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# POSITIVE LYAPUNOV EXPONENTS FOR LORENZ-LIKE FAMILIES WITH CRITICALITIES

*by*

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*Dedicated to Adrien Douady on the occasion of his* $60^{th}$ *birthday*

***Abstract.*** — We introduce a class of one-parameter families of real maps extending the classical geometric Lorenz models. These families combine singular dynamics (discontinuities with infinite derivative) with critical dynamics (critical points) and are based on the behaviour displayed by Lorenz flows over a fairly wide range of parameters. Our main result states that – nonuniform – expansion is the prevalent form of dynamics even after the formation of the criticalities.

## 1. Introduction and statement of results

Numerical analysis of the now famous system of differential equations

$$(1) \qquad \begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

for parameter values $r \approx 28$, $\sigma \approx 10$, $b \approx 8/3$, led Lorenz **[11]** to identify sensitive dependence of orbits with respect to the corresponding initial points as a main source of unpredictability in deterministic dynamical systems. His observations were then interpreted by **[1]**, **[6]**, who described expanding ("strange") attractors in certain geometric models for the behaviour of (1). Conjecturedly, such an attractor exists also for Lorenz' original equations, although this has not yet been proved.

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Further study of (1) revealed that relatively small variations of these parameter values may lead to quite different, albeit even more complex, dynamical features. Indeed, already as $r$ is increased past $r \approx 30$ Poincaré return maps cease to be described by the cusp-type pictures corresponding to the geometric models, instead they exhibit "folded cusps", or "hooks"; moreover, these hooks persist in a large window of values of $r$ (extending beyond $r \approx 50$), see **[15]** for a thorough discussion. Trying to understand this folding process and its effect on the behaviour of the flow was, in fact, a main motivation behind Hénon's model of strange attractor for maps in two dimensions, **[7]**, **[8]**. In constructing this model he focused on the dynamics near the fold, in particular disregarding trajectories which pass close to equilibrium points.

Here we aim at a more global understanding of the dynamics of Lorenz flows, accounting for the interaction between *singular behaviour* (corresponding to trajectories near equilibria) and *critical behaviour* (near folding regions). Indeed, we introduce a one-dimensional prototype for this problem, largely inspired by the observations in **[15]**, which we call Lorenz-like families with criticalities. Apart from their present motivation, these families of maps are also of interest in their own right, as models of rich nonsmooth dynamics in dimension one. Moreover, in an ongoing work we are further pushing the present constructions and conclusions to the context of smooth flows in three-dimensional space, *cf.* comments below.

Let us begin by explaining what we mean by *Lorenz-like families with criticalities.* We consider one-parameter families $\{\varphi_a\}$ of real maps of the form

$$\varphi_a(x) = \begin{cases} \varphi(x) - a & \text{if } x > 0 \\ -\varphi(-x) + a & \text{if } x < 0 \end{cases}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth and satisfies:

**L1 :** $\varphi(x) = \psi(x^\lambda)$ for all $x > 0$, where $0 < \lambda < 1/2$ and $\psi$ is a smooth map defined on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$;

**L2 :** there exists some $c > 0$ such that $\varphi'(c) = 0$;

**L3 :** $\varphi''(x) < 0$ for all $x > 0$.

As we already mentioned, this definition is motivated by a fair amount of numerical and analytical data concerning the behaviour of Lorenz flows. In particular, the condition $\lambda < 1/2$ corresponds to the fact that, for the parameter region we are interested in, the expanding eigenvalue $\lambda_u$ of (1) at the origin is more than twice stronger than the weakest contracting eigenvalue $\lambda_s$ (that is $\lambda_u + 2\lambda_s > 0$).

For small values of the parameter the maximal invariant set of $\varphi_a$ in the interval $[-a, a]$ is a hyperbolic Cantor set. Under certain natural conditions, implied by L4 and L5 below, the entire interval $[-a, a]$ becomes forward invariant as $a$ crosses some value $a_1 > 0$. This situation persists for a certain range of parameter values and corresponds to the class of maps usually associated to the "Lorenz attractor" (see **[5]**, **[6]**, **[1]**). The dynamics of such maps is relatively well understood: they admit an

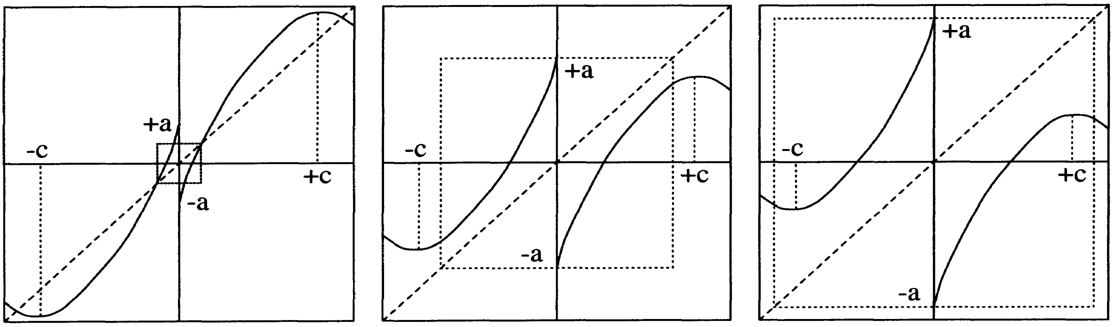


FIGURE 1. Lorenz-like families with criticalities

invariant measure which is absolutely continuous with respect to Lebesgue and has positive metric entropy; they are not structurally stable but are fully persistent in the sense that any small perturbation also admits an absolutely continuous invariant measure of positive entropy.

We are mainly interested in studying the bifurcation which occurs as the parameter crosses the value $a = c$. With this in mind, we add two natural assumptions on $\varphi$ which ensure that a Lorenz attractor persists for all $a < c$.

Let $x_{\sqrt{2}}$ denote the unique point in $(0, c)$ such that $\varphi'(x_{\sqrt{2}}) = \sqrt{2}$; sometimes we also write $a_2 = x_{\sqrt{2}}$. Then we suppose

**L4** : $0 < \varphi_a(x_{\sqrt{2}}) < \varphi_a(a) < x_{\sqrt{2}}$ for all $a \in (a_2, c]$.

The last inequality implies that given any $y$ with $|y| \in [x_{\sqrt{2}}, a)$ there exists a unique $x \in [-a, a]$ such that $\varphi_a(x) = y$. Note that $x$ and $y$ have opposite signs. Moreover, the first inequality implies that $|x| < x_{\sqrt{2}}$. Our last assumption is

**L5** : $|(\varphi_c^2)'(x)| > 2$ for all $x \in [-c, c] \setminus \{0\}$ such that $|\varphi_c(x)| \in [x_{\sqrt{2}}, c]$.

Observe that this is automatic if $\varphi_c(x) = x_{\sqrt{2}}$ (because $|x|$ is strictly smaller than $x_{\sqrt{2}}$, by the previous remarks) and also if $\varphi_c(x)$ is close to $c$ (then $x$ is close to zero and so $|(\varphi_c^2)'(x)| \approx |x|^{2\lambda-1} \approx \infty$).

It is straightforward to check that L1-L5 are satisfied by a nonempty open set of one-parameter families, where openness is meant with respect to the $C^2$ topology in the space of real maps $\psi$. Moreover, we shall show that these hypotheses do imply that $\varphi_a$ is essentially uniformly expanding for all parameters up to $c$:

***Proposition 1.1.*** — *Given any $a \in [a_1, c]$,*

(1) *the interval $[-a, a]$ is forward invariant and $\varphi|[-a, a]$ is transitive*
(2) $|(\varphi_a^n)'(x)| \geq \min\{\sqrt{2}, |\varphi_a'(x)|\}(\sqrt{2})^{n-1}$ *for all $x \in [-a, a]$ such that $\varphi_a^j(x) \neq 0$ for every $j = 0, 1, \ldots, n-1$.*

After the bifurcation $a = c$ such uniform expansivity is clearly impossible, due to the presence of the critical point in the domain of the map. However, our main result states that – nonuniform – expansivity persists, in a measure-theoretical sense, and is even the prevalent form of dynamics after the bifurcation. We denote $c_j(a) = \varphi_a^j(c)$ for each $j \geq 1$.

***Theorem.*** — *Let $\{\varphi_a\}$ be a Lorenz-like family satisfying conditions L1-L5. Then there are $\sigma > 0$ and $\mathcal{A}^+ \subset \mathbb{R}$ such that $|(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j}$ for all $a \in \mathcal{A}^+$ and $j \geq 1$ and*

$$\lim_{\varepsilon \to 0} \frac{m(\mathcal{A}^+ \cap [c, c+\varepsilon])}{\varepsilon} = 1 \qquad (m = \text{Lebesgue measure on } \mathbb{R}).$$

*Moreover, there is $\sigma_1 > 0$ such that if $a \in \mathcal{A}^+$ then for $m$-almost all $x \in [-a, a]$ we have $\limsup \frac{1}{n}|\log (\varphi_a^n)'(x)| \geq \sigma_1$ as $n \to \infty$.*

Measure theoretic persistence of positive Lyapunov exponents (outside the class of uniformly expanding maps) was first proved by Jakobson **[9]**, for maps in the quadratic family $f_a(x) = 1 - ax^2$ close to parameter values $\bar{a}$ satisfying (**[12]**)

$$\inf_{j \in \mathbb{N}} \left| f_{\bar{a}}^j(c) - c \right| > 0 \qquad (c = \text{critical point} = 0). \tag{2}$$

There exist today many proofs of this theorem, *e.g.* **[4]**, **[2]**, as well as generalizations to families of smooth maps with finitely many critical points **[16]**, and to families of maps in which a single discontinuity coincides with the critical point **[14]**. A number of differences should be pointed out in this setting, between smooth maps and our Lorenz-like maps.

While all proofs of Jakobson's theorem in the smooth context rely in one way or the other on the nonrecurrence condition (2), here we need no assumption on the orbits of the critical points for $a = c$. Instead, we simply take advantage of the strong expansivity estimates given by Proposition 1.1 for that parameter value.

Various technical complications arise in the present situation from the existence of discontinuities and of regions where the derivative has arbitrarily large norm. Several estimates (including distortion bounds), which in the smooth case rely on the boundedness and Lipschitz continuity of the derivative, now require nontrivial reformulations together with a detailed study of the recurrence near the discontinuity (and not only near the critical points).

Lorenz-like families with criticalities undergo codimension-one bifurcations which mark a direct transition from uniformly expanding dynamics (for $a < c$), to nonuniformly expanding dynamics (for $a \in \mathcal{A}^+$), a kind of bifurcation which does not seem to be known in the smooth one-dimensional context. The fact that the bifurcation parameter $a = c$ is a Lebesgue density point for $\mathcal{A}^+$ is related to the strong form of expansivity exhibited by $\varphi_c$. An interesting question is whether some characterization of the density points of $\mathcal{A}^+$ can be given in terms of special hyperbolicity features of the corresponding maps (*e.g.* uniformly hyperbolic structure on periodic orbits ?).

We remark here that the symmetry inherent in our definition of Lorenz-like maps, though partly justified by the symmetry which exists in Lorenz' system of equations, is not strictly necessary for the proof of the theorem. We carry out the proof in the symmetric case in order to simplify the exposition (in particular we shall often discuss some construction or result with explicit reference to only one of the critical points with the implicit understanding that the same statements apply to the other one as well, by symmetry) but all the arguments hold, up to minor modifications, in a nonsymmetric setting.

Closing this section, we observe that the dynamics of the Lorenz flows in the parameter range we want to consider cannot be expected to fully reduce to that of one-dimensional maps (as happens for the geometric models of the Lorenz attractor mentioned previously). Indeed, the very phenomenon of "folding" which we want to encompass in our description, is also an obstruction to the existence of invariant foliations transverse to the flow. Nevertheless, drawing on the results obtained in this article we are developing, in a forthcoming paper by the same authors **[10]**, a natural extension of those geometric models to this wider range of parameters. Such Lorenz-like flows are amongst the simplest systems in which behaviour arising from the presence of equilibria interacts with dynamical features related to the presence of criticalities (homoclinic and heteroclinic tangencies). The understanding of the bifurcations taking place in this model is probably a necessary step towards a global description of the dynamics of flows, in the spirit of the program proposed a few years ago by Palis, see **[13]**.

The proof of our main result is organized as follows. In Section 2 we identify a pair of conditions on the parameter $a$ which ensure that $a \in \mathcal{A}^+$. Sections 3 and 4 are then devoted to showing that the set of parameters for which such conditions are satisfied is large in the sense of the statement of the theorem. The whole global approach is inspired on **[3]**.

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## 2. Positive Lyapunov Exponents

We begin by proving Proposition 1.1. In doing this we focus only on $a \in [a_2, c]$: the case $a < a_2$ corresponds to the situation in **[6]**, and it also follows from (simpler versions of) these same arguments.

**2.1. Proof of Proposition 1.1.** — The invariance of $[-a,a]$ is an immediate consequence of $\lim_{x\to 0}|\varphi_a(x)| = a$, $|\varphi_a(\pm a)| < x_{\sqrt{2}} \le a$ (recall L4), and the monotonicity of $\varphi_a$ on $(-a,0)$ and $(0,a)$.

Next, let $x$ and $1 \le j \le n-1$ be as in part (2). If $|\varphi_a^j(x)| \le x_{\sqrt{2}}$ then we have $|\varphi_a'(\varphi_a^j(x))| \ge \sqrt{2}$. If $|\varphi_a^j(x)| > x_{\sqrt{2}}$ then, by L4, there exists a unique $z \in [-c,c]$ such that $\varphi_c(z) = \varphi_a^j(x) = \varphi_a(\varphi_a^{j-1}(x))$. Moreover, $z$ and $\varphi_a^{j-1}(x)$ have the same sign (opposite to that of $\varphi_a^j(x)$) and $|z| \ge |\varphi_a^{j-1}(x)|$, because $a \le c$. Using L5 we get

$$\begin{aligned}|(\varphi_a^2)'(\varphi_a^{j-1}(x))| =& |\varphi'(\varphi_a^{j-1}(x))\,\varphi'(\varphi_a^j(x))| \\ \ge& |\varphi'(z)\,\varphi'(\varphi_a^j(x))| = |(\varphi_c^2)'(z)| > 2.\end{aligned}$$

Part (2) follows directly from these remarks and it is easy to see that one even gets a somewhat better bound, with $\sqrt{2}$ replaced by some slightly larger constant $\theta$.

To prove transitivity, we let $U_0, V_0 \subset [-a,a]$ be arbitrary open sets and show that $\varphi_a^n(U_0) \cap V_0 \ne \varnothing$ for some $n > 0$. Suppose, without loss of generality, that $0 \notin U_0$ and $U_0 \subset (-x_{\sqrt{2}}, x_{\sqrt{2}})$ (recall L4). As long as $0 \notin \varphi_a^j(U_0)$, write $U_j = \varphi_a^j(U_0)$ and notice that $|U_j| \ge \theta^j |U_0|$. Thus we must have $0 \in \varphi_a(U_{k_1-1})$ for some $k_1 \ge 1$. Let $U_{k_1}$ denote the largest connected component of $\varphi_a(U_{k_1-1}) \setminus \{0\}$ and observe that $|U_{k_1}| \ge \frac{1}{2}|\varphi_a(U_{k_1-1})| \ge \frac{1}{2}\theta^{k_1}|U_0|$. Suppose first that $U_{k_1} \subset (z^-, z^+)$, where $z^- < 0 < z^+$ are the preimages of zero under $\varphi_a$; observe that $|z^\pm| < x_{\sqrt{2}}$ as a consequence of the first inequality in L4. Then we proceed as before, with $U_0$ replaced by $U_{k_1}$. More precisely, we define $U_{k_1+j} = \varphi_a^j(U_{k_1})$ until the first iterate $k_2 > k_1$ for which $0 \in \varphi_a(U_{k_2-1})$; at that point we take $U_{k_2}$ to be the largest component of $\varphi_a(U_{k_2-1})$ and repeat the whole procedure again. As long as $U_{k_i} \subset (z^-, z^+)$ we have $k_{i+1} \ge k_i + 2$, hence

$$|U_{k_{i+1}}| \ge |\varphi_a(U_{k_i-1})|/2 \ge \theta^{k_{i+1}-k_i}|U_{k_i}|/2 \ge \theta^2|U_{k_i}|/2$$

grows exponentially with $i$. Thus, one eventually reaches some $k = k_j$ for which $U_k$ contains either $(z^-,0)$ or $(0,z^+)$. In the first case $\varphi_a(U_k)$ contains $(0,a) \supset (0,z^+)$ and then $\varphi_a^2(U_k)$ contains $(-a,0)$, which ensures that either $\varphi_a(U_k)$ or $\varphi_a^2(U_k)$ intersect $V_0$. The second case is entirely analogous so the proof of the proposition is complete.

Now we fix a number of constants to be used in the sequel of our argument. Recall that $0 < \lambda < 1/2$. We take $\sigma_0 > 0$ and $\sigma > 0$ such that $0 < 2\sigma < \sigma_0 < \log\sqrt{2}$ and also choose

$$\gamma > 1 \text{ and } \delta, \iota > 0 \text{ such that } 1 < \gamma + \delta + \iota < 1/2\lambda.$$

We will be choosing $\delta$ small with respect to $\lambda$ and $\iota$ small with respect to $\delta$. We remark for future reference that this implies $\gamma + \delta + \iota < 1/\lambda - 1$. Then, we let $0 < \alpha < \beta$ be small, depending on the previous constants (the precise conditions are stated throughout the proof wherever they are required).

By conditions L1-L3 there exist $\eta_1, \eta_2 > 0$ such that

$$\lim_{x\to 0} \frac{|\varphi(x)|}{|x|^\lambda} = \eta_1 \qquad \text{and} \qquad \lim_{x\to c} \frac{|\varphi(x) - \varphi(c)|}{|x-c|^2} = \eta_2.$$

For each $i = 1, 2$, we fix constants $\eta_i^- = \eta_i - \upsilon$ and $\eta_i^+ = \eta_i + \upsilon$, where $\upsilon$ is some small positive number (once more, precise conditions are to be stated along the way). Then we have

**M1 :** for all $x \neq 0$ close enough to the origin,

$$\begin{aligned} \eta_1^-|x|^\lambda \le \varphi_a(x) + a \le \eta_1^+|x|^\lambda \qquad & \text{if } x > 0,\\ -\eta_1^+|x|^\lambda \le \varphi_a(x) - a \le -\eta_1^-|x|^\lambda \qquad & \text{if } x < 0,\\ \text{and} \qquad \eta_1^-\lambda|x|^{\lambda-1} \le |\varphi_a'(x)| \le \eta_1^+\lambda|x|^{\lambda-1}; & \end{aligned}$$

**M2 :** for all $x$ close enough to the critical point $c$

$$\begin{aligned} \eta_2^-(x-c)^2 \le |\varphi_a(x) - \varphi_a(c)| \le \eta_2^+(x-c)^2 \\ \text{and} \qquad 2\eta_2^-|x-c| \le |\varphi_a'(x)| \le 2\eta_2^+|x-c| \end{aligned}$$

and a similar fact holds for all $x$ close enough to $-c$.

Now, for each small $\varepsilon > 0$ we let $\Delta_+^0$, $\Delta_+^c$, $\Delta_+^{-c}$ denote the $\varepsilon^\gamma$- neighbourhoods of the origin and of the critical points $c$ and $-c$, respectively. We define partitions of $\Delta_+^0$ and $\Delta_+^{\pm c}$ by writing $I_r = [\varepsilon^\gamma e^{-r}, \varepsilon^\gamma e^{-r+1})$ and

$$\Delta_+^0 = \{0\} \cup \bigcup_{|r|\ge 1} I_r^0 \qquad \text{and} \qquad \Delta_+^{\pm c} = \{\pm c\} \cup \bigcup_{|r|\ge 1} I_r^{\pm c},$$

where $I_r^0 = I_r$ and $I_{-r}^0 = -I_r$, for each $r \ge 1$, and the $I_r^{\pm c} = I_r^0 \pm c$ are simply the translates of the $I_r^0$. We shall always assume that $\varepsilon > 0$ is small enough so that $\Delta_+^0$ and $\Delta_+^{\pm c}$ are contained in the regions for which M1 and M2 are valid. Moreover, we let $r_\varepsilon = [\delta \log \varepsilon^{-1}]$ (here $[x]$ is the integer part of $x$) and we consider restricted neighbourhoods

$$\Delta^0 = \{0\} \cup \bigcup_{|r|\ge r_\varepsilon+1} I_r^0 \qquad \text{and} \qquad \Delta^{\pm c} = \{\pm c\} \cup \bigcup_{|r|\ge r_\varepsilon+1} I_r^{\pm c}$$

(of radius $\approx \varepsilon^{\gamma+\delta}$) of the origin and the critical points. We shall also need an even smaller neighbourhood (of radius $\approx \varepsilon^{2(\gamma+\delta)+\iota}$) of the origin. So let

$$\Delta_{r_s}^0 = \{0\} \cup \bigcup_{|r|\ge r_s+1} I_r^0, \qquad r_s = [(\gamma + 2\delta + \iota)\log 1/\varepsilon].$$

We shall prove below that the preimages of the critical neighbourhoods $\Delta_+^{\pm c}$ are always contained in this smallest neighbourhood of the origin, *i.e.* $\varphi_a^{-1}(\Delta_+^{\pm c}) \subset \Delta_{r_s}^0$.

**2.2. Breaking the hyperbolic structure.** — The loss of expansivity occurring after the bifurcation $a = c$ and caused by the critical points entering the domain of the map is, in some sense, local: for $c \leq a \leq c + \varepsilon$, it occurs only in a neighbourhood of the critical points of size $\varepsilon^{\gamma+\delta} \ll \varepsilon$. More precisely, any piece of orbit that does not intersect $\Delta^{\pm c}$ has an exponentially growing derivative. Proving this fact requires two preliminary lemmas. First we determine the position and size of the preimage of $\Delta^{\pm c}$ for a convenient range of parameter values. Then we estimate the accumulated derivative of points which pass close to the discontinuity or to the critical points. In all that follows we write $\rho = 2^{-\lambda}$.

***Lemma 2.1.*** — *If $\varepsilon > 0$ is sufficiently small then $(\varepsilon/\eta_1^+)^{1/\lambda} \leq |\varphi_{c+\varepsilon}^{-1}(\pm c)| \leq (\varepsilon/\eta_1^-)^{1/\lambda}$ and*

$$\frac{1}{e} \leq \frac{|\varphi_a^{-1}(y)|}{|\varphi_{c+\varepsilon}^{-1}(\pm c)|} \leq e$$

*for every $y \in \Delta_+^{\pm c}$ and $a \in [c + \rho\varepsilon, c + \varepsilon]$.*

*Proof.* — By symmetry it suffices to consider the preimages of points in $\Delta_+^c$. Using the second inequality in M1,

$$-\eta_1^+ |\varphi_{c+\varepsilon}^{-1}(c)|^\lambda \leq c - (c+\varepsilon) \leq -\eta_1^- |\varphi_{c+\varepsilon}^{-1}(c)|^\lambda,$$

which immediately gives the first claim. To prove the second one notice that, for any $y$ and $a$ as in the statement, $y - a = (c - a) - (c - y) \in [-\varepsilon - \varepsilon^\gamma, -\rho\varepsilon + \varepsilon^\gamma]$ and so, using M1 in the same way as before,

$$\left(\rho\varepsilon - \frac{\varepsilon^\gamma}{\eta_1^+}\right)^{1/\lambda} \leq |\varphi_a^{-1}(y)| \leq \left(\frac{\varepsilon + \varepsilon^\gamma}{\eta_1^-}\right)^{1/\lambda}.$$

Combining with the first part of the present lemma, we get

$$\left(\frac{\eta_1^-}{\eta_1^+}\frac{\rho\varepsilon - \varepsilon^\gamma}{\varepsilon}\right)^{1/\lambda} \leq \left|\frac{\varphi_a^{-1}(y)}{|\varphi_{c+\varepsilon}^{-1}(c)}\right| \leq \left(\frac{\eta_1^+}{\eta_1^-}\frac{\varepsilon + \varepsilon^\gamma}{\varepsilon}\right)^{1/\lambda}.$$

The left hand side is close to 1/2, and hence larger than $1/e$, if $\varepsilon$ is small (and $v$ has been fixed sufficiently small, recall the definition of $\eta_i^\pm$). Analogously, the right hand side is smaller than $e$ if $\varepsilon$ and $v$ are small enough. The proof is complete. □

Now we define $r_c = r_c(\varepsilon) \geq 1$ by the condition $\varphi_{c+\varepsilon}^{-1}(c) \in I_{-r_c}^0$. Observe that

$$fracle\left(\frac{\varepsilon}{\eta_1^+}\right)^{1/\lambda} \leq \varepsilon^\gamma e^{-r_c} \leq \left(\frac{\varepsilon}{\eta_1^-}\right)^{1/\lambda} \tag{3}$$

by the first part of the previous lemma. Moreover, the second part gives

$$\varphi_a^{-1}(\Delta_+^c) \subset I_{-r_c+1}^0 \cup I_{-r_c}^0 \cup I_{-r_c-1}^0, \quad \text{for every } a \in [c + \rho\varepsilon, c + \varepsilon]. \tag{4}$$

Notice that we have from (3) $e^{-r_c} \leq \varepsilon^{1/\lambda - \gamma}(\eta_1^-)^{1/\lambda}$ which yields

$$r_c \geq (1/\lambda - \gamma)\log 1/\varepsilon + 1/\lambda \log \eta_1^- > (\gamma + 2\delta + \iota)\log 1/\varepsilon$$

if $\varepsilon$ is small enough. This shows, as promised above, that $\varphi_a^{-1}(\Delta_+^{\pm c}) \subset \Delta_{r_s}^0$.

***Lemma 2.2***. — *For every $a \in [c+\rho\varepsilon, c+\varepsilon]$ and $x \in I_r^0$, $|r| \geq 1$,*

(1) *if $\varphi_a(x) \notin \Delta^{\pm c}$ then $|(\varphi_a^2)'(x)| \geq \mathrm{const}(\varepsilon^\gamma e^{-r})^{2\lambda-1} \geq e^{2\sigma_0}e^{\beta|r|}$;*
(2) *if $\varphi_a(x) \in \Delta_+^{\pm c}$, with $\varphi_a(x) \in I_{\widetilde{r}}^{\pm c}$,*

*then*

$$|\varphi_a'(x)| \geq \mathrm{const}(\varepsilon^\gamma e^{-r})^{\lambda-1} > e^{\sigma_0}e^{\beta|r|}$$

*and*

$$|(\varphi_a^2)'(x)| \geq \varepsilon^{\gamma+1-1/\lambda}e^{-|\widetilde{r}|} > e^{2\sigma_0}e^{-|\widetilde{r}|}.$$

*Proof*. — We consider $r, \widetilde{r} \geq 1$, the other cases being entirely analogous. For the sake of clearness let us split the proof of (1) into three different cases.

Suppose first that $r < r_c - 2$. Then, in view of (4), $|x - \varphi_a^{-1}(\pm c)| \geq |I_{r+1}| = (1-1/e)\varepsilon^\gamma e^{-r}$. Thus, by the mean value theorem,

$$\begin{aligned}|\varphi_a(x) \pm c| &\geq |x - \varphi_a^{-1}(\pm c)| \cdot \inf\{\varphi'(z) : z \in [x, \varphi_a^{-1}(c)]\}\\ &\geq (1-1/e)\varepsilon^\gamma e^{-r}\eta_1^-\lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1}\\ &\geq k_1\varepsilon^{\gamma\lambda}e^{-r\lambda},\end{aligned}$$

with $k_1 = (1-1/e)\eta_1^-\lambda e^{\lambda-1}$. It follows, using M1, M2,

$$\begin{aligned}|(\varphi_a^2)'(x)| &\geq |\varphi_a'(x)||\varphi_a'(\varphi_a(x))| && \geq \eta_1^-\lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1}2\eta_2^-k_1\varepsilon^{\gamma\lambda}e^{-r\lambda}\\ &\geq k_2\varepsilon^{-\gamma(1-2\lambda)}e^{r(1-2\lambda)} && \geq e^{2\sigma_0}e^{\beta r},\end{aligned}$$

where $k_2 = 2\eta_1^-\lambda e^{\lambda-1}\eta_2^-k_1$ and, for the last inequality, we suppose $\beta < 1-2\lambda$ and $\varepsilon$ sufficiently small.

Clearly, exactly the same argument works for $|r| \geq r_c - 2$ where we have even greater expansion.

Finally, suppose that $\varphi_a(x) \in I_{\widetilde{r}}^{\pm c} \subset \Delta_+^{\pm c}$. Clearly,

$$|\varphi_a'(x)| \geq \eta_1^-\lambda(\varepsilon^\gamma e^{-r+1})^{\lambda-1} \geq \eta_1^-\lambda e^{\lambda-1}\varepsilon^{-\gamma(1-\lambda)}e^{r(1-\lambda)} \geq e^{\sigma_0}e^{\beta r}$$

if $\beta$ and $\varepsilon$ are small. Moreover, by (4), $\varepsilon^\gamma e^{-r_c-1} \leq |x| \leq \varepsilon^\gamma e^{-r_c+2}$, which gives

$$\begin{aligned}|(\varphi_a^2)'(x)| &\geq |\varphi_a'(x)||\varphi_a'(\varphi_a(x))| && \geq \eta_1^-\lambda(\varepsilon^\gamma e^{-r_c+2})^{\lambda-1}2\eta_2^-\varepsilon^\gamma e^{-\widetilde{r}}\\ &\geq k_5\varepsilon^{\gamma\lambda}e^{(1-\lambda)r_c}e^{-\widetilde{r}} && \geq k_5\varepsilon^{\gamma\lambda+(1-\lambda)(\gamma-1/\lambda)}(\eta_1^-)^{(1/\lambda)-1}e^{-\widetilde{r}}\\ &\geq k_6\varepsilon^{\gamma-(1/\lambda)+1}e^{-\widetilde{r}} && \geq e^{2\sigma_0}e^{-\widetilde{r}},\end{aligned}$$

where $k_5 = 2\eta_1^-\lambda e^{2(\lambda-1)}\eta_2^-$ and $k_6 = k_5(\eta_1^-)^{(1/\lambda)-1}$ and we use the relation (3) in the fourth inequality. □

***Lemma 2.3***. — *For any $a \in [c+\rho\varepsilon, c+\varepsilon]$ and $x \in [-a, a]$,*

(1) *if $\{\varphi_a^j(x)\}_{j=0}^{n-1} \cap \Delta^{\pm c} = \varnothing$ then $|(\varphi_a^n)'(x)| \geq \min\{e^{\sigma_0}, |\varphi_a'(x)|\}e^{\sigma_0(n-1)}$;*
(2) *if, in addition, $\varphi_a^n(x) \in \Delta_+^{\pm c}$ then $|(\varphi_a^n)'(x)| \geq e^{\sigma_0 n}$.*

*Proof.* — Denote $x_j = \varphi_a^j(x)$, for $0 \le j \le n-1$. We claim that given any $j \ge 1$ either $|\varphi_a'(x_j)| \ge e^{\sigma_0}$ or $|(\varphi_a^2)'(x_{j-1})| \ge e^{2\sigma_0}$. This is obvious if $|x_j| \le x_{\sqrt{2}}$, because we get $|\varphi_a'(x_j)| = |\varphi'(x_j)| > e^{\sigma_0}$. From now on we consider $x_j \ge x_{\sqrt{2}}$, the case $x_j \le -x_{\sqrt{2}}$ being entirely analogous. If $x_{j-1} \in \Delta_+^0$ then $|(\varphi_a^2)'(x_{j-1})| \ge e^{2\sigma_0}$ by part (1) of the previous lemma. Therefore, we may suppose $x_{j-1} \notin \Delta_+^0$, that is $|x_{j-1}| \ge \varepsilon^\gamma$. Then, recall M1, M2, $c - \varphi_c(x_{j-1}) \ge \eta_1^- \varepsilon^{\gamma\lambda}$ and so $|\varphi'(\varphi_c(x_{j-1}))| \ge 2\eta_1^-\eta_2^-\varepsilon^{\gamma\lambda}$. Hence, using also $\varphi_a(x_{j-1}) - \varphi_c(x_{j-1}) = a - c \le \varepsilon$, we get

$$\begin{aligned}\frac{|(\varphi_a^2)'(x_{j-1})|}{|(\varphi_c^2)'(x_{j-1})|} &= \frac{|\varphi'(\varphi_a(x_{j-1}))|}{|\varphi'(\varphi_c(x_{j-1}))|}\\ &\ge 1 - \frac{|\varphi'(\varphi_a(x_{j-1})) - \varphi'(\varphi_c(x_{j-1}))|}{|\varphi'(\varphi_c(x_{j-1}))|} \ge 1 - k_7\varepsilon^{1-\lambda\gamma},\end{aligned}$$

where $k_7 = k/(2\eta_1^-\eta_2^-)$, with $k$ a Lipschitz constant for $\varphi'$ on $\{x \ge x_{\sqrt{2}} - \varepsilon_0\}$ ($\varepsilon_0$ is some small constant, we take $\varepsilon \le \varepsilon_0$). Since $1 - \lambda\gamma > 0$, the left hand term is larger that $e^{2\sigma_0}/2$ if $\varepsilon$ is small enough and then the claim follows from L5. Moreover, the first statement in the lemma is a direct consequence of our claim (*cf.* the proof of lemma 2.1).

In order to deduce the second part of the lemma we may suppose $|\varphi_a'(x)| \le e^{\sigma_0}$, for otherwise there is nothing to prove. Observe also that if $\varphi_a^n(x) \in \Delta_+^{\pm c}$ then, by (3), (4), we have $|\varphi_a'(\varphi_a^{n-1}(x))| \ge k_8\varepsilon^{1-1/\lambda}$, with $k_8 = \lambda e^{2(\lambda-1)}(\eta_1^-)^1/\lambda$.

Moreover, by hypothesis, $x \notin \Delta^{\pm c}$ and so $|\varphi_a'(x)| \ge \eta_1^-\lambda\varepsilon^{\gamma+\delta}$. Altogether, writing $k_9 = \eta_1^-\lambda k_8$,

$$\begin{aligned}|(\varphi_a^n)'(x)| &\ge |\varphi_a'(x)|e^{\sigma_0(n-2)}|\varphi'(\varphi_a^{n-1}(x))| &&\ge k_9\varepsilon^{\gamma+\delta}e^{\sigma_0(n-2)}\varepsilon^{1-1/\lambda}\\ &\ge k_9\varepsilon^{\gamma+\delta+1-1/\lambda}e^{\sigma_0(n-2)} &&\ge e^{\sigma_0 n},\end{aligned}$$

if $\varepsilon$ is small enough. □

**2.3. Recovering expansion.** — Now we deal with the expansion losses occurring when trajectories pass close to some of the critical points $\pm c$. More precisely, we consider points $x \in \Delta^{\pm c}$. Assuming that the critical trajectories satisfy (exponential) expansivity and bounded recurrence conditions (during a convenient number of iterates, depending on $|x \pm c|$), we show that the small value of $\varphi_a'(x)$ is fully compensated in the subsequent iterates, during which the trajectory of $x$ remains close to that of the critical point (and so exhibits rapidly increasing derivative).

For each $j \ge 0$ let $c_j = c_j(a) = \varphi_a^j(\pm c)$ and denote $d(c_j) = \min\{|c_j|, |c_j \pm c|\}$. In what follows $\varepsilon > 0$ is fixed and we suppose $a \in [c + \rho\varepsilon, c + \varepsilon]$.

***Lemma 2.4.*** — *There exists $\theta = \theta(\beta - \alpha) > 0$ such that the following estimates hold. Let $x \in I_r^{\pm c}$ for some $|r| \ge r_\varepsilon$. Suppose that there is $n \ge |r|/\alpha$ such that*

$$(5) \qquad d(c_j) \ge \varepsilon^\gamma e^{-\alpha j} \quad \textit{and} \quad |(\varphi_a^j)'(c_1)| \ge e^{\sigma j}, \quad \textit{for all } 1 \le j \le n-1.$$

*Then there exists an integer $p = p(x) \ge 1$ such that*

(1) *For all* $y_1, z_1 \in [\varphi_a(x), \varphi_a(\pm c)]$ *and for all* $1 \leq k \leq p$,

$$\frac{1}{\theta} \leq \frac{|(\varphi_a^k)'(z_1)|}{|(\varphi_a^k)'(y_1)|} \leq \theta$$

(2) $p \leq (2|r| + \frac{3}{2}\gamma \log 1/\varepsilon)/\sigma \leq n-1$;

(3) (a) $|(\varphi_a^{p+1})'(x)| \geq \varepsilon^{2\beta\gamma/\sigma} e^{(1-2\beta/\sigma)r}$;

(b) $|(\varphi_a^{p+1})'(x)| \geq \varepsilon^{-\beta/\sigma} e^{|r|-(\alpha n/2)}$;

(c) $|(\varphi_a^{p+1})'(x)| \geq e^{\sigma_0 + \beta p} \geq e^{\beta(p+1)}$.

*Proof*. — We suppose $x \in I_r^c$ with $r \geq 1$, the remaining cases being treated in precisely the same way. Define $p = p(x) \geq 1$ as the maximum integer such that

$$|x_i - c_i| \leq \varepsilon^\gamma e^{-\beta i} \quad \text{for all } 1 \leq i \leq p \tag{6}$$

where $x_i = \varphi_a^i(x)$. Recall that we fix $\beta > \alpha$. Therefore, (6) and the first condition in (5) ensure that the intervals $[x_i, c_i]$, $1 \leq i \leq p$, do not contain the origin nor any of the critical points $\pm c$. Therefore, $\varphi_a^i : [x_1, c_1] \to [x_{i+1}, c_{i+1}]$ is a diffeomorphism for all $1 \leq i \leq p$. In particular, given any $y_1, z_1 \in [x_1, c_1]$ we have $y_i, z_i \in [x_i, c_i]$ for $1 \leq i \leq p$, where $y_i = \varphi_a^i(y)$ and $z_i = \varphi_a^i(z)$. By the chain rule,

$$\left|\frac{(\varphi_a^k)'(z_1)}{(\varphi_a^k)'(y_1)}\right| = \prod_{i=1}^{k} \left|\frac{\varphi_a'(z_i)}{\varphi_a'(y_i)}\right| = \prod_{i=1}^{k} \left|1 + \frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right|$$

and so part (1) will follow if we show that

$$\sum_{i=1}^{k} \left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| \tag{7}$$

is bounded by some constant depending only on $\beta - \alpha$. By the mean value theorem there exists, for each $1 \leq i \leq k$ some $\xi_i \in [z_i, y_i]$ s.t.

$$\left|\frac{\varphi_a'(z_i) - \varphi_a'(y_i)}{\varphi_a'(y_i)}\right| = \left|\frac{|z_i - y_i|\varphi_a''(\xi_i)}{\varphi_a'(y_i)}\right| \leq \varepsilon^\gamma e^{-\beta i} \left|\frac{\varphi_a''(\xi_i)}{\varphi_a'(y_i)}\right|.$$

Thus it is sufficient to show that $|\varphi_a''(\xi_i)/\varphi_a'(y_i)| \leq \text{const}\, \varepsilon^{-\gamma} e^{\alpha i}$ to conclude that the terms of the sum (7) are decreasing exponentially and so the entire sum is bounded by a constant independent of $k$. We fix some small constant $\varepsilon' > 0$ independent of $\varepsilon$. The norm of $\varphi_a''(x)$ is bounded above and below outside $(-\varepsilon', \varepsilon')$ by some constant $C = \sup\{|\varphi_a''(x)| : x \notin (-\varepsilon', \varepsilon')\}$. For simplicity, and without loss of generality, we shall assume that this supremum is actually achieved at $\varepsilon'$. Inside $(-\varepsilon', \varepsilon')$ we have by the form of the map $\varphi$ that $|\varphi_a''(x)| \leq \eta^+ \lambda(\lambda - 1)|x|^{\lambda-2}$.

We distinguish two cases. If $[x_i, c_i] \cap (-\varepsilon', \varepsilon') = \varnothing$ then we have

$$|\varphi_a'(y_i)| \geq 2\eta_2^- |y_i - c| \geq 2\eta_2^- \varepsilon^\gamma(\varepsilon^{-\alpha i} - e^{-\beta i}) \geq 2\eta_2^-(1 - e^{\alpha-\beta})\varepsilon^\gamma e^{-\alpha i}$$

and so

$$\left|\frac{\varphi_a''(\xi_i)}{\varphi_a'(y_i)}\right| \leq \frac{C}{2\eta_2^-(1 - e^{\alpha-\beta})}\varepsilon^{-\gamma} e^{\alpha i}$$

as desired. If $[x_i, c_i] \cap (-\varepsilon', \varepsilon') \neq \varnothing$ we have the following estimates. To simplify the notation we shall suppose that $[x_i, c_i] \subset (0, c)$. The other case $[x_i, c_i] \subset (-c, 0)$ is dealt with similarly. Taking $\epsilon'$ small and since $|x_i - c_i| \leq \varepsilon^\gamma$ we can suppose that $[x_i, c_i]$ is contained in the neighbourhood of 0 for which conditions M1 and M2 hold. We have

$$|\varphi_a'(y_i)| \geq |\varphi_a'(c_i + \varepsilon^\gamma e^{-\beta i})| \geq \eta_1^- \lambda (c_i + \varepsilon^\gamma e^{-\beta i})^{\lambda - 1}$$

and

$$|\varphi_a''(\xi_i)| \leq |\varphi_a''(c_i - \varepsilon^\gamma e^{-\beta i})| \leq \eta_1^+ \lambda(\lambda - 1)(c_i - \varepsilon^\gamma e^{-\beta i})^{\lambda - 2}.$$

This gives

$$\left|\frac{\varphi_a''(\xi_i)}{\varphi_a'(y_i)}\right| \leq \frac{\eta_1^+(\lambda - 1)}{\eta_1^-} \left(\frac{c_i - \varepsilon^\gamma e^{-\beta i}}{c_i + \varepsilon^\gamma e^{-\beta i}}\right)^{\lambda - 1} (c_i - \varepsilon^\gamma e^{-\beta i})^{-1} \leq \operatorname{const}(\varepsilon^\gamma e^{-\alpha i})^{-1}.$$

This follows from the fact that $|c_i| \geq \varepsilon^\gamma e^{-\alpha i}$ and therefore

$$(c_i - \varepsilon^\gamma e^{-\beta i})^{-1} \leq (\varepsilon^\gamma e^{-\alpha i})^{-1}(1 - e^{(\alpha - \beta)i})^{-1} \leq \operatorname{const}(\varepsilon^\gamma e^{-\alpha i})^{-1}$$

and that $(c_i - \varepsilon^\gamma e^{-\beta i})/(c_i + \varepsilon^\gamma e^{-\beta i}) \leq$ const. Indeed this last fact follows from observing that

$$c_i - \varepsilon^\gamma e^{-\beta i} \geq \varepsilon^\gamma e^{-\alpha i} - \varepsilon^\gamma e^{-\beta i} \geq \varepsilon^\gamma e^{-\alpha i}(1 - e^{(\alpha - \beta)i}) \geq (1 - e^{\alpha - \beta})\varepsilon^\gamma e^{-\alpha i}$$

and similarly

$$c_i + \varepsilon^\gamma e^{-\beta i} \leq (1 + e^{\alpha - \beta})\varepsilon^\gamma e^{-\alpha i}$$

which together give

$$\frac{c_i - \varepsilon^\gamma e^{-\beta i}}{c_i + \varepsilon^\gamma e^{-\beta i}} \leq \frac{1 - e^{\alpha - \beta}}{1 + e^{\alpha - \beta}} = \operatorname{const}.$$

This proves (1).

Starting the proof of (2), let $q = \min\{p, n - 1\}$. As $x \in I_r^c$, we have $|x - c| \geq \varepsilon^\gamma e^{-r}$ and so $|x_1 - c_1| \geq \eta_2^- \varepsilon^{2\gamma} e^{-2r}$. Then, in view of the second condition in (5) and the distortion estimate we have just proved, the mean value theorem yields $\eta_2^- \varepsilon^{2\gamma} e^{-2r} \theta^{-1} e^{\sigma(q-1)} \leq |x_q - c_q| \leq \varepsilon^\gamma e^{-\beta q}$. Thus

$$q \leq \frac{2r + \gamma \log(1/\varepsilon) + \sigma - \log(\eta_2^-/\theta)}{\sigma + \beta} \leq \left(2r + \frac{3}{2}\gamma \log 1/\varepsilon\right)/\sigma$$

as long as $\varepsilon$ is sufficiently small. Since we also take $\alpha n \geq r \geq [\delta \log 1/\varepsilon] \gg 1$, we find that $q \leq (2\alpha n + 3\gamma\alpha n/2\delta)/\sigma < n$ (if $\alpha$ is small), so that it must be $q = p$. In this way we have proved that $p \leq (2r + \frac{3}{2}\gamma \log 1/\varepsilon)/\sigma < n$, as claimed in part (2) of the lemma.

Now, by the definition of $p$ we have $|x_{p+1} - c_{p+1}| \geq \varepsilon^\gamma e^{-\beta(p+1)}$. Thus, using part (1) in conjunction with the mean value theorem,

$$|(\varphi_a^p)'(x_1)| \geq \frac{1}{\theta}\frac{|x_{p+1} - c_{p+1}|}{|x_1 - c_1|} \geq \frac{\varepsilon^\gamma e^{-\beta(p+1)}}{\theta \eta_2^- \varepsilon^{2\gamma} e^{-2r+2}} \geq \operatorname{const} \varepsilon^{-\gamma} e^{2r - \beta p}.$$

Since $|\varphi_a'(x)| \geq 2\eta_2^- \varepsilon^\gamma e^{-r}$, we find

$$|(\varphi_a^{p+1})'(x)| \geq \operatorname{const} e^{r-\beta p}. \tag{8}$$

Using part (2) we immediately get

$$e^{r-\beta p} \geq e^{r-\frac{\beta}{\sigma}(2r+\frac{3}{2}\gamma\log 1/\varepsilon)} \geq e^{(1-\frac{2\beta}{\sigma})r-\frac{3\beta\gamma}{2\sigma}\log 1/\varepsilon} \geq \varepsilon^{\frac{3\beta\gamma}{2\sigma}} e^{(1-\frac{2\beta}{\sigma})r}.$$

This proves the first statement in part (3) since

$$|(\varphi_a^{p+1})'(x) \geq \operatorname{const} \varepsilon^{3\beta\gamma/2\sigma} e^{(1-2\beta/\sigma)r} \geq \varepsilon^{2\beta\gamma/\sigma} e^{(1-2\beta/\sigma)r}.$$

Using part (2) again together with $\alpha n \geq r \geq [\delta \log 1/\varepsilon] \gg 1$, we get

$$\beta p \leq \frac{2\beta}{\sigma}\Big(\alpha n + \frac{\gamma}{\delta}2\alpha n + \frac{1}{\delta}2\alpha n - \log 1/\varepsilon\Big) \leq \frac{4\beta}{\sigma\delta\lambda}\alpha n - \frac{2\beta}{\sigma}\log 1/\varepsilon \leq \frac{1}{2}\alpha n - \frac{2\beta}{\sigma}\log 1/\varepsilon$$

as long as we take $\beta < \sigma\delta\lambda/8$ (we also used $\delta + \gamma + 1 < 1/\lambda$). Replacing in (8), and supposing $\varepsilon$ sufficiently small, we get the second statement in part (3). Finally notice that $|r| \geq [\delta \log 1/\varepsilon]$ and therefore $p \leq 2/\sigma(r + \gamma \log 1/\varepsilon) \leq (1+\gamma/\delta)|r|$ and $|r| \geq p/(1+\gamma/\delta)$. therefore we have

$$|(\varphi_a^{p+1})'(x)| \geq \operatorname{const} e^{r-\beta p} \geq \operatorname{const} e^{(\frac{1}{1+\gamma/\delta}-\beta)p} \geq e^{\beta(p+1)}$$

if $\beta$ is small. This completes the proof of part (3) and of the lemma. □

**2.4. Proving positive Lyapunov exponents.** — We can now state the main results of this section, asserting that, under two convenient assumptions on the parameter $a$ to be stated below, the critical trajectories exhibit exponential growth of the derivative and, in fact, the same is true for most trajectories of $\varphi_a$.

As before, we write $c_j = c_j(a) = \varphi_a^j(c)$, for $j \geq 1$. For the time being we fix some $n \geq 1$ and assume that

**CP1**$(n)$ : $d(c_j) = \min\{|c_j|, |c_j \pm c|\} \geq \varepsilon^\gamma e^{-\alpha j}$ for all $1 \leq j \leq n$.

and

**EG**$(n-1)$ : $|(\varphi_a^j)'(c_1)| \geq e^{\sigma j}$ for all $1 \leq j \leq n-1$.

Then we define sequences of integers $\nu_i$, $p_i$, by $\nu_1 = \inf\{\nu \geq 1 : c_\nu \in \Delta^{\pm c}\}$ and

(i) $p_i = p(c_{\nu_i})$, as given by lemma 2.4;

(ii) $\nu_{i+1} = \inf\{\nu > \nu_i + p_i : c_\nu \in \Delta^{\pm c}\}$.

(CP1$(n)$ ensures that $c_{\nu_i} \in I_r^{\pm c}$ for some $|r| \leq \alpha\nu_i \leq \alpha n$). We take $s \geq 0$ maximum such that $\nu_s \leq n$. Then either $\nu_s \leq n < \nu_s + p_s$ or $\nu_s < \nu_s + p_s \leq n$. Now we define $P_n = p_1 + \cdots + p_{s-1}$ in the first case and $P_n = p_1 + \cdots + p_{s-1} + p_s$ in the second one. Then we further assume that

**CP2**$(n)$ : $P_j \leq j/2$ for all $1 \leq j \leq n$.

All iterates occurring during a binding period $[\nu_i + 1, \nu_i + p_i]$ are called *bound iterates.* All others (including returns $\nu_i$) are called *free* iterates.

***Lemma* 2.5.** — *Suppose that some parameter* $a \in [c+\rho\varepsilon, c+\varepsilon]$ *satisfies CP1(n), CP2(n), and EG(n − 1). Then it also satisfies*

$$\mathbf{EG}(n) \; : \; |(\varphi_a^j)'(c_1)| \geq e^{\sigma j} \text{ for all } 1 \leq j \leq n.$$

*Proof.* — We let $\nu_i$, $p_i$, be as above and define $q_0 = \nu_1 - 1$ and $q_i = \nu_{i+1} - (\nu_i + p_i + 1)$ for $1 \leq i \leq s-1$. If $n \geq \nu_s + p_s$ we also write $q_s = n - (\nu_s + p_s)$. Then

$$(9) \quad |(\varphi_a^n)'(c_1)| = |(\varphi_a^{q_0})'(c_1)| \prod_{i=1}^{s-1} (|(\varphi_a^{p_i+1})'(c_{\nu_i})||(\varphi_a^{q_i})'(c_{\nu_i+p_i+1})|) \, |(\varphi_a^{n-\nu_s+1})'(c_{\nu_s})|.$$

The first factor on the right can be estimated as follows. Since $\varphi_a(c_{q_0}) = c_{\nu_1} \in \Delta^{\pm c}$, relations (3) and (4) yield $|\varphi_a'(c_{q_0})| \geq \text{const}(\varepsilon^\gamma e^{-r_c})^{\lambda-1} \geq \text{const}\, \varepsilon^{(1/\lambda)-1}$. Hence, using also the first part of lemma 2.3,

$$|(\varphi_a^{q_0})'(c_1)| = |(\varphi_a^{q_0-1})'(c_1)||(\varphi_a)'(c_{q_0})| \geq \text{const}\, e^{\sigma_0 q_0} \varepsilon^{1-1/\lambda}$$

(note that the last inequality in L4 implies $|c_1| < x_{\sqrt{2}}$ and so $|\varphi_a'(c_1)| > \sqrt{2} > e^{\sigma_0}$ for all $a$ close to $c$). On the other hand, lemma 2.4(3) and lemma 2.3(2) give, for $1 \leq i \leq s-1$,

$$|(\varphi_a^{p_i+1})'(c_{\nu_i})| \geq e^{\sigma_0 + \beta p_i} \quad \text{and} \quad |(\varphi_a^{q_i})'(c_{\nu_i+p_i+1})| \geq e^{\sigma_0 q_i}.$$

For estimating the last factor in (9), we distinguish two cases. If $n \geq \nu_s + p_s$ then we use lemmas 2.4(3) and 2.3(1) once more and get

$$\begin{aligned} |(\varphi_a^{n-\nu_s+1})'(c_{\nu_s}) &= |(\varphi_a^{p_s+1})'(c_{\nu_s})| \cdot |(\varphi_a^{q_s})'(c_{\nu_s+p_s+1})| \\ &\geq e^{\sigma_0+\beta p_s} \min\{e^{\sigma_0}, |\varphi_a'(c_{\nu_s+p_s+1})|\} e^{\sigma_0(q_s-1)} \\ &\geq \text{const}\, e^{\sigma_0+\beta p_s} \varepsilon^{\gamma+\delta} e^{\sigma_0 q_s} \end{aligned}$$

(the final bound remains valid when $n = \nu_s + p_s$, *i.e.* $q_s = 0$). Replacing in (9),

$$(10) \qquad |(\varphi_a^n)'(c_1)| \geq \text{const}\, \varepsilon^{1-(1/\lambda)+\gamma+\delta} e^{(\sum_{i=0}^s \sigma_0 q_i + \sum_{i=1}^s (\sigma_0+\beta p_i))}.$$

Now, CP2($n$) implies (recall that we take $\sigma_0 > 2\sigma$)

$$\sum_{i=0}^s \sigma_0 q_i + \sum_{i=1}^s (\sigma_0 + \beta p_i) \geq \sigma_0(n - P_n) + \beta P_n \geq \sigma_0 \frac{n}{2} > \sigma n$$

and the lemma follows by replacing this in (10) and assuming $\varepsilon$ sufficiently small.

Suppose now that $\nu_s \leq n < \nu_s + p_s$. In this case we cannot take advantage of the estimates in lemma 2.4(3), as we did before. Instead, we use CP1($n$), EG($n-1$), and the distortion estimate in lemma 2.4(1), to conclude that

$$|(\varphi_a^{n-\nu_s+1})'(c_{\nu_s})| = |\varphi'(c_{\nu_s})| \cdot |(\varphi_a^{n-\nu_s})'(c_{\nu_s+1})| \geq \text{const}\, \varepsilon^\gamma e^{-\alpha\nu_s} \; \text{const}\, e^{\sigma(n-\nu_s)}.$$

This gives

$$(11) \qquad |(\varphi_a^n)'(c_1)| \geq \text{const}\, \varepsilon^{1-(1/\lambda)+\gamma} e^{(\sum_{i=0}^{s-1} \sigma_0 q_i + \sum_{i=1}^{s-1} (\sigma_0+\beta p_i) - \alpha\nu_s + \sigma(n-\nu_s))}.$$

Now,

$$\sum_{i=0}^{s-1}\sigma_0 q_i+\sum_{i=1}^{s-1}(\sigma_0+\beta p_i)-\alpha\nu_s+\sigma(n-\nu_s)$$
$$\geq \sigma_0(\nu_s-P_{\nu_s})-\alpha\nu_s+\sigma(n-\nu_s)\geq \frac{\sigma_0}{2}\nu_s-\alpha\nu_s+\sigma(n-\nu_s)\geq\sigma n$$

as long as we take $2\alpha<\sigma_0-2\sigma$. Replacing in (11) (and assuming $\varepsilon$ small) we get the conclusion of the lemma also in this case. Our argument is complete. □

***Proposition 2.6.*** — *Suppose that some parameter $a\in[c+\rho\varepsilon,c+\varepsilon]$ satisfies CP1(n) and CP2(n) for all $n\geq 1$. Then $|(\varphi_a^n)'(c_1)|\geq e^{\sigma n}$ for all $n\geq 1$. Moreover, there is $\sigma_1>0$ such that for any $x\in[-a,a]$ satisfying $\varphi_a^j(x)\notin\{0,\pm c\}$ for all $j\geq 0$ we have* $\limsup\frac{1}{n}\log|(\varphi_a^n)'(x)|\geq\sigma_1$.

*Proof.* — The first claim follows directly from the previous lemma, by induction on $n$. Observe that the step $n=1$ is an immediate consequence of L4 which, as we already remarked, implies $|c_1|<x_{\sqrt{2}}$ and so $|\varphi_a'(c_1)|>\sqrt{2}>e^{\sigma}$.

For the second statement we distinguish two cases according as to whether the orbit of $x$ accumulates one of the critical points or not. If it does not, the result follows immediately from the previous lemmas which guarantee that $|(\varphi_a^j)'(x)|\geq Ce^{\beta j}$, $\forall\, j\geq 0$ which immediately implies the result taking $\sigma_1<\beta$. If the orbit of $x$ does accumulate one of the critical points then we claim that for every $N>0$ there exists an $n\geq N$ such that $|(\varphi_a^n)'(x)|\geq e^{\beta n}$. This claim clearly implies the desired statement. Let $\mu_1<\mu_2<\cdots<\mu_k\leq N$ be all the returns of $x$ to $\Delta^{\pm c}$ before time $N$ and let $p_1,p_2,\ldots$ be the lengths of the corresponding binding periods. If $N\geq\mu_k+p_k$ then we have by the same arguments used in the proof of lemma 2.5 that $|(\varphi_a^N)'(x)|\geq e^{\beta N}$ which proves the claim in this case. If $N\in(\mu_k,p_k)$, just take $n=\mu_k+p_k+1$ and repeat the argument above. This completes the proof of the claim and of the proposition. □

***Remark 2.1.*** — A refinement of the previous arguments permits to show a stronger statement: $\liminf\frac{1}{n}\log|(\varphi_a^n)'(x)|\geq\sigma_2$ for some $\sigma_2>0$ and almost every point $x$. First one notes that this holds whenever $x$ satisfies

$$d(\varphi^j(x))=\min\{|\varphi^j(x)|,|\varphi^j(x)\pm c_j|\}\geq e^{-\alpha j}$$

for all $j\geq 0$, by using essentially the same argument as we did above for the critical orbits. Then, using the distortion bounds we have been deriving, one shows that for Lebesgue almost every point $y$ there is some $k\geq 0$ such that $x=\varphi^k(y)$ is as above.

Finally, we make the simple, yet useful observation.

***Lemma 2.7.*** — *Suppose that some parameter $a\in[c+\rho\varepsilon,c+\varepsilon]$ satisfies CP1(n) and CP2(n) for some $n\geq 1$. Let $1\leq\mu_1\leq\mu_2\leq n$ be free iterates for the orbit of c. Then*

$$|(\varphi_a^{\mu_2-\mu_1})'(c_{\mu_1}(a))|\geq\min\{|\varphi_a'(c_{\mu_1(a)})|,e^{\sigma_0}\}e^{\beta(\mu_2-\mu_1-1)}.$$

*In particular, if* $|\varphi_a'(c_{\mu_1}(a))| \geq e^{\beta}$, *then* $|(\varphi_a^{\mu_2-\mu_1})'(c_{\mu_1}(a))| \geq e^{\beta(\mu_2-\mu_1)}$.

*Proof.* — The proof follows easily by arguments almost identical to those used in the proof of lemma 2.5. □

## 3. Partitions and distortion estimates

**3.1. Preliminary distortion estimates.** — In this section we set up the machinery which will enable us, in the next section, to estimate the size of the set of parameters satisfying conditions CP1($n$) and CP2($n$) for all $n \geq 1$. Most of this analysis deals with properties of the family of maps

$$c_j : \omega_0 \longrightarrow [-c-\varepsilon, c+\varepsilon], \qquad c_j(a) = \varphi_a^j(c),$$

where $\omega_0 = [c+\rho\varepsilon, c+\varepsilon]$ and $j \geq 1$. Our first result implies that the derivatives $c_j'(a)$ of such maps grow exponentially fast with $j$, as long as the phase-space derivatives $(\varphi_a^j)'(c_1(a)) = (\partial_x \varphi_a^j)(c_1(a))$ do.

***Lemma 3.1.*** — *There is* $\eta > 1$ *such that if* $|(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j}$ *for* $1 \leq j \leq n$, *then*

$$\frac{1}{\eta} \leq \frac{|c_{j+1}'(a)|}{|(\varphi_a^j)'(c_1(a))|} \leq \eta \qquad \textit{for all } 1 \leq j \leq n$$

*Proof.* — The arguments are fairly standard. Using the chain rule we can write

$$c_{j+1}' = \partial_a \varphi_a(c_j) + \partial_x \varphi_a(c_j)\partial_a \varphi_a(c_{j-1}) + \cdots + \partial_x \varphi_a^{j-1}(c_2)\partial_a \varphi_a(c_1) + \partial_x \varphi_a^j(c_1)c_1'$$

and so

$$\frac{c_{j+1}'}{(\varphi_a^j)'(c_1)} = \sum_{i=1}^{j} \frac{\partial_a \varphi_a(c_i)}{(\varphi_a^i)'(c_1)} + c_1'. \tag{12}$$

Note that $c_1' = 1$ and $|\partial_a \varphi_a(c_i)| = 1$ for each $i$. Hence, our hypothesis implies that (12) is bounded from above (in norm) by $\eta = \sum_{i=0}^{\infty} e^{-\sigma i}$.

The bound from below requires a more careful analysis. For the time being we restrict to $a = c$ and note that the first two terms in (12) are both positive (L4 gives $c_1 > 0$ and so $\partial_a \varphi_a(c_1) > 0$, $\varphi_a'(c_1) > 0$, and $\partial_a \varphi_a(c_1) \cdot (\varphi_a^2)'(c_1) > 0$). We distinguish three cases.

If $|\varphi_a'(c_2)| \geq \sqrt{2}$ and $|\varphi_a'(c_3)| \geq \sqrt{2}$ (*i.e.* $|c_i| \leq x_{\sqrt{2}}$ for $i = 2, 3$) then

$$|(\varphi_a^{2k+1})'(c_1)| \geq (\sqrt{2})^{2k}|\varphi_a'(c_1)| \quad \text{and} \quad |(\varphi_a^{2k+2})'(c_1)| \geq (\sqrt{2})^{2k}|(\varphi_a^2)'(c_1)|, \tag{13}$$

by Proposition 1.1. It follows that

$$\sum_{1 \leq 2k+1 \leq j} \frac{\partial_a \varphi_a(c_{2k+1})}{(\varphi_a^{2k+1})'(c_1)} \geq \frac{1}{\varphi_a'(c_1)}\left(1 - \sum_{k=1}^{\infty} 2^{-k}\right) \geq 0$$

and a similar estimate holds for the sum over even indices. Thus the quotient in (12) is bounded from below by $c_1'(a) = 1$, which proves the lemma in this case.

Suppose now that $|\varphi_a'(c_2)| \geq \sqrt{2}$ and $|\varphi_a'(c_3)| < \sqrt{2}$. Clearly, we still have the first estimate in (13) and so the sum of all the terms in (12) corresponding to odd values of $i$ is again nonnegative. In order to bound the sum of the even terms we use Proposition 1.1 once more and get (recall that $|\varphi_a'(c_1)| \geq \sqrt{2}$ by L4) $|(\varphi_a^{2k})'(c_1)| \geq (\sqrt{2})^{2k}$ for each $k$. It follows that

$$\sum_{1\leq 2k\leq j} \frac{\partial_a \varphi_a(c_{2k})}{(\varphi_a^{2k})'(c_1)} \geq 0 - \sum_{k=2}^{\infty} (\sqrt{2})^{-2k} \geq -\frac{1}{2}.$$

Hence (12) is bounded from below by 1/2.

The case $|\varphi_c'(c_2)| < \sqrt{2}$ and $|\varphi_c'(c_3)| \geq \sqrt{2}$ is quite similar to the previous one. The second estimate in (13) is valid here and so the total contribution of the even terms in (12) is nonnegative. On the other hand, $|(\varphi_a^{2k-1})'(c_1)| \geq (\sqrt{2})^{2k-1}$ yields

$$\sum_{1\leq 2k-1\leq j} \frac{\partial_a \varphi_a(c_{2k-1})}{(\varphi_a^{2k-1})'(c_1)} \geq 0 - \sum_{k=2}^{\infty} (\sqrt{2})^{1-2k} \geq -\frac{1}{\sqrt{2}}$$

and so (12) is bounded from below by $1 - 1/\sqrt{2} > 1/4$.

Note also that we cannot have $|\varphi_a'(c_2)| < \sqrt{2}$ and $|\varphi_a'(c_3)| < \sqrt{2}$ simultaneously, by L5. This means that the lemma is proved for $a = c$. The general case now follows easily. Fix $l \geq 1$ large so that $\sum_{i>l} e^{-\sigma i} < 1/10$ and take $\varepsilon$ to be small enough so that

$$\left| \frac{c_{j+1}'(a)}{(\varphi_a^j)'(c_1(a))} - \frac{c_{j+1}'(c)}{(\varphi_c^j)'(c_1(c))} \right| \leq \frac{1}{10} \quad \text{for all } j \leq l \text{ and all } a \in [c, c+\varepsilon].$$

It follows, immediately, that $c_{j+1}'(a)/(\varphi_a^j)'(c_1(a)) \geq (1/4) - (1/10) = (3/20)$ for all $j \leq l$. Moreover, for $j > l$ we have

$$\left| \frac{c_{j+1}'(a)}{(\varphi_a^j)'(c_1(a))} - \frac{c_{l+1}'(a)}{(\varphi_a^l)'(c_1(a))} \right| = \left| \sum_{i=l+1}^{j} \frac{\partial_a \varphi_a(c_i)}{(\varphi_a^i)'(c_1)} \right| \leq \sum_{l+1}^{\infty} e^{-\sigma i} < \frac{1}{10}$$

and so $c_{j+1}'(a)/(\varphi_a^j)'(c_1(a)) \geq (3/20) - (1/10) = (1/20)$. □

It follows from this lemma that as long as the space derivatives $(\varphi_a^j)'(c_1(a))$ are growing exponentially for all $a$ belonging to some interval $\omega$ of parameter values, the maps $c_j : \omega \to c_j(\omega)$ are diffeomorphisms, since $|c_j'(a)| \geq 1/\eta |(\varphi_a^j)'(c_1(a))| \neq 0$. In particular the same is true for the maps $\Phi : c_i(\omega) \to c_j(\omega)$ defined by

$$x \to c_j \circ c_i^{-1}(x)$$

for $0 \leq i \leq j$, even though the space derivatives may not be growing exponentially between time $i$ and time $j$. We have that

$$\Phi'(c_i(a)) = \frac{|c_j'(a)|}{|c_i'(a)|}, \qquad a \in \omega. \tag{14}$$

Thanks to this fact we can still estimate the average expansion of the "intermediate" images of $\omega$.

***Lemma 3.2.*** — *Suppose that we have* $|(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j}$ *for all* $1 \leq j \leq n$ *and* $a \in \omega$. *Then, for all integers* $1 \leq k \leq l \leq n$ *we have for some* $\xi \in \omega$

$$\frac{1}{\eta^2}|(\varphi_\xi^{l-k})'(c_k(\xi))| \leq \frac{|c_l(\omega)|}{|c_k(\omega)|} \leq \eta^2 |(\varphi_\xi^{l-k})'(c_k(\xi))|.$$

*Proof.* — Defining $\Phi : c_k(\omega) \to c_l(\omega)$ as above, we have by the Mean Value theorem $|c_l(\omega)| = |\Phi'(c_k(\xi))||c_k(\omega)|$ for some $\xi \in \omega$. Thus from Lemma 3.1 and the formula above for the derivative of $\Phi$ we have immediately the statement in the lemma. □

**3.2. Partitions.** — We shall now use the family of maps $\{c_j\}$, together with Lemma 3.1, to construct two nested families $\{F_n\}_{n\in\mathbb{N}}$ and $\{E_n\}_{n\in\mathbb{N}}$ of subsets of $\omega_0$

$$\cdots \subseteq E_n \subseteq F_n \subseteq E_{n-1} \subseteq \cdots \subseteq \omega_0$$

and a monotone sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of families of subintervals of $\omega_0$

$$\cdots \succ \mathcal{P}_n \succ \mathcal{P}_{n-1} \succ \cdots \succ \{\omega_0\} \quad (\text{given } \omega \in \mathcal{P}_n \ \exists\, \omega' \in \mathcal{P}_{n-1} \text{ with } \omega \subset \omega')$$

as follows. All the parameters belonging to $F_n$ satisfy CP1($n$): $d(c_j) \geq \varepsilon^\gamma e^{-\alpha j}$ for all $1 \leq j \leq n$. Each $\mathcal{P}_n$ is a partition of $F_n$ into intervals. Moreover, $E_n$ is a union of elements $\omega \in \mathcal{P}_n$ such that CP2($n$): $P_j \leq j/2$ for all $1 \leq j \leq n$ and $a \in \omega$ holds. In view of Proposition 2.6, the parameters in $\mathcal{A}_\varepsilon^+ = \bigcap_{n\in\mathbb{N}} E_n$ will satisfy EG for all times (we shall take $\mathcal{A}^+ = \cup_\varepsilon \mathcal{A}_\varepsilon^+$, where the union is over all small values of $\varepsilon > 0$).

The construction of the objects described above is carried out inductively. As the first step of the induction we simply set $E_0 = F_0 = \omega_0$ and $\mathcal{P}_0 = \{\omega_0\}$. Now suppose that $F_{n-1}, E_{n-1}$, $\mathcal{P}_{n-1}$ have been defined and let us explain how parameters are excluded at the $n$th stage and the partition $\mathcal{P}_n$ and the sets $F_n$, $E_n$ are constructed. For that we consider separately the cases $n < r_s/\alpha$ and $n \geq r_s/\alpha$, see Remark 3.5 below. In what follows we denote $\Delta_+ = \Delta_+^0 \cup \Delta_+^c \cup \Delta_+^{-c}$ and $\Delta = \Delta^0 \cup \Delta^c \cup \Delta^{-c}$. For $r > 0$ we let $\Delta_r$ denote the $\varepsilon^\gamma e^{-r}$-neighbourhood of the origin and of the critical points. Recall also that $r_\varepsilon = [\delta \log 1/\varepsilon]$ and $r_s = [(\gamma + 2\delta + \iota) \log 1/\varepsilon]$.

Suppose first that $n < r_s/\alpha$. Given any $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$, there are two possibilities:

**1** : If $c_n(\omega)$ does not intersect $\Delta_{[\alpha n]}$ then, by definition, $\omega$ is also an element of $\mathcal{P}_n$ and it is contained in $F_n$ and in $E_n$.

**2** : If $c_n(\omega) \cap \Delta_{[\alpha n]} \neq \varnothing$ then parameters have to be thrown out in order that CP1($n$) hold. We write $\omega_e' = c_n^{-1}(\Delta_{[\alpha n]} \cap c_n(\omega))$ and we also let $\omega_e''$ be the union of those connected components of $\omega \setminus \omega_e'$ whose image under $c_n$ is completely contained in $\Delta_{[\alpha n]-1}$. Both these sets of parameters are excluded from the sequel of the argument: by definition

$$E_n \cap \omega = F_n \cap \omega = \omega \setminus (\omega_e' \cup \omega_e'')$$

and the elements of $\mathcal{P}_n$ contained in $\omega$ are precisely the connected components of $\omega \setminus (\omega_e' \cup \omega_e'')$. We observe, for future reference, that any such component $\tilde{\omega}$

contains an interval of the form $I_r^0$ with $|r| = [\alpha n] - 1$. We call this interval the *host interval* of $\widetilde{\omega}$ at the return $n$. Moreover this immediately implies that $|c_n(\widetilde{\omega})| \geq \varepsilon^\gamma e^{-[\alpha n]}$.

Thus we have for each $k < r_s/\alpha$ and $\omega \in \mathcal{P}_k$ a nested sequence of intervals

$$\omega = \omega_k \subseteq \omega_{k-1} \subseteq \cdots \subseteq \omega_1 \subseteq \omega_0 \text{ with } \omega_i \in \mathcal{P}_i, i = 0, \ldots, k$$

and a sequence of *escape times*

$$1 \leq \nu_1 < \nu_2 < \cdots < \nu_s \leq k$$

defined by the fact that the situation described in case 2 occurs precisely at these times. The corresponding components $\omega_{\nu_i} \in \mathcal{P}_{\nu_i}$ are called *escaping components.* By definition we also call the original parameter interval $\omega_0 = (c + \varepsilon/\rho, c + \varepsilon)$ an escaping component. Notice that case 1 and 2 describe the only situations which can occur before time $r_s/\alpha$. Since case 1 does not involve making any changes to existing parameter intervals (*i.e.* $\omega_i = \omega_{i-1}$ if $c_i(\omega_{i-1})$ satisfies the conditions described in 1) we have $\omega_{\nu_{i+1}} \subset \omega_{\nu_{i+1}-1} = \omega_{\nu_{i+1}-2} = \cdots = \omega_{\nu_i}$. As we shall see this is not true in general for iterates larger that $r_s/\alpha$.

Now we treat the case $n \geq r_s/\alpha$. In order to define $\mathcal{P}_n$, $F_n$, $E_n$, we need a refinement of the partitions $\{I_r^*\}$, $* = 0, \pm c$, introduced in the previous section: for each $|r| \geq 1$ we let $\{I_{r,l}^*\colon 1 \leq l \leq r^2\}$ be the partition of $I_r^*$ into $r^2$ intervals of equal length. We suppose that $l$ is increasing in the same direction as $r$, *i.e.* as we get closer to the singularity or the critical points. As above we associate to every $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$ a nested sequence of intervals $\omega = \omega_{n-1} \subseteq \cdots \subseteq \omega_0$ and a sequence of escape times

$$1 \leq \nu_1 < \cdots < \nu_l < r_s/\alpha \leq \nu_{l+1} < \cdots < \nu_s \leq n - 1.$$

Each $\omega_i$, $0 \leq i \leq n-1$, is just the element of $\mathcal{P}_i$ that contains $\omega$. For $j \leq l$ the $\nu_j$ are exactly the escape times described above. However, for $l \leq j \leq s-1$ we also have between two consecutive escape times a (possibly empty) sequence of *return times*

$$\nu_j < \mu_{0,j} < \mu_{1,j} < \cdots < \mu_{q(j),j} < \nu_{j+1}$$

and similarly for $j = s$ we have

$$\nu_s < \mu_{0,s} < \cdots < \mu_{q(s),s} \leq n - 1.$$

Moreover to each such sequence of return times is associated a sequence of integers $p_{0,j}, p_{1,j}, \ldots, p_{q(j),j} \geq 0$.

As part of the inductive step of our construction we also explain when and how $\nu_{s+1}, \mu_{q(s)+1,s}$ and $p_{q(s)+1,s}$ are introduced, assuming that such sequences are defined for all iterates up to $n-1$. We consider four cases separately:

**3 :** If $q(s) > 0$ and $\mu_{q(s)+1,s} \leq n \leq \mu_{q(s),s} + p_{q(s),s}$ then $\omega \in \mathcal{P}_n$ and $\omega \subset E_n \subset F_n$. Moreover, we leave the sequences unchanged.

**4 :** If $c_n(\omega)$ does not intersect $\Delta^{\pm c} \cup \Delta^0_{r_s}$ or if this intersection is completely contained in one of the extreme subintervals of $\Delta^{\pm c} \cup \Delta^0_{r_s}$, *i.e.*

$$c_n(\omega) \cap (\Delta^{\pm c} \cup \Delta^0_{r_s}) \subseteq I^0_{\pm(r_s+1),(r_s+1)^2} \cup I^{\pm c}_{\pm(r_\varepsilon+1),(r_\varepsilon+1)^2},$$

then once more we let $\omega \in \mathcal{P}_n$ and $\omega \subset E_n \subset F_n$, and we keep the sequences unchanged.

From now on we assume that neither 3 nor 4 hold.

**5 :** If $c_n(\omega)$ intersects $\Delta^{\pm c} \cup \Delta^0_{r_s}$ but does not properly contain any subinterval $I^{\pm c}_{r,l}$ with $|r| > r_\varepsilon$ or $I^0_{r,l}$ with $|r| > r_s$, we set $\omega \in \mathcal{P}_n$ and $\omega \subset F_n$. Moreover, we let $\mu_{q(s)+1} = n$ and $p_{q(s)+1} = \min\{p(c_n(a)) \colon a \in \omega\}$, recall Lemma 2.4, if $c_n(\omega) \cap \Delta^0_{r_s} = \varnothing$, and $p_{q(s)+1} = 0$ otherwise. We say that $n$ is an *(inessential) return* for $\omega \in \mathcal{P}_n$ and that $p_{q(s)+1}$ is the length of the associated *binding period*. Note that $c_n(\omega)$ is contained in the union of at most two $I^*_{s,m}$. We shall prove in Lemma 3.4 that CP1($n$) is automatically satisfied in this case. Then we take $\omega \subset E_n$ if all $a \in \omega$ satisfy CP2($n$) and $E_n \cap \omega = \varnothing$ otherwise.

**6 :** If $c_n(\omega)$ intersects $\Delta^{\pm c} \cup \Delta^0_{r_s}$ and contains some $I^{\pm c}_{r,l}$ with $|r| > r_\varepsilon$ or $I^0_{r_s}$ with $|r| > r_s$, we carry out the following construction. We start by excluding the parameters which do not satisfy CP1($n$). More precisely, we let

$$\omega'_e = c_n^{-1}(\Delta_{[\alpha n]} \cap c_n(\omega))$$

and $\omega''_e$ be the union of the connected components of $\omega \setminus \omega'_e$ whose image under $c_n$ contains no subinterval $I^*_{s,m}$. By definition $F_n \cap \omega = \overline{\omega} = \omega \setminus (\omega'_e \cup \omega''_e)$. Then we partition $\overline{\omega}$ into subintervals $\overline{\omega} = (\cup_{r,l}\omega_{r,l}) \cup (\cup_i \widetilde{\omega}_i)$ where the first union runs over some subset of pairs $(r, l)$ with $r_\varepsilon < |r| \le [\alpha n]$ or $r_s < |r| \le [\alpha n]$ (depending on whether $n$ is a return to $\Delta^{\pm c}$ or $\Delta^0$), the second one involves at most two $\widetilde{\omega}_i$, and

**a :** $c_n(\omega_{r,l}) \supset I^*_{r,l}$ but contains no other interval $I^*_{s,m}$ (thus it is contained in the union of $I^*_{r,l}$ with the two $I^*_{s,m}$ adjacent to it). We call $I^*_{r,l}$ the *host interval* associated to $\omega_{r,l}$ at time $n$.

**b :** $c_n(\widetilde{\omega}_i)$ is disjoint from $\Delta^{\pm c} \cup \Delta^0_{r_s}$ but contains some $I^{\pm c}_{r,1}$ with $|r| = r_\varepsilon$ or $I^0_{r,1}$ with $|r| = r_s$. Again we call this interval the *host interval* associated to $\widetilde{\omega}$ and time $n$.

The elements of $\mathcal{P}_n$ contained in $\omega$ are precisely these $\omega_{r,l}$ and $\widetilde{\omega}_i$. For $\omega_{r,l}$ we let $\mu_{q(s)+1} = n$ and $p_{q(s)+1} = \min\{p(c_n(a)) \colon a \in \omega_{r,l}\}$ if $* = \pm c$ and $p_{q(s)+1} = 0$ if $* = 0$. In particular $n$ is an *essential return* time for each $\omega_{r,l} \in \mathcal{P}_n$. For the intervals $\widetilde{\omega}_i$ described in 6b we let $\nu_{s+1} = n$. In particular $n$ is an *escape time* for $\widetilde{\omega} \in \mathcal{P}_n$ and these intervals are *escaping components*. Finally, $E_n \cap \omega$ consists of the union of the intervals described above which satisfy CP2($n$).

This completes the inductive definition of the sets $E_n, F_n$, the partitions $\mathcal{P}_n$, and the sequences $\nu_j, \mu_j$, and $p_j$.

***Remark 3.1.*** — Host intervals can give some indication as to the type of situation we are dealing with. So, if $\omega \in \mathcal{P}_i$ has, at time $i$, an associated host interval of the form $I^0_r$, $|r| = [\alpha n] - 1$ then this implies that $i = \nu_j$ is an escape time and that $\omega$ is created at time $\nu_j$ as a consequence of a situation like the one described in case 2. Similarly if the host interval is of the form $I^0_{r,1}$, $|r| = r_s$ or $I^{\pm c}_{r,1}$, $|r| = r_\varepsilon$ then $i = \nu_j$ is also an escape time as described in 6b. On the other hand if the host interval is of the form $I^0_{r,l}$ with $|r| \geq r_s + 1$, $1 \leq l \leq r^2$ or $I^{\pm c}_{r,l}$ with $|r| \geq r_\varepsilon + 1$, $1 \leq l \leq r^2$ then $i = \mu_{k,j}$ for some $k, j$ and $\mu_{k,j}$ is either an inessential return as described in case 5 or an essential return as described in case 6a.

A main difference between the latter two cases is that for essential returns we have upper and lower bounds for the length of $c_{\mu_{k,j}}(\omega)$ in terms of the associated host interval. More precisely, recall case 6a, $c_{\mu_{k,j}}(\omega)$ contains some $I^*_{r,l}$ and is contained in the union of $I^*_{r,l}$ and its two adjacent intervals of the form $I^*_{s,m}$. Therefore we have $\varepsilon^\gamma e^{-r}/r^2 \leq |c_{\mu_{k,j}}(\omega)| \leq 10\varepsilon^\gamma e^{-r}/r^2$. In the case of inessential returns we have the same upper bound but no a priori lower bound.

***Remark 3.2.*** — We will sometimes talk about the sequence of escape times and returns associated to a single parameter value $a$ or a subinterval $\omega' \subset \omega \in \mathcal{P}_n$ which does not itself necessarily belong to any partition (although we will always consider subsets of intervals which do belong to some $\mathcal{P}_n$). In these cases the sequences are just those associated to the the interval $\omega \in \mathcal{P}_n$ to which $a$ or $\omega'$ belong.

***Remark 3.3.*** — We will frequently talk about *returning situations* or *escape times* at time $n$ for some interval $\omega_{n-1} \in \mathcal{P}_{n-1}$ (and *not* in $\mathcal{P}_n$). In these cases we will just be referring to the fact that $c_n(\omega_{n-1})$ intersects a neighbourhood of the origin or of the critical points in a way which is described in one of cases 2,5 or 6 above. Therefore, at time $n$ some action may be required (parameter exclusions, subdivision of $\omega_{n-1}$ into smaller intervals) which yields the final classification of the surviving pieces of $\omega_{n-1}$ into pieces (now belonging to $\mathcal{P}_n$) for which $n$ is either a return (essential or inessential) or an escape time.

***Remark 3.4.*** — Notice that the definition of the binding period $p = p(\omega)$ given here does not completely coincide with the definition given in Lemma 2.4 for a fixed parameter value $a$. However all the estimates obtained in that lemma continue to hold for the slightly shorter binding period defined here.

To simplify the exposition we will often refer to a generic host interval of the form $I^*_{r,l}$. This will include the host intervals which occur in case 2 which, strictly speaking, are of the form $I^*_r$. Moreover we shall often suppose that $r > 0$ since most of the times we are only interested in the norm of $r$ and not its sign.

***Remark 3.5.*** — The condition $n < r_s/\alpha$ means that $\Delta_{[\alpha n]} \supset \Delta_{r_s}$ and so

$$c_n(\widetilde{\omega}) \cap \Delta^{\pm c} = \varnothing \quad \text{for all } \widetilde{\omega} \in \mathcal{P}_n.$$

Indeed, $c_n(a) \in \Delta^{\pm c}_{+}$ would imply $c_{n-1}(a) \in I^0_r$ with $|r| \approx r_c \approx ((1/\lambda) - \gamma) \log \varepsilon^{-1} \gg (2(\gamma + \delta) + \iota) \log \varepsilon^{-1} \approx r_s$, recall (3) and (4), and so $c_{n-1}(a) \in \Delta^0_{r_s}$. This is a contradiction, as such a parameter $a$ would have been excluded already at time $n-1$. By construction (and the remark we have just made) all the elements of $\mathcal{P}_n$ obtained in that case contain some interval $I^0_r$ with $|r| = [\alpha n]$. Combined with Lemma 2.3 this gives

$$|(\varphi^j_a)'(c_1)| \geq e^{\sigma_0 j} \quad \text{for all } 1 \leq j \leq n \text{ and } a \in F_n.$$

Finally, $r_s/\alpha$ can be made arbitrarily large by fixing $\varepsilon$ and $\alpha$ sufficiently small.

***Remark 3.6***. — We shall show below that for $\omega \in \mathcal{P}_n$ the distortion

$$\sup\{|(\varphi^n_a)'(c_1(a))/(\varphi^n_b)'(c_1(b))| : a, b \in \omega\}$$

is bounded above by some constant. It is crucial to the overall argument that this constant is independent of $\omega$ as well as of $n$. The proof of this fact relies on the exponential growth of the intervals $c_j(\omega)$ as well as on the fact that $c_j(\omega)$ is small compared to its distance from the critical points and the discontinuity, where the distortion explodes. This is why no action is required in case 5 above whereas we need to cut $\omega$ into smaller pieces in case 6.

Throughout the rest of the paper we will use $C$ (resp. $\widetilde{C}$) to denote a generic small (resp. large) positive constant independent of $\varepsilon, \omega$ or the iterate under consideration.

***Lemma 3.3***. — *Let $\omega \in \mathcal{P}_{n-1}, \omega \subset E_{n-1}$ and suppose that $n$ is a return for $\omega \in \mathcal{P}_{n-1}$. Let $\nu \leq n-1$ be the last essential return or escape time of $\omega$ before time $n$ and let $I^*_{r,l}$ be the host interval associated to $\nu$. Then we have the following estimates.*

(1) *If $* = 0$,* i.e. *if $\nu$ is a return or escape time associated to $\Delta^0$, and*
   (a) *$n = \nu + 1$ is a return to $\Delta^{\pm c}$: $|c_n(\omega)| \geq \varepsilon^{1+\iota}$;*
   (b) *$n > \nu + 1$ is a return to $\Delta^0$:*
   $$|c_n(\omega)| \geq C(\varepsilon^\gamma e^{-r})^{2\lambda}/r^2 \geq \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r};$$
   (c) *$n > \nu + 1$ is a return to $\Delta^{\pm c}$:*
   $$|c_n(\omega)| \geq C(\varepsilon^\gamma e^{-r})^{2\lambda} \varepsilon^{1-1/\lambda}/r^2 \geq \varepsilon^{1-1/\lambda} \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r}.$$

(2) *If $* = \pm c$,* i.e. *if $\nu$ is a return or an escape time associated to $\Delta^{\pm c}$, and $r_\varepsilon \leq r \leq (\gamma + \delta + \iota) \log 1/\varepsilon$ then $n$ is* necessarily *a return to $\Delta^0$ and*
$$|c_n(\omega)| \geq \varepsilon^{-\iota/2} \varepsilon^\gamma e^{-r_s} \geq \varepsilon^{2(\gamma+\delta)+\iota/2}.$$

(3) *If $* = \pm c$ and $r \geq (\gamma + \delta + \iota) \log 1/\varepsilon$ and*
   (a) *$n$ is a return to $\Delta^0$:*
   $$|c_n(\omega)| \geq \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma} e^{-2\beta r/\sigma}/r^2 \geq \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r};$$
   (b) *$n$ is a return to $\Delta^{\pm c}$:*
   $$|c_n(\omega)| \geq \varepsilon^{1-1/\lambda+2\gamma+\delta+2\beta\gamma/\sigma} e^{-2\beta r/\sigma}/r^2 \geq \varepsilon^{1-1/\lambda} \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r}.$$

*Proof.* — The basic strategy of the proof is to estimate $|(\varphi_a^{n-\nu})'(c_\nu(a))|$ for elements $a \in \omega$. Applying Lemma 3.2 and the Mean Value theorem we can then carry this expansion over to parameter space and get

$$|c_n(\omega)| \geq \inf\{|(\varphi_a^{n-\nu})'(c_\nu(a))| : a \in \omega\}|c_\nu(\omega)|/\eta^2. \tag{15}$$

Keeping in mind that the definition of host interval implies $|c_\nu(\omega)| \geq \varepsilon^\gamma e^r/r^2$ we shall get the desired result in each case.

Suppose first that $\nu$ is a return to $\Delta^0$ and that $n = \nu + 1$ is a return to $\Delta^{\pm c}$. Then $c_\nu(\omega) \subset \varphi_a^{-1}(\Delta^{\pm c})$ for some $a \in \omega$ and Lemma 2.2 gives

$$|\varphi_a'((c_\nu(a))| \geq C(\varepsilon^\gamma e^{-r})^{\lambda-1} \geq \varepsilon^{1-1/\lambda}$$

since $r \approx (1/\lambda - \gamma)\log 1/\varepsilon$. Moreover $|c_\nu(\omega)| \geq C\varepsilon^{1/\lambda}/((1/\lambda + \gamma)\log 1/\varepsilon)^2$ and so

$$|c_\nu(\omega)| \geq C\varepsilon/((1/\lambda + \gamma)\log 1/\varepsilon)^2 \geq \varepsilon^{1+\iota}$$

for small $\iota > 0$. This proves (1a). Now let $\nu$ be a return to $\Delta^0$ and consider first the situation in which $|r| \geq r_s + 1$. We have that for all $a \in \omega$, by Lemmas 2.2 and 2.7 we have

$$|(\varphi_a^2)'(c_\nu(a))| \geq (\varepsilon^\gamma e^r)^{2\lambda-1} \text{ and } |(\varphi_a^{n-\nu-2})'(c_{\nu+2}(a))| \geq Ce^{\beta(n-\nu-2)}.$$

For this last statement we have used the fact that $c_\nu \in \Delta^0$ implies $|c_{\nu+1}| > x_{\sqrt{2}}$ and this in turn implies $|c_{\nu+2}| < x_{\sqrt{2}}$ (see condition L4) which in particular means that $|\varphi'(c_{\nu+2})| > e^\beta$. Applying Lemma 3.2 we get $|c_n(\omega)| \geq C(\varepsilon^\gamma e^{-r})^{2\lambda}/\eta^2 r^2$ proving the first inequality in (1b). The second inequality just follows by taking $\beta/\sigma < 1 - 2\lambda$. If $n$ is a return to $\Delta^{\pm c}$ then we have an additional factor of $\varepsilon^{1-1/\lambda}$ coming from the large derivative in $\varphi^{-1}(\Delta^{\pm c})$ and we get (1c). If $r \leq r_s$ we apply the same arguments to the subinterval $\overline{\omega} \subset \omega$ where $c_\nu(\overline{\omega}) = I^*_{r_s,1}$ and get $|c_n(\overline{\omega})| \geq C(\varepsilon^\gamma e^{-r})^{2\lambda}/\eta^2 r^2$ respectively $|c_n(\omega)| \geq C\varepsilon^{1-1/\lambda}(\varepsilon^\gamma e^{-r})^{2\lambda}/\eta^2 r^2$ which yields the desired result since $|c_n(\omega)| \geq |c_n(\overline{\omega})|$.

Now suppose that $\nu$ is a return to $\Delta^{\pm c}$. If $|r| = r_\varepsilon$ then the binding period has zero length by definition and we simply have, defining $\overline{\omega} = c_\nu^{-1}(I^{\pm c}_{r_\varepsilon,1}) \subset \omega$ ,

$$|c_n(\omega)| \geq |c_n(\overline{\omega})| \geq C\varepsilon^{\gamma+\delta}(\varepsilon^\gamma e^{-r_\varepsilon})/\eta^2 r_\varepsilon^2 \geq C\varepsilon^{2(\gamma+\delta)}/\eta^2 r_\varepsilon^2 \geq \varepsilon^{-\iota/2}\varepsilon^\gamma e^{r_s}.$$

If $|r| \geq r_\varepsilon + 1$ we have by by Lemmas 2.4 and 2.7 that, for all $a \in \omega$,

$$|(\varphi_a^{n-\nu})'(c_\nu(a))| \geq \varepsilon^{2\beta\gamma/\sigma}e^{(1-2\beta/\sigma)r} \cdot \inf\{\varphi_a'(c_{\nu+p+1}(a)), e^{\sigma_0}\}e^{\beta(n-\nu-p-1)} \tag{16}$$

$$\geq \varepsilon^{\gamma+\delta+2\beta\gamma/\sigma}e^{(1-2\beta/\sigma)r} \tag{17}$$

if $n$ is a return to $\Delta^0$ and $|(\varphi_a^{n-\nu})'(c_\nu(a))| \geq \varepsilon^{1-1/\lambda+\gamma+\delta+2\beta\gamma/\sigma}e^{(1-2\beta/\sigma)r}$ if it is a return to $\Delta^{\pm c}$. Notice that in both cases we have used $|\varphi'(c_{\nu+p+1}(a))| \geq \varepsilon^{\gamma+\delta}$. Now applying Lemma 3.2 and equations (15)(16) we get

$$|c_n(\omega)| \geq \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma}e^{-2\beta r/\sigma}/\eta^2 r^2. \tag{18}$$

Now if $r \le (\gamma+\delta+\iota)\log 1/\varepsilon$ then we have from (18)

$$|c_n(\omega)| \ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma+2\beta(\gamma+\delta+\iota)/\sigma}/r^2 \ge \varepsilon^{(2\gamma+\delta)+\iota/2} \ge \varepsilon^{-\iota/2}\varepsilon^{\gamma}e^{r_s}.$$

If $r > (\gamma+\delta+\iota)\log 1/\varepsilon$ then we write $e^{-2\beta r/\sigma} = e^{-(1-\beta/\sigma)r}e^{(1-\beta/\sigma)r}$ and $e^{(1-\beta/\sigma)r} \ge \varepsilon^{-(1-\beta/\sigma)(\gamma+\delta+\iota)}$ and therefore, using (18),

$$|c_n(\omega)| \ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma-(\gamma+\delta+\iota)+\beta/\sigma(\gamma+\delta+\iota)}e^{-(1-\beta/\sigma)r} \ge \varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)r}.$$

This concludes the proof of the lemma. □

We now formulate some easy consequences of these estimates.

***Lemma 3.4.*** — *If $n$ is an inessential return for some $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$, then CP1($n$) holds for every $a \in \omega$.*

*Proof.* — We claim that

$$(19) \qquad |c_n(\omega)| \ge 6\varepsilon^{\gamma}e^{-[\alpha n]}.$$

This implies the statement in the lemma for the following reason. Suppose by contradiction that some $a \in \omega$ did not satisfy CP1($n$), *i.e.* $d(c_n(a)) \le \varepsilon^{\gamma}e^{-\alpha n}$ and in particular $c_n(\omega) \cap \Delta^*_{[\alpha n]} \ne \varnothing, * = \pm c, 0$. Then, either $c_n(\omega) \subset \Delta^*_{[\alpha n]-1}$ or $c_n(\omega) \supset I^*_{[\alpha n]}$. The second alternative is not possible since it would contradict the fact that $n$ is an inessential return. The first alternative cannot happen either since the claim implies $|c_n(\omega)| > 2e\varepsilon^{\gamma}e^{[\alpha n]-1} = |\Delta^*_{[\alpha n]-1}|$. Thus we have reduced the proof of the lemma to that of (19). However this is an easy consequence of Lemma 3.3. Indeed recall that $r \le \alpha n$ and therefore if the last essential return $\nu \le n-1$ occurred in $\Delta^0$ we have

$$|c_n(\omega)| \ge C\varepsilon^{2\lambda\gamma}e^{-2\lambda r}/r^2 \gg 6\varepsilon^{\gamma}e^{-[\alpha n]}.$$

If $\nu$ is return to $\Delta^{\pm c}$ and $|r| = r_\varepsilon$ then the result follows immediately from part (2) in Lemma 3.3 keeping in mind that returns to $\Delta^{\pm c}$ can occur only for iterates $n \ge \alpha/r_s$. If $|r| \ge r_\varepsilon + 1$ we distinguish two cases. Suppose first that $|r| \ge (\gamma+\delta+\iota)\log 1/\varepsilon$. Then

$$\begin{aligned}|c_n(\omega)| &\ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma}e^{-r}e^{1-2\beta/\sigma}r/r^2 \ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma-(1-2\beta/\sigma)(\gamma+\delta+\iota)}e^{-r}\\ &\ge \varepsilon^{\gamma+2\beta\gamma/\sigma-\iota+2\beta(\gamma+\delta+\iota)/\sigma}e^{-r} \ge \varepsilon^{-\iota/2}\varepsilon^{\gamma}e^{-r} \gg 6\varepsilon^{\gamma}e^{-[\alpha n]}.\end{aligned}$$

Now suppose that $r < (\gamma+\delta+\iota)\log 1/\varepsilon$. Since $\nu \le n-1$ is a return to $\Delta^{\pm c}$ we also have $n \ge r_s/\alpha = ([\gamma+2\delta+\iota]/\alpha)\log 1/\varepsilon$. Therefore it is sufficient to show that $|c_n(\omega)| \ge 6\varepsilon^{\gamma}e^{-r_s} \ge \varepsilon^{2(\gamma+\delta)+\iota}$. This follows from part (2) of Lemma 3.3 which gives

$$|c_n(\omega)| \ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma}e^{-2\beta r/\sigma} \ge \varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma+2\beta(\gamma+\delta+\iota)/\sigma} \gg 6\varepsilon^{2(\gamma+\delta)+\iota}.$$

This concludes the proof of the lemma. □

***Lemma 3.5.*** — *Suppose that $\omega \in \mathcal{P}_{n-1}, \omega \subset E_{n-1}$ is an escaping component created at some time $\nu \le n-1$. Then, if $n$ is a return for $\omega$ we have*

$$|c_n(\omega)| \ge \varepsilon^{-\iota/2}\varepsilon^{\gamma}e^{-r_s}.$$

*In particular $n$ is a return to $\Delta^0$ for $\omega$ and there is a component $\widetilde{\omega} \subset \omega$ for which $n$ is an escape time.*

*Proof.* — The statement in the lemma follows immediately from Lemma 3.3. If $\nu$ is a return to $\Delta^{\pm c}$ the result is part (2). If $\nu$ is a return to $\Delta^0$ then we have $|c_n(\omega)| \geq C(\varepsilon^\gamma e^{-r_s})^{2\lambda}/r_s^2 \gg \varepsilon^{-\iota/2}\varepsilon^\gamma e^{-r_s}$. □

**3.3. More distortion estimates.** — For the following lemma we fix $\varepsilon'$ independent of $\varepsilon$. Let $\Delta_{\varepsilon'}$ and $\Delta_{2\varepsilon'}$ denote respectively $\varepsilon'$ and $2\varepsilon'$ neighbourhoods of the origin and of the critical points. We suppose that $\varepsilon'$ is chosen sufficiently small so that conditions M1 and M2 hold in $\Delta_{2\varepsilon'}$. Let $I \subset [-a, a]$ be an interval. For each $x \in I$ define $d(x) = \min\{|x|, |x \pm c|\}$ and $d(I) = \inf\{d(x) : x \in I\}$. Finally let $\widetilde{D}(I) = \sup\{|\varphi''(x)/\varphi'(y)| : x, y \in I\}$. We call an interval *admissible* if $I \cap \Delta_{\varepsilon'} \neq \varnothing$ implies $|I| \leq \varepsilon'$.

***Lemma 3.6.*** — *For any constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that if $I$ is an admissible interval then*

$$|I| \leq C_1 d(I) \implies \widetilde{D}(I) \leq C_2/d(I).$$

*Proof.* — If $I \cap \Delta_{\varepsilon'} = \varnothing$ then both $\varphi'(x)$ and $\varphi''(x)$ are bounded above and below by constants which depend only on the map and on $\varepsilon'$, and the statement in the lemma follows immediately. So suppose that $I \cap \Delta_{\varepsilon'} \neq \varnothing$. Then, since $I$ is an admissible interval we have $|I| \leq \varepsilon'$ and in particular $I \subset \Delta_{2\varepsilon'}$. Therefore either $I \subset \Delta^0_{2\varepsilon'}$ or $I \subset \Delta^{\pm c}_{2\varepsilon'}$. If $I \subset \Delta^0_{2\varepsilon'}$ then, by condition M1,

$$|\varphi''(x)| \leq \widetilde{C} d(I)^{\lambda-2}$$

and

$$|\varphi'(x)| \geq C(d(I) - |I|)^{\lambda-1} \geq C((1-C_1)d(I))^{\lambda-1} \geq C d(I)^{\lambda-1}.$$

and so $\widetilde{D}(I) \leq C_2/d(I)$ for some constant $C_2 > 0$. If $I \subset \Delta^{\pm c}_{2\varepsilon'}$ then

$$|\varphi''(x)| \leq \widetilde{C} \text{ and } |\varphi'(x)| \geq C d(I)$$

which immediately gives $\widetilde{D}(I) \leq C_2/d(I)$. □

***Lemma 3.7.*** — *There exists a constant $A > 1$ (independent of $\varepsilon$ or $n$) such that if $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$ and $n$ is a return to $\Delta^{\pm c}$ then*

$$\left| \frac{(\varphi^k_{\overline{a}})'(c_1(\overline{a}))}{(\varphi^k_a)'(c_1(a))} \right| \leq A \quad \textit{for all } \overline{a}, a \in \omega \textit{ and all } 0 \leq k \leq n-1.$$

*If $n$ is a return to $\Delta^0$ we have the same result for any $\overline{a}, a$ belonging to a subinterval $\overline{\omega} \subset \omega$ with $|c_n(\overline{\omega})| \leq \max\{(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}, \varepsilon^{2(\gamma+\delta)}\}$.*

*Proof.* — We shall prove the result for the case $k = n - 1$. It will be apparent from the proof that the result holds for all other values of $k$ as well. Write $c_i = \varphi_a^i(c)$ and $\overline{c}_i = \varphi_{\overline{a}}^i(c)$. By the chain rule

$$\left|\frac{(\varphi_{\overline{a}}^k)'(c_1(\overline{a}))}{(\varphi_a^k)'(c_1(a))}\right| = \prod_{i=1}^k \left|1 + \frac{\varphi_{\overline{a}}'(\overline{c}_i) - \varphi_a'(c_i)}{\varphi_a'(c_i)}\right| \le \prod_{i=1}^k 1 + |A_i|,$$

letting $A_i = (\varphi_{\overline{a}}'(\overline{c}_i) - \varphi_a'(c_i))/\varphi_a'(c_i)$, and thus the proof reduces to showing that $\sum_{i=1}^k |A_i|$ is bounded above by some constant independent of $\varepsilon$, $n$, or $\omega$. By the Mean Value theorem we also have $|\varphi_{\overline{a}}'(\overline{c}_i) - \varphi_a'(c_i)| \le |\overline{c}_i - c_i|\varphi''(\xi)$ for some $\xi \in c_i(\omega)$. Therefore we have

$$|A_i| \le |c_i(\omega)| \cdot \widetilde{D}(c_i\omega). \tag{20}$$

Let $1 < \mu_1 < \mu_2 < \cdots < \mu_s < \mu_{s+1} = n$ be the essential and inessential returns (cases 5 and 6a) of $\omega$ in the time interval $[1, n]$. Notice that we do not include escape times in this list. Let $p_1, \ldots, p_s$ be the corresponding binding periods as defined in the previous subsection (recall that $p_j = 0$ if $\mu_j$ is a return close to $\Delta^0$) and $r_1, \ldots, r_s$ be the values associated to the corresponding time intervals.

We start by considering the case in which the sequence of essential and inessential returns is empty, *e.g.* if $n < r_s/\alpha$. Then $n$ is necessarily a return to $\Delta^0$ for otherwise $n - 1$ would have been such a return. We suppose without loss of generality that $|c_n(\omega)| \le (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$, for otherwise we could restrict ourselves to some subinterval $\overline{\omega}$ for which this condition is satisfied. Let $\varepsilon'$ be the constant fixed in Lemma 3.6 and suppose first that $c_i(\omega) \cap \Delta_{\varepsilon'} = \varnothing$. Then $d(c_i(\omega)) \ge \varepsilon'$ and $|\varphi_a'(c_i(\omega))| \ge C\varepsilon'$ for all $a \in \omega$, and therefore by the standard arguments which we have used repeatedly above we get $|c_i(\omega)| \le \widetilde{C}e^{-\sigma_0(n-1)}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda} \le \widetilde{C}d(c_i(\omega))$. Then by Lemma 3.6 we get $|A_i| \le \widetilde{C}e^{-\sigma_0(n-1)}$. Now suppose that $c_i(\omega) \cap \Delta_{\varepsilon'}^0 \neq \varnothing$. A preliminary estimate for the length of $c_i(\omega)$ is given by

$$|c_i(\omega)| \le e^{-\sigma_0(n-1)}|c_n(\omega)| \le e^{-\sigma_0(n-1)}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda} \ll \varepsilon'. \tag{21}$$

This shows that $c_i(\omega)$ is an admissible interval. Now we need to obtain a stronger estimate to show that it actually satisfies the hypothesis of Lemma 3.6. We distinguish two cases according as to whether $d(c_i(\omega)) > (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$ or $d(c_i(\omega)) \le (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$. In the first case we have from (21) that the hypothesis of the lemma are satisfied and $|A_i| \le \widetilde{C}|c_i(\omega)|/d(c_i(\omega)) \le \widetilde{C}e^{-\sigma_0(n-1)}$. In the second case we have that the *maximum* distance between $c_i(\omega)$ and the origin, for $a \in \omega$ is

$$|c_i(\omega)| + d(c_i(\omega)) \le 2(\varepsilon^\gamma e^{-\alpha n})^{2\lambda} \ll \varepsilon'.$$

Therefore $c_i(\omega)$ is entirely contained in the region in which condition M1 applies and it is easy to see by a simple variation of the argument in the proof of Lemma 2.2 that we have, for any $a \in \omega$,

$$|(\varphi_a^2)'(c_i(a))| \ge (\varepsilon^\gamma e^{-\alpha n})^{2\lambda(2\lambda-1)}.$$

From this we obtain an improved estimate for the size of $c_i(\omega)$, namely

$$|c_i(\omega)| \leq e^{-\sigma_0(n-1)}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda(2\lambda-1)}|c_n(\omega)| \leq e^{-\sigma_0(n-1)}(\varepsilon^\gamma e^{-\alpha n})^{(2\lambda)^2}.$$

Then, from this and Lemma 3.6, $|A_i| \leq |c_i(\omega)|/d(c_i(\omega)) \leq Ce^{-\sigma_0(n-1)}$. Now suppose that $c_i(\omega) \cap \Delta_{\varepsilon'}^{\pm c} \neq \varnothing$. Since $|\varphi'(c_i(a))| \approx d(c_i(\omega)$ we have

$$|c_i(\omega)| \leq e^{-\sigma_0(n-1)}(d(c_i(\omega)))^{-1}|c_n(\omega)| \leq e^{-\sigma_0(n-1)}(d(c_i(\omega)))^{-1}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}.$$

Moreover, in this case we necessarily have $d(c_i(\omega)) \geq (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$ for the following reason. Since $\omega \subset E_{n-1}$ and $i \leq n-1$ we have that, by definition, all $a \in \omega$ satisfy condition CP1 up to time $n-1$ and, in particular, at time $i-1$. Therefore $d(c_{i-1}(\omega)) \geq \varepsilon^\gamma e^{-\alpha(i-1)} \geq \varepsilon^\gamma e^{-\alpha n}$. Then because of the form of the map near the origin (condition M1) this implies $d(c_i(\omega)) \geq (\varepsilon^\gamma e^{-\alpha(i-1)})^\lambda \gg (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$. Therefore $c_i(\omega)$ is an admissible interval and we have

$$|c_i(\omega)| \leq e^{-\sigma_0(n-i)}(\varepsilon^\gamma e^{-\alpha n})^{-\lambda}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$$

which give $|A_i| \leq |c_i(\omega)|/d(c_i(\omega)) \leq Ce^{\sigma_0(n-i)}$. Therefore we can sum over all iterates to get

$$\sum_{i=0}^{n-1}|A_i| \leq C.$$

This proves the lemma if there are no essential or inessential returns for $\omega$ before time $n$.

Now we consider the cases in which there is a non empty sequence of returns. We start by estimating the values $|A_{\mu_j}|$ for $j = 1, \ldots, s$. Since $c_{\mu_j}$ is contained in the union of three intervals of the form $I^*_{r,l}$ we have $|c_{\mu_j}(\omega)| \leq \widetilde{C}\varepsilon^\gamma e^{-r_j}/r_j^2 \leq \widetilde{C}d(c_{\mu_j}(\omega))$ and therefore by Lemma 3.6,

$$\widetilde{D}(c_{\mu_j}(\omega)) \leq \widetilde{C}/d(c_{\mu_j}(\omega)) \leq \widetilde{C}\varepsilon^\gamma e^{r_j}.$$

Substituting in (20) we have

$$|A_{\mu_j}| \leq \widetilde{C}/r_j^2.$$

We now consider $A_i$ where $i$ is not a return iterate. Notice first of all that any return $\mu_{j+1}$ to $\Delta^{\pm c}$ is *immediately preceded* by a return $\mu_j = \mu_{j+1}-1$ to $\Delta^0$. Therefore if $i \in (\mu_j, \mu_{j+1})$ we necessarily have that $\mu_{j+1}$ is a return to $\Delta^0_{r_s}$. Therefore we only need to distinguish two cases according as to whether $\mu_j$ is a return to $\Delta^0$ or to $\Delta^{\pm c}$. For the moment we also restrict our attention to values of $j \leq s-1$.

Suppose first that $\mu_j$ is a return to $\Delta^0$. We distinguish two subcases: either $|\varphi'(c_i)| \geq e^\beta$ for all $a \in \omega$ or there is some $a \in \omega$ for which $|\varphi'(c_i(a))| < e^\beta$. Then Lemma 3.2 and Lemma 2.3 give $|(\varphi^{\mu_{j+1}-i})'(c_i(a))| \geq e^{\beta(\mu_{j+1}-i)}$ in the first subcase and $|(\varphi^{\mu_{j+1}-i})'(c_i(a))| \geq \varepsilon^{\gamma+\delta}e^{\beta(\mu_{j+1}-i-1)}$ in the second subcase. Moreover applying Lemma 3.2 this gives

$$|c_i(\omega)| \leq \eta^2 e^{-\beta(\mu_{j+1}-i)}|c_{\mu_{j+1}}(\omega)| \leq \eta^2 e^{-\beta(\mu_{j+1}-i)}\varepsilon^{2(\gamma+\delta)+\iota}/r_{j+1}^2$$

and

$$|c_i(\omega)| \le \eta^2 \varepsilon^{-\gamma+\delta} e^{-\beta(\mu_{j+1}-i-1)} |c_{\mu_{j+1}}(\omega)| \le \eta^2 e^{-\beta(\mu_{j+1}-i-1)} \varepsilon^{\gamma+\delta+\iota}/r_{j+1}^2$$

respectively, using the fact that $|c_{\mu_{j+1}(\omega)}| \le \varepsilon^\gamma e^{-r_{j+1}}/r_{j+1}^2$ and that $r_{j+1} > r_s$ since, as we mentioned above, $\mu_{j+1}$ is necessarily a return to $\Delta^0_{r_s}$. Moreover we have that, in the first case, since $i$ is not a return iterate, $d(c_i(\omega)) \ge \varepsilon^{2(\gamma+\delta)+\iota}$. In the second case we know (from the fact that $c_i(\omega)$ is small and $|\varphi'(c_i(a))| < e^\beta$ for some $a \in \omega$) that $c_i(\omega)$ is relatively far from $\Delta^0$ and relatively close to $\pm c$ and therefore since $i$ is not a return iterate, $d(c_i(\omega)) \ge \varepsilon^{\gamma+\delta}$. In both cases we have that $c_i(\omega)$ is an admissible interval and applying Lemma 3.6 and substituting in (20) we get $|A_i| \le Ce^{-\beta(\mu_{j+1}-i)}/r_{j+1}^2$ in each case. Moreover we can sum over all $i \in (\mu_j, \mu_{j+1})$ to get

$$\sum_{i=\mu_j+1}^{\mu_{j+1}-1} |A_i| \le C/r_{j+1}^2.$$

Now suppose that $\mu_j$ is a return to $\Delta^{\pm c}$. Then there follows a binding period $(\mu_j, \mu_j + p_j]$ and a (possibly empty) free period $(\mu_j + p_j + 1, \mu_{j+1})$. For iterates $i \in (\mu_j + p_j + 1, \mu_{j+1})$ the situation is exactly as in the case considered above and we have $|A_i| \le Ce^{-\beta(\mu_{j+1}-i)}/r_{j+1}^2$ and $\sum_{i=\mu_j+p_j+1}^{\mu_{j+1}-1} |A_i| \le C/r_{j+1}^2$. So it just remains to consider *bound* iterates $i \in (\mu_j, \mu_j + p_j]$.

First of all recall that $d(c_{\mu_j}(\omega)) \approx \varepsilon^\gamma e^{-r_j}$ and in particular, for all $a \in \omega$,

$$|c_1(a) - c_{\mu_j+1}(a)| \le \widetilde{C}(\varepsilon^\gamma e^{-r_j})^2.$$

By the mean value theorem we have $|c_i(\omega)| = |c_{\mu_{j+1}}(\omega)| \cdot |(\varphi^{i-\mu_j-1})'(\zeta)|$ for some $\zeta \in c_{\mu_j+1}(\omega)$. Then using the bounded distortion estimate in Lemma 2.4(1) and the fact that $|c_i(\omega)| \le \varepsilon^\gamma e^{\beta(\mu_j-i)}$ by the definition of binding period we get

$$|(\varphi^{i-(\mu_j-1)})'(\zeta)| \le \frac{\varepsilon^\gamma e^{-\beta(\mu_j-1)}}{|c_{\mu_{j+1}}(\omega)|} \le \frac{\varepsilon^\gamma e^{-\beta(\mu_j-1)}}{(\varepsilon^\gamma e^{-r_j})^2}.$$

Since $|\varphi'(x)| \approx \varepsilon^\gamma e^{-r_j}$ for all $x \in c_{\mu_j}(\omega)$ we then have

$$|(\varphi^{i-\mu_j})'(x)| \le \varepsilon^\gamma e^{-\beta(\mu_j-1)}/\varepsilon^\gamma e^{-r_j}.$$

By Lemma 3.2 this gives

$$|c_i(\omega)| \le \frac{\varepsilon^\gamma e^{-\beta(\mu_j-1)}}{\varepsilon^\gamma e^{-r_j}} |c_{\mu_j}(\omega)| \le \varepsilon^\gamma e^{-\beta(\mu_j-1)}/r_j^2.$$

Moreover $d(c_i(\omega)) \ge \varepsilon^\gamma e^{-\alpha(i-\mu_j)} - e^{-\beta(i-\mu_j)} \ge C\varepsilon^\gamma e^{-\alpha(i-\mu_j)}$ and therefore, substituting in (20) we get $|A_i| \le Ce^{(\alpha-\beta)(i-\mu_j)}/r_j^2$ which also yields

$$\sum_{\mu_j+1}^{\mu_j+p_j} |A_i| \le C/r_j^2.$$

Finally we consider the last piece of orbit $(\mu_s + p_s, \mu_{s+1} - 1]$. This interval can be empty if the return $\mu_{s+1}$ occurs immediately after the end of the binding period, *i.e.* if $\mu_{s+1} = \mu_s + p_s + 1$, or if $\mu_{s+1}$ is a return to $\Delta^{\pm c}$. Indeed, in the latter case we also have $\mu_{s+1} = \mu_s + p_s + 1$ keeping in mind that $\mu_s$ is necessarily a return to $\Delta^0$ and that therefore $p_s = 0$ by definition. So suppose that $i \in (\mu_s + p_s, \mu_{s+1} - 1]$. By the comments above this implies that $\mu_{s+1}$ is a return to $\Delta^0$. Moreover we are assuming, by the hypotheses in the lemma, that $c_{\mu_{s+1}} \subset \Delta^0_{2(\gamma+\delta)}$. Suppose first that $|\varphi'(c_i(a))| < e^{\beta}$ for some $a \in \omega$. Then, repeating the exact same arguments used above we have

$$|c_i(\omega)| \leq C\varepsilon^{-(\gamma+\delta)}e^{-\beta(n-i-1)}|\Delta^0_{2(\gamma+\delta)}| \leq Ce^{-\beta(n-i-1)}\varepsilon^{\gamma+\delta}.$$

Moreover, from Lemma 3.6 we have $\widetilde{D}(c_i(\omega)) \leq C\varepsilon^{-(\gamma+\delta)}$ and so

$$|A_i| \leq |c_i(\omega)| \cdot \widetilde{D}(c_i(\omega)) \leq Ce^{-\beta(n-i-1)}.$$

Now suppose that $|\varphi'(c_i)| \geq e^{\beta}$. We distinguish two further subcases according as to whether $d(c_i(\omega)) \geq \varepsilon^{2(\gamma+\delta)}$ or not. Suppose first that $d(c_i(\omega)) \geq \varepsilon^{2(\gamma+\delta)}$, *i.e.* $c_i(\omega) \cap \Delta^0_{2(\gamma+\delta)} = \varnothing$. Then we have $|c_i(\omega)| \leq e^{-\beta(n-i)}\varepsilon^{2(\gamma+\delta)}$ and applying Lemma 3.6,

$$|A_i| \leq |c_i(\omega)| \cdot \widetilde{D}(c_i(\omega)) \leq Ce^{-\beta(n-i)}.$$

If $d(c_i(\omega)) < \varepsilon^{2(\gamma+\delta)}$ then we still have $|c_i(\omega)| \leq e^{-\beta(n-i)}\varepsilon^{2(\gamma+\delta)}$ and, applying Lemma 3.6

$$\widetilde{D}(c_i(\omega)) \leq C/d(c_i(\omega)) \leq C\varepsilon^{-2(\gamma+\delta)-\iota}.$$

Notice however that since $c_i(\omega)$ is small ($\leq \varepsilon^{2(\gamma+\delta)}$) and $d(c_i(\omega)) < \varepsilon^{(\gamma+\delta)}$, $c_i(\omega)$ is completely contained in a small neighbourhood of the origin, say $c_i(\omega) \subset \Delta^0$ where the derivative is very large. In particular we have from Lemma 2.2 that $|(\varphi^2)'(c_i(a))| \geq \varepsilon^{(\gamma+\delta)(1-2\lambda)}$ for any $a \in \omega$. Therefore arguing as above we can obtain a much stronger bound on the size of $c_i(\omega)$, more precisely,

$$|c_i(\omega)| \leq e^{-\beta(n-i+2)}\varepsilon^{(\gamma+\delta)(1-2\lambda)}|\Delta_{2(\gamma+\delta)}| \leq Ce^{-\beta(n-i+2)}\varepsilon^{(\gamma+\delta)(1-2\lambda)}\varepsilon^{2(\gamma+\delta)}.$$

and therefore substituting in (20) we get

$$|A_i| \leq C\varepsilon^{\iota}e^{-\beta(n-i)}.$$

Finally, let $R(q)$ be the set of indices $j$ for which $|r_j| = q$ and when $R(q)$ is nonempty we denote by j(q) the largest of its elements. Notice that for all $j \in R(q)$ we have $c_{\mu_j}(\omega) \leq Ce^{-\beta(\mu_{j(q)}-\mu_j)}|c_{\mu_{j(q)}}|$ and therefore we have

$$\sum_{1}^{n-1}|A_i| = \sum_{1}^{\mu_s+p_s}|A_i| + \sum_{\mu_s+p_s+1}^{n-1}|A_i| \leq C\sum_{q:R(q)\neq\varnothing}q^{-2} + C \leq A.$$

This completes the proof of the lemma. □

***Lemma 3.8.*** — *There exists a constant $B > 1$ (independent of $\varepsilon$ or $n$) such that if $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$ and $n$ is a return to $\Delta^{\pm c}$ then*

$$\left|\frac{c_k'(\overline{a})}{c_k'(a)}\right| \leq B \quad \textit{for all } \overline{a}, a \in \omega \textit{ and all } 0 \leq k \leq n-1.$$

*If $n$ is a return to $\Delta^0$ we have the same result for any $\overline{a}, a$ belonging to a subinterval $\overline{\omega} \subset \omega$ with $|c_n(\overline{\omega})| \leq \max\{(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}, \varepsilon^{2(\gamma+\delta)}\}$.*

*Proof.* — This is a direct consequence of Lemmas 3.1 and 3.7, just take $B = A\eta$. □

## 4. Parameter exclusions

We now wish to estimate the total measure of the parameters excluded during every step of the induction. We shall treat separately the exclusions due to each one of the two conditions on the parameter. We shall start by showing that for some positive constants $\delta_1, \alpha_1$ we have

$$|E_{n-1} \setminus F_n| \leq \varepsilon^{\delta_1} e^{-\alpha_1 n} |E_{n-1}| \leq \varepsilon^{\delta_1} e^{-\alpha_1 n} |E_0|$$

*i.e.* the proportion of parameters excluded by CP1 at each iteration is exponentially small as $n \to \infty$ and as $\varepsilon \to 0$. Recall that there are no binding periods and therefore no exclusions due to CP2 for iterates $n \leq N = [r_s/\alpha]$ and that $N$ can be made arbitrarily large by taking $\varepsilon$ small. Thus we have $E_n = F_n$ for all $n \leq N$ and so

$$|E_{n-1} \setminus E_n| \leq \varepsilon^{\delta_1} e^{-\alpha_1 n} |E_0| \quad \forall n \leq N \tag{22}$$

and, inductively,

$$|E_n| \geq |E_{n-1}| - \varepsilon^{\delta_1} e^{-\alpha_1 n} |E_0| \geq |E_0|(1 - \sum_{i=0}^{n} \varepsilon^{\delta_1} e^{-\alpha_1 i}) \quad \forall n \leq N.$$

For general $n$ we shall show below that $|F_n \setminus E_n| \leq e^{-\alpha_1 n}|E_0|$ and therefore we get from (22), $|E_{n-1} \setminus E_n| \leq 2e^{-\alpha_1 n}|E_0|$. This then gives, for $n > N$,

$$|E_n| \geq |E_N| - |E_0| \sum_{i=N+1}^{n} 2e^{-\alpha_1 i} \geq |E_0| \left(1 - \sum_{i=0}^{N} \varepsilon^{\delta_1} e^{-\alpha_1 i} - \sum_{i=N+1}^{n} 2e^{-\alpha_1 i}\right)$$

and

$$|E_\infty| \geq |E_0| \left(1 - \sum_{i=0}^{N} \varepsilon^{\delta_1} e^{-\alpha_1 i} - \sum_{i=N+1}^{\infty} 2e^{-\alpha_1 i}\right) \geq |E_0|(1 - C(\varepsilon))$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

**4.1. Exclusions due to CP1.** — Recall that for each $n$ we need to throw away parameters $a$ for which $c_n(a)$ falls into the $\varepsilon^\gamma e^{-\alpha n}$-neighbourhood of the origin and of the critical points, which we denoted $\Delta_{\alpha n}$. Moreover, if exclusion of these parameters leads to the formation of small connected components in parameter space (smallness being expressed in terms of their image under $c_n$) then such components are also excluded.

Given an interval $\omega \in \mathcal{P}_{n-1}$ with $\omega \subset E_{n-1}$, the subset $\widehat{\omega} \subset \omega$ of parameters which get thrown out at the $n^{th}$ iteration satisfies $c_n(\widehat{\omega}) \subset \Delta_{[\alpha n]-1}$. The aim of this section is to show that the ratio $|\widehat{\omega}|/|\omega|$ is exponentially small for small $\varepsilon$ and large $n$. In principle this is achieved by estimating the ratio $|c_n(\widehat{\omega})|/|c_n(\omega)|$ and then invoking the bounded distortion estimates in Lemma 3.8 to show that this ratio is essentially preserved when pulled back by $c_n^{-1}$. This works well for returns to $\Delta^{\pm c}$ as the bounded distortion estimates (*cf.* Lemma 3.8) can then be applied to the entire interval $\omega$. For returns to $\Delta^0$ we face the problem that the bounded distortion estimates only apply to intervals $\overline{\omega}$ which satisfy $|c_n(\overline{\omega})| \leq \max\{(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}, \varepsilon^{2(\gamma+\delta)}\}$. Nevertheless the next lemma show that this is sufficient for our purposes.

***Lemma 4.1.*** — *There exist constants $\delta_1 > 0$ and $\alpha_1 > 0$ (independent of $n$ or $\varepsilon$) such that*

$$\frac{|E_{n-1}|}{|F_n|} \geq 1 - \varepsilon^{\delta_1} e^{-\alpha_1 n} \quad \textit{for any } \omega \in \mathcal{P}_{n-1} \textit{ with } \omega \subset E_{n-1}.$$

*Proof.* — Consider an element $\omega \in \mathcal{P}_{n-1}, \omega \subset E_{n-1}$. Let $\widehat{\omega} \subset \omega$ be those parameters which get excluded at time $n$ for failing to satisfy CP1. Clearly it is enough to estimate $|\widehat{\omega}/|\omega|$ and we can suppose that $n$ is a returning situation for $\omega$ otherwise the statement would be trivially true.

Suppose first that if $n$ is not a return to $\Delta^{\pm c}$ then $|c_n(\omega)| < (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$, in particular the hypotheses of Lemma 3.8 are satisfied. Then we have

$$|\widehat{\omega}|/|\omega| \leq B|\Delta_{[\alpha n]-1}|/|c_n(\omega)| \leq 2e^2 B \varepsilon^\gamma e^{-\alpha n}/|c_n(\omega)|.$$

The estimates for $c_n(\omega)$ have all been obtained in Lemma 3.3 and we just need to consider the various cases. If $n$ is a return to $\Delta^{\pm c}$ we have either $|c_n(\omega)| \geq \varepsilon^{1+\iota}$ from (1a) or $|c_n(\omega)| \geq \varepsilon^{1-1/\lambda} \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r}$ from (1c) and (3b). Using the fact that $r \leq \alpha n$ we clearly get the desired estimate in this case. If $n$ is a return to $\Delta^0$ then (1b) and (3a) give $|c_n(\omega)| \geq \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r}$ which again yields the statement in the lemma since $r \leq \alpha n$. Finally case (2) gives $|c_n(\omega)| \geq \varepsilon^{-\iota/2} \varepsilon^\gamma e^{-r_s}$. Notice that this case can only occur after a return to $\Delta^{\pm c}$ and such returns can only occur for large values of $n$, more precisely for $n \geq r_c/\alpha \gg r_s/\alpha$. Therefore we have

$$\varepsilon^\gamma e^{-\alpha n}/\varepsilon^{-\iota/2}\varepsilon^\gamma e^{-r_s} \leq \varepsilon^{\iota/2} e^{-\alpha n + r_s} \leq \varepsilon^{\iota/2} e^{-n(\alpha - r_s/n)} \leq \varepsilon^{\iota/2} e^{-\alpha' n}$$

which proves the result in this case also.

Finally suppose that $|c_n(\omega)| \geq (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$. Let $\overline{\omega} \subset \omega$ be such that $|c_n(\overline{\omega})| = (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}$, in particular, the hypotheses of Lemma 3.8 are satisfied by $\overline{\omega}$ and we have

$$\frac{|\widehat{\omega}|}{|\omega|} \leq \frac{|\widehat{\omega}|}{|\overline{\omega}|} \leq B\frac{|\Delta_{[\alpha n]-1}|}{|c_n(\overline{\omega})|} \leq 2e\beta\frac{\varepsilon^\gamma e^{-\alpha n}}{(\varepsilon^\gamma e^{-\alpha n})^{2\lambda}} \leq 2eB(\varepsilon^\gamma e^{-\alpha n})^{1-2\lambda}.$$

This completes the proof of the lemma. □

**4.2. Exclusions due to CP2.** — In this section we consider elements $\omega \subset F_n, \omega \in \mathcal{P}_n$ which satisfy CP1($n$). We will set up a statistical argument to show that most of these elements (in measure theoretical terms) have spent a small proportion of their time in binding periods and therefore also satisfy CP2($n$).

Recall from Section 3.2 that a sequence of escape times $0 = \nu_0 < \cdots < \nu_{s+1} = n$ is associated to each element $\omega \in \mathcal{P}_n$. Here we set $\nu_0 = 0$ and $\nu_{s+1} = n$ for notational convenience and we call these escape times as well. Between any two escape times we have a (possibly empty) sequence of essential and inessential returns $\mu_0 < \cdots < \mu_q$ with $\mu_i = \mu_{i,j}$ and $q = q(j)$. To each such return $\mu_i$ is associated a positive integer $p_i \geq 0$, the length of the associated binding period, and an integer $r_i > 0$ determined by the associated host interval. We let $\widetilde{P}_j = p_{0,j} + \cdots + p_{q,j}$, $R_j = r_{0,j} + \cdots + r_{q,j}$, $\widetilde{P}_+ = \widetilde{P}_1 + \cdots + \widetilde{P}_s$ and $R_+ = R_1 + \dots R_s$. In particular $\widetilde{P}_+ = P_n$, the total number of iterates before time $n$ belonging to binding periods. From Lemma 2.4 we immediately get the following

***Lemma 4.2.*** — *For $\omega \in \mathcal{P}_n, \omega \subset F_n$ we have*

$$P_n \leq \frac{2(R_+ + 3/2\log 1/\varepsilon)}{\sigma} \leq \frac{2(1+\gamma/\delta)R_+}{\sigma}.$$

In particular we can formulate an alternative condition

**CP2' :** $R_+ \leq \sigma n/4(1+\gamma/\delta)$

which immediately implies CP2: $P_n \leq n/2$.

***Lemma 4.3.*** — *Let $\omega_{\nu_j} \in \mathcal{P}_{\nu_j}$ and $\omega_{\nu_{j+1}} \in \mathcal{P}_{\nu_j}$ be escaping components with $\omega_{\nu_j} \subset \omega_{\nu_{j+1}}$ and suppose that there exists a non empty sequence $\mu_0 < \cdots < \mu_q$ of* essential *returns between time $\nu_j$ and time $\nu_{j+1}$. Then*

$$|\omega_{\nu_{j+1}}| \leq \varepsilon^\iota e^{-\iota R}|\omega_{\nu_j}|.$$

*Proof.* — Let $\omega_{\mu_i} \in \mathcal{P}_{\mu_i}$ be the subintervals of $\omega_{\mu_j}$ corresponding to the returns $\mu_i$, $i = 0, \dots, q$. We write

$$\frac{|\omega_{\nu_{j+1}}|}{|\omega_{\nu_j}|} = \frac{|\omega_{\mu_0}|}{|\omega_{\nu_j}|}\frac{|\omega_{\mu_1}|}{|\omega_{\mu_0}|}\cdots\frac{|\omega_{\mu_q}|}{|\omega_{\mu_{q-1}}|}\frac{|\omega_{\nu_{j+1}}|}{|\omega_{\mu_q}|} \tag{23}$$

and begin by estimating $|\omega_{\mu_0}|/|\omega_i|$. We have $|c_{\mu_0}(\omega_i)| \geq \varepsilon^{-\iota/2}\varepsilon^\gamma e^{-r_s}$, by Lemma 3.5. By the definition of the components $\omega_{\mu_i}$, notice that the first return after an escape time is always a return to $\Delta^0$, we have $|c_{\mu_o}(\omega_{\mu_0})| \leq 10\varepsilon^\gamma e^{r_0}/r_0^2$ for some $r_0 \geq r_s$. We distinguish two cases. Suppose first that $c_{\mu_0}(\omega_i) \subset \Delta_{2(\gamma+\delta)}$. Then we can apply the bounded distortion estimates and we have

$$\frac{|\omega_{\mu_0}|}{\omega_{\nu_j}} \leq \frac{B|c_{\mu_0}(\omega_{\mu_0})|}{|c_{\mu_0}(\omega_{\nu_j})|} \leq 10B\varepsilon^{-\iota/2}e^{r_s-r_0}/r_0^2. \tag{24}$$

If $c_{\mu_0}(\omega_i)$ is not completely contained in $\Delta_{2(\gamma+\delta)}$ then, using the fact that $\varepsilon^{\gamma}e^{-r_s} \approx \varepsilon^{2(\gamma+\delta)}$ we have $|c_{\mu_0}(\overline{\omega}_{\nu_j})| \geq \varepsilon^{2(\gamma+\delta)} - \varepsilon^{\gamma}e^{-r_s} \geq \varepsilon^{2(\gamma+\delta)}(1-\varepsilon^{\iota})$ where $\overline{\omega} \subset \omega_i$ is that subinterval of $\omega_{\nu_j}$ whose image is completely contained in $\Delta_{2(\gamma+\delta)}$. Then

$$\begin{aligned}\frac{|\omega_{\mu_0}|}{|\omega_{\nu_j}|} \leq \frac{|\omega_{\mu_0}|}{|\overline{\omega}_{\nu_j}|} \leq \frac{B|c_{\mu_0}(\omega_{\mu_0})|}{|c_{\mu_0}(\overline{\omega}_i)|} &\leq \frac{10B\varepsilon^{\gamma}e^{-r_0}}{r_0^2\varepsilon^{2(\gamma+\delta)}(1-\varepsilon^{\iota})} \\ &\leq 10B\varepsilon^{-(\gamma+2\delta)}e^{-r_0}/r_0^2 \ll 10B\varepsilon^{-\iota/2}e^{r_s-r_0}.\end{aligned}$$

We now turn to estimating the ratios $|\omega_{\mu_j}|/|\omega_{\mu_{j-1}}|$ for $j = 1,\ldots,q$. Each time we need to distinguish as above the cases in which $c_{\mu_{j-1}}(\omega_{\mu_{j-1}})$ is completely contained in $\Delta_{2(\gamma+\delta)}$ so that we can apply the bounded distortion estimates, and the cases in which this is not true. However, repeating the argument above we see that the estimates which we obtain in the former cases are always satisfied in the latter. Thus we shall consider in detail only the situation in which $c_{\mu_{j-1}}(\omega_{\mu_{j-1}}) \subset \Delta_{2(\gamma+\delta)}$. Suppose that $\mu_{j-1}$ is a return to $\Delta^0$. Then we have

$$\frac{|\omega_{\mu_j}|}{|\omega_{\mu_{j-1}}|} \leq \frac{10Br_{j-1}^2\varepsilon^{\gamma}e^{-r_j}}{r_j^2\varepsilon^{2\lambda\gamma}e^{-2\lambda r_{j-1}}} \leq 10Br_{j-1}^2\varepsilon^{(1-2\lambda)\gamma}e^{2\lambda r_{j-1}-r_j} \tag{25}$$

If $\mu_{j-1}$is a return to $\Delta^{\pm c}$ then we have

$$\frac{|\omega_{\mu_j}|}{|\omega_{\mu_{j-1}}|} \leq \frac{10Br_{j-1}^2\varepsilon^{\gamma}e^{-r_j}}{r_j^2\varepsilon^{2\gamma+\delta+2\beta\gamma/\sigma}e^{-2\beta\gamma/\sigma r_{j-1}}} \leq 10Br_{j-1}^2\varepsilon^{-(\gamma+\delta+2\beta\gamma/\sigma)}e^{2\beta r_{j-1}/\sigma-r_j}. \tag{26}$$

Recall moreover that if $\mu_j$ is a return to $\Delta^{\pm c}$ we gain an extra factor of $\varepsilon^{1-1/\lambda}$ on the right hand side of (25) and (26). Finally we have $|\omega_{i+1}|/|\omega_{\mu_q}| \leq 1$.

Now let $\widetilde{q}$ denote the number of returns between $\mu_0$ and $\mu_{q-1}$ (inclusive) which occur in $\Delta^0$ and $\widehat{q}$ denote the number of those which occur in $\Delta^{\pm c}$. We do not include $\mu_q$ in this count and therefore we have $q = \widetilde{q} + \widehat{q} + 1$. Let $\widetilde{R} = \sum \widetilde{r}_i$ and $\widehat{R} = \sum \widehat{r}_i$ and $R = \sum_{i=0}^{q} r_i = \widetilde{R} + \widehat{R} + r_q$. Then we have from (23)(24)(25)(26)

$$\frac{|\omega_{i+1}|}{|\omega_i|} \leq (10B)^{q+1}\varepsilon^{\iota/2}e^{r_s}\varepsilon^{(1-2\lambda)\gamma\widetilde{q}}e^{2\lambda\widetilde{R}}\varepsilon^{(1/\lambda-1)\widehat{q}}\varepsilon^{-(\gamma+\delta+2\beta\gamma/\sigma)\widehat{q}}e^{2\beta\widehat{R}/\sigma}e^{-R}. \tag{27}$$

To simplify this expression we make the following three observations:

(i) $e^{2\lambda\widetilde{R}} = e^{(\lambda+1/2)\widetilde{R}}e^{-(1/2-\lambda)\widetilde{R}}$;

(ii) $(1/\lambda - 1) > \gamma + \delta + 2\beta\gamma/\sigma$ and therefore

$$\varepsilon^{(1/\lambda-1-\gamma-\delta-2\beta\gamma/\sigma)\widehat{q}}e^{2\beta\widehat{R}/\sigma} \leq e^{2\beta\widehat{R}/\sigma} \leq e^{(\lambda+1/2)\widehat{R}};$$

(iii) $\mu_0$ is a return to $\Delta^0$ and therefore $\widetilde{q} \geq 1$ and $\widetilde{R} \geq r_s$.
From (27) and these observations we get

$$\frac{|\omega_{i+1}|}{|\omega_i|} \leq (10B)^{q+1}\varepsilon^{\iota/2}e^{r_s}\varepsilon^{(1-2\lambda)\gamma}e^{-(1/2-\lambda)r_s-(\lambda+1/2)r_q}e^{(\lambda+1/2)(\widetilde{R}+\widehat{R}+r_q)-R}.$$

Now using the fact that $r_s = [(\gamma + 2\delta + \iota)\log 1/\varepsilon$ and $r_q \geq r_\varepsilon = [(\gamma+\delta)\log 1/\varepsilon]$ we see that

$$\begin{aligned} e^{(\lambda-1/2)r_s-(\lambda+1/2)r_q} &\leq \varepsilon^{(\lambda-1/2)(\gamma+2\delta+\iota)+(\lambda+1/2)(\gamma+\delta)} \\ &\leq \varepsilon^{\lambda(2\gamma+3\delta+\iota)-(\delta+\iota)/2} \leq \varepsilon^{\iota/2}. \end{aligned}$$

Thus we have

$$\frac{|\omega_{i+1}|}{|\omega_i|} \leq (10B)^{q+1}\varepsilon^{\iota}e^{(\lambda+1/2-1)R}.$$

Finally, since $q \leq R/r_\varepsilon$ we have $(10B)^q \leq ((10B)^{1/r_\varepsilon})^R$ and so, taking $\varepsilon > 0$ small we get

$$\frac{|\omega_{i+1}|}{|\omega_i|} \leq \varepsilon^{\iota}e^{-\iota R}.$$

This concludes the proof. □

Now let $\eta_q(R)$ denote the number of possible sequences $r_1, \ldots, r_q$ with $r_i \geq \delta \log \varepsilon^{-1}$ and $r_i + \cdots + r_q = R$ and let $\eta(R) = \sum_{q\geq 0} \eta_q(R)$.

***Lemma 4.4.*** — *For $\varepsilon > 0$ sufficiently small we have that, for all $R \in \mathbb{N}$,*

$$\eta(R) \leq e^{\iota R/2}.$$

*Proof.* — The result is purely combinatorial. We want to estimate, for each fixed $q$,

$$\eta_q(R) \leq \binom{R}{q} \leq \frac{R!}{q!(R-q)!}.$$

Using Stirling's approximation formula for factorials:

$$\sqrt{2\pi k}k^k e^{-k} \leq k! \leq \sqrt{2\pi k}k^k e^{-k}(1+1/4k),$$

we get

$$\begin{aligned} \eta_q(R) &\leq \frac{\sqrt{2\pi R}R^R e^{-R}(1+1/4R)}{\sqrt{2\pi q}q^q e^{-q}\sqrt{2\pi(R-q)}(R-q)^{R-q}e^{-(R-q)}} \\ &\leq 2\frac{R^R}{q^q(R-q)^{R-q}} \qquad \text{for small } \varepsilon > 0 \text{ (and therefore } R \text{ large)} \\ &\leq 2\left(\frac{R}{q}\right)^q\left(\frac{R}{R-q}\right)^{R-q} \leq 2\left[\left(\frac{1}{q/R}\right)^{q/R}\left(\frac{1}{1-q/R}\right)^{1-q/R}\right]^R. \end{aligned}$$

Now since $q/R \leq 1/\delta \log 1/\varepsilon \to 0$ we have that $(1/(q/R))^{q/R}$ and $(1/(1-q/R))^{1-q/R}$ both tend to 1 as $\log \varepsilon \to 0$. Thus we get $\eta_q(R) \leq e^{(\iota R/2)/2}$. Notice that the value of

$q$ is bounded by $q \leq R/(\delta \log 1/\varepsilon) \leq R$, for each $R$, and so, summing over all possible values of $q$ we get $\eta(R) \leq \sum_{q \leq R} \eta_q(R) \leq e^{\iota R}$. □

Now let $\omega_{\nu_j} \in \mathcal{P}_{\nu_j}$ be an escaping component and let $\widetilde{\eta}(R)$ denote the total number of subintervals $\omega \subset \omega_{\nu_j}$ which are escaping components of the form $\omega_{\nu_{j+1}}$, $\nu_{j+1} = \nu_{j+i}(\omega)$ which undergo a sequence of returns $\mu_0, \ldots, \mu_q$ between time $\nu_j$ and time $\mu_{j+1}$ with $R_j = R$. Then we have

***Lemma 4.5***. — *For all $R \in \mathbb{N}$,*

$$\widetilde{\eta}(R) \leq e^{2\iota R}.$$

*Proof.* — Notice that the subintervals $\omega$ can be indexed in a unique way by the sequence of host intervals corresponding to the returns $\mu_0, \ldots, \mu_q$, *i.e.* by a sequence $(*_0, r_0, l_0) \ldots (*_q, r_q, l_q)$ where $* = 0, \pm c*$ and $1 \leq l_i \leq r_i^2$. It follows that for each $r$ there exist at most $6r^2$ intervals with $r_i = r$ and therefore the previous lemma immediately gives $\widetilde{\eta}(R) \leq 6R^2 e^{\iota R} \leq e^{2\iota R}$. □

For the final step of our argument we introduce the following notation. For $\omega \in \mathcal{P}_n$ let $\omega_{\nu_j}$, $j = 0, \ldots, s$ be the escaping components containing $\omega$ as defined above. Then define for $j = s+1, \ldots, n$, $\omega_{\nu_j} = \omega$ and call these escaping components as well. Thus we have a formally defined sequence of nested intervals

$$\omega = \omega_n = \cdots = \omega_{\nu_{s+1}} \subset \omega_{\nu_s} \subset \cdots \subset \omega_{\nu_0} = \omega_0$$

associated to each $\omega \in \mathcal{P}_n$. For each such $\omega_{\nu_j}$ let $\langle \omega_{\nu_j} \rangle = \omega_{\nu_j} \cap F_n$ and let $Q_j$ denote the union of all the escaping components of the form $\omega_{\nu_j}$. Notice that $Q_0 = \omega_0$ and $Q_n = F_n$.

***Lemma 4.6***. — *For every $n \geq 1$, we have*

$$\int_{F_n} e^{\iota R/4} da \leq (1 + \varepsilon^{\iota})^n |\omega_0|.$$

*Proof.* — For a given $\omega_{\nu_j}$ we have

$$\begin{aligned}\int_{<\omega_{\nu_j}>} e^{\iota R/4} da &= |\Omega_0| + \sum_{R \geq r_s} |\Omega(R)| e^{\iota R/4} \leq |\omega_{\nu_j}| + \sum_{R \geq r_s} \varepsilon^{\iota} e^{-\iota R/4} |\omega_i| \\ &\leq (1 + \varepsilon^{\iota} \sum_{R \geq r_s} e^{-\iota R/4}) |\omega_{\nu_j}| \leq (1 + \varepsilon^{\iota}) |\omega_{\nu_j}|.\end{aligned}$$

Clearly this implies

$$\int_{Q_i} e^{\iota R/4} da \leq (1 + \varepsilon^{\iota/2}) Q_i$$

and, inductively, the statement in the lemma. □

We are now in a position to estimate the proportion of parameters satisfying $CP2'$ : $R_+ \leq \sigma n/4(1+\gamma/\delta)$. This condition is equivalent to $e^{\iota R/4} \leq e^{\iota\sigma n/16(1+\gamma/\delta)} \leq e^{\xi n}$ where $\xi = \iota\sigma/16(1+\gamma/\delta)$. Thus we have

$$m\{a \in F_n : e^{\iota R/4} \geq e^{\xi n}\}e^{\xi n} \leq \int_{F_n} e^{\iota R/4} da \leq (1+\varepsilon^{\iota/2})^n |\omega_0|$$

which gives

$$m\{E_n \setminus F_n\} \leq (1+\varepsilon^{\iota})^n e^{-\xi n}|\omega_0| \leq e^{-\xi n/2}|\omega_0|$$

taking $\varepsilon$ small.

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