Astérisque

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Appendix A. Limit of hyperbolicity for spherical 3-orbifolds

Astérisque, tome 272 (2001), p. 173-178
<http://www.numdam.org/item?id=AST_2001__272__173_0>

APPENDIX A

LIMIT OF HYPERBOLICITY FOR SPHERICAL 3-ORBIFOLDS

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Let $\mathcal{O}$ be a spherical 3-orbifold of cyclic type. We denote the ramification locus by $\Sigma \subset \mathcal{O}$; it is a k-component link $\Sigma := \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$. During this appendix we assume that the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure of finite volume. For $t > 0$ small enough, let $C(t\alpha)$ be the hyperbolic cone manifold with topological type $(|\mathcal{O}|, \Sigma)$ and cone angles $t\alpha = (t2\pi/m_1, \ldots, t2\pi/m_k)$, where m_i is the ramification index along the component Σ_i (see Chapter 2, Proposition 2.2.4). Let t_∞ be the limit of hyperbolicity, i.e. $C(t\alpha)$ is a hyperbolic cone manifold for all $t \in J := [0, t_\infty)$.

The aim of this appendix is to prove that the hyperbolic cone manifolds $C(t\alpha)$ cannot degenerate directly to the spherical orbifold $\mathcal{O}$, i.e. we shall prove:

Main Proposition. — *Let $\mathcal{O}$ be a spherical 3-orbifold of cyclic type. If the complement $\mathcal{O} - \Sigma$ of the branching locus admits a complete hyperbolic structure of finite volume then the limit of hyperbolicity t_∞ is contained in the open interval $(0, 1)$.*

We obtain the following corollary from this proposition:

Corollary. — *The cone manifold $C(t_\infty\alpha)$ is Euclidean.*

Proof of the corollary. — By the main proposition we have $0 < t_\infty < 1$. Proceeding as in the proof of Proposition 2.3.1 of Chapter 2, we see that $C(t_\infty\alpha)$ is a Euclidean cone manifold with the same topological type as $\mathcal{O}$ and with cone angles $(t_\infty\alpha_1, \ldots, t_\infty\alpha_k)$. □

Remark A.0.1. — The main proposition does not follow from Proposition 5.2.1 of Chapter 5. The proof of the "Collapsing Case" requires the use of the simplicial volume and does not give information about the collapse itself. If we had a method to describe the collapse at the angle π in a geometric way we could probably see directly that a hyperbolic cone manifold cannot degenerate to a spherical orbifold.

Example A.0.2. — Let $\mathcal{O}(\alpha,\beta;n)$ be the 3-orbifold whose ramification locus is the 2-bridge knot or link $b(\alpha,\beta)\subset S^3$ and with branching index n. The orbifold $\mathcal{O}(\alpha,\beta;2)$ is spherical, and the 2-fold branched covering of $(S^3, b(\alpha,\beta))$ is the lens space $L(\alpha,\beta)$ which itself is a spherical space form. The complement of the branching locus supports a complete hyperbolic metric of finite volume iff $|\beta|>1$. The orbifold $\mathcal{O}(5,3;3)$ is Euclidean, and the orbifolds $\mathcal{O}(5,3;n)$, $n>3$, are hyperbolic. Note that $\mathcal{O}(\alpha,\beta;n)$, $n\geq 3$, is hyperbolic if $\alpha>5$ and $|\beta|>1$. These orbifolds and their limits of hyperbolicity were studied in **[HLM2]**.

The *strategy of the proof* of the main proposition is the following. We assume that $t_\infty = 1$ and we seek a contradiction. We consider a sequence $t_n\to 1$ in $J=[0,1)$ and the corresponding sequence of holonomy representations (ρ_n). By the construction in Chapter 2, we may assume that the sequence (ρ_n) belongs to an algebraic curve $\mathcal{C}$. This curve $\mathcal{C}$ has a natural compactification $\overline{\mathcal{C}}$ that consists in adding some ideal points. Up to a subsequence, ρ_n converges to a point in the compactification $\rho_\infty\in\overline{\mathcal{C}}$.

In Lemma A.1.1 we show that ρ_∞ is not an ideal point (i.e. ρ_∞ is a representation), by using Culler-Shalen theory about essential surfaces associated to ideal points and Lemma A.0.3. In fact, we prove that ρ_∞ is an irreducible representation into $SU(2)$ (Lemma A.1.2). Then we prove that ρ_∞ is μ-regular (see Definition A.0.4), which implies that, for n large, ρ_n is conjugate to a representation into $SU(2)$. We have obtained a contradiction, because the holonomy representation of a hyperbolic cone manifold of finite volume cannot be contained in $SU(2)$.

Spherical 3-orbifolds. — Let $\mathcal{O}$ be a spherical 3-orbifold. Then $\mathcal{O}=S^3/G$, where $G\subset SO(4)$ is finite. The orbifold $\mathcal{O}$ is very good, its universal covering is S^3, and its fundamental group $\pi_1(\mathcal{O})$ is the group of covering transformations, i.e. $\pi_1(\mathcal{O})=G$ is a finite group. There is a surjection $\pi_1(\mathcal{O})\to\pi_1(|\mathcal{O}|)$ where $|\mathcal{O}|$ is the underlying manifold (see **[DaM]**). The 3-manifold $|\mathcal{O}|$ is hence a rational homology sphere which contains the link Σ. Note that $\Sigma\subset|\mathcal{O}|$ is a prime link.

We denote respectively by $\mu_1,\ldots,\mu_k$ and $m_1,\ldots,m_k$ the meridians and ramification indices of the components $\Sigma_1,\ldots,\Sigma_k$ of Σ. We assume that each meridian μ_i is represented by a simple closed curve in $\partial\mathcal{N}(\Sigma)$ which bounds a properly embedded orbifold disc in $\mathcal{N}(\Sigma)$. Here $\mathcal{N}(\Sigma)$ denotes a tubular neighborhood of $\Sigma\subset\mathcal{O}$.

In what follows, we shall make use of the following lemma:

Lemma A.0.3. — *Let $F\subset\mathcal{O}-\mathcal{N}(\Sigma)$ be a properly embedded orientable surface (so ∂F may be empty). If F is incompressible and non boundary-parallel, then there is a meridian μ_i such that $\partial F\cap\mu_i\neq\varnothing$.*

Proof. — Let F be a properly embedded, orientable, incompressible, non boundary-parallel surface in $\mathcal{O}-\mathcal{N}(\Sigma)$. If ∂F has empty intersection with the meridians of Σ then we obtain a closed surface $\overline{F}\subset|\mathcal{O}|$, and hence the link $\Sigma\subset|\mathcal{O}|$ is sufficiently large (following the definition in [**CS**, § 5.1]). This is impossible because S^3 is a regular branched covering of $(|\mathcal{O}|, \Sigma)$ and, according to [**CS**, Thm. 5.1.1], such a covering contains either an incompressible surface of higher genus or a non-separating sphere (see also [**GL**, Theorem 1]). □

Varieties of representation and characters. — Let Γ be a finitely generated group. The *variety of characters* $X(\Gamma)$ is the quotient in the algebraic category of the action of $SL_2(\mathbb{C})$ by conjugation on the variety of representations $R(\Gamma) = \mathrm{Hom}(\Gamma, SL_2(\mathbb{C}))$. Following [**CS**], $X(\Gamma)$ is an affine complex variety, but it is not necessarily irreducible. For a representation $\rho \in R(\Gamma)$, its projection onto $X(\Gamma)$, denoted by χ_ρ, is called the character of ρ. The character χ_ρ may be interpreted as a map:

$$\begin{aligned}\chi_\rho : \Gamma &\longrightarrow \mathbb{C}\\ \gamma &\longmapsto \mathrm{tr}(\rho(\gamma)).\end{aligned}$$

For any $\gamma \in \Gamma$, the trace function $\tau_\gamma \colon R(\Gamma) \to \mathbb{C}$, $\tau_\gamma(\rho) = \mathrm{tr}(\rho(\gamma))$, is invariant under conjugation. Therefore, it factors through $R(\Gamma) \to X(\Gamma)$ to the rational function

$$\begin{aligned}I_\gamma : X(\Gamma) &\longrightarrow \mathbb{C}\\ \chi_\rho &\longmapsto \chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma)).\end{aligned}$$

We use the notation $X(\mathcal{O} - \Sigma) = X(\pi_1(\mathcal{O} - \Sigma))$.

Definition A.0.4. — Let $\rho : \pi_1(\mathcal{O} - \Sigma) \to SL_2(\mathbb{C})$ be an irreducible representation such that $\rho(\mu_1) \neq \pm \mathrm{Id}, \dots, \rho(\mu_k) \neq \pm \mathrm{Id}$. We say that ρ is *μ-regular* if the following conditions are satisfied:

i) $H_1(\mathcal{O} - \Sigma; \mathrm{Ad}\,\rho) \cong \mathbb{C}^k$, where k is the number of components of Σ.
ii) The function $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(\mathcal{O} - \Sigma) \to \mathbb{C}^k$ is locally biholomorphic at χ_ρ.

The following lemma is proved in [**Po1**, Prop. 5.24] and is going to be used at the end of the proof of the main proposition.

Lemma A.0.5. — *Let $\rho : \pi_1(\mathcal{O} - \Sigma) \to SU(2)$ be an irreducible representation such that $\mathrm{tr}(\rho(\mu_i)) \neq \pm 2$ for all $i = 1, \dots, k$.*

If ρ is μ-regular then there exists an open neighborhood $U \subset R(\mathcal{O} - \Sigma)$ of ρ such that for every representation $\rho' \in U$, ρ' is conjugate to a representation into $SU(2)$ if and only if $\mathrm{tr}(\rho'(\mu_i)) \in \mathbb{R}$ for all $i = 1, \dots, k$. □

A.1. Proof of the main proposition

Beginning of the proof. — Let $t_n \in [0, 1)$ be a sequence that converges to t_∞. We choose a lift $\rho_n \in R(\mathcal{O} - \Sigma)$ of the holonomy representation of the hyperbolic cone manifold $C(t_n\alpha)$. We may assume that the sequence (ρ_n) is contained in a complex curve $\mathcal{C} \subset R(\mathcal{O} - \Sigma) \subset \mathbb{C}^N$ (see the proof of Lemma 2.3.2). Now let $\overline{\mathcal{C}} \subset \mathbb{P}^N$ be the projective closure of $\mathcal{C}$ and let $\widetilde{\mathcal{C}}$ be the non-singular projective curve whose function field F is isomorphic to that of $\mathcal{C}$ (see **[CS]** for details). Following **[CS]**, we call the points of $\widetilde{\mathcal{C}}$ which correspond to points of $\overline{\mathcal{C}} - \mathcal{C}$ *ideal points*. We might assume (by passing to a subsequence) that (ρ_n) is contained in the non-singular part of $\mathcal{C}$ and hence we have that $(\rho_n) \subset \widetilde{\mathcal{C}}$. The sequence (ρ_n) converges since $\widetilde{\mathcal{C}}$ is compact.

Each point $\widetilde{x} \in \widetilde{\mathcal{C}}$ gives us a discrete valuation $\nu_{\widetilde{x}} \colon F^* \to \mathbb{Z}$ with valuation ring A. The ring A consists exactly of those functions which do not have a pole at $\widetilde{x}$.

The curve $\mathcal{C} \subset R(\mathcal{O} - \Sigma)$ gives us a *tautological* representation $P \colon \pi_1(\mathcal{O} - \Sigma) \to SL_2(F)$ (see **[CS]**). For a fixed point $\widetilde{x} \in \widetilde{\mathcal{C}}$ we obtain therefore a representation $P \colon \pi_1(\mathcal{O} - \Sigma) \to SL_2(F)$ where F is a field with a discrete valuation. The group $\pi_1(\mathcal{O} - \Sigma)$ acts hence on the associated Bass–Serre–Tits tree which will be denoted by T. An element $\gamma \in \pi_1(\mathcal{O} - \Sigma)$ fixes a point of T if and only if $\widetilde{\tau}_\gamma$ does not have a pole at $\widetilde{x}$ where $\widetilde{\tau}_\gamma \colon \widetilde{\mathcal{C}} \to \mathbb{P}^1$ denotes the rational function determined by τ_γ.

Lemma A.1.1. — *The sequence (ρ_n) does not converge to an ideal point if $t_\infty = 1$.*

Proof. — Assume that $t_\infty = 1$ and that (ρ_n) converges to an ideal point $\widetilde{x} \in \mathcal{C}$.

Let $\mu_1, \dots, \mu_k$ be the meridians of Σ. Since $\mathrm{tr}(\rho_n(\mu_i)) = \pm 2\cos(t_n\pi/m_i)$ converges to $\pm 2\cos(\pi/m_i)$, it follows that $\widetilde{\tau}_{\mu_i}$ does not have a pole at $\widetilde{x}$. The image $P(\mu_i)$ is therefore contained in a vertex stabilizer of T. We obtain hence an incompressible non boundary-parallel surface $F \subset \mathcal{O} - \mathcal{N}(\Sigma)$ such that $F \cap \mu_i = \varnothing$ for $i = 1, \dots, k$ (see **[CS**, Prop. 2.3.1]). This surface cannot exist by Lemma A.0.3. ☐

Lemma A.1.2. — *If $t_\infty = 1$ then the sequence (ρ_n) converges to a representation $\rho_\infty \in R(\mathcal{O} - \Sigma)$ which has the following properties:*

i) ρ_∞ factors through a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$;

ii) ρ_∞ is conjugate to a representation into $SU(2)$;

iii) ρ_∞ is irreducible.

Proof. — The sequence (ρ_n) converges to a representation $\rho_\infty \in R(\mathcal{O} - \Sigma)$ by Lemma A.1.1 and we have:

$$\mathrm{tr}(\rho_\infty(\mu_i)) = \pm 2\cos(\pi/m_i), \quad \text{for } i = 1, \dots, k.$$

In particular $\rho_\infty(\mu_i^{m_i}) = \pm\,\mathrm{Id}$ and therefore ρ_∞ factors trough $\pi_1(\mathcal{O} - \Sigma) \to \pi_1(\mathcal{O})$ to a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$. Note that $\pi_1(\mathcal{O})$ is the quotient of $\pi_1(\mathcal{O} - \Sigma)$ by the normal subgroup generated by $\{\mu_1^{m_1}, \dots, \mu_k^{m_k}\}$. This proves i).

Assertion ii) follows from i): $\pi_1(\mathcal{O})$ is finite and up to conjugation $SU(2)$ is the only maximal compact subgroup of $SL_2(\mathbb{C})$.

Assume that ρ_∞ is reducible. It follows from ii) that ρ_∞ is abelian because every reducible representation into $SU(2)$ is conjugate to a diagonal representation. The representations ρ_n are all irreducible (see **[Po1**, Prop. 5.4]). The abelian representation ρ_∞ is therefore the limit of irreducible representations which implies the existence of a reducible metabelian (but not abelian) representation $\rho'_\infty \in R(\mathcal{O} - \Sigma)$ such that

$\operatorname{tr}(\rho_\infty(g)) = \operatorname{tr}(\rho'_\infty(g))$ for all $g \in G$ (see **[HLM2]**). Since $\rho'_\infty(\mu_i) = \pm 2\cos(\pi/m_i)$ it follows that the image of ρ'_∞ is finite. We obtain that ρ'_∞ is conjugate to a representation into $SU(2)$. This contradicts the fact that ρ'_∞ is metabelian and non-abelian. Hence the lemma is proved. □

With the help of the next lemma we are able to prove the main proposition.

Lemma A.1.3. — *If $t_\infty = 1$ then the limit representation ρ_∞ is μ-regular.*

End of the proof of the main proposition. — Assume that $t_\infty = 1$. The sequence (ρ_n) converges to an irreducible representation $\rho_\infty\colon \pi_1(\mathcal{O} - \Sigma) \to SU(2)$ such that $\operatorname{tr}(\rho_\infty(\mu_i)) \neq \pm 2$. The representation ρ_∞ is μ-regular by Lemma A.1.3 and hence we can apply Lemma A.0.5.

The image of ρ_n is contained in $SU(2)$ up to conjugation if n is sufficiently large by Lemma A.0.5; note that $\operatorname{tr}(\rho_n(\mu_i)) \in \mathbb{R}$. This contradicts the fact that ρ_n is the holonomy of a compact hyperbolic cone manifold (see [**Po1**, Prop. 5.4]). □

It remains to prove Lemma A.1.3. Before we start with the proof, we briefly recall how to define the homology of an orbifold $\mathcal{O}$ with twisted coefficients $\operatorname{Ad}\rho$. Let ρ be a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$ and let K be a CW-complex whose underlying space is the orbifold $\mathcal{O}$ such that the ramification locus Σ is contained in the 1-skeleton. The CW-complex K lifts to a $\pi_1(\mathcal{O})$-equivariant CW-complex $\widetilde{K}$ over the universal covering of $\mathcal{O}$. Set:

$$C_*(K; \operatorname{Ad}\rho) = sl_2(\mathbb{C}) \otimes_{\pi_1(\mathcal{O})} C_*(\widetilde{K}; \mathbb{Z}),$$

where $\gamma \in \pi_1(\mathcal{O})$ acts on the right on the Lie algebra $sl_2(\mathbb{C})$ via the adjoint of $\rho(\gamma^{-1})$. Note that $C_*(\widetilde{K}; \mathbb{Z})$ is not a free $\pi_1(\mathcal{O})$–module (see [**Po1**, Section 1.2] for the details). There is a natural boundary map $\partial_i : C_i(K; \operatorname{Ad}\rho) \to C_{i-1}(K; \operatorname{Ad}\rho)$ (induced by the boundary operator on $C_*(\widetilde{K}; \mathbb{Z})$) and the homology $H_*(\mathcal{O}; \operatorname{Ad}\rho)$ is defined. This homology does not depend on the CW-complex and on the conjugacy class of ρ. When $\Sigma = \varnothing$ (i.e. $\mathcal{O}$ is a manifold), this is the usual homology with twisted coefficients.

Proof of Lemma A.1.3. — In order to compute $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$ we consider the homology of the orbifold. Note that, since ρ_∞ induces a representation of $\pi_1(\mathcal{O})$ into $PSL_2(\mathbb{C})$, the adjoint representation of $\pi_1(\mathcal{O})$ into the endomorphism group of the Lie algebra $sl_2(\mathbb{C})$ is well defined.

Step 1. — $H_*(\mathcal{O}, \operatorname{Ad}\rho_\infty) \cong 0$.

The universal covering of $\mathcal{O}$ is S^3. The projection $\pi\colon S^3 \to \mathcal{O}$ induces a map

$$\pi_* : H_*(S^3, sl_2(\mathbb{C})) \to H_*(\mathcal{O}, \operatorname{Ad}\rho_\infty)$$

where $H_*(S^3, sl_2(\mathbb{C})) \cong H_*(S^3, \mathbb{C}) \otimes_{\mathbb{C}} sl_2(\mathbb{C})$ is the homology of S^3 with non-twisted coefficients $sl_2(\mathbb{C}) \cong \mathbb{C}^3$. Since we work over $\mathbb{C}$, we can construct a right inverse to

π_* by using the transfer map (see [**Bre**, Chapter III])

$$s_*\colon\ H_*(\mathcal{O}, \operatorname{Ad}\rho_\infty) \to H_*(S^3, sl_2(\mathbb{C})),$$

i.e. $\pi_* \circ s_* = \operatorname{Id}$. In particular s_* is injective and its image is invariant by the action of $\pi_1(\mathcal{O})$. The homology $H_*(S^3, sl_2(\mathbb{C}))$ is only non-trivial in dimensions 0 and 3. Since ρ_∞ is irreducible, the subspace of $sl_2(\mathbb{C})$ invariant by $\pi_1(\mathcal{O})$ is trivial, hence $H_*(\mathcal{O}, \operatorname{Ad}\rho_\infty) \cong 0$.

Step 2. — $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty) \cong \mathbb{C}^k$.

We apply a Mayer-Vietoris argument (adapted to the orbifold situation) to the pair $(\mathcal{N}(\Sigma), \mathcal{O} - \mathcal{N}(\Sigma))$, where $\mathcal{N}(\Sigma)$ is a tubular neighborhood of Σ. Since $H_*(\mathcal{O}, \operatorname{Ad}\rho) \cong 0$, we have a natural isomorphism induced by the inclusion maps:

$$H_1(\mathcal{N}(\Sigma); \operatorname{Ad}\rho_\infty) \oplus H_1(\mathcal{O} - \mathcal{N}(\Sigma); \operatorname{Ad}\rho_\infty) \cong H_1(\partial\mathcal{N}(\Sigma); \operatorname{Ad}\rho_\infty) \tag{A.1}$$

The homology groups $H_1(\mathcal{N}(\Sigma), \operatorname{Ad}\rho_\infty)$ and $H_1(\partial\mathcal{N}(\Sigma), \operatorname{Ad}\rho_\infty)$ are easily computed, and they have dimension k and $2k$ over $\mathbb{C}$ respectively (see [**Po1**, Lemma 2.8 and Prop. 3.18] for instance). Therefore $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty) \cong \mathbb{C}^k$.

Step 3. — χ_{ρ_∞} is a smooth point of $X(\mathcal{O} - \Sigma)$ with local dimension k.

By an estimate of Thurston [**Thu1**, Thm. 5.6], see also [**CS**, Thm. 3.2.1], the dimension of $X(\mathcal{O} - \Sigma)$ at χ_{ρ_∞} is $\geq k$. In addition, since $H^1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$ contains the Zariski tangent space $T_{\chi_\rho}X(\mathcal{O} - \Sigma)$, and $H^1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$ is dual to the space $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$, $\dim(T_{\chi_\rho}X(\mathcal{O} - \Sigma)) \leq k$. Thus $\dim(T_{\chi_\rho}X(\mathcal{O} - \Sigma)) = k$ and χ_{ρ_∞} is a smooth point.

Step 4. — $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k}) : X(\mathcal{O} - \Sigma) \to \mathbb{C}^k$ is locally biholomorphic at χ_{ρ_∞}.

Viewing $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$ as the Zariski cotangent space $T_{\chi_\rho}X(\mathcal{O} - \Sigma)$, the proof consists in finding a basis for $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty)$ that can be interpreted as the set of differential forms $\{dI_{\mu_1}, \dots, dI_{\mu_k}\}$.

Let $\Sigma_1 \sqcup \cdots \sqcup \Sigma_k = \Sigma$ be the decomposition of Σ in connected components. Choose $\lambda_1, \dots, \lambda_k \in \pi_1(\mathcal{O} - \Sigma)$ such that λ_i, μ_i generate $\pi_1(\partial(\mathcal{N}(\Sigma_i))) \cong \mathbb{Z} \oplus \mathbb{Z}$, for $i = 1, \dots, k$. Since $\operatorname{tr}(\rho(\mu_i)) \neq \pm 2$, we may assume that $\operatorname{tr}(\rho(\lambda_i)) \neq \pm 2$, up to replacing λ_i by $\lambda_i\mu_i$ if necessary. If we identify homology groups with cotangent spaces, then the differential form dI_{λ_i} generates $H_1(\mathcal{N}(\Sigma_i); \operatorname{Ad}\rho_\infty) \cong \mathbb{C}$ and $\{dI_{\lambda_i}, dI_{\mu_i}\}$ is a basis for $H_1(\partial\mathcal{N}(\Sigma_i); \operatorname{Ad}\rho_\infty) \cong \mathbb{C}^2$ (see for instance [**Po1**, Lemma 3.20] or [**Ho2**] for these computations). It follows from the Mayer-Vietoris isomorphism (A.1) that $\{dI_{\mu_1}, \dots, dI_{\mu_k}\}$ is a basis for $H_1(\mathcal{O} - \Sigma; \operatorname{Ad}\rho_\infty) \cong T_{\chi_\rho}X(\mathcal{O} - \Sigma)$. Therefore $I_\mu = (I_{\mu_1}, \dots, I_{\mu_k})$ is locally biholomorphic at χ_{ρ_∞} and ρ_∞ is μ-regular. $\square$