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A RIGID ANALYTIC APPROXIMATION THEOREM

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1. Introduction

The main result of this paper is Theorem 5.1, which gives a global Artin Approximation Theorem between a "Henselization" $H_{m,n}$ of a ring T_{m+n} of strictly convergent power series and its "completion" $S_{m,n}$. These rings will be defined precisely in Section 2

A normed ring (A, v) is a ring A together with a function $v : A \to \mathbb{R}_+$ such that $v(a) = 0$ if, and only if, $a = 0$; $v(1) = 1$; $v(ab) \le v(a)v(b)$ and $v(a+b) \le v(a) + v(b)$. For example, when K is a complete, non-Archimedean valued field, the ring

$$K\langle \xi_1, \ldots, \xi_m \rangle := \left\{ \sum a_\mu \xi^\mu : |a_\mu| \to 0 \text{ as } |\mu| = \mu_1 + \ldots \mu_m \to \infty \right\}$$

of strictly convergent power series endowed with the Gauss norm

$$\left\| \sum a_\mu \xi^\mu \right\| := \max_\mu |a_\mu|$$

(see **[6]** or Section 2, below) is a complete normed ring. Another example may be obtained by endowing a Noetherian integral domain A with the I-adic norm induced by a proper ideal I of A.

An extension $A \subset \widehat{A}$ of normed rings is said to have the Approximation Property iff the following condition is satisfied:

Let $f_1, \ldots, f_r \in A[X_1, \ldots, X_s]$ be polynomials. For any $\hat{x}_1, \ldots, \hat{x}_s \in \widehat{A}$ such that $f(\hat{x}) = 0$ and for any $\varepsilon > 0$, there exist $x_1, \ldots, x_s \in A$ such that $f(x) = 0$ and $\max_{1 \le i \le s} v(\hat{x}_i - x_i) < \varepsilon$.

Let $\mathbb{C}[\![\xi]\!]$ be the ring of formal power series and $\mathbb{C}\{\xi\}$ the ring of convergent power series in several variables ξ, with complex coefficients. The prototype of the result proved in this paper is the theorem of Artin **[1]** that the extension $\mathbb{C}\{\xi\} \subset \mathbb{C}[[\xi]\!]$ has the Approximation Property with respect to the (ξ)-adic norm, which answered a conjecture of Lang **[9]**.

In [4], Bosch showed that the extension $K\langle\langle\xi\rangle\rangle \subset K\langle\xi\rangle$ has the Approximation Property with respect to the Gauss norm, where $K\langle\langle\xi\rangle\rangle$ denotes the ring of overconvergent power series

$$K\langle\langle\xi\rangle\rangle := \left\{\sum a_\mu \xi^\mu \in K[\![\xi_1,\ldots,\xi_m]\!] : \text{for some } \varepsilon > 1,\ \lim_{|\mu|\to\infty} |a_\mu|\,\varepsilon^{|\mu|} = 0\right\},$$

and $K\langle\xi\rangle$ is the ring of strictly convergent power series defined above. (In fact, Bosch's result is much stronger.) From this result, he recovered the result of [5] that $K\langle\langle\xi\rangle\rangle$ is algebraically closed in $K\langle\xi\rangle$, which generalized [15].

In this paper we prove another approximation property possessed by the rings of strictly convergent power series. Namely, the extension $H_{m,n} \subset S_{m,n}$ (for definitions, see Section 2, below) has the Approximation Property with respect to the (ρ)-adic norm (Theorem 5.1, below). From Theorem 5.1 it follows that $H_{m,n}$, defined as a "Henselization" of the ring $T_{m+n} = K\langle\xi_1,\ldots,\xi_m;\rho_1,\ldots,\rho_n\rangle$, is in fact the algebraic closure of T_{m+n} in the ring $S_{m,n} = K\langle\xi\rangle[\![\rho]\!]_s$ of separated power series (see [11, Definition 2.1.1]). Moreover, from Theorem 5.1 and the fact that the $S_{m,n}$ are UFDs, it follows that the $H_{m,n}$ are also UFDs.

The following is a summary of the contents of this paper.

In Section 2, we define the rings $H_{m,n}$ of Henselian power series. We also summarize (from [11]) the definition and some of the properties of the rings $S_{m,n}$ of separated power series.

In Section 3, we use a flatness property of the inclusion of a Tate ring T_{m+n} into a ring $S_{m,n}$, together with work of Raynaud [13], to deduce a Nullstellensatz for $H_{m,n}$.

In Section 4, we show that $H_{m,n}$ is excellent and that the inclusion $H_{m,n} \to S_{m,n}$ is a regular map of Noetherian rings. We define auxiliary rings $H_{m,n}(B,\varepsilon)$ and $S_{m,n}(B,\varepsilon)$ that in their (ρ)-adic topologies are, respectively, Henselian and complete. The inclusion $H_{m,n}(B,\varepsilon) \to S_{m,n}(B,\varepsilon)$ is a regular map of Noetherian rings. These auxiliary rings play a key role in the proof of the Approximation Theorem.

Section 5 contains the proof that the pair $H_{m,n} \subset S_{m,n}$ has the (ρ)-adic Approximation Property. The proof uses Artin smoothing (see [14]) and the fact that the rings $H_{m,n}(B,\varepsilon) \subset S_{m,n}(B,\varepsilon)$ have the (ρ)-adic Approximation Property.

I am happy to thank Leonard Lipshitz, who posed the question of an Approximation Property of the sort proved in this paper, and Mark Spivakovsky for helpful discussions.

2. The Rings of Henselian Power Series

Throughout this paper, K denotes a field of any characteristic, complete with respect to the non-trivial ultrametric absolute value $|\cdot| : K \to \mathbb{R}_+$. By K°, we denote the valuation ring of K, by $K^{\circ\circ}$ its maximal ideal, and by $\widetilde{K}$ the residue field. For integers $m, n \in \mathbb{N}$, we fix variables $\xi = (\xi_1, \ldots, \xi_m)$ and $\rho = (\rho_1, \ldots, \rho_n)$, thought (usually) to range, respectively, over K° and $K^{\circ\circ}$.

Let E be an ultrametric normed ring, let $E[\![\xi]\!]$ denote the formal power series ring in m variables over E, and by $E\langle\xi\rangle$ denote the subring

$$E\langle\xi\rangle := \left\{ f = \sum_{\mu\in\mathbb{N}^m} a_\mu \xi^\mu \in E[\![\xi]\!] : \lim_{|\mu|\to\infty} a_\mu = 0 \right\}.$$

The ring $K\langle\xi\rangle$ is called the ring of *strictly convergent power series* over K, which we often denote by T_m. The rings T_m are Noetherian ([**6**, Theorem 5.2.6.1]) and excellent ([**3**, Satz 3.3.3] and [**8**, Satz 3.3]). Moreover, they possess the following Nullstellensatz ([**6**, Proposition 7.1.1.3] and [**6**, Theorem 7.1.2.3]): For every $\mathfrak{M} \in \operatorname{Max} T_m$, the field $T_m/\mathfrak{M}$ is a finite algebraic extension of the field K. Let $|\cdot|$ denote the unique extension of the absolute value on the complete field K to one on a finite algebraic extension of K, and by $\bar{\cdot}$ denote the canonical map of a ring into a quotient ring. Then the maximal ideals of T_m are in bijective correspondence with those maximal ideals $\mathfrak{m}$ of the polynomial ring $K[\xi]$ that satisfy $|\bar{\xi}_i| \le 1$ in $K[\xi]/\mathfrak{m}$, $1 \le i \le m$, via $\mathfrak{m} \mapsto \mathfrak{m} \cdot T_m$. Moreover, any prime ideal $\mathfrak{p} \in \operatorname{Spec} T_m$ is an intersection of maximal ideals of T_m.

There is a natural K-algebra norm on T_m, called the *Gauss norm*, given by

$$\Big\| \sum_{\mu\in\mathbb{N}^m} a_\mu \xi^\mu \Big\| := \max_{\mu\in\mathbb{N}^m} |a_\mu|.$$

Put

$$\begin{aligned} T_m^\circ &:= \{f \in T_m : \|f\| \le 1\}, \\ T_m^{\circ\circ} &:= \{f \in T_m : \|f\| < 1\}, \\ \widetilde{T}_m &:= T_m^\circ / T_m^{\circ\circ} = \widetilde{K}[\xi]. \end{aligned}$$

The rings T_m are the rings of power series over K which converge on the "closed" unit polydisc $(K^\circ)^m$.

The rings $S_{m,n}$ of *separated power series* (see [**10**], [**11**] and [**2**]) are rings of power series which represent certain bounded analytic functions on the polydisc $(K^\circ)^m \times (K^{\circ\circ})^n$. When the ground field is a perfect field K of mixed characteristic, there is a complete, discretely valued subring $E \subset K^\circ$ whose residue field $\widetilde{E} = \widetilde{K}$. Then an example of a ring of separated power series is given by

$$S_{m,n} := K \widehat{\otimes}_E E\langle\xi\rangle[\![\rho]\!],$$

where $\widehat{\otimes}_E$ is the complete tensor product of normed E-modules (see [**6**, Section 2.1.7]). Clearly $T_{m+n} \subset S_{m,n}$. In this paper $S_{m,n}$ plays the role of a kind of completion of T_{m+n}.

In general the rings of separated power series are defined by

$$\begin{aligned} S_{m,n} &:= K \otimes_{K^\circ} S^\circ_{m,n} \subset K[\![\xi,\rho]\!]\,, \\ S^\circ_{m,n} &:= \varinjlim_{B\in\mathfrak{B}} B\langle\xi\rangle[\![\rho]\!]\,, \end{aligned}$$

where $\mathfrak{B}$ is a certain directed system (under inclusion) of complete, quasi-Noetherian rings $B \subset K^\circ$. (For the definition and basic properties of *quasi-Noetherian* rings, see [**6**, Section 1.8].) The elements $B \in \mathfrak{B}$ are obtained as follows. Let E be a complete, quasi-Noetherian subring of K°, which we assume to be fixed throughout. When $\operatorname{Char} K \neq 0$, we take E to be a complete DVR. (If, for example, K is a perfect field of mixed characteristic, we may take E to be the ring of Witt vectors over $\widetilde{K}$.) Then a subring $B \subset K^\circ$ belongs to $\mathfrak{B}$ iff there is a zero sequence $\{a_i\}_{i\in\mathbb{N}} \subset K^\circ$ such that B is the completion in $|\cdot|$ of the local ring

$$E[a_i : i \in \mathbb{N}]_{\{b\in E[a_i:i\in\mathbb{N}]:|b|=1\}}\,.$$

It follows from the results of [**6**, Section 1.8], that each $B \in \mathfrak{B}$ is quasi-Noetherian; in particular, the value semigroup $|B \setminus \{0\}| \subset \mathbb{R}_+ \setminus \{0\}$ is discrete. It is easy to see that $\mathfrak{B}$ forms a direct system under inclusion and that $\varinjlim_{B\in\mathfrak{B}} B = K^\circ$. Furthermore, for a fixed $\varepsilon \in K^\circ \setminus \{0\}$ and for any $B \in \mathfrak{B}$, there is some $B' \in \mathfrak{B}$ such that $K^\circ \cap \varepsilon^{-1}\cdot B \subset B'$; indeed, this is an immediate consequence of the fact that the ideal $\{b \in B : |b| \leq |\varepsilon|\} \subset B$ is quasi-finitely generated. It follows that $T_{m+n} \subset S_{m,n}$, and $S_{m,0} = T_m$.

By $\widetilde{B}$ denote the residue field of the local ring B. If $\widetilde{E} = \widetilde{K}$, then $\widetilde{B} = \widetilde{K}$ for all $B \in \mathfrak{B}$. In any case, $\{\widetilde{B}\}_{B\in\mathfrak{B}}$ forms a direct system under inclusion and $\varinjlim_{B\in\mathfrak{B}} \widetilde{B} = \widetilde{K}$. We will need certain residue modules obtained from an element $B \in \mathfrak{B}$. Since the value semigroup of B is discrete, there is a sequence $\{b_p\}_{p\in\mathbb{N}} \subset B \setminus \{0\}$ with $|B \setminus \{0\}| = \{|b_p|\}_{p\in\mathbb{N}}$ and $1 = |b_0| > |b_1| > \cdots$. The sequence of ideals

$$B_p := \{a \in B : |a| \leq |b_p|\},\ p \in \mathbb{N},$$

is called the *natural filtration* of B. For $p \in \mathbb{N}$, put $\widetilde{B}_p := B_p/B_{p+1}$; then $\widetilde{B} = \widetilde{B}_0 \subset \widetilde{K}$. By $\sim : K^\circ \to \widetilde{K}$ denote the canonical residue epimorphism. Then for $p \in \mathbb{N}$, we may identify the $\widetilde{B}$-vector space $\widetilde{B}_p$ with the $\widetilde{B}$-vector subspace $(b_p^{-1}B_p)^\sim$ of $\widetilde{K}$ via the map $(a + B_{p+1}) \mapsto (b_p^{-1}a)^\sim$. This yields a residue map

$$\pi_p : B_p \longrightarrow \widetilde{B}_p \subset \widetilde{K} : a \mapsto (b_p^{-1}a)^\sim\,.$$

When $p > 0$, the above identification of $\widetilde{B}_p$ with a $\widetilde{B}$-vector subspace of $\widetilde{K}$ is useful, though not canonical.

There is a natural K-algebra norm on $S_{m,n}$, also called the *Gauss norm*, given by

$$\Big\| \sum_{\substack{\mu\in\mathbb{N}^m \\ \nu\in\mathbb{N}^n}} a_{\mu\nu}\xi^\mu\rho^\nu \Big\| := \max_{\mu,\nu} |a_{\mu,\nu}|\,.$$

We have $S^{\circ}_{m,n} = \{f \in S_{m,n} : \|f\| \le 1\}$, and, unless K is discretely valued, this ring is not Noetherian. Put

$$\begin{aligned} S^{\circ\circ}_{m,n} &:= \{f \in S_{m,n} : |f| < 1\}, \text{ and} \\ \widetilde{S}_{m,n} &:= S^{\circ}_{m,n}/S^{\circ\circ}_{m,n} = \varinjlim_{B\in\mathfrak{B}} \widetilde{B}[\xi][\![\rho]\!] . \end{aligned}$$

Note that if $\widetilde{E} = \widetilde{K}$ then $\widetilde{S}_{m,n} = \widetilde{K}[\xi][\![\rho]\!]$. In any case, by [**11**, Lemma 2.2.1], $\widetilde{S}_{m,n}$ is Noetherian, $(\rho)\cdot\widetilde{S}_{m,n} \subset \operatorname{rad}\widetilde{S}_{m,n}$ and $\widetilde{K}[\xi][\![\rho]\!]$, the (ρ)-adic completion of $\widetilde{S}_{m,n}$, is faithfully flat over $\widetilde{S}_{m,n}$. It follows by descent that $\widetilde{S}_{m,n}$ is a flat $\widetilde{T}_{m,n}$-algebra.

We recall here some basic facts about the rings $S_{m,n}$. The rings $S_{m,n}$ are Noetherian ([**11**, Corollary 2.2.4]). Moreover, let $M \subset (S_{m,n})^r$ be an $S_{m,n}$-submodule, and put

$$M^{\circ} := (S^{\circ}_{m,n})^r \cap M, \quad M^{\circ\circ} := (S^{\circ\circ}_{m,n})^r \cap M, \quad \widetilde{M} := M^{\circ}/M^{\circ\circ} \subset (\widetilde{S}_{m,n})^r .$$

Lift a set $\widetilde{g}_1, \ldots, \widetilde{g}_s$ of generators of $\widetilde{M}$ to elements $g_1, \ldots, g_s$ of M°. Then for every $f \in M$, there are $h_1, \ldots, h_s \in S_{m,n}$ such that

$$f = \sum_{i=1}^{s} h_i g_i \quad \text{and} \quad \max_{1\le i\le s} \|h_i\| = \|f\| ;$$

in particular, $g_1, \ldots, g_s$ generate the $S^{\circ}_{m,n}$-module M° ([**11**, Lemma 3.1.4]). Note that the above holds also in $T_m = S_{m,0}$.

The rings $S_{m,n}$ satisfy the following Nullstellensatz ([**11**, Theorem 4.1.1]): For every $\mathfrak{M} \in \operatorname{Max} S_{m,n}$, the field $S_{m,n}/\mathfrak{M}$ is a finite algebraic extension of K. The maximal ideals of $S_{m,n}$ are in bijective correspondence with those maximal ideals $\mathfrak{m}$ of $K[\xi,\rho]$ that satisfy $|\overline{\xi}_i| \le 1$, $|\overline{\rho}_j| < 1$ in $K[\xi,\rho]/\mathfrak{m}$, $1 \le i \le m$, $1 \le j \le n$, via $\mathfrak{m} \mapsto \mathfrak{m}\cdot S_{m,n}$. Moreover, any prime ideal of $S_{m,n}$ is an intersection of maximal ideals. It follows that $T_{m+n} \cap \mathfrak{M} \in \operatorname{Max} T_{m+n}$ for any $\mathfrak{M} \in \operatorname{Max} S_{m,n}$. Finally, for any $\mathfrak{M} \in \operatorname{Max} S_{m,n}$, the natural inclusion $T_{m+n} \to S_{m,n}$ induces an isomorphism

$$(T_{m+n})\widehat{_{\mathfrak{m}}} \xrightarrow{\sim} (S_{m,n})\widehat{_{\mathfrak{M}}},$$

where $\mathfrak{m} := T_{m+n} \cap \mathfrak{M}$ and $\widehat{\ }$ denotes completion of a local ring in its maximal-adic topology ([**11**, Proposition 4.2.1]). Since $S_{m,n}$ is Noetherian, it follows from [**12**, Theorem 8.8] by faithfully flat descent that $S_{m,n}$ is a flat T_{m+n}-algebra.

Definition 2.1. — The ring $A_{m,n}$ $(n \ge 1)$ is given by

$$A_{m,n} := K \otimes_{K^{\circ}} A^{\circ}_{m,n} \subset S_{m,n}, \quad A^{\circ}_{m,n} := \left(T^{\circ}_{m+n}\right)_{1+(\rho)} \subset S^{\circ}_{m,n}.$$

We have $A^{\circ}_{m,n} = \{f \in A_{m,n} : \|f\| \le 1\}$. Put

$$A^{\circ\circ}_{m,n} := \{f \in A_{m,n} : \|f\| < 1\}, \quad \widetilde{A}_{m,n} := A^{\circ}_{m,n}/A^{\circ\circ}_{m,n} = \left(\widetilde{T}_{m+n}\right)_{1+(\rho)} .$$

Note that $(\rho)\cdot A^{\circ}_{m,n} \subset \operatorname{rad} A^{\circ}_{m,n}$. By [**13**, Chapitre XI], there is a Henselization $(H^{\circ}_{m,n}, (\rho))$ of the pair $(A^{\circ}_{m,n}, (\rho))$, but unless K is discretely valued, $H^{\circ}_{m,n}$ is not

Noetherian. Finally, the ring $H_{m,n}$ of *Henselian power series* is defined by

$$H_{m,n} := K \otimes_{K^\circ} H^\circ_{m,n}.$$

3. Flatness

In this section, we show that $H_{m,n}$ is a regular ring of dimension $m+n$ and that $H_{m,n}$ satisfies a Nullstellantz similar to that for $S_{m,n}$. The main result is Theorem 3.3: the canonical $A_{m,n}$-morphism $H_{m,n} \to S_{m,n}$ is faithfully flat.

The next lemma will allow us to effectively apply the results of [**13**].

Lemma 3.1. — *The following natural inclusions are flat.*

(i) $T^\circ_{m+n} \longrightarrow S^\circ_{m,n}$.

(ii) $A^\circ_{m,n} \longrightarrow S^\circ_{m,n}$.

(iii) $A_{m,n} \longrightarrow S_{m,n}$.

Moreover, the maps in (ii) *and* (iii) *are even faithfully flat.*

Proof. — Suppose we knew that $T^\circ_{m+n} \hookrightarrow S^\circ_{m,n}$ were flat; then since $(\rho) \cdot S^\circ_{m,n} \subset \operatorname{rad} S^\circ_{m,n}$, also $A^\circ_{m,n} \hookrightarrow S^\circ_{m,n}$ would be flat by [**12**, Theorem 7.1]. The induced map

$$K^\circ\langle\xi\rangle = A^\circ_{m,n}/(\rho) \longrightarrow S^\circ_{m,n}/(\rho) = K^\circ\langle\xi\rangle$$

is an isomorphism. Since $(\rho) \cdot A^\circ_{m,n} \subset \operatorname{rad} A^\circ_{m,n}$, it follows that no maximal ideal of $A^\circ_{m,n}$ can generate the unit ideal of $S^\circ_{m,n}$; hence $A^\circ_{m,n} \hookrightarrow S^\circ_{m,n}$ is faithfully flat by [**12**, Theorem 7.2]. This proves (ii).

By faithfully flat base-change

$$A_{m,n} = K \otimes_{K^\circ} A^\circ_{m,n} \longrightarrow \left(K \otimes_{K^\circ} A^\circ_{m,n}\right) \otimes_{A^\circ_{m,n}} S^\circ_{m,n} = S_{m,n}$$

is faithfully flat. This proves (iii).

It remains to show that $T^\circ_{m+n} \hookrightarrow S^\circ_{m,n}$ is flat.

***Claim* (A)**. — *Let $M \subset (T_m)^r$ be a T_m-module, and put*

$$M^\circ := (T^\circ_m)^r \cap M, \quad M^{\circ\circ} := (T^{\circ\circ}_m)^r \cap M, \quad \widetilde{M} := M^\circ/M^{\circ\circ} \subset (\widetilde{T}_m)^r.$$

Suppose $\widetilde{g}_1, \dots, \widetilde{g}_s \in \widetilde{M}$ generate the $\widetilde{T}_m$-module $\widetilde{M}$, and find $g_1, \dots, g_s \in M^\circ$ that lift the $\widetilde{g}_i$. Put

$$N := \left\{(f_1, \dots, f_s) \in (T_m)^s : \sum_{i=1}^{s} f_i g_i = 0\right\},$$

$$N' := \left\{(\widetilde{f}_1, \dots, \widetilde{f}_s) \in (\widetilde{T}_m)^s : \sum_{i=1}^{s} \widetilde{f}_i \widetilde{g}_i = 0\right\}.$$

Then $N' = \widetilde{N}$.

Clearly, $\widetilde{N} \subset N'$. Let $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_s) \in N'$ and find $h = (h_1, \ldots, h_s) \in (T_m^\circ)^s$ that lifts $\widetilde{f}$. Since $\|\sum_{i=1}^s h_i g_i\| < 1$, and since the $\widetilde{g}_i$ generate $\widetilde{M}$, by [**11**, Lemma 3.1.4], there is some $h' = (h'_1, \ldots, h'_s) \in (T_m^{\circ\circ})^s$ such that

$$\sum_{i=1}^s h'_i g_i = \sum_{i=1}^s h_i g_i.$$

Put $f := h - h'$; then $f \in N^\circ$ and f lifts $\widetilde{f}$. This proves the claim.

***Claim* (B).** — *Let $M \subset (T_{m+n})^r$ be a T_{m+n}-module and put $L := M \cdot S_{m,n} \subset (S_{m,n})^r$. Then $L^\circ = M^\circ \cdot S_{m,n}^\circ$.*

Find generators $\widetilde{g}_1, \ldots, \widetilde{g}_s$ of $\widetilde{M}$ and, using [**11**, Lemma 3.1.4], lift them to generators $g_1, \ldots, g_s$ of the T_{m+n}°-module M°. Let N and $N' = \widetilde{N}$ be the corresponding modules, as in Claim A. (It follows from [**11**, Lemma 3.1.4], that N° is a finitely generated T_{m+n}°-module.) Suppose $f_1, \ldots, f_s \in S_{m,n}^\circ$; by [**11**, Lemma 3.1.4], we must find elements $h_1, \ldots, h_s$ of $S_{m,n}^\circ$ such that

$$\sum_{i=1}^s f_i g_i = \sum_{i=1}^s h_i g_i \quad \text{and} \quad \max_{1 \le i \le s} \|h_i\| \le \Big\|\sum_{i=1}^s f_i g_i\Big\|.$$

For this, we may assume that

$$\max_{1 \le i \le s} \|f_i\| > \Big\|\sum_{i=1}^s f_i g_i\Big\| > 0. \tag{3.1}$$

Let $B \in \mathfrak{B}$ (see Section 2 for the definition of $\mathfrak{B}$) be chosen so that $f_1, \ldots, f_s \in B\langle\xi\rangle[\![\rho]\!]$, $g_1, \ldots, g_s \in (B\langle\xi, \rho\rangle)^r$, and $(B\langle\xi, \rho\rangle)^s$ contains generators of the T_{m+n}°-module N° (hence by Claim A, $(\widetilde{B}[\xi, \rho])^s$ contains generators of N'). Since the value semigroup $|B \setminus \{0\}| \subset \mathbb{R}_+ \setminus \{0\}$ is discrete, it suffices to show that there are $h_1, \ldots, h_s \in B\langle\xi\rangle[\![\rho]\!]$ with

$$\sum_{i=1}^s f_i g_i = \sum_{i=1}^s h_i g_i \quad \text{and} \quad \max_{1 \le i \le s} \|h_i\| < \max_{1 \le i \le s} \|f_i\|. \tag{3.2}$$

Let $B = B_0 \supset B_1 \supset \cdots$ be the natural filtration of B and find $p \in \mathbb{N}$ so that

$$(f_1, \ldots, f_s) \in (B_p\langle\xi\rangle[\![\rho]\!])^s \setminus (B_{p+1}\langle\xi\rangle[\![\rho]\!])^s.$$

By $\pi_p : B_p \to \widetilde{B}_p \subset \widetilde{K}$ denote the B-module residue epimorphism $a \mapsto (b_p^{-1}a)^\sim$ and write $\widetilde{K} = \widetilde{B}_p \oplus V$ for some $\widetilde{B}$-vector space V. By (3.1), $\sum_{i=1}^s \pi_p(f_i)\widetilde{g}_i = 0$. Since $\widetilde{K}[\xi, \rho] \hookrightarrow \widetilde{S}_{m,n}$ is flat (see Section 2), by [**12**, Theorem 7.4(i)],

$$(\pi_p(f_1), \ldots, \pi_p(f_s)) \in N' \cdot \widetilde{S}_{m,n}.$$

Since

$$\widetilde{K}[\xi][\![\rho]\!] = \widetilde{B}_p[\xi][\![\rho]\!] \oplus V[\xi][\![\rho]\!]$$

as $\widetilde{B}[\xi][\![\rho]\!]$-modules, and since $(\widetilde{B}[\xi,\rho])^s$ contains generators of N', we must have

$$(\pi_p(f_1),\ldots,\pi_p(f_s)) \in \left(\left(\widetilde{B}[\xi,\rho]\right)^s \cap N'\right)\cdot \widetilde{B}_p[\xi][\![\rho]\!]\,.$$

Thus by Claim A, there is some $(f_1',\ldots,f_s') \in (B_p\langle\xi\rangle[\![\rho]\!])^s$ such that

$$\sum_{i=1}^{s} f_i' g_i = 0 \quad \text{and} \quad f_i - f_i' \in B_{p+1}\langle\xi\rangle[\![\rho]\!]\,,\ 1\le i\le s.$$

Putting $h_i := f_i - f_i'$, $1\le i\le s$, satisfies (3.2). This proves the claim.

Now let $g_1,\ldots,g_r \in T^\circ_{m+n}$ and put

$$M := \{(f_1,\ldots,f_r)\in (T_{m+n})^r : \sum_{i=1}^{r} f_i g_i = 0\},$$

$$N := \{(f_1,\ldots,f_r)\in (S_{m,n})^r : \sum_{i=1}^{r} f_i g_i = 0\}.$$

By [**12**, Theorem 7.6], to show that $T^\circ_{m+n} \hookrightarrow S^\circ_{m,n}$ is flat, we must show that $N^\circ = M^\circ\cdot S^\circ_{m,n}$. But since $T_{m+n} \hookrightarrow S_{m,n}$ is flat (see Section 2,) this is an immediate consequence of Claim B. □

By [**13**, Exemple XI.2.2], the pairs $(B\langle\xi\rangle[\![\rho]\!],(\rho))$ are Henselian. Since the pair $(S^\circ_{m,n},(\rho))$ is the direct limit of the Henselian pairs $(B\langle\xi\rangle[\![\rho]\!],(\rho))$, $B\in\mathfrak{B}$, it follows [**13**, Proposition XI.2.2] that $(S^\circ_{m,n},(\rho))$ is Henselian. By the Universal Mapping Property of Henselizations ([**13**, Definition XI.2.4]), it follows that there is a canonical $A^\circ_{m,n}$-algebra morphism $H^\circ_{m,n}\to S^\circ_{m,n}$. We wish to show that this morphism is faithfully flat. It then follows from [**12**, Theorem 7.5], that, in particular, we may regard $H^\circ_{m,n}$ as a subring of $S^\circ_{m,n}$.

Lemma 3.2 **(*cf.*** [**13**, Proposition VII.3.3]). — *Let (A,I) be a pair with $I\subset \operatorname{rad} A$. Then the following are equivalent:*

(i) *(A,I) is Henselian.*

(ii) *If (E,J) is a local-étale neighborhood of (A,I), then $A\to E$ is an isomorphism.*

Proof

(ii)⇒(i). Let (A',I') be an étale neighborhood of (A,I). By [**13**, Proposition XI.2.1], we must show that there is an A-morphism $A'\to A$. Put $E := A'_{1+I'}$, $J := I'\cdot E$; then (E,J) is a local-étale neighborhood of (A,I). Hence the map $\varphi : A\to E$ is an isomorphism, and the composition

$$A'\to A'_{1+I'} = E \xrightarrow{\varphi^{-1}} A$$

is an A-morphism, as required.

(i)⇒(ii). Let (E,J) be a local-étale neighborhood of (A,I); then there is an étale neighborhood (A',I') of (A,I) such that $E = A'_{1+I'}$, $J = I'\cdot E$. By [**13**, Proposition XI.2.1], there is an A-morphism $\varphi : A'\to A$. Since $\varphi(I') = I\subset \operatorname{rad} A$, φ extends to an A-morphism $\psi : E \to A$, and we must show that $\operatorname{Ker}\psi = (0)$. For this, it suffices to show that the image of $\operatorname{Ker}\psi$ in $E_{\mathfrak{n}}$ is (0) for every maximal ideal $\mathfrak{n}$ of E.

Let $\mathfrak{n} \in \operatorname{Max} E$; then there is some $\mathfrak{m} \in \operatorname{Max} A$ such that $\mathfrak{n} = \psi^{-1}(\mathfrak{m})$. (Indeed, since $J \subset \psi^{-1}(I)$, ψ induces an A-morphism

$$A/I \cong A'/I' \cong E/J \longrightarrow A/I,$$

which must be an isomorphism; but $J \subset \operatorname{rad} E$ and $I \subset \operatorname{rad} A$.) It therefore suffices to show for each $\mathfrak{m} \in \operatorname{Max} A$ that the map

$$A'_{\mathfrak{m}'} \longrightarrow A_{\mathfrak{m}}$$

induced by φ is an isomorphism, where $\mathfrak{m}' := \varphi^{-1}(\mathfrak{m})$.

We now apply the Jacobian Criterion ([**13**, Théorème V.2.5]). Write

$$A' = A[Y_1, \ldots, Y_N]/\mathfrak{a}$$

for some finitely generated ideal $\mathfrak{a}$ of $A[Y]$, and by $\mathfrak{b}$ denote the inverse image of $\operatorname{Ker}\varphi$ in $A[Y]$. Then $\mathfrak{a} \subset \mathfrak{b}$. Let $\mathfrak{m} \in \operatorname{Max} A$, put $\mathfrak{m}' := \varphi^{-1}(\mathfrak{m})$ and let $\mathfrak{M}$ be the inverse image of $\mathfrak{m}'$ in $A[Y]$. We conclude the proof by showing that $\mathfrak{a} \cdot A[Y]_{\mathfrak{m}} = \mathfrak{b} \cdot A[Y]_{\mathfrak{M}}$. Since A' is étale over A, there are $f_1, \ldots, f_N \in \mathfrak{a}$ such that the images of $f_1, \ldots, f_N$ in $A[Y]_{\mathfrak{M}}$ generate $\mathfrak{a} \cdot A[Y]_{\mathfrak{M}}$ and $\det(\partial f_i/\partial Y_j) \notin \mathfrak{M}$. Then since $f_1, \ldots, f_N \in \mathfrak{b}$ and since $A[Y]/\mathfrak{b} = A$ is étale over A, the images of $f_1, \ldots, f_N$ in $A[Y]_{\mathfrak{M}}$ also generate $\mathfrak{b} \cdot A[Y]_{\mathfrak{M}}$; i.e., $\mathfrak{a} \cdot A[Y]_{\mathfrak{M}} = \mathfrak{b} \cdot A[Y]_{\mathfrak{M}}$. □

Theorem 3.3. — *The canonical $A^\circ_{m,n}$-morphism $H^\circ_{m,n} \to S^\circ_{m,n}$ is faithfully flat; it follows by faithfully flat base-change that $H_{m,n} \to S_{m,n}$ is also faithfully flat.*

Proof. — It suffices to prove that $S^\circ_{m,n}$ is flat over $H^\circ_{m,n}$. Indeed, since $(\rho) \cdot H^\circ_{m,n} \subset \operatorname{rad} H^\circ_{m,n}$, and since the induced map

$$K^\circ\langle\xi\rangle = H^\circ_{m,n}/(\rho) \longrightarrow S^\circ_{m,n}/(\rho) = K^\circ\langle\xi\rangle$$

is an isomorphism, this is a consequence of [**12**, Theorem 7.2].

Now, $H^\circ_{m,n}$ is a direct limit of local-étale neighborhoods (E, I) of $(A^\circ_{m,n}, (\rho))$ by [**13**, Théorème XI.2.2]. Therefore, it suffices to show that the induced map $E \to S^\circ_{m,n}$ is flat.

Since by Lemma 3.1 $S^\circ_{m,n}$ is a flat $A^\circ_{m,n}$-algebra, the map

$$E \longrightarrow (S^\circ_{m,n} \otimes_{A^\circ_{m,n}} E)_{1+(\rho)}$$

induced by $1 \otimes \mathrm{id}$ is flat. It therefore suffices to show that the map

$$\mu : (S^\circ_{m,n} \otimes_{A^\circ_{m,n}} E)_{1+(\rho)} \longrightarrow S^\circ_{m,n}$$

induced by $\sum f_i \otimes g_i \mapsto \sum f_i g_i$ is an isomorphism.

Now, since $(S^\circ_{m,n}, (\rho))$ is a Henselian pair, by Lemma 3.2, it suffices to show that $((S^\circ_{m,n} \otimes_{A^\circ_{m,n}} E)_{1+(\rho)}, J)$ is a local-étale neighborhood of $(S^\circ_{m,n}, (\rho))$, where $J :=$

$(\rho)\cdot(S^\circ_{m,n}\otimes_{A^\circ_{m,n}}E)_{1+(\rho)}$. For some étale neighborhood (E',I') of $(A^\circ_{m,n},(\rho))$, we have

$$(E,I)=(E'_{1+I'},I'\cdot E'_{1+I'}),$$

where $I'=(\rho)\cdot E'$. Since localization commutes with tensor product, it suffices to show that

$$(S^\circ_{m,n}\otimes_{A^\circ_{m,n}}E',(\rho)\cdot(S^\circ_{m,n}\otimes_{A^\circ_{m,n}}E'))$$

is an étale neighborhood of $(S^\circ_{m,n},(\rho))$. But this is immediate from [**13**, Proposition II.2]. □

From now on, we regard $H_{m,n}$ as a subring of $S_{m,n}$. In particular, the Gauss norm $\|\cdot\|$ is defined on $H_{m,n}$.

Corollary 3.4. — $H^\circ_{m,n}=\{f\in H_{m,n}:\|f\|\le 1\}$.

Proof. — We must show that $H^\circ_{m,n}=S^\circ_{m,n}\cap H_{m,n}$. Clearly, $H^\circ_{m,n}\subset S^\circ_{m,n}\cap H_{m,n}$; we prove $\supset$. Let $f\in S^\circ_{m,n}\cap H_{m,n}$; then for some $\varepsilon\in K^\circ\setminus\{0\}$, $\varepsilon f\in H^\circ_{m,n}$. But by [**12**, Theorem 7.5], $\varepsilon H^\circ_{m,n}=H^\circ_{m,n}\cap\varepsilon S^\circ_{m,n}$. It follows that $f\in H^\circ_{m,n}$. □

Since $S_{m,n}$ is a faithfully flat $H_{m,n}$-algebra, any strictly increasing chain of ideals of $H_{m,n}$ extends to a strictly increasing chain of ideals of $S_{m,n}$. Since $S_{m,n}$ is Noetherian, we obtain the following.

Corollary 3.5. — *$H_{m,n}$ is a Noetherian ring.*

Theorem 3.3 on the faithful flatness of $H^\circ_{m,n}\to S^\circ_{m,n}$ allows us to pull back to $H_{m,n}$ information from $S_{m,n}$ on the structure of maximal ideals and completions with respect to maximal-adic topologies.

***Corollary 3.6* (Nullstellensatz for $H_{m,n}$)**. —*For every $\mathfrak{m}\in\operatorname{Max}H_{m,n}$, the field $H_{m,n}/\mathfrak{m}$ is a finite algebraic extension of K. The maximal ideals of $H_{m,n}$ are in bijective correspondence with those maximal ideals $\mathfrak{n}$ of $K[\xi,\rho]$ that satisfy*

$$|\overline{\xi}_i|\le 1,\ |\overline{\rho}_j|<1,\quad 1\le i\le m,\ 1\le j\le n \tag{3.3}$$

in $K[\xi,\rho]/\mathfrak{n}$ via the map $\mathfrak{n}\mapsto\mathfrak{n}\cdot H_{m,n}$. Moreover, each prime ideal of $H_{m,n}$ is an intersection of maximal ideals.

Proof. — Let $\mathfrak{n}\in\operatorname{Max}K[\xi,\rho]$ satisfy (3.3), and put $\mathfrak{m}:=\mathfrak{n}\cdot H_{m,n}$, $\mathfrak{M}:=\mathfrak{n}\cdot S_{m,n}$. Since $H_{m,n}\to S_{m,n}$ is faithfully flat, $\mathfrak{m}=H_{m,n}\cap\mathfrak{M}$; hence $H_{m,n}/\mathfrak{m}\to S_{m,n}/\mathfrak{M}$ is injective. Since $K\subset H_{m,n}$ and $S_{m,n}/\mathfrak{M}$ is a finite algebraic extension of K, by [**12**, Theorem 9.3], $\mathfrak{m}\in\operatorname{Max}H_{m,n}$. Moreover, $H_{m,n}/\mathfrak{m}$ is a finite algebraic extension of K.

Let $\mathfrak{m}\in\operatorname{Max}H_{m,n}$ be arbitrary. Since $H_{m,n}\to S_{m,n}$ is faithfully flat, there is some $\mathfrak{M}\in\operatorname{Max}S_{m,n}$ with $\mathfrak{M}\supset\mathfrak{m}\cdot S_{m,n}$ and $\mathfrak{m}=H_{m,n}\cap\mathfrak{M}$. By the Nullstellensatz for $S_{m,n}$, $\mathfrak{M} = \mathfrak{n} \cdot S_{m,n}$ for some $\mathfrak{n} \in \operatorname{Max} K[\xi, \rho]$ satisfying (3.3). Since $\mathfrak{n} \subset \mathfrak{m}$, it follows that $\mathfrak{m} = \mathfrak{n} \cdot H_{m,n}$, as desired.

Now let $\mathfrak{p} \in \operatorname{Spec} H_{m,n}$ and put

$$\mathfrak{q} := \bigcap_{\substack{\mathfrak{m} \in \operatorname{Max} H_{m,n} \\ \mathfrak{m} \supset \mathfrak{p}}} \mathfrak{m}, \qquad \mathfrak{Q} := \bigcap_{\substack{\mathfrak{M} \in \operatorname{Max} S_{m,n} \\ \mathfrak{M} \supset \mathfrak{p} \cdot S_{m,n}}} \mathfrak{M};$$

we must show that $\mathfrak{p} \supset \mathfrak{q}$. Let $f \in \mathfrak{q} \subset \mathfrak{q}$. By the Nullstellensatz for $S_{m,n}$, $f^\ell \in \mathfrak{p} \cdot S_{m,n}$ for some $\ell \in \mathbb{N}$. Since $H_{m,n} \to S_{m,n}$ is faithfully flat, $f^\ell \in \mathfrak{p}$, and since $\mathfrak{p}$ is prime, $f \in \mathfrak{p}$. □

Corollary 3.7. — *Let $\mathfrak{M} \in \operatorname{Max} S_{m,n}$ and consider the maximal ideals put $\mathfrak{m} := H_{m,n} \cap \mathfrak{M}$, $\mathfrak{n} := A_{m,n} \cap \mathfrak{M}$ and $\mathfrak{p} := K[\xi, \rho] \cap \mathfrak{M}$. Then the inclusions $K[\xi, \rho] \hookrightarrow A_{m,n} \hookrightarrow H_{m,n} \hookrightarrow S_{m,n}$ induce isomorphisms*

$$K[\xi, \rho]\widehat{_\mathfrak{p}} \cong (A_{m,n})\widehat{_\mathfrak{n}} \cong (H_{m,n})\widehat{_\mathfrak{m}} \cong (S_{m,n})\widehat{_\mathfrak{M}},$$

where $\widehat{\ }$ denotes the maximal-adic completion of a local ring. Moreover $H_{m,n}$ is a regular ring of Krull dimension $m + n$.

Proof. — It follows by descent, from Lemma 3.1 and Theorem 3.3, that each of the inclusions $A_{m,n} \to H_{m,n} \to S_{m,n}$ is faithfully flat. Let $\ell \in \mathbb{N}$. Since by [**11**, Theorem 4.1.1] $\mathfrak{M} = \mathfrak{p} S_{m,n}$, each of $\mathfrak{p}^\ell$, $\mathfrak{n}^\ell$, $\mathfrak{m}^\ell$ and $\mathfrak{M}^\ell$ is generated by the monomials of degree ℓ in the generators of $\mathfrak{p}$, it follows that the natural maps

$$(A_{m,n})\widehat{_\mathfrak{n}} \longrightarrow (H_{m,n})\widehat{_\mathfrak{m}} \longrightarrow (S_{m,n})\widehat{_\mathfrak{M}}$$

are injective. But by [**11**, Proposition 4.2.1], $(A_{m,n})\widehat{_\mathfrak{n}} \to (S_{m,n})\widehat{_\mathfrak{M}} \cong K[\xi, \rho]\widehat{_\mathfrak{p}}$ is surjective; thus also $(H_{m,n})\widehat{_\mathfrak{m}} \to (S_{m,n})\widehat{_\mathfrak{M}} \cong K[\xi, \rho]\widehat{_\mathfrak{p}}$ is surjective. By Hilbert's Nullstellensatz $\mathfrak{p}$ can be generated by $m + n$ elements, and $\dim K[\xi, \rho]_\mathfrak{p} = m + n$. In particular $K[\xi, \rho]\widehat{_\mathfrak{p}}$ is a regular local ring of dimension $m + n$. Since $\mathfrak{m} = \mathfrak{p} H_{m,n}$ and $(H_{m,n})\widehat{_\mathfrak{m}} = K[\xi, \rho]\widehat{_\mathfrak{p}}$, it follows that $(H_{m,n})_\mathfrak{m}$ is a regular local ring of dimension $m+n$. Moreover by [**12**, Theorem 19.3], $H_{m,n}$ is a regular ring. □

4. Regularity

To obtain our Approximation Theorem, we will apply [**14**, Theorem 1.1]. For that, we need to know that certain maps are regular maps of Noetherian rings.

Proposition 4.1. — *$H_{m,n}$ is excellent; in particular it is a G-ring.*

Proof. — By [**12**, Theorem 32.4], to show that $H_{m,n}$ is a G-ring, it suffices to show that the map

$$(H_{m,n})_\mathfrak{m} \longrightarrow (H_{m,n})\widehat{_\mathfrak{m}}$$

is regular for each $\mathfrak{m} \in \operatorname{Max} H_{m,n}$. Fix $\mathfrak{m} \in \operatorname{Max} H_{m,n}$, and $\mathfrak{q} \in \operatorname{Spec}(H_{m,n})_\mathfrak{m}$; we must show that

$$\widehat{H}(\mathfrak{q}) := \widehat{(H_{m,n})_\mathfrak{m}} \otimes_{(H_{m,n})_\mathfrak{m}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$, the field of fractions of $(H_{m,n})_\mathfrak{m}/\mathfrak{q}$.

Since $A_{m,n}$ is a localization of the excellent ring $T_{m,n}$, it is a G-ring. In particular, by Corollary 3.7,

$$\widehat{H}(\mathfrak{p}) := \widehat{(H_{m,n})_\mathfrak{m}} \otimes_{(A_{m,n})_\mathfrak{n}} \kappa(\mathfrak{p}) = \widehat{(A_{m,n})_\mathfrak{n}} \otimes_{(A_{m,n})_\mathfrak{n}} \kappa(\mathfrak{p})$$

is geometrically regular over $\kappa(\mathfrak{p})$, where $\mathfrak{n} := A_{m,n} \cap \mathfrak{m}$ and $\mathfrak{p} := (A_{m,n})_\mathfrak{n} \cap \mathfrak{q} \in \operatorname{Spec}(A_{m,n})_\mathfrak{n}$. Suppose we knew: (i) that $\widehat{H}(\mathfrak{q})$ were a localization of $\widehat{H}(\mathfrak{p})$, and (ii) that $\kappa(\mathfrak{q})$ were separably algebraic over $\kappa(\mathfrak{p})$. Then by (i), we would have (i′) $\widehat{H}(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{p})$, and by (ii), we would have (ii′) $\Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})} = (0)$ by [**12**, Theorem 25.3], (where $\Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})}$ is the module of differentials of $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$).

Let $\mathfrak{a}$ be a maximal ideal of $\widehat{H}(\mathfrak{q})$; then by (i′), $\widehat{H}(\mathfrak{q})_\mathfrak{a}$ is geometrically regular over $\kappa(\mathfrak{p})$. By [**12**, Theorem 28.7], $\widehat{H}(\mathfrak{q})_\mathfrak{a}$ must be $\mathfrak{a}$-smooth over $\kappa(\mathfrak{p})$. Hence by (ii′) and [**12**, Theorem 28.6], $\widehat{H}(q)_\mathfrak{a}$ is $\mathfrak{a}$-smooth over $\kappa(\mathfrak{q})$. By [**12**, Theorem 28.7], this implies that $\widehat{H}(\mathfrak{q})_\mathfrak{a}$ is geometrically regular over $\kappa(\mathfrak{q})$. Since this holds for every maximal ideal $\mathfrak{a}$ of $\widehat{H}(\mathfrak{q})$, $\widehat{H}(\mathfrak{q})$ must be geometrically regular over $\kappa(\mathfrak{q})$. The proposition follows.

It remains to prove (i) and (ii). By [**13**, Théorème XI.2.2], $(H^\circ_{m,n}, (\rho))$ is a direct limit of local-étale neighborhoods (E, I) of $(A^\circ_{m,n}, (\rho))$; thus $(H_{m,n})_\mathfrak{m}$ is a local-ind-étale $(A_{m,n})_\mathfrak{n}$-algebra. By [**13**, Théorème VIII.4.3],

$$H(\mathfrak{p}) := (H_{m,n})_\mathfrak{m} \otimes_{(A_{m,n})_\mathfrak{n}} \kappa(\mathfrak{p}) = \big((H_{m,n})_\mathfrak{m}/\mathfrak{p} \cdot (H_{m,n})_\mathfrak{m}\big)_\mathfrak{p}$$

is a finite product of separable algebraic extensions of $\kappa(\mathfrak{p})$. It follows that $\kappa(\mathfrak{q})$ is the localization of $H(\mathfrak{p})$ at the maximal ideal $\mathfrak{q} \cdot H(\mathfrak{p})$, and that $\kappa(\mathfrak{q})$ is a separable algebraic extension of $\kappa(\mathfrak{p})$. This proves (ii). Note that

$$\widehat{H}(\mathfrak{q}) = \widehat{(H_{m,n})_\mathfrak{m}} \otimes_{(H_{m,n})_\mathfrak{m}} H(\mathfrak{p})_{\mathfrak{q}\cdot H(\mathfrak{p})},$$

which is a localization of

$$\widehat{H}(\mathfrak{p}) = \widehat{(H_{m,n})_\mathfrak{m}} \otimes_{(A_{m,n})_\mathfrak{n}} \kappa(\mathfrak{p}) = \widehat{(H_{m,n})_\mathfrak{m}} \otimes_{(H_{m,n})_\mathfrak{m}} H(\mathfrak{p}),$$

proving (i). □

Theorem 4.2. — *The inclusion $H_{m,n} \to S_{m,n}$ is a regular map of Noetherian rings.*

Proof. — Let $\mathfrak{M} \in \operatorname{Max} S_{m,n}$ and put $\mathfrak{m} := H_{m,n} \cap \mathfrak{M}$; we remark that

$$(H_{m,n})_\mathfrak{m} \longrightarrow (S_{m,n})_\mathfrak{M} \tag{4.1}$$

is regular. Indeed, since $(S_{m,n})_\mathfrak{M} \to \widehat{(S_{m,n})_\mathfrak{M}}$ is faithfully flat, [**12**, Theorem 8.8], by [**12**, Theorem 32.1], it suffices to show that $(H_{m,n})_\mathfrak{m} \to \widehat{(S_{m,n})_\mathfrak{M}}$ is regular. But by Corollary 3.7 $\widehat{(H_{m,n})_\mathfrak{m}} = \widehat{(S_{m,n})_\mathfrak{M}}$, hence this follows from Proposition 4.1.

Let $\mathfrak{p} \in \operatorname{Spec} H_{m,n}$. Since $S_{m,n}$ is flat over $H_{m,n}$ (Theorem 3.3), to show that $H_{m,n} \to S_{m,n}$ is regular, we must show that $S(\mathfrak{p}) := S_{m,n} \otimes_{H_{m,n}} \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. Let $\mathfrak{q} \in \operatorname{Spec} S(\mathfrak{p})$; it suffices to show that $S(\mathfrak{p})_\mathfrak{q}$ is geometrically regular over $\kappa(\mathfrak{p})$. Put $\mathfrak{P} := S_{m,n} \cap \mathfrak{q}$ and let $\mathfrak{M} \in \operatorname{Max} S_{m,n}$ be a maximal ideal containing $\mathfrak{P}$. Put $\mathfrak{m} := H_{m,n} \cap \mathfrak{M}$ and

$$S_\mathfrak{M}(\mathfrak{p}) := (S_{m,n})_\mathfrak{M} \otimes_{(H_{m,n})_\mathfrak{m}} \kappa(\mathfrak{p} \cdot (H_{m,n})_\mathfrak{m}).$$

Note that $S_\mathfrak{M}(\mathfrak{p}) = (S(\mathfrak{p}))_\mathfrak{M}$ and that $\mathfrak{q} = \mathfrak{P} \cdot S(\mathfrak{p})$. Since $\mathfrak{M} \supset \mathfrak{P}$, it follows that $S(\mathfrak{p})_\mathfrak{q}$ is a localization of $S_\mathfrak{M}(\mathfrak{p})$, which, by the regularity of (4.1) is geometrically regular over $\kappa(\mathfrak{p} \cdot (H_{m,n})_\mathfrak{m}) = \kappa(\mathfrak{p})$. Therefore, $S(\mathfrak{p})_\mathfrak{q}$ is geometrically regular over $\kappa(\mathfrak{p})$, as desired. □

Let $B \in \mathfrak{B}$, let $\varepsilon \in K^{\circ\circ} \setminus \{0\}$ and let $I(B,\varepsilon)$ be the ideal

$$I(B,\varepsilon) := \{b \in B : |b| \le |\varepsilon|\} \subset B.$$

It follows from the definition of quasi-Noetherian rings (see Section 2 and [**6**, Section 1.8]) that $B/I(B,\varepsilon)$ is Noetherian. Put

$$T_{m+n}(B) := B\langle \xi, \rho\rangle\,, \quad A_{m,n}(B) := T_{m+n}(B)_{1+(\rho)} \text{ and } S_{m,n}(B) := B\langle\xi\rangle[\![\rho]\!].$$

Note that

$$T_{m+n}(B,\varepsilon) := \big(B/I(B,\varepsilon)\big)\,[\xi,\rho]$$

is Noetherian, and

$$A_{m,n}(B,\varepsilon) := T_{m+n}(B,\varepsilon)_{1+(\rho)},$$

being a localization of a Noetherian ring, is Noetherian as well. Moreover, $(\rho) \cdot A_{m,n}(B,\varepsilon) \subset \operatorname{rad} A_{m,n}(B,\varepsilon)$. Let $(H_{m,n}(B,\varepsilon),(\rho))$ be a Henselization of the pair $(A_{m,n}(B,\varepsilon),(\rho))$.

The (ρ)-adic completion of $A_{m,n}(B,\varepsilon)$ is

$$S_{m,n}(B,\varepsilon) := \big(B/I(B,\varepsilon)\big)\,[\xi][\![\rho]\!],$$

which must coincide with the (ρ)-adic completion of $H_{m,n}(B,\varepsilon)$.

(Indeed, $(A_{m,n}(B,\varepsilon)/(\rho)^\ell,(\rho))$ being (ρ)-adically complete, is a Henselian pair by [**13**, Exemple XI.2.2]. If (E,I) is a local-étale neighborhood of $(A_{m,n}(B,\varepsilon),(\rho))$, then by [**13**, Proposition II.2], $(E/(\rho)^\ell, I \cdot E/(\rho)^\ell)$ is a local-étale neighborhood of $(A_{m,n}(B,\varepsilon)/(\rho)^\ell,(\rho))$. By Lemma 3.2, $E/(\rho)^\ell$ is isomorphic to $A_{m,n}(B,\varepsilon)/(\rho)^\ell$. Since $H_{m,n}(B,\varepsilon)$ is a direct limit of local-étale neighborhoods of $A_{m,n}(B,\varepsilon)/(\rho)$, the (ρ)-adic completions of $H_{m,n}(B,\varepsilon)$ and $A_{m,n}(B,\varepsilon)$ coincide.)

Since the rings $A_{m,n}(B,\varepsilon)$ and $H_{m,n}(B,\varepsilon)$ are both Noetherian, $S_{m,n}(B,\varepsilon)$ is faithfully flat over both $A_{m,n}(B,\varepsilon)$ and $H_{m,n}(B,\varepsilon)$ by [**12**, Theorem 8.14]. Therefore, by [**12**, Theorem 7.5], we may regard $H_{m,n}(B,\varepsilon)$ as a subring of $S_{m,n}(B,\varepsilon)$.

Proposition 4.3. — *Fix $B \in \mathfrak{B}$ and $\varepsilon \in K^{\circ\circ} \setminus \{0\}$. The inclusion $H_{m,n}(B,\varepsilon) \to S_{m,n}(B,\varepsilon)$ is a regular map of Noetherian rings.*

Proof. — Find $\varepsilon' \in K^{\circ\circ}\setminus\{0\}$ such that $|\varepsilon'| = \max\{|b| : b \in B \cap K^{\circ\circ}\}$. For convenience of notation, put

$$\begin{aligned} A &:= A_{m,n}(B,\varepsilon), & H &:= H_{m,n}(B,\varepsilon), & S &:= S_{m,n}(B,\varepsilon) \\ \widetilde{A} &:= A_{m,n}(B,\varepsilon'), & \widetilde{H} &:= H_{m,n}(B,\varepsilon'), & \widetilde{S} &:= S_{m,n}(B,\varepsilon'). \end{aligned}$$

Note that

$$\widetilde{A} = \widetilde{B}[\xi,\rho]_{1+(\rho)} \quad \text{and} \quad \widetilde{S} = \widetilde{B}[\xi][\![\rho]\!],$$

where $\widetilde{B}$ is the residue field of the local ring B. Furthermore, by the Krull intersection theorem [**12**, Theorem 8.10], ideals of A, H and S are closed in their radical-adic topologies. It follows that

$$\widetilde{A} = A/I(B,\varepsilon')\cdot A, \quad \widetilde{H} = H/I(B,\varepsilon')\cdot H, \quad \widetilde{S} = S/I(B,\varepsilon')\cdot S.$$

Let $\mathfrak{p} \in \operatorname{Spec} H$; we must show that $S \otimes_H \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. Each element of $I(B,\varepsilon')\cdot H$ is nilpotent; hence $I(B,\varepsilon')\cdot H \subset \mathfrak{p}$. Let $\widetilde{\mathfrak{p}} \in \operatorname{Spec}\widetilde{H}$ denote the image of $\mathfrak{p}$ in $\widetilde{H}$. Then

$$S \otimes_H \kappa(\mathfrak{p}) = \widetilde{S} \otimes_{\widetilde{H}} \kappa(\widetilde{\mathfrak{p}}),$$

and it suffices to show that $\widetilde{S} \otimes_{\widetilde{H}} \kappa(\widetilde{\mathfrak{p}})$ is geometrically regular over $\kappa(\widetilde{\mathfrak{p}})$.

We note the following facts. (i) The maps $\widetilde{\mathfrak{M}} \mapsto \widetilde{\mathfrak{M}}\cdot\widetilde{A} + (\rho)$, $\widetilde{\mathfrak{M}} \mapsto \widetilde{\mathfrak{M}}\cdot\widetilde{H} + (\rho)$, $\widetilde{\mathfrak{M}} \mapsto \widetilde{\mathfrak{M}}\cdot\widetilde{S} + (\rho)$ are bijections between the elements of $\operatorname{Max}\widetilde{B}[\xi]$ and the elements, respectively, of $\operatorname{Max}\widetilde{A}$, $\operatorname{Max}\widetilde{H}$ and $\operatorname{Max}\widetilde{S}$. (ii) Let $\widetilde{\mathfrak{M}} \in \operatorname{Max}\widetilde{S}$, $\widetilde{\mathfrak{M}} := \widetilde{H}\cap\widetilde{\mathfrak{M}} \in \operatorname{Max}\widetilde{H}$ and $\widetilde{\mathfrak{n}} := \widetilde{A}\cap\widetilde{\mathfrak{M}} \in \operatorname{Max}\widetilde{A}$; then $\widetilde{A} \to \widetilde{H} \to \widetilde{S}$ induces isomorphisms

$$\widehat{\widetilde{A}_{\widetilde{\mathfrak{n}}}} \cong \widehat{\widetilde{H}_{\widetilde{\mathfrak{M}}}} \cong \widehat{\widetilde{S}_{\widetilde{\mathfrak{M}}}}$$

(iii) The ring $\widetilde{A}$, being a localization of the excellent ring $\widetilde{B}[\xi,\rho]$ is excellent, and in particular, a G-ring.

Arguing just as in the proof of Proposition 4.1, we show that $\widetilde{H}$ is a G-ring. Then we argue as in Theorem 4.2 to show that $\widetilde{S} \otimes_{\widetilde{H}} \kappa(\widetilde{\mathfrak{p}})$ is geometrically regular over $\kappa(\widetilde{\mathfrak{p}})$. □

5. Approximation

***Theorem 5.1* (Approximation Theorem).** — *For a given system of polynomial equations with coefficients in $H_{m,n}$, any solution over $S_{m,n}$ can be approximated by a solution over $H_{m,n}$ arbitrarily closely in the (ρ)-adic topology.*

Proof. — Let $Y = (Y_1,\ldots,Y_N)$ be variables, let J be an ideal of $H_{m,n}[Y]$, and consider the finitely generated $H_{m,n}$-algebra $C := H_{m,n}[Y]/J$. Suppose we have a homomorphism $\widehat{\varphi} : C \to S_{m,n}$; then $\widehat{\varphi}(Y)$ is a solution over $S_{m,n}$ of the system of polynomial equations with coefficients in $H_{m,n}$ given by generators of the ideal J. Fix

$\ell \in \mathbb{N}$. We wish to demonstrate the existence of a homomorphism $\varphi : C \to H_{m,n}$ such that each $\varphi(Y_i) - \widehat{\varphi}(Y_i) \in (\rho)^\ell \cdot S_{m,n}$.

Since $H_{m,n} \to S_{m,n}$ is a regular map of Noetherian rings, by [**14**, Theorem 1.1], we may assume that C is smooth over $H_{m,n}$. Let E be the symmetric algebra of the C-module J/J^2. By Elkik's Lemma ([**7**, Lemme 3]), $\operatorname{Spec} E$ is smooth over $\operatorname{Spec} H_{m,n}$ of constant relative dimension N, there is a surjection

$$H_{m,n}[Y_1, \ldots, Y_{2N+r}] \to E$$

for some $r \in \mathbb{N}$, and there are elements $g_1, \ldots, g_{N+r}, h \in H_{m,n}[Y]$ such that

$$\left(H_{m,n}[Y]/I\right)_h \cong E,$$

where $I := (g_1, \ldots, g_{N+r})$, and

$$(1) = h \cdot H_{m,n}[Y] + I.$$

Since $\operatorname{Spec} E$ is smooth of relative dimension N over $\operatorname{Spec} H_{m,n}$, $\Omega_{E/H_{m,n}}$ is locally free of rank N. It follows that

$$h^d \in \mathfrak{M} + I$$

for some $d \in \mathbb{N}$, where $\mathfrak{M}$ is the ideal in $H_{m,n}[Y]$ generated by all $(N+r) \times (N+r)$ minors of the matrix

$$M(Y) := \left(\frac{\partial g_i}{\partial Y_j}\right)_{\substack{1 \le i \le N+r \\ 1 \le j \le 2N+r}}.$$

We may extend $\widehat{\varphi}$ to E; in particular, $g(\widehat{\varphi}(Y)) = 0$. Replacing Y by $\alpha^{-1}Y$ for a suitably small scalar $\alpha \in K^\circ \setminus \{0\}$ and normalizing by another scalar, we may assume $g_1, \ldots, g_{N+r}, h \in H^\circ_{m,n}[Y]$, $\widehat{\varphi}(Y) \in (S^\circ_{m,n})^{2N+r}$, and

$$\varepsilon \in h \cdot H^\circ_{m,n}[Y] + \sum_{i=1}^{N+r} g_i H^\circ_{m,n}[Y] \tag{5.1}$$

$$\varepsilon h^d \in \mathfrak{M}^\circ + \sum_{i=1}^{N+r} g_i H^\circ_{m,n}[Y]. \tag{5.2}$$

for a suitably small $\varepsilon \in K^{\circ\circ} \setminus \{0\}$, where $\mathfrak{M}^\circ$ is the ideal in $H^\circ_{m,n}[Y]$ generated by all $(N+r) \times (N+r)$ minors of the matrix M, above.

For each $B \in \mathfrak{B}$, let $(H_{m,n}(B), (\rho))$ be a Henselization of the pair $(A_{m,n}(B), (\rho))$. Since $A^\circ_{m,n} = \varinjlim A_{m,n}(B)$, we have a canonical isomorphism $\varinjlim H_{m,n}(B) \cong H^\circ_{m,n}$. Find $B \in \mathfrak{B}$ such that

$$\widehat{\varphi}(Y_1), \ldots, \widehat{\varphi}(Y_{2N+r}) \in S_{m,n}(B) := B\langle\xi\rangle[\![\rho]\!],$$

and such that $g_1, \ldots, g_{N+r} \in H_{m,n}(B)[Y]$. Consider the commutative diagram

$$\begin{array}{ccccc} A_{m,n}(B) & \longrightarrow & H_{m,n}(B) & \longrightarrow & S_{m,n}(B) \\ \downarrow & & \downarrow & & \downarrow \\ A_{m,n}(B,\varepsilon^{2d+2}) & \longrightarrow & H_{m,n}(B,\varepsilon^{2d+2}) & \longrightarrow & S_{m,n}(B,\varepsilon^{2d+2}), \end{array}$$

where the two outer vertical arrows represent reduction modulo $I(B,\varepsilon^{2d+2})$ and the other arrows represent the canonical morphisms. It follows from the Universal Mapping Property for Henselizations that all the vertical arrows must be surjective. Thus by Proposition 4.3 and [**14**, Theorem 11.3], there are $\eta_1,\ldots,\eta_{2N+r} \in H^\circ_{m,n}$ such that $\eta_i - \widehat{\varphi}(Y_i) \in (\rho)^{2\ell+1} \cdot S^\circ_{m,n}$, $1 \le i \le 2N+r$, and $\|g_i(\eta)\| \le \left|\varepsilon^{2d+2}\right|$, $1 \le i \le N+r$.

Replacing Y by η in (5.1), we find $g', h' \in H^\circ_{m,n}$ such that $h(\eta)h' = \varepsilon(1-\varepsilon^{2d+1}g')$. It follows that there is some $\delta \in K^\circ \setminus \{0\}$ with $|\delta| \ge |\varepsilon|$ and some unit h'' of $H^\circ_{m,n}$ such that $h(\eta) = \delta h''$. Replacing Y by η in (5.2), we find some $g'' \in H^\circ_{m,n}$ such that $\varepsilon^{d+1}((h'')^d - \varepsilon^{d+1}g'') \in \mathfrak{M}^\circ(\eta)$, where $\mathfrak{M}^\circ(\eta)$ is the ideal of $H^\circ_{m,n}$ generated by all $(N+r)\times(N+r)$ minors of the matrix $M(\eta)$. Since h'' is a unit of $H^\circ_{m,n}$, it follows that

$$\varepsilon^{d+1} \in \mathfrak{M}^\circ(\eta).$$

We follow the proof of Tougeron's Lemma given in [**7**] to obtain $y_1,\ldots,y_{2N+r} \in H^\circ_{m,n}$ such that $y_i - \eta_i \in (\rho)^\ell \cdot H^\circ_{m,n}$, $1 \le i \le 2N+r$, and $g_1(\eta) = \cdots = g_{N+r}(\eta) = 0$.

Let $\mu_1,\ldots,\mu_s$ denote the monomials in ρ of degree ℓ. Since the ideal generated by the $(N+r)\times(N+r)$ minors of $M(\eta)$ contains the $\varepsilon^{d+1}\mu_i$, there are $(2N+r)\times(N+r)$ matrices $N_1,\ldots,N_s$ such that

$$M(\eta)N_i = \varepsilon^{d+1}\mu_i \mathrm{Id}_{N+r},$$

where Id_{N+r} is the $(N+r)\times(N+r)$ identity matrix. We will find elements $u_i = (u_{i,1},\ldots,u_{i,2N+r}) \in ((\rho)\cdot H^\circ_{m,n})^{2N+r}$, $1 \le i \le s$, such that

$$g_j(\eta + \sum_{i=1}^s \varepsilon^{d+1}\mu_i u_i) = 0, \qquad 1 \le j \le N+r.$$

We have the Taylor expansion

$$\begin{bmatrix} g_1(\eta + \sum_{i=1}^s \varepsilon^{d+1}\mu_i u_i) \\ \vdots \\ g_{N+r}(\eta + \sum_{i=1}^s \varepsilon^{d+1}\mu_i u_i) \end{bmatrix} = \begin{bmatrix} g_1(\eta) \\ \vdots \\ g_{N+r}(\eta) \end{bmatrix} + \sum_{i=1}^s \varepsilon^{d+1}\mu_i M(\eta) \begin{bmatrix} u_{i,1} \\ \vdots \\ u_{i,2N+r} \end{bmatrix} + $$

$$+ \sum_{i,j} \varepsilon^{2d+2}\mu_i\mu_j P_{ij},$$

where each P_{ij} is a column vector whose components are polynomials in the u_i of order at least 2. We must solve

$$(5.3) \qquad 0 = \begin{bmatrix} g_1(\eta) \\ \vdots \\ g_{N+r}(\eta) \end{bmatrix} + \sum_{i=1}^{s} \varepsilon^{d+1} \mu_i M(\eta) \begin{bmatrix} u_{i,1} \\ \vdots \\ u_{i,2N+r} \end{bmatrix} + \sum_{i,j} \varepsilon^{2d+2} \mu_i \mu_j P_{ij}.$$

Since $\|g_i(\eta)\| \leq |\varepsilon^{2d+2}|$ and $g_i(\eta) \in (\rho)^{2\ell+1} \cdot H^\circ_{m,n}$, we have

$$\begin{bmatrix} g_1(\eta) \\ \vdots \\ g_{N+r}(\eta) \end{bmatrix} = \sum_{i,j} (\varepsilon^{d+1}\mu_i)(\varepsilon^{d+1}\mu_j) \begin{bmatrix} f_{ij1} \\ \vdots \\ f_{ijN+r} \end{bmatrix},$$

where the $f_{ijk} \in (\rho) \cdot H^\circ_{m,n}$. Thus (5.3) becomes

$$0 = \sum_{i=1}^{s} \varepsilon^{d+1} \mu_i M(\eta) \left(\sum_{j=1}^{s} N_j \begin{bmatrix} f_{ij1} \\ \vdots \\ f_{ijN+r} \end{bmatrix} \right) + \sum_{i=1}^{s} \varepsilon^{d+1} \mu_i M(\eta) \begin{bmatrix} u_{i,1} \\ \vdots \\ u_{i,2N+r} \end{bmatrix} + $$
$$+ \sum_{i=1}^{s} \varepsilon^{d+1} \mu_i M(\eta) \left(\sum_{j=1}^{s} N_j P_{ij} \right),$$

and it suffices to solve

$$(5.4) \qquad 0 = \begin{bmatrix} u_{i,1} \\ \vdots \\ u_{i,2N+r} \end{bmatrix} + \sum_{j=1}^{s} N_j \left(P_{ij} + \begin{bmatrix} f_{ij1} \\ \vdots \\ f_{ijN+r} \end{bmatrix} \right), \quad 1 \leq i \leq s.$$

Since 0 is a solution of this system modulo (ρ), and since its Jacobian at 0 is 1, the system (5.4) represents an étale neighborhood of $(H^\circ_{m,n}, (\rho))$, hence has a true solution (u_{ij}). Putting

$$(y_i) := (\eta_i) + \sum_{j=1}^{s} \varepsilon^{d+1} \mu_j u_j,$$

we obtain a solution in $H^\circ_{m,n}$ of the system $g = 0$ which agrees with $\widehat{\varphi}(Y)$ up to order ℓ in ρ. □

Corollary 5.2. — *$H_{m,n}$ is a UFD.*

Proof. — Let $f \in H_{m,n}$ be irreducible. We must show that $f \cdot H_{m,n}$ is a prime ideal. Since $S_{m,n}$ is a faithfully flat $H_{m,n}$-algebra (Theorem 3.3), and since $S_{m,n}$ is a UFD ([**11**, Theorem 4.2.7]), it suffices to show that f is an irreducible element of $S_{m,n}$. That is a consequence of Theorem 5.1. □

References

[1] M. Artin. — *On the solutions of analytic equations.* Invent. Math., 5 (1968), 277–291.

[2] W. Bartenwerfer. — *Die Beschränktheit der Stückzahl der Fasern K-analytischer Abbildungen.* J. reine angew. Math., 416 (1991), 49-70.

[3] R. Berger, R. Kiehl, E. Kunz and H-J. Nastold. — *Differential Rechnung in der Analytischen Geometrie.* Springer Lecture Notes in Math, 38, 1967.

[4] S. Bosch. — *A rigid analytic version of M. Artin's theorem on analytic equations.* Math. Ann., 255 (1981), 395–404.

[5] S. Bosch, B. Dwork and P. Robba. — *Un théorème de prolongement pour des fonctions analytiques.* Math. Ann., 252 (1980), 165–173.

[6] S. Bosch, U. Güntzer and R. Remmert. — *Non-Archimedean Analysis.* Springer-Verlag, 1984.

[7] R. Elkik. — *Solutions d'équations a coefficients dans un anneau Hensélien.* Ann. Scient. Éc. Norm. Sup., 4e série, 6 (1973) 553–604.

[8] R. Kiehl. — *Ausgezeichnete Ringe in der nicht-Archimedischen analytischen Geometrie.* J. Reine Angew. Math., 234 (1969), 89–98.

[9] S. Lang. — *On quasi-algebraic closure.* Ann. Math., 55 (1952), 373–390.

[10] L. Lipshitz. — *Isolated points on fibers of affinoid varieties.* J. reine angew. Math., 384 (1988), 208-220.

[11] L. Lipshitz and Z. Robinson. — *Rings of separated power series.* This volume.

[12] H. Matsumura. — *Commutative Ring Theory.* Cambridge University Press, 1989.

[13] M. Raynaud. — *Anneaux Locaux Henséliens.* Springer Lecture Notes in Mathematics, 169, 1970.

[14] M. Spivakovsky. — *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms.* J. Amer. Math. Soc., 12 (1999), 381-444.

[15] L. van den Dries. — *A specialization theorem for p-adic power series converging on the closed unit disc.* J. Algebra, 73 (1981), 613-623.