q)$	q	$S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$	$M_{p,q}$
$Sp(n,\mathbb{R})$	n	$GL(n,\mathbb{C})$	Sym_n
$SO^*(2n)$	$[\frac{n}{2}]$	$GL(n,\mathbb{C})$	Alt_n

Here the $K_{\mathbb{C}}$-action on $\mathfrak{p}_+$ is given as

$$g \cdot X = g_1 X g_2^{-1}, \quad g = (g_1, g_2) \in S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C})), \ X \in M_{p,q} \tag{5.2}$$

for $G = SU(p,q)$, and

$$g \cdot X = g X {}^t g, \quad g \in GL(n, \mathbb{C}), \ X \in \mathrm{Sym}_n \text{ or } \mathrm{Alt}_n$$

for $G = Sp(n,\mathbb{R})$ or $SO^*(2n)$. For this, see also [**27**, 7.1].

For every positive integer k, we realize the compact group G' as

$$\begin{cases} U(k) = \{g \in GL(k,\mathbb{C}) \mid g {}^t\overline{g} = I_k\} & \text{for } G = SU(p,q), \\ O(k) = U(k) \cap GL(k,\mathbb{R}) & \text{for } G = Sp(n,\mathbb{R}), \\ Sp(k) = \{g \in U(2k) \mid {}^t g J_k g = J_k\} & \text{for } G = SO^*(2n). \end{cases}$$

The complexification of G' will be denoted by $G'_{\mathbb{C}}$, i.e., $G'_{\mathbb{C}} = GL(k,\mathbb{C})$, $O(k,\mathbb{C})$, $Sp(k,\mathbb{C})$ respectively. Define a space M of complex matrices by

$$M := \begin{cases} M_{n,k} \ (n := p+q) & \text{for } G = SU(p,q), \\ M_{n,k} & \text{for } G = Sp(n,\mathbb{R}), \\ M_{n,2k} & \text{for } G = SO^*(2n). \end{cases}$$

For $G = SU(p,q)$, the elements $Z \in M$ will be written as

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{with} \quad A \in M_{p,k}, B \in M_{q,k}.$$

The group $K_{\mathbb{C}} \times G'_{\mathbb{C}}$ acts on M by

$$(g,g') \cdot Z := \begin{pmatrix} g_1 A g'^{-1} \\ {}^t g_2^{-1} B \, {}^t g' \end{pmatrix} \quad \text{with } g = (g_1, g_2), \tag{5.3}$$

for $G = SU(p,q)$, and by

$$(g,g') \cdot Z := g Z g'^{-1} \tag{5.4}$$

for $G = Sp(n,\mathbb{R})$ or $SO^*(2n)$, where $(g,g') \in K_{\mathbb{C}} \times G'_{\mathbb{C}}$ and $Z \in M$.

We now prepare some notation to describe the oscillator representation. Let ψ be a map from M to $\mathfrak{p}_+$ such that

$$\psi(Z) := \begin{cases} A \, {}^t B & \text{for } G = SU(p,q), \\ \frac{1}{2} Z \, {}^t Z & \text{for } G = Sp(n,\mathbb{R}), \\ \frac{1}{2} Z J_k \, {}^t Z & \text{for } G = SO^*(2n). \end{cases} \tag{5.5}$$

Note that $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant polynomial map of degree two, where we let $G'_{\mathbb{C}}$ act on $\mathfrak{p}_+$ trivially. For each $Y \in \mathfrak{p}_-$, let h_Y be a polynomial on M defined by

$$h_Y(Z) := B(\psi(Z), Y) \quad (B \text{ the Killing form of } \mathfrak{g}).$$

We set for $g \in K_{\mathbb{C}}$,

$$\delta_k(g) := \begin{cases} (\det g_1)^{-k} & \text{for } G = SU(p,q), \\ (\det g)^{-k/2} & \text{for } G = Sp(n,\mathbb{R}), \\ (\det g)^{-k} & \text{for } G = SO^*(2n), \end{cases} \tag{5.6}$$

where $g = (g_1, g_2)$ as in (5.2) for $G = SU(p,q)$. If $G = Sp(n,\mathbb{R})$, the function δ_k is two-valued on $K_{\mathbb{C}} = GL(n,\mathbb{C})$. We need to go up to the two fold cover of $K_{\mathbb{C}}$ in order that δ_k determines a genuine character of the group. Hereafter, we replace K and $K_{\mathbb{C}}$ by their two fold covering groups when $G = Sp(n,\mathbb{R})$. By abuse of notation, the latter covering groups will be denoted by K and $K_{\mathbb{C}}$ again.

Let $\mathbb{C}[M]$ denote the ring of polynomial functions on the complex vector space M. One can define a $(\mathfrak{g}, K)$-representation ω on $\mathbb{C}[M]$ in the following fashion. First, the $\mathfrak{p}_-$ action on $\mathbb{C}[M]$ is given by multiplication:

$$\omega(Y)f(Z) := h_Y(Z)f(Z), \quad Y \in \mathfrak{p}_-, \tag{5.7}$$

for $f \in \mathbb{C}[M]$. Second, $\mathfrak{p}_+$ acts by differentiation:

$$\omega(X)f(Z) := \kappa(h_{\overline{X}}(\partial)f)(Z), \quad X \in \mathfrak{p}_+.$$

Here $h_{\overline{X}}(\partial)$ stands for the constant coefficient differential operator on M defined by the polynomial $h_{\overline{X}}$, and the constant κ depends only on the Lie algebra $\mathfrak{g}_0$ of G. Third, the complexification $K_{\mathbb{C}}$ acts on $\mathbb{C}[M]$ holomorphically as

$$\omega(g)f(Z) := \delta_k(g)f((g^{-1}, e) \cdot Z), \quad g \in K_{\mathbb{C}}.$$

On the other hand, $\mathbb{C}[M]$ has a natural $G'_{\mathbb{C}}$-module structure through

$$R(g')f(Z) := f((e, {g'}^{-1}) \cdot Z), \quad g' \in G'_{\mathbb{C}}.$$

Then it is easily seen that these two representations ω and R commute with each other. The resulting $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-representation (ω, R) on $\mathbb{C}[M]$ will be called the Fock model of the (infinitesimal) *oscillator representation* of the pair (G, G').

It should be mentioned that the above oscillator representation ω of the pair (G, G') comes from the *Weil representation* of a metaplectic group. In fact, $G \times G'$ forms a reductive dual pair in a real symplectic group $Sp(N,\mathbb{R})$. Consider the Weil representation Ω (cf. [**12**]) of the metaplectic group $Mp(N, R)$, which is the two fold cover of $Sp(N,\mathbb{R})$. Restrict Ω to the metaplectic cover $\tilde{G} \times \tilde{G}'$ of $G \times G'$, and then twist it by a certain one-dimensional character of the compact group $\tilde{G}'$. One thus gets ω.

5.2. Unitarizable highest weight modules $L[\sigma]$. — Let (σ, V_σ) be an irreducible (finite-dimensional) unitary representation of the compact group G'. Extend σ to a holomorphic representation of $G'_{\mathbb{C}}$ in the canonical way. We set

$$L[\sigma] := \operatorname{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]), \tag{5.8}$$

which turns to be a $(\mathfrak{g}, K)$-module through the representation ω on $\mathbb{C}[M]$. Let $\Sigma(k)$ denote the totality of equivalence classes of irreducible unitary representations σ of G' such that $L[\sigma] \neq \{0\}$. Note that the $G'_{\mathbb{C}}$-action on $\mathbb{C}[M]$ is locally finite since $G'_{\mathbb{C}}$ preserves each subspace of homogeneous polynomials of any fixed degree. Then one gets

$$\mathbb{C}[M] \simeq \bigoplus_{\sigma \in \Sigma(k)} L[\sigma] \otimes V_\sigma \quad \text{as } (\mathfrak{g}, K) \times G'_{\mathbb{C}}\text{-modules.} \tag{5.9}$$

The isomorphism is given by

$$L[\sigma] \otimes V_\sigma \ni T \otimes v \longmapsto T(v) \in \mathbb{C}[M],$$

on each $G'_{\mathbb{C}}$-isotypic component $L[\sigma] \otimes V_\sigma$.

The following theorem states the celebrated Howe duality correspondence associated to (G, G').

Theorem 5.1 **([12], [6], [7]; cf. [3, §7])**

(1) *$L[\sigma]$ is an irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module for every $\sigma \in \Sigma(k)$. In particular,* (5.9) *gives the irreducible decomposition of the $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]$.*

(2) *Let $\sigma_1, \sigma_2 \in \Sigma(k)$. Then, $V_{\sigma_1} \simeq V_{\sigma_2}$ as $G'_{\mathbb{C}}$-modules if and only if $L[\sigma_1] \simeq L[\sigma_2]$ as $(\mathfrak{g}, K)$-modules.*

(3) *If $G = SU(p, q)$ or $Sp(n, \mathbb{R})$, any irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module is isomorphic to an $L[\sigma]$, where $\sigma \in \Sigma(k)$ for some positive integer k.*

Let $\tau[\sigma]$ denote the extreme K-type of highest weight $(\mathfrak{g}, K)$-module $L[\sigma]$, i.e., $L[\sigma] = L(\tau[\sigma])$. We note that the correspondence $\sigma \leftrightarrow \tau[\sigma]$ can be explicitly described in terms of their highest weights. For this, see the articles cited in the above theorem.

It follows from the standard argument in linear algebra that each $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+$ (see 3.1) consists of all the matrices in $\mathfrak{p}_+ = M_{p,q}, \mathrm{Sym}_n$ (resp. Alt_n) of rank m (resp. $2m$) for $G = SU(p, q), Sp(n, \mathbb{R})$ (resp. $SO^*(2n)$). Let $E_{s,t}(i, j)$ denote the (i, j)-matrix unit of size $s \times t$ whose (k, l)-matrix entry e_{kl} is equal to 1 if $(k, l) = (i, j)$; $e_{kl} = 0$ otherwise. We put

$$I_{s,t}(m) := \sum_{i=1}^{m} E_{s,t}(i, i) \in M_{s,t} \quad (m = 0, \ldots, \min(s, t)), \tag{5.10}$$

where $I_{s,t}(0) := 0$. Then, we take an element $X(m) \in \mathcal{O}_m$ explicitly as

$$X(m) := \begin{cases} I_{p,q}(m) & \text{for } G = SU(p, q), \\ I_{n,n}(m)/2 & \text{for } G = Sp(n, \mathbb{R}), \\ \sum_{i=1}^{m} (E_{n,n}(i, m+i) - E_{n,n}(m+i, i))/2 & \text{for } G = SO^*(2n). \end{cases} \tag{5.11}$$

Now, it is easily seen that the image $\psi(M)$ of the $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant map $\psi : M \to \mathfrak{p}_+$ in (5.5) is a $K_{\mathbb{C}}$-stable, irreducible algebraic variety described as

$$\psi(M) = \overline{\mathcal{O}_{m_k}} \quad \text{with} \quad m_k := \min(k, r), \tag{5.12}$$

where M and ψ depend on k. By (5.7) and (5.9), the annihilator ideal in $S(\mathfrak{p}_-)$ of $L[\sigma]$ ($\sigma \in \Sigma(k)$) consists exactly of all the elements $D \in S(\mathfrak{p}_-) = \mathbb{C}[\mathfrak{p}_+]$ vanishing on $\psi(M)$. In this way we have shown the following well-known fact.

***Proposition 5.2* (cf. [3, §12]).** — *For any $\sigma \in \Sigma(k)$, the associated variety of unitarizable highest weight module $L[\sigma]$ is equal to the closure of the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m_k} = \mathrm{Ad}(K_{\mathbb{C}})X(m_k)$. More precisely, $\mathrm{Ann}_{S(\mathfrak{p}_-)}L[\sigma]$ coincides to the prime ideal I_{m_k} defining $\overline{\mathcal{O}_{m_k}}$ (cf. Theorem 3.7).*

5.3. Variety $\mathcal{V}_k$ and ideal $\omega(\mathfrak{m})\mathbb{C}[M]$. — Now we consider the maximal ideal:

$$\mathfrak{m} := \mathfrak{m}(X(m_k)) = \sum_{Y \in \mathfrak{p}_-} (Y - B(X(m_k), Y))S(\mathfrak{p}_-) \subset S(\mathfrak{p}_-) \quad (\text{cf. (3.9)}),$$

for each positive integer k. For $m = 0, \dots, r$, let $K_{\mathbb{C}}(m) := K_{\mathbb{C}}(X(m))$ be the isotropy subgroup of $K_{\mathbb{C}}$ at $X(m) \in \mathcal{O}_m$. We want to describe the $K_{\mathbb{C}}(m_k)$-modules

$$\mathcal{W}[\sigma] := \mathcal{W}(X(m_k), \tau[\sigma]) = L[\sigma]/\mathfrak{m}L[\sigma] \quad (\sigma \in \Sigma(k)).$$

In view of (5.8) and (5.9), one gets an isomorphism

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]) \quad \text{as } K_{\mathbb{C}}(m_k)\text{-modules}. \tag{5.13}$$

So, our task is to decompose the quotient $\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]$ as $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$-modules.

To do this, we note that, by virtue of (5.7), $\omega(\mathfrak{m})\mathbb{C}[M]$ is equal to the ideal of $\mathbb{C}[M]$ generated by all matrix entries of the following polynomial function of degree two:

$$M \ni Z \longmapsto \psi(Z) - X(m_k) \in \mathfrak{p}_+. \tag{5.14}$$

We consider the corresponding affine algebraic variety $\mathcal{V}_k$ of M:

$$\mathcal{V}_k := \{Z \in M \mid \psi(Z) = X(m_k)\} = \psi^{-1}(X(m_k)),$$

which is the inverse image of $X(m_k)$ by ψ. Clearly, the variety $\mathcal{V}_k$ is stable under the action of $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$. Note that the codimension of $\mathcal{V}_k$ is given as

$$\dim M - \dim \mathcal{V}_k = \dim \mathcal{O}_{m_k} = \dim \mathfrak{p}_-(m_k) \quad (\text{cf. Lemma 4.2})$$

by virtue of (5.12).

Now, let us give the $G'_{\mathbb{C}}$-orbit decomposition of $\mathcal{V}_k$ for each group G separately, where $G'_{\mathbb{C}}$ is identified with the subgroup $\{e\} \times G'_{\mathbb{C}}$ of $K_{\mathbb{C}} \times G'_{\mathbb{C}}$. We define a subgroup $G'_{\mathbb{C}}(k - r)$ ($r = \mathbb{R}\text{-rank}\, G$) of $G'_{\mathbb{C}}$ by

$$G'_{\mathbb{C}}(k - r) := \begin{cases} \{I_k\} \text{ (the unit group)} & \text{if } k \leqslant r, \\ \left\{ \begin{pmatrix} I_r & O \\ O & h \end{pmatrix} \in G'_{\mathbb{C}} \right\} & \text{if } k > r, \end{cases} \tag{5.15}$$

for $G = SU(p,q)$, $Sp(n,\mathbb{R})$, and by

(5.16)
$$G'_{\mathbb{C}}(k-r) := \begin{cases} \{I_{2k}\} \text{ (the unit group)} & \text{if } k \leqslant r, \\ \left\{ \begin{pmatrix} I_k & O & O & O \\ O & h_{11} & O & h_{12} \\ O & O & I_k & O \\ O & h_{21} & O & h_{22} \end{pmatrix} \in G'_{\mathbb{C}} \;\middle|\; h_{ij} \in M_{k-r,k-r} \right\} & \text{if } k > r, \end{cases}$$

for $G = SO^*(2n)$. Note that if $k > r$, the group $G'_{\mathbb{C}}(k-r)$ is naturally isomorphic to $GL(k-r,\mathbb{C})$, $O(k-r,\mathbb{C})$ or $Sp(k-r,\mathbb{C})$ according as $G = SU(p,q)$, $Sp(n,\mathbb{R})$ or $SO^*(2n)$ respectively.

First, the following lemma for the case $SU(p,q)$ is due to Tagawa.

Lemma 5.3 ([**31**, 3.5 and 3.8]). — *Assume that $G = SU(p,q)$ ($r = q$, $G'_{\mathbb{C}} = GL(k,\mathbb{C})$).*

(1) *If $k \leqslant q$, the group $G'_{\mathbb{C}} = GL(k,\mathbb{C})$ acts on $\mathcal{V}_k$ simply transitively. One gets*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{p,k}(k) \\ I_{q,k}(k) \end{pmatrix} \simeq G'_{\mathbb{C}} \quad \text{as } G'_{\mathbb{C}}\text{-sets,} \tag{5.17}$$

where the matrices $I_{p,k}(k), I_{q,k}(k)$ are as in (5.10).

(2) *If $k > q$ and $p = q$, then the $G'_{\mathbb{C}}$-action on $\mathcal{V}_k$ is still transitive, and it holds that*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{q,k}(q) \\ I_{q,k}(q) \end{pmatrix} \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-q) \quad \text{as } G'_{\mathbb{C}}\text{-sets.}$$

Here $G'_{\mathbb{C}}(k-q)$ coincides with the isotropy subgroup of $G'_{\mathbb{C}}$ at $\begin{pmatrix} I_{q,k}(q) \\ I_{q,k}(q) \end{pmatrix} \in \mathcal{V}_k$.

(3) *If $k > q$ and $p > q$, $\mathcal{V}_k$ is no longer $G'_{\mathbb{C}}$-homogeneous. In fact, let $\tilde{M}_{p-q,k-q}$ be the subspace of M defined by*

$$\tilde{M}_{p-q,k-q} := \left\{ \tilde{U} = \begin{pmatrix} I_q & O \\ O & U \\ I_q & O \end{pmatrix} \;\middle|\; U \in M_{p-q,k-q} \right\}.$$

Then $\mathcal{V}_k$ is decomposed as

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{p-q,k-q} = \coprod_{\tilde{U} \in \Lambda} G'_{\mathbb{C}} \cdot \tilde{U}, \tag{5.18}$$

where Λ denotes a complete system of representatives in $\tilde{M}_{p-q,k-q}$ of the $G'_{\mathbb{C}}(k-q)$-orbit space

$$\tilde{M}_{p-q,k-q}/G'_{\mathbb{C}}(k-q) \simeq M_{p-q,k-q}/GL(k-q,\mathbb{C}).$$

Second, the structure of $G'_{\mathbb{C}}$-variety $\mathcal{V}_k$ is much simpler for $Sp(n,\mathbb{R})$. This is because the corresponding Hermitian symmetric space is always of tube type.

Lemma 5.4. — *Assume that $G = Sp(n,\mathbb{R})$ ($r = n$ and $G'_{\mathbb{C}} = O(k,\mathbb{C})$). Then it holds that*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot I_{n,k}(m_k) \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-n) \quad \text{as } G'_{\mathbb{C}}\text{-sets},$$

Here $m_k = \min(k,n)$, and the isotropy subgroup of $G'_{\mathbb{C}}$ at $I_{n,k}(m_k)$ is equal to the group $G'_{\mathbb{C}}(k-n)$ in (5.15).

Third, one obtains the following lemma for $SO^*(2n)$.

Lemma 5.5. — *Assume that $G = SO^*(2n)$ ($r = [n/2]$ and $G'_{\mathbb{C}} = Sp(k,\mathbb{C})$).*

(1) *If $k \leqslant r$, one has*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot I_{n,2k}(2k) \simeq G'_{\mathbb{C}} \quad \text{as } G'_{\mathbb{C}}\text{-sets}. \tag{5.19}$$

(2) *If $k > r = n/2$ with even integer n, the variety $\mathcal{V}_k$ is described as*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-r), \tag{5.20}$$

where $G'_{\mathbb{C}}(k-r) \simeq Sp(k-r,\mathbb{C})$ (cf. (5.16)) coincides with the isotropy subgroup of $G'_{\mathbb{C}}$ at the matrix $\begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}$ in $M = M_{2r,2k}$.

(3) *If $k > r = (n-1)/2$ with odd integer n, $\mathcal{V}_k$ consists of two $G'_{\mathbb{C}}$-orbits. In fact, we set*

$$(z_1,z_2)^{\sim} := \begin{pmatrix} I_r & O & O & O \\ O & O & I_r & O \\ o & z_1 & o & z_2 \end{pmatrix}$$

for $(z_1,z_2) \in M_{1,2(k-r)} = M_{1,k-r} \times M_{1,k-r}$. Then $\mathcal{V}_k$ decomposes as

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{1,2(k-r)} = G'_{\mathbb{C}} \cdot (0\ldots 0, 0\ldots 0)^{\sim} \coprod G'_{\mathbb{C}} \cdot (1\ 0\ldots 0, 0\ldots 0)^{\sim}, \tag{5.21}$$

where $\tilde{M}_{1,2(k-r)} := \{(z_1,z_2)^{\sim} \mid z_1, z_2 \in M_{1,k-r}\}$.

We give below a proof of Lemma 5.5 for $G = SO^*(2n)$. Lemmas 5.3 and 5.4 can be shown in the same way (so we omit the proofs of these two lemmas).

Proof of Lemma 5.5. — (1) Suppose $k \leqslant r = [n/2]$. In view of (5.5) and (5.11), one observes that an element

$$Z = \begin{pmatrix} C \\ D \end{pmatrix} \in M \quad \text{with} \quad C \in M_{2k,2k},\ D \in M_{n-2k,2k}$$

belongs to $\mathcal{V}_k$ if and only if

$$CJ_k{}^tC = J_k, \quad CJ_k{}^tD = O, \quad \text{and} \quad DJ_k{}^tD = O,$$

which means that $C \in Sp(k,\mathbb{C})$ and $D = O$. We thus get (5.19).

(2) Consider the case $k > r = n/2$ with even integer n. Take any matrix Z in $\mathcal{V}_k$. Let $c_i \in M_{1,2k} = \mathbb{C}^{2k}$ ($i = 1,\ldots,n$) denote the i-th row vector of Z. Set $d_i := c_{r+i}$ ($i = 1,\ldots,r$). By the condition $ZJ_k{}^tZ = J_r$ ($\Leftrightarrow \psi(Z) = X(r)$), we can extend $\{c_1,\ldots,c_r,d_1,\ldots,d_r\}$ to a symplectic basis $\{c_1,\ldots,c_k,d_1,\ldots,d_k\}$ of $\mathbb{C}^{2k}$ with

respect to the nondegenerate alternating form defined by J_k. Then, there exists an element $g' \in G'_{\mathbb{C}} = Sp(k,\mathbb{C})$ such that $e_i g' = c_i, e_{k+i} g' = d_i$ $(i = 1,\dots,k)$, where $e_1,\dots,e_{2k}$ denotes the standard basis of $\mathbb{C}^{2k}$. This implies that

$$Z = g'^{-1} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}.$$

Furthermore, $g' \in G'_{\mathbb{C}}$ fixes the above matrix if and only if $e_i g' = e_i$ and $e_{k+i} g' = e_{k+i}$ for all $i = 1,\dots,r$, or equivalently, $g' \in G'_{\mathbb{C}}(k-r)$.

(3) Suppose that $k > r = (n-1)/2$ with odd integer n. Just as in (2), one can show that any element in $\mathcal{V}_k$ lies in the $G'_{\mathbb{C}}$-orbit through a matrix Z of the form

$$Z = \begin{pmatrix} e_1 \\ \vdots \\ e_r \\ e_{k+1} \\ \vdots \\ e_{k+r} \\ z \end{pmatrix} \quad \text{for some } z \in \mathbb{C}^{2k}.$$

Then the condition $\psi(Z) = X(r)$ imposes

$$e_i J_k {}^t z = e_{k+i} J_k {}^t z = 0 \quad \text{for} \quad i = 1,\dots,r.$$

Hence one finds that $Z = (z_1,z_2)^{\sim}$ for some $(z_1,z_2) \in M_{1,2(k-r)}$, i.e., $\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{1,2(k-r)}$.

Finally, observe that two matrices $(z_1,z_2)^{\sim}$ and $(z'_1,z'_2)^{\sim}$ in $\mathcal{V}_k$ belong to the same $G'_{\mathbb{C}}$-orbit if and only if the corresponding vectors (z_1,z_2) and (z'_1,z'_2) in $M_{1,2(k-r)}$ are conjugate under the action of $Sp(k-r,\mathbb{C})$. This yields the second equality in (5.21), by noting that $Sp(k-r,\mathbb{C})$ acts on $M_{1,2(k-r)} \backslash \{0\}$ transitively. □

The above three lemmas imply in particular the following

Proposition 5.6. — *The affine algebraic variety $\mathcal{V}_k$ is irreducible except the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$.*

Remark 5.7. — If $G = Sp(n,\mathbb{R})$ with $k \leqslant n$, then $\mathcal{V}_k \simeq O(k,\mathbb{C})$ has two irreducible components according as the coset decomposition $O(k,\mathbb{C}) = SO(k,\mathbb{C}) \cup g' SO(k,\mathbb{C})$ with $g' \in O(k,\mathbb{C}) \backslash SO(k,\mathbb{C})$.

Proof of Proposition 5.6. — Let $(G'_{\mathbb{C}})_o = GL(k,\mathbb{C})$, $SO(k,\mathbb{C})$ or $Sp(k,\mathbb{C})$ be the identity component of the complex classical group $G'_{\mathbb{C}} = GL(k,\mathbb{C})$, $O(k,\mathbb{C})$ or $Sp(k,\mathbb{C})$ respectively. Under the hypothesis of the proposition, we find from Lemmas 5.3–5.5 that $\mathcal{V}_k$ is the image of an irreducible variety $(G'_{\mathbb{C}})_o$ or $(G'_{\mathbb{C}})_o \times \mathbb{C}^p$ (for some $p > 0$) by a continuous map (with respect to the Zariski topology) between two affine spaces over $\mathbb{C}$. This proves the proposition. □

The next proposition is important to specify our $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$.

Proposition 5.8. — *The ideal $\omega(\mathfrak{m})\mathbb{C}[M]$ of $\mathbb{C}[M]$ coincides with the defining ideal of $\mathcal{V}_k$ in $\mathbb{C}[M]$:*

$$\omega(\mathfrak{m})\mathbb{C}[M] = \{f \in \mathbb{C}[M] \mid f(Z) = 0 \quad \textit{for all } Z \in \mathcal{V}_k\}. \tag{5.22}$$

Hence one gets a natural isomorphism

$$\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M] \simeq \mathbb{C}[\mathcal{V}_k] \quad \textit{as} \quad K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}\textit{-modules}, \tag{5.23}$$

where $\mathbb{C}[\mathcal{V}_k]$ denotes the affine coordinate ring of $\mathcal{V}_k$ consisting of all functions on $\mathcal{V}_k$ given by restricting polynomials on M to $\mathcal{V}_k$.

Proof. — We write $\mathcal{I}_k$ for the defining ideal of $\mathcal{V}_k$, the right hand side of (5.22). By definition one has $\omega(\mathfrak{m})\mathbb{C}[M] \subset \mathcal{I}_k$. So we want to show the converse inclusion $\mathcal{I}_k \subset \omega(\mathfrak{m})\mathbb{C}[M]$.

First, we prove the inclusion in question when the variety $\mathcal{V}_k$ is irreducible. Namely, we exclude the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$ exactly (see Proposition 5.6). Take any basis $Y_1, \ldots, Y_t$ of the vector space $\mathfrak{p}_-(m_k) = [\mathfrak{k}, Y(m_k)]$ (cf. Lemma 4.2). We define $f_1, \ldots, f_t \in \omega(\mathfrak{m})\mathbb{C}[M]$ by

$$f_i(Z) := B(\psi(Z) - X(m_k), Y_i) \quad \text{for} \quad Z \in M.$$

Lemma 4.13 together with (5.9) yields

$$\mathcal{V}_k = \{Z \in M \mid f_i(Z) = 0 \quad (i = 1, \ldots, t)\}.$$

By case-by-case examination, we can find an element $Z_0 \in \mathcal{V}_k$ on which the differentials $(df_i)_{Z_0}$ $(i = 1, \ldots, t)$ are linearly independent. In fact, the "identitylike" matrices given in Lemmas 5.3–5.5 satisfy this requirement if $\mathcal{V}_k$ is a single $G'_{\mathbb{C}}$-orbit. Otherwise, one can choose Z_0 as

$$\begin{cases} \tilde{O} \in \tilde{M}_{p-q,k-q} & (SU(p,q),\ k > q,\ p > q), \\ (0\ldots 0, 0\ldots 0)^{\sim} \in \tilde{M}_{1,2(k-r)} & (SO^*(2n), k > (n-1)/2 \text{ with odd } n). \end{cases}$$

Thus we get $(f_1, \ldots, f_t) = \mathcal{I}_k$, by applying Lemma 4 of [**17**, page 345]. This shows $\mathcal{I}_k \subset \omega(\mathfrak{m})\mathbb{C}[M]$ as desired.

Second, consider the case $G = Sp(n,\mathbb{R})$ with $k \leqslant n$. Then we know $\mathcal{V}_k \simeq O(k,\mathbb{C})$ by Lemma 5.4, and hence the equality $\omega(\mathfrak{m})\mathbb{C}[M] = \mathcal{I}_k$ is an easy consequence of a classical theorem of Weyl [**37**, Theorem (5.2.C)].

Now the equality (5.22) and so the isomorphism (5.23) have been proved completely. □

5.4. $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$. — We are now in a position to specify the $K_{\mathbb{C}}(m_k)$-modules $\mathcal{W}[\sigma]$ for every $\sigma \in \Sigma(k)$ $(k = 1, 2, \ldots)$. First, we prepare some notation to state the results in a unified form. Let $G'_{\mathbb{C}}(k-r)$ $(r = \mathbb{R}\text{-rank}\, G)$ be the subgroup of $G'_{\mathbb{C}}$ in (5.15) and (5.16). With Lemmas 5.3–5.5 in mind, we introduce a $G'_{\mathbb{C}}(k-r)$-stable subvariety $\mathcal{U}_k$ of $\mathcal{V}_k$ as follows. We set

$$\mathcal{U}_k := \begin{cases} \left\{ \begin{pmatrix} I_{p,k}(m_k) \\ I_{q,k}(m_k) \end{pmatrix} \right\} & (k \leqslant q \ \text{ or } \ k > q = p), \\ \tilde{M}_{p-q,k-q} & (k > q \ \text{ and } \ p \neq q) \end{cases}$$

for $G = SU(p, q)$, and

$$\mathcal{U}_k := \{I_{n,k}(m_k)\} \quad (k = 1, 2, \ldots) \quad \text{for } G = Sp(n, \mathbb{R}),$$

where $m_k = \min(k, r)$ as before. The variety $\mathcal{U}_k$ for $G = SO^*(2n)$ is defined to be

$$\mathcal{U}_k := \begin{cases} \{I_{n,2k}(2k)\} & (k \leqslant r = [n/2]) \\ \left\{ \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \right\} & (k > r = n/2 \ \text{ with } \ n \text{ even}), \\ \tilde{M}_{1,2(k-r)} & (k > r = (n-1)/2 \ \text{ with } \ n \text{ odd}). \end{cases}$$

Then, Lemmas 5.3–5.5 imply that

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \mathcal{U}_k, \tag{5.24}$$

and that the $G'_{\mathbb{C}}$-orbits $\mathcal{X}$ in $\mathcal{V}_k$ are in one-one correspondence with the $G'_{\mathbb{C}}(k-r)$-orbits $\mathcal{X} \cap \mathcal{U}_k$ in $\mathcal{U}_k$.

Definition 5.9. — We say that the pair (G, G') is of type (S$\vee$T) if the pair (G, G') is in the stable range with smaller member G' (i.e., $k \leqslant r$), or the symmetric space $K\backslash G$ is of tube type (i.e., $G = SU(p, q)$ with $p = q$, $Sp(n, \mathbb{R})$, or $SO^*(2n)$ with n even). This happens exactly when $\mathcal{U}_k$ consists of a single $G'_{\mathbb{C}}(k-r)$-fixed point, say Z_0. We call it the case (S$\vee$T), too.

Now Proposition 5.8 allows us to deduce the following

Proposition 5.10. — *Under the above notation, let $\mathbb{C}[\mathcal{U}_k]$ be the coordinate ring of $G'_{\mathbb{C}}(k-r)$-stable variety $\mathcal{U}_k$ viewed as a $G'_{\mathbb{C}}(k-r)$-module in the canonical way. Then one has a linear isomorphism*

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k]) \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)} \quad (\sigma \in \Sigma(k)). \tag{5.25}$$

In particular, it holds that

$$\mathcal{W}[\sigma] \simeq (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)} \quad \textit{for the case } (\mathrm{S}\vee\mathrm{T}). \tag{5.26}$$

Here $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ denotes the subspace of $V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k]$ of $G'_{\mathbb{C}}(k-r)$-fixed vectors, and the right hand side of (5.26) *turns to be V_σ^* if $k \leqslant r$.*

Proof. — We know the $K_{\mathbb{C}}(m_k)$-isomorphism $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k])$ thanks to (5.13) and (5.23). Let T be a $G'_{\mathbb{C}}$-homomorphism from V_σ to $\mathbb{C}[\mathcal{V}_k]$. Set $T_0(v) := T(v)|\mathcal{U}_k$, the restriction of $T(v) \in \mathbb{C}[\mathcal{V}_k]$ to $\mathcal{U}_k$, for each $v \in V_\sigma$. Then T_0 gives a homomorphism of $G'_{\mathbb{C}}(k-r)$-modules from V_σ to $\mathbb{C}[\mathcal{U}_k]$. By using Lemmas 5.3–5.5 (see also (5.24)), it is standard to verify that the assignment $T \mapsto T_0$ sets up a linear isomorphism

$$\mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k]) \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k]), \tag{5.27}$$

which is a variant of the Frobenius reciprocity. We thus obtain (5.25) (the second isomorphism is a natural one). (5.26) follows from (5.25) immediately, since $\mathbb{C}[\mathcal{U}_k]$ is the one-dimensional trivial $G'_{\mathbb{C}}(k-r)$-module for the case (S$\vee$T). □

Remark 5.11. — For the case $G = SU(p,q)$, the above proposition is due to Tagawa [**31**, Th.3.10.1].

Remark 5.12. — The irreducible decomposition of $G'_{\mathbb{C}}(k-r)$-module $\mathbb{C}[\mathcal{U}_k]$ is well-known even if $\mathcal{U}_k$ is not a variety of single point. Indeed, $\mathbb{C}[\mathcal{U}_k]$ is isomorphic to the natural $GL(k-q,\mathbb{C})$-module $\mathbb{C}[M_{p-q,k-q}]$ (resp. such $Sp(k-r,\mathbb{C})$-module $\mathbb{C}[M_{1,2(k-r)}]$) when $G = SU(p,q)$ with $p > q$ and $k > q$ (resp. $G = SO^*(2n)$ with $k > (n-1)/2$ and n odd). On one hand, the $GL_{p-q} \times GL_{k-q}$ duality can be used to decompose $\mathbb{C}[M_{p-q,k-q}]$ into irreducibles. On the other hand, the space $S^l(M_{1,2(k-r)})$ of homogeneous polynomials on $M_{1,2(k-r)}$ of any fixed degree l turns to be an irreducible $Sp(k-r,\mathbb{C})$-module with highest weight $(l,0,\ldots,0)$. This yields the irreducible decomposition

$$\mathbb{C}[M_{1,2(k-r)}] = \oplus_{l\geqslant 0} S^l(M_{1,2(k-r)})$$

as $Sp(k-r,\mathbb{C})$-modules.

Hence the right hand side of (5.25) can be described concretely by a combinatorial method, once one knows the branching rule of irreducible representations of $G'_{\mathbb{C}}$ restricted to the subgroup $G'_{\mathbb{C}}(k-r)$ (cf. [**19**], [**30**]). Although we do not discuss it in this paper, the author would like to thank K. Koike for kind communication on the branching rule of finite-dimensional representations of complex classical groups.

In view of Corollary 3.9, we get a direct consequence of Proposition 5.10 as follows.

Corollary 5.13. — *Let σ be in $\Sigma(k)$. Then, the multiplicity* $\mathrm{mult}_{I_{m_k}}(L[\sigma])$ *of irreducible highest weight module $L[\sigma]$ at the defining ideal I_{m_k} of the associated variety $\mathcal{V}(L[\sigma])$ coincides with the dimension of vector space $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$. Especially, one gets* $\mathrm{mult}_{I_{m_k}}(L[\sigma]) = \dim\sigma$ *if $k \leqslant r$ (cf.* [**27**, Th.9.1]).

At the end, we are going to clarify how the isotropy subgroup $K_{\mathbb{C}}(m_k)$ acts on the space $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$. To do this, we note that the elements g of the subgroup $K_{\mathbb{C}}(m)$ $(0 \leqslant m \leqslant r)$ of $K_{\mathbb{C}}$ (see Table (5.1)) are written for each group $G = SU(p,q)$, $Sp(n,\mathbb{R})$ and $SO^*(2n)$ respectively as follows.

$$g = \begin{cases} \left(\begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix}, \begin{pmatrix} g_{11} & O \\ g_{43} & g_{44} \end{pmatrix}\right) \in K_{\mathbb{C}} \text{ with } g_{11} \in GL(m,\mathbb{C}) & (SU(p,q)), \\[2ex] \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} \text{ with } g_{11} \in O(m,\mathbb{C}) & (Sp(n,\mathbb{R})), \\[2ex] \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} \text{ with } g_{11} \in Sp(m,\mathbb{C}) & (SO^*(2n)). \end{cases}$$

This enables us to define a group homomorphism

$$\alpha : K_{\mathbb{C}}(m_k) \to G'_{\mathbb{C}}, \quad g \mapsto \alpha(g)$$

by

$$\alpha(g) := \begin{pmatrix} g_{11} & O \\ O & I_{k-r} \end{pmatrix} \quad \text{for} \quad SU(p,q) \text{ or } Sp(n,\mathbb{R}),$$

and by

$$\alpha(g) := \begin{pmatrix} p_{11} & O & p_{12} & O \\ O & I_{k-r} & O & O \\ p_{21} & O & p_{22} & O \\ O & O & O & I_{k-r} \end{pmatrix} \quad \text{with } g_{11} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad \text{for } SO^*(2n).$$

Here p_{ij} is a matrix of size k, and $\alpha(g)$ should be understood as g_{11} if $k \leqslant r$. Note that the elements of $\alpha(K_{\mathbb{C}}(m_k))$ commute with those of the subgroup $G'_{\mathbb{C}}(k-r)$.

Now we can deduce

Theorem 5.14 **(Case (S$\vee$T)).** — *Assume that the pair (G,G') is of type* (S$\vee$T) *in Definition 5.9. Then it holds that*

$$(5.28) \qquad \mathcal{W}[\sigma] \simeq (\delta_k \otimes (\sigma^* \circ \alpha), \ \ (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)}) \quad \textit{as } K_{\mathbb{C}}(m_k)\textit{-modules},$$

where δ_k is the character of $K_{\mathbb{C}}$ in (5.6). In particular, $\mathcal{W}[\sigma]$ is an irreducible $K_{\mathbb{C}}(m_k)$-module if $k \leqslant r$.

Proof. — Let Z_0 be the unique element of $\mathcal{U}_k$. By noting that

$$g \cdot Z_0 = \alpha(g)^{-1} \cdot Z_0 \quad (g \in K_{\mathbb{C}}(m_k)),$$

it is a routine task to transfer the $K_{\mathbb{C}}(m_k)$-action on $\mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[\mathcal{V}_k])$ to that on $(V_\sigma^*)^{G'_{\mathbb{C}}(k-r)} \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$ through the isomorphism (5.27). We thus get (5.28). If $k \leqslant r$, the homomorphism α is surjective. Hence (5.28) implies the irreducibility of $\mathcal{W}[\sigma]$ for $k \leqslant r$. $\square$

Next we consider the remaining case, and assume that (G, G') is not of type (S∨T). Then one has $k > r$ and so $m_k = r$. Set $l := p - q$ for $G = SU(p,q)$ $(p > q)$, and $l = 1$ for $G = SO^*(2n)$ (n odd). Then, $\beta(g) := g_{22}$ $(g \in K_{\mathbb{C}}(r))$ defines a group homomorphism β from $K_{\mathbb{C}}(r)$ to $GL(l, \mathbb{C})$. The group $K_{\mathbb{C}}(r)$ acts on

$$\mathbb{C}[\mathcal{U}_k] \simeq \mathbb{C}[M_{l,\epsilon(k-r)}]$$

naturally through the left multiplication composed with β, where $\epsilon := 1$ for $G = SU(p,q)$, and $\epsilon := 2$ for $G = SO^*(2n)$. We denote by ν the resulting representation of $K_{\mathbb{C}}(r)$ on $\mathbb{C}[\mathcal{U}_k]$. Note that ν as well as $\sigma^* \circ \alpha$ commutes with the $G'_{\mathbb{C}}(k-r)$-action.

Theorem 5.15 **(Non (S∨T) case).** — *Under the above assumption and notation, the reductive part of $K_{\mathbb{C}}(r)$ acts on $\mathcal{W}[\sigma] \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ by the representation $\delta_k \otimes (\sigma^* \circ \alpha) \otimes \nu$.*

Proof. — This theorem can be proved just as in the proof of Theorem 5.14 by noting that

$$g \cdot \tilde{U} = \alpha(g)^{-1} \cdot (\beta(g)U)^{\sim} \quad (U \in M_{l,\epsilon(k-r)})$$

holds if $g \in K_{\mathbb{C}}(r)$ lies in the reductive part of $K_{\mathbb{C}}(r)$, i.e., $g_{12} = 0$. We omit the detail of the proof. □

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THE DEGREES OF ORBITS OF THE MULTIPLICITY-FREE ACTIONS

by

Shohei Kato & Hiroyuki Ochiai

Abstract. — We give a formula for the degrees of orbits of the irreducible representations with multiplicity-free action. In particular, we obtain the Bernstein degree and the associated cycle of the irreducible unitary highest weight modules of the scalar type for arbitrary hermitian Lie algebras.

***Résumé* (Degrés des orbites nilpotentes des représentations irréductibles sans multiplicité)**
Nous donnons une formule pour les degrés des orbites nilpotentes des représentations irréductibles sans multiplicité. Nous obtenons les degrés de Bernstein et les cycles associés des représentations irréductibles unitaires de plus haut poids de type scalaire pour des algèbres de Lie hermitiennes.

1. Introduction

Let K be a connected reductive complex algebraic group, and V an irreducible representation of K. We assume that the action of K is multiplicity-free; that is, each irreducible representation of K occurs at most once in the polynomial ring $\mathbb{C}[V]$. We also assume that the image of K in $GL(V)$ contains all nonzero scalar matrices $\mathbb{C}^\times \mathrm{id}_V$. Such representations have been classified by Kac [**10**]. There are eight families and five exceptional representations.

In this paper, we determine the degree of each closed K-stable subset Y of V. We establish a method by which we can express some asymptotic behavior of the dimension of the filtered module in terms of a definite integral. This is a generalization of the technique presented in Ref. [**19**]. As a corollary, a formula for the degree of each K-stable closed subset can be obtained (Theorem 2.5). The multiplicity-free action contains an important family coming from the hermitian symmetric spaces. Such representations consist of four families and two exceptionals of the classification

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mentioned above. Using the detailed structure of the restricted root system, we can obtain a formula in these hermitian symmetric cases that is more concise than that obtained in the general case (Theorem 3.2). This formula unifies three kinds (i.e., homomorphism, symmetric endomorphism and skew-symmetric endomorphism) of Giambelli formulas, as well as the corresponding formula for the exceptional Lie algebras. The formula for the degree of the closure of the orbit immediately gives the Bernstein degree of the irreducible unitary highest weight module of the scalar type(Corollary 4.1). For three families of classical Lie algebras $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{u}(p,q)$ and $\mathfrak{o}^*(2n)$, this result is obtained in Section 7 of Ref. **[19]** through case analysis. In the final section, we give two examples demonstrating the calculation of the Bernstein degree of the unitary highest weight modules of the non-scalar type. These are also derived from Theorem 2.3. In the Appendix, we list the explicit values for the degree of the closure of the orbits for all thirteen families of multiplicity-free actions, with some comment on the structure of the orbits.

A part of this paper is taken from the master thesis of the first author **[12]**.

2. Degree of the multiplicity-free action

2.1. Degree. — Let V be a finite-dimensional complex vector space, $\mathbb{C}[V]$ the ring of polynomials on V, and M a finitely-generated $\mathbb{C}[V]$-module. By a standard procedure, we can associate two additive, numerical invariants, the dimension and the multiplicity of M. This procedure is briefly summarized in Section 1 of Ref. **[19]** in this volume.

Let Y be a closed conic subvariety of V, and let $\mathbf{I}(Y)$ be the defining ideal of Y;

$$\mathbf{I}(Y) = \{p \in \mathbb{C}[V] \mid p(y) = 0 \text{ for all } y \in Y\}.$$

We define $\mathbb{C}[Y] = \mathbb{C}[V]/\mathbf{I}(Y)$. Defined in this manner, $\mathbb{C}[Y]$ is the coordinate ring of Y. Since $\mathbf{I}(Y)$ is a (reduced) graded ideal of $\mathbb{C}[V]$, $\mathbb{C}[Y]$ is naturally a graded $\mathbb{C}[V]$-module. The multiplicity of $\mathbb{C}[Y]$ is called the degree of Y, and is denoted by $\deg(Y)$. It is known that the degree of a complete intersection is elementary.

Lemma 2.1

(i) *If Y is a complete intersection, then the degree of Y is the product of the degrees of the defining equations of the irreducible components of Y.*
(ii) *If Y is a hypersurface, then the degree of Y is the degree (as a homogeneous polynomial) of the defining equation of Y.*
(iii) *If Y is a linear subspace of V, then the degree of Y is* 1.

The assertion (iii) is a special case of (ii), and (ii) is a special case of (i). The assertion (i) is found in standard textbooks, such as Ref. **[4]**. On the other hand, if the variety Y is not a complete intersection, such as a determinantal variety, its degree is non-trivial, as can be seen from the Giambelli formula.

2.2. Asymptotic behavior of some graded module. — Let K be a connected reductive complex algebraic group, and let V be a finite dimensional representation of K. Let $\mathbb{C}[V]^i$ be the set of homogeneous polynomials in $\mathbb{C}[V]$ of degree i. We assume that the image of K in $GL(V)$ contains all nonzero scalar matrices. Then, there exists an element $Z \in \mathrm{Lie}(K)$ such that $Z \cdot p = ip$ for all $p \in \mathbb{C}[V]^i$. This element is called the degree operator (or Euler operator). We denote the natural action of K on the graded algebra $\mathbb{C}[V]$ by Ad. We call M a $(\mathbb{C}[V], K)$-module if M is a $\mathbb{C}[V]$-module and is a completely reducible K-module with the compatibility condition $k \cdot (p \cdot (k^{-1} \cdot m)) = (\mathrm{Ad}\,(k)(p)) \cdot m$ for all $k \in K$, $p \in \mathbb{C}[V]$ and $m \in M$. We denote the decomposition into K-isotypic components by $M = \oplus_\mu M_\mu$. Assume that there exists some isotypic component M_λ generating M as a $\mathbb{C}[V]$-module. Such a component is unique if it exists. We define a graded component by $M^i = \mathbb{C}[V]^i M_\lambda$ for $i \in \mathbb{Z}_{\geqslant 0}$. Then $M = \oplus_i M^i$ is a graded $\mathbb{C}[V]$-module, and each graded component is given by

$$M^i = \{m \in M \mid Z \cdot m = (\lambda(Z) + i)m\}.$$

We assume, moreover, that M has a multiplicity-free decomposition

$$M = \bigoplus_{\varphi \in \Lambda(M)} F(\lambda + \varphi),$$

where $F(\mu)$ is a (finite-dimensional) irreducible K-module whose highest weight is μ, and that there exists linearly independent weights $\varphi_1, \ldots, \varphi_m$ such that

$$\Lambda(M) = \{n_1\varphi_1 + \cdots + n_m\varphi_m \mid n_i \in \mathbb{Z}_{\geqslant 0}\}.$$

In this case, the graded component M^i is given by

$$M^i = \oplus F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m),$$

where the summation is over $(n_1, \ldots, n_m) \in \mathbb{Z}^m_{\geqslant 0}$ with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) = i$. We will determine the asymptotic of the dimension of the graded component for large i.

Using the Weyl dimension formula, it can be shown that $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m)$ is a polynomial in $(n_1, \ldots, n_m)$. To be more explicit, let Δ_K^+ be the set of positive roots of the Lie algebra of K, and let ρ_K be the half sum of positive roots. We define

$$\Delta_M^+ = \Delta_K^+ \setminus \{\alpha \in \Delta_K^+ \mid \langle \alpha, \varphi_i \rangle = 0 \text{ for all } i = 1, \ldots, m\}$$

and

$$f(x_1, \ldots, x_m) = \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times$$
$$\times \prod_{\alpha \in \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle + x_1 \langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}.$$

Then $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m) = f(n_1, \ldots, n_m)$. The degree of the polynomial f is equal to the number $|\Delta_M^+|$ of roots in Δ_M^+, and the leading term, which we denote by $\bar{f}$, is

$$\bar{f}(x_1, \ldots, x_m) = \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}.$$

We define a filtered module $M_l = \oplus_{i=0}^{l} M^i$. This $\{M_l\}_l$ gives the filtration of M; and the dimension of the filtered component is

$$\dim M_l = \sum f(n_1, \ldots, n_m),$$

where the summation is over $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_{\geqslant 0}^m$, with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) \leqslant l$. We express this condition as $|\mathbf{n}| \leqslant l$ for short.

Lemma 2.2. — *Let $d = m + |\Delta_M^+|$. Then*

$$\lim_{l \to \infty} l^{-d} \sum_{|\mathbf{n}| \leqslant l} f(\mathbf{n}) = \int \bar{f}(x) dx_1 \cdots dx_m,$$

where the domain of integration is the simplex

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

Summarizing the above, we have the following theorem:

Theorem 2.3. — *If l is large, then*

$$\dim M_l = c \cdot l^d / d! + (\text{lower order terms}),$$

where $d = m + |\Delta_M^+|$ and

$$c = d! \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \int \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle} dx_1 \cdots dx_m,$$

with the domain of integration

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

2.3. Multiplicity-free action. — Let V and K be as in the Introduction. That is, in addition to the assumption made in the previous subsection, we assume that the representation V is irreducible and that $\mathbb{C}[V]$ is multiplicity-free. The set of highest weights of K-types arising in $\mathbb{C}[V]$ is a free semigroup. We denote the set of generators by $P\hat{A}^+(V)$.

Let Y be a closed irreducible K-stable subset of V. Since V has a finite number of K-orbits, Y is the closure of a K-orbit on V. We set $M = \mathbb{C}[Y]$. As in § 2.1, M can naturally be considered the quotient ring of $\mathbb{C}[V]$, and thus it inherits the natural grading from $\mathbb{C}[V]$. Then the $(\mathbb{C}[V], K)$-module M satisfies the first assumption in § 2.2, with the weight λ taken to be zero.

Lemma 2.4. — *Suppose Y is an irreducible closed K-stable subset of V. Then there exists a free semigroup $\Lambda(Y)$ such that $\mathbb{C}[Y] = \oplus_{\varphi\in\Lambda(Y)} F(\varphi)$ as a K-module. The generators of the free semigroup $\Lambda(Y)$ form a subset of $P\hat{A}^+(V)$. This subset is denoted by $P\hat{A}^+(Y) \subset P\hat{A}^+(V)$.*

A proof of this lemma is given in Ref. **[6]**. Also appearing there is the explicit form of the subset generating the subsemigroup, which we use in an application below.

We denote the number of elements of $P\hat{A}^+(Y)$ by m, and we set $P\hat{A}^+(Y) = \{\varphi_1,\dots,\varphi_m\}$. For a weight α, we define the vector $(\alpha_1,\dots,\alpha_m) \in \mathbb{R}^m$ by $(\langle\alpha,\varphi_1\rangle,\dots,\langle\alpha,\varphi_m\rangle)$. We define

$$\Delta_Y^+ = \{\alpha \in \Delta^+ \mid (\alpha_1,\dots,\alpha_m) \neq 0\}$$

and $k_i = \varphi_i(Z) \in \mathbb{Z}_{>0}$. Then, the K-type $F(\varphi_i)$ appears in the homogeneous component $\mathbb{C}[Y]^{k_i}$. With this notation, we can give the degree of Y.

Theorem 2.5. — *The dimension of Y is $m + |\Delta_Y^+|$, and the degree of Y is*

$$\frac{(m+|\Delta_Y^+|)!}{\prod_{\alpha\in\Delta_Y^+}\langle\alpha,\rho_K\rangle} \times \int \prod_{\alpha\in\Delta_Y^+} (\alpha_1x_1 + \cdots + \alpha_m x_m)\, dx_1\cdots dx_m,$$

where the domain of the integration is the simplex

$$\{(x_1,\dots,x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0,\dots,x_m \geqslant 0,\ k_1x_1 + \cdots + k_mx_m \leqslant 1\}.$$

Proof. — Applying Theorem 2.3 with

$$\bar{f}(x_1,\dots,x_m) = \prod_{\alpha\in\Delta_Y^+} \frac{\alpha_1x_1 + \cdots + \alpha_m x_m}{\langle\alpha,\rho_K\rangle},$$

we obtain the result. □

3. Hermitian symmetric case

In this section, we consider the subclass of the multiplicity-free actions consisting of the holomorphic tangent spaces of the hermitian symmetric spaces. In this case, we can obtain a more sophisticated formula for the degree by using the structure of the restricted root system.

3.1. Hermitian Lie algebra. — We first recall some standard notation of Lie algebras, root systems and weights.

Let $\mathfrak{g}_0$ be a non-compact real simple Lie algebra. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We assume that the center $\mathfrak{c}_0$ of $\mathfrak{k}_0$ is non-zero, that is, that $\mathfrak{g}_0$ is of the hermitian type. Then $\mathfrak{c}_0$ is one dimensional. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Then $\mathfrak{t}_0$ is a compact Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{t}$ denote the respective complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{t}_0$. We denote the Killing form by $B(\cdot,\cdot)$. The restriction of the Killing form on $\mathfrak{t}$ is a non-degenerate symmetric bilinear form.

Using this, we identify $\mathfrak{t}$ with its dual $\mathfrak{t}^*$, and introduce the non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{t}^*$. Let Δ be the root system of $(\mathfrak{g},\mathfrak{t})$, and $\mathfrak{g}_\alpha$ the root space corresponding to the root $\alpha\in\Delta$. A root α is said to be compact (resp., non-compact) if $\mathfrak{g}_\alpha\subset\mathfrak{k}$ (resp., $\mathfrak{g}_\alpha\subset\mathfrak{p}$). Let Δ_c (resp., Δ_n) denote the set of all compact (resp., non-compact) roots in Δ. We have the disjoint decomposition $\Delta=\Delta_c\cup\Delta_n$.

There exists an element $Y_0\in\sqrt{-1}\mathfrak{c}_0$ such that $\gamma(Y_0)=\pm1$ for any $\gamma\in\Delta_n$. This Y_0 is called the characteristic element. We set $\Delta_n^\pm=\{\alpha\in\Delta\mid\alpha(Y_0)=\pm1\}$. Then $\Delta_c=\{\alpha\in\Delta\mid\alpha(Y_0)=0\}$, and we have the disjoint decomposition $\Delta=\Delta_n^+\cup\Delta_c\cup\Delta_n^-$. Then $\mathfrak{k}=\mathfrak{t}\oplus(\oplus_{\alpha\in\Delta_c}\mathfrak{g}_\alpha)$ gives the root space decomposition, and if we set $\mathfrak{p}^\pm=\oplus_{\alpha\in\Delta_n^\pm}\mathfrak{g}_\alpha$, then we have the triangular decomposition $\mathfrak{g}=\mathfrak{p}^+\oplus\mathfrak{k}\oplus\mathfrak{p}^-$. We choose an ordering of Δ such that the set Δ^+ of all positive roots satisfies the condition $\Delta_n^+\subset\Delta^+$. Let $\Delta_c^\pm=\Delta^\pm\cap\Delta_c$.

As in Ref. **[1]**, we construct a maximally strongly orthogonal subset $\{\gamma_1,\dots,\gamma_r\}\subset\Delta_n^+$ such that γ_i is the smallest element of the subset of elements in Δ_n^+ orthogonal to $\gamma_1,\dots,\gamma_{i-1}$. Then γ_1 is the unique simple non-compact root. For a $\lambda\in\mathfrak{t}^*$, we define $H_\lambda\in\mathfrak{t}$ by $B(H_\lambda,h)=\lambda(h)$ for all $h\in\mathfrak{t}$, or equivalently, $\lambda'(H_\lambda)=\langle\lambda,\lambda'\rangle$ for all $\lambda'\in\mathfrak{t}^*$. Let $\mathfrak{t}^-=\sum_{i=1}^r\mathbb{C}H_{\gamma_i}$. Then $\{H_{\gamma_1},\dots,H_{\gamma_r}\}$ forms a basis of $\mathfrak{t}^-$. Then, letting $\mathfrak{t}^+=\{H\in\mathfrak{t}\mid\gamma_i(H)=0\text{ for all }i=1,\dots,r\}$, we have $\mathfrak{t}=\mathfrak{t}^+\oplus\mathfrak{t}^-$.

We summarize several facts on strongly orthogonal roots (see, e.g., **[23]**, **[24]**). Note that the strongly orthogonal roots $\{\gamma_i\}$ here are taken from the minimal γ_1, while those of **[24]** in this volume are taken from the maximal γ_r.

Lemma 3.1

(1) *For $1\leqslant i<j\leqslant r$, γ_i and γ_j are strongly orthogonal: $\gamma_i\pm\gamma_j\notin\Delta$.*

(2) *The number r of maximally strongly orthogonal roots is equal to the split rank of $\mathfrak{g}_0$.*

(3) *If $\alpha\in\Delta_c^+$, then the restriction $\alpha|_{\mathfrak{t}^-}$ takes one of the following possible forms:*
- $-\bar\gamma_i/2$ *for some $i=1,\dots,r$.*
- $-(\bar\gamma_k-\bar\gamma_l)/2$ *for some $1\leqslant k<l\leqslant r$.*
- 0.

(4) *If $\alpha\in\Delta_n^+$, then the restriction $\alpha|_{\mathfrak{t}^-}$ takes one of the following possible forms:*
- $\bar\gamma_i/2$, $\bar\gamma_i$ *for some $i=1,\dots,r$.*
- $(\bar\gamma_k+\bar\gamma_l)/2$ *for some $1\leqslant k<l\leqslant r$.*

(5) *The set of non-zero restrictions of $\Delta(\mathfrak{g},\mathfrak{t})$ to $\mathfrak{t}^-$ is one of the following two:*
- $\Delta(\mathfrak{g},\mathfrak{t}^-)=\{\pm\bar\gamma_i,\pm(\bar\gamma_k\pm\bar\gamma_l)/2\mid1\leqslant i\leqslant r,1\leqslant k<l\leqslant r\}$ *: type C_r,*
- $\Delta(\mathfrak{g},\mathfrak{t}^-)=\{\pm\bar\gamma_i/2,\pm\bar\gamma_i,(\bar\gamma_k\pm\bar\gamma_l)/2\mid1\leqslant i\leqslant r,1\leqslant k<l\leqslant r\}$ *: type BC_r.*

The root system is of type C_r if and only if the hermitian Lie algebra $\mathfrak{g}_0$ is of the tube type.

(6) *By the Cayley transformation, the toral subalgebra $\mathfrak{t}^-$ is isomorphic to the complexification of a split Cartan subalgebra of $\mathfrak{g}_0$. This implies that the root system $\Delta(\mathfrak{g},\mathfrak{t}^-)$ coincides with the restricted root system of $\mathfrak{g}_0$.*

(7) *The dimension of root spaces has the following properties:*
– $\dim \mathfrak{g}(\mathfrak{t}^-, \pm\bar{\gamma}_i) = 1$.
– *The dimension* $\dim \mathfrak{g}(\mathfrak{t}^-, \pm(\bar{\gamma}_k \pm \bar{\gamma}_l)/2)$ *does not depend on k or l. This dimension is called the multiplicity of middle roots.*
– *The dimension* $\dim \mathfrak{g}(\mathfrak{t}^-, \pm\bar{\gamma}_i/2)$ *does not depend on i. This dimension is called the multiplicity of short roots. The multiplicity of short roots is zero if and only if the Lie algebra $\mathfrak{g}_0$ is of the tube type.*

(8) *The number of compact roots has the following properties:*
– *The cardinality of the set* $\{\alpha \in \Delta_c^+ \mid \alpha|_{\mathfrak{t}^-} = -(\bar{\gamma}_k - \bar{\gamma}_l)/2\}$ *is equal to the multiplicity of middle roots.*
– *The cardinality* $\#\{\alpha \in \Delta_c^+ \mid \alpha|_{\mathfrak{t}^-} = -\bar{\gamma}_i/2\} = \#\{\alpha \in \Delta_n^+ \mid \alpha|_{\mathfrak{t}^-} = -\bar{\gamma}_i/2\}$ *is equal to half of the multiplicity of short roots.*

(9) *Let* $\Delta_0 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{t}^-} = 0\}$. *Then Δ_0 is a subset of Δ_c and is the root system corresponding to the reductive subalgebra* $Z_{\mathfrak{k}}(\mathfrak{t}^-) = \{X \in \mathfrak{k} \mid [X, H] = 0,$ *for all* $H \in \mathfrak{t}^-\}$.

(10) *The strongly orthogonal roots $\gamma_1, \dots, \gamma_r$ are long roots and have the same length.*

We recall the classification of the hermitian Lie algebra $\mathfrak{g}_0$ and some relevant information which we will use later.

	CI	AIII	DIII	BI, DI	EIII	EVII
$\mathfrak{g}_0$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{su}(p,q)$	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2,n)$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{7(-25)}$
r	n	$\min(p,q)$	$[n/2]$	2	2	3
c	1/2	1	2	$(n-2)/2$	3	4
middle	1	2	4	$n-2$	6	8
short	0	$2\|p-q\|$	0 or 4	0	8	0

Here, we follow the notation of Ref. [**1**]. The split rank r of $\mathfrak{g}_0$ is denoted by t in Table 1 of Ref. [**2**]. The length of the interval c of the Wallach set is given in Table 2.9 of Ref. [**1**]. It is denoted by $\varepsilon = \varepsilon_{\mathfrak{g},\alpha}$ in Table 1 of Ref. [**2**]. Then c is equal to half of the multiplicity of the middle roots. The entries in the row labeled 'middle' (resp. 'short') are the root multiplicities of the restricted root system of $\mathfrak{g}_0$. These values are quoted from [**5**](Table VI, Ch.X). The multiplicity of short roots for type DIII is zero (resp., four) if n is even (resp., odd).

3.2. Degree of the orbit. — Let $G_{\mathbb{C}}$ be a connected linear Lie group with Lie algebra $\mathfrak{g}$, and let $G_{\mathbb{R}}$, K and $K_{\mathbb{R}}$ be the connected analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}_0$, $\mathfrak{k}$ and $\mathfrak{k}_0$, respectively. The restriction of the adjoint action of $G_{\mathbb{C}}$ on $\mathfrak{g}$ to the subgroup K preserves the subspaces $\mathfrak{k}$ and $\mathfrak{p}^{\pm}$. We now recall the orbit decomposition of the action of K on $\mathfrak{p}^+$. (See Section 3.1 of Ref. [**24**].) In this decomposition, the closure relation of the orbits is a linear ordering, and the number of K-orbits on $\mathfrak{p}^+$ is $r+1$. Then we can enumerate orbits $\mathcal{O}_m$ with $m = 0, 1, \ldots, r$ so that the closure is given by $\overline{\mathcal{O}_m} = \mathcal{O}_m \cup \cdots \cup \mathcal{O}_1 \cup \mathcal{O}_0$. Any K-stable closed subset of $\mathfrak{p}^+$ is irreducible and of the form $\overline{\mathcal{O}_m}$.

We define $\varphi_i = -(\gamma_1 + \cdots + \gamma_i)$. Then, with the notation of Lemma 2.4, $P\hat{A}^+(V) = \{\varphi_1, \ldots, \varphi_r\}$ and $P\hat{A}^+(\overline{\mathcal{O}_m}) = \{\varphi_1, \ldots, \varphi_m\}$.

We define the following definite integral:

$$I^\alpha(s, m) = \int_{D_m} (x_1 x_2 \cdots x_m)^s \prod_{1 \leqslant i < j \leqslant m} |x_i - x_j|^\alpha \, dx_1 \cdots dx_m.$$

Here the parameters α and s are positive real numbers, and the domain of the integration D_m is the simplex

$$D_m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqslant 0, \ x_1 + \cdots + x_m \leqslant 1\}.$$

This integral is evaluated in Ref. [**11**] (see also Example VI.10.7(c) of Ref. [**16**] and Theorem 2.2 of Ref. [**18**]). The result is

$$I^\alpha(s, m) = \frac{m! \prod_{i=1}^m \Gamma(i\alpha/2)}{\Gamma(\alpha/2)^m} \frac{\prod_{i=1}^m \Gamma((s+1) + (i-1)\alpha/2)}{\Gamma(1 + m(s+1) + (m-1)m\alpha/2)}.$$

Let s_r (resp., c) be equal to half of the root multiplicity of the short (resp., middle) roots of the restricted root system. In particular, $s_r = 0$ for the tube type. For $m = 0, \ldots, r$, we define $s_m = s_r + 2c(r-m)$, $d_m = m(s_m + 1) + (m-1)mc$, and

$$\Delta^+_{c,m} = \{\alpha \in \Delta^+_c \mid \langle \alpha, \gamma_i \rangle = 0 \text{ for all } i = 1, \ldots, m\}$$

Theorem 3.2

(1) *The dimension of the orbit $\mathcal{O}_m$ is d_m. This is the homogeneous degree of the integrand of the integral $I^{2c}(s_m, m)$.*
(2) *The degree of $\overline{\mathcal{O}_m}$ is given by*

$$\deg(\overline{\mathcal{O}_m}) = d_m! \times \frac{(\langle \gamma_1, \gamma_1 \rangle / 2)^{d_m - m}}{\prod_{\alpha \in \Delta^+_c \setminus \Delta^+_{c,m}} \langle \alpha, \rho_c \rangle} \times \frac{1}{m!} I^{2c}(s_m, m).$$

The explicit values of degrees are given in the Appendix. We remark that the degrees of almost all orbits in the present case can be obtained without using the above theorem, as they can be obtained from the previously obtained results appearing in many detailed works. This theorem, however, gives a unified formula for the degree in terms of K-types corresponding to the orbits.

3.3. Proof of Theorem 3.2. — We first apply Theorem 2.5. Let $V = \mathfrak{p}^+$ and $Y = \overline{\mathcal{O}_m}$ with $0 \leqslant m \leqslant r$. Then $\Delta^+_K = \Delta^+_c$ and $\Delta^+_Y = \Delta^+_c \setminus \Delta^+_{c,m}$. Using the Killing form, we can identify the dual of $\mathfrak{p}^+$ with $\mathfrak{p}^-$. Then $\mathbb{C}[V]$ is isomorphic to the symmetric algebra $S(\mathfrak{p}^-)$. The degree operator Z is $-Y_0$, where Y_0 is the characteristic element. Since $\gamma_i \in \Delta^+_n$, we have $\varphi'_i(Z) = 1$ for all i.

We employ a new set of coordinates y_i, defined in terms of the original coordinates by $y_i = x_i + \cdots + x_m$, in the integral in Theorem 2.5. We define $\varphi'_1 = \varphi_1$ and $\varphi'_i = \varphi_i - \varphi_{i-1}$ for $i \geqslant 2$. Since $\varphi_i = -(\gamma_1 + \cdots + \gamma_i)$, $\varphi'_i = -\gamma_i$. Then the semigroup $\Lambda(M)$ can be expressed as

$$\Lambda(M) = \{n'_1\varphi'_1 + \cdots + n'_m\varphi'_m \mid n'_i \in \mathbb{Z}_{\geqslant 0},\ n'_1 \geqslant n'_2 \geqslant \cdots \geqslant n'_m\}.$$

Next, we define $(\alpha'_1, \ldots, \alpha'_m) = (\langle\alpha, \varphi'_1\rangle, \ldots, \langle\alpha, \varphi'_m\rangle)$ for $\alpha \in \Delta^+_M$. Then $\alpha_1 x_1 + \cdots + \alpha_m x_m = \alpha'_1 y_1 + \cdots + \alpha'_m y_m$. Clearly, the integral

$$\int \prod_{\alpha\in\Delta^+_M} (\alpha_1 x_1 + \cdots + \alpha_m x_m) dx_1 \cdots dx_m$$

over the domain

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}$$

is equal to

$$\int \prod_{\alpha\in\Delta^+_M} (\alpha'_1 y_1 + \cdots + \alpha'_m y_m) dy_1 \cdots dy_m \tag{1}$$

over the domain

$$\{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m \geqslant 0,\ y_1\varphi'_1(Z) + \cdots + y_m\varphi'_m(Z) \leqslant 1\}.$$

We next determine Δ^+_Y and $(\alpha'_1, \ldots, \alpha'_m)$ for each $\alpha \in \Delta^+_Y$.

Lemma 3.3. — *Let e_i be the i-th unit vector in $\mathbb{R}^m$.*

(1) *For any $\alpha \in \Delta^+_c$, $(\alpha'_1, \ldots, \alpha'_m) = (\langle\alpha, -\gamma_1\rangle, \ldots, \langle\alpha, -\gamma_m\rangle)$ takes one of the following forms:*
- *$(\langle\gamma_1, \gamma_1\rangle/2)(e_k - e_l)$ with some $1 \leqslant k < l \leqslant m$.*
- *$(\langle\gamma_1, \gamma_1\rangle/2)e_i$ with some $1 \leqslant i \leqslant m$.*
- *0.*

(2) *For each $1 \leqslant k < l \leqslant m$, the number of $\alpha \in \Delta^+_c$ satisfying the condition $(\alpha'_1, \ldots, \alpha'_m) = (\langle\gamma_1, \gamma_1\rangle/2)(e_k - e_l)$ is equal to the root multiplicity $2c$ of the middle roots.*

(3) *For each i, the number of $\alpha \in \Delta^+_c$ satisfying $(\alpha'_1, \ldots, \alpha'_m) = (\langle\gamma_1, \gamma_1\rangle/2)e_i$ is equal to s_m.*

Proof. — For $m = r$, the assertion follows from the identity $\mathfrak{t}_r = \mathfrak{t}^-$. We next consider the case for general m.

We define $a = (\langle\gamma_1, \gamma_1\rangle/2)$ for convenience. For $1 \leqslant k < l \leqslant m$, we have

$$\{\alpha \in \Delta^+_c \mid (\alpha'_1, \ldots, \alpha'_m) = a(e_k - e_l)\} = \{\alpha \in \Delta^+_c \mid (\alpha'_1, \ldots, \alpha'_r) = a(e_k - e_l)\}.$$

This demonstrate the assertion for the middle roots. For $1 \leqslant i \leqslant m$, the set

$$\{\alpha \in \Delta^+_c \mid (\alpha'_1, \ldots, \alpha'_m) = ae_i\}$$

is the disjoint union of

$$\{\alpha \in \Delta_c^+ \mid (\alpha_1', \dots, \alpha_r') = ae_i\}$$

and

$$\bigsqcup_{j=m+1}^{r} \{\alpha \in \Delta_c^+ \mid (\alpha_1', \dots, \alpha_r') = a(e_i - e_j)\}.$$

This shows that $s_m = s_r + 2c(r-m)$. □

We now complete the proof of Theorem 3.2. By Lemma 3.3, the number of elements in Δ_Y^+ is $2c \times m(m-1)/2 + ms_m$. This implies the formula $d_m = m(s_m+1)+(m-1)mc$. Also by this lemma, we have the following formula for $\bar{f}'$:

$$\begin{aligned}\bar{f}'(y_1, \dots, y_m) &= C \prod_{i=1}^{m} (ay_i)^{\dim \mathfrak{k}(\mathfrak{t}_m, e_i)} \prod_{1 \leqslant k < l \leqslant m} (ay_k - ay_l)^{\dim \mathfrak{k}(\mathfrak{t}_m, e_k - e_l)} \\ &= Ca^{d_m - m} (y_1 \cdots y_m)^{s_m} \prod_{1 \leqslant k < l \leqslant m} (y_k - y_l)^{2c},\end{aligned}$$

where we denote

$$C = \frac{1}{\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle}.$$

The domain of integral in (1) is

$$(2) \qquad D_m' = \{(y_1, \dots, y_m) \in \mathbb{R}^m \mid y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m \geqslant 0,\ y_1 + \cdots + y_m \leqslant 1\}.$$

Since the integrand of $I^\alpha(s,m)$ is symmetric with respect to permutations of the variables $(y_1, \dots, y_m)$, the integral $\int_{D_m'} \bar{f}'(y)dy$ is equal to $\dfrac{1}{m!}\int_{D_m} \bar{f}'(y)dy$. Hence the degree of $\overline{\mathcal{O}_m}$ is equal to $d_m! C a^{d_m - m} I^{2c}(s_m, m)/m!$. This completes the proof of Theorem 3.2. □

4. Unitary highest weight modules of the scalar type

4.1. Highest weight modules. — We keep the notation of Section 3. We define $\mathfrak{p}^+ = \oplus_{\alpha \in \Delta_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \oplus_{\alpha \in \Delta_n^-} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ is a graded Lie algebra with a characteristic element Y_0. We next define $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Then $\mathfrak{q}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with the commutative nilpotent radical $\mathfrak{p}^+$. Every maximal parabolic subalgebra with a commutative nilpotent radical arises in this way.

A weight $\lambda \in \mathfrak{t}^*$ is said to be a Δ_c^+-dominant integral weight if $2\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}_{\geqslant 0}$ for all $\alpha \in \Delta_c^+$. We denote the set of all Δ_c^+-dominant integral weights of $\mathfrak{t}^*$ by P_c^+. Also, we denote the fundamental weight corresponding to the non-compact simple root γ_1 by ζ. In other words, the element $\zeta \in \mathfrak{t}^*$ is characterized by the conditions

$$\langle \zeta, \alpha \rangle = 0 \text{ for all } \alpha \in \Delta_c, \qquad \text{and } \langle \zeta, \gamma_1 \rangle = \langle \gamma_1, \gamma_1 \rangle / 2.$$

Let ρ_c be equal to half of the sum of roots in Δ_c^+ and ρ that of Δ^+.

We denote the irreducible finite dimensional representation of $\mathfrak{k}$ with the highest weight $\lambda \in P_c^+$ by $F(\lambda)$. Through the Levi decomposition $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$, a $\mathfrak{k}$-module is considered as a $\mathfrak{q}$-module on which $\mathfrak{p}^+$ acts trivially. We define the generalized Verma module (or induced module) by

$$N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F(\lambda),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. By definition, $N(\lambda)$ is a highest weight $\mathfrak{g}$-module. It is well known that $N(\lambda)$ has a unique simple quotient $L(\lambda)$. Note that, as in the definition in Ref. **[1]**, we employ no rho-shift in our definition of an irreducible highest weight module $L(\lambda)$. The infinitesimal character of $N(\lambda)$ and $L(\lambda)$ is $\lambda + \rho \in \mathfrak{t}^*$.

The Poincaré-Birkhoff-Witt theorem implies the isomorphism $N(\lambda) \cong U(\mathfrak{p}^+) \otimes_{\mathbb{C}} F(\lambda)$ as a $(U(\mathfrak{p}^+), K)$-module. Note that $\mathfrak{p}^+$ is commutitive and that the enveloping algebra $U(\mathfrak{p}^+)$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{p}^+)$. It is significant that the module $N(\lambda)$ together with $L(\lambda)$ is not only filtered by $U(\mathfrak{g})$ but also is graded by the action of the characteristic element.

4.2. Unitary highest weight modules. — An irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$ is called *unitarizable* if it has a $\mathfrak{g}_0$-invariant positive definite sesqui-linear form. The set of irreducible unitary highest weight modules consists of two classes; one is the set of induced modules (irreducible generalized Verma modules), and the other is the set of irreducible unitary highest weight modules which is not induced. In particular, the latter class with one-dimensional lowest K-types is called the Wallach set. It is easy to see (e.g., Section 2.2 of Ref. **[19]**) that the associated cycle of the generalized Verma module $N(\lambda)$ is $(\dim F(\lambda)) \cdot [\mathfrak{p}^+]$. In what follows, we consider the representation which is not induced.

Let us recall the number c introduced in Section 3.1. For unitary highest weight modules of the *scalar* type $L(z\zeta)$, the Wallach set corresponds to the set of parameters $z = 0, -c, \ldots, -(r-1)c$. It is shown in Ref. **[2]** that the annihilator is

$$\mathrm{Ann}_{U(\mathfrak{p}^-)} L(-mc\zeta) = \mathbf{I}(\overline{\mathcal{O}_m})$$

for $m = 0, \ldots, r$. Since for $m = r$ the Verma module $N(-rc\zeta)$ is irreducible, the unitarizable $L(-rc\zeta)$ does not belong to the Wallach set. However, since the situation is the same for the case $m = r$, we do not exclude the case $m = r$. As a $(U(\mathfrak{p}^-), K)$-module, we have the isomorphism

$$L(-mc\zeta) = U(\mathfrak{p}^-)/\mathbf{I}(\overline{\mathcal{O}_m}) \cong \mathbb{C}[\overline{\mathcal{O}_m}].$$

Thus, the associated variety of $L(-mc\zeta)$ is $\overline{\mathcal{O}_m}$, and the associated cycle of $L(-mc\zeta)$ is $[\overline{\mathcal{O}_m}]$. The Gelfand-Kirillov dimension of $L(-mc\zeta)$ is the dimension of the variety $\overline{\mathcal{O}_m}$, and the Bernstein degree of $L(-mc\zeta)$ is the degree of $\overline{\mathcal{O}_m}$.

As a direct consequence of Theorem 3.2, we can determine the Gelfand-Kirillov dimension and the Bernstein degree of the unitary highest weight module $L(-mc\zeta)$ of the scalar K-type.

Corollary 4.1. — *Let s_m, d_m and $\Delta^+_{c,m}$ be the same as in Theorem 3.2. We consider the representation $L(-mc\zeta)$ with $m = 0, 1, \ldots, r$.*

(1) *The Gelfand-Kirillov dimension of $L(-mc\zeta)$ is d_m.*

(2) *The Bernstein degree of $L(-mc\zeta)$ is*

$$d_m! \frac{(\langle \gamma_1, \gamma_1 \rangle / 2)^{d_m - m}}{\prod_{\alpha \in \Delta^+_c \setminus \Delta^+_{c,m}} \langle \alpha, \rho_c \rangle} \times \frac{1}{m!} I^{2c}(s_m, m).$$

We note that K-type decompositions like that in Lemma 2.4 are given in Ref. **[21]** for the generalized Verma module and in Theorem 5.10 of Ref. **[23]** for the module $L(-mc\zeta)$ in the Wallach set.

5. Further example of the degree of unitary highest weight modules

In the previous section we saw the method introduced in Section 2 is effective for modules of the scalar type. We now consider its application to modules of non-scalar type. In this section, we give calculations of the degrees of some unitary highest weight modules of non-scalar type. These examples are based on the examples in Ref. **[1]**, and we follow the notation used there for the root system.

Let $\{\alpha_1, \ldots, \alpha_l\} \subset \Delta^+$ be the set of simple roots and $\{\omega_1, \ldots, \omega_l\}$ the set of the corresponding fundamental weights.

5.1. EIII, case II, "the last unitarizable place".

— Let $\mathfrak{g}_0$ be of type EIII. The corresponding multiplicity-free action is of the type (xi) in the Appendix. The compact root system Δ_c is of type D_5. Let us consider the Δ^+_c-dominant integral weight of the form

$$\lambda = a\omega_6 + (-a - 4)\omega_1,$$

with positive integer $a \in \mathbb{Z}_{>0}$. Here, the simple root α_1 is taken to be non-compact, and the fundamental weight ω_1 is perpendicular to Δ_c. The weight of this form is referred to in Ref. **[1]** as "the last unitarizable place of Case II". The set $\Delta_\lambda = \{\alpha \in \Delta_c \mid \langle \lambda, \alpha \rangle = 0\}$ is the root system of type D_4 whose simple system is $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

We consider the unitary highest weight representation $L(\lambda)$. This is the only unitary highest weight module $L(\lambda)$ of non-scalar type which is neither induced nor at 'the first reduction point'.

Proposition 5.1. — *For $L(a\omega_6 + (-a - 4)\omega_1)$, the Gelfand-Kirillov dimension is* 16 *and the Bernstein degree is* 1.

Proof. — From the K-type decomposition

$$L(\lambda) = \bigoplus_{n_1 \geqslant n_2 \geqslant 0, n_i \in \mathbb{Z}} F(\lambda - n_1\gamma_1 - n_2\gamma_2)$$

given in Proposition 12.5 of Ref. [1], we have $m = 2$. Using the realization of the root system in the standard Euclidean space [1], we have $\Delta_c^+ = \{\pm e_i + e_j \mid 1 \leqslant i < j \leqslant 5\}$. We calculate

$$\Delta_{c,m}^+ = \{e_1 + e_i \mid i = 2, 3, 4\} \cup \{e_j - e_k \mid 2 \leqslant k < j \leqslant 4\},$$

which is the root system of type A_3 with the simple system $\{\alpha_2, \alpha_4, \alpha_5\}$. This implies that $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_{c,m}^+$. Then

$$\prod_{\alpha \in \Delta_{c,m}^+} \frac{\langle \alpha, \lambda + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = 1.$$

Hence, by Theorem 3.2, the asymptotic of the dimension of the filtered pieces of $L(\lambda)$ is identical to that of the scalar case with $m = 2$. The proposition thus follows from (xi) in the Appendix or (iii) of Lemma 2.1. $\square$

5.2. EVII, case II, the last unitarizable place. — Let $\mathfrak{g}_0$ be a Lie algebra of type EVII. The corresponding multiplicity-free action is given in (xiii) in the Appendix. The root system Δ_c is of type E_6. Let us consider the weight

$$\lambda = k\omega_6 + (-2k - 8)\omega_7,$$

with positive integer k. The fundamental weight ω_7 corresponds to the non-compact simple root α_7. The weight of this form is called "the last unitarizable place of Case II". The subset $\Delta_\lambda = \{\alpha \in \Delta_c \mid \langle \lambda, \alpha \rangle = 0\}$ is the root system of type D_5 whose simple system is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

We consider the representation $L(\lambda)$. This is the only unitary highest weight module of non-scalar type which is neither induced nor at the first reduction point.

Proposition 5.2. — *For $L(k\omega_6 + (-2k - 8)\omega_7)$, the Gelfand-Kirillov dimension is 26 and the Bernstein degree is*

$$\frac{3(2k+7)\prod_{i=1}^{6}(k+i)}{7!}.$$

Proof. — The K-type decomposition is given in Proposition 13.10 of Ref. [1]. We have

$$L(\lambda) = \bigoplus_{\substack{n_i \in \mathbb{Z} \\ n_1 \geqslant n_2 \geqslant 0 \\ 0 \leqslant n_3 \leqslant k}} F(\lambda - n_1\gamma_1 - n_2\gamma_2 - n_3\delta),$$

where $\delta = \alpha_6 + \alpha_7$. We apply Theorem 2.3 with $m = 2$ and with λ replaced by $\lambda - n_3\delta$ for each $n_3 = 0, \ldots, k$. For $m = 2$, $\Delta_{c,m}^+ = \{\pm e_i + e_j \mid 1 \leqslant i < j \leqslant 4\}$, which is the root system of type D_4 with the simple roots $\{\alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. This implies that $\langle \alpha, \lambda \rangle = 0$ for all $\alpha \in \Delta_{c,m}^+$. Then, for each n_3, the contribution to the degree is

$$(3) \qquad \deg(\overline{\mathcal{O}_2}) \times \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, \lambda - n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = \deg(\overline{\mathcal{O}_2}) \times \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, -n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle},$$

Here, the root systems Δ_c^+ and Δ_{c2}^+ have the following significant relation.

Lemma 5.3. — *For any $\alpha \in \Delta_{c2}^+$, we have*

$$\langle \alpha, \delta \rangle = \langle \alpha, \omega_{1,D_4} \rangle \text{ and } \langle \alpha, \rho_c \rangle = \langle \alpha, \rho_{c,D_4} \rangle.$$

Here, $\rho_{c,D_4} = e_2 + 2e_3 + 3e_4$ is equal to half of the sum of the roots in Δ_{c2}^+. The fundamental weight $\omega_{1,D_4} = e_4$ of the natural representation of $\mathfrak{so}(8)$ corresponds to the simple root α_5.

This lemma implies that the quantity (3) is equal to the dimension of the irreducible finite-dimensional representation $F(\mathfrak{so}(8), n_3\omega_{1,D_4})$ of the Lie algebra $\mathfrak{so}(8)$ with the highest weight $n_3\omega_{1,D_4}$. Its value is

$$(n_3+1)(n_3+2)(n_3+3)^2(n_3+4)(n_3+5)/(3\cdot 5!) = \binom{n_3+6}{6} + \binom{n_3+5}{6}.$$

Then, the degree of the representation $L(\lambda)$ is $d = \deg(\overline{\mathcal{O}_2})$ multiplied by the quantity

$$\sum_{n_3=0}^{k} \prod_{\alpha \in \Delta_{c2}^+} \frac{\langle \alpha, -n_3\delta + \rho_c \rangle}{\langle \alpha, \rho_c \rangle} = \binom{k+7}{7} + \binom{k+6}{7} = \frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!},$$

as is required in the proposition. □

Corollary 5.4. — *The associated cycle of $L(k\omega_6 + (-2k-8)\omega_7)$ is*

$$\frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!} \times [\overline{\mathcal{O}_2}].$$

Remark 5.5. — Vogan **[22]** has introduced the *isotropy representation* of the isotropy subgroup of the generic point of the associated variety on the space of the multiplicity of a given $(\mathfrak{g}, K)$-module. In our case, the Lie algebra of the Levi part of the isotropy subgroup of a point of the nilpotent orbit $\mathcal{O}_2$ in K is isomorphic to $\mathfrak{so}(9)$. Let ω_{1,B_4} be the fundamental weight corresponding to the natural (vector) representation of $\mathfrak{so}(9)$, and $F(\mathfrak{so}(9), k\omega_{1,B_4})$ the irreducible finite-dimensional representation of $\mathfrak{so}(9)$ with highest weight $k\omega_{1,B_4}$. It is easy to see, by the Weyl dimension formula, that

$$\frac{(2k+7)\prod_{i=1}^{6}(k+i)}{7!} = \dim F(\mathfrak{so}(9), k\omega_{1,B_4}).$$

Since the restriction of the irreducible representation $F(\mathfrak{so}(9), k\omega_{1,B_4})$ to the subalgebra $\mathfrak{so}(8)$ is decomposed as $\oplus_{n_3=0}^{k} F(\mathfrak{so}(8), n_3\omega_{1,D_4})$, the proof above may suggest interpreting the number as the dimension of the representation as above. Hence, it is suggested that the isotropy representation attached to the representation $L(k\omega_6 + (-2k-8)\omega_7)$ is precisely $F(\mathfrak{so}(9), k\omega_{1,B_4})$.

6. Appendix : List of degrees of orbits

We define the orbit $\mathcal{O}_0 = \{0\}$. The orbit $\mathcal{O}_{\max}$ is open dense.

6.1. Hermitian symmetric case. — The following case (i), (ii), (iii), (iv), (xi), or (xiii) corresponds to the case with Cartan label AIII, CI, DIII, (BI and DI), EIII, or EVII, respectively. (c.f. Table in § 3.1.) We use Theorem 3.2. In the following, we normalize the inner product $\langle\cdot,\cdot\rangle$ so that the restriction on Δ_c is induced from the Killing form on $\mathfrak{k}$. For example, $\langle\gamma_i,\gamma_i\rangle = 4$ for the case (i), while $\langle\gamma_i,\gamma_i\rangle = 2$ for other five cases.

(i) $GL_p \times GL_q$ with $p \geqslant q$: Here the orbits are parametrized by $\{0,1,\ldots,q\}$. We apply the following identifications to Theorem 2.5: Δ_c is of type $A_{p-1} \times A_{q-1}$, $\Delta_c(\mathfrak{t}_m)$ is of type $A_{p-1-m} \times A_{q-1-m}$, the denominator of the formula is $\prod_{\alpha\in\Delta_c^+\setminus\Delta_{c,m}^+}\langle\alpha,\rho_c\rangle = \prod_{i=1}^m((p-i)!(q-i)!)$, and $s_m = p+q-2m$. In this case,

$$\dim(\overline{\mathcal{O}_m}) = m(p+q) - m^2$$
$$\deg(\overline{\mathcal{O}_m}) = \frac{0!1!\cdots(m-1)! \times (p+q-2m)!\cdots(p+q-m-2)!(p+q-m-1)!}{(p-m)!(p-m+1)!\cdots(p-1)! \times (q-m)!(q-m+1)!\cdots(q-1)!}.$$

This coincides with the Giambelli formula.

(ii) S^2GL_n: Here the orbits are parametrized by $\{0,1,\ldots,n\}$. In this case, Δ_c is of type A_{n-1}, $\Delta_c(\mathfrak{t}_m)$ is of type A_{n-1-m}, $\prod_{\alpha\in\Delta_c^+\setminus\Delta_{c,m}^+}\langle\alpha,\rho_c\rangle = \prod_{i=1}^m(n-i)!$, and $s_m = n-m$. We then obtain

$$\dim(\overline{\mathcal{O}_m}) = mn - (m-1)m/2$$
$$\deg(\overline{\mathcal{O}_m}) = \frac{0!1!\cdots(m-1)!}{0!!1!!\cdots(m-1)!!}\,\frac{(2n-2m)!!(2n-2m+1)!!\cdots(2n-m+1)!!}{(n-m)!(n-m+1)!\cdots(n-1)!},$$

where $l!! = l(l-2)\cdots4\cdot2$ for an even integer l, and $l!! = l(l-2)\cdots3\cdot1$ for odd l. This coincides with the Giambelli formula.

(iii) Λ^2GL_n: In this case, we parameterize the orbits by $\{0,1,2,\ldots,[n/2]\}$, not by $\{0,2,4,\ \ldots,2[n/2]\}$, since our numbering should be compatible with the enumeration of the Wallach set for the unitary highest weight module of the scalar K-type. Here, Δ_c is of type A_{n-1}, $\Delta_c(\mathfrak{t}_m)$ is of type $A_{n-1-2m} \times A_1^m$, $\prod_{\alpha\in\Delta_c^+\setminus\Delta_{c,m}^+}\langle\alpha,\rho_c\rangle = \prod_{i=1}^{2m}(n-i)!$, and $s_m = 2n-4m$. We have

$$\dim(\overline{\mathcal{O}_m}) = 2mn - (2m+1)m$$
$$\deg(\overline{\mathcal{O}_m}) = \frac{1!3!\cdots(2m-1)! \times (2n-4m)!(2n-4m+2)!\cdots(2n-2m-2)!}{(n-2m)!\cdots(n-m+1)!(n-m)!\cdots\cdots(n-1)!}$$

This coincides with the Giambelli formula.

(iv) $O_n \times GL_1$: In this case, the orbits are parametrized by $\{0,1,2\}$. Then, Δ_c is the root system of $\mathfrak{so}(n)$, $\Delta_c(\mathfrak{t}_2) = \Delta_c(\mathfrak{t}_1)$ is the root system of $\mathfrak{so}(n-2)$, and for $m = 1$ the denominator of the formula is $\prod_{\alpha\in\Delta_c^+\setminus\Delta_{c,m}^+}\langle\alpha,\rho_c\rangle = (n-3)!(\frac{1}{2}n-1) =$

$(n-2)!/2$. Here we have

$$\dim(\overline{\mathcal{O}_m}) = 0, n-1, n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 2, 1 \qquad \text{for } m = 0, 1, 2, \text{ respectively.}$$

Since the closure of the orbit $\overline{\mathcal{O}_1}$ is a quadratic hypersurface, the formula above follows from Lemma 2.1.

(xi) $Spin_{10} \times GL_1$: Here the orbits are parametrized by $\{0, 1, 2\}$, and in this case $\Delta_c = \{\alpha_i \mid 2 \leqslant i \leqslant 6\}$ is of type D_5, $\Delta_c(\mathfrak{t}_1) = \{\alpha_i \mid i = 2, 4, 5, 6\}$ is of type A_4, and $\Delta_c(\mathfrak{t}_2) = \{\alpha_i \mid i = 2, 4, 5\}$ is of type A_3. The denominator $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle$ for $m = 1$ is $7!5!/2$, and that for $m = 2$ is $7!5!4!/2$. We have

$$\dim(\overline{\mathcal{O}_m}) = 0, 11, 16,$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 12, 1 \qquad \text{for } m = 0, 1, 2, \text{ respectively.}$$

(xiii) $E_6 \times GL_1$: Here the orbits are parametrized by $\{0, 1, 2, 3\}$ and in this case $\Delta_c = \{\alpha_i \mid 1 \leqslant i \leqslant 6\}$ is of type E_6, $\Delta_c(\mathfrak{t}_1) = \{\alpha_i \mid 1 \leqslant i \leqslant 5\}$ is of type D_5, and $\Delta_c(\mathfrak{t}_2) = \Delta_c(\mathfrak{t}_3) = \{\alpha_i \mid 2 \leqslant i \leqslant 5\}$ is of type D_4. The denominator $\prod_{\alpha \in \Delta_c^+ \setminus \Delta_{c,m}^+} \langle \alpha, \rho_c \rangle$ for $m = 1$ is $11!8!/6$, and that for $m = 2, 3$ is $2 \cdot 11!8!7!/3$.

$$\dim(\overline{\mathcal{O}_m}) = 0, 17, 26, 27,$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 78, 3, 1 \qquad \text{for } m = 0, 1, 2, 3, \text{ respectively.}$$

Since the hermitian symmetric space of type EVII is of the tube type, it is known that the orbit $\overline{\mathcal{O}_2}$ is a hypersurface, and that the defining equation, which is the basic relative invariant of the corresponding prehomogeneous vector space, is cubic. The degree here was known previously, except for the case $m = 1$.

6.2. Non-hermitian case

(v) $Sp_{2n} \times GL_1$: In this case, the orbit structure is the same as that for $GL_{2n} \times GL_1$, which is a special case of case (i).

(ix) $Spin_7 \times GL_1$: In this case, the orbit structure is the same as that for $O(7) \times GL_1$, which is a special case of case (iv).

(xii) $G_2 \times GL_1$: Here, the orbit structure is the same as that for $O(7) \times GL_1$, which is a special case of case (iv).

(vi) $Sp_{2n} \times GL_2$: In this case, the orbits are parametrized by $\{(0,0), (1,0), (2,0), (2,2)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}\}$ of case (i) $GL_{2n} \times GL_2$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{vi})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{vi})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{vi})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{vi})}.$$

We also know that $\overline{\mathcal{O}_{(2,0)}}$ is a quadratic hypersurface. Then the formula for the degrees of the closure of orbits can be reduced to that in known cases.

$$\dim(\overline{\mathcal{O}_m}) = 0, 2n+1, 4n-1, 4n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 2n, 2, 1.$$

(vii) $Sp_{2n} \times GL_3$: In this case, the orbits are parametrized by $\{(0,0),(1,0),(2,0),(2,2),(3,0),(3,2)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}, \mathcal{O}_3^{(\mathrm{i})}\}$ of the case (i) $GL_{2n} \times GL_3$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{vii})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{vii})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{vii})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{vii})}, \mathcal{O}_3^{(\mathrm{i})} = \mathcal{O}_{(3,0)}^{(\mathrm{vii})} \cup \mathcal{O}_{(3,2)}^{(\mathrm{vii})}.$$

It is not difficult to see that the variety $\overline{\mathcal{O}_{(3,0)}}$ is the complete intersection of three quadratic hypersurfaces. Thus the degree of the variety for this case, except for $\overline{\mathcal{O}_{(2,0)}}$, was known previously. We have

$$\dim(\overline{\mathcal{O}_m}) = 0, 2n+2, 4n+1, 4n+2, 6n-3, 6n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, n(2n+1), 4n(n-1), n(2n-1), 8, 1$$

(viii) $Sp_4 \times GL_n$: Here, the orbits are parametrized by $\{(0,0),(1,0),(2,0),(2,2),(3,2),(4,4)\}$. Comparing with the orbits $\{\mathcal{O}_0^{(\mathrm{i})}, \mathcal{O}_1^{(\mathrm{i})}, \mathcal{O}_2^{(\mathrm{i})}, \mathcal{O}_3^{(\mathrm{i})}, \mathcal{O}_4^{(\mathrm{i})}\}$ of the case (i) $GL_4 \times GL_n$, we have

$$\mathcal{O}_0^{(\mathrm{i})} = \mathcal{O}_{(0,0)}^{(\mathrm{viii})}, \mathcal{O}_1^{(\mathrm{i})} = \mathcal{O}_{(1,0)}^{(\mathrm{viii})}, \mathcal{O}_2^{(\mathrm{i})} = \mathcal{O}_{(2,0)}^{(\mathrm{viii})} \cup \mathcal{O}_{(2,2)}^{(\mathrm{viii})}, \mathcal{O}_3^{(\mathrm{i})} = \mathcal{O}_{(3,2)}^{(\mathrm{viii})}, \mathcal{O}_4^{(\mathrm{i})} = \mathcal{O}_{(4,4)}^{(\mathrm{viii})}.$$

In this case the degree of the variety, except for $\overline{\mathcal{O}_{(2,0)}}$, was known previously. Here we have

$$\dim(\overline{\mathcal{O}_m}) = 0, n+3, 2n+3, 2n+4, 3n+1, 4n$$
$$\deg(\overline{\mathcal{O}_m}) = 1, n(n+1)(n+2)/6, (n-1)n(n+1)/3, (n-1)n^2(n+1)/12,$$
$$(n-2)(n-1)n/6, 1, \text{ respectively.}$$

(x) $Spin_9 \times GL_1$: Here, the orbits are parametrized by $\{0,1,2,2'\}$. This representation is equivalent to the isotropy representation on the tangent space of the Riemannian symmetric space $F_4/Spin_9$ of rank one. Thus it has an invariant quadratic form. Comparing the orbits $\{\mathcal{O}_0^{(\mathrm{iv})}, \mathcal{O}_1^{(\mathrm{iv})}, \mathcal{O}_2^{(\mathrm{iv})}\}$ of the case (iv) $O_{16} \times GL_1$, we find

$$\mathcal{O}_0^{(\mathrm{iv})} = \mathcal{O}_0^{(\mathrm{x})}, \mathcal{O}_1^{(\mathrm{iv})} = \mathcal{O}_1^{(\mathrm{x})} \cup \mathcal{O}_2^{(\mathrm{x})}, \mathcal{O}_2^{(\mathrm{iv})} = \mathcal{O}_{2'}^{(\mathrm{x})}.$$

On the other hand, the representation (x) is the restriction of the representation (xi). The correspondence between orbits is

$$\mathcal{O}_0^{(\mathrm{xi})} = \mathcal{O}_0^{(\mathrm{x})}, \mathcal{O}_1^{(\mathrm{xi})} = \mathcal{O}_1^{(\mathrm{x})}, \mathcal{O}_2^{(\mathrm{xi})} = \mathcal{O}_2^{(\mathrm{x})} \cup \mathcal{O}_{2'}^{(\mathrm{x})}.$$

Then the formula for the degrees of the closure of orbits can be reduced to that in cases (iv) and (xi). We obtain

$$\dim(\overline{\mathcal{O}_m}) = 0, 11, 15, 16$$
$$\deg(\overline{\mathcal{O}_m}) = 1, 12, 2, 1, \text{ respectively.}$$

6.3. Examples. — Let us illustrate the calculations necessary to obtain the above results by considering the spaces (vii) and (viii). In these cases, the group K is a direct product, say, $K = {}^lK \times {}^rK$ with ${}^lK = Sp(n, \mathbb{C}) = Sp_{2n}$ and ${}^rK = GL(n', \mathbb{C})$. We consider the case in which $(n, n') = (n, 3)$ or $(2, n')$. We use the superscript l or r to indicate an object corresponding to ${}^lK$ or ${}^rK$. The root system ${}^l\Delta_K^+$ is of type C_n, and ${}^r\Delta_K^+$ is of type $A_{n'-1}$. In the standard realization,

$$\begin{aligned} {}^l\Delta_K^+ &= \{e_i \pm e_j \mid 1 \leqslant i < j \leqslant n\} \cup \{2e_i \mid 1 \leqslant i \leqslant n\}, \\ \text{and} \qquad {}^r\Delta_K^+ &= \{e_i - e_j \mid 1 \leqslant i < j \leqslant n'\}. \end{aligned}$$

We denote the weight $e_1 + \cdots + e_i = (1, \dots, 1, 0, \dots, 0)$ of K_l (resp., K_r) by l_i (resp., r_i).

First, we consider the space $Sp_{2n} \times GL_3$. Here, the set of primitive weights arising in V is $P\hat{A}^+(V) = \{\psi_1, \psi_2, \psi_3, \psi_2', \psi_3', \psi_4'\}$, where $\psi_1 = (l_1; r_1)$, $\psi_2 = (l_2; r_2)$, $\psi_3 = (l_3; r_3)$, $\psi_2' = (0; r_2)$, $\psi_3' = (l_1; r_3)$ and $\psi_4' = (l_2; r_1 + r_3)$. The generators of the subsemigroups corresponding to orbits are known:

$$P\hat{A}^+(\overline{\mathcal{O}_{(0,0)}}) = \varnothing; \qquad P\hat{A}^+(\overline{\mathcal{O}_{(1,0)}}) = \{\psi_1\}; \qquad P\hat{A}^+(\overline{\mathcal{O}_{(2,0)}}) = \{\psi_1, \psi_2\};$$
$$P\hat{A}^+(\overline{\mathcal{O}_{(2,2)}}) = \{\psi_1, \psi_2, \psi_2'\}; \quad P\hat{A}^+(\overline{\mathcal{O}_{(3,0)}}) = \{\psi_1, \psi_2, \psi_3\}.$$

We consider the degree of the closure of the orbit $Y = \overline{\mathcal{O}_{(3,0)}}$. First, we note that the lattice $\hat{A}^+(\overline{\mathcal{O}_{(3,0)}})$ is

$$\begin{aligned} &\{n_1\psi_1 + n_2\psi_2 + n_3\psi_3 \mid n_i \in \mathbb{Z}_{\geqslant 0}\} \\ &\qquad = \{n_1'\psi_1 + n_2'(\psi_2 - \psi_1) + n_3'(\psi_3 - \psi_2) \mid n_1' \geqslant n_2' \geqslant n_3' \geqslant 0\}. \end{aligned}$$

Next, for each α, we set $(\alpha_1', \alpha_2', \alpha_3') = (\langle \alpha, \psi_1 \rangle, \langle \alpha, \psi_2 - \psi_1 \rangle, \langle \alpha, \psi_3 - \psi_2 \rangle)$. Then Theorem 2.5 implies that the degree of Y is

$$\text{(4)} \qquad \frac{d!}{\prod_{\alpha \in {}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K \rangle \prod_{\alpha \in {}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K \rangle} \int_{D_m'} {}^l\bar{f}(x_1, \dots, x_m) \cdot {}^r\bar{f}(x_1, \dots, x_m) dx_1 \cdots dx_m,$$

where $m = 3$, $d = m + |{}^l\Delta_Y^+| + |{}^r\Delta_Y^+|$, ${}^l\bar{f} = \prod_{\alpha \in {}^l\Delta_Y^+} (\alpha_1' x_1 + \cdots + \alpha_m' x_m)$, and ${}^r\bar{f}$ is similar to ${}^l\bar{f}$. The domain of integration is D_3' of (2), since the degree of $\psi_1, \psi_2 - \psi_1$ and $\psi_3 - \psi_2$ is 1. From the explicit form of ψ_i, we know that ${}^l\Delta_K^+ \setminus {}^l\Delta_Y^+$ is the positive root system of type C_{n-3} and ${}^l\Delta_Y^+ = \{e_1 \pm e_j, e_2 \pm e_j \mid 3 \leqslant j \leqslant n\} \cup \{e_1 \pm e_2, 2e_1, 2e_2\}$. We also have ${}^r\Delta_Y^+ = {}^r\Delta_K^+$. Then, with some calculation, we obtain the denominator of the formula as

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 8(2n-1)!(2n-3)!(2n-5)!, \quad \text{and} \quad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = 2.$$

The leading polynomials here are

$${}^l\bar{f}(x_1,x_2,x_3) = (x_1x_2x_3)^{2n-6}(x_1^2-x_2^2)(x_1^2-x_3^2)(x_2^2-x_3^2)(2x_1)(2x_2)(2x_3),$$
$${}^r\bar{f}(x_1,x_2,x_3) = (x_1-x_2)(x_1-x_3)(x_2-x_3).$$

Finally, we recall the evaluation of the integral

$$(5)\quad d!\int_{D'_m}(x_1\cdots x_m)^r\Big(\prod_{1\leqslant i<j\leqslant m}(x_i-x_j)\Big)^2 s(x)dx_1\cdots dx_m$$
$$= 2^{m(m-1)/2}(m-1)!\cdots 1!\cdot r!(r+2)!\cdots(r+2m-2)!,$$

where $d = m(r+1)+3m(m-1)/2$, and $s(x) = \prod_{1\leqslant i<j\leqslant m}(x_i+x_j)$ is the Schur function attached to the staircase partition $(m-1,\dots,1,0)$. This is a special case of the formula in Example VI.10.7(c) of Ref. **[16]**. The right-hand side of (5) equals $2r!(r+2)!$ for $m=2$, and $16r!(r+2)!(r+4)!$ for $m=3$. We use the formula (5) for $m=3$ and $r=2n-5$. With this information, we conclude that the degree of $\overline{\mathcal{O}_{(3,0)}}$ is 8.

We now consider another orbit, $Y=\overline{\mathcal{O}_{(2,0)}}$. The lattice $\hat{A}^+(\overline{\mathcal{O}_{(2,0)}})$ is $\{n_1\psi_1+n_2\psi_2 \mid n_i\in\mathbb{Z}_{\geqslant 0}\} = \{n_1'\psi_1+n_2'(\psi_2-\psi_1)\mid n_1'\geqslant n_2'\geqslant 0\}$. Here, we can again use (4), with $m=2$. The denominators and the leading polynomials in this case are

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 4(2n-1)!(2n-3)!, \qquad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = 2.$$
$${}^l\bar{f} = (x_1x_2)^{2n-4}(x_1-x_2)(x_1+x_2)(2x_1)(2x_2), \qquad {}^r\bar{f} = x_1x_2(x_1-x_2).$$

Then, the integral is of the form (5) with $m=2$ and $r=2n-2$. Hence, we conclude that the degree of $\overline{\mathcal{O}_{(2,0)}}$ is $4n(n-1)$.

Finally, we consider the orbit $Y=\overline{\mathcal{O}_{(2,0)}}$ of the space $Sp_4\times GL_{n'}$. In this case, the primitive weights are known to be $\psi_1=(l_1;r_1)$, $\psi_2=(l_2;r_2)$, $\psi_3=(l_1;r_3)$, $\psi_4=(0;r_4)$, $\psi_2'=(0;r_2)$, and $\psi_4'=(l_2;r_1+r_3)$. We also know that $P\hat{A}^+(V) = \{\psi_1,\psi_2,\psi_3,\psi_4,\psi_2',\psi_4'\}$, $P\hat{A}^+(\mathcal{O}_{(0,0)}) = \varnothing$, $P\hat{A}^+(\mathcal{O}_{(1,0)}) = \{\psi_1\}$, $P\hat{A}^+(\mathcal{O}_{(2,0)}) = \{\psi_1,\psi_2\}$, $P\hat{A}^+(\mathcal{O}_{(2,2)}) = \{\psi_1,\psi_2,\psi_2'\}$, and $P\hat{A}^+(\mathcal{O}_{(3,2)}) = \{\psi_1,\psi_2,\psi_3,\psi_2',\psi_4'\}$. Then, we see that ${}^l\Delta_Y^+ = {}^l\Delta_K^+$ and that ${}^r\Delta_K^+\setminus{}^r\Delta_Y^+$ is a positive system of type $A_{n'-3}$. We then find that the denominators and the leading polynomials in the formula giving the degree are

$$\prod_{\alpha\in{}^l\Delta_Y^+} \langle \alpha, {}^l\rho_K\rangle = 4, \qquad \prod_{\alpha\in{}^r\Delta_Y^+} \langle \alpha, {}^r\rho_K\rangle = (n'-1)!(n'-2)!,$$
$${}^l\bar{f} = (2x_1)(2x_2)(x_1-x_2)(x_1+x_2), \qquad {}^r\bar{f} = (x_1x_2)^{n'-2}(x_1-x_2).$$

Thus we can apply the integral formula (5) with $m=2$ and $r=n'-1$. Hence, we conclude that the degree of $\overline{\mathcal{O}_{(2,0)}}$ is $(n'-1)n'(n'+1)/3$.

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CONCLUDING REMARKS

Here, we shall pose some problems and consequences of our results in this volume. We mainly follow the notation in [**Intro**].

1. Theta correspondence and associated cycles

Let (G, G') be a reductive dual pair in the stable range with G' the smaller member. We do *not* assume that G' is compact. Let $\widetilde{G}$ and $\widetilde{G'}$ be the inverse images in the metaplectic cover. Take an irreducible unitary representation σ of $\widetilde{G'}$ and denote its theta lift by $\theta(\sigma) \in \mathrm{Irr}(\widetilde{G})$. Then $\pi = \theta(\sigma)$ is known to be unitary (see [**3**]) (or possibly zero).

If G' is compact, the results in this volume tell us that

$$\mathcal{AC}_{\sigma} = \dim\sigma \cdot [\{0\}] \quad \text{and} \quad \mathcal{AC}_{\pi} = \dim\sigma \cdot [\overline{\mathcal{O}_\pi}] \quad (\pi = \theta(\sigma)).$$

From this fact, we expect the following. For non-compact G', take an irreducible unitary representation σ of $\widetilde{G'}$ whose associated variety is irreducible, i.e., $\mathcal{AV}_{\sigma} = \overline{\mathcal{O}'_\sigma}$ for some nilpotent $K'_{\mathbb{C}}$-orbit $\mathcal{O}'_\sigma$. Then it is expected that the associated variety of $\pi = \theta(\sigma)$ is also irreducible: $\mathcal{AV}_{\pi} = \overline{\mathcal{O}_\pi}$ (cf. [**7**]). Therefore, the associated cycles of σ and π can be written as

$$\mathcal{AC}_{\sigma} = m_\sigma[\overline{\mathcal{O}'_\sigma}] \quad \text{and} \quad \mathcal{AC}_{\pi} = m_\pi[\overline{\mathcal{O}_\pi}]$$

with multiplicities m_σ and m_π.

Problem A

(1) *When does the equality* $m_\sigma = m_\pi$ *hold?*
(2) *Is there a good description for* $\deg\overline{\mathcal{O}'_\sigma}$ *and* $\deg\overline{\mathcal{O}_\pi}$*?*

2. Whittaker vectors and associated cycles

We make here some remarks on the relationship between $C^{-\infty}$-Whittaker vectors and associated cycles of large representations, which are consequences of the results of Matumoto [**4**] and Schmid-Vilonen [**6**]. To be more precise, let G be a real reductive linear Lie group, and K be a maximal compact subgroup of G. The corresponding Lie algebras are denoted by $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{k}_{\mathbb{R}}$ respectively, and we keep the notation fixed in [**Intro**]. Let $G = KAN$ (resp. $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$) be an Iwasawa decomposition of G (resp. $\mathfrak{g}_{\mathbb{R}}$). The Harish-Chandra module X_π of an irreducible admissible representation π of G is called *large* if X_π has the largest possible Gelfand-Kirillov dimension, i.e., $\operatorname{Dim} X_\pi = \dim N$. We know that X_π is large if and only if the action on X_π of the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n} = \mathfrak{n}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is torsion free (see [**2**, Th.3.4]).

Now we assume that X_π is large. Let X_π^∞ be the Fréchet G-module consisting of all smooth vectors for π. We write $X_\pi^{-\infty}$ for the continuous dual space of X_π^∞. For each principal nilpotent G-orbit $\mathcal{O}^{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$, we take an admissible unitary character $\psi_{\mathcal{O}^{\mathbb{R}}}$ of the maximal nilpotent Lie subalgebra $\mathfrak{n}_{\mathbb{R}}$ such that $\operatorname{Ad}(G)\psi_{\mathcal{O}^{\mathbb{R}}} = \sqrt{-1}\mathcal{O}^{\mathbb{R}}$, where $\psi_{\mathcal{O}^{\mathbb{R}}}$ is looked upon as an element of $\sqrt{-1}\mathfrak{g}_{\mathbb{R}}$ canonically through the Killing form. Let us consider the space

$$\mathrm{Wh}^\infty_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi) := \{T \in X_\pi^{-\infty} \mid T \circ Z = -\psi_{\mathcal{O}^{\mathbb{R}}}(Z)T \quad (Z \in \mathfrak{n}_{\mathbb{R}})\}$$

of all $C^{-\infty}$-Whittaker vectors T for π of type $\psi_{\mathcal{O}^{\mathbb{R}}}$. In [**4**, Th.5.5.1], Matumoto proved that the asymptotic cycle of X_π is equal to

$$\sum_{\mathcal{O}^{\mathbb{R}}} \dim \mathrm{Wh}^\infty_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi) \cdot \mu(\mathcal{O}^{\mathbb{R}}).$$

Here $\mathcal{O}^{\mathbb{R}}$ runs over the principal nilpotent G-orbits in $\mathfrak{g}_{\mathbb{R}}$, and $\mu(\mathcal{O}^{\mathbb{R}})$ is a G-invariant measure on $\mathcal{O}^{\mathbb{R}}$ with a suitable normalization. This together with a recent result of Schmid-Vilonen [**6**, Th.1.4] and also with Theorem 1.4 (2) in [**NOT**] implies the following

Theorem B. — *The associated cycle and the Bernstein degree of large $(\mathfrak{g}, K)$-module X_π are described respectively as*

$$\mathcal{AC}_\pi = \sum_{\mathcal{O}^{\mathbb{R}}} \dim \mathrm{Wh}^\infty_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi) \cdot [\overline{\mathcal{O}}],$$

$$\operatorname{Deg}\pi = \frac{w_G}{l_G} \sum_{\mathcal{O}^{\mathbb{R}}} \dim \mathrm{Wh}^\infty_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi).$$

Here $\mathcal{O}$ denotes the principal nilpotent $K_{\mathbb{C}}$-orbit corresponding to $\mathcal{O}^{\mathbb{R}}$ through the Kostant-Sekiguchi correspondence, l_G is the number of principal nilpotent G-orbits, and we write w_G for the order of the little Weyl group of G.

This theorem says that the dimension of the space $\mathrm{Wh}^\infty_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi)$ of $C^{-\infty}$-Whittaker vectors gives the multiplicity of π at $\mathcal{O}$. We note that if $\pi = \operatorname{Ind}_P^G(\sigma)$ is a principal series induced from an irreducible finite-dimensional representation σ of a minimal parabolic subgroup P of G, one gets

$$\dim \mathrm{Wh}^{\infty}_{\psi_{\mathcal{O}^{\mathbb{R}}}}(X_\pi) = \dim \sigma$$

for every principal nilpotent orbit $\mathcal{O}^{\mathbb{R}}$. In this case, two equalities in Theorem B turn out to be

$$\mathcal{AC}_{\operatorname{Ind}_P^G(\sigma)} = \sum_{\mathcal{O}} \dim \sigma \cdot [\overline{\mathcal{O}}],$$

where $\mathcal{O}$ ranges over all principal nilpotent $K_{\mathbb{C}}$-orbits, and

$$\operatorname{Deg} \operatorname{Ind}_P^G(\sigma) = w_G \cdot \dim \sigma.$$

See also [**NOT**, Section 2.6].

For quasi-split case, Theorem B combined with Shalika's multiplicity one theorem gives the following remarkable conclusion.

Theorem C. — *Suppose that the group G is quasi-split. Let X_π be an irreducible large $(\mathfrak{g}, K)$-module. We write l_π for the number of principal nilpotent $K_{\mathbb{C}}$-orbits contained in the associated variety $\mathcal{AV}_\pi$ of π. Then, the Bernstein degree of X_π equals $w_G l_\pi / l_G$. In particular, one gets* $\operatorname{Deg} \pi = w_G / l_G$ *if the variety $\mathcal{AV}_\pi$ is irreducible.*

We note that, as mentioned in [**Intro**] and in [**Y**, Introduction], Matumoto established some interesting results on the "holonomicity" of the space of generalized Whittaker vectors for non-large irreducible representations π, in connection with the associated variety of the primitive ideal or the wave front set of π.

Last in this section, we should like to say that the present work [**Y**] reveals a stronger relationship, similar to the one given in Theorem B, between generalized Whittaker vectors and associated cycles for unitary highest weight representations, which are rather *small* irreducible representations of G.

3. Summation formula for stable branching coefficients

There is a feedback to the summation formula of stable branching coefficients from our results. We explain it in this section based on an example.

Let $(G, G') = (Sp(2n, \mathbb{R}), O(m))$. Consider a spherical pair $(\mathbf{L}, \mathbf{H}) = (GL_m, O_m = G'_{\mathbb{C}})$ over $\mathbb{C}$. Then we have

$$\Phi^+ = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Phi \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m\}, \quad \Phi = \mathbb{Z}^m$$
$$\Phi^+(H) = \{2\lambda \mid \lambda \in \Phi\}, \quad \Phi(H) = (2\mathbb{Z})^m.$$

For simplicity, we assume that $m = 2k+1$ is odd. Then $O_m \simeq SO_m \times \mathbb{Z}_2$ is a direct product, and $\sigma \in \operatorname{Irr}(O_m)$ is parametrized as

$$\sigma = \sigma^{\varepsilon}_{\mu}$$
$$(\varepsilon \in \{\pm 1\} = \mathbb{Z}_2 \text{ and } \mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k, \mu_1 \geqslant \cdots \geqslant \mu_k \geqslant 0),$$

where μ denotes the highest weight of $\sigma|_{SO_m}$ and ε is a character of $\mathbb{Z}_2$. Put $\pi = \theta(\sigma)$, the theta lift of σ. Then **[Intro**, Theorem E] tells us that the multiplicity m_π in $\mathcal{AC}_\pi$ is given by

$$m_\pi = \sum_{[\lambda]\in\Lambda_n^+/(2\mathbb{Z})^m} m([\lambda],\sigma),$$

since $r([\lambda]) = 1$ in this case (see also Corollary 8.4 in [**NOT**]). If $n \geqslant m$, then $\Lambda_n^+ = \Lambda_m^+$ and $\Lambda_m^+/(2\mathbb{Z})^m \simeq \Phi^+/\Phi(H)$ holds. So, the above formula is equal to $\dim\sigma$ by Sato's summation formula.

For $n < m$, we have the following theorem.

Theorem D. — *Assume that $n < m$. Let $\Lambda_n^+ = \{\lambda \in \Phi^+ \mid \lambda_{n+1} = \cdots = \lambda_m = 0\}$ and $\sigma = \sigma_\mu^\varepsilon \in \mathrm{Irr}(G')$ as above. Then we have*

$$\sum_{[\lambda]\in\Lambda_n^+/(2\mathbb{Z})^n} m([\lambda],\sigma) = \dim\sigma^{O(m-n)},$$

where $\sigma^{O(m-n)}$ is the space of $O(m-n)$-spherical vectors, and the subgroup $O(m-n) \subset O(m)$ is realized as the lower principal diagonal subgroup (cf. [**Y**, Eq. (5.15)]).

Remark. — If $n < m/2$, then $\dim\sigma^{O(m-n)} = \dim\tau_\mu$ holds, where τ_μ is the irreducible finite dimensional representation of GL_n with highest weight μ. Thus we have

$$\sum_{[\lambda]\in\Lambda_n^+/(2\mathbb{Z})^n} m([\lambda],\sigma) = \dim\tau_\mu,$$

which is due to Gelbart ([**1**]; see also Remark (1) after Theorem 2 in [**5**]).

Proof. — The left hand side of the formula is equal to the multiplicity m_π at $\mathcal{O}_\pi$ in the associated cycle $\mathcal{AC}_\pi$ by [**Intro**, Theorem E]. The right hand side is also equal to m_π by Theorem 5.14 in [**Y**]. □

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