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TAMAGAWA MEASURES ON UNIVERSAL TORSORS AND POINTS OF BOUNDED HEIGHT ON FANO VARIETIES

by

Per Salberger

Abstract. — Let X be a Fano variety over a number field. An Arakelov system of v-adic metrics on the anticanonical line bundle on X gives rise to a height function on the set of rational points and to a new kind of adelic measures on the universal torsors over X.

The aim of the paper is to relate the asymptotic growth of the number of rational points of bounded height on X to volumes of adelic spaces corresponding to the universal torsors over X.

Introduction

One important but very difficult problem in diophantine geometry is to count the number $f(B)$ of rational points of height at most B on a projective variety over a number field k and to study the asymptotic growth of the counting function when $B \to \infty$. If $A \subset \mathbb{P}^n$ is an abelian variety, then it was proved by Néron (cf. [**62**]) that $f(B)/(\log B)^{\operatorname{rk} A(k)/2}$ converges to a constant depending on $A \subset \mathbb{P}^n$ and the height function. There is a precise adelic conjecture about the rank of $A(k)$, but this has only been established for classes of elliptic curves E over $\mathbb{Q}$ for which $E(\mathbb{Q})$ is of rank 0 or 1.

For Fano varieties there is a (mostly conjectural) theory of counting functions. This theory was initiated by Manin who made some striking observations (cf. [**23**], [**3**], [**42**], [**43**]) about the counting functions for special classes of Fano varieties X under their anticanonical embeddings. Similar observations for other linear systems were made by Batyrev and Manin [**3**]. The heights involved are the "usual" multiplicative heights of Weil depending on the ground field k and the choice of coordinates (see [**37**, p. 50]). There may exist accumulating closed subsets with many rational points (e.g. lines on cubic surfaces). Manin therefore counts the number $f_U(B)$ of rational points of height at most B on sufficiently small Zariski open k-subsets U of X. He

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notices that for a number of Fano varieties X there are open k-subsets U of X such that

$$(*) \qquad f_U(B) = CB(\log B)^{\mathrm{rk}\,\mathrm{Pic}\,X-1}(1+o(1))$$

for an anticanonical height function. Very recently Batyrev and Tschinkel [**5**] have found that this cannot be true for all Fano varieties. But there are important classes of Fano varieties (cf. [**23**], [**52**], [**7**], [**4**]) for which asymptotic formulas of the above form have been established, and it is interesting to find common features of these results. The motivation for this paper is to obtain a better understanding of the constant C in these formulas and to develop a framework which might be useful for the study of counting functions of other classes of Fano varieties.

The first systematic attempt to understand the constant C in the asymptotic formulas is due to Peyre [**52**], who introduced several new ideas. He defined Tamagawa numbers for Fano varieties, thereby generalizing the classical Tamagawa numbers studied by Weil [**67**]. To define these, Peyre uses a system of ν-adic "metrics" for the places ν of k on the analytic anticanonical line bundle over $X(k_\nu)$ satisfying an adelic condition. He associates to any such "adelic metric" a height function H on $X(k)$ and a measure on the adelic space $X(A_k)$ and suggests that the constant C in $(*)$ should be equal to the product of the Tamagawa number $\tau(X)$ of $X(A_k)$ and an invariant $\alpha(X)$ depending only on effective cone in $\mathrm{Pic}\,X$.

Peyre assumes that the Picard group of X_K for an algebraic closure K of k contains a $\mathbb{Z}$-basis which is invariant under the action of the Galois group $\mathrm{Gal}(K/k)$. It is clear from the work of Batyrev and Tschinkel in [**7**] and [**4**] that some restriction of this kind is needed. They prove $(*)$ for toric varieties and obtain the constant

$$(**) \qquad C = \alpha(X)\tau(X)h^1(X)$$

where $h^1(X)$ is the order of $H^1(\mathrm{Gal}(K/k), \mathrm{Pic}\,X_K)$.

We shall in the paper "explain" the appearance of $h^1(X)$ in $(**)$ by means of a new kind of Tamagawa numbers for universal torsors over X. The main idea is that the constant C is related to the volumes of some adelic spaces defined by the universal torsors $\mathcal{T}$ over X. Universal torsors were introduced by Colliot-Thélène and Sansuc [**15**] as a generalization of the classical descent varieties of elliptic curves studied by Fermat, Mordell and Weil. The main applications of this theory so far have been in the study of the Hasse principle and weak approximation for various classes of rational varieties.

We shall use universal torsors as a natural tool when counting rational points on Fano varieties. One central idea will be to extend the height function on $X(k)$ to suitable subquotients of the adelic spaces $\mathcal{T}(A_k)$ by means of certain adelic splittings associated to the universal torsors $\pi : \mathcal{T} \to X$. For toric varieties this reduces the original counting problem to an adelic lattice point problem. We obtain thereby another proof of the asymptotic formula of Batyrev and Tschinkel [**4**] for toric varieties over $\mathbb{Q}$.

We now give a description of the content of the 11 sections of the paper.

In the first section, we study in detail the analytic manifold structure $X_{\mathrm{an}}(k_\nu)$ on $X(k_\nu)$ and metrics (which we will call *norms* from now on) on analytic line bundles on $X_{\mathrm{an}}(k_\nu)$. For anticanonical line bundles with a norm we recall the measure constructed by Peyre and explain its relation to the classical construction of a measure from a global differential form. For submersions

$$Y_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu)$$

we give a relative version of Peyre's construction and associate a positive linear map

$$\Lambda : C_c(Y_{\mathrm{an}}(k_\nu)) \longrightarrow C_c(X_{\mathrm{an}}(k_\nu))$$

to norms on relative anticanonical line bundles.

In section 2, we assume that ν is non-archimedean and study norms for the compact open subsets $\Xi_\nu(o_\nu) \subseteq X_{\mathrm{an}}(k_\nu)$ defined by models Ξ_ν of $X \times k_\nu$ over the valuation ring o_ν in k_ν. We relate in (2.14) the volume of $\Xi_\nu(o_\nu)$ with respect to the measure determined by the above norm to the density of the reduction of $\Xi_\nu(o_\nu)$ modulo finite powers of the maximal ideal m_ν in o_ν. As a consequence, we get an explicit formula (cf. (2.14)(b)) for the volume of $\Xi_\nu(o_\nu)$ with respect to any measure defined by a norm on $\Xi_\nu(o_\nu)$. This formula holds also when Ξ_ν has bad reduction and X is non proper. If X is proper, then $\Xi_\nu(o_\nu) = X_{\mathrm{an}}(k_\nu)$ and we get a formula for the volume of $X_{\mathrm{an}}(k_\nu)$. But there are also important applications of this formula to universal torsors and other non-proper varieties.

In section 3, we study invariant norms over local fields on the relative anticanonical line bundles for X-torsors $\pi : \mathcal{T} \to X$ under arbitrary algebraic groups G. We then concentrate on the norms defined by relative differential forms. If G is a torus T, then there is a canonical norm of this kind which we will baptize the *order norm*. Now using this relative norm we obtain for each norm on $X_{\mathrm{an}}(k_\nu)$ an "induced" norm on $\mathcal{T}_{\mathrm{an}}(k_\nu)$ which in its turn defines an "induced" measure on $\mathcal{T}_{\mathrm{an}}(k_\nu)$.

In section 4, we consider varieties X over number fields k and the adelic topological space $X(A_k)$. We have not found any modern rigorous version of Weil's account **[66]** and we therefore explain how to use schemes of finite presentation in EGA to develop the foundations for adelic spaces. We then generalize Peyre's notion of "adelic metric" and his adelic measures in many ways. We introduce e.g. relative adelic norms for smooth morphisms $\pi : Y \to X$ over X and positive linear maps $\Lambda : C_c(Y(A_k)) \to C_c(X(A_k))$. It is thereby necessary to consider convergence factors which vary among the fibres and to consider fibres over A_k not defined over k even if $\pi : Y \to X$ is defined over k.

In section 5, we restrict to torsors $p : \mathcal{T} \to X$ for varieties over number fields and study adelic norms and measures for them. When X is smooth and proper with

$$H^1_{\mathrm{Zar}}(X, \mathcal{O}_X) = H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$$

and with torsion-free Néron-Severi group, then there is a notion of universal torsors $\pi : \mathcal{T} \to X$. The ν-adic order norms form an adelic norm which in its turn gives rise to a positive linear map

$$\Lambda : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k)).$$

This map depends on the choice of convergence factors. But one can choose these to be inverse to the convergence factors for $X(A_k)$ so that the induced measure on $C_c(\mathcal{T}(A_k))$ requires no convergence factors. Using this measure on $\mathcal{T}(A_k)$, we define Tamagawa numbers for universal torsors.

In section 6, we recall the Brauer group and torsor obstructions to weak approximation of Manin, Colliot-Thélène and Sansuc. By using their theory together with results of Ono on the arithmetic of tori, we relate our Tamagawa numbers for universal torsors to the Tamagawa numbers of Peyre. The factor $h^1(X)$ enters naturally.

In section 7, we modify the original growth conjectures of Manin in order to exclude the Fano varieties containing infinitely many weakly accumulating subvarieties. This is not so original and closely related to notions of Manin and Peyre (cf. **[51]**). We also generalize and refine Peyre's Tamagawa conjecture for Fano varieties by means of our Tamagawa numbers for universal torsors.

In section 8, we study the geometry of universal torsors $\pi : \mathcal{T} \to X$ over smooth complete toric varieties which are trivial over the unit element of the k-torus U in X. We identify them with the toric varieties studied by Cox in his article **[16]** and find that they are open subsets of affine spaces.

In section 9, we consider toric varieties over local fields k_ν. We give an explicit description of the norms for universal torsors obtained by inducing the norms of Batyrev-Tschinkel **[7]** and of the corresponding measures (cf. (9.12)). Another central idea is the introduction of a *canonical toric splitting*

$$\check{\psi}_\nu : X(k_\nu) \longrightarrow \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$

of the map

$$\check{\pi}_\nu : \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X(k_\nu).$$

induced by the principal universal torsor $\pi : \mathcal{T} \to X$. Here $T(k_\nu)_{\mathrm{cp}}$ is the maximal compact subgroup of the analytic group $T(k_\nu)$ defined by the Néron-Severi torus.

In section 10, we consider toric varieties over number fields and the induced adelic norm on the universal torsor obtained from the induced ν-adic norms in section 9. The product map of all $\check{\psi}_\nu$ gives rise to a continuous canonical toric splitting

$$\check{\psi}_A : X(A_k) \longrightarrow \mathcal{T}(A_k)/T(A_k)_{\mathrm{cp}}$$

of the map from $\mathcal{T}(A_k)/T(A_k)_{\mathrm{cp}}$ to $X(A_k)$ induced by π. By means of $\check{\psi}$ we give a new torsor theoretic interpretation of the heights of Batyrev-Tschinkel **[7]** and an interpretation of the constant $C = \alpha(X)\tau(X)h^1(X)$. There are several analogies with Bloch's use of torsors (cf. **[8]**, **[47]**) to interpret the Néron-heights and the Birch/Swinnerton-Dyer/Tate conjecture for abelian varieties. The universal torsors over toric varieties play the same role as the biextensions do for abelian varieties. The constant $\alpha(X)\tau(X)h^1(X)$ is interpreted by means of the Laurent expansion of an adelic integral which is a continuous approximation on the universal torsors of the zeta functions for X considered by Manin, Batyrev and Tschinkel. We also indicate how these zeta functions can be studied for toric varieties without making use of group structure of the torus.

In section 11, we give a proof of the conjectures of Manin and Peyre for split toric Fano varieties over $\mathbb{Q}$ (first proved in **[4]**). Our method is essentially an extension of the method developed by Schanuel **[56]** for projective spaces and by Peyre **[52]** for certain special blow-ups $X \to \mathbb{P}^n$ of projective spaces. But we make a more systematical use of the universal torsors than in **[52]** and we use geometric invariants of the fans in the study of the main terms and the error terms. Our asymptotic formulas are slightly more precise than in **[52]**, **[7]**, **[4]** and of the type

$$CB(\log B)^{r-1} + O(B(\log B)^{r-3/2+\varepsilon}), \quad r = \operatorname{rk}\operatorname{Pic} X.$$

while the formulas in (op. cit.) are of the type $CB(\log B)^{r-1}(1+o(1))$. But the real motivation for giving another proof of the theorem of Batyrev and Tschinkel is that many of the arguments can be used for other classes of Fano varieties like moduli spaces of ordered sets of points on the projective line.

The use of universal torsors in the study of Manin's conjectures is quite natural. We used them to give an upper bound for the counting functions of del Pezzo surfaces of degree 5 in a talk at Bern at the Borel seminars in the summer semester 1993. Some months later the author received a preliminary version of Peyre's important paper **[52]** which has had a strong influence on this paper. There he made implicit use of descent varieties for some classes of toric varieties without emphasizing their toric structure and without making use of the descent theory of Colliot-Thélène and Sansuc. Most of the theory of this paper has been developed in an attempt to use universal torsors to count points on toric varieties and to refine Peyre's Tamagawa number conjecture for general Fano varieties. It is thus not surprising that Peyre himself recently considered universal torsors. We understood from his visit at ETH 1996 that he has also found that the use of universal torsors leads to a term $h^1(X)$ in the conjectured asymptotic formula by means of adelic zeta-functions on the universal torsors. His line of thoughts was somewhat different, however, and did not use the induced measures of this paper. We refer to the paper of Peyre **[51]** in this volume for his vision of the role of universal torsors in the theory of counting functions for Fano varieties. We have only given some brief comments here about his work since we received it when this paper was almost completed. There is some overlapping between **[51]** and sections 5, 6 and 10 of this paper. But there are also many differences. Peyre consider certain equivariant partial compactifications of the universal torsors while we systematically avoid compactifications. He does not consider adelic splittings of the universal torsors, but uses instead the notion of "system of heights" for arbitrary linear systems. He uses complete intersections as a prototype for Fano varieties while we have chosen to dicuss toric varieties in detail.

We have learned much about the arithmetic of toric varieties from the impressive papers of Batyrev and Tschinkel [**7**] and [**4**]. Their approach using adelic harmonic analysis for Manin's zeta functions is elegant and conceptual, but depends strongly on the underlying group action on the toric variety. The advantage with universal torsors is that they exist for all Fano varieties. One can e.g. prove the best upper bound $O(B(\log B)^4)$ for del Pezzo surfaces of degree 5 by means of the descent theoretic approach in this paper.

We have after this paper was completed received the paper [**6**] of Batyrev and Tschinkel. The measure theory of our paper (e.g. (1.22) and (4.25)-(4.28)) has applications to the theory of L-primitive fibrations in [**6**]. One can also avoid the use of compactifications and the reference to Denef's work in (op. cit.) and give more intrinsic constructions of the measures involved by means of the results here.

We would like to thank Batyrev, Manin, Peyre and Tschinkel for discussions on counting problems for Fano varieties.

1. Analytic manifolds over locally compact fields

Throughout this section k denotes a non-discrete locally compact field of characteristic zero. It is well known (cf. [**66**]) that k is either totally disconnected (in which case it is a finite extension of the p-adic number field $\mathbb{Q}_p$) or connected (in which case it is either $\mathbb{R}$ or $\mathbb{C}$). Denote by $|\ |: k \to \mathbb{R}$ the absolute value *normalized* in the following way. If $k = \mathbb{R}$, let $|\alpha| = \max(\alpha, -\alpha)$ and if $k = \mathbb{Q}_p$, let $|up^l| = p^{-l}$ for units u in $\mathbb{Z}_p$. If k is a finite extension of $k_0 = \mathbb{R}$ or $k_0 = \mathbb{Q}_p$, let $|\ |: k \to \mathbb{R}$ be the map obtained by composing the norm $N: k \to k_0$ with $|\ |: k_0 \to \mathbb{R}$. The topology of k is induced by the absolute value $|\ |: k \to \mathbb{R}$ and k is complete with respect to $|\ |: k \to \mathbb{R}$ in all cases. We shall assume that all norms and all seminorms on vector spaces over k are compatible with the normalized absolute values $|\ |: k \to \mathbb{R}$ under scalar multiplication ([**36**, p. 33], [**38**, p. 44]).

If k is a finite extension of $k_0 = \mathbb{Q}_p$, let o be the maximal $\mathbb{Z}_p$-order (cf. [**54**]) in k. Then o is a complete discrete valuation ring. The inverse different o^D is defined by

$$o^D := \{\alpha \in k : \operatorname{Tr}(\alpha\beta) \in \mathbb{Z}_p \text{ for all } \beta \in o\}.$$

Denote by μ the Haar measure on the additive locally compact group k normalized in the following way. If $k = \mathbb{R}$, let μ be the usual Lebesgue measure dx and if $k = \mathbb{C}$, let $\mu = dx$ be the measure $2dudv$ for the real (resp. imaginary) part u and v. If k is a finite extension of $\mathbb{Q}_p$, choose μ to be self-dual as in [**66**, Ch. VII, §2]. This means that $\mu(o)\mu(o^D) = 1$. Hence $\mu(o) = 1$ if and only if k is a unramified over $\mathbb{Q}_p$. The Haar measure is sometimes (cf. e.g. [**67**], [**52**]) normalized such that $\mu(o) = 1$ for all finite extensions of $\mathbb{Q}_p$. The normalization chosen here implies that we get no discriminant factors for volumes of adelic measures. If K is a number field, and A_K the adelic group [**66**] of K, then $\mu(A_K/K) = 1$ for the restricted product measure of the local measures above (cf. [**67**, 2.1.3] for more details). Also, we get no discriminant factor (cf. (5.21), (6.12)) in our definition of Tamagawa numbers. Instead, we get a discriminant factor in the local results in (2.14) and (2.15).

Most of the theory of analytic manifolds over locally compact fields are obvious generalizations of the theory of analytic manifolds over $\mathbb{R}$ and $\mathbb{C}$ in differential and analytic geometry. We first recall some definitions (1.1)-(1.3) from Serre's book [**59**, Part II].

Definition 1.1. — Let $V \subset k^n$ be open and let $f : V \to k$ be a function. Then f is said to be analytic in V if for each point $P = (a_1, a_2, \ldots, a_n) \in V \subset k^n$, we may find an open neighbourhood of P

$$N_{P,r} = \{(x_1, x_2, \ldots, x_n) \in k^n : |x_i - a_i| < r, \quad i = 1, \ldots, n\} \subset V$$

such that f is defined by a convergent power series in $k\,[\![x_1 - a_1, \ldots, x_n - a_n]\!]$ on $N_{P,r}$.

Definition 1.2. — Let T be a topological space.

(a) An *n-chart* (or *n-coordinate map*) on T is a homeomorphism $\phi : U \to V$ between open subsets $U \subset T$ and $V \subset k^n$.
(b) Two n-charts $\phi : U \to V$ and $\phi' : U' \to V'$ are *compatible* if the maps $\phi' \circ \phi^{-1}|\phi(U \cap U')$ and $\phi \circ \phi'^{-1}|\phi'(U \cap U')$ are analytic.
(c) An *n-atlas* A on T is a family of n-charts $\phi_j : U_j \to V_j$, $j \in J$ such that $T = \cup_{j \in J} U_j$ and such that ϕ_{j_1} and ϕ_{j_2} are *compatible* for any $j_1, j_2 \in J$.
(d) Two n-atlases

$$A_i := \{\psi_i : U_i \to V_i,\ i \in I\}, A_j := \{\phi_j : U_j \to V_j,\ j \in J\}$$

are said to be *compatible* if ψ_i and ϕ_j are compatible for any $\psi_i \in A_i$, $\phi_j \in A_j$.

Compatibility of n-atlases is an equivalence relation (cf. LG 3.2 in op. cit.).

Definition 1.3. — Let T be a topological space. An analytic n-manifold structure on T (over k) is an equivalence class of compatible n-atlases on T.

Note that the analytic manifold structures considered in this paper are such that the dimension n is the same at all points of the manifold.

Let M_1 and M_2 be two analytic manifolds (over k) of possibly different dimensions. A continuous map $f : M_1 \to M_2$ is said to be an analytic morphism if it is "locally given by analytic functions" (cf. LG 3.6 in op. cit. for a precise definition).

Definition 1.4. — An *analytic vector bundle of rank r over an n-manifold M* consists of a family $\{E_P\}_{P \in M}$ of r-dimensional vector spaces over k parametrized by M, together with an analytic $(n+r)$-manifold structure on $E = \cup_{P \in M} E_P$ such that:

(i) The projection map $\pi : E \to M$ taking E_P to P is analytic.
(ii) For every $P_0 \in M$, there is an open neighbourhood $U \subset M$ of P_0 and an analytic isomorphism $\phi_U : \pi^{-1}(U) \to U \times k^r$ which restricts to a linear homeomorphism from E_P to $\{P\} \times k^r$ for each $P \in U$.

The tangent bundle $\mathrm{Tan}(M) \to M$ and the cotangent bundle $\mathrm{Cot}(M) \to M$ are analytic bundles of rank n on an analytic n-manifold M (see [**12**], [**38**, Ch. XXII] for the global properties and [**59**, Ch. 3, §8] for the local properties of these bundles).

Let $p : E \to M$ be a vector bundle and let $f : N \to M$ be an analytic morphism. The *fibre product*

$$N \times_M E = \{(Q, s) \in N \times E : f(Q) = p(s)\}$$

is an analytic submanifold of $N \times E$ (see [**59**, LG 3.26]) and the projection map

$$p_N : N \times_M E \longrightarrow N$$

is a vector bundle of the same rank as $p : E \to M$. (The trivializations of $p_N : N \times_M E \to N$ and their transition functions are defined by pulling back the trivializations of the bundle on M and their transition functions, cf. e.g. [**25**, p. 68] for the case $k = \mathbb{C}$.) We shall in the sequel write f^*E for $N \times_M E$ and call $p_N : N \times_M E \to N$ the *pullback bundle* of $p : E \to M$ under $f : N \to M$.

Operations on vector spaces induce operations on vector bundles (cf. e.g. [**25**, pp. 66-67]). There exists thus for each vector bundle $p : E \to M$ a *dual* vector bundle $p^* : E^* \to M$ and exterior product bundles $\Lambda^s E \to M$. The *cotangent bundle* $\mathrm{Cot}(M) \to M$ is dual to the *tangent bundle* $\mathrm{Tan}(M) \to M$. We shall write $\mathbf{det}\, E$ for the line bundle $\Lambda^r E$ when $r = \mathrm{rk}\, E$ and call $\mathbf{det}\,\mathrm{Cot}(M)$ the *canonical bundle* and $\mathbf{det}\,\mathrm{Tan}(M)$ the *anticanonical bundle* of M.

If $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are two analytic vector bundles, then there are analytic bundles $E_1 \oplus E_2 \to M$ and $E_1 \otimes E_2 \to M$. The analytic manifold $E_1 \oplus E_2$ is equal to the fibre product $E_1 \times_M E_2$ and hence an analytic submanifold of $E_1 \times E_2$ (see [**59**, LG 3.26]). If $g : E_1 \to E_2$ is an analytic map with $\pi_1 = \pi_2 g$ such that corresponding maps between the fibres $g_P : E_{1,P} \to E_{2,P}$ are linear for each $P \in M$, then we define vector bundles $\ker g \to M$ (resp. $\mathrm{coker}\, g \to M$) when g is surjective (resp. injective).

Definition 1.5. — Let $p : E \to M$ be an (analytic) vector bundle over an analytic manifold M (over k). A *norm* (resp. *seminorm*) on $p : E \to M$ is a continuous map

$$\|\ \| : E \longrightarrow [0, \infty)$$

such that the restriction to E_P is a norm (resp. seminorm) on the vector space E_P (over k) for each $P \in M$. A *normed vector bundle* is a vector bundle with a norm.

Remarks 1.6

(a) Silverman [**63**] and Peyre [**52**] define a metric

$$\| \ \| : E \longrightarrow [0, \infty)$$

on a line bundle $p : E \to M$ to be a collection of vector space norms

$$\| \ \|_x : E_x \longrightarrow [0, \infty), \quad x \in M$$

for which the map

$$P \in U \longrightarrow \|s(P)\| \in [0, \infty)$$

is continuous for any local analytic section $s : U \to p^{-1}(U)$ of $p : E \to M$. It is obvious that a norm on a line bundle $p : E \to M$ is a metric. Conversely, each metric $\| \ \| : E \to [0, \infty)$ is a norm. To see this, one works locally at M and reduces to the case $E = M \times k$ by means of (1.4)(ii). Then use the fact

$$P \in M \longrightarrow s(P) = (P, 1) \in E$$

is an analytic section of $pi : E \to M$ with

$$\|(P, \alpha)\| = |\alpha| \|s(P)\|$$

for all $P \in M$, $\alpha \in k$.

(b) Suppose we are given one norm $\| \ \|_1$ and one seminorm $\| \ \|_2$ on a line bundle $p : E \to M$ over an analytic manifold M. Then there is a unique continuous function $r : M \to [0, \infty)$ such that

$$\|s\|_2 = r(p(s)) \|s\|_1$$

for all $s \in E$. If M is compact and $\| \ \|_2$ is a norm then there are positive constants C and D such that

$$C \|s\|_1 \leq \|s\|_2 \leq D \|s\|_1$$

for all $s \in E$.

Any vector space norm on k^r defines a norm on the trivial vector bundle $M \times k^r$ by means of the projection $M \times k^r \to k^r$. We now give some further examples of norms and seminorms on (analytic) vector bundles.

Examples 1.7

(a) Let $p : E \to M$ be a line bundle and let ω be a continuous global section of the dual line bundle $p^\vee : E^\vee \to M$. Then ω may be regarded as a continuous map $E \to M \times k$ which restricts to the linear map $E_P \to P \times k$ for each $P \in M$. We obtain a seminorm $\| \ \| : E \to [0, \infty)$ by sending $s \in E$ to the absolute value of the image of $\omega(s)$ under the projection $M \times k \to k$. In particular, any continuous differential form ω of maximal degree on M defines a seminorm on the anticanonical line bundle $\mathbf{det}\,\mathrm{Tan}(M)$. This seminorm is a norm if and only if ω vanishes nowhere.

(b) Let $p_1 : E_1 \to M$ and $p_2 : E_2 \to M$ be two line bundles and

$$\|\ \|_1 : E_1 \longrightarrow [0,\infty), \qquad \|\ \|_2 : E_2 \longrightarrow [0,\infty)$$

be seminorms on E_1 resp. E_2. Then there exists a product seminorm

$$\|\ \| : E_1 \otimes E_2 \longrightarrow [0,\infty)$$

such that

$$\|(s_1 \otimes s_2)\| = \|s_1\|_1 \cdot \|s_2\|_2$$

for two sections $s_1 \in E_1$, $s_2 \in E_2$ with $p_1(s_1) = p_2(s_2)$. This seminorm $\|\ \|$ is a norm if and only if $\|\ \|_1$ and $\|\ \|_2$ are norms.

(c) Let $p : E \to M$ be a vector bundle and let $p_N : N \times_M E \to N$ be the pullback bundle of $p : E \to M$ under $f : N \to M$. Then there is a seminorm

$$f^*\|\ \| : N \times_M E \longrightarrow [0,\infty)$$

on the pullback bundle associated to each seminorm $\|\ \|$ on $p : E \to M$. This is obtained by composing the projection map $N \times_M E \to E$ with $\|\ \| : E \to [0,\infty)$. The seminorm $f^*\|\ \|$ is a norm if $\|\ \|$ is a norm. We shall then call $f^*\|\ \|$ the pullback norm of $\|\ \|$.

In the next example we shall need the concept of a lattice in a vector bundle.

Definition 1.8. — Let o be a complete discrete valuation ring as above with quotient field k. Let M be an analytic manifold over k and $p : E \to M$ be an analytic vector bundle of rank r. An *analytic lattice* of $p : E \to M$ is a submanifold L of E such that

(a) $L_P := L \cap E_P$ is an o-lattice in E_P for each $P \in M$.

(b) For every $P_0 \in M$, there is an open neighbourhood $U \subset M$ of P_0 and an analytic isomorphism $\lambda_U : L \cap p^{-1}(U) \to U \times o^r$ which restricts to an o-module isomorphism from L_P to $\{P\} \times o^r$ for each $P \in U$.

Example 1.9. — Let $p : E \to M, L, o, k$ be as in (1.8) and let p be a uniformizing parameter of o. If $P \in M$ and $s \in E_P$, let

$$\|s\| = \inf\{|\pi^m| : \pi^m s \in L_P\}.$$

Then,

$$\|\ \| : E \longrightarrow [0,\infty)$$

is a norm. (The verification is local on the base so it suffices (see (1.8)(b) to treat the trivial case where p is the projection from $E = M \times k^r$ to M and $L = M \times o^r$.)

The following result is a reformulation of a result due to Peyre [**52**].

Theorem 1.10. — *Let M be an analytic Hausdorff n-manifold and let $\| \ \|$ be a seminorm on the anticanonical bundle* $\mathbf{det}\,\mathrm{Tan}(M)$ *of M. Then there is a unique positive linear functional Λ on the vector space $C_c(M)$ of real valued continuous functions with compact support on M such that for any n-chart*

$$\phi : U \longrightarrow V \subset k^n,$$

$$\phi(x) = (\phi_1(x), \dots, \phi_n(x)) = (x_1, \dots, x_n), x \in U \subset M$$

and any $f \in C_c(M)$ with support in U the following equality holds:

$$\Lambda f = \int_V f(\phi^{-1}(x_1, \dots, x_n)) \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \right\| dx_1 dx_2 \cdots dx_n.$$

Moreover, if $\| \ \|$ is a norm and $f : M \to \mathbb{R}_{\geq 0}$ is a non-negative function in $C_c(M)$, then $\Lambda f = 0$ if and only if $f = 0$.

Proof. — Let $\phi_\alpha : U_\alpha \to V_\alpha \subset k^n$, $\phi_\beta : U_\beta \to V_\beta \subset k^n$ be two charts and suppose that $\phi \in C_c(M)$ have support in $U_a \cap U_b$. Then Peyre (op. cit.) deduces from the Jacobian formula for change of variables in a multiple integral ([**67**, p. 14]) that the two integrals over V_a and V_b coincide so that Λf is well defined. His argument uses only that $\| \ \|$ is a seminorm although it is formulated for a metric (cf. (1.6)(a)). Now once the compatibility of integrals under transition of charts has been established, one concludes by a standard argument with partitions of unity (see th. 5.1 in [**38**, Ch. IX] and note that no paracompactness assumption is needed).

To prove the last statement, it suffices by partition of unity to consider the case when there is an isomorphism $\phi : M \to V$ onto an open subset V of k^n. But then the assertion follows from the integral formula above, thereby completing the proof. □

Since M is locally compact and Hausdorff there exists according to Riesz representation theorem [**38**, Ch. IX, §2] a unique positive Borel measure m on M such that:

1.11 (i) If W is open, then

$$m(W) = \sup \Lambda f,$$

where $f \in C_c(M)$ runs over all functions with support in W such that $0 \leq f(x) \leq 1$ for all $x \in M$.

1.11 (ii) If B is a Borel set, then $m(B) = \inf m(W)$, W open $\supset M$.

This measure m has the following properties:

1.11 (iii) If K is compact, then $m(K)$ is finite.

1.11 (iv) If B is open or a union of countably many Borel sets of finite measure, then $m(B) = \sup m(K)$, K compact $\subset B$.
1.11 (v) $\Lambda f = \int_M f dm$ for any $f \in C_c(M)$.

Definition 1.12. — We shall call m *the positive Borel measure on M determined by the seminorm* $\| \ \| : \mathbf{det}\, \mathrm{Tan}(M) \to [0, \infty)$ and $\Lambda : C_c(M) \to \mathbb{R}$ the *positive functional determined* by $\| \ \| : \mathbf{det}\, \mathrm{Tan}(M) \to [0, \infty)$.

If $\| \ \|$ is a norm, then it follows from (1.11) and the last assertion in (1.10) that $m(W) > 0$ for all non-empty open subsets W of M.

Example 1.13. — Let $\| \ \| : \mathbf{det}\, \mathrm{Tan}(M) \to [0, \infty)$ be a seminorm defined by a continuous differential form ω on M (see 1.7). Then the measure m is just the "usual" positive Borel measure $|\omega|$ on M determined by ω (see e.g. [**38**, Ch. XXIII, §3]) for the case $k = \mathbb{R}$ and [**12**, §10] for general locally compact fields k). The construction of m in (1.10), (1.11) is thus a generalization of the classical volume form construction.

Definition 1.14. — A positive Borel measure m on a locally compact Hausdorff space M will be called σ-regular if it satisfies the properties (1.11)(ii), (1.11)(iii) and (1.11)(iv) in (1.11). It will be called regular if it satisfies (1.11)(ii), (1.11)(iii) and the following stronger version of (1.11)(iv):
$(*)$ If W is a Borel set, then $m(W) = \sup m(K)$, K compact W.

The space M is called σ-finite with respect to m if M is a union of countably many Borel sets $B \subseteq M$ of finite measure $m(B)$.

Remarks 1.15

(a) It follows from the definitions that (cf. [**38**, p. 257]) a σ-regular positive Borel measure m on M is regular if M is σ-finite with respect to m. The σ-finiteness condition is satisfied (see (1.11)(iii)) if M is σ-compact (i.e. a union of countably many compact subsets). It is sometimes useful for integration with respect to product measures.

(b) The positive Borel measure m determined by a seminorm

$$\| \ \| : \mathbf{det}\, \mathrm{Tan}(M) \to [0, \infty)$$

on an analytic Hausdorff n-manifold M is σ-regular (by 1.11) and regular if M is σ-compact.

Lemma 1.16. — *Let M_1 and M_2 be two locally compact Hausdorff spaces and let m_1 (resp. m_2) be a σ-regular positive Borel measure on M_1 (resp. M_2). Then there* *exists a unique σ-regular positive Borel measure m on $M := M_1 \times M_2$ such that*

$$\int_M hdm = \int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$$

for any function $h \in C_c(M)$ which is a product

$$h(P_1, P_2) = h_1(P_1)h_2(P_2)$$

of two functions $h_1 \in C_c(M_1), h_2 \in C_c(M_2)$.

Moreover, if $h(P_1, P_2) \in C_c(M)$, then

$$g(P_2) := \int_{M_1} h(P_1, P_2)dm_1 \in C_c(M_2)$$

and

$$\int_M hdm = \int_{M_2} gdm_2 = \int_{M_2} \left(\int_{M_1} h(P_1, P_2)dm_1 \right) dm_2.$$

Proof. — This follows from [**11**, Ch. III, §5] and Riesz representation theorem. □

Let M_1 and M_2 be two analytic manifolds over k. The topological space $M := M_1 \times M_2$ carries a natural analytic manifold structure described in [**59**, p. LG 3.7]. Let $pr_i : M \to M_i$, $i = 1, 2$ be the two projections and let $pr_i^* \operatorname{Tan}(M_i)$, $i = 1, 2$ be the pull back bundles (cf. (1.7(c)) of the tangent bundles on M_1 and M_2. Then there exists a natural identification

$$\operatorname{Tan}(M) = pr_1^* \operatorname{Tan}(M_1) \oplus pr_2^* \operatorname{Tan}(M_2)$$

of vector bundles over M which induces a canonical isomorphism

$$\mathbf{det} \operatorname{Tan}(M) = pr_1^*(\mathbf{det} \operatorname{Tan}(M_1)) \otimes pr_2^*(\mathbf{det} \operatorname{Tan}(M_2)).$$

Suppose that we are given seminorms:

$$\| \ \|_{M_i} : \mathbf{det} \operatorname{Tan}(M_i) \longrightarrow [0, \infty), \quad i = 1, 2.$$

Then there are pullback seminorms $pr_i^* \| \ \|_{M_i}$ on $pr_i^*(\mathbf{det} \operatorname{Tan}(M_i))$, $i = 1, 2$ (cf. (1.7)(c)) which are norms if (and only if) $\| \ \|_{M_i}$ are norms.

Theorem 1.17. — *Let M_1 and M_2 be two Hausdorff analytic manifolds over k and let M be the product manifold $M = M_1 \times M_2$. Let*

$$\| \ \|_{M_i} : \mathbf{det} \operatorname{Tan}(M_i) \longrightarrow [0, \infty), \quad i = 1, 2$$

be two seminorms and let

$$\| \ \|_M : \mathbf{det} \operatorname{Tan}(M) \longrightarrow [0, \infty)$$

be the product seminorm of the two pullback seminorms

$$pr_i^* \| \ \|_{M_i} : pr_i^*(\mathbf{det} \operatorname{Tan}(M_i)) \longrightarrow [0, \infty).$$

Let m_i, $i = 1, 2$ (resp. m) be the σ-regular positive Borel measures determined by $\| \|_{M_i}$, $i = 1, 2$ (resp. $\| \|_M$). Then the following assertions hold.

(a)
$$\int_M hdm = \int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$$
for any function $h \in C_c(M)$ which is a product $h(P_1, P_2) = h_1(P_1)h_2(P_2)$ of two functions $h_1 \in C_c(M_1)$, $h_2 \in C_c(M_2)$.

(b) *If $h(P_1, P_2) \in C_c(M)$, then*
$$g(P_2) = \int_{M_1} h(P_1, P_2) dm_1 \in C_c(M_2)$$
and
$$\int_M hdm = \int_{M_2} gdm_2 = \int_{M_2} \left(\int_{M_1} h(P_1, P_2) dm_1 \right) dm_2.$$

Proof

(a) Using charts and partitions of unity and the definition of the functionals on $C_c(M_1)$, $C_c(M_2)$ and $C_c(M)$ corresponding to m_i, $i = 1, 2$ (see 1.10) it is clear that it suffices to treat the case when $M_1 = k^q$ and $M_2 = k^t$, $q, t \in \mathbb{Z}_{>0}$.

Let us choose coordinates $(x_1, \ldots, x_q)$ for M_1 and $(x_{q+1}, \ldots, x_{q+t})$ for M_2. Then by definition we have that

$$\int_M hdm =$$
$$\int_{k^{q+t}} h(x_1, \ldots, x_{q+t}) \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_q} \right\| \left\| \frac{\partial}{\partial x_{q+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{q+t}} \right\| dx_1 dx_2 \cdots dx_{q+t}$$

where the seminorms should be read as $\| \|_{M_1}$ resp. $\| \|_{M_2}$.

Let
$$f_1(x_1, \ldots, x_q) = h_1(x_1, \ldots, x_q) \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_q} \right\|$$
and
$$f_2(x_{q+1}, \ldots, x_{q+t}) = h_2(x_{q+1}, \ldots, x_{q+t}) \left\| \frac{\partial}{\partial x_{q+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{q+t}} \right\|.$$

Then (cf. (1.6)(a)) $f_1 \in C_c(k^q)$ and $f_2 \in C_c(k^t)$. Therefore, by an elementary version of Fubini's theorem it follows that

$$\int_M hdm$$
$$= \int_{k^q} f_1(x_1, \ldots, x_q) dx_1 dx_2 \cdots dx_q \int_{k^{q+t}} f_2(x_{q+1}, \ldots, x_{q+t}) dx_{q+1} \cdots dx_{q+t}$$

where the right hand side is equal to $\int_{M_1} h_1 dm_1 \int_{M_2} h_2 dm_2$ by definition.

(b) This follows from (a) and the previous lemma. □

Now let $f : N \to M$ be an analytic morphism from an m-manifold to an n-manifold. Then $f^* \operatorname{Tan}(M) = N \times_M \operatorname{Tan}(M)$ is an analytic vector bundle over N of rank m. It is endowed with a canonical morphism (cf. [**59**, LG 3.12] and [**12**])

$$f' : \operatorname{Tan}(N) \longrightarrow N \times_M \operatorname{Tan}(M)$$

of vector bundles over N. The analytic morphism $N \to M$ is called an *immersion* (resp. a *submersion*) if it "locally looks like" a linear injection (resp. surjection) $k^n \to k^m$ (cf. LG 3.12-14 in (op. cit.) for the precise meaning of this phrase). There is also a proof in (op. cit.) that f is an immersion (resp. a submersion) if and only if f' is injective (resp. surjective).

Definition 1.18

(a) If $f : N \to M$ is an immersion, then the *normal bundle* $\operatorname{Nor}(N/M)$ is defined to be the cokernel of $f' : \operatorname{Tan}(N) \to N \times_M \operatorname{Tan}(M)$.

(b) If $f : N \to M$ is a submersion, then the *relative tangent bundle* $\operatorname{Tan}(N/M)$ is defined to be the kernel of f(see [**12**, 8.1]). The *relative cotangent bundle* $\operatorname{Cot}(N/M)$ is the dual vector bundle of $\operatorname{Tan}(N/M)$.

If $f : N \to M$ is a submersion, then $\mathbf{det}\operatorname{Tan}(N/M)$ is an (analytic) line bundle over N, which we shall call the *relative anticanonical line bundle* of $f : N \to M$. The restriction of $\mathbf{det}\operatorname{Tan}(N/M)$ to a fibre $N_P, P \in M$ of $f : N \to M$ is equal to the anticanonical line bundle $\mathbf{det}\operatorname{Tan}(N_P)$ of N_P. There is also a canonical isomorphism

$$\mathbf{det}\operatorname{Tan}(N) = \mathbf{det}(\operatorname{Tan}(N/M)) \otimes f^*(\mathbf{det}\operatorname{Tan}(M)) \tag{1.19}$$

of line bundles over N induced by the exact sequence of vector bundles over N

$$0 \longrightarrow \operatorname{Tan}(N/M) \longrightarrow \operatorname{Tan}(N) \xrightarrow{f'} N \times_M \operatorname{Tan}(M) \longrightarrow 0. \tag{1.20}$$

Notation 1.21. — Let $f : N \to M$ be a submersion between two analytic Hausdorff manifolds and let

$$\|\ \|_{N/M} : \mathbf{det}\operatorname{Tan}(N/M) \longrightarrow [0,\infty)$$

be a seminorm on $\mathbf{det}\operatorname{Tan}(N/M) \to N$.

If $P \in f(N)$, let

$$\Lambda_P : C_c(N_P) \longrightarrow \mathbb{R}$$

be the positive linear functional on the fibre N_P of $f : N \to M$ at P determined by the restriction

$$\|\ \|_P : \mathbf{det}\operatorname{Tan}(N_P) \longrightarrow [0,\infty)$$

of $\| \ \|_{N/M}$ to N_P (see (1.10)) and let $h_P \in C_c(N_P)$ be the restriction of $h \in C_c(N)$. Then we denote by

$$\Lambda_{N/M}(h) : M \longrightarrow \mathbb{R}$$

the function with value $\Lambda_P(h_P)$ for $P \in f(N)$ and with value 0 if $P \notin f(N)$.

A linear map

$$\Lambda_{N/M} : C_c(N) \longrightarrow C_c(M)$$

is said to be positive if any non-negative function $h \in C_c(N;\mathbb{R})$ is sent to a non-negative function $g \in C_c(M;\mathbb{R})$.

The following result is stated (without proof) by Serre [**61**, p. 83] in the special case of seminorms $\| \ \|_{N/M}$ and $\| \ \|_M$ defined by global sections of $\mathbf{det}\,\mathrm{Cot}(N/M)$ and $\mathbf{det}\,\mathrm{Cot}(M)$ (cf. (1.7)(a)).

Theorem 1.22. — *Let $f : N \to M$ be a submersion between two analytic Hausdorff manifolds. Let*

$$\begin{aligned} \| \ \|_{N/M} &: \mathbf{det}\,\mathrm{Tan}(N/M) \longrightarrow N \\ \| \ \|_{M} &: \mathbf{det}\,\mathrm{Tan}(M) \longrightarrow M \end{aligned}$$

be seminorms and let

$$\| \ \|_N : \mathbf{det}\,\mathrm{Tan}(N/M) \otimes f^*\,\mathbf{det}(\mathrm{Tan}(M)) \longrightarrow [0,\infty)$$

be the seminorm on $\mathbf{det}\,\mathrm{Tan}(N) \to N$ obtained by taking the product of $\| \ \|_{N/M}$ and $f^*\| \ \|_M$ (cf. (1.7)(b), (1.7)(c) and (1.19)).

Then the following holds,

(a) *$\Lambda_{N/M}$ is a positive linear map from $C_c(N)$ to $C_c(M)$.*

(b) *Let $\Lambda_M : C_c(M) \to \mathbb{R}$ (resp. $\Lambda_N : C_c(N) \to \mathbb{R}$) be the positive functionals determined by $\| \ \|_M$ (resp. $\| \ \|_N$). Then $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.*

(c) *Let m resp. n be the positive σ-regular Borel measures on M (resp. N) determined by $\| \ \|_M$ (resp. $\| \ \|_N$. Let $\theta(P)$, $P \in f(M)$ be the positive σ-regular Borel measure on N_P (cf. (1.11)) corresponding to the positive functional $\Lambda_P : C_c(N_P) \to \mathbb{R}$ determined by the restriction to the fibre N_P of $f : N \to M$ at P of $\| \ \|_{N/M}$ to N_P (cf. (1.10), (1.21)). Then,*

$$\int_N h dn = \int_M \Lambda_{N/M}(h) dm = \int_{P\in f(M)} \left(\int_{N_P} h_P d\theta(P) \right) dm$$

for any $h \in C_c(N)$. The integral over N_P is defined to be 0 if N_P is empty.

Proof

(a) It is obvious from the definition in (1.21) that $\Lambda_{N/M}$ is linear and positive. It thus remains to show that $\Lambda_{N/M}(h) \in C_c(M)$ for $h \in C_c(N)$. To show this, we first note that $f(N)$ is an open subset of M (since any submersion is open) and that the support of $\Lambda_{N/M}(h)$ is contained in the compact subset $f(\operatorname{Supp} h)$ of $f(N)$. It is therefore sufficient to prove that $\Lambda_{N/M}(h)$ is continuous in an open neighbourhood of $P = f(Q) \in M$ for each $Q \in N$.

Let t resp. s be the dimensions of N resp. M. Then since f is a submersion there exist open analytic neighbourhoods U of Q, V of P and W of 0 in k^{t-s} and an analytic isomorphism $\psi : U \to V \times W$ such that $f(U) = V$ and the following diagram commutes (cf. [**59**, LG 3.16])

(1.23)
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi"'] & V \\ V \times W \arrow[ur, "pr_1"'] & \end{tikzcd}$$

If we cover N with such subsets U and apply the linearity of Λ_P for all $P \in M$, then it is clear from the "usual" argument with partitions of unity [**38**, p. 270] that it suffices to show that $g := \Lambda_{N/M}(h) \in C_c(M)$ for $h \in C_c(N)$ with support in an open subset of U as above. Further, if $h_U \in C_c(U)$ is the restriction of h, then $\Lambda_{N/M}(h) = \Lambda_{U/V}(h_U)$ on V and $\Lambda_{N/M}(h) = 0$ outside V. We may thus assume that $N = U$ and $M = V$.

Since ψ is an isomorphism and (1.23) commutes and the constructions of Λ_P (see (1.10)) are functorial under isomorphisms we may and shall further assume that $U = V \times W$ and $f = pr_1$. Then $N = M \times W$ and $\mathbf{det}\,\mathrm{Tan}(N/M) = pr_2^*(\mathbf{det}\,\mathrm{Tan}(W))$ for the projection $pr_2 : M \times W \to W$.

If $\|\ \|_{N/M} : \mathbf{det}\,\mathrm{Tan}(N/M) \to [0, \infty)$ is the pullback $pr_2^*\|\ \|_W$ of a seminorm $\|\ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, then it follows from (1.17)(b) that $\Lambda_{N/M}$ is a positive linear map from $C_c(N)$ to $C_c(M)$.

In the general case, choose a norm $\|\ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, and let $\lambda_{N/M}$ be the linear map $C_c(M) \to C_c(N)$ determined by the norm $pr_2^*\|\ \|_W$ on

$$\mathbf{det}\,\mathrm{Tan}(N/M) \to N.$$

Then there exists a unique continuous function $r : N \to [0, \infty)$ such that $\|s\|_{N/M}$ is equal to $r(Q)pr_2^*\|s\|_W$ for any section $s \in \mathbf{det}\,\mathrm{Tan}(N/M)$ above $Q \in N$ (see (1.6)(b)). Therefore, $\Lambda_{N/M}(h) = \lambda_{N/M}(rh)$ for all $h \in C_c(N)$. Therefore, the general case follows from the already known case where $\|\ \|_{N/M} = pr_2^*\|\ \|_W$.

(b) One reduces immediately to the case $N = M \times W$ and $f = pr_1$ by means of the same arguments as in the proof of (a). If $\|\ \|_{N/M}$ is the pullback $pr_2^*\|\ \|_W$ of a seminorm $\|\ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$, then it follows from (1.17)(b) that $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.

In the general case, choose a norm $\| \ \|_W$ on $\mathbf{det}\,\mathrm{Tan}(W) \to W$ and define the linear map $\Lambda_{N/M} : C_c(M) \to C_c(N)$ as above. Let $r : N \to [0, \infty)$ be the function described in the proof of (a) and let $\Lambda_N : C_c(N) \to \mathbb{R}$ be the functional defined by the product seminorm $\| \ \|_{M\times W}$ of $pr_1^*\| \ \|_M$ and $pr_2^*\| \ \|_W$. Then $\|s\|_N = r(Q)\|s\|_{M\times W}$ for any section $s \in \mathbf{det}\,\mathrm{Tan}(N)$ above. Therefore $\Lambda_N(h) = \lambda_N(rh)$ and $\Lambda_{N/M}(h) = \lambda_{N/M}(rh)$ for all $h \in C_c(N)$ (see (a)). This combined with $\lambda_N = \Lambda_M \circ \lambda_{N/M}$ (see (1.17)(b)) implies that $\Lambda_N = \Lambda_M \circ \Lambda_{N/M}$.

(c) This follows from (b) and Riesz representation theorem. □

2. Measures and densities for algebraic varieties over local fields

Let k denote a non-discrete locally compact field of characteristic zero and let X be a smooth k-variety. It is well known that the set of k-points $X(k)$ on X can be given a natural analytic manifold structure $X_{\mathrm{an}}(k)$ and that this construction is functorial. The best reference for the applications here seems to be chapter III of the book [**53**].

We first define the underlying topology of $X(k)$. If X is an affine k-scheme of finite type, then the k-topology on $X(k)$ is defined to be the coarsest topology for which all maps $X(k) \to k$ defined by regular functions on X are continuous. We shall say that a subset of $X(k)$ is k-open (resp. k-closed) if it is open (resp. closed) in the k-topology.

In the following result (2.1) we always refer to the k-topology when regarding the set of k-points $X(k)$ on an affine k-scheme X as a topological space.

Proposition 2.1

(a) *Let $g : X \to Y$ be a closed k-immersion of two affine schemes of finite type over k. Then g induces a closed immersion $X(k) \to Y(k)$ of topological spaces*

(b) *Let $Z = X \times_k Y$ be a product of two affine schemes of finite type over k. Then $Z(k) = X(k) \times Y(k)$ as topological spaces.*

(c) *Let $g : X \to Y$ be an open k-immersion of two affine schemes of finite type over k. Then g induces an open immersion $X(k) \to Y(k)$ of topological spaces.*

(d) *The k-topology on $A_k^r(k)$ is equal to the direct product topology on $k^r = A_k^r(k)$.*

Proof. — (a) and (b) are obvious and the proofs left to the reader.

(c) X may be identified with the closed k-subvariety of $Y \times A_k^1 = Y \times \mathrm{Spec}\, k[T]$ defined by the equation $hT = 1$ for some invertible function h in the coordinate ring of Y. By (a) and (b) one gets that $X(k)$ is the closed subset of the product space $Y(k) \times A_k^1(k)$ defined by $hT = 1$. Now use the fact that the projection of $X(k) \subset Y(k) \times A_k^1(k)$ onto $Y(k)$ is an open immersion of topological spaces.

(d) It suffices by (b) to consider the case $X = A_k^1 = \operatorname{Spec} k[T]$. Then any regular function on X is a polynomial in T with coefficients in k. This combined with the fact that k is a topological ring implies that the k-topology on $X(k)$ is the coarsest topology for which $T : X(k) \to k$ is continuous. Hence $T : X(k) \to k$ is a homeomorphism, as was to be proved. □

Proposition and definition 2.2. — *Let X be a separated scheme of finite type over k. Then the family of all k-open subsets of sets of k-points on all Zariski open affine k-subschemes form the base of a topology of $X(k)$. If X is affine this topology is the k-topology above. If X is not affine we call this topology the k-topology of $X(k)$.*

Proof. — The statement follows immediately from (2.1)(c). □

It is obvious from the definition that a k-morphism $g : Y \to X$ between two separated schemes of finite type over k induces a continuous map $Y(k) \to X(k)$ with respect to the k-topologies. It is also clear that the assertions in (2.1)(a)-(c) hold in the non-affine case.

Proposition 2.3

(a) *$X(k)$ is a Hausdorff space with respect to the k-topology.*
(b) *There is a countable base of open subsets with compact support for the k-topology. In particular, $X(k)$ is σ-compact and paracompact with respect to the k-topology.*
(c) *If $X = P_k^n$, then the k-topology of $X(k)$ is the quotient topology arising from the canonical map $k^{n+1}\backslash(0) \to P_k^n$.*
(d) *If X is a closed k-subscheme of P_k^n for some n, then $X(k)$ is compact with respect to the k-topology.*
(e) *If X is a smooth, proper and geometrically connected k-scheme, then $X(k)$ is compact with respect to the k-topology.*

Proof

(a) X is Zariski closed in $X \times_k X$ by assumption. The diagonal of $X(k) \times X(k)$ is therefore closed in the k-topology.

(b) It suffices to prove this for a finite set of open affine subvarieties in a covering of X. To show this, use (2.1) and the fact that the locally compact field k has such a base.

(c) This is checked locally using the standard covering of P_k^n by $n + 1$ affine n-spaces.

(d) It suffices by the non-affine version of (2.1)(a) to prove this in the case $X = P_k^n$. Then use the fact that the compact subset of $k^{n+1}\backslash(0)$ consisting of $(n+1)$-tuples $(x_0, x_1, \ldots, x_n)$ with $\sup_i |x_i| = 1$ is mapped onto P_k^n under the canonical map from $k^{n+1}\backslash(0)$ to P_k^n.

(e) There exists by Chow's lemma [**32**, p 107] a proper k-morphism $g : Y \to X$ from a projective k-variety Y and a Zariski non-empty open k-subvariety U of X such that g induces an isomorphism of $g^{-1}(U)$ to U. This implies by (a) and(c) that $g(Y(k))$ is a k-closed subset of $X(k)$ containing $U(k)$. But X is smooth and hence $U(k)$ dense in $X(k)$ (cf. [**53**, lemma 3.2]). Therefore $g(Y(k)) = X(k)$, thereby completing the proof. □

The smoothness condition in (2.3)(e) is not necessary, but will be satisfied for all the applications in this paper. One can prove that any proper morphism $g : Y \to X$ between k-varieties gives rise to a proper continuous map $Y(k) \to X(k)$ between topological spaces as remarked by Serre in [**61**, Ch. 2].

Next, let o be a complete discrete valuation ring and $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type. We shall in the sequel write $\Xi(o)$ for the set of o-morphisms $\sigma : \operatorname{Spec} o \to \Xi$ such that $\Phi\sigma$ is the identity morphism on $\operatorname{Spec} o$.

Proposition 2.4. — *Let k be the quotient field of a complete discrete valuation ring o. Let $\iota : \Xi_1 \to \Xi_2$ be a closed immersion of separated o-schemes of finite type over o and let $\Xi_1 \to \Xi_2$ be the closed immersion of their generic fibres. Then the natural maps $\Xi_i(o) \to X_i(k)$, $i = 1, 2$ are injective and $\Xi_1(o) = X_1(k) \cap \Xi_2(o)$ in $X_2(k)$.*

Proof. — The assertion follows from Grothendieck's valuative criterion for separated resp. proper morphisms (cf. e.g. [**32**, Ch. 2 §4]). If $\Xi_1 \to \Xi_2$ is an arbitrary proper o-morphism between separated schemes of finite type over o, then the diagram

$$\begin{array}{ccc} \Xi_1(o) & \longrightarrow & \Xi_2(o) \\ \downarrow & & \downarrow \\ X_1(k) & \longrightarrow & X_2(k) \end{array}$$

is set-theoretically cartesian. □

Corollary 2.5. — *Let k be a finite extension of $\mathbb{Q}_p$, and let o be the maximal $\mathbb{Z}_p$-order in k. Also, let Ξ be a separated scheme of finite type over o. Then the following holds.*

(a) *$\Xi(o)$ is a compact and open subset of $X(k)$ in the k-topology.*
(b) *Let X_{cl} be the scheme theoretical closure of X in Ξ. Then the natural map from $X_{\mathrm{cl}}(o)$ to $\Xi(o)$ is bijective.*

Proof

(a) There is a covering of Ξ by finitely many Zariski open affine o-schemes. We may therefore assume that Ξ is affine over o and choose a closed o-immersion ι : $\Xi \to A_o^n$. We may then by (2.1) and (2.4) reduce to the obvious case $\Xi = A_o^n$.

(b) X_{cl} is also separated and of finite type over o with X as generic fibre. The assertion therefore follows from (2.4). □

We shall not need (2.5)(b) in the sequel, but it makes it possible to reduce to the case of reduced models over o.

Definition 2.6. — A *k-variety* X is a geometrically connected, separated scheme of finite type over k.

We shall in this paper use the word k-variety in this sense also for other fields than locally compact fields. Thus a smooth k-variety will always be assumed to be geometrically integral in this paper.

The definition of the k-topology works for arbitrary separated schemes of finite type over k. For the definition of an analytic manifold structure on $X(k)$ we assume from now on that X is a *smooth* k-variety.

To define an n-chart ($n = \dim X$) around a k-point X of $X(k)$ choose an affine neighbourhood U equipped with an étale k-morphism $h : U \to A_k^n$. It follows from the inverse function theorem [**59**, part II, Ch. 3, §9] that there is a k-open neighbourhood N_x in $U(k)$ of X such that the restriction of h to N_x defines an homeomorphism between N_x and a k-open neighbourhood of $y = h(x)$ in $k^n = A_k^n(k)$. Moreover, any two such n-charts defined in this way are compatible (see [**53**, p. 111]). Also, any two n-atlases consisting of such n-charts are compatible (in the sense of (1.2)). There is therefore a (canonical) analytic manifold structure on $X(k)$ (see p. 112 in op. cit.). If k is a complete discrete valuation field, then it follows from (2.3)(b) and a result of Serre [**58**] that $X_{\mathrm{an}}(k)$ is a disjoint union of "balls".

Each k-morphism $g : Y \to X$ gives rise to a (unique) analytic morphism g_{an} : $Y(k) \to X(k)$ between the corresponding analytic manifolds so that one becomes a covariant functor from the category of smooth algebraic k-varieties to the category of analytic manifolds over k (cf. e.g. [**62**] for the case $k = \mathbb{C}$).

The following result is well-known.

Proposition 2.7

(a) *Let $g : Y \to X$ be a smooth k-morphism between smooth k-varieties of constant relative dimension d. Then $g_{\mathrm{an}} : Y(k) \to X(k)$ is a submersion of relative dimension d. In particular, g_{an} is open.*

(b) *Let $g : Y \to X$ be a closed immersion of smooth k-varieties of codimension d. Then $g_{\mathrm{an}} : Y(k) \to X(k)$ is a closed immersion of codimension d.*

Proof

(a) Let $\Omega_{Y/X}$ (resp. $\Omega_{Y/k}$) be the relative cotangent sheaf of Y/X (resp. Y/k). Then the natural sequence

$$0 \longrightarrow g^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow \Omega_{Y/X} \longrightarrow 0$$

is an exact sequence of locally free coherent sheaves on Y which corresponds to an exact sequence of vector bundles over Y (cf. [**30**, 16.5.12 and 17.2.3])

$$0 \longrightarrow V(\Omega_{Y/X}) \longrightarrow V(\Omega_{Y/k}) \longrightarrow V(\Omega_{X/k}) \times_X Y \longrightarrow 0$$

where $V(\mathcal{E})$ means the spectrum of the quasi-coherent symmetric algebra $S(\mathcal{E})$ on $\mathcal{E}$ as in [**29**, 1.7.8]. Now note that the analytic tangent bundle $\mathrm{Tan}(Y_{\mathrm{an}}(k)) \to Y_{\mathrm{an}}(k)$ is equal to the analytic morphism $V(\Omega_{Y/k})_{\mathrm{an}}(k) \to Y_{\mathrm{an}}(k)$ associated to the algebraic tangent bundle $V(\Omega_{Y/k}) \to Y$ [**53**, p. 113]. Hence the last sequence gives rise to an exact sequence of analytic vector bundles as in (1.20) with $N = Y_{\mathrm{an}}(k)$ and $M = X_{\mathrm{an}}(k)$. This implies that g is a submersion by the criterion mentioned just before (1.18).

(b) The proof is similar, but one uses the exact sequence (cf. [**30**, 17.2.5])

$$0 \longrightarrow V(\Omega_{Y/k}) \longrightarrow V(\Omega_{Y/k}) \times_X Y \longrightarrow V(\mathcal{C}_{Y/X}) \longrightarrow 0$$

where $V(\mathcal{C}_{Y/X})$ is the normal bundle of Y/X corresponding to the algebraic conormal sheaf of $g : Y \to X$. □

The main application of (2.7)(a) will be in connection with the integral formula in (1.22).

For the rest of this section we assume the following:

(A) k is a finite extension of $\mathbb{Q}_p$, o is the maximal $\mathbb{Z}_p$-order in k

(B) $\Xi : X \to \operatorname{Spec} o$ is a separated morphism of finite type (e.g. a proper morphism) such that the generic fibre X is smooth and geometrically connected.

Then, by (2.5)(a) there exists a natural compact analytic manifold structure over k on the k-open subset $\Xi(o)$ of $X_{\mathrm{an}}(k)$. We shall denote this compact analytic manifold by $\Xi_{\mathrm{an}}(o)$.

Next, let $\Omega_{\Xi/o}$ be the relative algebraic cotangent sheaf of $\Phi : \Xi \to \operatorname{Spec} o$. It is a coherent (but not necessarily locally free) sheaf on X and one may form the spectrum $V(\Omega_{\Xi/o})$ of the quasi-coherent symmetric $\mathcal{O}_\Xi$-algebra $S(\Omega_{\Xi/o})$ of $\Omega_{\Xi/o}$ [**29**, 1.7]. Following [**30**, 16.5.12] we will write $T_{\Xi/o} = V(\Omega_{\Xi/o})$. The morphism $T_{\Xi/o} \to \Xi$ is affine, hence separated. The composite map $T_{\Xi/o} \to \operatorname{Spec} o$ is thus also separated and its generic fibre is equal to the algebraic tangent bundle $T_{X/k} = V(\Omega_{X/k})$ of X/k.

The analytic tangent bundle $\mathrm{Tan}(X_{\mathrm{an}}(k)) \to X_{\mathrm{an}}(k)$ is equal to the analytic morphism $T_{X/k}(k) \to X(k)$ associated to the algebraic tangent bundle $T_{X/k} \to X$ (cf.

[**53**, p. 113]). Also, the analytic tangent bundle $\mathrm{Tan}(\widetilde{M})$ over $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ is equal to the inverse image of the analytic tangent bundle $\mathrm{Tan}(X_{\mathrm{an}}(k)) \to X_{\mathrm{an}}(k)$ over $\widetilde{M}$. One may therefore regard $T_{\Xi/o}(o)$ as a compact open subset of the analytic manifold $\mathrm{Tan}(\widetilde{M})$.

Proposition 2.8. — *Let Ξ/o be as in (A) and (B) and let $\mathrm{Tan}(\widetilde{M}) \to \widetilde{M}$ denote the analytic tangent bundle of $\widetilde{M} = \Xi_{\mathrm{an}}(o) \subseteq X_{\mathrm{an}}(k)$. Then $L := T_{\Xi/o}(o)$ is an analytic o-lattice of $\mathrm{Tan}(\widetilde{M})$ in the sense of (1.18).*

Proof. — The question is local so we may reduce to the affine case. Choose a closed o-immersion $\iota : \Xi \to A_o^m$ and make use of the natural epimorphism (see [**32**, II.8])

$$\iota^*\Omega_{A_o^m/o} \longrightarrow \Omega_{\Xi/o}.$$

This defines in its turn (cf. [**30**, 16.5.12]) a closed immersion

$$T_{\Xi/o} \longrightarrow T_{A_o^m/o} \times_{A_o^m} \Xi$$

and hence by (2.5) an equality of subsets of $T_{A_k^m/k}$:

$$T_{\Xi/o}(o) = T_{X/k}(k) \cap T_{A_o^m/o}.$$

But it is noted in [**57**, 3.3] that the right hand side is "un champ localement constant de résaux" for affine k-varieties $X \subseteq A_o^m \times_o k$. This means in our language that it is an analytic o-lattice in $\mathrm{Tan}(\widetilde{M})$. The proof is thus complete. □

By taking exterior products of L one gets an analytic o-lattice $\mathbf{det}\, L$ in $\mathbf{det}\,\mathrm{Tan}(\widetilde{M})$.

Definition 2.9. — We shall call

$$\mathbf{det}\, L \subset \mathbf{det}\,\mathrm{Tan}(\widetilde{M})$$

the analytic o-lattice defined by Ξ/o. The norm

$$\|\ \| : \mathbf{det}\,\mathrm{Tan}(\widetilde{M}) \longrightarrow [0,\infty)$$

determined by $\mathbf{det}\, L \subset \mathbf{det}\,\mathrm{Tan}(\widetilde{M})$ (cf. (1.9)) will be called the *model norm determined by Ξ/o.* The positive Borel measure (see (1.12)) m on $\widetilde{M}$ defined by the model norm

$$\|\ \| : \mathbf{det}\,\mathrm{Tan}(\widetilde{M})[0,\infty)$$

above will be called the *model measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ determined by Ξ/o.*

Remark 2.10. — It follows from simple functoriality properties of algebraic cotangent sheaves that the analytic o-lattice in $\mathrm{Tan}(M)$ is uniquely determined by the closure X_{cl} of X in Ξ. This implies that the model norms and the model measures on $\widetilde{M} = \Xi_{\mathrm{an}}(o) = X_{\mathrm{cl,an}}(o)$ determined by Ξ/o and X_{cl}/o coincide (cf. (2.5)(b)).

We now partition $\Xi(o)$ into congruence classes. Let π be a uniformizing parameter of o and let $o_r := o/(\pi^r)$ for each positive integer r. Denote by $\Xi(o_r)$ the set of morphisms $\sigma_r : \mathrm{Spec}(o_r) \longrightarrow \Xi$ such that $\Phi\sigma_r : \mathrm{Spec}(o_r) \to \mathrm{Spec}(o)$ corresponds to the reduction map $o \to o_r$. We shall also write F for o_1 and $\Xi(F)$ for $\Xi(o_1)$.

Lemma 2.11. — *Let $y \in \Xi(F)$. Then there is a natural bijection between sections $\sigma \in \Xi(o)$ which are sent to $y \in \Xi(F)$ under the reduction map $\Xi(o) \to \Xi(F)$ and morphisms* $\mathrm{Spec}\, o \to \mathrm{Spec}\, \mathcal{O}_{\Xi,y}$ *which are sections of the morphism* $\mathrm{Spec}\, \mathcal{O}_{\Xi,y} \to \mathrm{Spec}\, o$ *induced by Φ. This is also true if we replace the stalk $\mathcal{O}_{\Xi,y}$ by the completion along its maximal ideal.*

Proof. — By definition of morphisms of schemes any morphism $\sigma : \mathrm{Spec}\, o \to \Xi$ which is a lifting of $y \in \Xi(F)$ factorizes over an affine morphism $\mathrm{Spec}\, o \to \mathrm{Spec}\, \mathcal{O}_{\Xi,y}$ which is a section of the morphism $\mathrm{Spec}\, \mathcal{O}_{\Xi,y} \to \mathrm{Spec}\, o$ induced by Φ. This proves the first statement. The second follows from the first and the fact that o is complete. $\square$

We shall need the following explicit version of the inverse function theorem on power series $h(y_1,\ldots,y_m) \in o[\![y_1,\ldots,y_m]\!]$. We will write $y = (y_1,\ldots,y_m)$, $y^\varepsilon = \prod_{i=1}^m y_i^{\varepsilon_i}$ for $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_m) \in \mathbb{Z}^{(m)}$ and $\deg(\varepsilon)$ for the degree $\varepsilon_1+\cdots+\varepsilon_m$.

Lemma 2.12. — *Let*

$$h_j(y_1,\ldots,y_m) \in o[\![y_1,\ldots,y_m]\!]\,, \quad j = 1,\ldots,m$$

be formal power series of the form

$$h_j(y) = y_j + \sum \alpha_\varepsilon y^\varepsilon$$

where $\alpha_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in \mathbb{Z}^{(m)}$ runs over exponents $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_m)$ with $\varepsilon_1,\ldots,\varepsilon_m \geq 0$ of degree ≥ 2.

Then there are unique formal power series

$$\gamma_j(y) = y_j + \sum \beta_\varepsilon y^\varepsilon \in o[\![y_1,\ldots,y_m]\!]$$

such that:

$$h_j(\gamma_1(y_1,\ldots,y_m),\ldots,\gamma_m(y_1,\ldots,y_m)) = y_j$$

for $j = 1,\ldots,m$.

Moreover, $\varepsilon \in \mathbb{Z}^{(m)}$ only runs over exponents $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_m)$; $\varepsilon_1,\ldots,\varepsilon_m \geq 0$ of degree $\varepsilon_1+\cdots+\varepsilon_m \geq 2$ and the coefficients $\beta_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$. In particular, all power series converge.

Proof. — This is an immediate consequence of a more general result proved on pp. 11-12 in chapter LG 2 of **[59]**. See in particular the statements 1. and 2. on the top of page LG 2.12. $\square$

Theorem 2.13. — *Let $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type with smooth geometrically connected generic fibre of dimension d and let m be the model measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$. Let $\sigma_0 \in \Xi(o)$ and let e be the largest integer such that $H^0(\operatorname{Spec} o, \sigma_0^*\Omega_{\Xi/o})$ contains a torsion subgroup isomorphic to $o/(\pi^e)$. Finally, let $D(\sigma_0, \Xi, r) \subset \Xi_{\mathrm{an}}(o)$ be the k-open subset of all $\sigma \in \Xi(o)$ with the same reduction as σ_0 in $\Xi_r(o_r)$.*

Then,

$$m(D(\sigma_0, \Xi, r)) = (\mu(o)/\operatorname{Card}(o_r))^d$$

for $r > e$.

Proof. — It follows from (2.11) that $D(\sigma_0, \Xi, 1)$ is contained in $\sum(o)$ for each Zariski open neighbourhood $\sum$ of σ_0 in Ξ. It is therefore sufficient to treat the affine case and we shall choose a closed immersion $\iota : \Xi \to A_o^m$ and coordinates $x_1, \dots, x_m$ such that

$$\sigma_0\iota : \operatorname{Spec} o[x_1, \dots, x_m]/(x_1, \dots, x_m) \longrightarrow \operatorname{Spec} o[x_1, \dots, x_m]$$

is the obvious morphism. Then $F := H^0(\operatorname{Spec} o, (\sigma_0\iota)^*\Omega_{A_o^m/o})$ is a free o-module generated by $dx_1, \dots, dx_m$.

Let $(f_1, \dots, f_k) \in o[x_1, \dots, x_m], k \geq m - d$ generate the ideal defining Ξ. Then $W := H^0(\operatorname{Spec} o, \sigma_0^*\Omega_{\Xi/o})$ is the quotient module F/R of F by the o-submodule R generated by the k elements:

(i) $$df_j = (\partial f_j/\partial x_1)dx_1 + \cdots + (\partial f_j/\partial x_m)dx_m, \quad j = 1, \dots, k$$

The generic fibre is smooth of dimension d and the o-module R is therefore generated by $m - d$ of these elements, say $df_1, \dots, df_{m-d}$. By the elementary divisor theorem we may after a linear coordinate change of write:

(ii) $$f_j(x_1, \dots, x_m) = \pi^{e(j)}x_{d+j} + \sum \xi_\varepsilon x^\varepsilon, \xi_\varepsilon \in o$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{Z}^{(m)}$ runs over exponents $\varepsilon_1, \dots, \varepsilon_m \geq 0$ of total degree $\varepsilon_1 + \cdots + \varepsilon_m \geq 2$ and where $\varepsilon = \varepsilon(1) \geq \varepsilon(2) \geq \cdots \geq \varepsilon(m - d) \geq 0$.

The polynomials $f_j(x_1, \dots, x_m)$, $j = 1, \dots, m - d$ define $\Xi \subset A_o^m$ in an open Zariski neighbourhood of σ_0, which by (2.11) contains $D(\sigma_0, 1, \Xi)$. We may and shall therefore assume that $\Xi \subset A_o^m$ is defined by the polynomials

$$f_j(x_1, \dots, x_m) = 0, \quad j = 1, \dots, m - d.$$

We now consider the analytic map

$$p : D(\sigma_0, \Xi, r) \longrightarrow D(0, A_o^d, r)$$

obtained by projecting onto the d first coordinates $x_1, \ldots, x_d$. Let us first show that this map is an analytic isomorphism when $r > e$ by means of an explicit construction of the implicit functions involved.

Consider the morphism $H : A_o^m \to A_o^m$ sending $(y_1, \ldots, y_m)$ to $(h_1, \ldots, h_m)$ where

$$\begin{aligned} h_j(y_1, \ldots, y_m) &= y_j & 1 \le j \le d \\ h_j(y_1, \ldots, y_m) &= \pi^{-e(j)-r} f_{j-d}(\pi^r y_1, \ldots, \pi^r y_m) & d+1 \le j \le m \end{aligned}$$

Then each $h_j(y)$, $j = 1, \ldots, m$ is of the form

(iii) $$h_j(y) = y_j + \sum \alpha_\varepsilon y^\varepsilon \in o[y_1, \ldots, y_m]$$

where $\alpha_\varepsilon (\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in Z^{(m)}$ runs over non-negative m-tuples of degree ≥ 2.

We may thus apply the inverse function theorem described in (2.12) and find unique convergent power series

$$\gamma_j(y_1, \ldots, y_m) \in o\,[\![y_1, \ldots, y_m]\!]\,, \quad j = 1, \ldots, m$$

such that

(iv) $$\gamma_j(y) = y_j + \sum \beta_\varepsilon y^\varepsilon$$

where $\beta_\varepsilon \in (\pi^{\deg(\varepsilon)-1})$ and $\varepsilon \in \mathbb{Z}^{(m)}$ runs over non-negative m-tuples of degree ≥ 2

(v) $$h_j(\gamma_1(y_1, \ldots, y_m), \ldots, \gamma_m(y_1, \ldots, y_m)) = y_j$$

for $j = 1, \ldots, m$.

We now consider $\Gamma : A_o^m \to A_o^m$ sending $(y_1, \ldots, y_m)$ to $(\gamma_1, \ldots, \gamma_m)$. Then (v) says that $H\Gamma$ is the identity map. By applying (2.12) to Γ instead of H one gets an analytic map $\Psi : A_o^m \to A_o^m$ such that $\Gamma\Psi$ is the identity map. But then $H = H\Gamma\Psi = \Psi$ such that:

(vi) $$\gamma_j(h_1(y_1, \ldots, y_m), \ldots, h_m(y_1, \ldots, y_m)) = y_j$$

for $j = 1, \ldots, m$.

From $h_j(y_1, \ldots, y_m) = y_j$ for $j = 1, \ldots, d$ and (vi) one deduces further that:

(vii) $$\gamma_j(y_1, \ldots, y_d, 0, \ldots, 0) = y_j, \quad j = 1, \ldots, m$$

so that $h_j(y_1, \ldots, y_m) = 0$ for $j = d+1, \ldots, m$. Set

$$\phi_j(x_1, \ldots, x_d) = \pi^r \gamma_j(\pi^{-r} x_1, \ldots, \pi^{-r} x_d, 0, \ldots, 0)$$

for $j = 1, \ldots, m$. Then, from (iv) (resp. (v), resp. (viii)) it follows that:

(viii) f_j is analytic on $D(0, A_o^d, r)$ with image in $D(0, A_o^1, r)$.

(ix) $f_j(\phi_1(x_1, \ldots, x_d), \ldots, \phi_m(x_1, \ldots, x_d)) = 0$ for $j = 1, \ldots, m-d$.

(x) $\phi_j(x_1, \ldots, x_d) = x_j$ for $j = 1, \ldots, m$ and $(x_1, \ldots, x_m) \in D(\sigma_0, \Xi, r)$.

Let

$$\Phi(x_1, \ldots, x_d) = (\phi_1(x_1, \ldots, x_d), \ldots, \phi_m(x_1, \ldots, x_d)).$$

Then from (viii), (ix) and (x) it follows that Φ defines a bijective analytic map

$$D(0, A_o^d, r) \longrightarrow D(\sigma_0, \Xi, r)$$

which is inverse to the projection map

$$p : D(\sigma_0, \Xi, r) \longrightarrow D(0, A_o^d, r).$$

We now prove that the analytic isomorphism given by Φ and p between $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$ is an isometry.

Let $\sigma \in D(\sigma_0, \Xi, r)$. Then it follows from (i) that $H^0(\operatorname{Spec} o, \sigma^*\Omega_{\Xi/o})$ is the o-module generated by $dx_1, \ldots, dx_m$ with relations

$$df_j = (\partial f_j/\partial x_1)dx_1 + \cdots + (\partial f_j/\partial x_m)dx_m, \quad j = 1, \ldots, m-d.$$

Moreover, by (ii) we obtain that

(xi) $$df_j = \pi^{e(j)}dx_{d+j} \quad (\text{modulo } \pi^r), \quad j = 1, \ldots, m-d.$$

where by assumption $r > e(j)$ for $j = 1, \ldots, m-d$. This implies that the isomorphism between the analytic tangent bundles of $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$ induced by p and Φ preserves the o-lattices of these tangent bundles described in (2.8). Hence (see (2.9)) the analytic isomorphism induced by p and Φ respects the model measures on $D(\sigma_0, \Xi, r)$ and $D(0, A_o^d, r)$. It is therefore sufficient to prove the theorem for $\Xi = A_o^d$ and $\sigma_0 = 0$. But then $m(D(0, A_o^d, r)) = (\mu(o)/\operatorname{Card}(o_r))^d$ for $r > 0$. This completes the proof. □

Theorem 2.14. — *Let $\Phi : \Xi \to \operatorname{Spec} o$ be a separated morphism of finite type with smooth geometrically connected generic fibre of dimension d and let m be the model measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$. Then there is an integer E such that the torsion part of*

$$H^0(\operatorname{Spec} o, \sigma^*\Omega_{\Xi/o})$$

is annihilated under multiplication by π^E for all $\sigma \in \Xi(o)$. Moreover, if $r > E$, then

(a)

$$m(\Xi_{\mathrm{an}}(o)) = \operatorname{Card}(\operatorname{Im}(\Xi(o) \to X(o_r)))(\mu(o)/\operatorname{Card}(o_r))^d$$

(b)

$$\int_{\Xi(o)} h\,dm = \sum_{P \in \operatorname{Im}(\Xi(o) \to X(o_r))} h(P)(\mu(o)/\operatorname{Card}(o_r))^d$$

for any $h \in C_c(\Xi_{\mathrm{an}}(o))$ which is constant on the fibres of the reduction map from $\Xi(o)$ to $\Xi(o_r)$.

Proof. — The first statement is an obvious consequence of the assumptions on Φ and the reader may find a proof of a stronger result in [**10**, 3.3.3]. The fact that (a) and (b) hold is a consequence of the previous theorem. $\square$

It follows from (1.6)(b) that any measure on $\widetilde{M} = \Xi_{\mathrm{an}}(o)$ determined by a semi-norm on $\Xi_{\mathrm{an}}(o)$ on $\mathbf{det}\,\mathrm{Tan}(M)$ is of the form hdm for a non-negative continuous function $h \in C_c(X_{\mathrm{an}}(o))$. We may thus use (2.14)(b) to compute the volume of $\Xi_{\mathrm{an}}(o)$ with respect to any such measure.

The theorem (2.14)(a) is due to Serre [**57**, p. 147] (cf. also [**47**]) in case Ξ is affine and $o = \mathbb{Z}_p$. He proves that the equality holds for all sufficiently large r without giving any explicit bound for those r. The following result is due to Peyre [**52**, 2.2.1] in case Φ is projective and $r = 1$.

Corollary 2.15. — *Let $\Phi : \Xi \to \mathrm{Spec}\, o$ be a smooth separated morphism of finite type with geometrically connected generic fibre of dimension d. Then,*

$$m(\Xi_{\mathrm{an}}(o)) = \mathrm{Card}(\Xi(o_r))(\mu(o)/\,\mathrm{Card}(o_r))^d$$

for all $r > 0$.

Proof. — It follows from the smoothness of Φ that $H^0(\mathrm{Spec}\, o, \sigma^*\Omega_{Xi/o})$ is torsion-free for all $\sigma \in \Xi(o)$ (see [**10**, 3.3.1]). Therefore the equality $(*)$ above holds for all $r > 0$. To complete the proof, note that $\Xi(o) \to \Xi_r(o_r)$ is surjective by Hensel's lemma. $\square$

Our next goal is to study families of measures along the fibres of an analytic morphism $\pi_{\mathrm{an}} : Y_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ associated to a smooth morphism $\pi : Y \to X$ of k-varieties. We shall thereby write $\Omega_{Y/X}$ for the relative cotangent sheaf and $T_{Y/X} = V(\Omega_{Y/X})$ for the relative tangent bundle (cf. [**30**, 16.5.12] and [**29**, 1.7.8] for the affine morphism $V(\mathcal{E}) \to Z$ associated to a coherent sheaf $\mathcal{E}$ on a scheme Z).

Proposition 2.16. — *Let $\widetilde{X}$ (resp. $\widetilde{Y}$) be a smooth separated o-scheme of finite type with geometrically connected fibre X (resp. Y). Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be a smooth o-morphism of constant relative dimension d and let $\pi : Y \to X$ be the corresponding k-morphism between the generic fibres. Then the following holds.*

(a) *The relative cotangent sheaf $\Omega_{\widetilde{Y}/\widetilde{X}}$ is locally free of constant rank d and its restriction to the generic fibre of $\widetilde{Y}/o$ is equal to $\Omega_{Y/X}$.*

(b) *The relative tangent bundles define a commutative diagram with injective vertical maps*

$$\begin{array}{ccc} T_{\widetilde{Y}/\widetilde{X}}(o) & \longrightarrow & \widetilde{Y}(o) \\ \big\downarrow & & \big\downarrow \\ T_{Y/X}(k) & \longrightarrow & Y(k) \end{array}$$

(c) *Let* $N := Y_{\mathrm{an}}(k)$, $M := X_{\mathrm{an}}(k)$ *and* $\pi_{\mathrm{an}} : N \to M$ *be the analytic map defined by* π. *Then* π_{an} *is an analytic submersion of relative dimension* d *and the relative analytic tangent bundle* $\mathrm{Tan}(N/M) \to N$ *is equal to the algebraic map* $T_{Y/X}(k) \to Y(k)$ *in (b).*

(d) *Let* $\widetilde{N} = \widetilde{Y}(o)$ *and let* $\widetilde{M} = \widetilde{X}(o)$. *Then* $\widetilde{N}$ *(resp.* $\widetilde{M}$*) is a compact open subset of* N *(resp.* M*) and* π_{an} *restricts to an analytic submersion* $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *of relative dimension* d. *Further,* $T_{\widetilde{Y}/\widetilde{X}}(o)$ *is an analytic* o*-lattice of the relative analytic tangent bundle* $\mathrm{Tan}(\widetilde{N}/\widetilde{M})$.

Proof

(a) This is well known (cf. [**30**, Ch. XVI]).

(b) $V(\Omega_{\widetilde{Y}/\widetilde{X}}) \to \widetilde{Y}$ is separated since it is affine and $\widetilde{Y}$ is separated over o by assumption. Hence $T_{\widetilde{Y}/\widetilde{X}} := V(\Omega_{\widetilde{Y}/\widetilde{X}})$ is also separated over o. Further, $V(\Omega_{Y/X}) = V(\Omega_{\widetilde{Y}/\widetilde{X}}) \times_{\widetilde{Y}} Y$ by (a).

(c) The submersion statement follows from the implicit function theorem. The last assertion follows from a straightforward comparison of the definitions of algebraic and analytic relative tangent bundles.

(d) The first assertion is proved in (2.5)(a) and the submersion statement also follows from (2.5)(a). The o-lattice statement is local and easy to deduce from (a) and (b). It suffices thereby to treat the case where $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is an affine o-morphism between two affine o-schemes. □

Now set $\widetilde{L} = T_{\widetilde{Y}/\widetilde{X}}(o)$. Then $\mathbf{det}\,\widetilde{L} \to \widetilde{N}$ is an analytic o-lattice of the relative anticanonical line bundle $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to \widetilde{N}$.

Definition 2.17. — Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ and $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ be as in (2.16) and let $\widetilde{L} = T_{\widetilde{Y}/\widetilde{X}}(o)$. Then $\mathbf{det}\,\widetilde{L}$ will be called *the analytic o-lattice of* $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to \widetilde{N}$ *defined by* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. The norm

$$\| \; \|_{\widetilde{N}/\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \longrightarrow [0, \infty)$$

determined by this o-lattice (cf. (1.9)) will be called *the relative model norm determined by* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$.

Note that the restriction of $\| \ \|_{\widetilde{N}/\widetilde{M}}$ to the tangent bundle of a fibre $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ is equal to the (absolute) model norm for this fibre.

Proposition 2.18. — *Let* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ *and* $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *be as in (2.16) and let* $\| \ \|_{\widetilde{N}/\widetilde{M}}$ *be the relative model norm determined by* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. *Further, let*

$$\| \ \|_{\widetilde{N}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}) \longrightarrow [0,\infty)$$

$$\| \ \|_{\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{M}) \longrightarrow [0,\infty)$$

be the model norms determined by $\widetilde{Y}/o$ *resp.* $\widetilde{X}/o$ *and*

$$\widetilde{\pi}^*_{\mathrm{an}}\| \ \|_{\widetilde{M}} : \widetilde{\pi}^*_{\mathrm{an}}(\mathbf{det}\,\mathrm{Tan}(\widetilde{M})) \longrightarrow [0,\infty)$$

be the pullback norm of $\| \ \|_{\widetilde{M}}$. *Then* $\| \ \|_{\widetilde{N}}$ *corresponds to the product norm of* $\| \ \|_{\widetilde{N}/\widetilde{M}}$ *and* $\widetilde{\pi}^*_{\mathrm{an}}\| \ \|_{\widetilde{M}}$ *under the canonical isomorphism (cf. (1.19))*

$$\mathbf{det}\,\mathrm{Tan}(\widetilde{N}) = \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \otimes \widetilde{\pi}^*_{\mathrm{an}}(\mathbf{det}\,\mathrm{Tan}(\widetilde{M})).$$

Proof. — The smoothness of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ and $\widetilde{X}/o$ implies that the canonical sequence of coherent sheaves on:

$$0 \longrightarrow \widetilde{\pi}^*\Omega_{\widetilde{X}/o} \longrightarrow \Omega_{\widetilde{Y}/o} \longrightarrow \Omega_{\widetilde{Y}/\widetilde{X}} \longrightarrow 0 \tag{2.19}$$

is exact and that all sheaves involved are locally free.

There is therefore a “dual” exact sequence of (algebraic) vector bundles over

$$0 \longrightarrow T_{\widetilde{Y}/\widetilde{X}} \longrightarrow T_{\widetilde{Y}/o} \longrightarrow T_{\widetilde{X}/o} \times_{\widetilde{X}} \widetilde{Y} \longrightarrow 0 \tag{2.20}$$

and an exact sequence of analytic o-lattices over

$$0 \longrightarrow T_{\widetilde{Y}/\widetilde{X}}(o) \longrightarrow T_{\widetilde{Y}/o}(o) \longrightarrow (T_{\widetilde{X}/o} \times_{\widetilde{X}} \widetilde{Y})(o) \longrightarrow 0 \tag{2.21}$$

inside the canonical exact sequence of analytic vector bundles over $\widetilde{N}$ described in (1.20). The assertion is now an obvious consequence of the compatibility between these two exact sequence. □

Corollary 2.22. — *Let* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ *and* $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *be as in (2.16). Let*

$$\Lambda_{\widetilde{M}} : C_c(\widetilde{M}) \longrightarrow \mathbb{R}$$

$$\Lambda_{\widetilde{N}} : C_c(\widetilde{N}) \longrightarrow \mathbb{R}$$

be the positive functional (cf. (1.10)) defined by the model norms ($\| \ \|_{\widetilde{M}}$ *resp.* $\| \ \|_{\widetilde{N}}$) *and let*

$$\Lambda_{\widetilde{N}/\widetilde{M}} : C_c(\widetilde{N}) \longrightarrow C_c(\widetilde{M})$$

be the positive linear map (cf. (1.22)(a)) defined by the relative model norm $\| \; \|_{\widetilde{N}/\widetilde{M}}$. *Then,*

$$\Lambda_{\widetilde{N}} = \Lambda_{\widetilde{M}} \circ \Lambda_{\widetilde{N}/\widetilde{M}}.$$

Proof. — This follows from (2.18) and (1.22)(b). □

The following corollary of (2.15) will be useful in the theory of adelic measures.

Corollary 2.23. — *Let* $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$, $\pi : Y \to X$, $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *and* $\pi_{\mathrm{an}} : N \to M$ *be as in (2.16) and hence of constant relative dimension* d. *Let* $\Lambda_{\widetilde{N}} : C_c(\widetilde{N}) \to \mathbb{R}$, $\Lambda_{\widetilde{M}} : C_c(\widetilde{M}) \to \mathbb{R}$ *and* $\Lambda_{\widetilde{Y}/\widetilde{X}} : C_c(\widetilde{N}) \to C_c(\widetilde{M})$ *be the positive linear maps defined by the model norms.*

Let F *be the (finite) residue field of* o *and* $1_{\widetilde{N}} \in C_c(\widetilde{N})$ *resp.* $1_{\widetilde{M}} \in C_c(\widetilde{M})$ *be the constant functions with value* 1. *Suppose that all fibres of* π *are geometrically connected and that all fibres of* $\widetilde{\pi}$ *over* F*-points have the same positive number* r *of* F*-points. Then,*

$$\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}}) = r(\mu(o)/\operatorname{Card}(F))^d 1_{\widetilde{M}} = (\Lambda_{\widetilde{N}}(1_{\widetilde{N}})/\Lambda_{\widetilde{M}}(1_{\widetilde{M}}))1_{\widetilde{M}}.$$

Proof. — Let $P \in \widetilde{M} = \widetilde{X}(o)$, $\widetilde{N}_P$ be the fibre of $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ over P. Then the restriction of $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M})$ over $\widetilde{N}_P$ is equal to $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}_P) \to \widetilde{N}_P$ and the restriction of the relative model norm $\| \; \|_{\widetilde{N}/\widetilde{M}}$, to $\mathbf{det}\,\mathrm{Tan}(\widetilde{N}_P)$ is equal to the model norm defined $\| \; \|_P$ by the smooth o-model $\widetilde{Y}_P$ of Y_P. This implies (cf. (1.21)) that the value of $\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}})$ at P is equal to $n_P(\widetilde{N}_P)$ for the model measure n_P on $\widetilde{N}_P$ determined by $\widetilde{Y}_P$. But it follows from (2.15) and the assumptions that $n_P(\widetilde{N}_P) = r(\mu(o)/\operatorname{Card}(F))^d$. Hence $\Lambda_{\widetilde{N}/\widetilde{M}}(1_{\widetilde{N}})$ is constant on $\widetilde{M}$ and it is then a formal consequence of (2.22) that it is equal to $(\Lambda_{\widetilde{N}}(1_{\widetilde{N}})/\Lambda_{\widetilde{M}}(1_{\widetilde{M}}))1_{\widetilde{M}}$. This completes the proof. □

3. Invariant norms on torsors over local fields

The purpose of this section is to study norms and measures on torsors which are invariant under the group action. The word K-variety will always mean a *geometrically connected* separated scheme of finite type over a field K. We shall use the following assumptions and notations throughout the section.

3.1 (a) K is an arbitrary field with separable closure $\overline{K}$.
3.1 (b) X is a smooth K-variety with structure morphism $h : X \to \operatorname{Spec} K$.
3.1 (c) G is a smooth K-variety which is an algebraic group over K.

We shall write $e : \operatorname{Spec} K \to G$ for the unit section and

$$c : G \times_K G \longrightarrow G$$

for the morphism defining the group multiplication. We shall for a K-scheme Y write

$$e_Y : Y \longrightarrow G \times_K Y$$

for the morphism obtained from e by base extension.

Definition 3.2

(a) Let $\mathcal{T}$ be a K-scheme. A K-morphism $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ is a (left) G-action on $\mathcal{T}$ if the following conditions hold.

(i) The composition $\sigma e_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ is the identity map.

(ii) The following diagram commutes

$$\begin{tikzcd} G \times_K G \times_K \mathcal{T} \arrow[r, "\mathrm{id} \times \sigma"] \arrow[d, "c \times \mathrm{id}"'] & G \times_K \mathcal{T} \arrow[d, "\sigma"] \\ G \times_K \mathcal{T} \arrow[r, "\sigma"] & \mathcal{T} \end{tikzcd}$$

(b) Let $\pi : \mathcal{T} \to X$ be a K-morphism. Then a K-morphism $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ is a (left) G-action on the fibres of $\pi : \mathcal{T} \to X$ if (i) and (ii) hold and if the following diagram commutes:

(iii)
$$\begin{tikzcd} G \times_K \mathcal{T} \arrow[r, "\sigma"] \arrow[d, "pr_2"'] & \mathcal{T} \arrow[d, "\pi"] \\ \mathcal{T} \arrow[r, "\pi"] & X \end{tikzcd}$$

The map pr_2 is the projection map onto the second factor.

Definition 3.3. — By a (left) X-torsor under G (with respect to the fppf-topology) we shall mean a K-morphism

$$\pi : \mathcal{T} \longrightarrow X$$

endowed with a G-action

$$\sigma : G \times_K \mathcal{T} \longrightarrow \mathcal{T}$$

on the fibres of $\pi : \mathcal{T} \to X$ satisfying the following conditions.

(a) The structural morphism $\pi : \mathcal{T} \to X$ is faithfully flat and locally of finite presentation.

(b) The morphism

$$\rho = (\sigma, pr_2) : G \times_K \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

is an isomorphism.

Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{T}$ endowed with an isomorphism $\phi : \sigma^*\mathcal{E} \to pr_2^*\mathcal{E}$ of sheaves over $G \times_K \mathcal{T}$ (cf. (3.2)(iii)). From ϕ we get the following three isomorphisms of sheaves over $G \times_K G \times_K \mathcal{T}$:

$$(c \times \mathrm{id}_{\mathcal{T}})^*\phi : (\mathrm{id}_G \times \sigma)^*(\sigma^*\mathcal{E}) = (c \times \mathrm{id}_{\mathcal{T}})^*(\sigma^*\mathcal{E}) \longrightarrow (c \times \mathrm{id}_{\mathcal{T}})^*(pr_2^*\mathcal{E})$$

$$(\mathrm{id}_G \times \sigma)^*\phi : (\mathrm{id}_G \times \sigma)^*(\sigma^*\mathcal{E}) \longrightarrow (\mathrm{id}_G \times \sigma)^*(pr_2^*\mathcal{E}) = p_{23}^*(\sigma^*\mathcal{E}) \text{ (cf. (3.2)(ii))}$$

$$p_{23}^*\phi : p_{23}^*(\sigma^*\mathcal{E}) \longrightarrow p_{23}^*(pr_2^*\mathcal{E}) = (c \times \mathrm{id}_{\mathcal{T}})^*(pr_2^*\mathcal{E})$$

where p_{23} is the projection onto the last two factors of $G \times_K G \times_K \mathcal{T}$.

ϕ is called an S-linearization of $\mathcal{E}$ (see [**45**, Ch I §3]) if

$$(c \times \mathrm{id}_{\mathcal{T}})^*\phi = (p_{23}^*\phi) \circ (\mathrm{id}_G \times \sigma)^*\phi.$$

The symmetric algebra $S(\mathcal{E})$ on $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_{\mathcal{T}}$-algebra and defines a vector bundle

$$q : V(\mathcal{E}) := \operatorname{Spec} S(\mathcal{E}) \longrightarrow \mathcal{T}$$

(see [**29**, 1.7.8] or [**32**, pp. 128-9]). The isomorphisms

$$\phi : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

of locally free sheaves over $G \times_K \mathcal{T}$ corresponds canonically to vector bundle isomorphisms over $G \times_K \mathcal{T}$

$$\Phi : (G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E}) \longrightarrow (G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E})$$

where the maps from $G \times_K \mathcal{T}$ to $\mathcal{T}$ in the fibre products

$$(G \times_K \mathcal{T}) \times_{\mathcal{T}} V(\mathcal{E})$$

are given by pr_2 resp. σ. Let

$$\Sigma : G \times_K V(\mathcal{E}) \longrightarrow V(\mathcal{E})$$

be the composition of Φ with the projection onto $V(\mathcal{E})$. Then there is a commutative diagram

(3.4)
$$\begin{array}{ccc} G \times_K V(\mathcal{E}) & \xrightarrow{\Sigma} & V(\mathcal{E}) \\ {\scriptstyle \mathrm{id} \times \pi}\downarrow & & \downarrow{\scriptstyle q} \\ G \times_K \mathcal{T} & \xrightarrow{\sigma} & \mathcal{T} \end{array}$$

The cocycle condition on ϕ is equivalent to the condition that Σ is a G-action on $V(\mathcal{E})$. This gives (see [**45**, p. 32]) a bijection between G-linearizations of $\mathcal{E}$ and liftings of the G-action on $\mathcal{T}$ to G-actions on $V(\mathcal{E})$.

The isomorphism

$$\rho = (\sigma, pr_2) : G \times_K \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

defines a bijection between covering data η of $\mathcal{E}$ (cf. [**10**, p. 133]) and isomorphisms

$$\phi = \rho^*\eta : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

such that η is a descent datum (cf. [**10**, 6.1]) if and only if ϕ is an G linearization of $\mathcal{E}$. There exists therefore by Grothendieck's theory of faithfully flat descent [**28**, exp. VIII] an equivalence between the categories of G-linearized locally free sheaves $(\mathcal{E}, \phi)$ on $\mathcal{T}$ and locally free sheaves ϕ on X (this is indicated at the end on p. 32 in [**45**]). Alternatively, one may use descent for vector bundles [**45**, I. 2.23] and establish the following result.

Lemma 3.5. — *There is an equivalence of the category of vector bundles*

$$q : V(\mathcal{E}) \longrightarrow \mathcal{T}$$

endowed with a G-action

$$\Sigma : G \times_K V(\mathcal{E}) \longrightarrow V(\mathcal{E})$$

lifting

$$\sigma : G \times_K \mathcal{T} \longrightarrow \mathcal{T}(cf.\ (3.4))$$

and the category of vector bundles

$$p : V(\mathcal{F}) \longrightarrow X.$$

The pair (q, Σ) is obtained from (p, σ) by means of base extension under $\pi : \mathcal{T} \to X$. Conversely, $V(\mathcal{F})$ is the geometric quotient $V(\mathcal{E})/G$ (cf. [**45**, p. 4]*) and p is the map*

$$V(\mathcal{E})/G \longrightarrow \mathcal{T}/G$$

induced by q.

We shall in the sequel by a G-vector bundle over $\mathcal{T}$ mean a vector bundle $q : V(\mathcal{E}) \to \mathcal{T}$ endowed with a G-action Σ as in (3.5).

Example 3.6. — Let $\mathcal{J}_X$ (resp. $\mathcal{J}_\mathcal{T}$) be the ideal sheaves of the closed immersions

$$e_X : X \longrightarrow G \times_K X$$

$$e_\mathcal{T} : \mathcal{T} \longrightarrow G \times_K \mathcal{T}$$

and let

$$\pi_G : G \times_K X \longrightarrow G \times_K X$$

be the morphism induced by π under the base extension from K to G. Then

$$e_X\pi = \pi_G e_\mathcal{T}$$

and

$$\mathcal{J}_\mathcal{T}/\mathcal{J}_\mathcal{T}^2 = \pi_G^*(\mathcal{J}_X/\mathcal{J}_X^2).$$

The conormal sheaf

$$\mathcal{E} = e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2)$$

of $e_{\mathcal{T}}$ is therefore equal to the inverse image $\mathcal{E} = \pi^*\mathcal{F}$ of the conormal sheaf

$$\mathcal{F} = e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)$$

of e_X. The corresponding G-linearization

$$\phi : \sigma^*\mathcal{E} \longrightarrow pr_2^*\mathcal{E}$$

of sheaves over $G \times_K \mathcal{T}$ is the identity map beetween the inverse images of $\mathcal{J}_X/\mathcal{J}_X^2$ along $\pi_G e_{\mathcal{T}}\sigma = \pi_G e_{\mathcal{T}} pr_2$.

Since $\pi : \mathcal{T} \to X$ is locally of finite presentation it is also separated. Write $\mathcal{J}$ for the ideal sheaf on $\mathcal{T} \times_X \mathcal{T}$ of the closed diagonal immersion

$$\delta : \mathcal{T} \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

and $\Omega_{\mathcal{T}/X}$ for the relative cotangent sheaf $\delta^*(\mathcal{J}/\mathcal{J}^2)$. Then the following identities hold.

3.7 (a) $\delta = \rho e_{\mathcal{T}}$
3.7 (b) $\rho^*(\mathcal{J}/\mathcal{J}^2) = \mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2 = \pi_G^*(\mathcal{J}_X/\mathcal{J}_X^2)$
3.7 (c) $\Omega_{\mathcal{T}/X} = e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2) = \pi^*(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$

We now consider the tangent bundle (cf. [**30**, 16.5.12]) $V(\Omega_{\mathcal{T}/X})$ of $\pi : \mathcal{T} \to X$.

Proposition 3.8

(a) *There exists a canonical isomorphism*

$$V(\Omega_{\mathcal{T}/X}) = V(e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2))$$

between the tangent bundle of $\pi : \mathcal{T} \to X$ and the normal bundle of $e_{\mathcal{T}} : \mathcal{T} \to G \times_K \mathcal{T}$.

(b) *There exists a canonical $\mathcal{T}$-isomorphism*

$$V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)) = \mathcal{T} \times_X V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$$

for the normal bundle $V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$ of $e_X : X \to G \times_K X$.

Proof. — This follows from (3.7)(c) and the contravariant equivalence between locally free sheaves and "geometric" vector bundles described in [**32**, p. 129]. □

It follows from (3.8) that there is a natural G-action on the tangent bundle

$$V(\Omega_{\mathcal{T}/X}) \to \mathcal{T}$$

given by the G-linearization of $e_{\mathcal{T}}^*(\mathcal{J}_{\mathcal{T}}/\mathcal{J}_{\mathcal{T}}^2) = \pi^*(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2))$ in (3.6).

Proposition 3.9. — *Let $\mathcal{J}$ be the ideal sheaf of the closed immersion $e : \operatorname{Spec} K \to G$ and let*

$$V(e^*(\mathcal{J}/\mathcal{J}^2))$$

be the tangent space of G at e regarded as a K-variety. Then there is a canonical isomorphism

$$V(e_X^*(\mathcal{J}_X/\mathcal{J}_X^2)) = X \times_K V(e^*(\mathcal{J}/\mathcal{J}^2))$$

from the normal bundle of $e_X : X \to G \times_K X$.

Proof. — Let $h_G : G \times_K X \to G$ be the first projection map. Then $\mathcal{J}_X/\mathcal{J}_X^2 = h_G^*(\mathcal{J}/\mathcal{J}^2)$ and $h_G e_X = eh$. Hence $e_X^*(\mathcal{J}_X/\mathcal{J}_X^2) = h^*(e^*(\mathcal{J}/\mathcal{J}^2))$ as was to be proved. □

Now let k denote a non-discrete locally compact field of characteristic zero and let $\pi : \mathcal{T} \to X$ be a torsor over a smooth k-variety X under an algebraic k-group G as in (3.1).

The algebraic morphism π gives rise to an analytic morphism $\pi_{\text{an}} : \mathcal{T}_{\text{an}}(k) \to X_{\text{an}}(k)$. Set

$$N = \mathcal{T}_{\text{an}}(k),$$
$$M = \pi_{\text{an}}(\mathcal{T}_{\text{an}}(k)).$$

Then M is a k-open subset of $X_{\text{an}}(k)$ by the implicit function theorem. It is thus endowed with a natural analytic manifold structure. We shall from now on change notation and write

$$\pi_{\text{an}} : N \longrightarrow M$$

for the *surjective* analytic morphism sending $Q \in N$ to $\pi_{\text{an}}(Q)$. This map is a submersion in the sense of [**59**, LG 3.16].

The fibre product

$$N \times_M N := \{(Q_1, Q_2) \in N \times N : \pi_{\text{an}}(Q_1) = \pi_{\text{an}}(Q_2)\}$$

is an analytic submanifold of $N \times N$ since π_{an} is a submersion (cf. [**59**, LG3.26]). It is the analytic immersion defined by the algebraic immersion $\mathcal{T} \times_X \mathcal{T} \to \mathcal{T} \times_k \mathcal{T}$.

Let $\Gamma = G_{\text{an}}(k)$. The group law $c : G \times_k G \to G$ gives rise to an analytic morphism

$$c_{\text{an}} : \Gamma \times \Gamma \longrightarrow \Gamma$$

which makes Γ to an analytic group in the sense of [**59**, Chap. IV] with $e \in G(k)$ as neutral element.

The G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$ (cf. (3.4)) induces an analytic (left) G-action

$$\sigma_{\text{an}} : \Gamma \times N \longrightarrow N$$

as in [**59**, LG 4.11]. This Γ-action satisfies $\pi_{\text{an}}(\sigma_{\text{an}}(g, n)) = \pi_{\text{an}}(n)$ and we shall therefore say that σ_{an} is a Γ-action on $\pi_{\text{an}} : N \to M$ (cf. (3.2)(iii)).

Definition 3.10. — A submersion

$$\alpha : N \longrightarrow M$$

between analytic manifolds with an analytic G-action

$$\beta : \Gamma \times N \longrightarrow N$$

on α is called an analytic M-torsor under Γ if the map

$$\gamma : \Gamma \times N \longrightarrow N \times_M N$$

sending (g, n) to $(\beta(g, n), n)$ is an isomorphism.

The torsor isomorphism $\rho : G \times_k \mathcal{T} \to \mathcal{T} \times_X \mathcal{T}$ induces an analytic isomorphism

$$\rho_{\mathrm{an}} : \Gamma \times N \longrightarrow N \times_M N$$

such that the pair $(\pi_{\mathrm{an}} : N \to M,\ \sigma_{\mathrm{an}} : \Gamma \times N \to N)$ obtained from the pair $(\pi : \mathcal{T} \to X, \sigma : G \times_k \mathcal{T} \to \mathcal{T})$ is an analytic M-torsor under Γ.

Proposition 3.11. — *Let $\alpha : N \to M$ be an analytic M-torsor under the Γ-action $\beta : \Gamma \times N \to N$.*

(a) *There exists for each $P \in M$ an analytic section $s : U \to N$ of α for some open neighbourhood $U \subset M$ of P.*

(b) *Let $V = \alpha^{-1}(U)$ for the open subset $U \subset M$ in (a). Then the restriction of β to $\Gamma \times s(U)$ defines an analytic isomorphism between $\Gamma \times s(U)$ and V.*

Proof

(a) This follows from the implicit function theorem (cf. [**59**, LG 3.1.6]).

(b) ρ_{an} restricts to an analytic isomorphism between $\Gamma \times s(U)$ and $V \times_U s(U)$ which preserves the second coordinate. The projection map $V \times_U s(U) \to V$ is also an isomorphism. This completes the proof. □

By (3.11) any analytic M-torsor under Γ is a locally trivial analytic principal G-bundle over M as defined in [**59**, Ch. IV]. The converse is also true. Note that (3.11) implies that M as a topological space is homeomorphic to the quotient space N/Γ.

Let $q : V(\mathcal{E}) \to \mathcal{T}$ be an (algebraic) vector bundle as in (3.5) equipped with a G-action $\Sigma : G \times_k V(\mathcal{E}) \to V(\mathcal{E})$ lifting $\sigma : G \times k\mathcal{T} \to \mathcal{T}$. Let $E = V(\mathcal{E})_{\mathrm{an}}(k)$ and let $q_{\mathrm{an}} : E \to N$ be the analytic vector bundle defined by q. Then Σ defines an analytic G-action

$$\Sigma_{\mathrm{an}} : \Gamma \times E \longrightarrow E$$

which lifts the Γ-action on N. We shall in the sequel write gs instead of $\Sigma_{\mathrm{an}}(g, s)$ for each pair $(g, s) \in \Gamma \times E$ whenever it is clear what the Γ-action is. A seminorm

$$\|\ \| : E \longrightarrow [0, \infty)$$

on $q_{\mathrm{an}} : E \to N$ (cf. (1.5)) will be called G-invariant if $\|gs\| = \|s\|$ for all $(g, s) \in \Gamma \times E$.

Let $p : V(\mathcal{F}) \to X$ be a vector bundle corresponding to the G-vector bundle $q : V(\mathcal{E}) \to \mathcal{T}$ under the equivalence of categories in (3.5) and let us identify $V(\mathcal{E})$ and $\mathcal{T} \times_X V(\mathcal{F})$. To p one may associate an analytic vector bundle

$$p_{\mathrm{an}} : V(\mathcal{F})_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k).$$

Let $F \subseteq V(\mathcal{F})_{\mathrm{an}}(k)$ be the inverse image of $M := \pi_{\mathrm{an}}(\mathcal{T}_{\mathrm{an}}(k))$ under p_{an}. Then p_{an} restricts to an analytic vector bundle $p_{\mathrm{an},M} : F \to M$. The following result is trivial.

Proposition 3.12

(a) *The analytic bundle $q_{\mathrm{an}} : E \to N$ is the pullback bundle of $p_{\mathrm{an},M} : F \to M$ under $\pi_{\mathrm{an}} : N \to M$.*

(b) *The pullback seminorm on $q_{\mathrm{an}} : E \to N$ (cf. (1.7)(c)) of a seminorm on $p_{\mathrm{an},M} : F \to M$ is G-invariant.*

(c) *Any G-invariant seminorm (resp. norm) on $q_{\mathrm{an}} : E \to N$ is the pullback of a unique seminorm (resp. norm) on $p_{\mathrm{an},M} : F \to M$.*

Now recall that $V(\Omega_{\mathcal{T}/X}) = \mathcal{T} \times_k V(e^*(\mathcal{J}/\mathcal{J}^2))$ is endowed with a G-action (cf. (3.5), (3.8)). As a corollary of the corresponding algebraic results result we obtain:

Corollary 3.13. — *The analytic tangent bundle*

$$\mathrm{Tan}(N/M) \longrightarrow N$$

is canonically isomorphic to the pullback of the analytic normal bundle

$$\mathrm{Nor}(M/\Gamma \times M) \longrightarrow M$$

along $\pi_{\mathrm{an}} : N \to M$.

There is a natural analytic G-action on $\mathrm{Tan}(N/M) \to N$. *The normal bundle*

$$\mathrm{Nor}(M/\Gamma \times M) \longrightarrow M$$

is canonically isomorphic to the constant bundle

$$M \times V \longrightarrow M$$

for the analytic tangent space V of G at e.

By considering the corresponding determinant bundles one deduces immediately the following result.

Main proposition 3.14. — *Let $\pi_{\mathrm{an}} : N \to M$ be an analytic torsor under the analytic group G as above and let V be the analytic tangent space V of Γ at e. Then the following holds:*

(a) *The analytic anticanonical bundle*

$$\mathbf{det}\,\mathrm{Tan}(N/M) \longrightarrow N$$

is canonically isomorphic to the constant bundle

$$N \times \mathbf{det}\, V \longrightarrow N.$$

(b) *Let*

$$\|\ \|_M : M \times \mathbf{det}\, V \longrightarrow [0, \infty)$$

be a norm on $M \times \mathbf{det}\, V \to M$. *Then the pullback norm*

$$\pi_{\mathrm{an}}^* \|\ \|_M : N \times \mathbf{det}\, V \longrightarrow [0, \infty)$$

is a Γ*-invariant norm on* $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$.

(c) *Any* Γ*-invariant norm on* $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$ *is the pullback norm of a unique norm on* $M \times \mathbf{det}\, V \to M$.

Definition 3.15. — A constant norm on $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$ is a pullback norm of a vector space norm on $\mathbf{det}\, V$ under the projection

$$\mathbf{det}\,\mathrm{Tan}(N/M) = N \times \mathbf{det}\, V \to \mathbf{det}\, V.$$

The constant norms on $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$ are clearly Γ-invariant and unique up to multiplication by a positive real number.

Remark 3.16. — Let ω_0 be a non-zero section of $\mathbf{det}\, V^\vee$ for the cotangent space $V^\vee$ of Γ at e. Then ω_0 defines a vector space norm $\|\ \| : \mathbf{det}\, V \to [0, \infty)$ (cf. (1.7)) and any norm on $\mathbf{det}\, V$ is obtained in this way. Further, the bundle

$$\mathbf{det}\,\mathrm{Cot}(N/M) \to N$$

is canonically isomorphic to the constant bundle $N \times \mathbf{det}\, V^\vee \to N$ by (3.14)(a). There is thus a global Γ-invariant section ω of $\mathbf{det}\,\mathrm{Cot}(N/M) \to N$ corresponding to the pullback of ω_0 along the projection $N \times \mathbf{det}\, V^\vee \to N$. The constant norm on $\mathbf{det}\,\mathrm{Tan}(N/M) \to N$ corresponding to $\|\ \| : \mathbf{det}\, V \to [0, \infty)$ is the norm associated to ω in (1.7).

We now formulate an analog of (3.7) for schemes satisfying the following hypothesis.

3.17 (a) o is a henselian discrete valuation ring, o^{sh} is a strict henselization of o.

3.17 (b) $\widetilde{h} : \widetilde{X} \to \mathrm{Spec}\, o$ is a smooth separated morphism of finite type with geometrically connected fibre.

3.17 (c) $\widetilde{G}$ is a smooth separated group scheme of finite type over o with geometrically connected fibres.

There is thus a unit section $\widetilde{e} : \operatorname{Spec} o \to \widetilde{G}$ and a morphism

$$\widetilde{c} : \widetilde{G} \times_o \widetilde{G} \longrightarrow \widetilde{G}$$

defining the group multiplication. We denote by $\widetilde{\mathcal{J}}$ the ideal sheaf on $\widetilde{G}$ of the closed immersion $\widetilde{e}$ and by $\widetilde{\mathcal{J}}_{\widetilde{X}}$ the ideal sheaf on $\widetilde{G}_{\widetilde{X}}$ of the closed immersion

$$\widetilde{e}_{\widetilde{X}} : \widetilde{X} \longrightarrow \widetilde{G} \times_o \widetilde{X}.$$

Let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ be an o-morphism between o-schemes. A $\widetilde{G}$-action

$$\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{T}}$$

on the fibres of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is defined just as in (3.2)(b). A (left) $\widetilde{X}$-torsor under $\widetilde{G}$ (with respect to the fppf-topology) is an o-morphism

$$\widetilde{\pi} : \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{X}}$$

with an action

$$\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \longrightarrow \widetilde{\mathcal{T}}$$

on the fibres of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ satisfying the following conditions.

3.18 (a) The structural morphism $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is faithfully flat and locally of finite presentation.

3.18 (b) The morphism $\widetilde{\rho} = (\widetilde{\sigma}, pr_2) : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} \times_{\widetilde{X}} \widetilde{\mathcal{T}}$ is an isomorphism.

Proposition 3.19. — *Assume (3.17) and let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ be an $\widetilde{X}$-torsor under $\widetilde{G}$. Then there is a canonical $\widetilde{\mathcal{T}}$-isomorphism*

$$V(\Omega_{\widetilde{\mathcal{T}}/\widetilde{X}}) = \widetilde{\mathcal{T}} \times_{\widetilde{X}} V(\widetilde{e}^*_{\widetilde{X}}(\widetilde{\mathcal{J}}_{\widetilde{X}}/\widetilde{\mathcal{J}}_{\widetilde{X}^2}))$$

and a canonical $\widetilde{X}$-isomorphism

$$V(\widetilde{e}^*_{\widetilde{X}}(\widetilde{\mathcal{J}}_{\widetilde{X}}/\widetilde{\mathcal{J}}_{\widetilde{X}^2})) = \widetilde{X} \times_o V(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2)).$$

Proof. — The proofs are almost identical to those of (3.8), (3.9). □

We now add the following hypothesis to (3.17).

3.20 (a) k is a finite extension of $\mathbb{Q}_p$, o is the maximal $\mathbb{Z}_p$-order in k.

3.20 (b) $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is an $\widetilde{X}$-torsor under with action $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$.

3.20 (c) The restrictions to the generic fibres of $\widetilde{h}$, $\widetilde{\pi}$ and $\widetilde{\sigma}$ are equal to the k-morphisms $h : X \to \operatorname{Spec} k$, $\pi : \mathcal{T} \to X$ and $\sigma : G \times_K \mathcal{T} \to \mathcal{T}$ in (3.1) and (3.3).

Then $\widetilde{N} = \widetilde{\mathcal{T}}(o)$ (resp. $\widetilde{\Gamma} = \widetilde{G}(o)$) is a compact open subset of $N = \mathcal{T}_{\mathrm{an}}(k)$ (resp. $\Gamma = G_{\mathrm{an}}(k)$) in the k-topology (cf. (2.5)(a)). Also, $\widetilde{M} = \widetilde{X}(o)$ is a compact open subset of $X_{\mathrm{an}}(k)$. This provides $\widetilde{M}$ and $\widetilde{N}$ with analytic manifold structures and $\widetilde{\Gamma}$ with an analytic group structure. The algebraic morphisms $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ and $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ induce analytic morphisms

$$\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \longrightarrow \widetilde{M},$$

$$\widetilde{\sigma}_{\mathrm{an}} : \widetilde{\Gamma} \times \widetilde{N} \longrightarrow \widetilde{N}.$$

Lemma 3.21

(a) $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *is a surjective submersion.*
(b) $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ *is an analytic* $\widetilde{M}$*-torsor under the* $\widetilde{\Gamma}$*-action given by* $\widetilde{\sigma}$*.*
(c) *There is a canonical isomorphism of analytic vector bundles over* $\widetilde{N}$*:*

$$\mathrm{Tan}(\widetilde{N}/\widetilde{M}) = \widetilde{N} \times V$$

for the analytic tangent space of Γ *at* $e \in \Gamma(k)$*. This isomorphism is compatible with the isomorphism*

$$\mathrm{Tan}(N/M) = N \times V$$

in (3.14).

Proof

(a) The pullback of $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ with respect to a section $\operatorname{Spec} o \to \widetilde{X}$ is a torsor over $\operatorname{Spec} o$ under $\widetilde{G}$. The closed fibre of such a torsor is a torsor under the *connected* algebraic group $\widetilde{G} \times_o o/m$ and therefore trivial by a well-known result of Lang (cf. e.g. [**64**]). It follows from Hensel's lemma that $\widetilde{\pi}_{\mathrm{an}} : \widetilde{\mathcal{T}}(o) \to \widetilde{X}(o)$ is surjective. It is thus a submersion since $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{X}}$ is smooth.

(b) The torsor isomorphism $\widetilde{\rho} = (\widetilde{\sigma}, pr_2) : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} \times_X \widetilde{\mathcal{T}}$ (cf. (3.18)(b)) induces an analytic isomorphism $\widetilde{\rho}_{\mathrm{an}} : \widetilde{\Gamma} \times \widetilde{N} \to \widetilde{N} \times_{\widetilde{M}} \widetilde{N}$.

(c) This follows from the fact that the relative analytic tangent bundle $\mathrm{Tan}(\widetilde{N}/\widetilde{M})$ is equal to the inverse image over $\widetilde{N}$ of $\mathrm{Tan}(N/M) \to N$. □

Definition 3.22. — The tangent lattice of $\widetilde{G}/o$ at $\widetilde{e} \in \widetilde{G}(o)$ is the o-module

$$L_1 := \mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2))(o)$$

of sections $\operatorname{Spec} o \to \mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2))$ of $\mathbf{V}(\widetilde{e}^*(\widetilde{\mathcal{J}}/\widetilde{\mathcal{J}}^2)) \to \operatorname{Spec} o$.

Note that L_1 is an o-lattice of the k-vector space $V = \mathbf{V}(\widetilde{e}^*(\mathcal{J}/\mathcal{J}^2))$.

Now let $T_{\widetilde{\mathcal{T}}/\widetilde{\mathcal{X}}} = \mathbf{V}(\Omega_{\widetilde{\mathcal{T}}/\widetilde{\mathcal{X}}})$ and recall that $T_{\widetilde{\mathcal{T}}/\widetilde{\mathcal{X}}}(o)$ is an analytic o-lattice of the vector bundle $\mathrm{Tan}(\widetilde{N}/\widetilde{M})$ (see (2.16)(d)).

Proposition 3.23. — *Assume (3.17) and (3.20) and let $\widetilde{N}$, $\widetilde{M}$ be as above. Let L_1 be the tangent lattice of $\widetilde{G}/o$ at $\widetilde{e}$ in the tangent space V of G/k at $\widetilde{e}$. Then $T_{\widetilde{\mathcal{T}}/\widetilde{X}}(o)$ is mapped isomorphically onto $\widetilde{N} \times L_1$ under the canonical isomorphism*

$$\mathrm{Tan}(\widetilde{N}/\widetilde{M}) = \widetilde{N} \times V.$$

of analytic vector bundles over in (3.21)(c).

Proof. — This follows from (3.19). □

Corollary 3.24. — *Assume (3.17) and (3.20) and let $\widetilde{N}$, $\widetilde{M}$ be as above. Then the relative model norm*

$$\| \; \|_{\widetilde{N}/\widetilde{M}} : \mathbf{det}\,\mathrm{Tan}(\widetilde{N}/\widetilde{M}) \to [0, \infty)$$

determined by $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ (cf. (2.17)) is equal to the constant norm

$$\widetilde{N} \times \mathbf{det}\, V \longrightarrow [0, \infty)$$

obtained by pulling back the vector space norm on $\mathbf{det}\, V$ given by $\mathbf{det}\, L_1$ for the tangent lattice L_1 of $\widetilde{G}/o$ at $\widetilde{e}$.

Proposition 3.25. — *Assume (3.17) and (3.20). Let μ be the model measure on $\Delta_{\mathrm{an}}(o)$ determined by Δ/o for any of the three smooth o-schemes $\Delta = \widetilde{X}$, $\widetilde{G}$ or $\widetilde{\mathcal{T}}$. Then*

$$\mu(\widetilde{\mathcal{T}}(o)) = \mu(\widetilde{X}(o))\mu(\widetilde{G}(o)).$$

Proof. — Let $\widetilde{N} = \widetilde{\mathcal{T}}(o)$, $\widetilde{M} = \widetilde{X}(o)$, $\widetilde{\Gamma} = \widetilde{G}(o)$ and $\widetilde{\pi}_{\mathrm{an}} : \widetilde{N} \to \widetilde{M}$ be as above. Then it follows from the theorem of Lang already used in (3.21) that each fibre of $\widetilde{\pi}$ over a F-point contains the same number of F-points as $\widetilde{G}$. This implies by (2.23) that $\mu(\widetilde{N})/\mu(\widetilde{M})$ is equal to $\mathrm{Card}\, \widetilde{G}(F)(\mu(o)/\,\mathrm{Card}(F))^{\dim G}$ which in its turn is equal to $\mu(\widetilde{G}(o))$ by (2.15). This finishes the proof. □

We now restrict to the case when G is a K-torus. This means that G is a commutative algebraic group which after a finite separable base field extension is isomorphic to the product of a finite number of $\mathbb{G}_m$. The K-torus is said to be split if there is such an isomorphism defined over K.

Let $\hat{G}$ be the group of characters defined over K. Then there is a homomorphism:

$$d\log : \hat{G} \longrightarrow H^0(G, \Omega^1_{G/K}) \tag{3.26}$$

which sends a character $f : G \to \mathbb{G}_m$ to the G-invariant differential form df/f.

If G is a product of r copies of $\mathbb{G}_m$, then $\hat{G}$ is a free $\mathbb{Z}$-module of rank r and one obtains by taking the r-th exterior products a homomorphism

$$\bigwedge^r d\log : \bigwedge^r \hat{G} \longrightarrow H^0(G, \Omega^1_{G/K}). \tag{3.27}$$

Definition 3.28. — Let G be a split K-torus of dimension r. Then the image of any generator of $\bigwedge^r \hat{G}$ under $\bigwedge^r d\log$ is called an algebraic differential r-form of minimal $d\log$-type. We shall also say that the corresponding analytic differential r-form is of minimal $d\log$-type.

Proposition and definition 3.29. — *Let k be a non-discrete locally compact field and let G be an r-dimensional k-torus. Then there exists a unique vector space norm $\| \ \|$ on $\bigwedge^r(T_{G,e}(k))$ for the tangent space $T_{G,e}(k)$ of G at e defined as follows.*

Let E be a finite field extension of k such that the absolute Galois group of E acts trivially on $\bigwedge^r(\hat{T})$ and let ω_1 be an differential r-form of minimal $d\log$-type on G_E. Then if $s \in \bigwedge^r(T_{G,e}(k))$, put

$$\|s\| := \sqrt[d]{|\omega_1(s)|_E}$$

where $d := \dim_k E$ and $| \ |_E$ is the normalized absolute value of E.

This definition does not depend on the choice of E and ω_1. The number $\|s\|$ will be called the order of s and

$$\| \ \| : \bigwedge^r (T_{G,e}(k)) \longrightarrow \mathbb{R}$$

the order norm.

Proof. — The proof of the independence statement is obvious and left to the reader. □

Definition 3.30. — Let k be a non-discrete locally compact field and let $\pi : \mathcal{T} \to X$ be a torsor under a k-torus G. Then the constant norm (cf. (3.15))

$$\| \ \|_{\mathcal{T}/X} : \mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k)) \longrightarrow [0, \infty)$$

obtained by pulling back the order norm on $\bigwedge^r(T_{G,e}(k))$ is called the order norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k))$. If

$$\| \ \|_X : \mathbf{det}\,\mathrm{Tan}(X(k)) \longrightarrow [0, \infty)$$

is a norm for X, then the product norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k))$ of the order norm $\| \ \|_{\mathcal{T}/X}$ and the pullback norm $\pi^*\| \ \|_X$ (cf. (1.19)) is called the norm on $\mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k))$ induced by $\| \ \|_X$.

We shall in the sequel write $\| \ \|_{X \to \mathcal{T}}$ for the norm induced by $\| \ \|_X$.

Remark 3.31. — Let k be the quotient of a complete discrete valuation ring o. Suppose that G extends to a commutative group scheme $\widetilde{G}$ and that π extends to a torsor $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ under $\widetilde{G}$ as in (3.20). Then the order norm

$$\| \ \|_{\mathcal{T}/X} : \mathbf{det}\,\mathrm{Tan}(\mathcal{T}(k)/X(k)) \to [0, \infty)$$

restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o)/\widetilde{X}(o))$. (To see this, use (3.24).) Hence, if a norm $\|\ \|_X$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{X}(o))$, then the induced norm $\|\ \|_{X\cdot\to\mathcal{T}}$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o))$ by (2.22).

4. Adelic norms and measures

The aim of the section is to define norms and measures on the adele spaces of varieties over number fields. This section is inspired by **[52]** but our adelic notions are more general than in that paper.

We shall in this section use the following notations.

Notations 4.1

(a) k denotes a number field and o denotes the maximal $\mathbb{Z}$-order in k.

(b) If ν is a non-archimedean place of k, then o_ν denotes the complete discrete valuation ring corresponding to ν and F_ν the residue field of o_ν.

(c) If Σ is a finite closed subset of $\mathrm{Spec}\, o$, then $o_{(\Sigma)}$ denotes the Dedekind domain of elements in k which are integral with respect to non-archimedean places outside Σ.

(d) W denotes the set of all places ν of k and W_∞ the subset of all archimedean places of k.

(e) W_{fin} denotes the set of all non-archimedean places of k, which we will identify with the set of closed points of $\mathrm{Spec}\, o$.

(f) If Σ be a finite subset of W_{fin}, then

$$A(\Sigma) = A_k(\Sigma) = \prod_{\nu\in T} k_\nu \times \prod_{\nu\in W-T} o_\nu, \qquad T := \Sigma \cup W_\infty.$$

(g) $A = A_k$ is the adele ring

$$A = \varinjlim A(S), \quad S \subset W_{\text{fin}}$$

where S is finite.

(h) Let $\Sigma \subseteq S$ be finite subsets of W_{fin}, $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{A} = A_k(\Sigma)$ and $\widetilde{X}$ (resp. Ξ) be a scheme over $\widetilde{o}$ (resp. $\widetilde{A}$). Then

$$\widetilde{X}_{(S)} := \widetilde{X} \times_{\widetilde{o}} o(S),$$

$$\Xi_{A(S)} := \Xi \times_{\widetilde{A}} A(S),$$

and

$$\Xi_A := \Xi \times_{\widetilde{A}} A.$$

(i) Let $\Sigma \subseteq S$ be finite subsets of W_{fin} and $\widetilde{P}$, $\widetilde{X}$ (resp. Π, Ξ) be schemes over $\widetilde{o} = o_{(\Sigma)}$ (resp. $\widetilde{A} = A_k(S)$). Then,

$$\mathrm{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)})$$

is the set of all $o_{(S)}$-morphisms from $\widetilde{P}_{(S)}$ to $\widetilde{X}_{(S)}$ and

$$\mathrm{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)})$$

the set of all $A(S)$-morphisms from $\Pi_{A(S)}$ to $\Xi_{A(S)}$.

We shall in the sequel also consider the embedding $k \subset A_k$ obtained from the inductive limit of the embeddings $o_{(S)} \subset A_{(S)}$ for all finite subsets S of W_{fin}.

Proposition 4.2

(a) *Let X be a scheme of finite type over k and let X_A be a scheme of finite presentation over A_k. Then there exists a scheme $\widetilde{X}$ (resp. Ξ) of finite presentation over $\widetilde{o} = o_{(\Sigma)}$ (resp. $\widetilde{A} = A_k(\Sigma)$) for some finite closed subset Σ of* Spec o, *such that*

$$X = \widetilde{X} \times_{\widetilde{o}} k$$

and

$$X_A = \Xi \times_{\widetilde{A}} A.$$

(b) *Let Σ be a finite closed subset of* Spec o. *Let $\widetilde{P}$, $\widetilde{X}$ be two schemes of finite type over $\widetilde{o} = o_{(\Sigma)}$ with generic k-fibres P and X and let Π, Ξ be two schemes of finite presentation over $\widetilde{A} = A_k(\Sigma)$ with*

$$P_A = \Pi \times_{\widetilde{A}} A,$$

$$X_A = \Xi \times_{\widetilde{A}} A.$$

Then the obvious maps

$$e : \varinjlim \mathrm{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)}) \longrightarrow \mathrm{Hom}_k(P, X), \quad \Sigma \subseteq S \subset W_{\text{fin}},$$

$$e_A : \varinjlim \mathrm{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)}) \longrightarrow \mathrm{Hom}_A(P_A, X_A), \quad \Sigma \subseteq S \subset W_{\text{fin}},$$

over finite subsets S are bijective.

(c) *Let*

$$(p_{(S)}) \in \varinjlim \mathrm{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)})$$

resp.

$$(\pi_{A(S)}) \in \varinjlim \mathrm{Hom}_{A(S)}(\Pi_{A(S)}, \Xi_{A(S)})$$

be the element corresponding to

$$p \in \mathrm{Hom}_k(P, X) \quad \textit{resp.} \quad \pi_A \in \mathrm{Hom}_A(P_A, X_A)$$

under the bijection e (resp. e_A) in (b). Let $\mathcal{P}$ be any of the following properties.

(i) *p resp. π_A is an isomorphism,*
(ii) *p resp. π_A is an open immersion,*
(iii) *p resp. π_A is a closed immersion,*
(iv) *p resp. π_A is separated,*
(v) *p resp. π_A is surjective,*
(vi) *p resp. π_A is affine,*
(vii) *p resp. π_A is proper,*
(viii) *p resp. π_A is projective,*
(ix) *p resp. π_A is quasi-projective,*
(x) *p resp. π_A is smooth.*

Then there exists a closed subset $T \supseteq \Sigma$ of $\operatorname{Spec} o$ *such that $\mathcal{P}$ holds for $p_{(S)}$ resp. $\pi_{A(S)}$ for all finite closed subsets S of W_{fin} containing T.*

Proof. — Let R be a Noetherian ring. Then a scheme of finite type over R is also of finite presentation over R (cf. [**31**, 6.3.7]). Therefore, (a) resp. (b) is a special case of part (ii) resp. (i) of Théorème 8.8.2 in [**30**] and part (i)-(ix) of (c) is a special case of Théorème 8.10.5 in (op. cit.) and (x) is a special case of Proposition 17.7.8 in (op. cit.). □

Lemma 4.3. — *Let Σ be a finite closed subset of* $\operatorname{Spec} o$ *and let $f : \widetilde{X} \to \operatorname{Spec} o_{(\Sigma)}$ be a morphism of finite type. Then there exists a finite closed subset $T \supseteq \Sigma$ of* $\operatorname{Spec} o$ *such that the morphism*

$$f_{(S)} : \widetilde{X}_{(S)} \longrightarrow \operatorname{Spec} o_{(S)}$$

is flat for all finite subsets $S \subset W_{\text{fin}}$ containing T.

Proof. — See Théorème 6.9.1 in [**30**]. □

Definition 4.4

(a) Let X be a separated scheme of finite type over k and let Σ be a finite closed subset of $\operatorname{Spec} o$. An *$o_{(\Sigma)}$-model of X* is an $o_{(\Sigma)}$-scheme $\widetilde{X}$ which is separated and of finite type over $o_{(\Sigma)}$ and for which the generic fibre of $\widetilde{X}/o_{(\Sigma)}$ is equal to X. A *model of X* is an $o_{(\Sigma)}$-model of X for some finite subset $\Sigma \subset W_{\text{fin}}$.

(b) Let X_A be an A_k-scheme which is separated and of finite presentation over A_k. Let Σ be a finite closed subset of $\operatorname{Spec} o$. An *$A_k(\Sigma)$-model of X_A/A_k* is an $A_k(\Sigma)$-scheme $\widetilde{X}$ which is separated and of finite presentation over $A_k(\Sigma)$. A *model of X_A/A_k* is an $A_k(\Sigma)$-model of X_A for some finite subset $\Sigma \subset W_{\text{fin}}$.

Remarks 4.5

(a) It follows from (4.2)(a) that any k-scheme X (resp. any A_k-scheme X) in (4.4) has a model. Also, by (4.3) there exists a model $\widetilde{X}$ of X which is flat over its base ring. If X/k (resp. X_A/A_k) is smooth, then there exists a model which is smooth over the base ring (cf. (4.2)(c)(x)). The models are not unique, but any two such models become isomorphic after a base extension to $o_{(S)}$ resp. $A_k(S)$ for a sufficiently large finite subset

$$S \subset W_{\text{fin}}$$

(see (b) and (c)(i) in (4.2)).

(b) Let X be a separated scheme of finite type over k and let $\widetilde{X}$ be a model over $\widetilde{o} = o_{(\Sigma)}$ of X. Further, let

$$\widetilde{A} = A_k(\Sigma),$$
$$\Xi = \widetilde{X} \times_{\widetilde{o}} \widetilde{A}$$

for the diagonal embeddings $\widetilde{o} \subset \widetilde{A}$, $k \subset A$ and

$$X_A = X \times k_A.$$

Then Ξ is a model of X_A/A_k.

The adele ring A_k is a locally compact topological ring. The underlying topological space is the restricted direct product

$$A_k = \prod\nolimits'_{\nu \in W} k_\nu$$

of the locally compact fields k_ν with respect to the compact integer rings

$$o_\nu \subset k_\nu, \quad \nu \in W_{\text{fin}}.$$

The subspace topology of $A_k(\Sigma)$ for finite subsets $\Sigma \subset W_{\text{fin}}$ is the product topology of the ν-adic topologies. The topology of A_k is therefore the inductive limit topology of the product topologies of $A_k(\Sigma)$ for finite subsets $\Sigma \subset W_{\text{fin}}$.

Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation. Write $X_A(k_\nu)$ (resp. $X_A(A_k)$) for the set of all A_k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X_A,$$
$$\operatorname{Spec} A_k \longrightarrow X_A.$$

The embedding

$$A_k \subset \prod_{\nu \in W} k_\nu$$

induces an injection

$$X_A(A_k) \subset \prod_{\nu \in W} X_A(k_\nu).$$

If $\Xi/\widetilde{A}$, $\widetilde{A} = A_k(\Sigma)$ is a model of X_A, then Ξ is the direct sum (cf. [**31**, 3.1]) of the o_ν-schemes

$$\Xi_\nu := \Xi \times_{\widetilde{A}} o_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

and of the k_ν-schemes

$$X_\nu := X_A \times_A k_\nu, \quad \nu \in \Sigma \cup W_\infty.$$

The set $\Xi(o_\nu)$ of all $\widetilde{A}$-morphisms

$$\operatorname{Spec} o_\nu \longrightarrow \widetilde{X}$$

form a compact open subset of $X_A(k_\nu)$ for $\nu \in W_{\text{fin}} \backslash \Sigma$ (cf. (2.6)).

Lemma 4.6. — *Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/\widetilde{A}$, $\widetilde{A} = A_k(\Sigma)$ be a model of X_A. Further, let*

$$\prod\nolimits'_{\nu\in W} X_A(k_\nu)$$

be the set of elements

$$\{P_\nu\} \in \prod_{\nu\in W} X_A(k_\nu)$$

such that $P_\nu \in \Xi(o_\nu)$ for all but finitely many places. Then $X_A(A_k)$ is mapped onto $\prod'_{\nu\in W} X_A(k_\nu)$ under the embedding

$$X_A(A_k) \subset \prod_{\nu\in W} X_A(k_\nu)$$

If $\Xi'/o_{(\Sigma')}$ is another model of X_A, then

$$\Xi(o_\nu) = \Xi'(o_\nu)$$

in $X(k_\nu)$ for all but finitely many places.

Proof. — The first statement is a consequence of the isomorphisms e_A (see (4.2)(b)) between $\varinjlim \operatorname{Hom}_{A(S)}(\operatorname{Spec} A(S), \Xi_{A(S)})$ and $\operatorname{Hom}_A(\operatorname{Spec} A, X_A)$. To prove the second assertion we may replace Σ and Σ' by $\Sigma \cup \Sigma'$ and reduce to the case $\Sigma = \Sigma'$. Then the assertion follows from (4.2)(c)(i) for $\Pi = \Xi'$. □

Definition and proposition 4.7. — *Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/A_k(\Sigma)$ be a model of X. The adelic topology on $X_A(A_k)$ is the restricted product topology on*

$$X_A(A_k) = \prod\nolimits'_{\nu\in W} X_A(k_\nu)$$

with respect to the compact open subsets

$$\Xi(o_\nu) \subset X(k_\nu), \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

This topology is independent of the choice of model.

If X is a separated scheme of finite type over k and $X(A_k)$ is the set of all k-morphisms

$$\operatorname{Spec} A_k \longrightarrow X,$$

then we define the adelic topology on $X(A_k)$ to be the topology induced by the adelic topology on $X_A(A_k)$ for

$$X_A = X \times_k A_k$$

under the obvious bijection between $X(A_k)$ and $X_A(A_k)$. The adelic space of X is the topological space $X(A_k)$ endowed with the adelic topology.

Proof. — The restricted topological product

$$\prod\nolimits'_{\nu \in W} X(k_\nu)$$

does not change if we change or omit $\widetilde{X}(o_\nu)$ (resp. $\Xi(o_\nu)$) at finitely many places. It therefore follows from (4.6) that the adelic topology is independent of the choice of model. $\square$

Examples 4.8

(a) Let X be a separated scheme of finite type over k and let $\widetilde{X}/o_{(\Sigma)}$ be a model of X. If $\nu \in W$ (resp. $\nu \in W_{\text{fin}}\backslash\Sigma$), let $X_A(k_\nu)$ (resp. $\widetilde{X}(o_\nu)$) denote the set of all k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X$$

resp. $o_{(\Sigma)}$-morphisms

$$\operatorname{Spec} o_\nu \longrightarrow \widetilde{X}.$$

Let

$$X(A_k) \subset \prod_{\nu \in W} X(k_\nu)$$

be the injection induced by the embedding

$$A_k \subset \prod_{\nu \in W} k_\nu.$$

Then, the adelic space of X is equal to the restricted topological product of all $X(k_\nu)$, $\nu \in W$ with respect to the compact open subsets $\widetilde{X}(o_\nu) \subseteq X(k_\nu)$ (cf. (2.5)) for all $\nu \in W_{\text{fin}}\backslash\Sigma$. In particular, $X(A_k)$ is locally compact.

(b) Let $X = \mathbb{A}_k^r$ and choose $\Xi = \mathbb{A}_o^r$ as o-model of X. The adelic space $X(A_k)$ may be identified with the topological product $A_k^{(r)}$ of r copies of A_k. If X is an affine k-variety, then the adelic topology of $X(A_k)$ is the coarsest topology such that all the maps $X(A_k) \to A_k$ determined by regular k-functions on X are continuous.

(c) Let X_A be a smooth proper A-scheme. There exists (cf. (4.2)) a proper model $\Xi/A_k(\Sigma)$ of X over $o_{(\Sigma)}$ for some finite subset Σ of W_{fin}. Moreover,

$$\Xi(o_\nu) = X(k_\nu)$$

for all $\nu \in W_{\text{fin}} \backslash \Sigma$ by Grothendieck's valuative criterion for properness so that

$$X_A(A_k) = \prod_{\nu \in W} X(k_\nu)$$

as a topological space. By Tychonoff's theorem we conclude from (2.3)(b) that

$$X_A(A_k) = \prod_{\nu \in W} X(k_\nu)$$

is compact if X is smooth and proper.

Remarks 4.9

(a) Let X_A be an A_k-scheme such that X_A/A_k is separated and of finite presentation and let $\Xi/A_k(\Sigma)$ be a model of X. Then it follows from the definition of the adelic topology that the sets of the form:

$$\prod_{\nu \in S} U_\nu \times \prod_{\nu \notin S} \Xi(o_\nu), \quad \Sigma \cup W_\infty \subseteq S \subseteq W$$

for finite S and open subsets

$$U_\nu \subset X(k_\nu), \quad \nu \in S$$

form an open base for the topology on $X_A(A_k)$. Also, there exists for each place $\nu \in W$ a countable base $\mathcal{F}_\nu$ of open subsets with compact support for the k_ν-topology on $X(k_\nu)$ (cf. (2.3)). Let $\mathcal{F}$ be the family of open subsets of $X(A_k)$ as above with the additional condition that $U_\nu \in \mathcal{F}$ for all $\nu \in S$. Then $\mathcal{F}$ is a countable base of open subsets with compact support for the adelic topology on $X_A(A_k)$. It follows that the adelic space $X_A(A_k)$ is a locally compact, σ-compact, paracompact Hausdorff space.

(b) Let $\pi_A : P_A \to X_A$ be an A-morphism between two A_k-schemes as in (a). Then π_A determines a map

$$\pi_A(A_k) : P(A_k) \longrightarrow X(A_k)$$

between the adelic spaces. It is an easy consequence of (4.2) and the continuity of the maps

$$P(k_\nu) \longrightarrow X(k_\nu)$$

that this map is continuous.

We now define adelic norms for A-schemes X_A, thereby generalizing Peyre's notion of adelic metrics [Pe1]. If $X_A = X \times_k A_k$ for a k-variety X, then

$$X_A \times_A k_\nu = X \times_k k_\nu$$

and $X_A(k_\nu)$ corresponds bijectively to the set $X(k_\nu)$ of all k-morphisms

$$\operatorname{Spec} k_\nu \longrightarrow X.$$

We shall by *abuse of notation* write

$$X_\nu = X_A \times_A k_\nu,$$
$$X(k_\nu) = X_A(k_\nu)$$

also in the case where X_A is *not* a fibre product $X \times_k A_k$ for a k-variety X.

The scheme X_A will from now on be smooth, separated and of finite type over A. This implies that X_A is of finite presentation over A since any smooth morphism is locally of finite presentation by definition. Hence there exist smooth models of X_A by (4.2).

The set $X(k_\nu) := X_A(k_\nu)$ is canonically isomorphic to $X_\nu(k_\nu)$ and will always be endowed with a manifold structure. We shall write $X_{\text{an}}(k_\nu)$ for this analytic manifold over k_ν. If X is an $A_k(\Sigma)$-model of X and $\nu \in W_{\text{fin}} \backslash \Sigma$, then the compact open subset

$$\Xi(o_\nu) \subseteq X(k_\nu)$$

inherits an analytic manifold structure from $X(k_\nu)$, which we denote by $\Xi_{\text{an}}(o_\nu)$.

Definition 4.10

(a) A smooth A_k-variety is an A_k-scheme which is smooth, separated and of finite type over $A = A_k$ such that X_ν is geometrically connected for all $\nu \in W$.

(b) Let X_A be a smooth A-variety. An *adelic norm for* X_A is a family of norms

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\, \operatorname{Tan} X_{\text{an}}(k_\nu) \longrightarrow [0,\infty), \quad \nu \in W\}$$

on the analytic tangent bundles of $X_{\text{an}}(k_\nu)$, $\nu \in W$ with the following property. There exists a finite subset Σ of W_{fin} and an $A_k(\Sigma)$-model Ξ of X such that the restriction of $\| \ \|_\nu$ to the analytic anticanonical line bundle on $\Xi(o_\nu)$ is equal to the model norm determined by X_ν/o_ν for all $\nu \in W_{\text{fin}} \backslash \Sigma$.

If $X_A = X \times_k A_k$ for a smooth k-variety X, then $\| \ \|$ is said to be *an adelic norm for* X.

It is clear from (4.2) that the last condition in (4.10) is true for any model of X as soon as it is true for one model of X. We may therefore assume that Ξ is a smooth model.

Let M be a topological space. We shall in the sequel write $C(M)_{>0}$ for the vector space of all continuous functions

$$f : M \longrightarrow (0,\infty).$$

Definition 4.11. — Let X_A be smooth A-variety and let

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

be a set of positive continuous functions. Then γ is called a set of convergence factors for X_A if the following condition holds.

There exists a finite subset Σ of W_{fin} and an $A_k(\Sigma)$-model Ξ of X such that the product

$$\prod_{W_{\text{fin}}\setminus\Sigma} \left(\int_{\Xi(o_\nu)} \gamma_\nu d\mu_\nu \right)$$

converges absolutely to a positive real number for the model measures μ_ν on $\Xi(o_\nu)$ determined by Ξ_ν/o_ν (see (2.9)).

If all functions γ_ν are constant, then $\gamma = \{\gamma_\nu,\ \nu \in W\}$ is called a *constant set of convergence factors for* $X_A(A_k)$ if

$$\prod_{W_{\text{fin}}\setminus\Sigma} \gamma_\nu \mu_\nu(\Xi(o_\nu))$$

converges absolutely to a positive real number.

If $X_A = X \times_k A_k$ for a smooth k-variety X, then

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

is said to be a *set of convergence factors for* X.

Remarks 4.12

(a) The absolute convergence of the product is not affected by a change of finitely many γ_ν. We shall therefore use the term "set of convergence factors" also in cases where γ_ν is not defined for a finite set S of places of k. One can then put $\gamma_\nu = 1$ for $\nu \in S$.

(b) It is clear from (4.2)(c)(i) that the absolute convergence of the products above only depends on γ and not on the choice of model.

(c) If X is a smooth k variety and $\widetilde{X}$ is a model over $\widetilde{o} = o_{(\Sigma)}$, then $\widetilde{X}(o_\nu)$ is non-empty for almost all $\nu \in W_{\text{fin}}\setminus\Sigma$ (cf. (4.19)). One may thus find a set of constant convergence factors for X with

$$\gamma_\nu = \mu_\nu(\widetilde{X}(o_\nu))^{-1}$$

for almost all ν. This is not true for arbitrary smooth A_k-varieties since there are smooth A_k-varieties such that $X_A(k_\nu) = \varnothing$ for infinitely many places.

Definition 4.13

Let X_A be an A_k-variety as in (4.10).

(a) Let S be a finite set of places of k. Then X_S is the topological space

$$X_S = \prod_{\nu \in S} X(k_\nu)$$

endowed with the product topology of the analytic k_ν-topologies, $\nu \in S$. A subset B_S of X_S is said to be decomposable if it is of the form

$$B_S := \prod_{\nu \in S} B_\nu$$

for some Borel subsets

$$B_\nu \subset X_{\mathrm{an}}(k_\nu), \quad \nu \in S.$$

(b) A decomposable subset of $X_A(A_k)$ is a subset B of the form

$$B := \prod_{\nu \in W} B_\nu$$

for Borel subsets

$$B_\nu \subset X_{\mathrm{an}}(k_\nu)$$

such that

$$B_\nu = \Xi(o_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ for some model $\Xi / o_{(\Sigma)}$ of X.

(c) Let $B = \prod_{\nu \in W} B_\nu$ be a compact decomposable subset of $X(A_k)$. For each $\nu \in W$, let

$$f_\nu : X(k_\nu) \longrightarrow \mathbb{R}$$

be a continuous function with support in B_ν. Suppose that there exists a finite subset $T \subset W$ such that f_ν is (strictly) positive on B_ν for all $\nu \in W \backslash T$ and such that

$$\prod_{W \backslash T} f_\nu$$

converges absolutely to a strictly positive function on $\prod_{W \backslash T} B_\nu$. Let $f \in C_c(X_A(A_k))$ be the function with support in B, such that

$$f = \prod_{\nu \in W} f_\nu$$

on B. Then f is called *the restricted product of* $\{f_\nu,\ \nu \in W\}$ and any such function $f \in C_c(X_A(A_k))$ is said to be *decomposable*. If f_ν can be chosen to be the characteristic function of B_ν for almost all $\nu \in W$, then f is said to be *finitely decomposable*.

We now define adelic measures following Weil [**67**] (cf. also [**61**] and [**52**]).

Theorem 4.14. — *Let X_A be a smooth A-variety as in (4.10) and let*

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

be a set of convergence factors for $X(A_k)$. Let

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}\,X(k_\nu) \longrightarrow [0,\infty), \quad \nu \in W\}$$

be an adelic norm for X_A and let

$$\{m_\nu,\ \nu \in W\}$$

be the regular positive Borel measures on $X(k_\nu)$ determined by these norms (see (1.12), (1.14)). Then the following holds.

(a) *Let T be a finite subset of W. Then there exists a unique σ-regular positive Borel measure $m_{T,\gamma}$ on $X(k_\nu)$ such that*

$$m_{T,\gamma}(B) = \prod_{\nu \in T} \left(\int_{B_\nu} \gamma_\nu dm_\nu \right)$$

for any decomposable subset

$$B = \prod_{\nu \in T} B_\nu$$

of $X(k_\nu)$.

(b) *There exists a unique σ-regular positive Borel measure $m_{A,\gamma}$ on $X_A(A_k)$ such that*

$$(*) \qquad m_{A,\gamma}(B) = \prod_{\nu \in W} \left(\int_{B_\nu} \gamma_\nu dm_\nu \right)$$

for any decomposable subset

$$B = \prod_{\nu \in W} B_\nu$$

of $X_A(A_k)$ for which all $m_\nu(B_\nu)$ are finite.

(c) *Let $\{f_\nu \in C_c(X(k_\nu)),\ \nu \in W\}$, $T \subset W$, $f \in C_c(X(A_k))$ be as in (4.13)(c). Then,*

$$\prod_{W \setminus T} \left(\int_{X(k_\nu)} f_\nu \gamma_\nu dm_\nu \right)$$

converges absolutely to a positive real number and

$$\int_{X(A_k)} f dm_{A,\gamma} = \prod_{\nu \in W} \left(\int_{X(k_\nu)} f_\nu \gamma_\nu dm_\nu \right).$$

Proof

(a) Let $\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}\, X(k_\nu) \to [0,\infty)$, $\nu \in W$, be the new norms obtained by multiplying $\| \ \|_\nu$ with γ_ν and let m_ν be the Borel measure on $X(k_\nu)$ determined by $\| \ \|_\nu$. Then,

$$\int_{B_\nu} \gamma_\nu dm_\nu = m_\nu(B_\nu).$$

The assertion is therefore equivalent to the existence and uniqueness of the product measure of m_ν over all $\nu \in T$. This follows from repeated use of Theorem 8.2 in [**38**, Ch. VI] (the reader can also consult Chap. III, §5, No 4 in [**11**]).

(b) Let Ξ be an $A_k(\Sigma)$-model of X_A for some finite subset S of W_{fin} such that $\| \ \|_\nu$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}\,\Xi(o_\nu)$ for all $\nu \in W_{\mathrm{fin}}\backslash\Sigma$. We shall call a set of functions

$$\{g_\nu \in C_c(X(k_\nu),\ \nu \in W\}$$

adelic if g_ν is the characteristic function of

$$\Xi(o_\nu) \subseteq X(k_\nu)$$

for all but finitely many $\nu \in W_{\mathrm{fin}}\backslash\Sigma$. It is then by Riesz' representation theorem (cf. (1.11) and the references there) sufficient to show that there exists a unique positive functional Λ on $X_A(A_k)$ such that:

$$(") \qquad \Lambda(g) = \prod_{\nu\in W}\left(\int_{B_\nu} g_\nu\gamma_\nu dm_\nu\right)$$

for the restricted product g of any adelic set $\{g_\nu \in C_c(X(k_\nu)),\ \nu \in W\}$.

The product of two finitely decomposable function is again finitely decomposable. This implies that finite sums of decomposable functions form a subalgebra $\mathcal{A}$ of $C_c(X(A_k))$. Also, if $g_1, \dots, g_n$ are finitely decomposable functions on $X_A(A_k)$ with non-negative sum, then it follows from (a) that

$$\Lambda(g_1) + \cdots + \Lambda(g_n) \geq 0.$$

There is thus a unique well-defined positive functional Λ on $\mathcal{A}$ satisfying (").

Next, let $f \in C_c(X_A(A_k))$. There is then an open decomposable neighbourhood

$$U = \prod_{\nu\in W} U_\nu$$

of the support of f with compact closure

$$J = \prod_{\nu\in W} J_\nu$$

in $\prod_{\nu\in W} X(k_\nu)$.

The restrictions to J of functions in $C_c(X(A_k))$ form a subalgebra

$$\mathcal{A}(J) \subseteq C_c(J)$$

which separates points and contains the constant functions. There exists therefore by the Stone-Weierstrass theorem [**38**, p. 52] a sequence $(g_i)_{i=1}^{\infty}$ of functions in $\mathcal{A}(J)$ which converges uniformly to f on J. There exists further by Urysohn's lemma [**38**, IX. 2.1] a finitely decomposable function $H \in C_c(X(A_k))$ such that

$$\begin{aligned} 0 \leq H \leq 1 \quad & \text{on } X_A(A_k) \\ H = 0 \quad & \text{outside } U \\ H = 1 \quad & \text{on } \operatorname{Supp} f. \end{aligned}$$

Hence

$$(f_i)_{i=1}^{\infty} := (Hg_i)_{i=1}^{\infty}$$

is a sequence of functions in $\mathcal{A}$ with support in J which converges uniformly to f on $X(A_k)$. The same is true for $(H^2 g_i^2)_{i=1}^{\infty}$ with respect to f^2. We may thus for non-negative f assume that all functions in the sequence $(f_i)_{i=1}^{\infty}$ are non-negative.

Now let $(f_i)_{i=1}^{\infty}$ be an arbitrary sequence of functions in $\mathcal{A}$ which converges uniformly to f on $X(A_k)$ and such that there exists a compact subset K of $X(A_k)$ containing the supports of all f_i. There exists then by Urysohn's lemma a finitely decomposable non-negative function G such that $G = 1$ on K. Then,

$$-G \sup |f_i - f_j| \leq (f_i - f_j) \leq G \sup |f_i - f_j|$$

on $X_A(A_k)$ so that:

$$|\Lambda(f_i) - \Lambda(f_j)| = |\Lambda(f_i - f_j)| \leq \Lambda(G) \sup |f_i - f_j|$$

by the linearity and positivity of Λ for functions in $\mathcal{A}$. Also,

$$|\Lambda(f_i)| \leq \Lambda(G) \sup |f_i|$$

by the same argument.

This implies that $(\Lambda f_i)_{i=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ for any sequence $(f_i)_{i=1}^{\infty}$ as above and that $(\Lambda f_i)_{i=1}^{\infty}$ converges to zero if $(f_i)_{i=1}^{\infty}$ converges to zero. There exists therefore a unique extension of the positive functional Λ on $\mathcal{A}$ to a positive functional Λ on $C_c(X(A_k))$. This completes the proof of (b).

(c) Choose a bijective function $\mathbb{Z}_{>0} \to W \backslash T$ and write W_i for the image of $\{1, \ldots, i\}$. Let $g_i \in C_c(X(A_k))$, $i \in \mathbb{Z}_{>0}$ be the restricted product of f_ν for $\nu \in T \cup W_i$ and of the characteristic function of B_ν for $\nu \in W \backslash (T \cup W_i)$. Then $(g_i)_{i=1}^{\infty}$ is a sequence of functions of finitely decomposable functions with supports in B such that $(g_i)_{i=1}^{\infty}$ converges uniformly to f on $X(A_k)$. Therefore, $(\Lambda g_i)_{i=1}^{\infty}$

converges to Λf so that the result follows from the validity of the formula for finitely decomposable functions established in (b). This completes the proof. □

Remarks 4.15

(a) All the Borel measures in (4.14) are regular since $X_A(A_k)$ is σ-compact (see (1.15) and (4.9)).

(b) Suppose that $X_A := X \times_k A_k$. We may then regard $m_{A,\gamma}$ as a measure on the adelic space $X(A_k)$ for X (cf. (4.7) and (4.8)(a)).

The following example is due Peyre [**52**, 2.2] except for the fact that (2.15) makes it possible to consider non-projective varieties as well.

Example 4.16. — Let X be a smooth proper k-variety. Then there exists a smooth proper model $\widetilde{X}/o_{(\Sigma)}$ of X (cf. (4.2), (4.4)) such that all reductions

$$Y_\nu := \Xi \times F_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

are geometrically integral (cf. [**30**, 12.2.1] or (4.18) below).

Let $\overline{F}_\nu$ be an algebraic closure of F_ν, and let

$$\overline{Y}_\nu = \Xi \times \overline{F}_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

Then the geometric Frobenius Fr_ν acts on $\operatorname{Pic} \overline{Y} \otimes \mathbb{Q}$ and one defines the local L-function for $\nu \in W_{\text{fin}} \backslash \Sigma$ by:

$$L_\nu(s, \operatorname{Pic} \overline{Y}_\nu) = 1/\det(1 - q_\nu^{-s} \text{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q}).$$

Now let $H^1_{\text{Zar}}(X, \mathcal{O}_X) = H^2_{\text{Zar}}(X, \mathcal{O}_X) = 0$. Then

$$H^1_{\text{Zar}}(Y_\nu, \mathcal{O}_{Y_\nu}) = H^2_{\text{Zar}}(Y_\nu, \mathcal{O}_{Y_\nu}) = 0$$

for all but finitely many $\nu \in W_{\text{fin}} \backslash \Sigma$. Hence Deligne's theorem [**19**, 3.3.9] implies that (cf. [**52**, p. 117])

$$\operatorname{Card}(Y_\nu(F_\nu))/q_\nu^{\dim X} = 1 + \operatorname{Tr}(\text{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q})/q_\nu + O(1/q_\nu^{3/2})$$

for $\nu \in W_{\text{fin}} \backslash \Sigma$.

Moreover, by (2.15) one has

$$\mu_\nu(\Xi(o_\nu)) = \operatorname{Card}(Y_\nu(F_\nu))(\mu_\nu(o_\nu)/q_\nu)^{\dim X}$$

with $\mu_\nu(o_\nu) = 1$ for all but finitely many $\nu \in W_{\text{fin}}$.

Set

$$\gamma_\nu := \begin{cases} 1/L_\nu(1, \operatorname{Pic} \overline{Y}_\nu) & \text{for } \nu \in W_{\text{fin}} \backslash \Sigma \\ 1 & \text{for } \nu \in W_\infty \cup \Sigma. \end{cases}$$

Then it follows from the equalities above that (γ_ν) is a set of convergence factors for X.

Example 4.17. — Serre [**61**, p. 655] applies Deligne's theorem [**19**, 3.3.4] to smooth (not necessarily proper) varieties X for which the first two Betti numbers

$$B_i := \dim_{\mathbb{Q}} H^i_{\mathrm{Sing}}(X_{\mathrm{an}}(\mathbb{C}), \mathbb{Q}), \quad i = 1, 2$$

vanish (for some embedding $k \subset \mathbb{C}$). He concludes that any model $\Xi/o_{(\Sigma)}$ of X satisfies

$$\mathrm{Card}(\Xi(F_\nu))/q_\nu^{\dim X} = 1 + O(1/q_\nu^{3/2}), \quad \nu \in W_{\mathrm{fin}} \backslash \Sigma.$$

One may thus choose $\lambda_\nu := 1, \quad \nu \in W$ as a set of convergence factors for such X.

It is possible to formulate a result for smooth (not necessarily proper) varieties with $B_1 = 0$ which contains the results of Peyre and Serre as special cases. The convergence factors are given by $1/L_\nu(1)$ for local L-functions $L_\nu(s)$ defined by the means of the second ℓ-adic cohomology groups for a fixed ℓ. One expects that the local L-functions $L_\nu(s)$ are independent of the choice of ℓ, but this is known only in the proper case (cf. [**34**, pp. 27-28]).

Our next goal is to describe a relative version of theorem (4.14) for families of smooth varieties. We first need some results on models.

Lemma 4.18. — *Let Σ be a finite subset of W_{fin} and let $\widetilde{P}$, $\widetilde{X}$ be two schemes of finite type over $o_{(\Sigma)}$ with generic fibres P resp. X. Let $p : P \to X$ be a morphism such that all geometric fibres of p are irreducible (resp. connected, reduced or integral).*

Then there exists a finite subset $S \subset W_{\mathrm{fin}}$ containing Σ and an extension of $p : P \to X$ to a morphism.

$$p_{(S)} : \widetilde{P}_{(S)} \longrightarrow \widetilde{X}_{(S)}$$

for which all geometric fibres are irreducible (resp. connected, reduced or integral).

Proof. — Let

$$p_{(S)} : \widetilde{P}_{(S)} \longrightarrow \widetilde{X}_{(S)}, \quad \Sigma \subseteq S \subset W_{\mathrm{fin}}$$

be a morphism in the inductive limit (cf. (4.2))

$$(p_{(S)}) \in \varinjlim \mathrm{Hom}_{(S)}(\widetilde{P}_{(S)}, \widetilde{X}_{(S)}), \quad \Sigma \subseteq S \subset W_{\mathrm{fin}}$$

corresponding to p and let E be the set of closed points Q of $\widetilde{X}_{(S)}$ such that the geometric fibre of $p_{(S)}$ over Q is irreducible (resp. connected, reduced or integral).

Then $p_{(S)}$ is of finite presentation since: $\widetilde{P}_{(S)}$ and $\widetilde{X}_{(S)}$ are of finite presentation over $o_{(S)}$ [**31**, 6.3.7-8]. Hence by [**30**, Th. 9.7.7], E must be a locally constructible subset of $\widetilde{X}_{(S)}$. Then it follows from a theorem of Chevalley [**31**, 7.1.4] that $\widetilde{X}_{(S)} \backslash E$ is mapped onto a locally constructible subset of $\mathrm{Spec}\, o_{(S)}$ under the structure morphism of $\widetilde{X}_{(S)}/o_{(S)}$. But any locally constructible subset of an affine Dedekind scheme is either an open or closed subset. Thus since $\widetilde{X}_{(S)} \backslash E$ and the generic fibre of $\widetilde{X}_{(S)}/o_{(S)}$ are disjoint by assumption we conclude that $\widetilde{X}_{(S)}\backslash E$ is contained in finitely many closed fibres. This completes the proof. □

Recall that a k-variety X in this paper means a *geometrically connected* separated scheme of finite type over k. Hence if X is smooth, then it is geometrically integral.

Lemma 4.19. — *Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism of constant relative dimension on X with geometrically connected fibres. Then there exists two smooth models $\widetilde{Y}$, $\widetilde{X}$ over some integer ring $\widetilde{o} = o_{(\Sigma)}$ and an extension of π to a smooth $\widetilde{o}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ with the following properties:*

(i) all fibres of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ are smooth and geometrically integral,
(ii) $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is of constant relative dimension on $\widetilde{Y}$,
(iii) the map from $\widetilde{Y}(o_\nu)$ to $\widetilde{X}(o_\nu)$ defined by composition with $\widetilde{\pi}$ is surjective for all places.

Proof. — It follows from (4.2) that π has an extension to a smooth $\widetilde{o}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between smooth models over some integer ring $\widetilde{o} = o_{(\Sigma)}$. We now prove (i)-(iii)

(i) It follows from (4.18) that the fibres of $\widetilde{\pi}$ are geometrically integral after an enlargement of Σ since the fibres of $\pi : Y \to X$ are geometrically integral

(ii) It follows from the irreducibility of $\widetilde{Y}$ and the fact that $\widetilde{\pi}$ is flat and of locally finite presentation that $\widetilde{\pi}$ is of constant relative dimension on $\widetilde{Y}$ (cf. [**30**, 14.2.2 and 2.4.6]).

(iii) We may (and shall) assume that (i) holds and that there exists a prime number ℓ which is invertible in all residue fields F_ν of $\widetilde{o}$. Then by the Lefschetz formula of Grothendieck and Deligne (cf. [**34**]) one has for each variety Z over F_ν, $\nu \in W_{\text{fin}}\backslash\Sigma$ an equality

$$\operatorname{Card} Z(F_\nu) = \sum_i (-1)^i \operatorname{Tr}(F^n, H_c^i(Z \times \overline{F}_\nu, \mathbb{Q}_l)) = \sum_{i,j} (-1)^i \alpha_{i,j}^n$$

where $H_c^i(Z, \mathbb{Q}_l)$ are the ℓ-adic cohomology groups $0 \leq i \leq 2\dim Z$ with compact support and $\alpha_{i,j}$ are the eigenvalues of the Frobenius endomorphsim on $H_c^i(Z, \mathbb{Q}_l)$. Moreover, by the fundamental result of Deligne [**19**, 3.3.4] one has that $|\alpha_{i,j}| \leq q_\nu^{i/2}$ for $q_\nu = \operatorname{Card} F_\nu$ with a unique eigenvalue $\alpha_i = \pm q^{\dim Z}$ when $i = 2\dim Z$ (use Poincaré duality and the integrality of $Z \times \overline{F}_\nu$). Finally, since (cf. [**18**, 6.2]) $R^i\widetilde{\pi}_!\mathcal{F}$ is constructible for any constructible sheaf $\mathcal{F}$ on $\widetilde{Y}$ it follows that the dimensions of the ℓ-adic cohomology groups

$$H_c^i(Z \times \overline{F}_\nu, \mathbb{Q}_l)$$

are uniformly bounded for the closed fibres Z of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$. Therefore,

$$\operatorname{Card} Z(F_\nu) > 0$$

for all F_ν-fibres Z of $\widetilde{\pi}$ if q_ν is sufficiently large (cf. [**62**, p. 184] for the case $X = k$). To complete the proof, use Hensel's lemma. □

It is worth noting that it follows from (4.19) and (2.7)(a) that the map

$$\pi_A : Y(A_k) \longrightarrow X(A_k)$$

between adelic spaces is *open* for $\pi : Y \to X$ as in (4.19).

Definition 4.20. — Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. An *adelic norm for* $\pi : Y \to X$ is a family of norms

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\operatorname{Tan}(Y_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu)) \longrightarrow [0, \infty), \quad \nu \in W\}$$

on the relative analytic anticanonical bundles of $\pi_{\nu,\mathrm{an}} : Y_{\mathrm{an}}(k_\nu) \to X_{\mathrm{an}}(k_\nu)$, $\nu \in W$ with the following property.

There exists a finite subset Σ of W_{fin} and an extension of π to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between two smooth models $\widetilde{Y}$ and $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$ such that the restriction of $\| \ \|_\nu$ to the relative analytic anticanonical line bundle of $\widetilde{Y}(o_\nu) \to \widetilde{X}(o_\nu)$ is equal to the model norm determined by $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ (cf. (2.17)) for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

The notion of adelic norms is similar to notions in Arakelov theory and it is possible to use (4.2) to define adelic norms for arbitrary vector bundles. Here we introduce adelic norms only in order to define adelic measures.

Definition 4.21. — Let X be a smooth k-variety and $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Then a set

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

of positive continuous functions is called a *set of convergence factors for* $\pi : Y \to X$ if the following holds.

There exists an extension of $\pi : Y \to X$ to a smooth $o_{(\Sigma)}$-morphism

$$\widetilde{\pi} : \widetilde{Y} \longrightarrow \widetilde{X}$$

as in (4.19) satisfying the following condition:

($*$) The product

$$\prod_{W_{\mathrm{fin}} \backslash \Sigma} \left(\int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu \right)$$

converges absolutely to a positive real number for the model measure μ_ν on the o_ν-schemes

$$Z_\nu, \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

obtained by base extension of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ along any

$$\{P_\nu\} \in \prod_{W_{\text{fin}} \backslash \Sigma} \widetilde{X}(o_\nu).$$

Remarks 4.22

(a) The condition $(*)$ is clearly valid for *all* smooth morphisms $\widetilde{\pi}$ satisfying the conditions in (4.19) as soon as it is valid for one such morphism.

(b) Let $\nu \in W_{\text{fin}} \backslash \Sigma$. Then the map

$$P_\nu \longrightarrow \int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu$$

is a continuous real-valued function on the compact space $\widetilde{X}(o_\nu)$ (cf. (1.22)(a) and (2.5)). The product of integrals in $(*)$ defines therefore a continuous function on

$$\prod_{W_{\text{fin}} \backslash \Sigma} \widetilde{X}(o_\nu).$$

(c) One may construct constant sets of convergence factors for $\pi : Y \to X$ when the first two Betti numbers B_1 and B_2 vanish for all geometric fibres of $\pi : Y \to X$. This follows from the arguments in the proofs of (4.17) and (4.19)(iii).

Example 4.23. — Let $\pi : Y \to X$ be as in (4.21) and suppose in addition that X is proper. There is then an extension of $\pi : Y \to X$ to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ as in (4.19) such that $\widetilde{X}$ is proper over $o_{(\Sigma)}$. For $\nu \in W_{\text{fin}} \backslash \Sigma$, let

$$\widetilde{\Lambda}_\nu : C_c(\widetilde{Y}(o_\nu)) \longrightarrow C_c(\widetilde{X}(o_\nu))$$

be the positive linear map of $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ (cf. (1.22)(a)) defined by the relative model norm (2.17) and let

$$\varepsilon_\nu \in C_c(\widetilde{X}(o_\nu)) = C_c(X(k_\nu))$$

be the image of the constant map on $\widetilde{Y}(o_\nu)$ with value 1.

Then,

$$\varepsilon_\nu \in C(X(k_\nu))_{>0}$$

since

$$\widetilde{\pi}_\nu : \widetilde{Y}(o_\nu) \longrightarrow \widetilde{X}(o_\nu)$$

is surjective (cf. (4.19) and the last part of (1.10)). Hence any set

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

with

$$\beta_\nu = 1/(\varepsilon_\nu \circ \pi_\nu)$$

for all $\nu \in W_{\text{fin}} \backslash \Sigma$ form a set of convergence factors for $\pi : Y \to X$.

Note also that

$$\beta_\nu = 1/(\varepsilon_\nu \circ \pi_\nu), \quad \nu \in W_{\text{fin}} \backslash \Sigma$$

is constant on $Y(k_\nu)$ if the number of F_ν-points in the F_ν-fibres of $\widetilde{\pi}$ is a constant function on $\widetilde{X}(F_\nu)$.

Notation 4.24. — Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Let

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu)) \longrightarrow [0, \infty), \ \nu \in W\}$$

be an adelic norm and let

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0}, \ \nu \in W\}$$

be a set of convergence factors for $\pi : Y \to X$. If $P \in \pi(Y(A_k)) \subseteq X(A_k)$, let Z_P be the smooth A_k-variety which is the fibre product of $\pi : Y \to X$ and $P : \operatorname{Spec} A_k \to X(A_k)$. Then $\theta(P)$ is the positive σ-regular Borel measure on $Z_P(A_k)$ determined by the restrictions of $\| \ \|$ and β to Z_P (cf. (4.14)(b)).

Theorem 4.25. — *Let X, π, $\| \ \|$, β, $\theta(P)$, Z_P, $P \in \pi(Y(A_k))$ be as in (4.24). Let f_P be the restriction of $f \in C_c(Y(A_k))$ to $Z_A(A_k)$ and let $\Lambda_\beta(f) : X(A_k) \to \mathbb{R}$ be the function such that*

$$\Lambda_\beta(f)(P) = \int_{Z_P(A)} f_P d\theta(P)$$

for $P \in \pi(Y(A_k))$ and such that $\Lambda_\beta(f)(P) = 0$ for $P \notin \pi(Y(A_k))$.

Then Λ_β is a positive linear map from $C_c(Y(A_k))$ to $C_c(X(A_k))$.

Proof. — The proof is very similar to the proof of (4.14)(b) and we shall use the notions introduced there without further comments. In particular, $\mathcal{A}$ will denote the subalgebra of $C_c(Y(A_k))$ of finite sums of finitely decomposable functions. It is obvious from the definition that Λ_β is positive and linear. It is thus sufficient to prove that $\Lambda_\beta(f) \in C_c(X(A_k))$ for all $f \in C_c(Y(A_k))$. Moreover, since $\pi_A : Y(A_k) \to X(A_k)$ is continuous it suffices to show that $\Lambda_\beta(f)$ is continuous for all $f \in C_c(Y(A_k))$.

We first treat the case where $f \in C_c(Y(A_k))$ is the restricted product of an adelic set

$$\{f_\nu \in C_c(Y(k_\nu)), \ \nu \in W\}.$$

Choose a finite subset $\Sigma \subset W_{\text{fin}}$ and two smooth models $\widetilde{Y}$, $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$ and an extension of π to a smooth $\widetilde{o}$-morphism

$$\widetilde{\pi} : \widetilde{Y} \longrightarrow \widetilde{X}$$

satisfying all the conditions of (4.19) and such that f_ν is the characteristic function of

$$\widetilde{Y}(o_\nu) \subseteq Y(k_\nu)$$

for all $\nu \in W_{\text{fin}} \backslash \Sigma$.

Let

$$\Lambda_\nu : C_c(Y(k_\nu)) \longrightarrow C_c(X(k_\nu))$$

be the positive linear map corresponding to $\| \ \|_\nu$ and let

$$P = \{P_\nu\}_{\nu \in W} \in \widetilde{X}(A_k(\Sigma)) = \prod_{W_{\text{fin}} \backslash \Sigma} \widetilde{X}(o_\nu) \times \prod_{W_\infty \cup \Sigma} X(k_\nu).$$

Finally, let Z_ν, $\nu \in W_{\text{fin}} \backslash \Sigma$ be the o_ν-scheme obtained from $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ after base extension along P_ν and let μ_ν be the model measure on $Z_\nu(o_\nu)$. Then,

$$\Lambda_\beta(f)(P) = \prod_{W_{\text{fin}} \backslash \sigma} \left(\int_{Z_\nu(o_\nu)} \beta_\nu d\mu_\nu \right) \times \prod_{W_\infty \cup \Sigma} \Lambda_\nu(\beta_\nu f_\nu).$$

Hence by (4.22)(b) and (1.22)(a) we get that the restriction of $\Lambda_\beta(f)$ to $\widetilde{X}(A_k(\Sigma))$ is continuous. Moreover, since $\Lambda_\beta(f) = 0$ outside the compact open subset

$$\widetilde{X}(A_k(\Sigma)) \subseteq X(A_k)$$

we obtain that $\Lambda_\beta(f) \in C_c(X(A_k))$ for all decomposable functions $f \in C_c(Y(A_k))$. Moreover, by the additivity of Λ_β, one has the same result for any function f in $\mathcal{A}$.

For arbitrary $f \in C_c(Y(A_k))$, there exists by the proof of (4.14)(b) a compact subset K of $Y(A_k)$ and a sequence $(f_i)_{i=1}^\infty$ of functions in $\mathcal{A}$ with support in K which converges uniformly to f. There exists also a continuous decomposable non-negative function G in $C_c(Y(A_k))$ such that $G = 1$ on K. From the linearity and positivity of Λ_β one gets (cf. the proof of op. cit.) an inequality:

$$\sup |\lambda_\beta(f) - \Lambda_\beta(f_i)| = \sup |\lambda_\beta(f - f_i)| \leq \sup \lambda(G) \sup |(f - f_i)|$$

for all positive integers i. Therefore $(\Lambda_\beta(f_i))_{i=1}^\infty$ is a sequence in $C_c(X(A_k))$ which converges uniformly to $\Lambda_\beta(f)$. Hence $\Lambda_\beta(f) \in C_c(X(A_k))$, thereby completing the proof. □

Now recall the canonical isomorphism described in (1.22):

(4.26)

$$\mathbf{det}\,\mathrm{Tan}(Y_{\text{an}}(k_\nu)) = \mathbf{det}\,\mathrm{Tan}(Y_{\text{an}}(k_\nu)/X_{\text{an}}(k_\nu)) \otimes \pi^*_{\nu,\text{an}}(\mathbf{det}\,\mathrm{Tan}(X_{\text{an}}(k_\nu)))$$

Lemma 4.27. — *Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Let*

$$\| \ \|_X = \{\| \ \|_{X,\nu},\ \nu \in W\}$$

and

$$\| \ \|_{Y/X} = \{\| \ \|_{Y/X,\nu},\ \nu \in W\}$$

be adelic norms for X resp. $\pi : Y \to X$.

Finally, for $\nu \in W$, let

$$\| \ \|_{Y,\nu} = \| \ \|_{Y/X,\nu} \cdot \pi^*_{\nu,\mathrm{an}} \| \ \|_{X,\nu}$$

be the product norm of $\| \ \|_{Y/X,\nu}$ and the pullback norm $\pi^*_{\nu,\mathrm{an}}\| \ \|_{X,\nu}$ (cf. (1.7)(a), (b) and (4.26)). Then

$$\| \ \|_Y = \{\| \ \|_{Y,\nu},\ \nu \in W\}$$

is an adelic norm for Y.

Proof. — It is clear from (1.7) and (4.26) that $\| \ \|_{Y,\nu}$ is a norm on $\mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu))$ for each ν. Moreover, if $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is an extension of π to a smooth $o_{(\Sigma)}$-morphism $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ between two smooth models $\widetilde{Y}$, $\widetilde{X}$ over $\widetilde{o} = o_{(\Sigma)}$, then

$$\| \ \|_{Y,\nu} : \mathbf{det}\,\mathrm{Tan}(Y_{\mathrm{an}}(k_\nu)) \longrightarrow [0,\infty)$$

restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{Y}_{\mathrm{an}}(o_\nu))$ for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ by (2.18). This completes the proof. □

Theorem 4.28. — *Let X be a smooth k-variety and let $\pi : Y \to X$ be a smooth k-morphism with geometrically connected fibres. Let $\| \ \|_X$ (resp. $\| \ \|_{Y/X}$) be an adelic norm for X (resp. an adelic norm for $\pi : Y \to X$) and let*

$$\gamma = \{\gamma_\nu \in C(X(k_\nu))_{>0},\ \nu \in W\}$$

resp.

$$\beta = \{\beta_\nu \in C(Y(k_\nu))_{>0},\ \nu \in W\}$$

be a set of convergence factors for X (resp. $\pi : Y \to X$).

Let $m_{A,\gamma}$ be the positive σ-regular Borel measure on $X(A_k)$ determined by $\| \ \|_X$ and γ and let

$$\Lambda_\beta : C_c(Y(A_k)) \longrightarrow C_c(X(A_k))$$

be the positive linear map determined by $\| \ \|_{Y/X}$ and β (cf. (4.26)).

Finally, let

$$\| \ \|_Y = \{\| \ \|_{Y/X,\nu}\pi^*_{\nu,\mathrm{an}}\| \ \|_{X,\nu},\ \nu \in W\}$$

be the adelic product norm for Y constructed in (4.27).

Then the following holds.

(a) *The products*

$$\alpha_\nu = \beta_\nu(\gamma_\nu \circ \pi_\nu), \quad \nu \in W$$

form a set α of convergence factors for Y.

(b) *Let $n_{A,\alpha}$ be the positive σ-regular Borel measure on $Y(A_k)$ determined by $\| \ \|_Y$ and α. Then,*

$$\int_{Y(A_k)} f dn_{A,\alpha} = \int_{X(A_k)} \Lambda_\beta(f) dm_{A,\gamma}$$

for any $f \in C_c(Y(A_k))$.

Proof

(a) Let $\widetilde{\pi} : \widetilde{Y} \to \widetilde{X}$ be a smooth $o_{(\Sigma)}$-morphism between smooth $o_{(\Sigma)}$-schemes which extends $\pi : Y \to X$ and satisfies all the conditions in (4.19) and such that the restriction of $\| \ \|_\nu$ to $\mathbf{det}\, \mathrm{Tan}(\widetilde{Y}_{\mathrm{an}}(o_\nu)/\widetilde{X}_{\mathrm{an}}(o_\nu))$ is the model norm for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$.

Let $\nu \in W_{\mathrm{fin}} \backslash \Sigma$, and write μ_ν for all model measures. Let

$$\widetilde{\Lambda}_\nu : C_c(\widetilde{Y}(o_\nu)) \longrightarrow C_c(\widetilde{X}(o_\nu))$$

be the positive linear map defined by the relative model norm. Then, by (2.22),

$$\int_{\widetilde{Y}(o_\nu)} \alpha_\nu d\mu_\nu = \int_{\widetilde{X}(o_\nu)} \widetilde{\Lambda}(\alpha_\nu) d\mu_\nu = \int_{\widetilde{X}(o_\nu)} \gamma_\nu \widetilde{\Lambda}_\nu(\beta_\nu) d\mu_\nu.$$

Choose $P_\nu \in \widetilde{X}(o_\nu)$ for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ such that $|\log(\Lambda_\nu \beta_\nu(P_\nu))|$ is maximal. Then,

$$\left| \log \int_{\widetilde{Y}(o_\nu)} \alpha_\nu d\mu_\nu \right| \leq \left| \log \int_{\widetilde{X}(o_\nu)} \gamma_\nu d\mu_\nu \right| + |\log(\Lambda_\nu \beta_\nu(P_\nu))|$$

by the positivity of the model measure on $\widetilde{X}(o_\nu)$.

Now sum over all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ and use the assumptions on $\{\gamma_\nu\}$ and $\{\beta_\nu\}$.

(b) First, let f be finitely decomposable. One may adapt the choice of $\widetilde{Y}$, $\widetilde{X}$ and $\widetilde{\pi}$ in (a) to f such that in addition f is the restricted product of an adelic set

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\},$$

where f_ν is the characteristic function of

$$\widetilde{Y}(o_\nu) \subseteq Y(k_\nu)$$

for all $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Now combine the calculation of $\Lambda_\beta(f)$ in the proof of (4.25) with the formula in (4.14)(c) for the integral over $X(A_k)$ for a decomposable function. Then the formula for finitely decomposable functions follows from the corresponding assertion over local fields established in (1.22)(c).

Let $\mathcal{A}$ denote the subalgebra of $C_c(Y(A_k))$ of finite sums of finitely decomposable functions. By additivity of the integrals and Λ_β, we get the formula also for functions f in $\mathcal{A}$.

Finally, let f be an arbitrary function in $C_c(Y(A_k))$. Then, by the proof of (4.25), there exists a sequence $(f_i)_{i=1}^\infty$ of functions in $\mathcal{A}$ which converges uniformly to f on $X(A_k)$ and such that the supports of all f_i are contained in one common compact subset K of $X(A_k)$. Also, by (op. cit.), $(\Lambda_\beta(f_i))_{i=1}^\infty$ is a sequence in $C_c(X(A_k))$ with supports in $f(K)$ converging uniformly to $\Lambda_\beta(f)$. The uniform convergence and the compactness statement now implies that

$$\int_{Y(A_K)} f dn_{A,\alpha} = \lim_{i\to\infty}\int_{Y(A_K)} f_i dn_{A,\alpha}$$

$$\int_{X(A_K)} \Lambda_\beta(f) dm_{A,\gamma} = \lim_{i\to\infty}\int_{X(A_K)} \Lambda_\beta(f_i) dm_{A,\gamma}$$

so that the formula also holds for arbitrary functions $f \in C_c(Y(A_k))$. This completes the proof. □

5. Torsors over global fields and Tamagawa measures

The purpose of this section is to define Tamagawa measures on universal torsors. It is thereby important to generalize some of the constructions of Colliot-Thélène and Sansuc in **[15]** and to define universal torsors over schemes. We shall keep the notations in part 4 and use the word k-variety for a geometrically connected, separated scheme of finite type over k. Also, k will denote a number field and K a general field.

Hypothesis 5.1

(a) k is a number field
(b) X is a smooth k-variety with structure morphism $h : X \to \operatorname{Spec} k$
(c) G is a smooth k-variety which is an algebraic group over k
(d) $\pi : \mathcal{T} \to X$ is a (left) X-torsor under G (with respect to the fppf-topology) with G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$ (cf. (3.3))

Lemma 5.2. — *Assume (5.1). Then there exists schemes $\widetilde{X}$, $\widetilde{G}$, $\widetilde{\mathcal{T}}$ of finite presentation over $\widetilde{o} := o_{(\Sigma)}$ for some finite set Σ of closed points of* Spec o *with the following properties.*

(a) *$\widetilde{X}$ is a smooth $\widetilde{o}$-model of X (see (4.4)(a)).*
(b) *$\widetilde{h} : \widetilde{X} \to$ Spec $\widetilde{o}$ is a smooth separated morphism with geometrically connected fibres.*

(c) $\widetilde{G}$ *is a smooth separated group scheme over* $\widetilde{o}$ *with geometrically connected fibres.*

(d) $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ *is a (left)* $\widetilde{X}$*-torsor under* $\widetilde{G}$ *with* $\widetilde{G}$*-action* $\widetilde{\sigma} : \widetilde{G} \times_o \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$.

(e) *The generic fibres of* $\widetilde{X}$, $\widetilde{G}$, $\widetilde{\mathcal{T}}$ *are equal to* X, G *and* $\mathcal{T}$ *and the restrictions to the generic fibres of* $\widetilde{h}$, $\widetilde{\pi}$ *and* $\widetilde{\sigma}$ *are equal to the* k*-morphisms* h, π, σ *in (5.1).*

Any other set of such $\widetilde{o}$*-schemes and* $\widetilde{o}$*-morphisms are related to the given ones by means of a canonical isomorphism after base extension to* $o_{(S)}$ *for some finite set* $S \supseteq \Sigma$ *of places of* k.

Proof. — This result is a formal consequence of (4.2). The reader may also consult [**45**, III.4.3] and [**1**, VII.5]. □

We denote the unit sections of G/k and $\widetilde{G}/\widetilde{o}$ by e resp. $\widetilde{e}$. There is thus a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \widetilde{o} & \xrightarrow{\widetilde{e}} & \widetilde{G} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e} & G \end{array}$$

We shall in the sequel consider o_ν-schemes and o_ν-morphisms obtained by base extension from the $\widetilde{o}$-schemes and $\widetilde{o}$-morphisms in (5.2). We will add a lower index ν for places defined by prime ideals of $\widetilde{o}$ to indicate that we have made a base extension to the ν-adic completion o_ν of o and $\widetilde{o}$. We denote by F_ν the residue field of o_ν for $\nu \in W_{\text{fin}}$.

Let $\pi : \mathcal{T} \to X$ be an X-torsor under G as in (5.1) with G-action $\sigma : G \times_k \mathcal{T} \to \mathcal{T}$. Then σ determines a (continuous) left $G(A_k)$-action

$$\sigma_A : G(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k)$$

on the fibres of

$$\pi_A : \mathcal{T}(A_k) \longrightarrow X(A_k)$$

(cf. (3.2)(iii)).

Let

$$\mathcal{T}(A_k) \times_{X(A)} \mathcal{T}(A_k) \subset \mathcal{T}(A_k) \times \mathcal{T}(A_k)$$

be the inverse image of the diagonal under

$$(\pi_A, \pi_A) : \mathcal{T}(A_k) \times \mathcal{T}(A_k) \to X(A_k) \times X(A_k).$$

It is a closed subspace of $\mathcal{T}(A_k) \times \mathcal{T}(A_k)$ since $X(A_k)$ is a Hausdorff space. The condition (3.3)(b) for torsors implies that the map

(5.3) $$\rho_A := (\sigma_A, pr_2) : G(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k) \times_{X(A)} \mathcal{T}(A_k)$$

is a homeomorphism of adelic spaces.

Proposition 5.4. — *Let $\pi : \mathcal{T} \to X$ be a (left) X-torsor under G as in (5.1) and let $\pi_A : \mathcal{T}(A_k) \to X(A_k)$ be the continuous map induced by $\pi : \mathcal{T} \to X$. Then*

(a) *π_A is open.*

(b) *$\pi_A(\mathcal{T}(A_k))$ is homeomorphic to the homogeneous space*

$$\mathcal{T}(A_k)/G(A_k)$$

of $\mathcal{T}(A_k)$ with respect to σ_A.

Proof

(a) Let $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ be a torsor of $o_{(\Sigma)}$-schemes as in (5.2). Then the open decomposable sets of the form

$$B := \prod_{\nu \in W} B_\nu = \left(\prod_{\nu \notin S} \mathcal{T}(o_\nu)\right) \times \left(\prod_{\nu \in S} B_\nu\right), \quad S \supseteq \Sigma \cup W_\infty$$

for finite subsets S of W form a base for the adelic topology. Also, by Lang's theorem (cf. (3.21)(a)) one has

$$\pi_\nu(B_\nu) = \widetilde{X}(o_\nu)$$

for all places outside S. This combined with the openness of π_ν for $\nu \in S$ (cf. (3.11)(b)) implies that $\pi_A(B)$ is open in $X(A_k)$.

(b) π_A is open and continuous. A subset U of $\pi_A(\mathcal{T}(A_k))$ is therefore open in $X(A_k)$ if and only if $\pi_A^{-1}(U)$ is open in $X(A_k)$ (cf.[**38**, p. 311]). □

We shall now consider positive Borel measures on $\mathcal{T}(A_k)$ which are invariant under the left action of $G(A_k)$.

Recall (cf. (3.8), (3.9)) that the relative algebraic tangent bundle $T_{\mathcal{T}/X} \to \mathcal{T}$ is equal to the fibre product

$$\mathcal{T} \times_k T_{G,e} \longrightarrow \mathcal{T}$$

for the fibre $T_{G,e}$ at $e \in G(k)$ of the algebraic tangent bundle of G. There is also a canonical isomorphism where $T_{\widetilde{G},\widetilde{e}}$ is the fibre at $\widetilde{e} \in \widetilde{G}(\widetilde{o})$ of the algebraic tangent bundle of $\widetilde{G}/\widetilde{o} = o_{(\Sigma)}$. The $\widetilde{o}$-points of $T_{\widetilde{G},\widetilde{e}}$ form an $\widetilde{o}$-lattice $T_{\widetilde{G},\widetilde{e}}(\widetilde{o})$ in the tangent space $T_{G,e}(k)$ of G at e.

Definition 5.5. — A constant adelic norm for $\pi : \mathcal{T} \to X$ is a set of constant norms (cf. (3.15)

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}(\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu))) \longrightarrow [0,\infty), \nu \in W\}$$

such that $\| \ \|_\nu$ is the pullback of the ν-adic norm on $\mathbf{det}(T_{G,e}(k_\nu))$ determined by the o_ν-lattice $\mathbf{det}(T_{\widetilde{G},\widetilde{e}}(o_\nu))$ for all but finitely many places in $W_{\mathrm{fin}} \backslash \Sigma$.

Example 5.6. — Let ω_0 be a local non-zero section at e of the canonical line bundle of G. Then ω_0 defines ν-adic vector space norms on $\mathbf{det}(T_{G,e}(k_\nu))$ for all places ν which are equal to the lattice norms given by $\mathbf{det}(T_{\widetilde{G},\widetilde{e}}(o_\nu))$ for almost all ν. The constant norms

$$\| \|_\nu : \mathbf{det}(\mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k_\nu)/X_{\mathrm{an}}(k_\nu))) \longrightarrow [0,\infty), \quad \nu \in W$$

therefore form an adelic norm for $\pi : \mathcal{T} \to X$.

To define adelic measures, we shall need sets of convergence factors.

Lemma 5.7. — *Let $\pi : \mathcal{T} \to X$ be an X-torsor under G as in (5.1) and let*

$$\beta : W \longrightarrow \mathbb{R}_{>0}$$

be a set of constant convergence factors for G. Then β is also a set of convergence factors for $\pi : \mathcal{T} \to X$.

Proof. — Choose $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{G}/\widetilde{o}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2) and let $\nu \in W_{\mathrm{fin}}\backslash\Sigma$. Then $\widetilde{\pi}(\widetilde{\mathcal{T}}(o_\nu)) = \widetilde{X}(o_\nu)$ (cf. (3.21)). Moreover, if $P_\nu \in \widetilde{X}(o_\nu)$, then

$$\widetilde{\mathcal{T}} \times_{\widetilde{X}} P_\nu \approx \widetilde{G} \times_{\widetilde{o}} o_\nu$$

as o_ν-schemes by the torsor isomorphism (3.18)(b). This completes the proof. □

The following result is fundamental for the applications of torsors to problems concerning counting functions of rational points.

Theorem 5.8. — *Let $\pi : \mathcal{T} \to X$ be an X-torsor under G with left action $\sigma : G \times \mathcal{T} \to \mathcal{T}$ as in (5.1). Also, let*

$$\{\| \|_\nu,\ \nu \in W\}$$

be a constant adelic norm for $\pi : \mathcal{T} \to X$ defined by a local differential form $\omega_0 \neq 0$ of G at e and let

$$\{\Lambda_\nu : C_c(\mathcal{T}(k_\nu)) \longrightarrow C_c(X(k_\nu)), \quad \nu \in W\}$$

be the positive linear maps defined by these norms. Finally, let

$$\beta : W \to \mathbb{R}_{>0}$$

be a set of constant convergence factors for G. Then there exists a unique positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

which is independent of the choice of ω_0 such that

$$(**) \qquad \Lambda_\beta(f_A) = \prod_{\nu\in W} \beta_\nu \Lambda_\nu(f_\nu)$$

for the restricted product

$$f_A = \prod\nolimits'_{\nu \in W} f_\nu$$

of any adelic set of functions

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\}.$$

Moreover, if $\sigma(g) : \mathcal{T}(A_k) \to \mathcal{T}(A_k)$ is the left translation defined by an element $g \in G(A_k)$, then

$$\Lambda_\beta(f) = \Lambda_\beta(f \circ \sigma(g))$$

for any $f \in C_c(\mathcal{T}(A_k))$.

Proof. — The existence and the uniqueness of Λ_β follows from (4.25) and (5.7). To prove that Λ_β is independent of the choice of ω_0, let $\alpha \neq 0$ be an element of k. Then the corresponding adelic norms is given by $\{|\alpha|_\nu \| \ \|_\nu,\ \nu \in W\}$ and the corresponding ν-adic linear maps (cf. (1.21)) by $|\alpha|_\nu \Lambda_\nu$. The independence therefore follows from the formula

$$\prod_{\nu \in W} |\alpha|_\nu = 1.$$

To prove the last assertion use $(**)$ to reduce to the statement

$$\Lambda_\nu(f_\nu) = \Lambda_\nu(f_\nu \circ \sigma(g_\nu))$$

for the left translations

$$\sigma(g_\nu) : \mathcal{T}(k_\nu) \longrightarrow \mathcal{T}(k_\nu)$$

defined by $g_\nu \in G(k_\nu)$. The equality follows from the fact that $\| \ \|_\nu$ is invariant under the left action of $g_\nu \in G(k_\nu)$ (cf. (3.15)), thereby completing the proof. □

Remarks 5.9

(a) It is immediate from the definitions of Λ_ν that $\Lambda_\beta(f)$ has support in the open subset $\pi_A(\mathcal{T}(A_k))$ of $X(A_k)$. Therefore (cf. (5.4)) Λ_β may also be seen as a positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/G(A_k)).$$

This can be used to give another construction of Λ_β starting with a Haar measure on $G(A_k)$ (cf. [**11**, Ch. VII, §2, n°2]), thereby avoiding the general result in (4.25).

But we shall in other papers give applications of (4.25), which cannot be deduced from the theory of homogenous spaces in (op. cit). The results (4.25)-(4.28) can e.g. be used to answer a question in [**6**] about L-primitive fibrations. Also, even for torsors it is useful to consider measures which are not invariant under the group action and to define measures on (partial) compactifications of torsors (see [**51**]).

(b) Let G be an r-dimensional k-torus and let $\pi : \mathcal{T} \to X$ be an X-torsor under G. Let

$$\| \ \|_{\mathcal{T}/X,\nu} : \mathbf{det}(\mathrm{Tan}(\mathcal{T}(k_\nu)/X(k_\nu)) \longrightarrow [0,\infty), \quad \nu \in W$$

be the order norms (see (3.30)). It follows from (3.31) that these ν-adic norms form an adelic norm. There is, therefore, for each set $\{\beta_\nu,\ \nu \in W\}$ of constant convergence factors for G a unique positive linear map

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

satisfying the same property $(**)$ as in (5.8). This map coincides with the positive linear map in (5.8) for split k-tori G (since the order norms are defined by means of a differential form of minimal $d\log$-type). The two maps also coincide for non-split k-tori G since $\prod_{\nu\in W} \|s\|_\nu = 1$ for the order norms $\| \ \|_\nu : \bigwedge^r(T_{G,e}(k_\nu)) \to \mathbb{R}$ of a section $s \neq 0$ of $\bigwedge^r(T_{G,e}(k_\nu))$.

Definition 5.10

(a) A *torus T over a scheme B* is a commutative group scheme of finite type over B which locally in the fpqc-topology (cf. [**21**, exp. VIII]) is isomorphic to the group scheme

$$G_{m,B} \times_B \cdots \times_B G_{m,B}$$

over B for a finite number of copies $G_{m,B}$. The torus is said to be split over B if T is isomorphic to

$$G_{m,B} \times_B \cdots \times_B G_{m,B}.$$

(b) A *finitely generated torsion free twisted constant group scheme* Π *over B* is a commutative group scheme over B which locally in the fpqc-topology (cf. [**21**, exp. VIII]) is isomorphic to the constant group scheme

$$Z_B \times_B \cdots \times_B Z_B$$

over B for a finite number of copies Z_B. The f.g. torsion free twisted constant group scheme Π over B is said to be *split over B* if T is isomorphic to

$$Z_B \times_B \cdots \times_B Z_B.$$

It is known by a theorem of Grothendieck (cf. [**21**, exp. X]) that any torus and any twisted constant torsion free group scheme split after some surjective étale base extension. One can therefore replace the fpqc-topology by the *fppf-topology* or by the *étale topology* in the definition above. In particular, if $B = \mathrm{Spec}\, K$ for a field

K, then all tori and all f.g. torsion free twisted constant group schemes split over a separable closure $\overline{K}$ of K.

Notations 5.11. — If T is a B-torus, then

$$\hat{T} := \mathrm{Hom}_{Gr}(T, G_{m,B})$$

If Π is a f.g. torsion free twisted constant B-group scheme, then

$$\mathbf{D}(\Pi) := \mathrm{Hom}_{Gr}(\Pi, G_{m,B}).$$

Here $\mathrm{Hom}_{Gr}(\cdot, G_{m,B})$ means the group scheme representing the Hom-functor in the category of group schemes. It is easily seen that $\hat{T}$ is a f.g. torsion free twisted constant B-group scheme and that $\mathbf{D}(M)$ is a B-torus in (5.11).

These two constructions are inverse to each other in the sense that they define a contravariant equivalence of categories between tori T over B and f.g. twisted constant torsion free group schemes Π over B (cf. [**21**, X.5]).

If the base scheme B is a field K, and $\overline{K}$ is a separable closure over K, then the functor sending Π to the $\mathrm{Gal}(\overline{K}/K)$-module $P = \Pi(\overline{K})$ defines an equivalence between the category of f.g. torsion free twisted constant group schemes over K and the category of f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-modules (see [**45**, p. 52]). There is therefore in this case a contravariant equivalence between the category of K-tori and the category of f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-modules for which a K-torus is sent to its character $\mathrm{Gal}(\overline{K}/K)$-module $\hat{T} := \mathrm{Hom}(\overline{T}, G_{m,\overline{X}})$. Conversely, if M is a f.g. torsion free continuous discrete $\mathrm{Gal}(\overline{K}/K)$-module, form the K-Hopf algebra $\overline{K}[M]^G$ of $\mathrm{Gal}(\overline{K}/K)$-invariant elements in $\overline{K}[M]$ and let $D(P) = \mathrm{Spec}\,\overline{K}[M]^G$. Then $D(P)$ is an K-torus (cf. Ch. III, §8 in Borel's book [**9**]).

Now assume the following:

5.12 (a) B is a Noetherian scheme.

5.12 (b) $f : X \to B$ is a smooth proper surjective morphism of constant relative dimension with geometrically connected fibres.

5.12 (c) $R^2 f_* \mathcal{O}_X = 0$.

5.12 (d) The relative Picard functor (cf. [**10**, Ch. 8]) of f is representable by a f.g. torsion free twisted constant B-group scheme $\mathbf{Pic}_{X/B}$.

It follows from more general results of Grothendieck and Murre (cf. [**10**, pp. 210-211]) that $\mathbf{Pic}_{X/B}$ is representable when f is projective or B is the spectrum of a field. It is easy to show that the assumptions in (5.12) also hold for smooth proper toric schemes over B.

Let $g : T \to B$ be a B-torus. Let $\mathcal{F}_T$ (resp. $\mathcal{G}_T$) be the fppf-sheaves of abelian groups on $(\mathbf{Sch}\,/B)^{\mathrm{opp}}$ associated to the functor:

$$Y \longrightarrow H^1_{\mathrm{fppf}}(X \times_B Y, T_Y)$$

resp.

$$Y \longrightarrow \mathrm{Hom}_Z(\mathrm{Hom}_Y(T_Y, G_{m,Y}), H^1_{\mathrm{fppf}}(X \times_B Y, G_{m,Y}))$$

where Hom_Y means the abelian group of Y-homomorphisms in the category of group schemes over Y. The covariant functoriality of $H^1_{\mathrm{fppf}}(X \times_B Y, \cdot)$ under homomorphisms of commutative group schemes over $X \times_B Y$ gives rise to a morphism of functors:

$$\mathcal{F} \longrightarrow \mathcal{G}.$$

It is easy to verify that this is an isomorphism of functors since T is locally isomorphic to $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$ in the fppf-topology. One can also according to a theorem of Grothendieck [**27**, pp. 171-183] define $\mathcal{F}_T$ (resp. $\mathcal{G}_T$) by means of the étale topology instead of the fppf-topology.

If $T = \mathbb{G}_m$, then $\mathcal{F}_T$ coincides with the relative Picard functor. We may thus for an arbitrary torus $g : T \to B$ regard $\mathcal{G}_T$ as the fppf-sheaves of abelian groups on $(\mathbf{Sch}\,/B)^{\mathrm{opp}}$, where

$$Y \longrightarrow \mathrm{Hom}_Y(\hat{T}_Y, (\mathbf{Pic}_{X/B})_Y).$$

There is also an interpretation

$$\mathcal{F}_T(Y) = H^0(Y, R^1 g_*(T))$$

where the right hand side can be read both with respect to the fppf-topology and the étale topology (cf. [**10**, p. 202-203]). The isomorphism $\mathcal{F}_T(B) = \mathcal{G}_T(B)$ may thus be interpreted as an isomorphism:

$$H^0(B, R^1 g_*(T)) = \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}).$$

There is further by Leray's spectral sequence an exact sequence

$$0 \to H^1(B,T) \to H^1(X,T) \to H^0(B, R^1 g_*(T)) \to H^2(B,T) \to H^2(X,T)$$

of cohomology groups in the fppf-topology or the étale topology, which reduces to a short exact sequence

$$0 \longrightarrow H^1(B,T) \longrightarrow H^1(X,T) \longrightarrow H^0(B, R^1 g_*(T)) \longrightarrow 0$$

if there is a section $s : B \to X$ to $f : X \to B$.

There are thus isomorphic exact sequences generalizing the sequence of Colliot-Thélène and Sansuc (cf. [**15**, 2.0.2])

5.13 (a)

$$1 \to H^1_{\text{ét}}(B,T) \to H^1_{\text{ét}}(X,T) \xrightarrow{t} \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}) \to H^2_{\text{ét}}(B,T) \to H^2_{\text{ét}}(X,T),$$

5.13 (b)

$$1 \longrightarrow H^1_{\mathrm{fppf}}(B,T) \longrightarrow H^1_{\mathrm{fppf}}(X,T) \xrightarrow{t} \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B}) \longrightarrow H^2_{\mathrm{fppf}}(B,T) \longrightarrow H^2_{\mathrm{fppf}}(X,T).$$

The following definitions are equivalent to those of Colliot-Thélène and Sansuc **[15]** when B is the spectrum of a field. We shall write $[\mathcal{T}]$ for the class in $H^1_{\text{ét}}(X,T) = H^1_{\text{fppf}}(X,T)$ of an X-torsor $\pi : \mathcal{T} \to X$ under T.

Definition 5.14. — Assume (5.12).

(a) Let $\pi : \mathcal{T} \to X$ be a torsor under T. Then the image

$$t([\mathcal{T}]) \in \mathrm{Hom}_B(\hat{T}, \mathbf{Pic}_{X/B})$$

is called *the type of* $\pi : \mathcal{T} \to X$.

(b) The B-torus

$$g : \mathbf{D}(\mathbf{Pic}_{X/B}) \longrightarrow B$$

is called *the Néron-Severi torus of* $f : X \to B$.

(c) Let $T = \mathbf{D}(\mathbf{Pic}_{X/B})$ be the Néron-Severi torus of $f : X \to B$. A *universal torsor over* X is a torsor

$$\pi : \mathcal{T} \longrightarrow X$$

under T of identity type $t([\mathcal{T}]) : \hat{T} \to \mathbf{Pic}_{X/B}$.

(We use here the reflexive property of the group scheme $\mathbf{Pic}_{X/B}$, cf. **[21**, Exp. VIII**]**).

The following result in (op. cit.) is an easy consequence of (5.13)(a)

Proposition 5.15. — *Suppose that there exists a section $s : B \to X$ to $f : X \to B$. Then there are universal torsors over X. The isomorphism classes of the universal torsors over X are parametrized by the $H^1_{\text{ét}}(B,T)$-orbit in $H^1_{\text{ét}}(X,T)$ defined by $t^{-1}(\mathrm{id})$.*

Proof. — The map $H^2_{\text{ét}}(B,T) \to H^2_{\text{ét}}(X,T)$ in (5.13)(a) is the contravariant functorial map. There is also a contravariant map $H^2_{\text{ét}}(X,T) \to H^2_{\text{ét}}(B,T)$ induced by s. There are thus classes in $H^1_{\text{ét}}(X,T) = H^1_{\text{fppf}}(X,T)$ of identity type. It is well-known that these classes are represented by torsors (cf. **[45**, p. 121**]**). This completes the proof. □

For the rest of this section let k be a number field and let X be a smooth proper k-variety as in (5.13). Denote by T the Néron-Severi k-torus of X. The following result shows a remarkable property of universal torsors.

Lemma 5.16. — *Let k be a number field and let X be a smooth proper k-variety satisfying $H^1_{\text{Zar}}(X,\mathcal{O}_X) = H^2_{\text{Zar}}(X,\mathcal{O}_X) = 0$ and such that the Néron-Severi group of $\overline{X} := \overline{k} \times X$ is torsion-free. Let T be the Néron-Severi k-torus of X and let $\pi : \mathcal{T} \to X$ be a universal torsor or an arbitrary X-torsor under T. Then $\{\alpha_\nu = 1,\ \nu \in W\}$ form a set of convergence factors for $\mathcal{T}$.*

Proof. — Choose $\widetilde{o} = o_{(\Sigma)}$, $\widetilde{T}/\widetilde{o}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2). The image H of the representation

$$\rho : \mathrm{Gal}(\overline{k}/k) \longrightarrow Gl(\hat{T})$$

is finite and the subfield $K \subset \overline{k}$ of H-invariant elements is unramified over k at all places $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. For these places, define the local L-function by

$$L_\nu(s,T) = 1/\det(1 - q_\nu^{-s}\rho(\mathrm{Fr}_\nu) \mid \hat{T}).$$

The reduction $\widetilde{T} \times F_\nu$ is an F_ν-torus for each $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ and $\widetilde{T}(F_\nu)$ of cardinality $(\mathrm{Card}\, F_\nu)^{\dim T}/L_\nu(1,T)$ by a result of Ono [**49**, 3.3]. Further, by (2.15) one has

$$\mu_\nu(\widetilde{T}(o_\nu)) = \mathrm{Card}(\widetilde{T}(F_\nu)) \left(\mu_\nu(\widetilde{T}(o_\nu))/(\mathrm{Card}\, F_\nu)\right)^{\dim T}$$

for $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Therefore,

$$\beta_\nu = \begin{cases} L_\nu(1,T) = 1/\mu_\nu(\widetilde{T}(o_\nu)), & \nu \in W_{\mathrm{fin}} \backslash \Sigma \\ 1, & \nu \in W_\infty \cup \Sigma \end{cases}$$

form a set of convergence factors for T (this is due to Weil [**67**]).

Hence by (5.7) one gets that $\beta : W \to \mathbb{R}_{>0}$ is a set of convergence factors for $\pi : \mathcal{T} \to X$. But $\widehat{T} = \mathrm{Pic}(\overline{X})$ by assumption. Therefore,

$$\gamma_\nu = \begin{cases} 1/L_\nu(1,T), & \nu \in W_{\mathrm{fin}} \backslash \Sigma \\ 1, & \nu \in W_\infty \cup \Sigma \end{cases}$$

form a set of convergence factors for X by (4.16). Hence $\{\beta_\nu\gamma_\nu,\ \nu \in W\}$ form a set of convergence factors for $\mathcal{T}$, as was to be proved. □

Theorem 5.17. — *Let k be a number field and let X be a smooth proper k-variety as in (5.16). Let T be the Néron-Severi k-torus of X and let $\pi : \mathcal{T} \to X$ be a universal torsor or an arbitrary torsor under T. Let*

$$\|\ \|_X = \{\|\ \|_{X,\nu} : \mathbf{det}\, \mathrm{Tan}\, X(k_\nu) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X and let

$$\|\ \|_{X\to\mathcal{T}} = \{\|\ \|_{\mathcal{T},\nu} : \mathbf{det}\, \mathrm{Tan}(\mathcal{T}(k_\nu)) \longrightarrow [0,\infty),\ \nu \in W\}$$

be the induced adelic norm for (cf. (3.30), (3.31)). Let

$$\Lambda_\nu : C_c(\mathcal{T}(k_\nu)) \longrightarrow \mathbb{R}, \quad \nu \in W$$

be the positive functionals determined by the norms $\|\ \|_{\mathcal{T},\nu}$ (cf. (1.10)). Then there exists a unique positive linear map

$$\Lambda : C_c(\mathcal{T}(A_k)) \longrightarrow \mathbb{R}$$

such that

$$\Lambda(f_A) = \prod_{\nu \in W} \Lambda_\nu(f_\nu)$$

for the restricted product

$$f_A = \prod\nolimits'_{\nu \in W} f_\nu$$

of any adelic set of functions

$$\{f_\nu \in C_c(Y(k_\nu)),\ \nu \in W\}.$$

Proof. — It follows from (3.31) that the order norms $\{\| \ \|_{\mathcal{T}/X,\nu} : \nu \in W\}$ form an adelic norm for $\pi : \mathcal{T} \to X$. The assertion therefore follows from (5.8) and (5.16). □

The group of characters of the Néron-Severi torus T of X defined over k is canonically isomorphic to $\operatorname{Pic} X$. Therefore, the map

$$\{\alpha_\nu\}_{\nu \in W} \in G_{m,k}(A_k) \longrightarrow \log(\prod_{\nu \in W} |\alpha|_\nu) \in \mathbb{R}$$

induces a pairing $T(A_k) \times \operatorname{Pic} X \to \mathbb{R}$ which we may reinterpret as a continuous epimorphism

$$T(A_k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}).$$

Let $T^1(A_k)$ be the kernel of this map. Then there is an exact sequence:

(5.18) $$1 \longrightarrow T^1(A_k) \longrightarrow T(A_k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}) \longrightarrow 0$$

where $T(k) \subseteq T^1(A_k)$ by the Artin-Whaples product formula for number fields.

We now define a Haar measure $\overline{\Theta}^1_\Sigma$ on $T^1(A_k)/T(k)$. To do this, we recall the following result.

Lemma 5.19. — *Let G be a locally compact group and let N be a closed normal subgroup of G. Let Θ_N be a Haar measure on N and let $f \in C_c(G)$. Then there exists a unique function $f^N \in C_c(G/N)$ such that*

$$f^N(gN) := \int_N f(gn)d\Theta_N$$

for all $g \in G$.

Moreover, if Θ_H is a Haar measure on $H := G/N$ and Θ_G is a Haar measure on G, then there exists a unique positive real number $c > 0$ such that

(*) $$\int_G f(g)d\Theta_G = c\int_H f^N(h)d\Theta_H$$

for all $f \in C_c(G)$.

Proof. — See [**38**, Ch. XII]. □

We shall say that the three Haar measures $\Theta_G, \Theta_N, \Theta_H$ are compatible if $(*)$ holds with $c = 1$.

Now endow $T(k)$ with the counting measure and

$$V := T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

with the unique Haar measure such that $\mathrm{Vol}(V/L) = 1$ for the $\mathbb{Z}$-lattice

$$L := \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z})$$

in V. Then apply (5.19) twice to the chain of normal closed subgroups

$$T(k) \subseteq T^1(A_k) \subseteq T(A_k).$$

This defines a bijection between Haar measures on $T^1(A_k)/T(k)$ and Haar measures on $T(A_k)$.

Proposition 5.20. — *Let X be as in (5.1) and let B_A be a compact open subset of $X(A_k)$. Let T be the Néron-Severi torus and let $\pi : \mathcal{T} \to X$ be a universal torsor under T. Let $\| \ \|_X$ be an adelic norm for X and let m_ν, $\nu \in W$ (resp. n_ν, $\nu \in W$) be the Borel measure on $X(k_\nu)$ determined by by $\| \ \|_{X(k)}$ (resp. $\| \ \|_{X(k) \to \mathcal{T}(k)}$).*

Choose Σ, $\widetilde{o}$, $\widetilde{X}$, $\widetilde{T}$, $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$ as in (5.2) and such that there exists a compact open subset

$$B_S \subseteq \prod_{\nu \in S} X(k_\nu), \qquad S = W_\infty \cup \Sigma$$

for which

$$B_A = B_S \times \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} \widetilde{X}(o_\nu). \tag{\%}$$

Let Θ_S be the Haar measure on $T(A_k)$ given by the adelic order norm (cf. (5.9)(b) and the convergence factors

$$\begin{aligned} \beta_\nu &= 1/\mu_\nu(\widetilde{T}(o_\nu)), \qquad && \nu \in W_{\mathrm{fin}} \backslash \Sigma \\ \beta_\nu &= 1, && \nu \in W_\infty \cup \Sigma \end{aligned}$$

and let $\overline{\Theta}^1_\Sigma$ be the corresponding Haar measure on $T^1(A_k)/T(k)$ under the bijection above. Also, let $m_{A,\Sigma}$ be the Borel measures on $X(A_k)$ determined by $\| \ \|_X$ and $\gamma = \{1/\beta_\nu\}_{\nu \in W}$. Then the following holds.

(a) *$\pi_A(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$*

(b) *The product*

$$\tau_\varepsilon(X, B_A, \| \ \|_X) := \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k)))$$

is independent of the choices of Σ and $\widetilde{\pi} : \widetilde{\mathcal{T}} \to \widetilde{X}$.

(c) *Suppose that Σ, $\widetilde{o}$, $\widetilde{X}$ satisfy the additional condition that $\| \ \|_{X(k_\nu)}$ is the model norm defined by $\widetilde{X}_\nu/o_\nu$ for $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Then,*

$$\tau_\varepsilon(X, B_A, \| \ \|_X) = \overline{\Theta}^1_\Sigma(T^1_A/T(k)) m_S(B_S \cup \pi((\mathcal{T}_S))) \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} n_\nu(\widetilde{\mathcal{T}}(o_\nu)).$$

Proof

(a) It follows from the implicit function theorem that $\pi(\mathcal{T}(k_\nu))$ is an open subset of $X(k_\nu)$. By twisting the torsor with elements in $H^1_{\text{ét}}(k_\nu, T_\nu)$ one concludes (cf. [**15**]) that the functorial map $X(k_\nu) \to H^1_{\text{ét}}(k_\nu, T_\nu)$ defined by $\mathcal{T}_\nu$ is locally constant. Therefore, since $H^1_{\text{ét}}(k_\nu, T_\nu)$ is finite, it follows that $\pi(\mathcal{T}(k_\nu))$ is a closed subset of $X(k_\nu)$.

Also, $X(k_\nu)$ is compact since X is proper (see (2.3)) so that $\pi(\mathcal{T}(k_\nu))$ is compact for all places ν of k. Hence $\pi_A(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$.

(b) It suffices by the last assertion in (5.2) to show that the product does not change if we replace Σ by $\Phi = \Sigma \cup \{w\}$ and $\widetilde{\mathcal{T}}$ by $\widetilde{\mathcal{T}} \times o_{(\Phi)}$ for $w \in W_{\mathrm{fin}} \backslash \Sigma$. It follows from the definitions of Θ_Σ and $\overline{\Theta}^1_\Sigma$ that

$$\overline{\Theta}^1_\Phi(T^1(A_k)/T(k)) = \overline{\Theta}^1_\Sigma(T^1(A_k)/T(k)) \mu_w(\widetilde{T}(o_w)).$$

It is therefore sufficient to show that:

$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k))) \mu_w(\widetilde{T}(o_w)).$$

To show this, we first note that

$$\widetilde{\pi}(\widetilde{\mathcal{T}}(o_\nu)) = \widetilde{X}(o_\nu), \qquad \widetilde{X}(o_\nu) = X(k_\nu)$$

for $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ by (3.21) resp. the properness of $\widetilde{X}_w/o_w$. Hence

$$\pi(\mathcal{T}(k_\nu)) = \widetilde{X}(o_\nu)$$

for $\nu \in W_{\mathrm{fin}} \backslash \Sigma$ and

$$B_A \cap \pi_A(\mathcal{T}(A_k)) = B_S \cap \pi(\mathcal{T}_S) \times \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} \widetilde{X}(o_\nu)$$

for

$$\pi(\mathcal{T}_S) := \prod_{\nu \in S} \pi(\mathcal{T}(k_\nu)).$$

Now from (%) and the definition of $m_{A,\Sigma}$, we get

(5.21)
$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_S(B_S \cap \pi(\mathcal{T}_S)) \prod_{\nu \in W_{\mathrm{fin}} - \Sigma} \mu_\nu(\widetilde{T}(o_\nu)) m_\nu(\widetilde{X}(o_\nu))$$

where m_S is the product measure of m_ν for $\nu \in S$.

Let $F = S \cup \{w\} = \Phi \cup W_\infty$ and let m_F be the product measure of m_ν for $\nu \in F$. Moreover, let

$$\pi(\mathcal{T}_F) = \prod_{\nu \in F} \pi(\mathcal{T}(k_\nu)), \qquad B_F = \widetilde{X}(o_w) \times B_S.$$

Then, by the definition of $m_{A,\Phi}$, we get

$$m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_F(B_F \cap \pi(\mathcal{T}_F)) \prod_{\nu \in W_{\mathrm{fin}} - \Phi} \mu_\nu(\widetilde{T}(o_\nu)) \cdot m_\nu(\widetilde{X}(o_\nu)).$$

Now note that

$$B_F \cap \pi(\mathcal{T}_F) = \widetilde{X}(o_w) \times (B_S \cap \pi(\mathcal{T}_S))$$
$$m_F(B_F \cap \pi(\mathcal{T}_F)) = m_w(\widetilde{X}(o_w)) m_S(B_S \cap \pi(\mathcal{T}_F)).$$

Hence,

$$m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k))) = m_{A,\Phi}(B_A \cap \pi_A(\mathcal{T}(A_k)))\mu_w(\widetilde{T}(o_w))$$

thereby completing the proof of (b).

(c) Let $\nu \in W_{\mathrm{fin}} \backslash \Sigma$. Then $\| \ \|_{X(k_\nu) \to \mathcal{T}(k_\nu)}$ restricts to the model norm on $\mathbf{det}\,\mathrm{Tan}(\widetilde{\mathcal{T}}(o_\nu))$ by (3.21). Hence m_ν, n_ν are equal to the model measures on $\widetilde{X}(o_\nu)$ (resp. $\widetilde{\mathcal{T}}(o_\nu)$) and

$$\mu_\nu(\widetilde{T}(o_\nu)) m_\nu(\widetilde{X}(o_\nu)) = n_\nu(\widetilde{\mathcal{T}}(o_\nu))$$

by (3.25). Now apply (5.21) and the equality above. This finishes the proof. □

Definition 5.22. — Let X be as in (5.1) and let B_A be a compact and open subset of $X(A_k)$. Let $\varepsilon \in H^1_{\text{ét}}(X, T)$ be the class of a universal torsor $\pi : \mathcal{T} \to X$ and let $\| \ \|_X$ be an adelic norm for X. Then the *Tamagawa number* $\tau_\varepsilon(X, B_A, \| \ \|_X)$ is the product

$$\tau_\varepsilon(X, B_A, \| \ \|_X) = \overline{\Theta}_\Sigma^1(T^1(A_k)/T(k)) m_{A,\Sigma}(B_A \cap \pi_A(\mathcal{T}(A_k)))$$

in (5.20).

In order to understand this definition we shall need the following corollary of (4.28).

Corollary 5.23. — *Let $\pi : \mathcal{T} \to X$, $\| \ \|_X$, $\| \ \|_{X \to \mathcal{T}}$, $\beta = \{\beta_\nu\}_{\nu \in W}$ be as in (5.20). Let*

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \to C_c(X(A_k))$$

be the positive linear map described in (5.8) and let m_A resp. n_A be the Borel measures on $X(A_k)$ resp. $\mathcal{T}(A_k)$ determined by $\| \ \|_X$ and $\gamma = \{1/\beta_\nu\}_{\nu \in W}$ (cf. (4.14), (4.16)) resp. $\| \ \|_{X \to \mathcal{T}}$ and the convergence factors 1 (cf. (5.17)).

Let

$$\mathrm{Tr}: C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/T(k))$$

be the trace map obtained by summing over all $T(k)$-translates of a function in $C_c(\mathcal{T}(A_k))$. Then the following holds.

(a) *There exists a unique, positive σ-regular Borel measure $\overline{n}_A$ on $\mathcal{T}(A_k)/T(k)$ with the following property:*

If M is a Borel subset of $\mathcal{T}(A_k)$ with $tM \cap M = \varnothing$ for all $t \in T(k)\backslash\{1\}$ and $\overline{M}$ is the image of M in $\mathcal{T}(A_k)/T(k)$, then

$$\overline{n}_A(\overline{M}) = n_A(M).$$

(b) *There exists a unique positive linear map*

$$\overline{\Lambda}_\beta : C_c(\mathcal{T}(A_k)/T(k)) \longrightarrow C_c(X(A_k))$$

with $\Lambda_\beta = \overline{\Lambda}_\beta \circ \mathrm{Tr}$.

(c)

$$\int_{\mathcal{T}(A_k)/T(k)} f d\overline{n}_A = \int_{X(A_k)} \overline{\Lambda}_\beta(f) dm_{A,\gamma}$$

for each $f \in C_c(\mathcal{T}(A_k)/T(k))$.

Proof

(a) This follows from the $T(k)$-invariance of n_A (cf. e.g [**11**, Ch. VII, §2 prop. 4 b]).

(b) It suffices by the usual argument with partitions of unity [**38**, p. 270] to prove the existence and uniqueness of such a map $C_c(\overline{M}) \to C_c(X(A_k))$ for each open subset $\overline{M} \subset \mathcal{T}(A_k)/T(k)$ of the type described in (a).

(c) One reduces again to the case $f \in C_c(\overline{M})$ and applies (4.28)(b). □

Remark 5.24. — The motivation for the definition (5.22) is the following. The continuous map

$$\pi_A : \mathcal{T}(A_k) \longrightarrow X(A_k)$$

and the continuous $T(A_k)$-action

$$\sigma_A : T(A_k) \times \mathcal{T}(A_k) \longrightarrow \mathcal{T}(A_k)$$

determine (cf. (5.4)) a continuous map

$$\mathcal{T}(A_k)/T^1(A_k) \longrightarrow \pi_A(\mathcal{T}(A_k))$$

and a continuous $T(A_k)/T^1(A_k)$-action

$$T(A_k)/T^1(A_k) \times \mathcal{T}(A_k)/T^1(A_k) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

such that $\mathcal{T}(A_k)/T^1(A_k)$ becomes a topological $\pi_A(\mathcal{T}(A_k))$-torsor under

$$T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}).$$

Now assume that there exists a continuous section

$$\psi : \pi_A(\mathcal{T}(A_k)) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

of this map. This gives rise to an isomorphism:

$$I : \pi_A(\mathcal{T}(A_k)) \times \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}) \longrightarrow \mathcal{T}(A_k)/T^1(A_k)$$

of topological $\pi_A(\mathcal{T}(A_k))$-torsors under $T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$.

Let

$$\mathcal{F}_0 \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

be a fundamental domain (cf. e.g. [**53**, p. 163]) with respect to

$$\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z}) \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

and let $\mathcal{U}$ be the inverse image of

$$I((B_A \cap \pi_A(\mathcal{T}(A_k))) \times \mathcal{F}_0)$$

under the rest class map

$$\mathcal{T}(A_k)/T(k) \longrightarrow \mathcal{T}(A_k)/T^1(A_k).$$

Then $\mathcal{U}$ is a Borel subset of $\mathcal{T}(A_k)/T(k)$ which is invariant under $T^1(A_k)/T(k)$ for the action

$$\overline{\sigma}_A : T(A_k)/T(k) \times \mathcal{T}(A_k)/T(k) \longrightarrow \mathcal{T}(A_k)/T(k)$$

induced by σ_A. Moreover,

Assertion 5.25. — *We have*

$$\tau_\varepsilon(X, B_A, \|\ \|_X) = \overline{n}_A(\mathcal{U}).$$

To see this, one can assume that $\mathcal{F}_0$ has compact closure in $\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. Then $\mathcal{U}$ has compact closure $\mathcal{U}_c$ in $\mathcal{T}(A_k)/T(k)$ with

$$\overline{n}_A(\mathcal{U}) = \overline{n}_A(\mathcal{U}_c).$$

The characteristic function of

$$\mathcal{U}_c \subset \mathcal{T}(A_k)/T(k)$$

is the infimum of the set J of all $T^1(A_k)/T(k)$-invariant non-negative functions in $C_c(\mathcal{T}(A_k)/T(k))$ which are equal to 1 on $\mathcal{U}_c$ (cf. [**38**, p. 256]). Now regard these functions as functions on $\mathcal{T}(A_k)/T^1(A_k)$ and make use of (5.23)(c). Then,

$$\overline{n}_A(\mathcal{U}_c) = \inf_{h\in J} \int_{\mathcal{T}(A_k)/T(k)} h d\overline{n}_A = \int_{X(A_k)} (\inf_{h\in J} \overline{\Lambda}_\beta(h)) dm_{A,\gamma}.$$

Moreover, from the isomorphism I above, it follows that

$$\inf_{h\in J}\overline{\Lambda}_\beta(h)=\begin{cases}\overline{\Theta}_\Sigma^1(T^1(A_k)/T(k)) & \text{on } B_A\cap\pi_A(\mathcal{T}(A_k))\\ 0 & \text{otherwise.}\end{cases}$$

Hence,

$$\overline{n}_A(\mathcal{U})=\overline{\Theta}_\Sigma^1(T^1(A_k)/T(k))\cdot m_{A,\Sigma}(B_A\cap\pi_A(\mathcal{T}(A_k)))$$

as asserted.

We shall in section 10 construct a canonical "toric" section ψ as above for universal torsors over toric varieties. One can more generally construct a continuous section ψ as above for varieties with a "system of heights" in the sense of Peyre [**51**].

6. Reciprocity conditions and Tamagawa numbers

The aim of this section is to relate the Tamagawa numbers for the universal torsors defined in the previous section to the Tamagawa numbers defined by Peyre [**52**]. To do this, we shall need Manin's reciprocity condition for adelic points on X which is defined by means of étale cohomology.

If R is commutative ring and F is an abelian sheaf on the étale site of $\operatorname{Spec} R$ (cf. [**45**]), we shall write $H^i_{\text{ét}}(R,F)$ instead of $H^i_{\text{ét}}(\operatorname{Spec} R,F)$. If G_Y is a commutative group scheme over a scheme Y we let G_Y denote also the abelian sheaf F on the étale site $Y_{\text{ét}}$ associated to this group scheme (cf. [**45**, p. 52]). In particular, we shall write $H^1_{\text{ét}}(Y,G_Y)=H^1_{\text{ét}}(Y,F)$. If $G_Y=G_{m,Y}$ we omit the index Y.

There is a canonical isomorphsims of schemes:

$$\operatorname{Spec} A_k=\lim_{\leftarrow}\operatorname{Spec} A_k(\Sigma)$$

where Σ runs over all finite subsets of W_{fin}. Since "étale cohomology commutes with inverse limits of schemes" (cf. [**1**, VII.5.9]) we get:

$$H^2_{\text{ét}}(A_k,\mathbb{G}_m)=\lim_{\leftarrow}H^2_{\text{ét}}(A_k(\Sigma),\mathbb{G}_m).$$

Next, note that $\operatorname{Spec} A_k(\Sigma)$ is a direct sum (cf. [**31**, 3.1]) of schemes

$$\operatorname{Spec} A_k(\Sigma)=(\oplus_{\nu\in T}\operatorname{Spec} k_\nu)\oplus\left(\oplus_{\nu\in W\setminus T}\operatorname{Spec} o_\nu\right)$$

where $T:=\Sigma\cup W_\infty$. Hence:

$$H^2_{\text{ét}}(A_k(\Sigma),\mathbb{G}_m)=\left(\oplus_{\nu\in T}H^2_{\text{ét}}(k_\nu,\mathbb{G}_m)\right)\oplus\left(\oplus_{\nu\in W\setminus T}H^2_{\text{ét}}(o_\nu,\mathbb{G}_m)\right).$$

Moreover, $H^2_{\text{ét}}(\operatorname{Spec} o_\nu,\mathbb{G}_m)=0$ for $\nu\in W_{\text{fin}}$ (see [**45**, p. 108]). Hence,

$$H^2_{\text{ét}}(A_k,\mathbb{G}_m)=\oplus_{\nu\in W}H^2_{\text{ét}}(k_\nu,\mathbb{G}_m).$$

The Hasse invariant

$$i_\nu: H^2_{\text{ét}}(k_\nu,\mathbb{G}_m)\longrightarrow\mathbb{Q}/\mathbb{Z}$$

is defined as follows. If k_ν is the quotient field of a complete discrete valuation ring o_ν with residue field F_ν, then the Hasse-Witt residue map (cf. [**27**, III.2]) gives a canonical isomorphism

$$H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = H^1_{\text{ét}}(F_\nu, \mathbb{Q}/\mathbb{Z}).$$

The map i_ν is obtained by evaluating

$$H^1_{\text{ét}}(F_\nu, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G_\nu, \mathbb{Q}/\mathbb{Z})$$

at the Frobenius element of the absolute Galois group G_ν of k_ν.

If $k_\nu = \mathbb{R}$, then i_ν is the group monomorphism

$$H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

If $k_\nu = \mathbb{C}$, then $H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) = 0$.

There is a fundamental exact sequence of Albert-Brauer-Hasse-Noether (cf. e.g. [**65**, p, 196])

$$0 \longrightarrow H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(A_k, \mathbb{G}_m) \xrightarrow{i} \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{6.1}$$

The map from $H^2_{\text{ét}}(k, \mathbb{G}_m)$ to $H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is the functorial map and the reciprocity map

$$i : \oplus_{\nu \in W} H^2_{\text{ét}}(k_\nu, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the sum of the Hasse invariants. The fact that the sum of the Hasse invariants is zero for elements in $H^2_{\text{ét}}(k, \mathbb{G}_m)$ is called the reciprocity law.

The other ingredient in Manin's reciprocity condition is the (cohomological) Brauer group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of X. The contravariant functoriality of $H^2_{\text{ét}}(\cdot, \mathbb{G}_m)$ yields pairings:

6.2 (a) $X(k_\nu) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$
6.2 (b) $X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(A_k, \mathbb{G}_m)$.

Notation 6.3. — Let $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ be an element of the Brauer group of X.

(a) If $Q_\nu \in X(k_\nu)$, then $\mathcal{A}(Q_\nu) \in H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$ is the element defined by the pairing (6.2)(a).

(b) If $Q_A \in X(A_k)$, then $\mathcal{A}(Q_A) \in H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is the element defined by the pairing (6.2)(b).

Proposition 6.4. — *Let X be a smooth variety over a number field k and let $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. Then the following holds.*

(a) *The map which sends $Q_\nu \in X(k_\nu)$ to $\mathcal{A}(Q_\nu) \in H^2_{\text{ét}}(k_\nu, \mathbb{G}_m)$ is locally constant for all places $\nu \in W$ of k. It has finite image if X is proper.*

(b) *The map which sends $Q_A \in X(A_k)$ to $\mathcal{A}(Q_A) \in H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is locally constant. It has finite image if X is proper.*

Proof

(a) We change notation and write $k = k_\nu$, $X = X_\nu$, $\mathcal{A} = \mathcal{A}_\nu$. Also, let $\delta \in H^2_{\text{ét}}(k, \mathbb{G}_m)$ and choose $\mathcal{B} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ such that $\mathcal{B} - \mathcal{A}$ is equal to the image δ under the contravariant map

$$H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)$$

corresponding to the structure morphism from X to $\operatorname{Spec} k$.

Then the subset of $Q \in X(k)$ with $\mathcal{A}(Q) = \delta$ is equal to the kernel of the specialization map

$$X(k) \longrightarrow H^2_{\text{ét}}(k, \mathbb{G}_m)$$

defined by $\mathcal{B}$. To study this kernel, represent $\mathcal{B}$ locally (in the Zariski topology) by Azumaya algebras (cf. [**45**, IV.2.16]) and apply the implicit function theorem to the associated Severi-Brauer schemes. It then follows that the kernel of the specialization map defined by $\mathcal{B}$ is open in the k-topology (cf. sec. 2), thereby proving the first assertion.

The second assertion follows from the first and the fact that $X(k)$ is compact (see (2.3)(d)).

(b) Each element $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ is the restriction of an element $\widetilde{\mathcal{A}} \in H^2_{\text{ét}}(\widetilde{X}, \mathbb{G}_m)$ for some smooth $o_{(\Sigma)}$-model $\widetilde{X}$ of X (cf. [**45**, III.1.16]). This combined with the vanishing of $H^2_{\text{ét}}(o_\nu, \mathbb{G}_m)$ implies that

$$\mathcal{A}(Q_\nu) = 0$$

for

$$Q_\nu \in \operatorname{Im}(\widetilde{X}(o_\nu) \longrightarrow X(k_\nu)), \quad \nu \in W_{\text{fin}} \backslash \Sigma.$$

Hence by (a) the map from $X(A_k)$ to $H^2_{\text{ét}}(A_k, \mathbb{G}_m)$ is locally constant. The finiteness of the image follows from the fact (see (4.9)(c)) that $X(A_k)$ is compact. $\square$

The pairing in (6.2)(b) is part of a commutative diagram:

(6.5)
$$\begin{array}{ccc} X(A_k) \times H^2_{\text{ét}}(k, \mathbb{G}_m) & \longrightarrow & H^2_{\text{ét}}(k, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m) & \longrightarrow & H^2_{\text{ét}}(A_k, \mathbb{G}_m) \end{array}$$

where the top map is the projection onto the second factor.

By the reciprocity law (6.1), one deduces from (6.5) a pairing

(6.6)
$$X(A_k) \times H^2_{\text{ét}}(X, \mathbb{G}_m) / \operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which is a homomorphism with respect to the second factor.

Corollary 6.7. — *Suppose that $H^2_{\text{ét}}(X,\mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k,\mathbb{G}_m)$ is a finite group. Then the map*

$$X(A_k) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H^2_{\text{ét}}(X,\mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k,\mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z})$$

induced by (6.6) is locally constant. In particular, if X is proper, then the inverse image of any element of group on the right hand side is a compact open subset of $X(A_k)$.

Proof. — This is an immediate consequence of (6.4)(b). □

Notation 6.8. — $X(A_k)^0 \subseteq X(A_k)$ is the inverse image of 0 under the map in (6.7).

We now examine the group $H^2_{\text{ét}}(X,\mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k,\mathbb{G}_m)$.

Notation 6.9

(a) $H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{alg}} := \ker(H^2_{\text{ét}}(X,\mathbb{G}_m) \to H^2_{\text{ét}}(\overline{X},\mathbb{G}_m))$
(b) $H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{trans}} := H^2_{\text{ét}}(X,\mathbb{G}_m)/H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{alg}}$
(c) $H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}},\mathbb{Z})_{\text{tors}}$

is the torsion subgroup of $H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}},\mathbb{Z})$ where $X(\mathbb{C})$ depends on the choice of an embedding $k \subset \mathbb{C}$.

It follows from $H^2_{\text{ét}}(\overline{k},\mathbb{G}_m) = 0$, that the image of $H^2_{\text{ét}}(k,\mathbb{G}_m)$ in $H^2_{\text{ét}}(X,\mathbb{G}_m)$ is a subgroup of $H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{alg}}$. The following result is well-known (cf. **[40]**, **[41]** for (a))

Lemma 6.10. — *Let k be a number field and let X be a smooth (geometrically connected) proper k-variety satisfying $H^1_{\text{Zar}}(X,\mathcal{O}_X) = H^2_{\text{Zar}}(X,\mathcal{O}_X) = 0$ and for which the Néron-Severi group of $\overline{X} := \overline{k} \times X$ is torsion-free. Then*

(a) $H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{alg}}/\operatorname{Im} H^2_{\text{ét}}(k,\mathbb{G}_m) \cong H^1(\operatorname{Gal}(\overline{k}/k),\operatorname{NS}(\overline{X}))$
(b) $H^2_{\text{ét}}(\overline{X},\mathbb{G}_m) \cong H^3_{\text{sing}}(X(\mathbb{C})_{\text{an}},\mathbb{Z})_{\text{tors}}$ *for any embedding $k \subset \mathbb{C}$.*
(c) $H^2_{\text{ét}}(X,\mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k,\mathbb{G}_m)$ *is a finite group.*

Proof

(a) The spectral sequence (cf. **[27**, II.2.4**]**)

$$H^p(\operatorname{Gal}(\overline{k}/k), H^q_{\text{ét}}(\overline{X},\mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{ét}}(X,\mathbb{G}_m)$$

of Hochschild-Serre gives rise to an exact sequence

$$H^2_{\text{ét}}(k,\mathbb{G}_m) \to H^2_{\text{ét}}(X,\mathbb{G}_m)_{\text{alg}} \to H^1(\operatorname{Gal}(\overline{k}/k), H^1_{\text{ét}}(\overline{X},\mathbb{G}_m)) \to H^3_{\text{ét}}(k,\mathbb{G}_m).$$

For global fields k it is known from class field theory that $H^3_{\text{ét}}(k,\mathbb{G}_m) = 0$. Hence it only remains to note that $\operatorname{Pic}(\overline{X}) = \operatorname{NS}(\overline{X})$ by the assumption on X.

(b) This follows from results of Grothendieck (cf. [**45**, VI.4.3], [**27**, I.3.1] and [**26**, 1.35]). It is also known from the work of Artin-Mumford [**2**].

(c) This follows from (a) and (b). □

Peyre makes use of the following lemma (cf. [**52**, 2.1.1]) in his definition of Tamagawa numbers.

Lemma 6.11. — *Let k be a number field and let X be a k-variety as in (6.10). Then there is an $o_{(\Sigma)}$-model Ξ of X for some finite set $\Sigma \subset \operatorname{Spec} o$ such that Ξ is smooth and proper over $\operatorname{Spec} o_{(\Sigma)}$. Let $\overline{Y}_\nu := \Xi \times \overline{F}_\nu$ and let (cf. (4.16))*

$$L_\nu(s, \operatorname{Pic} \overline{Y}_\nu) = 1/\det(1 - q_\nu^{-s} \operatorname{Fr}_\nu \mid \operatorname{Pic} \overline{Y}_\nu \otimes \mathbb{Q}), \nu \in W_{\text{fin}} \backslash \Sigma.$$

Then,

$$L_\Sigma(s, \operatorname{Pic} \overline{X}) := \prod_{\nu \in W_{\text{fin}} \backslash \Sigma} L_\nu(s, \operatorname{Pic} \overline{Y}_\nu)$$

converges absolutely and uniformly on compact subsets of $\Re s > 1$ and defines a holomorphic function for $\Re s > 1$.

The function $L_\Sigma(s, \operatorname{Pic} \overline{X})$ has a meromorphic continuation to $\mathbb{C}$ with a pole of order $r := \operatorname{rk} \operatorname{Pic} X$ at $s = 1$.

Definition and proposition 6.12. — *Let k be a number field and let X be a k-variety as in (6.10). Let*

$$\| \ \|_X = \{\| \ \|_{X(k_\nu)} : \mathbf{det}(\operatorname{Tan} X(k_\nu)) \longrightarrow [0, \infty), \ \nu \in W\}$$

be an adelic norm for X. Then there is an $o_{(\Sigma)}$-model Ξ of X for some finite set $\Sigma \subset W_{\text{fin}}$ such that Ξ is smooth and proper over $\operatorname{Spec} o_{(\Sigma)}$ and such that $\| \ \|_{X(k_\nu)}$ is the model norm determined by Ξ_ν / o_ν for $\nu \in W_{\text{fin}} \backslash \Sigma$.

Let m_ν, $\nu \in W$, be the positive Borel measure on $X_\nu(k_\nu)$ defined by the ν-adic norm $\| \ \|_{X(k)}$ and let

$$\gamma_\nu = \begin{cases} 1/L_\nu(1, \operatorname{Pic} \overline{Y}_\nu) & \text{for } \nu \in W_{\text{fin}} \backslash \Sigma, \\ 1 & \text{for } \nu \in W_\infty \cup \Sigma. \end{cases}$$

If $X(A_k) \neq \varnothing$, then $\tau(X, \| \ \|)$ is defined to be the product:

$$\lim_{s \to 1} (s-1)^{\operatorname{rk} \operatorname{Pic} X} L_\Sigma(s, \operatorname{Pic} \overline{X}) (m_{A,\Sigma}(X(A_k)^0))$$

where $m_{A,\Sigma} := m_{A,\gamma}$ is the regular positive Borel measure described in (4.14)(b).

This number is positive and independent of the choices of Σ and Ξ.

Proof. — The proof of the statement is clear from the definitions of $L_\Sigma(s, \operatorname{Pic} \overline{X})$ and $m_{A,\gamma}$ (compare [**52**, 2.2.4], but note that there is an additional discriminant factor there due to differences in the normalization of the Haar measures). □

Remark 6.13. — Peyre [**52**] uses the topological closure $\overline{X(k)}$ of $X(k)$ instead of $X(A_k)^0$ in his definition of Tamagawa numbers. It follows from the reciprocity law that $X(k)$ is mapped into $X(A_k)^0$ under the functorial embedding $X(k) \subset X(A_k)$ (cf. [**40**], [**41**]).

$X(A_k)^0$ is a closed subset of $X(A_k)$ (see (6.7) and (6.10)(c)). Therefore, $\overline{X(k)} \subseteq X(A_k)^0$. It has been conjectured in [**15**] that $\overline{X(k)} = X(A_k)^0$ when X is rational. We shall in (7.8) explain why we find it more natural to consider $m_A(X(A_k)^0)$ than $m_A(\overline{X(k)})$.

The following result is due to Colliot-Thélène and Sansuc [**15**, §3] in the case of rational varieties.

Lemma 6.14. — *Let k be a number field and let X be a k-variety as in (6.10). Let $(Q_\nu)_{\nu \in W} \in X(A_k)$ be an adelic point on X. Then the following assertions are equivalent*

(i) $\sum_{\nu \in W} i_\nu(\mathcal{A}(Q_\nu)) = 0$ *for all* $\mathcal{A} \in H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{alg}}$.

(ii) There exists a universal torsor $\pi : \mathcal{T} \to X$ such that

$$(P_\nu)_{\nu \in W} \in \pi_A(\mathcal{T}(A_k)).$$

Proof. — This is proved in [**15**, §3] for rational varieties by means of an explicit computation of cocycles. An examination of the proof in (op. cit.) reveals that it works also under the weaker hypothesis in (6.10). For a conceptual proof, which does not use brutal force, see [**55**]. □

Lemma 6.15. — *Let k be a number field and let X be a k-variety as in (6.10). Let $\pi : \mathcal{T} \to X$ be a universal torsor. Then the restriction to*

$$\pi(\mathcal{T}(A_k)) \times H^2_{\text{ét}}(X, \mathbb{G}_m)$$

of Manin's pairing (6.3) factorizes to give a pairing

$$\pi(\mathcal{T}(A_k)) \times H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{trans}} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Proof. — This is a corollary of the previous lemma. □

Notation 6.16

$$Ш^1(k, T) := \ker(H^1_{\text{ét}}(k, T) \longrightarrow \prod_{\nu \in W} H^1_{\text{ét}}(k_\nu, T)).$$

It is known from class field theory that this (Shafarevich) group is finite.

Lemma 6.17. — *Let k be a number field and let X be a k-variety as in (6.10) with $X(A_k) \neq \varnothing$. Let*

$$\|\ \|_X = \{\|\ \|_{X(k_\nu)} : \mathbf{det}\,\mathrm{Tan}\,X(k) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X and let $m_{A,\gamma}$ be the Borel measure on $X(A_k)$ (cf. (4.14)(b)) determined by $\|\ \|_X$ and the convergence factors γ in (6.12). Let $I \subset H^1_{\text{ét}}(X,T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T} \to X$ such that $\pi(\mathcal{T}(A_k)) \cap X(A_k)^0 \neq \varnothing$. Then the following holds.

(a) *I is finite*

(b) *$\pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ is a compact open subset of $X(A_k)$.*

(c)

$$\mathrm{Card}(\text{Ш}^1(k,T))m_{A,\gamma}(X(A_k)^0)) = \sum_{\varepsilon\in I} m_{A,\gamma}(\pi_\varepsilon(\mathcal{T}_\varepsilon(A_k) \cap X(A_k)^0)).$$

Proof

(a) Let $J \subset H^1_{\text{ét}}(X,T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T}(A_k) \to X$ such that $\mathcal{T}(A_k) \neq \varnothing$. It then follows by a weak Mordell-Weil argument (cf. [**15**, Th. 2.7.3]) that J is finite. Hence $I \subseteq J$ is also finite.

(b) It was shown in the proof of (5.20) that $\pi(\mathcal{T}(k_\nu))$ is a compact open subset of $X(k_\nu)$ for all places ν of k and that $\pi(\mathcal{T}(k_\nu)) = X(k_\nu)$ for all but finitely many places. Hence $\pi(\mathcal{T}(A_k))$ is a compact open subset of $X(A_k)$. By (6.7) $X(A_k)^0$ is also a compact open subset of $X(A_k)$. Therefore $\pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ is a compact open subset of $X(A_k)$.

(c) Let $\pi_\alpha : \mathcal{T}_\alpha \to X$, $\pi_\beta : \mathcal{T}_\beta \to X$ be two universal torsors and α, β their classes in $H^1_{\text{ét}}(X,T)$. Then from (5.13), (5.15) it follows that:

$$\begin{aligned} \pi_\alpha(\mathcal{T}_\alpha(A_k)) = \pi_\beta(\mathcal{T}_\beta(A_k)) &\iff \alpha - \beta \in \mathrm{Im}(\text{Ш}^1(k,T) \to H^1_{\text{ét}}(X,T)),\\ \pi_\alpha(\mathcal{T}_\alpha(A_k)) \cap \pi_\beta(\mathcal{T}_\beta(A_k)) = \varnothing &\iff \alpha - \beta \notin \mathrm{Im}(\text{Ш}^1(k,T)). \end{aligned}$$

This combined with $\pi(\mathcal{T}(A_k)^0) = \pi(\mathcal{T}(A_k)) \cap X(A_k)^0$ implies that

$$\mathrm{Card}(\text{Ш}^1(k,T))m_{A,\gamma}(\bigcup_{\varepsilon\in I} \pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k)^0)) = \sum_{\varepsilon\in I} m_{A,\gamma}(\pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k)^0)).$$

But

$$X(A_k)^0 = \bigcup_{\varepsilon\in I} \pi_{\varepsilon,A}(\mathcal{T}_\varepsilon(A_k))^0$$

by (6.14). This completes the proof. □

Definition 6.18. — Let X be as in (6.10) and let $\varepsilon \in H^1_{\text{ét}}(X, T)$ be the class of a universal torsor $\pi : \mathcal{T} \to X$. Let $\| \ \|_X$ be an adelic norm for X. Then the Tamagawa number $\tau_\varepsilon(X, \| \ \|_X)$ is the number $\tau_\varepsilon(X, X(A_k)^0, \| \ \|_X)$ defined in (5.22).

The following theorem is one of the main results of this paper.

Theorem 6.19. — *Let k be a number field and let X be a k-variety as in (6.10). Let*

$$\| \ \|_X = \{\| \ \|_{X(k_\nu)} : \mathbf{det}(\operatorname{Tan} X(k_\nu)) \longrightarrow [0, \infty), \ \nu \in W\}$$

be an adelic norm for X. Then $\mathcal{T}(A_k) = \varnothing$ and $\tau_\varepsilon(X, \| \ \|_X) = 0$ for all but finitely many isomorphism classes $\varepsilon \in H^1_{\text{ét}}(X, T)$ of universal torsors $\pi_\varepsilon : \mathcal{T}_\varepsilon \to X$. Further,

$$(*) \qquad \sum_\varepsilon \tau_\varepsilon(X, \| \ \|_X) = \operatorname{Card} H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} \overline{X}) \tau(X, \| \ \|_X)$$

where $\varepsilon \in H^1_{\text{ét}}(X, T)$ runs over all elements of identity type $\chi(\varepsilon)$ in

$$\operatorname{Hom}_G(\hat{T}, H^1_{\text{ét}}(\overline{X}, \mathbb{G}_m)).$$

In particular, if $\operatorname{Pic}(\overline{X})$ is a direct summand of a permutation $\operatorname{Gal}(\overline{k}/k)$-module, then

$$\tau(X, \| \ \|) = \tau_\varepsilon(X, \| \ \|)$$

for the unique element $\varepsilon \in H^1_{\text{ét}}(X, T)$ of identity type.

Proof. — Let $I \subset H^1_{\text{ét}}(X, T)$ be the subset parametrizing isomorphism classes of universal torsors $\pi : \mathcal{T} \to X$ such that $\pi(\mathcal{T}(A_k))^0 \neq \varnothing$. Then I is finite by (6.17)(a). To prove $(*)$, choose Σ, $\widetilde{o} := o_{(\Sigma)}$, Ξ, $\widetilde{T}$, $\widetilde{\pi}_\varepsilon : \widetilde{\mathcal{T}}_\varepsilon \to \Xi$ as in (5.2) for each representative $\pi_\varepsilon : \mathcal{T}_\varepsilon \to X$ of an isomorphism class $\varepsilon \in I$ of universal torsors with $\mathcal{T}(A_k) \neq \varnothing$.

We may clearly choose the same finite set Σ and $\widetilde{o} = o_{(\Sigma)}$ for all $\varepsilon \in I$ since I is finite. Also, by the definition of adelic norms, we may assume that $\| \ \|_{X(k_\nu)}$ is the model norm determined by Ξ_ν/o_ν for $\nu W_{\text{fin}} \backslash \Sigma$ and by (6.7) we may chose the finite set $\Sigma \subset W_{\text{fin}}$ such that

$$X(A_k)^0 = X_S(A_k)^0 \times \prod_{\nu \notin S} X(k_\nu)$$

for some compact open subset

$$X_S(A_k)^0 \subseteq \prod_{\nu \in S} X(k_\nu), \qquad S := \Sigma \cup W_\infty.$$

By definition of $\tau_\varepsilon(X, \| \ \|_X)$ we have (cf. (5.20), (6.18)):

$$\tau_\varepsilon(X, \| \ \|_X) = \Theta^1_\Sigma(T^1(A_k)/T(k)) \cdot m_{A,\Sigma}(\pi_\varepsilon(\mathcal{T}_\varepsilon(A_k)^0)).$$

Hence by (6.17)(c) we get:

$$\sum_{\varepsilon\in I}\tau_\varepsilon(X,\|\ \|_X)=\Theta^1_\Sigma(T^1(A_k)/T(k))\cdot m_{A,\Sigma}(X(A_k)^0)\cdot \operatorname{Card}(\text{Ш}^1(k,T)).$$

By the main theorem of Ono [**50**, §5] on Tamagawa numbers for tori we have firther

$$\Theta^1_\Sigma(T^1(A_k)/T(k))=(h(T)/\operatorname{Card}\text{Ш}^1(k,T))\lim_{s\to 1}(s-1)^{\operatorname{rk}\operatorname{Pic}X}L_\Sigma(s,\operatorname{Pic}\overline{X})$$

where

$$h(T):=\operatorname{Card}H^1(\operatorname{Gal}(\overline{k}/k),\hat{T})=\operatorname{Card}H^1(\operatorname{Gal}(\overline{k}/k),\operatorname{Pic}\overline{X}).$$

Hence,

$$\sum_{\varepsilon\in I}\tau_\varepsilon(X,\|\ \|_X)=h(T)(\lim_{s\to 1}(s-1)^{\operatorname{rk}\operatorname{Pic}X}L_\Sigma(s,\operatorname{Pic}\overline{X})m_{A,\Sigma}(X(A_k)^0)$$

as was to be proved. □

7. Counting functions of Fano varieties

We shall in this section study the asymptotic growth of the number of rational points of bounded anticanonical height on Fano varieties X over number fields. This theory was initiated by Manin. He suggested (cf. [**23**], [**3**], [**42**], [**43**]) that the asymptotic growth of the counting function should be of the form

$$CB(\log B)^{\operatorname{rk}\operatorname{Pic}X-1}(1+o(1))$$

on sufficiently small Zariski open subsets of X.

Peyre [**52**] gave a conjectural interpretation of the constant C by means of his Tamagawa number $\tau(X,\|\ \|)$ and a geometrical invariant $\alpha(X)$ depending only on the effective cone in $\operatorname{Pic}X$. His interpretation of the constant C concerns the case where $\operatorname{Pic}\overline{X}$ is a permutation module. It does not cover the case of general toric varieties which have recently been studied by Batyrev and Tschinkel ([**7**], [**4**]). The aim of this section is to use the Tamagawa numbers of universal torsors over X to reinterpret and to refine the conjecture of Peyre on the value of C.

Definition 7.1. — Let k be a field and let X be a smooth proper geometrically connected scheme over k. Then X is said to be a Fano variety if the anticanonical sheaf ω_X^{-1} of X is very ample. A Fano variety of dimension 2 is called a del Pezzo surface.

It is usually only required that ω_X^{-1} is ample in the definition of Fano varieties and del Pezzo surfaces. In particular, "our" del Pezzo surfaces are always of degree ≥ 3 and we do not consider surfaces of degree 1 or 2 (cf. [**41**], [**44**]). It is known (cf. e.g. [**52**, 1.2.1]) that Fano varieties satisfy all the conditions in (6.10) and (5.12). We can therefore apply all the constructions and results made in the previous two sections to them.

We now turn to the arithmetic of Fano varieties and assume for the rest of this section that k is a number field. Let

$$\| \ \| = \{\| \ \|_\nu : \mathbf{det}\,\mathrm{Tan}\, X(k_\nu) \longrightarrow [0,\infty),\ \nu \in W\}$$

be an adelic norm for X. It is well-known (cf. e.g. [**63**]) that $\| \ \|$ defines a height function

$$H : X(k) \longrightarrow [0,\infty)$$

although the "metrics" $\| \ \|_\nu$ usually only occur explicitly for the archimedean places.

Definition 7.2. — The *height function* $H : X(k) \to (0,\infty)$ *defined by* $\| \ \|$ is given by

$$H(P) = \prod_{\nu \in W} \|s(P)\|_\nu^{-1}$$

where s is a local section of ω_X^{-1} at P with $s(P) \neq 0$.

This function is well defined since $H(P)$ does not depend on the choice of s by the product formula for number fields.

Remarks 7.3

(a) It follows from the compactness of all $X(k_\nu)$ that $\log H_1 - \log H_2$ is bounded on $X(k)$ for the height functions H_1 and H_2 defined by two adelic norms $\| \ \|_1$, $\| \ \|_2$ for X (cf. (1.6)(b), (2.3)(e) and (4.10)).

(b) Let X be a Fano variety. Then

$$\mathcal{C}(B) := \{P \in X(k) : H(P) \leq B\}$$

is *finite* for any height H defined by an adelic norm. It suffices by (a), to prove this for *one* adelic norm on X and it is natural to consider a norm defined by means of a finite set of global sections generating ω_X^{-1} (cf. [**52**, pp. 107-8]). One can then apply the classical arguments for heights defined by projective coordinates (cf. e.g. [**37**, Ch. 3, §1]).

Notation 7.4. — Let X be a Fano variety and let H be a height defined by an adelic norm $\| \ \|$. Let U be a constructible subset defined over k. Then,

(a)

$$\mathcal{C}_U(B, \| \ \|) = \mathrm{Card}\{P \in U(k) : H(P) \leq B\}.$$

(b)

$$\beta_U := \limsup_{B \to \infty} \log \mathcal{C}_U(B, \| \ \|)/\log B.$$

(c)
$$\gamma_U(\|\ \|) := \limsup_{B\to\infty} \mathcal{C}_U(B)/B(\log B)^{r-1} > 0$$
for $r = \operatorname{rk}\operatorname{Pic} X$.

It is clear from (7.3)(a) that β_U is independent of the choice of adelic norm $\|\ \|$. It also follows from (7.3)(a) that the condition $\gamma_U(\|\ \|) > 0$ is independent of the choice of adelic norm and we shall therefore write $\gamma_U > 0$ for this condition. The function sending B to $\mathcal{C}_U(B, \|\ \|)$ will be called the counting function of U with respect to $\|\ \|$ and β_U the growth order of U.

The following definitions are inspired by notions of Manin (cf. [**42**]) and [**51**]). But the reader should observe that our definitions are not identical with the definitions in (op. cit.) and that we only consider Fano varieties.

Definition 7.5. — Let X be a Fano variety.

(a) A closed proper subset $F \subset X$ of X is said to be accumulating if for each non-empty subset V of F one has $\beta_V > 1$.
(b) A closed proper subset $F \subset X$ of X is said to be weakly accumulating if $\gamma_V > 0$ for each non-empty open subset V of F.

We are now in a position to formulate a version of Manin's conjecture on $\mathcal{C}_U(H)$ (cf. [**23**], [**3**], [**42**], [**43**], [**52**], [**51**]).

Conjecture 7.6. — *Let X be a rational Fano variety over a number field k for which $X(A_k)^0$ is non-empty and let $\|\ \|$ be an adelic norm for X. Suppose that the complement U in X of the union of all weakly accumulating (proper) subsets is a Zariski open non-empty set defined over k. Then there is a positive constant $C = C(\|\ \|) > 0$ such that*
$$\lim_{B\to\infty} \mathcal{C}_U(B, \|\ \|)/B(\log B)^{r-1} = C \quad \textit{for } r = \operatorname{rk}\operatorname{Pic} X.$$

Manin assumes that $X(k)$ is Zariski dense instead of just assuming that $X(A_k)^0$ is non-empty. To compensate this we have added the condition that X is rational. This excludes cubic three-folds which are unirational but not rational. A more optimistic version of (7.6) would be to assume that X is unirational. This is still a restriction since some Fano varieties are not even uniruled [**35**, V.5].

We have in our definition of U used weakly accumulating (proper) subsets instead of accumulating (proper) subsets. It has recently been shown (cf. [**5**]) by Batyrev and Tschinkel that (7.6) is false when U is defined as the complement of the union of all accumulating subsets. The counterexamples are given by smooth hypersurfaces in $\mathbb{P}^3_k \times \mathbb{P}^n_k$ of bidegree $(3, 1)$.

To formulate a conjecture for the constant $C(\| \|) > 0$, Peyre [**52**, p. 120] defines for Fano varieties an invariant $\alpha(X)$ which we will often denote by $\alpha_{\text{Peyre}}(X)$. It depends only on the effective cone in $\operatorname{Pic} X$. To describe this invariant, let

$$L = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z})$$

$$V = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$$

and let $d\nu$ be the unique Haar measure on V such that volume $\operatorname{Vol}(V/L) = 1$.

Further, let $\sigma_{\text{eff}}(X) \subset V$ be the cone of all homomorphisms

$$\varphi : \operatorname{Pic} X \longrightarrow \mathbb{R}$$

such that

$$\varphi([D]) \geq 0$$

for the class $[D] \in \operatorname{Pic} X$ of each effective divisor on X. Let $\lambda : V \to \mathbb{R}$ be the linear form obtained by evaluating at the anticanonical class and let $V_x = \lambda^{-1}(x)$ for $x \in \mathbb{R}$ and note that $\lambda : V \to \mathbb{R}$ is a trivial analytic torsor under V_0.

There is then a unique positive linear map $\Lambda : C_c(V) \to C_c(\mathbb{R})$ such that:

$$\int_V g d\nu = \int_{\mathbb{R}} \Lambda(g) dx$$

for any function $g \in C_c(V)$. Λ restricts to a positive functional Λ_x on $V_x = \lambda^{-1}(x)$ for each $x \in \mathbb{R}$. Let $d\nu_x$ be the corresponding positive Borel measure on V_x. Then,

$$\alpha_{\text{Peyre}}(X) := \int_B d\nu_1 \quad \text{for } B := V_1 \cap \sigma_{\text{eff}}(X).$$

Batyrev and Tschinkel [**7**] define a similar invariant which they also call $\alpha(X)$, but which we will denote by $\alpha_{BT}(X)$. It follows from one of the lemmas in [**7**] that their invariant

$$\alpha_{BT}(X) = (r-1)! \cdot \alpha_{\text{Peyre}}(X).$$

The α_{BT}-invariant is multiplicative (see [Pe1, lemme 4.2])

$$\alpha_{BT}(X \times Y) = \alpha_{BT}(X)\alpha_{BT}(Y)$$

for varieties as in (6.10). It is more natural when one considers Manin s zeta-functions (see [**7**] and [**4**]).

The following conjecture is due to Peyre [**52**, 2.3.1].

Conjecture 7.7. — *Suppose that the assumptions in (7.6) are satisfied and suppose that* $\operatorname{Pic} \overline{X}$ *has a* $\mathbb{Z}$*-basis which is invariant under the action of the absolute Galois group* G_k *of* k*. Then the constant* $C(\| \|)$ *in Manin's conjecture is of the form:*

$$C(\| \|) = \alpha_{\text{Peyre}}(X)\tau(X, \| \|).$$

Remark 7.8. — Peyre defines (cf. (6.13)) "his" $\tau(X, \| \|)$ by means of $m_A(\overline{X(k)})$ instead of $m_A(X(A_k)^0)$ as we have done. The two definitions are expected (cf. (6.13)) to be identical when $\overline{X}$ is rational and perhaps also when $\overline{X}$ is unirational. We find it more natural to give an adelic interpretation of C (with an extra twist given by reciprocity pairing of Manin).

It is not difficult to compute $m_A(X(A_k)^0)$. The group

$$H^2_{\text{ét}}(X, \mathbb{G}_m)/\operatorname{Im} H^2_{\text{ét}}(k, \mathbb{G}_m)$$

is finite (cf. (6.10)) and it is clear from the proof of (6.4)(b) that it can be decided if an adelic point belongs to $X(A_k)^0$ by looking at finitely many places $S \subset W$. Also, if $\| \|_\nu$ is the model norm of a smooth model $\widetilde{X}/o_\nu$, then

$$m_\nu(X_\nu) = \operatorname{Card}(\widetilde{X}(F_\nu))(m_\nu(o_\nu)/\operatorname{Card}(F_\nu))^{\dim X}$$

by (2.15). It is thus possible, in principle, to compute $m_A(X(A_k)^0)$ in finitely many steps although the answer is given as an infinite product.

The only reasonable way to compute $m_A(\overline{X(k)})$, however, is to prove that $\overline{X(k)} = X^0_A$ and then use the computation for $m_A(X(A_k)^0)$. This is a much more difficult problem than the computation of $m_A(X(A_k)^0)$.

Fraenke, Manin and Tschinkel **[23]** proved (7.6) for generalized flag spaces (using Langlands' Eisenstein series). They also noticed that one can deduce (7.6) for smooth complete intersections whenever asymptotic results on the affine cone are available by the Hardy-Littlewood circle method. In these cases the asymptotic formulas hold for X itself. Peyre (op. cit) completed their results and proved his conjecture (7.7) for these varieties. He also proved (7.6) and (7.7) some toric surfaces over $\mathbb{Q}$. The open subset U is then the open subset defined by the underlying torus. These results are proved for counting functions defined by one special adelic norm.

Peyre formulates his conjecture (7.7) in the case where $\operatorname{Pic} \overline{X}$ is permutation G_k-module. It is natural to weaken this condition and also allow varieties for which $\operatorname{Pic} \overline{X}$ is a G_k-direct summand of a permutation G_k-module.

One cannot expect to omit such a condition completely. Batyrev and Tschinkel study in two recent papers **[7]**, **[4]** $\mathcal{C}_U(H)$ for smooth projective U-equivariant compactifications X of tori U. They choose a natural adelic norm (cf. (9.2) of this paper) and make strong use of the group action. Using the abstract Poisson formula, they conclude that (7.6) holds with the constant

$$C(\| \|) = \alpha_{\text{Peyre}}(X)\tau(X, \| \|)h^1(\operatorname{Pic} \overline{X}) \tag{7.9}$$

where $h^1(\operatorname{Pic} \overline{X}) = \operatorname{Card} H^1(G_k, \operatorname{Pic} \overline{X})$.

This does not contradict (7.7) since $H^1(G_k, \operatorname{Pic} \overline{X}) = 0$ when $\operatorname{Pic} \overline{X}$ is a G_k-direct summand of a permutation G_k-module. It is tempting to reformulate (7.7) and conjecture that (7.9) holds for general Fano varieties satisfying (7.6). Numerical work of Heath-Brown [33] on the two diagonal cubic surfaces defined by $x^3+y^3+z^3+2w^3$ and $x^3+y^3+z^3+3w^3$ seems to indicate that such a generalization of (7.9) is true. Note that

$$H^1(G_k, \operatorname{Pic} \overline{X}) \neq 0$$

for both these surfaces.

Definition 7.10. — Let G be a profinite group. Let M be a finitely generated torsion-free G-module and let $N = \operatorname{Hom}(M, \mathbb{Z})$. Then M is said to be flasque [14] if

$$H^1(H, N) = 0$$

for each closed subgroup H of G.

It is known from work of Colliot-Thélène /Sansuc [14] and Voskresenskii that $\operatorname{Pic} \overline{X}$ is a flasque G_k-module for toric varieties. It is obvious that $\operatorname{Pic} \overline{X}$ is flasque if it is a G_k-direct summand of a permutation G_k-module. The conjecture (7.6) has sofar only been verified for classes of k-varieties X for which $\operatorname{Pic} \overline{X}$ is a flasque G_k-module. This is not surprising since there are only finitely many isomorphism classes of universal torsors for such varieties. The asymptotic formulas obtained by Batyrev, Fraenke, Manin, Peyre and Tschinkel all satisfy the following conjecture:

Conjecture 7.11. — *Suppose that the assumptions in (7.6) are satisfied and suppose that* $\operatorname{Pic} \overline{X}$ *is a flasque* G_k*-module for the absolute Galois group* G_k *of* k*. Then the constant* $C(\| \ \|)$ *in Manin's conjecture is of the form:*

$$C(\| \ \|) = \alpha_{\text{Peyre}}(X)\tau(X, \| \ \|)h^1(\operatorname{Pic} \overline{X}).$$

This conjecture is compatible with products. To see this, one uses the arguments in [23] and [52] to prove compatibility of (7.6) and (7.7) under products. The only new ingredient needed is the equality

$$h^1(\operatorname{Pic} \overline{X \times Y}) = h^1(\operatorname{Pic} \overline{X})h^1(\operatorname{Pic} \overline{Y})$$

which follows from the canonical isomorphism

$$\operatorname{Pic} \overline{X \times Y} = (\operatorname{Pic} \overline{X}) \times (\operatorname{Pic} \overline{Y})$$

already used in [52, 4.1].

The following refinement of (7.6) and (7.11) is natural even if there is not much evidence for it.

Conjecture 7.12. — *Let* X *be a rational Fano variety over a number field* k *and suppose that* $\operatorname{Pic} \overline{X}$ *is a flasque* G_k*-module. Let* $\| \ \|$ *be an adelic norm for* X *and let* H *be the height function on* $X(k)$ *defined by* $\| \ \|$*. Let* $\varepsilon \in H^1_{\text{ét}}(X, T)$ *be the class of a universal torsor* $\pi : \mathcal{T} \to X$ *such that* $\mathcal{T}(A_k) \neq \varnothing$*. Suppose that the complement* U

in X of the union of all weakly accumulating subsets is Zariski open and non-empty and defined over k and let

$$\mathcal{C}_{U,\varepsilon}(B) = \mathrm{Card}\{P \in \pi(\mathcal{T}(k)) : H(P) \leq B\}.$$

Then

$$\lim_{B\to\infty} \mathcal{C}_{U,\varepsilon}(B)/B(\log B)^{r-1} = \alpha_{\mathrm{Peyre}}(X)\tau_{\varepsilon}(X, \|\ \|)$$

where $r = \mathrm{rk}\,\mathrm{Pic}\,X$.

It follows immediately from theorem (6.19) that (7.12) implies (7.11). If $\mathrm{Pic}\,\overline{X}$ is a direct summand of a permutation G_k-module, then there is only one isomorphism class of universal torsors since $H^1_{\text{ét}}(k,T) = 0$ (see (5.4)). Thus in this case (7.11) and (7.12) are equivalent by (6.19). If one examines the proof of conjecture (7.11) for toric varieties in [**7**] and [**4**] then it is likely that (7.12) will follow if the Poisson formula is applied to smaller discrete subgroups than in (op. cit.).

All numerical work on Manin's conjecture on (7.6) so far concerns surfaces. For surfaces the intersection pairing

$$\mathrm{Pic}\,\overline{X} \times \mathrm{Pic}\,\overline{X} \longrightarrow \mathbb{Z}$$

is perfect and induces a canonical isomorphism

$$\mathrm{Hom}(\mathrm{Pic}\,\overline{X}, \mathbb{Z}) = \mathrm{Pic}\,\overline{X}$$

of G_k-modules. In particular,

$$H^1(G_k, \mathrm{Pic}\,\overline{X}) = 0$$

if X is a surface and $\mathrm{Pic}\,\overline{X}$ is flasque.

The numerical work to date is insufficient to make any conjectures when $\mathrm{Pic}\,\overline{X}$ is not flasque The only work known to the author concerns the two diagonal cubic surfaces of Heath-Brown [**33**] described above.

8. Torsors over toric varieties

We shall in this section study universal torsors $\pi : \mathcal{T} \to X$ over toric varieties X. The aim is to prove that the universal torsors are the toric morphisms described by Cox in [**16**]. We shall in this section use the word complete instead of proper for toric varieties corresponding to complete fans.

Let M be a free finitely generated abelian group of rank $d \geq 1$ and let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice, with dual pairing denoted by $\langle\ ,\ \rangle$. Let us recall some basic facts on toric varieties and refer to [**24**] and the references there for more background.

Definition 8.1. — A finite set Δ consisting of convex rational polyhedral cones in $N_R = N \otimes \mathbb{R}$ is called a fan if the following conditions are satisfied

(i) Each cone in Δ contains $0 \in N_{\mathbb{R}}$;
(ii) Each face of a cone in Δ is also a cone in Δ;
(iii) The intersection of two cones in Δ is a face of both cones

A fan Δ in N is called complete (resp. regular) if

(a) $N_{\mathbb{R}}$ is the union of cones in Δ resp.
(b) Each cone in Δ is generated by a part of a $\mathbb{Z}$-basis of N.

Each cone σ in Δ determines a finitely generated commutative semigroup:

$$S_\sigma = \check{\sigma} \cap M = \{m \in M : \langle m, n\rangle \geq 0 \text{ for all } n \in \sigma\}.$$

The group ring $\mathbb{Q}[S_\sigma]$ is a finitely generated commutative $\mathbb{Q}$-algebra corresponding to an affine $\mathbb{Q}$-variety $U_\sigma = \operatorname{Spec}\mathbb{Q}[S_\sigma]$. If $\sigma = \{0\}$, then $S_\sigma = M$ is an abelian group and $\mathbb{Q}[S_\sigma]$ a Hopf algebra. This provides U_σ with a natural structure of algebraic group when $\sigma = \{0\}$ (cf. [**9**]) and we shall denote this $\mathbb{Q}$-torus by U. It is by definition the $\mathbb{Q}$-torus $\mathbf{D}(M)$ mentioned in section 5 and there is a canonical isomorphism between M and the group $\hat{U}$ of characters of U. We shall in the sequel write $\chi^m : U \to \mathbb{G}_m$ for the character corresponding to $m \in M$.

There is a natural U-action on U_σ for each cone σ in Δ and any inclusion $\rho \subset \sigma$ of cones corresponds to a U-equivariant open $\mathbb{Q}$-immersion $U_\rho \subset U_\sigma$. By gluing these one obtains for each complete regular fan Δ in $N := \operatorname{Hom}(M, \mathbb{Z})$ a smooth complete $\mathbb{Q}$-variety X_Δ containing $U = \operatorname{Spec}\mathbb{Q}[M]$ as an open Zariski-dense subvariety. This $\mathbb{Q}$-variety X_Δ is equipped with a $\mathbb{Q}$-morphism $U \times X_\Delta \to X_\Delta$ extending the group multiplication on U.

It is possible to do the whole construction over $\mathbb{Z}$ (cf. [**20**]) and start with the affine schemes $\widetilde{U}_\sigma = \operatorname{Spec}\mathbb{Z}[S_\sigma]$. If $\sigma = \{0\}$, then $\sigma = \operatorname{Spec}\mathbb{Z}[M]$ and any inclusion $\rho \subset \sigma$ of cones gives rise to an open embedding $\widetilde{U}_\rho \subset \widetilde{U}_\sigma$. We may therefore glue $\widetilde{U}_\sigma$ for all cones $\sigma \in \Delta$ and obtain a scheme $\widetilde{X}_\Delta$ which is smooth (resp. proper) over $\mathbb{Z}$ if Δ is regular (resp. complete). The open affine scheme $\widetilde{U} = \operatorname{Spec}\mathbb{Z}[M]$ of $\widetilde{X}_\Delta$ has a canonical structure of a $\mathbb{Z}$-torus which comes from the canonical isomorphism (cf. [**21**, exp. I, 4.4]) between $\operatorname{Spec}\mathbb{Z}[M]$ and $\mathbf{D}(M_{\mathbb{Z}}) = \operatorname{Hom}_{Gr}(M_{\mathbb{Z}}, \mathbb{G}_m)$ for the constant group scheme (cf. [**45**, p. 52])

$$M_{\mathbb{Z}} = \coprod_{m \in M} \operatorname{Spec}\mathbb{Z}.$$

The group scheme morphism $\widetilde{U} \times_{\mathbb{Z}} \widetilde{U} \to \widetilde{U}$ extends to a (left) $\widetilde{U}$- action

$$\widetilde{U} \times_{\mathbb{Z}} \widetilde{X}_\Delta \longrightarrow \widetilde{X}_\Delta.$$

We shall call a 1-dimensional cone a ray. If Δ is complete, then the set $\Delta(1)$ of rays of Δ spans $N_{\mathbb{R}}$. We shall for a given $\rho \in \Delta(1)$, let n_ρ denote the unique generator of $\rho \cap N$. We shall write $\sigma(1)$ for the set of one-dimensional faces of σ for any cone $\sigma \in \Delta$.

The affine toric variety U_ρ defined by a ray ρ of Δ has two U-orbits. Let D_ρ denote the Zariski closure of the orbit given by the complement of U in U_ρ. This defines a bijection between rays $\rho \in \Delta(1)$ and irreducible U-invariant Weil divisors D_ρ in X_Δ. The free abelian group of U-invariant Weil divisors will be denoted by $\mathbb{Z}^{\Delta(1)}$.

Any U-invariant Cartier divisor on the affine toric variety U_σ, $\sigma \in \Delta$ is represented by a character $\chi^{m(\sigma)} : U \to \mathbb{G}_m$, $m(\sigma) \in M$ which is unique up to an element in $M(\sigma) = \sigma^\perp \cap M$. This defines a canonical isomorphism between the group $\mathrm{Div}_U(X)$ of U-invariant Cartier divisor and $\lim_{\leftarrow} M/M(\sigma)$. If we order the maximal cones σ_i, $1 \leq i \leq s$ then the latter group is equal to (cf. [**24**, 3.3])

$$\ker(\oplus_i M/M(\sigma_i) \longrightarrow \oplus_{i<j} M/M(\sigma_i \cap \sigma_j)).$$

If Δ is complete, then all maximal cones of Δ are of the same dimension as $N_{\mathbb{R}}$. Hence $m(\sigma)$ is unique for each maximal cone σ of Δ in this case.

There is a commutative diagram with exact sequences [**24**, 3.4]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}[U]^*/\mathbb{Q}^* & \longrightarrow & \mathrm{Div}_U(X) & \longrightarrow & \mathrm{Pic}\, X & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Delta(1)} & \longrightarrow & \mathrm{CH}^1(X) & \longrightarrow & 0 \end{array} \tag{8.2}$$

The maps are defined as follows. The map from $\mathbb{Q}[U]^*/\mathbb{Q}^*$ to $\mathrm{Div}_U(X)$ is defined by representing a class in $\mathbb{Q}[U]^*/\mathbb{Q}^*$ by the unique character χ^m belonging to this class.

The map from $\mathrm{Div}_U(X) \to \mathrm{Pic}\, X$ is the standard map (cf. [**32**, 2.6]) sending a U-invariant Cartier divisor represented by $\chi^{m(\sigma)} : U \to \mathbb{G}_m$ on U_σ, $\sigma \in \Delta$ to the subsheaf of the constant sheaf $k(X)^*$ on X generated by $\chi^{m(\sigma)}$. We shall denote this invertible sheaf by $\mathcal{O}(D)$ where D is the associated Weil divisor of the second vertical map. If D is effective, then $\mathcal{O}(D)$ is the inverse of the ideal sheaf of D.

The map from M to $\mathbb{Z}^{\Delta(1)}$ is defined by sending $m \in M$ to $\sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho$ and the map from $\mathbb{Z}^{\Delta(1)}$ to $\mathrm{CH}^1(X)$ sends a divisor to its linear equivalence class.

The first vertical map is defined by sending $\chi^m \in \mathbb{Q}[U]^*/\mathbb{Q}^*$ to $m \in M$.

The second vertical map sends a U-invariant Cartier divisor defined by characters $\chi^{m(\sigma)} : U \to \mathbb{G}_m$ on U_σ, $\sigma \in \Delta$ to the Weil divisor $\Delta = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ where $a_\rho = \langle m(\sigma), n_\rho \rangle$ for any cone $\sigma \in \Delta$ for which $\rho \in \sigma(1)$. (This map is well-defined since $\langle m(\sigma), n_\rho \rangle$ is independent of the cone σ for which $\rho \in \sigma(1)$.) The map is the same as the "usual" map from Weil divisors to Cartier divisors [**32**, 2.6] since the order of vanishing of the Cartier divisor along D_ρ is equal to $\langle m(\sigma), n_\rho \rangle$ by [**24**, 3.3]. Any toric variety X_Δ is normal, so the map from $\mathrm{Div}_U(X)$ to $\mathbb{Z}^{\Delta(1)}$ is injective. It is also surjective when Δ is regular.

The third vertical map is defined by the exactness of the rows and the commutativity of the first square.

Definition 8.3

(a) Let $\phi : N' \to N$ be a homomorphism of lattices and Δ be a fan in N, Δ' be a fan in N' satisfying the condition:
For each cone σ' in Δ', there is some cone σ in Δ such that $\phi(\sigma') \subseteq \sigma$.
Then ϕ is called a *morphism of fans*.

(b) Let $\pi : X' \to X$ be a morphism of toric varieties and let $U' \to U$ be the corresponding homomorphism of tori obtained by restricting π. Then π is said to be a *toric morphism* if π is equivariant with respect to the toric actions of U' and U. This means that the following diagram commutes

$$\begin{tikzcd} U' \times X' \arrow[r] \arrow[d] & X' \arrow[d] \\ U \times X \arrow[r] & X \end{tikzcd}$$

It is easy to see that any homomorphism $\phi : N' \to N$ of lattices with $\phi(\sigma') \subseteq \sigma$ for two rational strongly convex polyhedral cones $\sigma' \subset N'_{\mathbb{R}}$ and $\sigma \subset N_{\mathbb{R}}$ determines a toric morphism between affine toric varieties $U_{\sigma'} \to U_\sigma$. It follows by gluing (cf. [**24**, p. 23]) that any morphism $(N', \Delta') \to (N, \Delta)$ of fans gives rise to a toric morphism from $X' = X_{\Delta'}$ to $X = X_\Delta$.

There is also a notion of toric morphism for toric schemes defined just as in (8.3)(b) and it is clear that the proof in (op. cit.) implies that any morphism of fans defines not only a toric morphism between toric varieties but also a toric morphism between toric schemes over $\mathbb{Z}$.

We now consider torsors over $X = X_\Delta$ for a regular complete fan Δ in $N :=$ $\mathrm{Hom}(M, \mathbb{Z})$. We shall by $D_\rho^\vee$, $\rho \in \Delta(1)$ denote the basis of $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$ which is dual to the base D_ρ, $\rho \in \Delta(1)$ of irreducible Weil divisors in $\mathbb{Z}^{\Delta(1)}$. Note that $D_\rho^\vee$ is sent to n_ρ under the map from $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$ to $N = \mathrm{Hom}(M, \mathbb{Z})$ induced by the inclusion $M \subseteq \mathbb{Z}^{\Delta(1)}$ in (8.2).

Proposition 8.4. — *Let $M_0 \subseteq \mathbb{Z}^{\Delta(1)}$ be a sublattice containing M and let $N_0 =$ $\mathrm{Hom}(M_0, \mathbb{Z})$. Let $n_{0,\rho} \in N_0$, $\rho \in \Delta(1)$ be the image of $D_\rho^\vee$ under the restriction map $\mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z}) \to N_0$. Moreover, if $\sigma \in \Delta$, let σ_0 be the cone in $N_{0,\mathbb{R}}$ generated by $n_{0,\rho}$ for all one-dimensional faces ρ of σ. Then the following holds.*

(a) *The set of all these cones $\sigma_0 \subseteq N_{0,\mathbb{R}}$ form a regular fan Δ_0 of $N_{0,\mathbb{R}}$ and any $\sigma_0 \in \Delta_0$ is sent isomorphically onto the cone $\sigma \in \Delta$ defining it under the restriction map from* $\mathrm{Hom}(M_0, \mathbb{R})$ *to* $N = \mathrm{Hom}(M, \mathbb{R})$.

(b) *Let $U_{0,\sigma} :=$ $\mathrm{Spec}[S_{\sigma_0}]$, $S_{\sigma_0} = \check{\sigma}_0 \cap M_0$ be the affine toric variety defined by $\sigma_0 \in \Delta_0$ and let $\pi_\sigma : U_{0,\sigma} \to U_\sigma$ be the toric morphism defined by the map from σ_0 to σ. Then these toric morphisms glue to a toric morphism $\pi : X_0 \to X$*

from the toric variety X_0 defined by (Δ_0, N_0) to the toric variety X defined by (Δ, N). This morphism is a torsor under the the torus $\mathbf{D}(M/M_0)$.

Proof

(a) This is a consequence of the fact that each cone $\sigma \in \Delta$ is generated by its one-dimensional faces.

(b) The last assertion is essentially a special case of the exercise on p. 41 in [**24**]. See also [**46**, §1.5]. □

Proposition 8.5. — *Let Δ be a complete regular fan in $N := \mathrm{Hom}(M, \mathbb{Z})$ and let (Δ_0, N_0) be the fan defined in (8.4) by a sublattice $M_0 \subseteq \mathbb{Z}^{\Delta(1)}$. Let P_0 be the image of M_0 in* Pic X *and let $S_0 = \mathbf{D}(P_0)$ be the dual $\mathbb{Q}$-torus. Then the corresponding morphism $X_0 \to X$ of toric varieties is an torsor under S_0 of type $P_0 \subseteq \mathrm{Pic}\, X$. In particular, if $M_0 = \mathbb{Z}^{\Delta(1)}$, then $X_0 \to X$ is a universal torsor.*

Proof. — The type of an S_0-torsor over X is uniquely determined by the types of the $\mathbb{G}_m$-torsors induced by the characters $S_0 \to \mathbb{G}_m$ of S_0. Using this and the functoriality of the construction of X_0 under restriction of the lattice M_0 to smaller lattices containing M, one reduces to the case $S_0 = \mathbb{G}_m$. Now note that by the completeness of Δ one may extend any group embedding $n : \mathbb{G}_m \to U$ to an equivariant closed embedding $\mathbb{P}^1 \to X$ of toric varieties. Since the one-parameter subgroups corresponds bijectively to elements of $N := \mathrm{Hom}(M, \mathbb{Z})$ it follows that a class in Pic X is determined by its restriction to these closed subschemes.

It may happen that P_0 is sent to zero under some of these restrictions, in which case the pullback of the $\mathbb{G}_m$-torsor is trivial for trivial reasons. It this is not the case, then we reduce to the case when M is of rank 1 and $X = \mathbb{P}^1_{\mathbb{Q}}$ and $P_0 = \mathrm{Pic}\, X$. Then (Δ_0, N_0) simply defines the affine cone of $\mathbb{P}^1_{\mathbb{Q}}$ and it is known and easy to prove that the affine cone of $\mathbb{P}^1_{\mathbb{Q}}$ is a universal torsor. This completes the proof. □

Remarks 8.6

(a) One can also give a proof of (8.5) based on purity and look at the restriction of the torsor over a toric variety defined by the cones in Δ of dimension ≤ 1.

(b) There is a version of (8.5) for smooth proper toric $\mathbb{Z}$-schemes $\widetilde{X}_\Delta$ obtained by gluing the affine schemes $\widetilde{U}_\sigma = \mathrm{Spec}\, \mathbb{Z}[S_\sigma]$ for the complete regular fan Δ in N. In particular, one can construct a universal torsor $\widetilde{X}_0 \to \widetilde{X}$ (cf. (5.14)) which extends the homomorphism of $\mathbb{Z}$-tori $\widetilde{U}_0 = \mathbf{D}(M_{0,\mathbb{Z}}) \to \widetilde{U} = \mathbf{D}(M_{\mathbb{Z}})$ associated to the monomorphism $M \subseteq M_0 = \mathbb{Z}^{\Delta(1)}$. We leave the details to the reader since the statement and the proof is almost identical to (8.5). One can now make base extensions and obtain versions of (8.5) for toric schemes over arbitrary base schemes B. One can e.g. choose $B = \operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$ and consider toric varieties over $\mathbb{Z}/p\mathbb{Z}$.

We shall in the sequel fix an arbitrary field k and let $X = X_\Delta$ be the k-variety determined by the complete regular fan (N, Δ).

Let $M_0 = \mathbb{Z}^{\Delta(1)}$ in (8.4). The fan Δ_0 and the morphism $(N_0, \Delta_0) \to (N, \Delta)$ of fans described there defines a toric variety X_0 and an equivariant morphism $\pi : X_0 \to X$ which according to (8.5) is a universal torsor. We shall call this universal torsor $\pi : X_0 \to X$ the *principal* universal X-torsor.

Let $N_1 = N_0$ and let Δ_1 be the fan in N_1 consisting of all cones generated by $n_{0,\rho} = D_\rho^\vee$, for ρ belonging to *any* subset of $\Delta(1)$. Then the toric variety X_1 determined by (N_1, Δ_1) is the affine n-space $\mathbb{A}^n$, $n = \operatorname{Card} \Delta(1)$ and the morphism of fans $(N_0, \Delta_0) \to (N_1, \Delta_1)$ defines an open equivariant embedding $X_0 \subseteq X_1$.

We now give a more concrete description of this embedding following Cox **[16]**. We introduce one variable x_ρ for each $\rho \in \Delta(1)$ and extend this to a bijection between monomials

$$x^D = \prod_{\rho\in\Delta(1)} x_\rho^{a_\rho}, \quad \rho \in \Delta(1)$$

and effective Weil divisors

$$D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$$

with support outside U. For a cone $\sigma \in \Delta$, let $\underline{\sigma}$ be the divisor

$$\underline{\sigma} = \sum_{\rho\notin\sigma(1)} D_\rho.$$

Then $U_{0,\sigma}$ (cf. (8.4)(b)) is the open subvariety of $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which $x^{\underline{\sigma}} \neq 0$.

The open affine toric subvarieties $U_{0,\sigma}$, $\sigma \in \Delta_{\max}$ form a covering of X_0. Hence X_0 is the open subvariety in $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which not all the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ vanish.

Proposition 8.7. — *Let Δ be a complete regular fan and let $\Delta_0 = \sum_{\rho\in\Delta(1)} D_\rho$. Let σ be a maximal cone of Δ and let $\chi^{m(\sigma)}$, $m(\sigma) \in M$ be the unique character of U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D_0)$ on U_σ. Let*

$$D(\sigma) = D_0 + \sum_{\rho\in\Delta(1)} \langle -m(\sigma), n_\rho \rangle D_\rho.$$

Then the following holds for any maximal cone σ of Δ.

(a) *If $\mathcal{O}(D_0)$ is generated by its global sections, then $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D_0)$ and $D(\sigma)$ is an effective divisor with support contained in $\cup_{\rho\notin\sigma(1)} D_\rho$.*

(b) *If $\mathcal{O}(D_0)$ is ample, then $\mathcal{O}(D_0)$ is very ample and $D(\sigma)$ an effective divisor with support $\cup_{\rho\notin\sigma(1)} D_\rho$.*

Proof

(a) If $\mathcal{O}(D_0)$ is generated by its global sections, then there exists for each $\sigma \in \Delta$ a global section of $\mathcal{O}(D_0)$ which generates $\mathcal{O}(D_0)$ on U_σ. But U_σ is an affine space for maximal cones σ in a complete fan Δ. There is thus up to multiplication with an element in k^* only one local section which generates $\mathcal{O}(D_0)$ on U_σ for $\sigma \in \Delta_{\max}$. This implies that for maximal cones σ, $\chi^{-m(\sigma)}$ is not only a local section on U_σ but a global section of $\mathcal{O}(D_0)$.

The Weil divisor of the rational function $\chi^{-m(\sigma)}$ is $\sum_{\rho\in\Delta(1)}\langle -m(\sigma), n_\rho\rangle D_\rho$ (cf. [**24**, p. 61]). Therefore, since $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D_0)$ we must have that $\langle -m(\sigma), n_\rho\rangle \geq -1$ for all rays ρ of Δ [**24**, p. 68]. Also, $\langle -m(\sigma), n_\rho\rangle = -1$ for $\rho \in \sigma(1)$, since $\chi^{-m(\sigma)}$ generates the fractional ideal $\mathcal{O}(D_0)$ on U_ρ so that $\langle -m(\sigma), n_\rho\rangle = -1$. Hence $\Delta(\sigma)$ is effective with support contained in $\cup_{\rho\notin\sigma(1)} D_\rho$.

(b) The very ampleness of $\mathcal{O}(D_0)$ is part of a more general result of Demazure [**24**, p. 71]. Also, if D_0 is ample, then the function $\psi : N_{\mathbb{R}} \to \mathbb{R}$ defined by $\psi(n) = \langle -m(\sigma), n\rangle$ is strictly convex [**24**, p. 70] and $\langle -m(\sigma), n_\rho\rangle > -1$ for all $\rho \notin \sigma(1)$. This completes the proof. □

Corollary 8.8. — *Let Δ be a complete regular fan such that $D_0 = \sum_{\rho\in\Delta(1)} D_\rho$ is ample. Let $D(\sigma)$, $\sigma \in \Delta_{\max}$ be the U-invariant Weil divisors on X described in (8.7).*

(a) *If $\mathcal{O}(D_0)$ is generated by its global sections and σ a maximal cone of Δ, then $x^{D(\sigma)} \neq 0$ on $U_{0,\sigma} \subseteq X_0$ (cf. (8.4)).*

(b) *If $\mathcal{O}(D_0)$ is ample and σ a maximal cone of Δ, then $U_{0,\sigma}$ is the open subset of $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ defined by $x^{D(\sigma)} \neq 0$. Hence X_0 is the open subvariety in $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$ for which not all $x^{D(\sigma)}$, $\sigma \in \Delta_{\max}$ vanish.*

Proof. — This follows from (8.7) and the description of $U_{0,\sigma} \subset X_1$ by means of $x^{\underline{\sigma}}$. □

The following lemma will be used to prove (9.10) and in the proof of the asymptotic formulas in section 11. Recall that a facet τ of a cone σ is a face of codimension one (cf. [**24**, Ch. 1]).

Lemma 8.9. — *Let Δ be a complete regular d-dimensional fan and let $\sigma^{(0)}$ be a maximal (and hence d-dimensional) cone of Δ with facets $\tau^{(1)}, \ldots, \tau^{(d)}$. Then* *there exist unique d-dimensional cones* $\sigma^{(1)}, \dots, \sigma^{(d)}$ *such that* $\tau^{(i)} = \sigma^{(0)} \cap \sigma^{(i)}$ *for* $i = 1, \dots, d$.

Moreover, for any such set of cones the following holds:

(i) There exists exactly one 1*-dimensional face* $\rho^{(i)}$ *of* $\sigma^{(0)}$ *such that* $\rho^{(i)} \cap \sigma^{(i)} = \{0\}$ *for each* $i = 1, \dots, d$. *Moreover, any one-dimensional face of* $\sigma^{(0)}$ *is equal to* $\rho^{(i)}$ *for exactly one integer* $i = 1, \dots, d$.

(ii) Let $\{n^{(i)} : 1 \le i \le d\}$ *be the* $\mathbb{Z}$*-basis of* N *defined by the generators of the rays* $\rho^{(1)}, \dots, \rho^{(d)} \in \sigma(1)$ *and let* $\{m^{(i)} : 1 \le i \le d\}$ *be the dual* $\mathbb{Z}$*-basis of* M. *Let* b_i *be the multiplicity of* D_i *in* $D(\sigma^{(i)})$, $1 \le i \le d$. *Then,*

$$m(\sigma^{(i)}) - m(\sigma^{(0)}) = b_i m^{(i)}, \quad i \in \{1, \dots, d\}.$$

(iii) Let $m = m^{(1)} + \dots + m^{(d)}$ *and let* $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$. *Then,*

$$D_0 - D(\sigma^{(0)}) = \sum_{\rho \in \Delta(1)} \langle m, n_\rho \rangle D_\rho.$$

(iv) Suppose that $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$ *is ample. Then* $b_i > 0$ *for* $i \in \{1, \dots, d\}$.

(v) Suppose that there is only one 1*-dimensional cone* $n^{(0)} \in \Delta$ *not contained in* $\sigma(0)$. *Then* $n^{(0)} + n^{(1)} + \dots + n^{(d)} = 0$.

Proof. — Let $\tau \in \Delta$ be a cone of dimension $d - 1$ and let H_τ be the hyperplane generated by τ and $-\tau$. Then τ is part of the boundary of any d-dimensional cone $\sigma \in \Delta$ containing τ [**24**, p. 10]. Hence any such cone $\sigma \in \Delta$ must lie in one of the closed half spaces of $N_\mathbb{R}$ defined by H_τ. Let R be one of these closed half spaces and let $\Omega \subset R$ be the open subset of interior points of R which do not lie on any cone of dimension $< d$. Then Ω is contained in the union $\sigma_1 \cup \dots \cup \sigma_s$ of the maximal cones $\sigma_1, \dots, \sigma_s$ with a point in Ω by completeness of Δ. Hence the closure R of Ω is also contained in the closed subset $\sigma_1 \cup \dots \cup \sigma_s$ so that $\tau = \tau \cap (\sigma_1 \cup \dots \cup \sigma_s)$ is a finite union of cones $(\tau \cap \sigma_1) \cup \dots \cup (\tau \cap \sigma_s)$. Therefore, $\tau = \tau \cap \sigma$ for some d-dimensional cone $\sigma \in \Delta$ with $\sigma \subset R$. This cone is clearly unique since any such cone σ must contain $O \cap R$ for any sufficiently small neighbourhood O around an interior point of τ. There are thus exactly two d-dimensional cones $\sigma \in \Delta$ containing τ.

The statement (i) is a trivial consequence of the regularity assumption on Δ. To prove (ii), let $m(\sigma) \in M$ be the unique element such that the character $\chi^{-m(\sigma)}$ of U generates $\mathcal{O}(D_0)$ on U_σ. Then (cf. (8.7)):

$$D(\sigma) := \sum_{\rho \in \Delta(1)} (1 + \langle -m(\sigma), n_\rho \rangle) D_\rho$$

where $\langle -m(\sigma), n_\rho \rangle = -1$ for any maximal cone $\sigma \in \Delta$ and any ray ρ of σ.

Hence
$$\langle m(\sigma^{(0)}) - m(\sigma^{(i)}), n^{(i)}\rangle = b_i$$
and
$$\langle m(\sigma^{(0)}) - m(\sigma^{(i)}), n^{(j)}\rangle = 0$$
for the generators $n^{(j)}, j \in \{1, \dots, d\}, j \neq i$ of the rays of $\mathcal{T}^{(i)}$. This proves (ii).

To show (iii), note that the multiplicity of the D_i in
$$D_0 - D(\sigma^{(0)}) = \sum_{\rho \in \Delta(1)} \langle m(\sigma^{(0)}), n_\rho\rangle D_\rho$$
and
$$\sum_{\rho \in \Delta(1)} \langle m, n_\rho\rangle D_\rho$$
is equal to 1 for $1 \leq i \leq d$. But then $\langle m(\sigma^{(0)}), n_\rho\rangle = \langle m, n_\rho\rangle$ for all rays ρ of Δ, since $\{n^{(i)} : 1 \leq i \leq d\}$ is a $\mathbb{Z}$-basis of N.

To show (iv), use the fact that $1 + \langle -m(\sigma), n_\rho\rangle > 0$ for any maximal cone $\sigma \in \Delta$ and any ray ρ of σ if D_0 is ample.

To show (v), note that any subset of d elements in $\{n^{(0)}, n^{(1)}, \dots, n^{(d)}\}$ form a $\mathbb{Z}$-basis of N by the regularity of the maximal cones $\{\sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(d)}\}$. Hence we have $\langle m^{(i)}, n^{(0)}\rangle = \pm 1$ for $1 \leq i \leq d$. Also, since $\sigma^{(i)} \neq \sigma^{(0)}$, we must have $\langle m^{(i)}, n^{(0)}\rangle \neq 1$ for $1 \leq i \leq d$. This completes the proof. □

Proposition 8.10. — *Let $D_0 = \sum_{\rho \in \Delta(1)} D_\rho$ and let $\omega_{X/k}$ be the canonical sheaf. Let du be a non-vanishing U-invariant section of $\omega_{X/k}$ on U. Then there exists a unique extension of du to a global section of $\omega_{X/k} \otimes \mathcal{O}(D_0)$. This global section generates $\omega_{X/k} \otimes \mathcal{O}(D_0)$.*

Proof. — This is proved in [**24**, 4.3] when du is of minimal $d\log$-type (cf. (3.28)). If $d\nu$ is another U-invariant section, then $d\nu/du$ is a U-invariant regular function on U and hence $d\nu = \alpha du$ for some $\alpha \in k$. This proves the assertion. □

We end this section with some comments on twisted toric varieties.

Suppose that $M_0 = \mathbb{Z}^{\Delta(1)}$ and let G be a finite group of automorphisms of (Δ, N). Then G acts also on the fan (Δ_0, N_0) so that we get a morphism of G-fans from (Δ_0, N_0) to (D, N). This implies that the corresponding morphism $\pi : X_0 \to X$ is a G-equivariant morphism between toric varieties with G-actions.

An important case is when G is the Galois group of a finite Galois extension $k \subset K$ of fields. If we regard $X = X_\Delta$ as a variety over the base field K, then the G-actions on (Δ, N) and K define a G-action on X which is compatible with the G-action on K. This may be interpreted as a descent datum (see [**10**, pp. 139-141]). The descent datum is effective if and only if each Galois orbit is contained in a quasi-affine variety (e.g. if X is projective). It then follows from results of Weil and Grothendieck (cf. op. cit) that there is a k-variety X^G with a G-equivariant K-isomorphism λ between $X^G \times_k K$ and X. The G-action on $X^G \times_k K$ is induced by the Galois action on K. The pair (X^G, λ) is unique up to unique isomorphism.

There is also a G-action of X_0 which defines a k-variety X_0^G and an isomorphism λ_0 between $X_0^G \times_k K$ and X_0. The corresponding descent datum is always effective since X_0 is quasi-affine. By Galois descent there exists further a unique k-morphism $\pi^G : X_0^G \to X^G$ such that the induced K-morphism $\pi^G \times K : X_0^G \times_k K \to X^G \times_k K$ belongs to the following commutative diagram of G-morphisms

$$\begin{array}{ccc} X_0^G \times_k K & \xrightarrow{\pi^G \times K} & X^G \times_k K \\ \lambda_0 \big\downarrow \wr & & \wr \big\downarrow \lambda \\ X_0 & \xrightarrow{\quad \pi \quad} & X \end{array} \tag{8.11}$$

Let $\pi_U : U_0 \to U$ be the restriction of π corresponding to the morphism of fans from $(\{0\}, N_0)$ to $(\{0\}, N)$. Then π^G restricts to a homomorphism $(\pi_U)^G : U_0^G \to U^G$ of k-tori which may also be obtained directly from the G-morphism of fans from $(\{0\}, N_0)$ to $(\{0\}, N)$. This morphism is nothing but the map from $\mathbf{D}(M_0) = U_0^G$ to $\mathbf{D}(M) = U^G$ dual to the G-monomorphism from $M = \ker(\mathbb{Z}^{\Delta(1)} \to \mathrm{CH}^1(X))$ to $M_0 = \mathbb{Z}^{\Delta(1)}$. The kernel of $(\pi_U)^G$ is thus the Néron-Severi k-torus T^G dual to the G-module $\mathrm{Pic}\, X = \mathrm{CH}^1(X)$.

The k-torus U_0^G (resp. U^G) acts on X_0^G (resp. X^G) so that X_0^G and X^G become twisted toric varieties. The k-morphism $\pi^G : X_0^G \to X^G$ is equivariant under these torus actions. It is clear from the G-equivariant commutative diagram (8.11) that $\pi^G : X_0^G \to X^G$ is a torsor under T^G. The type of this torsor is uniquely determined by the type of the torsor after a base extension. It must therefore be a universal torsor since $\pi^G : X_0^G \to X^G$ is a universal torsor. The fibre of this torsor at the neutral element e of $U^G \subset X^G$ is trivial since it contains the neutral element e_0 of U_0^G. We have thus (cf. (5.13)) determined the isomorphism class of the universal torsor $\pi^G : X_0^G \longrightarrow X^G$. We shall call this universal torsor *the principal* universal torsor.

9. Norms on toric varieties over local fields

Let $X = X_\Delta$ be a smooth complete toric variety over a locally compact field. There is a natural norm $\| \ \|_D$ for each U-invariant Weil divisor D on X described in [**7**]. We shall in this section give a new interpretation of these norms by means of a "canonical toric splitting" for the principal universal torsor $X_0 \to X$. This interpretation may be seen as an analog of Bloch's (cf. [**8**], [**48**]) approach to local Néron heights for abelian varieties. We shall also show that the induced norm on X_0 of $\| \ \|_D$ for the anticanonical system is very natural and simpler than $\| \ \|_D$.

We shall keep all the notations in section 8. We shall thus by X denote the smooth complete k-variety defined by a regular complete fan Δ in $N = \operatorname{Hom}(M, \mathbb{Z})$. The only difference is that k will denote a non-discrete *locally compact field* of characteristic 0 throughout the section.

Let $|\ | : k^* \to \mathbb{R}_{>0}$ be the normalized absolute value defined in section 1. The additive valuation $\log |\ | : k^* \to \mathbb{R}$ induces a homomorphism

$$L : U(k) = \operatorname{Hom}(M, k^*) \longrightarrow N_{\mathbb{R}} = \operatorname{Hom}(M, \mathbb{R})$$

from the multiplicative group $U(k)$ to the additive group $N_{\mathbb{R}}$. If $\sigma \in \Delta$ is a cone, then $L^{-1}(-\sigma)$ is a closed subset of $U(k)$ in the k-topology.

Notation 9.1. — Let $\sigma \in \Delta$. Then $C_\sigma(k)$ is the closure of $L^{-1}(-\sigma)$ in $X(k)$.

Batyrev and Tschinkel **[7]** use the compact subsets $C_\sigma(k)$, $\sigma \in \Delta$ to define a norm $\|\ \|_D$ on the line bundle $\mathcal{O}(D)$ for any U-invariant Weil divisor D. We now give a slightly different, but equivalent, definition of their norm.

Proposition and definition 9.2. — *Let D be a U-invariant Weil divisor on X and let s be a local analytic section of $\mathcal{O}(D)$ defined at $P \in X(k)$. Then any $P \in X(k)$ belongs to $C_\sigma(k)$ for some cone $\sigma \in \Delta$. Let $\chi^{m(\sigma)}$ be a character which on U_σ represents the Cartier divisor with Weil divisor D. Then the expression*

$$\|s(P)\|_D := |s(P)\chi^{m(\sigma)}(P)|$$

is independent of the choice of cone σ with $P \in C_\sigma(k)$. It defines a norm in the sense of (1.5) on the analytic line bundle $V(\mathcal{O}(-D))_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ of sections of $\mathcal{O}(D)$. If D is effective, then

$$|\chi^{m(\sigma)}(P)| \leq 1$$

and

$$\|s(P)\|_D \leq |s(P)|.$$

Proof. — It suffices to prove the first two statements for the dense subset $U(k)$. To show the first, use the completeness of Δ. For the second, note that

$$|\chi^{m(\sigma)}(P)| = \exp\langle m(\sigma), L(P)\rangle$$

for $P \in U(k)$. If $P \in C_\sigma(k) \cap C_\tau(k)$ for two cones σ and τ, then there are unique non-negative real numbers λ_ρ, $\rho \in (\sigma \cap \tau)(1)$ such that

$$-L(P) = \sum_{\rho \in (\sigma \cap \tau)(1)} \lambda_\rho n_\rho.$$

Therefore $|\chi^{m(\sigma)}(P)|$ and $|\chi^{m(\tau)}(P)|$ are equal since $\langle m(\sigma), n_\rho\rangle = \langle m(\tau), n_\rho\rangle$ is the multiplicity of D along D_ρ for each ray (cf. (8.2)). The third statement is obvious since $|s(P)\chi^{m(\sigma)}(P)|$ is a norm for the line bundle over $C_\sigma(k)$ for each cone σ. To prove the inequality, note that $\langle m(\sigma), L(P)\rangle \leq 0$ since $\langle m(\sigma), n_\rho\rangle \geq 0$ for each $\rho \in \sigma(1)$. This completes the proof. □

Let T be the Néron-Severi torus of X and let $\pi : \mathcal{T} \to X, \mathcal{T} = X_0$ be the principal universal X-torsor constructed from the fan (N_0, Δ_0) (see (8.5)). There is an analytic torsor

$$\pi_{\mathrm{an}} : \mathcal{T}_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k)$$

under $T_{\mathrm{an}}(k)$ associated to π. In particular, we may identify the topological space $X(k)$ with the quotient space $X_0(k)/T(k)$ (cf. (3.11)).

Notation 9.3

(a) $T(k)_{\mathrm{cp}}$ (resp. $U_0(k)_{\mathrm{cp}}$) is the maximal compact subgroup of the locally compact group $T(k)$ (resp. $U_0(k)$).

(b) $\breve{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$ is the unique continuous map sending the $T(k)_{\mathrm{cp}}$-orbit of a point $Q \in \mathcal{T}(k)$ to $\pi(Q)$.

The existence of the norms in (9.2) is related to the existence of a canonical splitting of $\breve{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$. To construct such a splitting, we first define local splittings of π_{an}.

Lemma 9.4. — *Let σ be a maximal cone of Δ and let $X_1 = \operatorname{Spec} k[x_\rho]$, $\rho \in \Delta(1)$. If D_ρ is a U-invariant Weil divisor on X corresponding to a ray $\rho \in \Delta(1)$, let $\chi_\rho^{m(\sigma)}$, $\sigma \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(-D_\rho)$ on U_σ. Let $\psi_\sigma : U_\sigma \to X_1$ be the affine k-morphism such that $\psi_\sigma^* x_\rho = \chi_\rho^{m(\sigma)}$, $\rho \in \Delta(1)$ and let $\pi_\sigma : U_{0,\sigma} \to U_\sigma$ be the open affine toric morphism described in (8.4)(b).*

Then, $\psi_\sigma(U_\sigma) \subseteq U_{0,\sigma}$ and $\pi_\sigma \circ \psi_\sigma : U_\sigma \to U_\sigma$ is the identity morphism.

Proof. — $\chi_\rho^{m(\sigma)}$ generates $\mathcal{O}(-D_\rho)$ on U_σ. Therefore, $x_\rho \neq 0$ on U_σ for all $\rho \notin \sigma(1)$. This proves the first statement since $U_{0,\sigma}$ is the open subset of X_1 such that $x_\rho \neq 0$ for all rays $\rho \in \Delta(1)$ not in σ.

The ring $k[U_\sigma]$ of regular functions is generated by characters $\chi^m \in \hat{U}$ for $m \in M$ in the dual cone $\sigma^\vee$ of σ. Let

$$D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}, \quad a_\rho = \langle m, n_\rho\rangle$$

be the corresponding Weil divisor (cf. (8.2)) and let

$$\begin{aligned} \pi_\sigma^\# : k[U_\sigma] &\longrightarrow k[U_{0,\sigma}], \\ \psi_\sigma^\# : k[U_{0,\sigma}] &\longrightarrow k[U_\sigma] \end{aligned}$$

be the homomorphisms defined by π_σ and ψ_σ. Then,

$$\pi_\sigma^\#(\chi^m) = x^D := \prod_{\rho\in\Delta(1)} x_\rho^{a_\rho}$$

and

$$\psi_\sigma^\#(x^D) = \prod_{\rho\in\Delta(1)} \chi_\rho^{m(\sigma)a_\rho} = \chi^m.$$

This completes the proof. □

Proposition 9.5

(a) *There exists a unique continuous map $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ such that $\check{\psi}(P)$ is the class of $\psi_\sigma(P) \in X_0(k)$ in $X_0(k)/T(k)_{\mathrm{cp}}$ for $P \in C_\sigma(k)$, $\sigma \in \Delta_{\max}$.*

(b) *The map $\check{\psi}$ is a section of the continuous map $\check{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$.*

Proof

(a) Suppose that $P \in C_\sigma(k) \cap C_\tau(k)$ for two maximal cones σ, τ of Δ. It then follows from the proof of (9.2) that $|\chi_\rho^{m(\sigma)}(P)| = |\chi_\rho^{m(\tau)}(P)|$ for each ray ρ of Δ. This is equivalent to the existence of an element $u \in U_0(k)_{\mathrm{cp}}$ such that $\psi_\sigma(P) = u\psi_\tau(P)$ under the toric action of $U_0(k)$ on $X_0(k)$. But $\pi(\psi_\sigma(P)) = \pi(\psi_\tau(P)) = P$ by the previous lemma. Therefore, since π is a toric morphism, $P = \pi(u)P$ for the toric action of $U(k)$ on $X(k)$. Hence $\pi(u) = 1$ in the group $U(k)$ and u an element of $T(k) = \ker(U_0(k) \to U(k))$. But then $u \in T(k) \cap U_0(k)_{\mathrm{cp}} = T(k)_{\mathrm{cp}}$ so that $\psi_\sigma(P) = \psi_\tau(P)$ in $\mathcal{T}(k)/T(k)_{\mathrm{cp}}$. This proves that $\check{\psi}$ is well defined.

The continuity of $\check{\psi}$ is immediate from the continuity of $\psi_{\sigma,\mathrm{an}}$ for each maximal cone σ of Δ.

(b) This follows from the corresponding property of ψ_σ (cf. (9.4)). □

We shall in the following sections call $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ *the canonical toric section* (or *splitting*) of $\check{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$.

Notation and remark 9.6

(a) Let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor on X. Then $|\ |_D : X_1(k) \to \mathbb{R}$ is the map which sends the n-tuple (β_ρ), $\rho \in \Delta(1)$ in $X_1(k)$ to $|\beta^D|$ where $\beta^D = \prod_{\rho\in\Delta(1)} \beta_\rho^{a_\rho}$. We shall also write $|\ |_D$ for the restrictions of this map to $\mathcal{T}(k) = X_0(k)$ and $U_0(k)$.

(b) $|\ |_D : X_1(k) \to \mathbb{R}$ is constant on the $U_0(k)_{\mathrm{cp}}$-orbits under the toric action of $U_0(k)$ on $X_1(k)$. Therefore $|\ |_D : \mathcal{T}(k) \to \mathbb{R}$ factorizes to give a map $\mathcal{T}(k)/T(k)_{\mathrm{cp}} \to \mathbb{R}$ which we also denote by $|\ |_D$ by abuse of notation.

Proposition 9.7. — *Let D_1, $D_2 \in \mathbb{Z}^{\Delta(1)}$ be two effective U-invariant Weil divisors on X and let $D = D_1 - D_2$. Then 1 is a local section of $\mathcal{O}(D)$ at each k-point $P \in X(k)$ outside the support of D_2. Moreover,*

$$\|1(P)\|_D = |\widetilde{\psi}(P)|_D$$

for each k-point outside the support of D_2.

Proof. — Let $\sigma \in \Delta$ be a maximal cone such that $P \in C_\sigma(k)$ and let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$. Further, let $\chi^{m(\sigma)}$ resp. $\chi_\rho^{m(\sigma)}$, $\rho \in \Delta(1)$ be the unique character on U which generates $\mathcal{O}(-D)$ resp. $\mathcal{O}(-D_\rho)$ on U_σ. Then,

$$\|1(P)\|_D = |\chi^{m(\sigma)}(P)| = |\prod_{\rho\in\Delta(1)} \chi_\rho^{m(\sigma)a_\rho}| = |\widetilde{\psi}(P)|_D$$

as was to be proved. □

Proposition 9.8. — *Let D be a U-invariant Weil divisor on X such that $\mathcal{O}(D)$ is generated by its global sections. Let $\chi^{-m(\sigma)}$, $\sigma \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(D)$ on U_σ. Then $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D)$ and $\chi^{-m(\sigma)}(P) \neq 0$ for points $P \in U_\sigma(k)$. If σ is a local section of $\mathcal{O}(D)$ defined at $P \in X(k)$, then*

$$\|s(P)\|_D = \inf_\sigma |s(P)\chi^{m(\sigma)}(P)|$$

where σ runs over all maximal cones in Δ.

Moreover, if D is ample and σ is a maximal cone in Δ, then $C_\sigma(k)$ is equal to the subset of $P \in X(k)$ such that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta$.

Proof. — It follows from the proof of (8.7)(a), that $\chi^{-m(\sigma)}$ is global section of $\mathcal{O}(D)$.

Next, let $D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho$. Then, by [**24**, p. 68], one has

(i) $$\langle m(\sigma), n_\rho\rangle \leq a_\rho \text{ for all } \sigma \in \Delta_{\max},\ \rho \in \Delta(1),$$

(ii) $$\langle m(\sigma), n_\rho\rangle = a_\rho \text{ for all } \sigma \in \Delta_{\max},\ \rho \in \sigma(1).$$

Here (i) follows from the fact that $\chi^{-m(\sigma)}$ is a global section of $\mathcal{O}(D)$ while (ii) follows from the assumption that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ.

To prove the formula for $\|s(P)\|_D$ it suffices by continuity to treat the case $P \in U(k)$. Let $\sigma \in \Delta_{\max}$ and suppose that $P \in C_\sigma(k) \cap U(k)$. Then $-L(P)$ is a linear combination $\sum_{\rho\in\sigma(1)} \lambda_\rho n_\rho$ with non-negative real coefficients λ_ρ (cf. (9.1)). By (i), (ii) we get that

$$\langle m(\sigma), -L(P)\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m(\sigma), n_\rho\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho a_\rho, \quad \sigma \in \Delta_{\max},$$

$$\langle m(\tau), -L(P)\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m(\tau), n_\rho\rangle \leq \sum_{\rho\in\sigma(1)} \lambda_\rho a_\rho, \quad \sigma, \tau \in \Delta_{\max}.$$

From this we conclude that if $P \in C_\sigma(k) \cap U(k)$, then

$$|\chi^{m(\sigma)}(P)| = \exp\langle m(\sigma), L(P)\rangle \leq \exp\langle m(\tau), L(P)\rangle = |\chi^{m(\tau)}(P)|$$

for any other maximal cone τ of Δ. This proves the first statement. Further, if $P \in C_\sigma(k)$, it follows by continuity that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta$.

Now assume that D is ample and that

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal cones $\tau \in \Delta_{\max}$ (cf. (8.9)). Then (cf. [**24**, p. 70]),

(iii) $$\langle m(\tau), n_\rho\rangle < a_\rho \text{ for all } \tau \in \Delta_{\max},\ \rho \notin \tau(1).$$

Therefore, if $P \in C_\sigma(k) \cap U(k)$ and $\tau \in \Delta_{\max}$, then

$$|\chi^{m(\sigma)}(P)| = |\chi^{m(\tau)}(P)| \Leftrightarrow \lambda_\rho = 0 \text{ for all } \rho \in \sigma(1)\backslash\tau(1) \Leftrightarrow P \in C_\tau(k).$$

Hence by continuity if $P \in C_\sigma(k)$ then $|\chi^{m(\sigma)-m(\tau)}(P)| = 1$ if and only if $P \in C_\tau(k)$. This completes the proof of the last assertion. □

We now concentrate on the anticanonical system.

Definition 9.9. — Let Δ be a complete regular fan and σ be a maximal cone. Then another maximal cone τ of Δ is said to be *adjacent* to σ if $\sigma \cap \tau$ is a facet of σ.

Lemma 9.10. — *Let Δ be a complete regular fan such that $D_0 = \sum_{\rho\in\Delta(1)} D_\rho$ is ample. Let σ be a maximal cone in Δ and let P be a k-point on $X_\Delta(k)$. Then $P \in C_\sigma(k)$ if and only if*

$$|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$$

for all maximal adjacent cones $\tau \in \Delta$.

Proof. — $-L(P)$ is a linear combination $\sum_{\rho\in\sigma(1)} \lambda_\rho n_\rho$ with real coefficients λ_ρ and $P \in C_\sigma(k)$ if and only if $\lambda_\rho \geq 0$ for all rays ρ of σ. Let $\{m_\rho,\ \rho \in \sigma(1)\}$ be the $\mathbb{Z}$-basis of M dual to the $\mathbb{Z}$-basis $\{n_\rho,\ \rho \in \sigma(1)\}$ of N. Let σ_ρ be the maximal adjacent cone corresponding to $\rho \in \sigma(1)$ under the bijection in (8.9). Then (cf. op. cit.) there exists a positive integer b_ρ, such that

$$b_\rho m_\rho = m(\sigma_\rho) - m(\sigma).$$

Hence,

$$\log|\chi^{m(\sigma)-m(\sigma_\rho)}(P)| = \langle m(\sigma) - m(\sigma_\rho), L(P)\rangle = -b_\rho\lambda_\rho$$

so that $\lambda_\rho \geq 0$ if and only if $|\chi^{m(\sigma)-m(\tau)}(P)| \leq 1$ for $\tau = \sigma_\rho$. This completes the proof. □

Proposition and definition 9.11

(a) *Let $X = X_\Delta$ be a smooth k-variety defined by a regular fan Δ and let $D = \sum_{\rho\in\Delta(1)} D_\rho$. Then there exists a unique extension of the order norm (3.30) on* $\mathbf{det}\,\mathrm{Tan}(U_{\mathrm{an}}(k)) \to U_{\mathrm{an}}(k)$ *to a norm*

$$\| \ \|^\# : \mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(D))_{\mathrm{an}}(k) \longrightarrow \mathbb{R}$$

which we shall call the order norm on

$$\mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(D))_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k)$$

and denote by $\| \ \|^\#$.

(b) *Let $X = X_\Delta$ be a smooth complete k-variety defined by a regular complete fan Δ and let $D = \sum_{\rho\in\Delta(1)} D_\rho$. Then we define the toric norm on*

$$\mathbf{det}(\mathrm{Tan}\, X_{\mathrm{an}}(k)) \longrightarrow X_{\mathrm{an}}(k)$$

to be the product norm of the order norm in (a) and the norm $\| \ \|_D$ on

$$V(\mathcal{O}(-D))_{\mathrm{an}}(k) \longrightarrow X_{\mathrm{an}}(k)$$

described in (9.2).

Proof. — The uniqueness follows from the density of $U(k)$ in $X_{\mathrm{an}}(k)$ (cf. (1.6)(b)). To show the existence, let du be an analytic differential form of minimal $d\log$-type on $U_{\mathrm{an}}(k)$ (3.28). Then du extends to a global nowhere vanishing section on $\mathbf{det}(\mathrm{Cot} X_{\mathrm{an}}(k)) \otimes V(\mathcal{O}(-D))_{\mathrm{an}}(k)$ by (8.10). We may thus define $\|s\|^\#$ to be the absolute value of $du(s) \in k$ (cf. (1.6)(a)). This completes the proof. □

The toric norm depends on the toric structure of $X = X_\Delta$, but is otherwise a "canonical" norm.

Choose an ordering $\rho_1, \dots, \rho_n$ of the one-dimensional cones in Δ and let $x_1, \dots, x_n$ be the corresponding variables. Recall that (8.7) the principal universal X-torsor X_0 is the open subset of $X_1 = \operatorname{Spec} k[x_1, \dots, x_n]$ for which not all the monomials

$$x^{\underline{\sigma}} := \prod_{\rho \notin \sigma(1)} x_\rho, \quad \sigma \in \Delta_{\max}$$

vanish.

Theorem 9.12. — *Let k be a non-discrete locally compact field of characteristic zero and let $X = X_\Delta$ be a toric variety over k obtained from a regular complete fan (N, Δ) such that the anticanonical sheaf is generated by its global sections. Let $D = \sum_{\rho \in \Delta(1)} D_\rho$ and let*

$$D(\sigma) := D + \sum_{\rho \in \Delta(1)} \langle -m(\sigma), n_\rho \rangle D_\rho, \qquad \sigma \in \Delta_{\max}$$

be the effective anticanonical U-invariant Weil divisors described in (8.7).

Let T be the Néron-Severi torus of X and let $\pi : \mathcal{T} \to X$, $\mathcal{T} = X_0$ be the universal torsor constructed from the fan (N_0, Δ_0) (see (8.4)).

Let $\| \ \| : \mathbf{det}\, \mathrm{Tan}(X_{\mathrm{an}}(k)) \to [0, \infty)$ be the toric norm on the analytic anticanonical line bundle and let $\| \ \|_0 : \mathbf{det}\, \mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k)) \to [0, \infty)$ be the induced norm on $\mathcal{T}_{\mathrm{an}}(k)$. Then,

$$\|s(P)\|_0 := |f(P)| / (\sup_\sigma |x^{D(\sigma)}(P)|), \quad \sigma \in \Delta_{\max}$$

for any local analytic section $s = f(x_1, \dots, x_n) \dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ defined at $P \in \mathcal{T}_{\mathrm{an}}(k)$.

Proof. — Let du be a U-invariant global section of $\mathbf{det}\, \Omega^1_{U/k}$ of minimal $d\log$-type (see (3.28)) and ω be a T-invariant global section of $\mathbf{det}\, \Omega^1_{U/k}$ of minimal $d\log$-type (cf. (3.16)). Then $du_0 := \omega \otimes \pi^* du$ is a U_0-invariant global section of $\mathbf{det}\, \Omega^1_{U_0/k}$ of minimal $d\log$-type. The order norm of $\mathbf{det}\, \mathrm{Tan}(U_{0,\mathrm{an}}(k))$ is therefore (cf. (3.30)) the product norm of the order norm on

$$\mathbf{det}\, \mathrm{Tan}(U_{0,\mathrm{an}}(k) / U_{\mathrm{an}}(k))$$

and the pullback norm of the order norm on $\mathbf{det}\, \mathrm{Tan}(U_{\mathrm{an}}(k))$. Also, by definition, $\| \ \|_0$ is the product norm on $\mathbf{det}\, \mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k)$ of the order norm $\| \ \|_{\mathcal{T}/X}$ on $\mathbf{det}\, \mathrm{Tan}(\mathcal{T}_{\mathrm{an}}(k) / X_{\mathrm{an}}(k))$ and the pullback norm $\pi^* \| \ \|$ on $\pi^*_{\mathrm{an}}(\mathbf{det}\, \mathrm{Tan}(X_{\mathrm{an}}(k)))$. Hence by the definition of $\| \ \|$ (cf. (9.11)) one concludes that the restriction of $\| \ \|_0$ to $\mathbf{det}\, \mathrm{Tan}(U_{0,\mathrm{an}}(k)) \subset \mathbf{det}\, \mathrm{Tan}(X_{0,\mathrm{an}}(k))$ is the product norm of the order norm

$\| \|_1$ on $\mathbf{det}\,\mathrm{Tan}(U_{0,\mathrm{an}}(k))$ and the restriction of $\pi_{\mathrm{an}}^* \| \|_D$ to the trivial line bundle over $U_{0,\mathrm{an}}(k)$.

Therefore, since $dx_1 \wedge \cdots \wedge dx_n / x^D$ is of minimal $d\log$-type it follows that:

$$\begin{aligned}\left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P) \right\|_0 &= \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P) \right\|_1 \pi_{\mathrm{an}}^* \|1(P)\|_D \\ &= \pi_{\mathrm{an}}^* \|1(P)\|_D.\end{aligned}$$

Hence, if $s = f(x_1, \ldots, x_n) \dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ is defined at $P \in U_0(k)$, then

$$\begin{aligned}\|s(P)\|_0 &= |f(P)| \left\| \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} x^D(P) \right\|_0 / |x^D(P)| \\ &= |f(P)| \|1(\pi_{\mathrm{an}}(P))\|_D / |x^D(P)|\end{aligned}$$

where $\|1(\pi(P))\|_D$ means the norm of 1 regarded as a section of the analytic line bundle $V(\mathcal{O}(-D))_{\mathrm{an}}(k) \to X_{\mathrm{an}}(k)$ at the point $\pi(P) \in U(k)$. But it follows from (9.8) that

$$\|1(\pi(P))\|_D = \inf_\sigma |\chi^{m(\sigma)}(\pi(P))| = \inf_\sigma |x^D(P)/x^{D(\sigma)}(P)|$$

where σ runs over all maximal cones in Δ.

Therefore, if $s = f(x_1, \ldots, x_n) \dfrac{\partial}{\partial x_1} \wedge \cdots \wedge \dfrac{\partial}{\partial x_n}$ is defined at $P \in U_0(k)$, then

$$\|s(P)\|_0 = |f(P)| \inf_\sigma |1/x^{D(\sigma)}(P)|, \quad \sigma \in \Delta_{\max}.$$

The same statement for $P \in X_0(k)$ follows by continuity since $U_0(k)$ is dense in $X_0(k)$ and both sides define a norm on the anticanonical line bundle on $\mathcal{T}_{\mathrm{an}}(k)$. This completes the proof. □

The preceding proposition is related to the following description of the canonical splitting $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ of $\check{\pi} : X_0(k)/T(k)_{\mathrm{cp}} \to X(k)$ (cf. (9.5)) for non-archimedean fields k.

Proposition 9.13. — *Let k be a finite extension of $\mathbb{Q}_p$, let o be the maximal $\mathbb{Z}_p$-order in k and let (Δ, N) be a complete regular fan. Let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the toric o-morphism of toric o-schemes defined by the obvious morphism $(\Delta_0, N_0) \to (\Delta, N)$ of fans (cf. (8.6)). Then the following holds.*

(a) *$T(k)_{\mathrm{cp}}$ is the image of $\widetilde{T}(o)$ under the natural embedding of $\widetilde{T}(o)$ in $T(k)$.*
(b) *The obvious maps $\widetilde{X}_0(o)/\widetilde{T}(o) \to \widetilde{X}(o)$ and $\widetilde{X}(o) \to X(k)$ are isomorphsisms. $\check{\psi} : X(k) \to X_0(k)/T(k)_{\mathrm{cp}}$ is the unique map such that the following diagram*

commutes

$$\begin{array}{ccc}\widetilde{X}(o) & \xleftarrow{\sim} & \widetilde{X}_0(o)/\widetilde{T}(o)\\ \wr\downarrow & & \downarrow\\ \check{X}(k) & \xrightarrow{\check{\psi}} & X_0(k)/T(k)_{\mathrm{cp}}\end{array}$$

Proof

(a) This is true for any split o-torus $\widetilde{T} = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ with generic fibre T.

(b) It follows from the assumptions on Δ that $\widetilde{X}$ is a smooth proper scheme over o and hence that $\widetilde{X}(o) = X(k)$. The morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ is a (universal) torsor under $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$. Thus, by Grothendieck's version of Hilbert 90, it follows that $\widetilde{\pi}(\widetilde{X}_0(o)) = \widetilde{X}(o)$ and $\widetilde{X}_0(o)/\widetilde{T}(o) = \widetilde{X}(o)$. The map $\check{\psi}$ is defined (cf. (9.5)) by gluing the restrictions of the maps

$$\psi_{\sigma,\mathrm{an}} : U_\sigma(k) \longrightarrow U_{0,\sigma}(k)$$

to $C_\sigma(k)$ modulo elements in $\widetilde{T}(o)$. But the algebraic maps $\psi_\sigma : U_\sigma \to U_{0,\sigma}$ and $\psi_\sigma : U_{0,\sigma} \to U_\sigma$ are actually defined over o (i.e. they extend to toric morphisms between affine toric schemes over o). Therefore, ψ_σ sends a point in $U_\sigma(k)$ to a lifting to $\widetilde{X}(o) \subset X_0(k)$ of the corresponding point in $\widetilde{X}(o) = X(k)$. But this defines its class in $X_0(k)/T(k)_{\mathrm{cp}}$ uniquely, thereby proving the assertion. □

Proposition 9.14. — *Let k be a finite extension of $\mathbb{Q}_p$ and let X, o, $\widetilde{X}$ be as above. Then the following holds.*

(a) *The toric norm (cf. (9.11)) of the analytical anticanonical line bundle on $\widetilde{X}(o) = X(k)$ is equal to the model norm determined by $\widetilde{X}/o$.*

(b) *The restriction to $\widetilde{X}_0(o)$ of the induced norm $\| \ \|_0$ for $X_0(k)$ of the toric norm $\| \ \|$ coincides with the model norm for $\widetilde{X}_0(o)$.*

Proof

(a) The quotients of two norms on the analytical anticanonical line bundle on $X(k)$ define a continuous function $f : X(k) \to (0, \infty)$ (cf. (1.6)(b)). Hence it suffices to prove that the two norms coincide for the local section 1 of $\mathcal{O}(D)$ on the dense set $U(k)$. Then,

$$\|1(P)\|_D = |\widetilde{\psi}(P)|_D$$

by (9.7) so that the statement is an immediate consequence of (9.13)(b).

(b) This follows from (a) and (3.31). □

Remarks 9.15

(a) It follows from (9.14)(a) that the toric ν-adic norms on a smooth complete (split) toric variety over a number field form an adelic norm which we shall call the *toric adelic norm.* Also, by (4.7) or (9.14)(b) we have that the induced ν-adic norms on the universal torsor form an adelic norm on the principal universal torsor which we shall call the *induced toric adelic norm.*

(b) One can define toric norms on *twisted* toric varieties (cf. sec. 8) over locally compact fields. Batyrev and Tschinkel [**7**] define their norms $\| \|_D$ for arbitrary (twisted) toric varieties by means of the corresponding norms for the (split) toric varieties obtained after a base extension. It is also clear that one can define order norms (cf. (9.11)(a)) on twisted toric varieties by means of $d\log$-forms of minimal type over a splitting field (cf. (3.29), (3.30) for the case of non-split tori). We may therefore, just as in (9.11)(b), define toric norms on twisted toric varieties over locally compact fields as a product of the Batyrev-Tschinkel norm $\| \|_D$ and the order norm. One can deduce from (9.14) that the toric norms form an adelic norm also for twisted toric varieties.

(c) One can construct a canonical toric splitting of the map $\tilde{\pi}$ (cf. (9.3)) for principal universal torsors over twisted toric varieties (cf. the end of section 8).

Proposition 9.16. — *Let $X = X_\Delta$ be a toric variety over k defined by a regular complete d-dimensional fan (N, Δ) and let m be the Borel measure on $X(k)$ determined by the toric norm. Then,*

(a) $m(X(k)) = 2^{\dim \Delta} \operatorname{Card}(\Delta_{\max})$ *for* $k = \mathbb{R}$,

(b) $m(X(k)) = (2\pi)^{\dim \Delta} \operatorname{Card}(\Delta_{\max})$ *for* $k = \mathbb{C}$.

Proof. — Let $\sigma \in \Delta$ be a maximal cone and let $\{m^{(j)},\ 1 \le j \le d\}$ be the $\mathbb{Z}$-basis of M which is dual to the $\mathbb{Z}$-basis $\{n^{(j)},\ 1 \le j \le d\}$ of N, consisting of generators of the rays $\rho_1, \dots, \rho_d$ of σ. Then the d characters $\chi^{(j)} \in k[U]$, $1 \le j \le d$ corresponding to $\{m^{(j)},\ 1 \le j \le d\}$ form a set of coordinates $(z_1, \dots, z_d)$ for the affine toric variety $U_\sigma = \operatorname{Spec} k[z_1, \dots, z_d]$ with $U = \operatorname{Spec} k[z_1, z_1^{-1}, \dots, z_d, z_d^{-1}]$. Moreover, by definition (cf. the proof of (9.10))

$$C_\sigma(k) = \{(z_1, \dots, z_d) \in k^{(d)} : |z_j| \le 1 \text{ for } j = 1, \dots, d\}.$$

Finally, note that $\prod_{j=1}^d z_i = \chi^{-m(\sigma)}$ by (8.9)(iv).

The norm $\| \|_D$ for $D = D_1 + \cdots + D_n$ is defined by (cf. (9.2))

$$\|s(P)\|_D = \left|s(P)\chi^{m(\sigma)}(P)\right| = \left|\left(s\prod_{j=1}^d z_i\right)(P)\right|$$

for a local section σ of $\mathcal{O}(D)$, $D = D_1 + \cdots + D_n$ defined at $P \in C_\sigma(k)$. This implies that the toric norm (cf. (9.11)) of the section $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_d}$ of $\mathbf{det}\, \mathrm{Tan}\, X_{\mathrm{an}}(k)$ is equal to 1 for all $P \in C_\sigma(k)$. Hence the Borel measure m determined by the toric norm (cf. (1.12)) is given by $dz_1 \cdots dz_d$, so that

$$(9.17) \qquad m(C_\sigma(k)) = \prod_{j=1}^{d} \int_{|z_j| \leq 1} dz_j = (\int_{|z| \leq 1} dz)^d = 2^d (resp.\ (2\pi)^d)$$

for $k = \mathbb{R}$ (resp. $k = \mathbb{C}$).

We now show that

$$(9.18) \qquad m(C_\sigma(k) \cap C_\tau(k)) = 0$$

for any pair of (different) maximal cones σ, τ of Δ. This implies that

$$(9.19) \qquad m(X(k)) = \sum_{\sigma \in \Delta_{\max}} m(C_\sigma(k))$$

since $\bigcup_{\sigma \in \Delta_{\max}} C_\sigma(k) = X(k)$.

Let $\chi^{-m(\sigma)}$ (resp. $\chi^{-m(\tau)}$) σ, $\tau \in \Delta_{\max}$ be the unique character on U which generates $\mathcal{O}(D)$ on U_σ (resp. U_τ). Then $\chi^{m(\sigma)-m(\tau)}$ is a Laurent monomial in $(z_1, z_1^{-1}, \ldots, z_d, z_d^{-1})$ with $|\chi^{m(\sigma)-m(\tau)}| = 1$ on $C_\sigma(k) \cap C_\tau(k)$ (cf. (9.2)).

Now suppose that $\sigma \neq \tau$. Then $\chi^{m(\sigma)-m(\tau)} \neq 1$ so that one of the variables, say z_1 occurs in the Laurent monomial $\chi^{m(\sigma)-m(\tau)}$. Let $r_j = |z_j|$ for $j = 1, \ldots, d$. Then there are rational numbers α_j for $j = 2, \ldots, d$ such that

$$r_1 = \prod_{j=2}^{d} r_j^{\alpha_j}$$

for each point on $C_\sigma(k) \cap C_\tau(k)$ with $\prod_{j=1}^{d} z_j \neq 0$. This implies that in case $k = \mathbb{R}$ there exists finitely many bounded subsets A_i of $\mathbb{R}^{d-1}$ and finitely many C^1-maps $f_i : A_i \to C_\sigma(\mathbb{R}) \cap C_\tau(\mathbb{R})$ satisfying a Lipschitz condition (cf. [**39**, Chap. XI, §1]) such that

$$C_\sigma(k) \cap C_\tau(k) = \bigcup_i f_i(A_i).$$

Therefore, $m(C_\sigma(k) \cap C_\tau(k)) = 0$ by lemma 1.3 in (op. cit).

For $k = \mathbb{C}$, we regard $U_\sigma(k) = \mathbb{C}^d$ as a real analytic manifold $\mathbb{R}^{2d}$ with coordinates $x_j = \Re z_j$, $y_j = \Im z_j$ and introduce polar coordinates $x_j = r_j \cos\theta_j$, $y_j = r_j \sin\theta_j$.

There is then a parametrization of the real analytic subset $C_\sigma(k) \cap C_\tau(k) \subset \mathbb{R}^{2d}$ by finitely many Lipschitz C^1-maps $f_i : A_i \to C_\sigma(k) \cap C_\tau(k)$ from bounded subsets A_i of $\mathbb{R}^{2d-1}$ so that $m(C_\sigma(k) \cap C_\tau(k)) = 0$ by lemma 1.3 in [**39**, Chap. XI]. Hence

(9.19) holds also for $k = \mathbb{C}$. To complete the proof of the proposition, combine (9.17) and (9.19). □

Note that $m(C_\sigma(k) \cap C_\tau(k)) \neq 0$ in the non-archimedean case. The volume $m(X(k))$ can then computed by means of (2.15) and (9.14)(a). We shall in (11.50) consider these volumes in a concrete case.

10. Toric height functions and Tamagawa volumes of universal torsors

Let $X = X_\Delta$ be a smooth complete toric variety over a number field k defined by a complete regular fan. We shall in this section use the canonical toric splittings of universal torsors over such varieties to define heights and to interpret the main term $\alpha(X)\tau_\varepsilon(X, \|\ \|)$ in the conjectured asymptotic formula (7.12) as an adelic volume using the induced measures on universal torsors. We shall not need to assume that $X = X_\Delta$ is a Fano variety. In (10.14) we assume that the anticanonical sheaf of $X = X_\Delta$ is generated by its global sections. This is used to give a very concrete description of the anticanonical height function and of the adelic volume corresponding to the main term $\alpha(X)\tau_\epsilon(X, \|\ \|)$. But the remaining arguments of the section go through without this assumption if one works with the original definition (10.4) of the anticanonical height function.

We shall in this section let k denote a number field and o denote the maximal $\mathbb{Z}$-order in k. Otherwise we will keep the notations in section 8. We shall thus by Δ denote a complete regular fan in $N = \mathrm{Hom}(M, \mathbb{Z})$ and put $M_0 = \mathbb{Z}^{\Delta(1)}$, $N_0 = \mathrm{Hom}(M_0, \mathbb{Z})$. We denote by Δ_0 the "pullback" fan in N_0 described in (8.4) and by Δ_1 the fan in $N_1 = N_0$ described after (8.5). There is a morphism of fans from (Δ_0, N_0) to (Δ, N) which defines a toric k-morphism $\pi : X_0 \to X$ of split toric k-varieties. This morphism is defined over $\mathbb{Q}$, but we shall in this section let $X = X_\Delta$ denote the corresponding (geometrically integral) k-variety obtained by base extension. The morphism π is by (8.5) a universal torsor over X and we shall also write $\mathcal{T}$ instead of X_0.

The morphism of fans from (Δ_0, N_0) to (Δ, N) defines also a toric morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ of toric schemes over $\mathbb{Z}$. But we shall in this section let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the toric o-morphism obtained by base extension (cf. (8.6)(b)). The restriction of $\widetilde{\pi}$ to the generic fibres over k is thus the morphism $\pi : X_0 \to X$ of toric varieties over k. The morphism $\widetilde{\pi}$ is by (8.5) a universal torsor over $\widetilde{X}$ under the Néron-Severi o-torus $\widetilde{T} = \mathbf{D}(\mathbf{Pic}_{\widetilde{X}/o})$ as defined in (5.14). We shall call $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ the *principal* universal torsor.

Recall (cf. (8.2)) that there is an exact sequence of finitely generated free $\mathbb{Z}$-modules

$$(10.1) \qquad 1 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow \mathrm{Pic}\, X \longrightarrow 1.$$

We may endow $\operatorname{Hom}(L,\mathbb{Z})$ with the trivial fan consisting of the cone $\{0\}$ for each of the three lattices $L = M$, $\mathbb{Z}^{\Delta(1)}$, $\operatorname{Pic} X$. The morphism between these fans defines toric k-morphisms between toric varieties over k as well as toric o-morphisms between toric o-schemes. We obtain thereby an exact sequence of split k-tori:

$$(10.2) \qquad 1 \longrightarrow T \longrightarrow U_0 \longrightarrow U \longrightarrow 1$$

as well as an exact sequence of split o-tori $G_{m,o} \times \cdots \times G_{m,o}$:

$$(10.3) \qquad 1 \longrightarrow \widetilde{T} \longrightarrow \widetilde{U}_0 \longrightarrow \widetilde{U} \longrightarrow 1.$$

The exact sequence of f.g. torsion-free abelian groups in (10.1) gives rise to an exact sequence of constant group schemes over k (resp. o). The sequences in (10.2) (resp (10.3)) are the "dual" exact sequences of tori over k (resp. o) under the contravariant equivalence between torsion-free constant group schemes and tori described in section 5.

The assumption that Δ is complete and regular implies that $\widetilde{X}/o$ is proper and smooth. There is a natural $\widetilde{U}_0$-equivariant open toric immersion of toric schemes $\widetilde{X}_0 \subset \widetilde{X}_1$ corresponding to the inclusion of fans $\Delta_0 \subset \Delta_1$. It was shown in section 8 that the generic fibre X_1 of $\widetilde{X}_1$ is canonically isomorphic to an affine space $\mathbb{A}_k^n = \operatorname{Spec} k[x_\rho]$ where n is the cardinality of $\Delta(1)$ and the variables are indexed by the rays ρ of Δ. Moreover, $X_0 \subset X_1 = \mathbb{A}_k^n$ is the open complement of the closed subscheme defined by the monomials

$$x^{\underline{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho, \qquad \sigma \in \Delta_{\max}.$$

There is a similar description of $\widetilde{X}_0 \subset \widetilde{X}_1$. The latter scheme is equal to $\mathbb{A}_o^n = \operatorname{Spec} o[x_\rho]$ where ρ runs over $\Delta(1)$ and $\widetilde{X}_0$ is the open complement in $\widetilde{X}_1$ of the closed subscheme defined by the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$.

Let W be the set of places of k and denote by $\|\ \|_\nu$, $\nu \in W$ the toric ν-adic norm (see (9.11)) of the analytic anticanonical line bundle of $X_{\text{an}}(k_\nu)$. It follows from (9.14) that the toric norms $\|\ \|_\nu$, $\nu \in W$ form an adelic norm $\|\ \|$ for X, which we shall call the *adelic toric norm* for $X = X_\Delta$.

The adelic toric norm $\|\ \|$ gives rise to a height function $H : X(k) \to (0,\infty)$ with

$$(10.4) \qquad H(P) = \prod_{\nu \in W} \|s(P)\|_\nu^{-1}$$

for a local section s of ω_X^{-1} at $P \in X(k)$ with $s(P) \neq 0$ (cf. (7.2)). We shall call this function the *toric anticanonical height function* or simply the *toric height function* when it is clear that we consider the anticanonical linear system.

We now give alternative descriptions of the toric height function. We first represent ω_X as the tensor product of the invertible sheaves

$$\omega_X = \mathcal{O}(-D_0) \otimes (\omega_X \otimes \mathcal{O}(D_0)), D_0 = \sum_{\rho \in \Delta(1)} D_\rho.$$

Let $\| \ \|_{D,\nu}$, $\nu \in W$ be the ν-adic norm on

$$V(\mathcal{O}(-D_0))_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu)$$

described in (9.2). The toric norm $\| \ \|_\nu$ on $\mathbf{det}\,\mathrm{Tan}\,X_{\mathrm{an}}(k_\nu) \to X_{\mathrm{an}}(k_\nu)$ is by definition the product of this norm and the order norm (cf. (9.11)) $\| \ \|_\nu^\#$ on

$$\mathbf{det}(\mathrm{Tan}\,X_{\mathrm{an}}(k_\nu)) \otimes V(\mathcal{O}(D_0))_{\mathrm{an}}(k_\nu) \longrightarrow X_{\mathrm{an}}(k_\nu).$$

Proposition 10.5. — *The toric height function $H : X(k) \to (0,\infty)$ is equal to the product*

$$\prod_{\nu \in W} \|s_1(P)\|_{D,\nu}^{-1}$$

where s_1 is a local section of $\mathcal{O}(D)$ for $D = D_0$ at $P \in X(k)$ with $s_1(P) \neq 0$.

Proof. — Choose local sections s_1 resp. s_2 at the k-point P (defined over k) of $\mathcal{O}(D)$ resp. $\omega_X^{-1} \otimes \mathcal{O}(-D)$ and put

$$H_1(P) = \prod_{\nu \in W} \|s_1(P)\|_{D,\nu}^{-1},$$

$$H_2(P) = \prod_{\nu \in W} \|s_2(P)\|_\nu^{\sharp -1}.$$

Then $H_1(P)$ and $H_2(P)$ are independent of the choice of s_1 and s_2 and give well defined height functions $H_1 : X(k) \to (0,\infty)$ and $H_2 : X(k) \to (0,\infty)$ such that $H(P) = H_1(P)H_2(P)$.

The invertible sheaf $\omega_X^{-1} \otimes \mathcal{O}(-D)$ is trivial and we may choose s_2 to be the dual of an algebraic differential form du of minimal $d\log$-type on $U_{\mathrm{an}}(k)$ (cf. (3.28)) for all points $P \in X(k)$. Then $\|s_2(P)\|_\nu^\sharp = |du(s_2(P))|_\nu = 1$ for all $\nu \in W$ (cf. the proof of (9.11)(a)) and $H_2(P) = 1$ for all $P \in X(k)$. This completes the proof. □

Definition 10.6

(a) Let $D \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor on X and $P \in X(k)$. Then,

$$H_D(P) := \prod_{\nu \in W} \|s(P)\|_\nu^{-1},$$

$$h_D(P) := \log H_D(P) = -\sum_{\nu \in W} \log \|s(P)\|_\nu$$

where s is a local section of $\mathcal{O}(D)$ at $P \in X(k)$ with $s(P) \neq 0$.

(b)
$$H_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow (0,\infty)$$
resp.
$$h_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{R}$$
is the pairing
$$H_\Delta(P, D) := H_D(P)$$
resp.
$$h_\Delta(P, D) := h_D(P).$$
Note that h_Δ is additive on the right hand side.

One can extend the restriction of H_D to $U(k)$ to a height function
$$F_{D,A} : U(A_k) \longrightarrow (0,\infty)$$
by choosing $s = 1$ as local section. To see this, we make use of the local ν-adic sections
$$\check{\psi}_\nu : X(k_\nu) \longrightarrow X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$$
of
$$\check{\pi}_\nu : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow X(k_\nu)$$
in (9.5) and the map
$$|\ |_{D,\nu} : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \longrightarrow \mathbb{R}$$
described in (9.6). Then (cf. (9.7)):

(10.7)
$$\|1(P_\nu)\|_{D,\nu}^{-1} = |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1}$$
for all $P_\nu \in U(k_\nu)$.

Let $\widetilde{X}$ be the smooth proper toric o-model above and let $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ be the principal universal torsor. This model can be used to define the splitting $\check{\psi}_\nu$ at non-archimedean places ν (see (9.13)). It is the unique map for which the following diagram commutes:

(10.8)
$$\begin{array}{ccc} \widetilde{X}(o_\nu) & \xleftarrow{\sim} & \widetilde{X}_0(o_\nu)/\widetilde{T}(o_\nu) \\ \wr\downarrow & & \downarrow \\ \check{X}(k_\nu) & \xrightarrow{\check{\psi}_\nu} & X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \end{array}$$

There is further an obvious commutative diagram:

(10.9)
$$\begin{array}{ccc} \widetilde{U}(o_\nu) & \xleftarrow{\sim} & \widetilde{U}_0(o_\nu)/\widetilde{T}(o_\nu) \\ \downarrow & & \downarrow \\ \widetilde{X}(o_\nu) & \xleftarrow{\sim} & \widetilde{X}_0(k_\nu)/\widetilde{T}(o_\nu) \end{array}$$

Hence, if $P_\nu \in \mathrm{Im}(\widetilde{U}(o_\nu) \to U(k_\nu))$, then

$$\check{\psi}(P_\nu) \in \mathrm{Im}(\widetilde{U}(o_\nu) \to \mathcal{T}(k_\nu)/T(k_\nu)_{\mathrm{cp}})$$

so that $\|1(P_\nu)\|_{D,\nu}^{-1} = |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1}$ by the remark in (9.6)(b). This implies the following result:

Proposition 10.10. — *Let $P_A \in U(A_k)$ be an adelic point and let*

$$\{P_\nu\}_{\nu\in W} \in \prod_{\nu\in W} U(k_\nu)$$

be the corresponding set of k_ν-points. Then the products

$$F_{D,A}(P_A) := \prod_{\nu\in W} \|1(P_\nu)\|_{D,\nu}^{-1} = \prod_{\nu\in W} |\check{\psi}_\nu(P_\nu)|_{D,\nu}^{-1}$$

are absolute convergent with at most finitely many factors different from 1.

Let

$$f_{D,A}(P_A) = \log F_{D,A}(P_A) = -\sum_{\nu\in W} \log \|1(P_\nu)\|_{D,\nu}.$$

It is an immediate consequence of the definition that the pairing

$$f_{\Delta,A} : U(A_k) \times \mathbb{Z}^{\Delta(1)} \to (0,\infty) \tag{10.11}$$

with $f_{\Delta,A}(P_A, D) := f_{D,A}(P_A)$ is additive on the right hand side. One has also the following result:

Proposition 10.12. — *Let $P \in U(k)$. Then $h_D(P)$ and $H_D(P)$, $D \in \mathbb{Z}^{\Delta(1)}$ depend only on the linear equivalence class of D in* $\mathrm{Pic}\, X$.

Proof. — h_Δ is additive with respect to $\mathbb{Z}^{\Delta(1)}$. It therefore suffices to prove that $H_D(P) = 1$ for principal divisors

$$D = \sum_{\rho\in\Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}, \quad a_\rho = \langle m, n_\rho\rangle,\ m \in M.$$

But $x^D(\check{\psi}_\nu(P)) = \chi^m(P)$ for all ν since $\check{\psi}_\nu$ is a section to $\check{\pi}_\nu$ (cf. the proof of (9.4)).

This implies in its turn that $|\check{\psi}_\nu(P_\nu)|_{D,\nu} = |\chi^m(P)|_\nu$ and hence that

$$H_D(P) := \prod_{\nu\in W} \|1(P)\|_{D,\nu}^{-1} = \prod_{\nu\in W} |\check{\psi}_\nu(P)|_{D,\nu}^{-1} = \prod_{\nu\in W} |\chi^m(P)|_\nu^{-1} = 1$$

by the product formula in algebraic number theory. This finishes the proof. □

Remarks 10.13

(a) The properties (10.11) and (10.12) of the height functions are essentially due to Batyrev and Tschinkel [**7**] (cf. also [**4**]) although they state them in a somewhat different language. They make essential use of these properties in their study of Manin's zeta functions for counting functions on $U(k)$ (cf. the comments after (10.28)). We shall not use the pairing in (10.11) further. Instead, we will extend the restriction of h_Δ to

$$U(k) \times \mathbb{Z}^{\Delta(1)} = U_0(k)/T(k) \times \mathbb{Z}^{\Delta(1)}$$

to a pairing

$$h_{\Delta,A} : X_0(A_k)/T(k) \times \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{R}$$

which factorizes over $X_0(A_k)/T(k) \times \operatorname{Pic} X$.

(b) The treatment of Tamagawa volumes and local heights by means of universal torsors and the canonical toric splitting of

$$\check{\psi}_\nu : X_0(k_\nu)/T(k_\nu)_{\mathrm{cp}} \to X(k_\nu)$$

has some similarities with Bloch's torsor theoretic approach to the Birch-Swinnerton-Dyer conjecture and to the local Néron symbols (cf. [**8**], [**48**]) for abelian varieties. The analogy is not perfect since the pairing $H_{\Delta,A} : U(A_k) \times \mathbb{Z}^{\Delta(1)} \to (0,\infty)$ is not multiplicative on the left hand side. But it is still suggestive to think of universal torsors for toric varieties as the analog of biextensions for abelian varieties.

Proposition 10.14. — *Let $X = X_\Delta$ be a smooth complete k-variety defined by a regular complete fan Δ. Let $D = \sum_{\rho\in\Delta(1)} D_\rho$ and suppose that $\mathcal{O}(D)$ is generated by its global sections. Let $\chi^{m(\sigma)}(P)$, $\sigma \in \Delta_{\max}$ be the unique character on U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D)$ on U_σ and let*

$$D(\sigma) = D + \sum_{\rho\in\Delta(1)} \langle -m(\sigma), n_\rho\rangle D_\rho$$

be the effective Weil divisors in (8.7). Let

$$H_0 : X_0(k) \longrightarrow (0,\infty)$$

be the composition of $\pi : X_0(k) \to X(k)$ and the toric anticanonical height function $H : X(k) \to (0,\infty)$. Then,

$$H_0(P_0) = \prod_{\nu\in W} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu .$$

Proof. — Let $P \in X(k)$ be the image of $P_0 \in X_0(k)$ and let $\tau \in \Delta_{\max}$ be chosen such that $P \in U_\tau$. Further, let du be a U-invariant global section of $\omega_{U/k}$ of minimal $d\log$-type regarded as a global section of $\omega_{X/k} \otimes \mathcal{O}(D)$ (cf. (8.8)). Then $\chi^{m(\tau)}du$ is a local section of $\omega_{X/k}$ which does not vanish at P. Let s be local section of ω_X^{-1} at P which is the inverse of $\chi^{m(\tau)}du$. Then, by definition of the toric ν-adic norm $\| \ \|$ (cf. (9.11)) and (9.8)), we have:

$$\|s(P)\|_\nu = \|\chi^{-m(\tau)}(P)\|_{D,\nu} = \inf_{\sigma\in\Delta_{\max}} |\chi^{m(\sigma)-m(\tau)}(P)|_\nu$$

for any $P \in X(k)$.

From $P \in U_\tau$ it follows that $x^{D(\tau)}(P_0) \neq 0$ (cf. (8.8)(a)). Hence

$$x^{D(\sigma)}(P_0)/x^{D(\tau)}(P_0)$$

is defined and equal to $\chi^{m(\tau)-m(\sigma)}(P)$. We may thus rewrite the preceding formula in the following form:

$$\|s(P)\|_\nu^{-1} = \sup_{\nu\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu/|x^{D(\tau)}(P_0)|_\nu.$$

By the Artin-Whaples product formula for number fields we have:

$$\prod_{\nu\in W} |x^{D(\tau)}(P_0)|_\nu = 1.$$

The assertion follows from the last two equalities. □

Notation 10.15

(a) Let B be a positive integer. Then

$$c(B) := \mathrm{Card}\{P \in U(k) : H(P) = B\}.$$

(b) Let $B \geq 1$ be a real number. Then

$$\mathcal{C}(B) := \mathrm{Card}\{P \in U(k) : H(P) \leq B\}.$$

(c) Let $B \geq 1$ be a real number. Then

$$\mathcal{D}(B) := \sum_{m=1}^{[B]} c(m)/m.$$

These numbers are finite (cf. the proof of (10.28) below).

Since X is a *split* toric variety, it follows that the Néron-Severi torus T is a split k-torus $T = \mathbb{G}_m^{(r)}$ and that $H^1_{\text{ét}}(k, T) = 0$. The principal universal torsor $\pi : X_0 \to X$ is thus the only universal torsor up to isomorphism. In particular, $\pi(X_0(k)) = X(k)$.

The conjecture (7.12) is therefore equivalent to the conjecture:

(10.16) $$\mathcal{C}(B) = \alpha(X)\tau_\varepsilon(X, \| \ \|)B(\log B)^{r-1}(1 + o(1))$$

where $r = \operatorname{rk}\operatorname{Pic} X$, $\alpha(X) = \alpha_{\text{Peyre}}(X)$ is the constant of Peyre (cf. the lines before (7.7)) and $\tau_\varepsilon(X, \| \ \|)$ is the Tamagawa number defined in (6.18) for the trivial (and only) class ε in $H^1_{\text{ét}}(k, T)$.

Moreover, $\tau_\varepsilon(X, \| \ \|) = \tau(X, \| \ \|)$, since $\operatorname{Pic}(\overline{X})$ is a Galois permutation module (cf. (6.19)). Therefore (cf. (5.21))

$$\tau_\varepsilon(X, \| \ \|) = \overline{\Theta}_\Sigma^{-1}(T(A_k)/T(k))m_S(\pi(\mathcal{T}_S)) \prod_{\nu \in W_{\text{fin}} - \Sigma} n_\nu(\widetilde{\mathcal{T}}(o_\nu)). \tag{10.17}$$

Peyre **[52]** verified that (7.12) for holds for some classes of toric varieties over $\mathbb{Q}$ and Batyrev and Tschinkel **[4]** establish a version of (7.12) for arbitrary toric varieties. Their proof depends on the claim that any effective divisor on X is linearly equivalent to an effective divisor with support outside U. This fact is well-known (cf. e.g. the lemma on p. 66 in **[24]**). We may therefore use the following equivalent definition of $\alpha(X)$ for smooth complete toric varieties.

Definition 10.18

Let $V = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$, let L be the $\mathbb{Z}$-lattice $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{Z})$ in V and let $d\nu$ be the unique Haar measure on V such that $\operatorname{Vol}(V/L) = 1$.

Let $\lambda : V \to \mathbb{R}$ be the linear form obtained by evaluating at the anticanonical class and let $\Lambda : C_c(V) \to C_c(\mathbb{R})$ be the unique positive linear map such that:

$$\int_V g d\nu = \int_{\mathbb{R}} \Lambda(g) dx$$

for each function $g \in C_c(V)$.

Let $V_x = \lambda^{-1}(x)$ for $x \in \mathbb{R}$ and let $d\nu_x$ be the positive Borel measure on V_x corresponding to the restriction $\Lambda_x : C_c(V_x) \to \mathbb{R}$ of Λ to V_x. Finally, let $\sigma_{\text{eff}}(X, U) \subset V$ be the cone of all homomorphisms $\varphi : \operatorname{Pic} X \to \mathbb{R}$ such that $\varphi([D]) \geq 0$ for the class $[D] \in \operatorname{Pic} X$ of each effective divisor on X with support outside U. Then,

$$\alpha(X) = \int_{D_1} g d\nu_1$$

for $D_1 = \sigma_{\text{eff}}(X, U) \cap V_1$.

Remark 10.19. — Let $x \geq 0$ and let $D_x = \sigma_{\text{eff}}(X, U) \cap V_x$. Then an easy substitution of variables implies that:

$$\int_{D_x} d\nu_x = x^{r-1}\alpha(X)$$

for $r = \operatorname{rk}\operatorname{Pic} X$.

Therefore, if E_b is the set of all $\varphi \in \sigma_{\text{eff}}(X,U)$ with $0 \le \lambda(\varphi) \le b$, and g is the characteristic function of E_b, then

$$\mathrm{Vol}(E_b) = \int_V g d\nu = \int_0^b \Lambda(g) dx = \alpha(X) \int_0^b x^{r-1} dx = \alpha(X) b^r / r.$$

In particular, $\alpha(X) = \mathrm{Vol}(E_1)/r$. This completes the remark.

Now suppose that (10.16) holds. Then, by partial summation we get that:

$$\begin{aligned} \mathcal{D}(B) &= \mathcal{C}(B)/(B+1) + \sum_{m=1}^{B} \mathcal{C}(m)/m(m+1) \\ &= \sum_{m=1}^{B} \mathcal{C}(m)/m^2 + (\log B)^{r-1} o(1). \end{aligned}$$

Also, when (10.16) can be proved one usually has enough information to deduce that

$$\sum_{m=1}^{B} \mathcal{C}(m)/m^2 = \alpha(X) \tau_\varepsilon(X, \| \ \|) \sum_{m=1}^{B} (\log m)^{r-1}/m + (\log B)^r o(1).$$

We may further replace

$$\sum_{m=1}^{B} (\log m)^{r-1}/m \quad \text{by} \quad \int_1^B (\log x)^{r-1} \frac{dx}{x}$$

and still have an error term of order $(\log B)^r o(1)$. This suggests the following conjecture, which in a sense is a weak version of (10.16):

(10.20) $$\mathcal{D}(B) = \mathrm{Vol}(E_1) \tau_\varepsilon(X, \| \ \|) (\log B)^{\mathrm{rk\,Pic}\, X} (1 + o(1)).$$

One advantage with this alternative conjecture is that it is easy to give an interpretation of

$$\mathrm{Vol}(E_1) \tau_\varepsilon(X, \| \ \|) (\log B)^r$$

as an adelic volume (cf. (10.22)) of a compact subset of $X_0(A_k)/X_0(k)$. To see this, we need some further notations.

Notations 10.21

(a)

$$T(A_k)_{\mathrm{cp}} = \prod_{\nu \in W} T(k_\nu)_{\mathrm{cp}}$$

is the maximal compact subgroup of $T(A_k)$.

(b)

$$\check{\psi}_A : X(A_k) \longrightarrow X_0(A_k)/T(A_k)_{\rm cp}$$

is the continuous product map of all $\check{\psi}_\nu : X(k_\nu) \to X_0(k_\nu)/T(k_\nu)_{\rm cp}$ (cf. (10.8)).

(c)

$$\check{\sigma}_\nu : T(k_\nu)/T(k_\nu)_{\rm cp} \times X_0(k_\nu)/T(k_\nu)_{\rm cp} \longrightarrow X_0(k_\nu)/T(k_\nu)_{\rm cp}$$

is the continuous map induced be the analytic action

$$\sigma_\nu : T(k_\nu) \times X_0(k_\nu) \longrightarrow X_0(k_\nu)$$

(cf. the lines before (3.10)).

(d)

$$\check{\rho}_\nu = (\check{\sigma}_\nu, pr_2) : T(k_\nu)/T(k_\nu)_{\rm cp} \times X_0(k_\nu)/T(k_\nu)_{\rm cp} \\ \longrightarrow X_0(k_\nu)/T(k_\nu)_{\rm cp} \times_{X(k_\nu)} X_0(k_\nu)/T(k_\nu)_{\rm cp}$$

is the homeomorphism induced by the analytic isomorphism

$$\rho_\nu = (\sigma_\nu, pr_2) : T(k_\nu) \times X_0(k_\nu) \longrightarrow X_0(k_\nu) \times_{X(k_\nu)} X_0(k_\nu)$$

(cf. (3.3)(b) and the lines before (3.11)).

(e)

$$\check{\xi}_\nu : X_0(k_\nu)/T(k_\nu)_{\rm cp} \longrightarrow T(k_\nu)/T(k_\nu)_{\rm cp}$$

is the continuous map composed of (cf. (9.5))

$$((\check{\psi}_\nu \check{\pi}_\nu), \mathrm{id}) : X_0(k_\nu)/T(k_\nu)_{\rm cp} \longrightarrow X_0(k_\nu)/T(k_\nu)_{\rm cp} \times_{X(k_\nu)} X_0(k_\nu)/T(k_\nu)_{\rm cp}$$

$\check{\rho}_\nu^{-1}$ and $pr_1 : T(k_\nu)/T(k_\nu)_{\rm cp} \times X_0(k_\nu)/T(k_\nu)_{\rm cp} \longrightarrow T(k_\nu)/T(k_\nu)_{\rm cp}$.

(f)

$$\check{\sigma}_A : T(A_k)/T(A_k)_{\rm cp} \times X_0(A_k)/T(A_k)_{\rm cp} \longrightarrow X_0(A_k)/T(A_k)_{\rm cp}$$

is the continuous map induced by (cf. (5.3))

$$\sigma_A : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k).$$

(g)

$$\check{\rho}_A = (\check{\sigma}_A, pr_2) : T(A_k)/T(A_k)_{\rm cp} \times X_0(A_k)/T(A_k)_{\rm cp} \\ \longrightarrow X_0(A_k)/T(A_k)_{\rm cp} \times_{X(A_k)} X_0(A_k)/T(A_k)_{\rm cp}$$

is the homeomorphism induced by the homeomorphism

$$\rho_A = (\sigma_A, pr_2) : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k) \times_{X(A_k)} X_0(A_k)$$

in (5.3).

(h)
$$\check{\xi}_A : X_0(A_k)/T(A_k)_{\mathrm{cp}} \longrightarrow T(A_k)/T(A_k)_{\mathrm{cp}}$$
is the continuous composite map obtained by replacing all the maps in (e) by their adelic counterparts.

(i)
$$\xi_A^1 : X_0(A_k)/T^1(A_k) \longrightarrow T(A_k)/T^1(A_k)$$
is the unique continuous map such that
$$\begin{tikzcd} \check{\xi}_A : X_0(A_k)/T(A_k)_{\mathrm{cp}} \arrow[r] \arrow[d] & T(A_k)/T(A_k)_{\mathrm{cp}} \arrow[d] \\ \xi_A^1 : X_0(A_k)/T^1(A_k) \arrow[r] & T(A_k)/T^1(A_k) \end{tikzcd}$$
commutes.

(j)
$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$
is the map obtained by composing the obvious map
$$X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T^1(A_k)$$
with
$$\xi_A^1 : X_0(A_k)/T^1(A_k) \longrightarrow T(A_k)/T^1(A_k)$$
and (cf. (5.18))
$$T(A_k)/T^1(A_k) = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}).$$

Proposition 10.22. — *Let B be a positive real number, $b = \log B$ and let $\mathbf{K}_B$ be the inverse image of $E_b \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$ under $\overline{\xi}_A : X_0(A_k)/T(k) \to \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. Let n_ν, $\nu \in W$ be the Borel measure on $X_0(k_\nu)$ determined by the induced norms of the toric norms on $X(k_\nu)$ (cf. (9.12)) and let n_A be the (restricted) product measure on $X_0(A_k)$ (cf. (5.16)) of these measures. Finally, let $\overline{n}_A$ be the quotient measure of n_A on $X_0(A_k)/T(k)$ (cf. (5.23)). Then,*
$$\overline{n}_A(\mathbf{K}_B) = \mathrm{Vol}(E_1)\tau_\varepsilon(X, \|\ \|)(\log B)^r.$$

Proof. — The translation invariance of n_A under $\sigma_A : T(A_k) \times X_0(A_k) \to X_0(A_k)$ implies that $\overline{n}_A$ is translation invariant under the action of $T(A_k)/T(k)$. Let $\mathcal{F} \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$ be a fundamental domain with respect to
$$\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{Z}) \subset \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}).$$
Then,
$$\overline{n}_A(\mathbf{K}_B)/\overline{n}_A(\overline{\xi}_A^{-1}(\mathcal{F})) = \mathrm{Vol}(E_{\log B})/\mathrm{Vol}(\mathcal{F}) = (\log B)^r\, \mathrm{Vol}(E_1).$$

Also, $\tau_\varepsilon(X, \| \ \|) = \overline{n}_A(\overline{\xi}_A^{-1}(\mathcal{F}))$ by (5.24). This completes the proof. $\square$

Thus, by (10.22), (10.20) is equivalent to the conjecture:

$$\mathcal{D}(B) = \overline{n}_A(\mathbf{K}_B)(1 + o(1)). \tag{10.23}$$

To understand the relation between the adelic volume $\overline{n}_A(\mathbf{K}_B)$ and the counting function

$$\mathcal{D}(B) := \sum_{\{P \in U(k): H(P) \leq B\}} 1/H(P)$$

we extend the toric height function

$$H : U(k) = U_0(k)/T(k) \longrightarrow \mathbb{R}$$

to a function on $X_0(A_k)/T(k)$ by means of the following result.

Proposition 10.24. — *Let $P \in U(k) = U_0(k)/T(k) \subset X_0(A_k)/T(k)$ and let $D \in \mathbb{Z}^{\Delta(1)}$ be a U-invariant Weil divisor. Let $V = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{R})$ and let $\lambda_D : V \to \mathbb{R}$ be the linear form obtained by evaluating at the class of D in* $\mathrm{Pic}\, X$. *Then,*

$$h_D(P) = \lambda_D \overline{\xi}_A(P).$$

In particular, $\overline{\xi}_A(P) \in \sigma_{\mathrm{eff}}(X, U)$.

Proof

$$H_D(P) = \prod_{\nu \in W} |\check{\psi}_\nu(P)|_{D,\nu}^{-1} = \prod_{\nu \in W} \|1(P)\|_{D,\nu}^{-1}$$

by (10.7) and (10.10). Let $P_0 \in U_0(k)$ be a lifting of P. Then $P_0 = \check{\xi}_\nu(P_0)\check{\psi}_\nu(P)$ in $U_0(k_\nu)/T(k_\nu)_{\mathrm{cp}}$ for each place ν of k by the definition of $\check{\xi}_\nu$. Therefore, since

$$\prod_{\nu \in W} |P_0|_{D,\nu}^{-1} = 1$$

(cf. (9.6)), we conclude that

$$H_D(P) = \prod_{\nu \in W} |\check{\xi}_\nu(P)|_{D,\nu}.$$

Hence,

$$h_D(P) = \log H_D(P) = \lambda_D(\overline{\xi}_A(P))$$

by the definitions of $| \ |_{D,\nu}$ (cf. (9.6)) λ_D and ξ_A^1. To prove the last statement, it suffices to show that $\log H_D(P) \geq 0$ for all effective divisors. But this follows from the first formula for $H_D(P)$ since $\|1(P)\|_D \leq 1$ by (9.2). This completes the proof. $\square$

To understand the conjecture $\lim_{B\to\infty} \mathcal{D}(B)/\overline{n}_A(\mathbf{K}_B) = 1$ in (10.23), recall that the measure $\overline{n}_A$ on $X_0(A_k)/T(k)$ is the quotient measure of the measure n_A on $X_0(A_k)$ which is the restricted product of the measures

$$dx_1 dx_2 \cdots dx_n/(\sup_\sigma |x^{D(\sigma)}(P_0)|_\nu), \sigma \in \Delta_{\max}, P_0 \in X_0(k_\nu)$$

with notations as in (9.12). Here we make use of the toric open embedding

$$X_0 \subset X_1 = \operatorname{Spec} k[x_\rho] = \mathbb{A}_k^n$$

and an ordering of the rays $\sigma \in \Delta(1)$.

Recall that

$$\mathcal{D}(B) := \sum 1/H_0(P_0)$$

where P_0 runs over all classes in $U_0(k)/T(k)$ such that (cf. (10.14)):

$$(') \qquad x_1 x_2 \cdots x_n \neq 0,$$

$$('') \qquad H_0(P_0) = \prod_{\nu \in W} \sup_{\sigma \in \Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu \leq B.$$

These two conditions can be reformulated by means of (10.24). Since $\overline{\xi}_A(P_0) \in \sigma_{\text{eff}}(X, U)$ for all $P_0 \in U_0(k)/T(k)$ one gets the following equivalent conditions:

10.25 (i) $x_1 x_2 \cdots x_n \neq 0$,

10.25 (ii) $P_0 \in \mathbf{K}_B$.

One can therefore regard $\overline{n}_A(\mathbf{K}_B)$ as an adelic integral approximating the sum $\mathcal{D}(B)$. To understand the link recall that the local measures in this paper are normalized (cf. sec.1) such that $\mu(A_k/k) = 1$ for the adelic product measure on A_k. This implies that the volume of the additive quotient group $X_1(A_k)/X_1(k)$ with respect to $dx_1 dx_2 \cdots dx_n$ is equal to 1.

One can now modify this discussion to give a similar interpretation of $\mathcal{C}(B)$ by means of (10.19)-(10.20). But it is better to introduce generating Dirichlet series following Manin, Batyrev and Tschinkel **[3]**, **[7]**, **[4]**.

Notation 10.26

(a) Let $\{1, \ldots, n\} \to \Delta(1)$ be a bijection (i.e. an ordering of the rays of Δ) sending i to ρ_i. Then we shall write

$$H_i : X(k) \longrightarrow (0, \infty)$$

for the height function defined by the irreducible U-invariant Weil divisor D_i (cf. (10.6)). If $D \in \mathbb{C}^{\Delta(1)}$ is a formal sum $s_1 D_1 + \cdots + s_n D_n$, $s_1, \ldots, s_n \in \mathbb{C}$ and $P \in U(k)$, put

$$h_D(P) = \sum_{i=1}^n s_i h_i(P),$$

$$H_D(P) = \prod_{i=1}^{n} H_i(P)^{s_i} = \exp h_D(P).$$

(b)

$$\begin{aligned} h_\Delta : X(k) \times \mathbb{C}^{\Delta(1)} &\longrightarrow \mathbb{C}, \\ H_\Delta : X(k) \times \mathbb{C}^{\Delta(1)} &\longrightarrow \mathbb{C} \end{aligned}$$

are the pairings for which

$$\begin{aligned} h_\Delta(D, P) &= h_D(P), \\ H_\Delta(D, P) &= H_D(P). \end{aligned}$$

Remarks 10.27

(a) The pairings in (10.26)(b) extend the pairings

$$\begin{aligned} h_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} &\longrightarrow \mathbb{R}, \\ H_\Delta : X(k) \times \mathbb{Z}^{\Delta(1)} &\longrightarrow (0, \infty) \end{aligned}$$

defined earlier.

(b) Suppose that $D = s_1 D_1 + \cdots + s_n D_n$ comes from $M_\mathbb{C} = M \otimes_\mathbb{Z} \mathbb{C}$ under the map induced by (8.2). Then $H_\Delta(D, P) = 0$ by the proof of (10.12). Therefore, the pairings in (10.26)(b) factorize over

$$U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

for $(\operatorname{Pic} X)_\mathbb{C} := \operatorname{Pic} X \otimes_\mathbb{Z} \mathbb{C}$. We shall by abuse of notation write

$$h_\Delta : U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

and

$$H_\Delta : U(k) \times (\operatorname{Pic} X)_\mathbb{C} \longrightarrow \mathbb{C}$$

also for these pairings.

Proposition 10.28. — *$Z_\Delta(s) := \sum_{P \in U(k)} H_\Delta(P, -s_1 D_1 - \cdots - s_n D_n)$ converges absolutely and uniformly in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1+\delta}$ for any $\delta > 0$.*

Proof. — This is due to Batyrev and Tschinkel [**4**, Ch. 4]. First note that

$$\begin{aligned} |H_\Delta(P, -s_1 D_1 - \cdots - s_n D_n)| &= \prod_{i=1}^{n} |H_i(P)|^{-\Re s_i} \\ &\leq |\prod_{i=1}^{n} H_i(P)|^{-(1+\delta)} \\ &= H(P)^{-(1+\delta)} \end{aligned}$$

where $H : X(k) \to (0,\infty)$ is the toric height function. Then choose a finite closed subset S of $\operatorname{Spec} o$, such that $o_{(S)}$ is a principal ideal domain and consider the toric $o_{(S)}$-morphism $\widetilde{\pi}_{(S)} : \widetilde{X}_{0(S)} \to \widetilde{X}_{(S)}$ obtained by base extension from the toric o-morphism $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$. The natural map from $\widetilde{X}_0(o_{(S)})$ to $X(k)$ is surjective and

$$H_0(P_0) = \prod_{\nu\in W} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu = \prod_{S\cup W_\infty} \sup_{\sigma\in\Delta_{\max}} |x^{D(\sigma)}(P_0)|_\nu \geq \prod_{S\cup W_\infty} |x^D|_\nu$$

for $D = D_1 + \cdots + D_n$ (cf. (11.20) for the inequality). Also, $x_1x_2\cdots x_n \neq 0$ for any lifting to $\widetilde{X}_0(o_{(S)})$ of a k-point in $U(k)$. Therefore,

$$|Z_\Delta(s)| \leq \Big(\sum_{x\in o_{(S)}\setminus\{0\}} \Big(\prod_{S\cup W_\infty} |x|_\nu^{-1-\delta}\Big)\Big)^n$$

so that the desired assertion follows from standard results on S-integers of bounded height (cf. e.g. [**37**, th. 5.2(i)] for the case $S = \varnothing$). □

The deepest contribution of the authors of [**7**] and [**4**] is the construction of a meromorphic continuation of $Z_\Delta(s)$ to a region with $\Re s \in \mathbb{R}^n_{>1-\delta}$ for some $\delta > 0$. They also prove that the function $\zeta_\Delta(s) := Z_\Delta(s,\ldots,s)$ has no other poles in the region $\Re s \in \mathbb{R}_{>1-\delta}$ than at $s = 1$ where it has a pole of order $r = \operatorname{rk}\operatorname{Pic} X$. Now once this is known it follows from the Tauberian theorem of Delange [**17**] that

(10.29) $$\mathcal{C}(B) = cB(\log B)^{r-1}(1 + o(1))$$

where $c = a_{-r}/(r-1)!$ for the leading coefficient in the Laurent expansion

$$\zeta_\Delta(s) = \sum_{k=-r}^{\infty} a_k(s-1)^k.$$

Batyrev and Tschinkel prove that $c = a_{-r}/(r-1)!$ is equal to the constant $\alpha(X)\tau_\varepsilon(X, \|\ \|)$ in (10.17). We shall here give an interpretation of the leading constant as an adelic density and describe an alternative approach to derive the meromorphic continuation of $Z_\Delta(s)$ which apart from the use of multidimensional zeta functions is closer to the method of Schanuel [**56**] and Peyre [**52**].

To describe the differences between the two methods, recall that there are (at least) two methods to find a meromorphic continuation of Riemann's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

to a region $\Re s > 1 - \delta$. The most elegant and group theoretic approach is to use the Poisson formula and integral transforms. This method was formulated in an adelic language by Tate and Iwasawa and this is the route followed by Batyrev and Tschinkel. But the use of harmonic analysis on $U(A_k)$ depends strongly on the group structure of $U(A_k)$ and it is hard to see how this "idelic" method can be applied for varieties without a transitive group action.

An alternative method to prove that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has a meromorphic continuation is by means of the integral representation

$$(10.30)\quad \sum_{n=1}^{\infty} n^{-s} = \int_1^{\infty} [x]^{-s} dx = \int_1^{\infty} x^{-s} dx + \sum_{n=1}^{\infty} \Big(\int_n^{n+1} ([x]^{-s} - x^{-s}) dx \Big)$$

where the last sum converges absolutely and uniformly on compact subsets with $\Re s > 0$. It defines therefore a holomorphic function in the right half plane. This should be seen as the "error term" while the main term $\int_1^{\infty} x^{-s} dx = 1/(s-1)$.

The idea is now to apply a multidimensional version of the last argument and approximate sums over lattices by integrals. This suffices over $\mathbb{Q}$ and the reader will find some of the techniques to bound the error terms in the next section although we shall not consider zeta functions there. Over arbitrary number fields, it is more systematical to use an adelic approach and approximate sums over the discrete set $U(k) = U_0(k)/T(k)$ by integrals over $X_0(A_k)/T(k)$. To find the adelic integral approximating $Z_\Delta(s)$ we extend the restrictions to $U(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$ of the pairings

$$\begin{aligned} h_\Delta : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C}, \\ H_\Delta = \exp h_\Delta : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C} \end{aligned}$$

in (10.27)(b) to pairings:

$$\begin{aligned} h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C}, \\ H_{\Delta,A} = \exp h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C}. \end{aligned}$$

To define these, we use the canonical toric splitting. Let

$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R})$$

be the continuous map described in (10.21)(i) and let $(\operatorname{Pic} X)_{\mathbb{R}} = \operatorname{Pic} X \otimes \mathbb{R}$. The obvious $\mathbb{R}$-linear pairing $\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R}) \times (\operatorname{Pic} X)_{\mathbb{R}} \to \mathbb{R}$ of dual real vector spaces extends uniquely to a pairing:

$$(10.31)\qquad \langle\ ,\ \rangle : \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic} X, \mathbb{R}) \times (\operatorname{Pic} X)_{\mathbb{C}} \longrightarrow \mathbb{C}$$

which is $\mathbb{C}$-linear on the right hand side.

Notations 10.32. — Let $(\overline{P}_0, [D]) \in X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$. Then,

$$\begin{aligned} h_{\Delta,A}(\overline{P}_0, [D]) &= \langle \overline{\xi}_A(\overline{P}_0), [D] \rangle, \\ H_{\Delta,A}(\overline{P}_0, [D]) &= \exp\langle \overline{\xi}_A(\overline{P}_0), [D] \rangle. \end{aligned}$$

Proposition 10.33. — *The pairings*

$$\begin{aligned} h_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C}, \\ H_{\Delta,A} : X_0(A_k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C} \end{aligned}$$

coincide on $U_0(k)/T(k) = U(k)$ *with the pairings*

$$\begin{aligned} h_\Delta : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C}, \\ H_\Delta : X(k) \times (\operatorname{Pic} X)_{\mathbb{C}} &\longrightarrow \mathbb{C} \end{aligned}$$

in (10.27)(b).

Proof. — The pairings $h_{\Delta,A}$ and h_Δ are $\mathbb{C}$-linear on the right hand side. It is therefore sufficient to prove that $h_{\Delta,A}(P,[D]) = h_\Delta(P,[D])$, $P \in U_0(k)/T(k) = U(k)$ for $D \in \mathbb{Z}^{\Delta(1)}$. But this has already been established in (10.24). □

Remark 10.34. — The restriction of $h_{\Delta,A}$ to

$$X_0(k)/T(k) \times (\operatorname{Pic} X)_{\mathbb{C}} = X(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$$

does not coincide with h_Δ outside $U(k) \times (\operatorname{Pic} X)_{\mathbb{C}}$. This is not surprising since the conjectures of Manin and Peyre according to our reinterpretation will give a link between the counting functions on $U(k)$ and adelic integrals on $X(A_k)/T(k)$ whereas on $X(k)\backslash U(k)$ there are accumulating subsets (cf. (7.5)).

To study the adelic integral which approximates $Z_\Delta(s)$, we shall need a positive linear map

$$\Lambda_\Delta : C_c(X_0(A_k)/T(k)) \longrightarrow C_c(\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}))$$

obtained by integrating along the fibres of

$$\overline{\xi}_A : X_0(A_k)/T(k) \longrightarrow \operatorname{Hom}(\operatorname{Pic} X, \mathbb{R}).$$

To define this map we choose (cf. the proof of (5.16)) two sets

$$\{\beta_\nu,\ \nu \in W\},$$

$$\{\gamma_\nu,\ \nu \in W\}$$

of convergence factors for T resp. X such that $\beta_\nu \gamma_\nu = 1$ for all $\nu \in W$. Let Θ_Σ be the Haar measure on $T(A_k)$ given by the adelic order norm (cf. (5.9)(b)) and the convergence factors $\{\beta_\nu,\ \nu \in W\}$ and let $\overline{\Theta}^1_\Sigma$ be the corresponding Haar measure on $T^1(A_k)/T(k)$ under the bijection between Haar measures on $T(A_k)$ and $T^1(A_k)/T(k)$ described after (5.19). Finally, let m be the measure on $X(A_k)$ given by the toric adelic norm and the convergence factors $\{\gamma_\nu,\ \nu \in W\}$.

Then, there are two positive linear maps

$$\begin{aligned} \overline{\Lambda}_\beta : C_c(X_0(A_k)/T(k)) &\longrightarrow C_c(X_0(A_k)/T^1(A_k)), \\ \Lambda_\gamma : C_c(X_0(A_k)/T^1(A_k)) &\longrightarrow C_c(\operatorname{Hom}(\operatorname{Pic} X, \mathbb{R})) \end{aligned}$$

defined as follows. Let $f \in C_c(X_0(A_k)/T(k))$ and $x \in X_0(A_k)/T(k)$ be a lifting of $x^1 \in X_0(A_k)/T^1(A_k)$. Then

$$\overline{\Lambda}_\beta(f)(x^1) := \int_{T^1(A_k)/T(k)} f(tx)d\overline{\Theta}^1_\Sigma. \tag{10.35}$$

Here we integrate with respect to $t \in T^1(A_k)/T(k)$. The notation $tx \in X_0(A_k)/T(k)$ refers to the the translation

$$T^1(A_k)/T(k) \times X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T(k)$$

induced by

$$\sigma_A : T(A_k) \times X_0(A_k) \longrightarrow X_0(A_k)$$

(cf. (5.3)). This map makes

$$X_0(A_k)/T(k) \longrightarrow X_0(A_k)/T^1(A_k)$$

into a topological torsor under the compact group $T^1(A_k)/T(k)$. We may therefore apply the results in [**11**, Ch. VII, §2 n°2] and conclude that $\overline{\Lambda}_\beta(f)(x^1)$ is a $T^1(A_k)/T(k)$-invariant function on $X_0(A_k)/T(k)$ with compact support. It is clear that $\overline{\Lambda}_\beta$ is a positive linear map.

To define Λ_γ, we make use of the canonical splitting

$$\xi^1_A : X_0(A_k)/T^1(A_k) \longrightarrow \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

in (10.21)(i) (cf. also (5.24)) and the corresponding canonical isomorphism

$$X_0(A_k)/T^1(A_k) = X(A_k) \times \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}). \tag{10.36}$$

Then each $g \in C_c(X_0(A_k)/T^1(A_k))$ can be regarded as a function on $X(A_k) \times \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. Further, by (1.16) and [**11**, Ch. III, §5], the integral

$$\int_{P \in X(A_k)} g(P, \lambda)dm, \quad \lambda \in \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

defines a continuous function on $\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$. There is, therefore, a positive linear map

$$\Lambda_\gamma : C_c(X_0(A_k)/T^1(A_k)) \longrightarrow C_c(\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})) \tag{10.37}$$

with

$$\Lambda_\gamma(g)(\lambda) = \int_{P \in X(A_k)} g(P, \lambda)dm$$

for $\lambda \in \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$.

Definition 10.38. — The *toric positive linear map*

$$\Lambda_\Delta : C_c(X_0(A_k)/T(k)) \longrightarrow C_c(\mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R}))$$

is the composition of $\overline{\Lambda}_\beta$ and Λ_γ.

The map Λ_Δ does not depend on the choice of the convergence factors as long as $\beta_\nu\gamma_\nu = 1$ for all $\nu \in W$. It depends only on the fan Δ.

Lemma 10.39. — *Let n_A be the measure on $\mathcal{T}(A_k) = X_0(A_k)$ determined by the induced adelic norm (cf. (5.16)) of the toric adelic norm on $X(A_k)$ and let $\overline{n}_A$ be the quotient measure of n_A on $X_0(A_k)/T(k)$ (cf. (5.23)). Let*

$$V = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{R}), \qquad L = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{Z}) \subset V$$

and $d\nu$ be the unique Haar measure on V such that the volume $\mathrm{Vol}(V/L) = 1$. *Then, the following holds.*

(a)

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{\mathrm{Hom}(\mathrm{Pic}\, X,\mathbb{R})} \Lambda_\Delta(g) d\nu$$

for any $g \in C_c(\mathcal{T}(A_k)/T(k))$.

(b) *Let $h \in C_c(V)$. Then,*

$$h \circ \overline{\xi}_A \in C_c(\mathcal{T}(A_k)/T(k)),$$

$$\Lambda_\Delta(h \circ \overline{\xi}_A) = \tau_\varepsilon(X, \|\ \|)h.$$

Proof

(a) Let

$$\mathrm{Tr} : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(\mathcal{T}(A_k)/T(k))$$

be the trace map obtained by summing over all $T(k)$-translates of a function in $C_c(\mathcal{T}(A_k))$. It suffices to prove the equality in the case $f = \mathrm{Tr}(g)$ for each $g \in C_c(\mathcal{T}(A_k))$. Let

$$\beta = \{\beta_\nu,\ \nu \in W\}, \qquad \gamma = \{\gamma_\nu,\ \nu \in W\}$$

be two sets of convergence factors for T resp. X such that $\beta_\nu\gamma_\nu = 1$ for all $\nu \in W$ and let

$$\Lambda_\beta : C_c(\mathcal{T}(A_k)) \longrightarrow C_c(X(A_k))$$

be the positive linear map defined by the adelic order norm on T and β (cf. (5.9)(b)). Then, by (5.23) and (4.28) there are equalities:

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{\mathcal{T}(A_k)} f dn_A = \int_{X(A_k)} \Lambda_\beta(f) dm_A \tag{10.40}$$

for the measure m_A on $X(A_k)$ defined by the toric norm and γ.

Now make use of the canonical isomorphism

$$\mathcal{T}(A_k)/T^1(A_k) = X(A_k) \times \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$$

in (10.36). Then by (1.16) we obtain a positive linear map

$$\Lambda^1 : C_c(\mathcal{T}(A_k)/T^1(A_k)) \longrightarrow C_c(X(A_k))$$

by integrating over the fibres of the projection $\mathcal{T}(A_k)/T^1(A_k) \to X(A_k)$. We may also change the order of integration (cf. op. cit.) such that

$$\int_{X(A_k)} \Lambda^1(h)dm_A = \int_{\mathrm{Hom}(\mathrm{Pic}\,X,\mathbb{R})} \Lambda_\gamma(h)d\nu \tag{10.41}$$

for any $h \in C_c(\mathcal{T}(A_k)/T^1(A_k))$. Finally, the following positive linear maps are equal:

$$\Lambda_\beta = \Lambda^1 \circ \overline{\Lambda}_\beta \circ \mathrm{Tr}\,. \tag{10.42}$$

This is a consequence of the compatibility of the Haar measures of

$$T(A_k), \quad T(A_k)/T^1(A_k), \quad T^1(A_k)/T(k) \quad \text{and} \quad T(k)$$

(cf. the discussion after (5.19)) and the alternative description of Λ_β. Now let $h = \overline{\Lambda}_\beta(g)$. Then, by (10.40) and (10.42) it follows that:

$$\int_{\mathcal{T}(A_k)/T(k)} g d\overline{n}_A = \int_{X(A_k)} \Lambda_\beta(f)dm_A = \int_{X(A_k)} \Lambda^1(h)dm_A. \tag{10.43}$$

Therefore, (a) follows from (10.43) and (10.41) and the identity $\Lambda_\gamma(h) = \Lambda_\Delta(g)$.

(b) It is clear that $h \circ \overline{\xi}_A \in C_c(\mathcal{T}(A_k)/T(k))$ since $\overline{\xi}_A : X_0(A_k)/T(k) \to V$ is continuous and proper (in the topological sense). The equality is a direct consequence of the definitions of Λ_Δ and $\tau_\varepsilon(X, \|\ \|)$ (see (5.24) for more details). This completes the proof. □

Notation 10.44. — Let $\sigma_{\mathrm{eff}}(X,U) \subset V = \mathrm{Hom}(\mathrm{Pic}\,X \to \mathbb{R})$ be the (dual) effective cone described in (10.18) and let $\overline{\xi}_A : X_0(A_k)/T(k) \to \mathrm{Hom}(\mathrm{Pic}\,X, \mathbb{R})$ be the map described in (10.21)(j). Then,

$$(X_0(A_k)/T(k))_{\mathrm{eff}} := \overline{\xi}_A^{-1}(\sigma_{\mathrm{eff}}(X,U)).$$

Definition and proposition 10.45. — *Let n_A be the (restricted) product measure on $X_0(A_k)$ of the induced adelic norm (cf. (5.16)) for X_0 of the toric adelic norm for X and let $\overline{n}_A$ be the quotient measure on $X_0(A_k)/T(k)$ (cf. (5.23)). Then,*

$$I_\Delta(s) := \int_{\overline{P}_0 \in (X_0(A_k)/T(k))_{\mathrm{eff}}} H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n)d\overline{n}_A$$

converges absolutely in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$. Moreover, if $d\nu$ is the normalized Haar measure of $V = \mathrm{Hom}(\mathrm{Pic}\,X, \mathbb{R})$ in (10.18), and

$$\langle\ ,\ \rangle : V \times (\mathrm{Pic}\,X)_{\mathbb{C}} \longrightarrow \mathbb{C}$$

is the obvious pairing (cf. (10.31)), then

$$I_\Delta(s) = \tau_\varepsilon(X, \| \ \|) \int_{\lambda \in \sigma_{\mathrm{eff}}(X,U)} \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n] \rangle d\nu \tag{10.46}$$

in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$.

Proof. — Fix $s \in \mathbb{R}^n_{>1}$ and let $f_s : V \to \mathbb{R}$ be the function with support in $\sigma_{\mathrm{eff}}(X, U)$ such that $f_s(\lambda) = \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n] \rangle$ for $\lambda \in \sigma_{\mathrm{eff}}(X, U)$. Then,

$$f_s(\overline{\xi}_A(\overline{P}_0)) = \begin{cases} H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n) & \text{if } \overline{P}_0 \in (X_0(A_k)/T(k))_{\mathrm{eff}}, \\ 0 & \text{if } \overline{P}_0 \notin (X_0(A_k)/T(k))_{\mathrm{eff}}. \end{cases}$$

so that

$$I_\Delta(s) := \int_{X_0(A_k)/T(k)} (f_s \circ \overline{\xi}_A) d\overline{n}_A.$$

Now recall that $C_c(V)$ is dense in $L^1(d\nu)$ (see [**38**, Ch. IX, §3]). There exists thus an L^1-Cauchy sequence $(h_i)_{i=1}^\infty$ in $C_c(V)$ converging to h. Hence, by (10.39), $(h_i \circ \overline{\xi}_A)_{i=1}^\infty$ is a L^1-Cauchy sequence in $C_c(X_0(A_k)/T(k))$ converging to $f_s \circ \overline{\xi}_A$ with

$$\lim_{i \to \infty} \int_{X_0(A_k)/T(k)} (h_i \circ \overline{\xi}_A) = \tau_\varepsilon(X, \| \ \|) \lim_{i \to \infty} \int_V h_i d\nu = \tau_\varepsilon(X, \| \ \|) \int_V f_s d\nu.$$

Hence $I_\Delta(s)$ converges absolutely and takes the value:

$$\begin{aligned} I_\Delta(s) &= \tau_\varepsilon(X, \| \ \|) \int_V f_s d\nu \\ &= \tau_\varepsilon(X, \| \ \|) \int_{\lambda \in \sigma_{\mathrm{eff}}(X,U)} \exp\langle \lambda, [-s_1D_1 - \cdots - s_nD_n] \rangle d\nu \end{aligned}$$

for $s \in \mathbb{R}^n_{>1}$. But then $I_\Delta(s)$ converges absolutely for $s \in \mathbb{C}^n$ with

$$(\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$$

since

$$|H_\Delta(\overline{P}_0, -s_1D_1 - \cdots - s_nD_n)| = H_\Delta(\overline{P}_0, -(\Re s_1)D_1 - \cdots - (\Re s_n)D_n).$$

It is also clear how to extend the proof to complex s by means of complex valued L^1-Cauchy sequences $(h_i)_{i=1}^\infty$ in $C_c(V, \mathbb{C})$. This completes the proof. □

Corollary 10.47. — *Let* $I_\Delta(\mathrm{s})$ *be as above and let* $Q_\Delta(\mathrm{s})$ *be a product of linear forms defining the codimension one faces of the cone in* $\operatorname{Pic} X$, *generated by the classes* $[D] \in \operatorname{Pic} X$ *of effective* U*-invariant Weil divisors* $D \in \mathbb{Z}^{\Delta(1)}$. *Then* $I_\Delta(\mathrm{s})Q_\Delta(\mathrm{s})$ *is a polynomial in* $s_1, \ldots, s_n$.

Proof. — This follows from a result of Batyrev and Tschinkel [**4**, prop. 5.4] applied to the characteristic transform:

$$\int_{\lambda \in \sigma_{\text{eff}}(X,U)} \exp\langle \lambda, [-s_1 D_1 - \cdots - s_n D_n]\rangle d\nu$$

of the effective cone in $\operatorname{Pic} X$. □

Now recall that $U(k) = U_0(k)/T(k)$ is a discrete subset of $(X_0(A_k)/T(k))_{\text{eff}}$ (cf. (10.24)) and that the volume of the additive quotient group $X_1(A_k)/X_1(k)$ with respect to $dx_1 dx_2 \cdots dx_n$ is equal to 1, because of the normalizations of the Haar measures in section 1. It is therefore reasonable to regard

$$I_\Delta(s) := \int_{\overline{P}_0 \in (X_0(A_k)/T(k))_{\text{eff}}} H_\Delta(\overline{P}_0, -s_1 D_1 - \cdots - s_n D_n) d\overline{n}_A$$

as a continuous approximation of

$$Z_\Delta(s) := \sum_{P \in U(k)} H_\Delta(P, -s_1 D_1 - \cdots - s_n D_n)$$

in the domain $\Re s = (\Re s_1, \ldots, \Re s_n) \in \mathbb{R}^n_{>1}$.

To make this precise, one has to choose a fundamental domain in $X_0(A_k)$ under the action of $T(k)$. Suppose for simplicity that the o is a principal domain. Then the natural map from $T(k)$ to the direct sum $\oplus T(k_\nu)/T(k_\nu)_{\text{cp}}$ over all non-archimedean places ν of k is surjective. This together with the canonical isomorphisms in (10.8) implies that there is a canonical isomorphism:

$$\left(\prod_{W_{\text{fin}}} \widetilde{X}_0(o_\nu) \times \prod_{W_\infty} X_0(k_\nu)\right) \Big/ T(o) = X_0(A_k)/T(k)$$

so that we are reduced to a choice of a fundamental domain modulo $T(o)$. We refer to the papers of Schanuel [**56**] and Peyre [**52**], [**51**] for more details about the choice of fundamental domains. The zeta function $Z_\Delta(s)$ may be reinterpreted as a sum over a subset of $U_0(k)$ lying in the fundamental domain $\mathcal{F}$ and the goal is to approximate this by the corresponding integral over the subset of $\mathcal{F}$ defined by the inverse image of $(X_0(A_k)/T(k))_{\text{eff}}$.

If we restrict to the diagonal $s_1 = \cdots = s_n$ and put $\zeta_\Delta(s) := Z_\Delta(s, \ldots, s)$ for $\Re s > 1$, this leads to proving that $\zeta_\Delta(s) - I_\Delta(s, \ldots, s)$ is small compared to $I_\Delta(s, \ldots, s)$.

Batyrev and Tschinkel prove in [**7**] that the characteristic transform in (10.46) has a pole of order r and that the leading coefficient in the Laurent expansion is equal to $(r-1)! \cdot \alpha_{\text{Peyre}}(X)$. If one applies the Tauberian theorem in $[D]$ to $\zeta_\Delta(s)$, one thus concludes that if

$$(s-1)^{r-1}(\zeta_\Delta(s) - I_\Delta(s, \ldots, s))$$

has a holomorphic continuation to a right half plane $\Re s > 1 - \delta$, $\delta > 0$, then (7.6) and (7.7) holds for the counting function defined by the toric height. (This is how Batyrev and Tschinkel proceeds in **[7]** and **[4]** although there is no volume-theoretic interpretation of I_Δ there.)

We shall in the next section prove (7.6) and (7.7) for toric varieties over $\mathbb{Q}$ without making use of Manin's zeta function $\zeta_\Delta(s)$. But the bounds of the error terms in section 11 can be reinterpreted as bounds for $\zeta_\Delta(s) - I_\Delta(s, \dots, s)$.

11. Asymptotic formulas for counting functions on toric $\mathbb{Q}$-varieties

We shall in this section study the asymptotic growth of counting functions for (split) toric varieties X_Δ over $\mathbb{Q}$ defined by complete regular fans Δ. We shall assume that the base field $k = \mathbb{Q}$, but otherwise keep the notations in section 10. We will thus write H for the toric (anticanonical) height function and $\mathcal{C}(B)$ for the number of rational points of toric height at most B on the open $\mathbb{Q}$-torus U in $X = X_\Delta$. Also, r will denote the rank of $\operatorname{Pic} X$ as in the previous sections. Hence $r = \operatorname{Card}\Delta(1) - \dim\Delta$ by (8.2). We assume throughout this section that ω_X^{-1} is generated by its global sections.

The aim of this section is to prove the asymptotic formula

$$\mathcal{C}(B) = CB(\log B)^{r-1} + O(B(1+\log B)^{r-3/2+\varepsilon}) \tag{11.1}$$

where

$$C = \alpha(X)\tau(X, \|\ \|) > 0$$

is the constant of Peyre (cf. (7.7)) and $\varepsilon > 0$ is a positive number which may be arbitrarily small. This improves somewhat upon the earlier results of Peyre **[52]** for special classes of toric $\mathbb{Q}$-varieties and the results of Batyrev and Tschinkel **[7]**, **[4]** for arbitrary toric varieties. They prove asymptotic formulas of the type

$$\mathcal{C}(B) = \alpha(X)\tau(X, \|\ \|)B(\log B)^{r-1}(1 + o(1))$$

Batyrev and Tschinkel deduce this result from the meromorphic continuation of the Manin zeta function $\zeta_\Delta(s)$ (cf. the end of sec. 10). By using their estimates for the growth of $\zeta_\Delta(s)$ along suitable vertical strips with $\Re s < 1$ and a method of Landau (cf. **[60]**), one can probably deduce the more precise result

$$\mathcal{C}(B) = BP(\log B) + O(B^{1-\delta})$$

for some real polynomial P and some positive number δ.

Our proof of (11.1) has its origin in the papers of Schanuel **[56]** and Peyre **[52]** on projective spaces resp. certain special blow-ups of projective spaces. The main idea is to count integer points on models of finite type over $\mathbb{Z}$ of universal torsors over X instead of rational points on X. If $r = 1$, then $X_\Delta = \mathbb{P}^n$ (cf. (8.9)(v) and **[24**, p. 22**]**) and the toric height is equal to the standard height. There is, then, only one (isomorphism class of) universal torsors given by the affine cone $\mathbb{A}^{n+1}\backslash(0, \dots, 0)$ of

$\mathbb{P}^n$. Hence our method reduces to Schanuel's method when $r = 1$. We shall therefore omit this trivial case in some lemmas.

Let W be the set of places of $\mathbb{Q}$ and let

$$\| \ \| = \{\| \ \|_\nu \ , \ \nu \in W\}$$

be the toric adelic norm for X. This adelic norm gives rise to the toric height function H on $X(\mathbb{Q})$ defined by

$$H(P) = \prod_{\nu \in W} \|s(P)\|_\nu^{-1} \tag{11.2}$$

where s is a local section of the anticanonical sheaf ω_X^{-1} at P with $s(P) \neq 0$ (cf. (7.2)).

Let $\pi : X_0 \to X$ (resp. $\pi : \widetilde{X}_0 \to \widetilde{X}$) be the principal universal torsor over $\mathbb{Q}$ (resp. $\mathbb{Z}$) described in the beginning of section 10. Let

$$D_0 = \sum_{\rho \in \Delta(1)} D_\rho.$$

If σ is a maximal cone of Δ, let $\chi^{m(\sigma)}$, $m(\sigma) \in M$ be the unique character of U such that $\chi^{-m(\sigma)}$ generates $\mathcal{O}(D_0)$ on U_σ.

Let

$$D(\sigma) = D_0 + \sum_{\rho \in \Delta(1)} \langle -m(\sigma), n_\rho \rangle, \quad \sigma \in \Delta_{\max}$$

be the Weil divisors described in (8.7) and recall that these are effective divisors when ω_X^{-1} is generated by its global sections. Let

$$x^D = \prod_{\rho \in \Delta(1)} x_\rho^{a_p}$$

be the monomial in the indeterminates x_ρ, $\rho \in \Delta(1)$ corresponding to the effective Weil divisor

$$D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}.$$

We now give a concrete description of the composite map $H_0 = H \circ \pi$ from $X_0(\mathbb{Q})$ to $\mathbb{R}_{>0}$ in the case where $\mathcal{O}(D_0)$ is generated by its global sections.

Proposition 11.3. — *Let P_0 be a $\mathbb{Q}$-point on X_0 which extends to a $\mathbb{Z}$-point on $\widetilde{X}_0$. Then*

$$H_0(P_0) = \sup_{\sigma \in \Delta_{\max}} |x^{D(\sigma)}(P_0)|$$

for the usual archimedean absolute value $| \ |$ of $\mathbb{R}$

Proof. — Let $\widetilde{P}_0 \in \widetilde{X}_0(\mathbb{Z})$ be the extension of P_0 and let $\widetilde{X}_0 \subset \widetilde{X}_1 = \operatorname{Spec} \mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$ be the toric open immersion described after 10.3. Let p be a prime and let $Y_0 \subset \operatorname{Spec} \mathbb{Z}/p\mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$ be the reduction modulo p of $\widetilde{X}_0$. Then, by applying (8.8)(a) to Y_0, we conclude that $x^{D(\sigma)}(\widetilde{P}_0) \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$ for some maximal cone σ of Δ. Hence the product formula for $H_0(P_0)$ in (10.14) reduces to its archimedian factor. This completes the proof. □

Lemma 11.4

(a) *Let* $c(m) = \operatorname{Card}\{P \in U(\mathbb{Q}) : H(P) = m\}$ *and*

$$c_0(m) = \operatorname{Card}\{P_0 \in \widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q}) : H_0(P_0) = m\}.$$

Then, $c(m) = c_0(m)/\operatorname{Card} \widetilde{T}(\mathbb{Z})$.

(b) *Let* $U_0(\mathbb{R})^+$ *be the real connected component of* $U_0(\mathbb{R})$ *containing* 1 *and let*

$$U_0(\mathbb{Q})^+ = U_0(\mathbb{Q}) \cap U_0(\mathbb{R})^+,$$

$$c_0(m)^+ = \operatorname{Card}\{P_0 \in \widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q})^+ : H_0(P) = m\}.$$

Then,

$$c_0(m) = \operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+) c_0(m)^+.$$

Proof

(a) It follows from Grothendieck's version of Hilbert 90 (cf. [**45**, p. 124]) and the fact that $\mathbb{Z}$ is a principal ideal domain that $H^1_{\text{ét}}(\operatorname{Spec} \mathbb{Z}, \widetilde{T}) = 0$. The fibres of $\widetilde{\pi} : \widetilde{X}_0 \to \widetilde{X}$ at $\mathbb{Z}$-points on $\widetilde{X}$ are therefore trivial $\operatorname{Spec} \mathbb{Z}$-torsors under $\widetilde{T}$. Hence $\widetilde{\pi}(\widetilde{X}_0(\mathbb{Z})) = \widetilde{X}(\mathbb{Z})$ and there are exactly $\operatorname{Card} \widetilde{T}(\mathbb{Z})$ points in $\widetilde{X}_0(\mathbb{Z})$ with a given image in $\widetilde{X}(\mathbb{Z})$.

(b) We regard $\widetilde{U}_0(\mathbb{Z})$ as a subgroup of $U_0(\mathbb{Q})$. Then, since the group action $U_0 \times U_0 \to U_0$ extends to a toric action $\widetilde{U}_0 \times \widetilde{X}_0 \to \widetilde{X}_0$ we conclude that $\widetilde{X}_0(\mathbb{Z}) \cap U_0(\mathbb{Q})$ is a union of $\widetilde{U}_0(\mathbb{Z})$-cosets in $U_0(\mathbb{Q})$. Also, each $\widetilde{U}_0(\mathbb{Z})$-coset in $U_0(\mathbb{Q})$ contains exactly one element in $U_0(\mathbb{Q})^+$ and the height function H_0 takes the same value for each $\mathbb{Q}$-point in a $\widetilde{U}_0(\mathbb{Z})$-orbit of $X_0(\mathbb{Q})$. Hence $c_0(m) = c_0(m)^+ \operatorname{Card} \widetilde{U}_0(\mathbb{Z})$. To complete the proof, note that there is a canonical group isomorphism between $U_0(\mathbb{R})$ and $\widetilde{U}_0(\mathbb{Z}) \times U_0(\mathbb{R})^+$. □

We may thus instead of counting $\mathbb{Q}$-points on U count $\mathbb{Z}$-points on the universal torsor $\widetilde{X}_0$ such that the corresponding $\mathbb{Q}$-points lie on $\pi^{-1}(U)$. The number of such $\mathbb{Z}$-points on the universal torsor $\widetilde{X}_0$ of a given height is the same in each real connected component of U_0.

To study $c_0(m)$, let $\rho_1, \dots, \rho_n$ be an ordering of the rays in Δ. Let $D_1, \dots, D_n$ be the corresponding U-invariant irreducible Weil divisors and let $x_1, \dots, x_n$ be the variables corresponding to these prime divisors. Each $x^{D(\sigma)}$, $\sigma \in \Delta_{\max}$ is a monomial in $x_1, \ldots, x_n$ with exponents $\varepsilon_1, \ldots, \varepsilon_n$ corresponding to the multiplicities of $D(\sigma)$ along $D_1, \ldots, D_n$. If $r = (r_1, \ldots, r_n)$ is an n-tuple with components in an arbitrary commutative ring R and $\sigma \in \Delta_{\max}$, let $r^{D(\sigma)}$ be the element in R obtained by evaluating $x^{D(\sigma)}$ at $x = r$.

Now recall (cf. the comments after (8.6) and (10.3)) that the open toric embedding

$$\widetilde{X}_0 \subset \widetilde{X}_1 = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$$

is the complement in $\widetilde{X}_1$ of the closed subscheme defined by the monomials $x^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$. This implies that $\widetilde{X}_0(\mathbb{Z}) \subset \widetilde{X}_1(\mathbb{Z}) = \mathbb{Z}^{(n)}$ is the subset of n-tuples $\mathbf{q}$ of integers for which the greatest common divisor

$$\gcd_{\sigma \in \Delta_{\max}} (\mathbf{q}^{\underline{\sigma}}) = 1.$$

Hence, by (11.3) it follows that $c_0(m)$ is the number of n-tuples $\mathbf{q}$ of non-zero integers satisfying the following two conditions

11.5 (i) $$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m,$$

11.5 (ii) $$\gcd_{\sigma \in \Delta_{\max}} (\mathbf{q}^{\underline{\sigma}}) = 1.$$

Moreover, $c_0(m)^+$ is the number of n-tuples $\mathbf{q}$ of *positive* integers satisfying (11.5).

If $\mathcal{O}(D_0)$ is generated by its global sections, then we have seen in the proof of (11.2) that (11.5)(ii) implies that

11.5 (ii)' $$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = 1,$$

If $\mathcal{O}(D_0)$ is ample, then these two conditions are equivalent (cf. (8.8)(b)).

It is difficult to make use of (11.5)(11.5 (ii)) directly. We shall therefore first study a simpler counting function $A_{\mathbf{d}}(H)$ and then make use of Möbius inversion. This idea is central in the papers of Schanuel **[56]** and Peyre **[52]**.

Notation 11.6

(a) $a(m)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m$$

$A(B)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B.$$

(b) Let $\mathbf{d}$ be an n-tuple of positive integers. Then, $a_{\mathbf{d}}(m)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| = m, \qquad \mathbf{d} \mid \mathbf{q}$$

(i.e. d_i divides q_i for $i = 1, \dots, n$) whereas $A_{\mathbf{d}}(B)$ is the number of n-tuples $\mathbf{q}$ of positive integers such that

$$\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B, \qquad \mathbf{d} \mid \mathbf{q}.$$

(c) $\mathcal{C}_0(B)$ (resp. $\mathcal{C}_0(B)^+$) is the set of all n-tuples $\mathbf{q}$ of non-zero integers (resp. the set of all n-tuples $\mathbf{q}$ of positive integers) such that

(i) $\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B$,

(ii) $\gcd_{\sigma\in\Delta_{\max}}(\mathbf{q}^{\underline{\sigma}}) = 1$.

(d)

$$\Pi(\mathbf{d}) := \prod_{i=1}^{n} d_i$$

for any n-tuple of integers $\mathbf{d} = (d_1, \dots, d_n)$.

All the cardinalities above are finite since (cf. (11.20))

$$\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \geq |\Pi(\mathbf{q})|$$

Lemma 11.7. — *Let* $\mathbf{d}$ *be an* n*-tuple of positive integers. Then,*

$$0 \leq \Pi(\mathbf{d})A_{\mathbf{d}}(B) \leq A(B).$$

Proof. — We introduce an equivalence relation $\sim$ on the set of all positive n-tuples $\mathbf{q} = (q_1, \dots, q_n)$ such that

$$\sup_{\sigma\in\Delta_{\max}} |\mathbf{q}^{D(\sigma)}| \leq B.$$

Let

$$\mathbf{q} = (q_1, \dots, q_n) \sim \mathbf{r} = (r_1, \dots, r_n)$$

if and only if

$$([(q_1 - 1)/d_1], \dots, [(q_n - 1)/d_n]) = ([(r_1 - 1)/d_1], \dots, [(r_n - 1)/d_n]).$$

An equivalence class consists of $\Pi(\mathbf{d})$ elements if and only if it contains a positive n-tuple $\mathbf{q} \in \mathbb{Z}^n_{>0}$ such that $\mathbf{d} \mid \mathbf{q}$. Also, no equivalence class contains more than one such n-tuple. Therefore, $A_{\mathbf{d}}(B)$ is the number of equivalence classes with $\Pi(\mathbf{d})$ elements. This implies the assertion. □

Notations 11.8

(a)
$$\chi : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$
is the function such that $\chi(\mathbf{e}) = 1$ if and only if the greatest common divisor of all $\mathbf{e}^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ is 1.

(b)
$$\chi^{(p)} : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$
is the function such that $\chi^{(p)}(\mathbf{e}) = 1$ if and only if the greatest common divisor of all $\mathbf{e}^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ is not divisible by p.

(c) Let $\mathbf{d} \in \mathbb{Z}^n_{>0}$. Then,
$$\chi_{\mathbf{d}} : \mathbb{Z}^n_{>0} \longrightarrow \{0,1\}$$
is the function such that $\chi_d(e) = 1$ if and only if $\mathbf{d} \mid \mathbf{e}$.

We have also denoted characters of tori by χ. But no confusion should occur since the indices are not the same.

We now define a Möbius function
$$\mu : \mathbb{Z}^n_{>0} \longrightarrow \mathbb{Z}$$
recursively with respect to the relation $|$. This Möbius function was defined by Peyre [**52**, 7.1.7] for special classes of toric varieties.

Definition and proposition 11.9. — *There exists a unique function*
$$\mu : \mathbb{Z}^n_{>0} \longrightarrow \mathbb{Z}$$
such that
$$\chi(\mathbf{e}) = \sum_{\mathbf{d}|\mathbf{e}} \mu(\mathbf{d})$$
for $\mathbf{d}$ *running over all* n*-tuples of positive integers dividing* $\mathbf{e} \in \mathbb{Z}^n_{>0}$.

Proof. — Use the finiteness of the set $\{\mathbf{d} \in \mathbb{Z}^n_{>0} : \mathbf{d} \mid \mathbf{e}\}$ for all $\mathbf{e} \in \mathbb{Z}^n_{>0}$. □

It is unfortunate that we also use the letter μ for measures, but it will be clear from the context if μ is a measure or a Möbius function.

From the definition of μ, it follows that:

$$\chi = \sum_{\mathbf{d}} \mu(\mathbf{d})\chi_{\mathbf{d}} \tag{11.10}$$

where $\mathbf{d}$ runs over all n-tuples of positive integers. Now consider the sum of the values of all $\mathbf{q} \in (\mathbb{Z}_{\neq 0})^n$ such that $H_0(\mathbf{q}) = m$. Then,

$$c_0(m)^+ = \sum_{\mathbf{d}} \mu(\mathbf{d}) a_{\mathbf{d}}(m), \mathbf{d} \in \mathbb{Z}^n_{>0}. \tag{11.11}$$

The sum is finite since $a_{\mathbf{d}}(m) = 0$ if $\sup(d_1, \ldots, d_n) > m$.

Notation 11.12. — Let p be a prime number. By

$$\sum_{\mathbf{d}}^{(p)}$$

we shall mean a sum over all n-tuples $\mathbf{d}$ of positive integers of the form

$$\mathbf{d} = (p^{e_1}, \ldots, p^{e_n})$$

for non-negative integers $e_1, \ldots, e_n$.

There is a local version of (11.10)

$$\chi^{(p)} = \sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d}) \chi_{\mathbf{d}} \tag{11.13}$$

where $\mu(\mathbf{d}) = 0$ if $p^2 \mid d_i$ for some component of $\mathbf{d} = (d_1, \ldots, d_n)$.

The values of the functions $\chi^{(p)}(\mathbf{q})$ and $\chi_{\mathbf{d}}(\mathbf{q})$ for n-tuples $\mathbf{d}$ as in (11.12), depend only on the residue class of $\mathbf{q} = (q_1, \ldots, q_n)$ in $(\mathbb{Z}/p\mathbb{Z})^n$. We can therefore compute the cardinality of

$$\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \{\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}/p\mathbb{Z})^n : \exists \sigma \in \Delta_{\max} \text{ with } \mathbf{r}^{\underline{\sigma}} \neq 0\}$$

by means of (11.13). If we consider the sum of the values of all $\mathbf{r} \in (\mathbb{Z}/p\mathbb{Z})^n$, then we obtain

$$\operatorname{Card} \widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d})(p^n/\Pi(\mathbf{d})). \tag{11.14}$$

Now write

$$\mathbf{de} := (d_1 e_1, \ldots, d_n e_n)$$

for n-tuples $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{e} = (e_1, \ldots, e_n)$ of integers.

The following result is due to Peyre for special classes of toric varieties (cf. **[52**, Ch. 7]).

Lemma 11.15

(a) *Let* $\mathbf{d} = (d_1, \ldots, d_n)$, $\mathbf{e} = (e_1, \ldots, e_n)$ *be n-tuples of positive integers. Let*

$$\delta = \gcd(\mathbf{d}^{\underline{\sigma}}), \quad \sigma \in \Delta_{\max},$$
$$\varepsilon = \gcd(\mathbf{e}^{\underline{\sigma}}), \quad \sigma \in \Delta_{\max}$$

and suppose that δ and ε are relatively prime. Then

$$\mu(\mathbf{de}) = \mu(\mathbf{d})\mu(\mathbf{e}).$$

(b) *Let* $(e_1, \ldots, e_n)$ *be an* n*-tuple of non-negative integers. Then* $\mu(p^{e_1}, \ldots, p^{e_n})$ *is independent of the prime number* p.

(c) *Let* $(e_1, \ldots, e_n)$ *be an* n*-tuple of non-negative integer, not all* 0 *and suppose that there exists a cone* $\sigma \in \Delta$ *such that* $e_i = 0$ *for all rays* $\rho_i \in \Delta$ *outside* σ. *Then*

$$\mu(p^{e_1}, \ldots, p^{e_n}) = 0.$$

(d) *Let* f *be the smallest integer such that there exist* f *rays of* Δ *not contained in a cone of* Δ. *Then the product*

$$\prod_p \left(\sum_{\mathbf{d}}^{(p)} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s \right)$$

over all prime numbers p *is absolute convergent for* $s > 1/f$.

(e) *The sum*

$$\sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s, \quad \mathbf{d} \in \mathbb{Z}^n_{>0}$$

is convergent for $s > 1/f$ *and equal to*

$$\prod_p \left(\sum_{\mathbf{d}}^{(p)} (|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s \right)$$

for $s > 1/f$.

(f) *The product* $\prod_p \left(\sum_{\mathbf{d}}^{(p)} \mu(\mathbf{d}) / \prod(\mathbf{d}) \right)$ *over all prime numbers* p *is absolute convergent and equal to* $\sum_{\mathbf{d}} \mu(\mathbf{d})/\Pi(\mathbf{d})$.

Proof. — (a) and (b) are easy consequences of the definition of μ.

(c) Let $\mathbf{q} = (p^{e_1}, \ldots, p^{e_n})$. Then the greatest common divisor of all integers $\mathbf{q}^{\underline{\sigma}}$, $\sigma \in \Delta_{\max}$ is 1 if and only if there exists a cone $\sigma \in \Delta$ with $e_i = 0$ for all rays $\rho_i \in \Delta$ outside σ (cf. the discussion before (11.5)). The desired statement now follows from the recursive definition of μ.

(d) It follows from (b) and (c) that there is a polynomial $Q(T)$ with non-negative integer coefficients such that $\sum_{\mathbf{d}}^{(p)} (|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s) = 1 + p^{-fs}Q(1/p^s)$ for all prime numbers p. Now use the fact that the sum of all $p^{-f}Q(1/p) \leq p^{-2}Q(1)$ is absolute convergent.

(e) This follows from (d) and the multiplicativity of $(|\mu(\mathbf{d})|/\Pi(\mathbf{d})^s)$ (cf. (a))

(f) This follows from (e) and the multiplicativity of $\mu(\mathbf{d})/\Pi(\mathbf{d})$. □

Lemma 11.16. — *Let $f : \mathbb{Z} > 0 \to \mathbb{R}$ be a function such that $f(B) > 0$ for all sufficiently large integers $B > 0$ and such that*

$$\Pi(\mathbf{d})A_{\mathbf{d}}(B) = f(B)(1 + o(1))$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$.

Then there exists a positive constant

$$\kappa = \prod_p \left(\sum_{\mathbf{d}}^{(p)} (\mu(\mathbf{d})/\Pi(\mathbf{d})) \right)$$

such that

$$\mathcal{C}_0(B)^+ = \kappa f(B)(1 + o(1)).$$

Proof. — Choose $k > 0$ such that $f(B)$ is positive for $B > k$ and let

$$g(\mathbf{d}, B) = |\mu(\mathbf{d})A_{\mathbf{d}}(B)/f(B) - \mu(\mathbf{d})/\Pi(\mathbf{d})|$$

for all $B > k$ and all $\mathbf{d} \in \mathbb{Z}^n_{>0}$. Then

$$|\mathcal{C}_0(B)^+/f(B) - \kappa| = \left| \sum_{\mathbf{d}} A_{\mathbf{d}}(B)/f(B) - \kappa \right| \leq \sum_{\mathbf{d}} g(\mathbf{d}, B)$$

for $B > k$. It is therefore sufficient to prove that

$$\lim_{B \to \infty} \sum_{\mathbf{d}} g(\mathbf{d}, B) = 0.$$

Also, since $0 \leq \Pi(\mathbf{d})A_{\mathbf{d}}(B) \leq A(B)$ there exists a constant $E > 0$ such that

$$|\Pi(\mathbf{d})A_{\mathbf{d}}(B)/f(B) - 1| \leq |A(B)/f(B)| + |A(B)/f(B) - 1| \leq E$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and all $B > k$.

Hence by (11.15)(f), it follows that $\sum_{\mathbf{d}} g(\mathbf{d}, B)$ converges uniformly with respect to $B > k$ (this means that it for each $\varepsilon > 0$ exists a finite subset S of $\mathbb{Z}^n_{>0}$ such that $\sum_{\mathbf{d} \notin S} g(\mathbf{d}, B) < \varepsilon$ for all $B > k$).

Therefore,

$$\lim_{B \to \infty} \sum_{\mathbf{d}} g(\mathbf{d}, B) = \sum_{\mathbf{d}} (\lim_{B \to \infty} g(\mathbf{d}, B)) = 0$$

as was to be proved. □

We can now extend the method of Peyre **[52]** for certain blow-ups of projective spaces to arbitrary toric varieties. It can be described as follows. One first proves asymptotic formulas of the type

$$\Pi(\mathbf{d})A_{\mathbf{d}}(B) = \varpi B(\log B)^{r-1}(1 + O(1/\log B)) \tag{11.17}$$

with the same positive constant ϖ for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and compares ϖ with a volume. Then from (11.16) one gets the asymptotic formula

$$(11.18) \qquad \mathcal{C}_0(B)^+ = \kappa\varpi B(\log B)^{r-1}(1+o(1))$$

with κ as in (op. cit.).

Peyre computes the main term $\kappa f(B)$ by Möbius inversion as in (11.11)-(11.15). He bounds the error term by means of an argument with uniform convergence as in (11.16) although he never mentions the role of uniform convergence. His proof of (11.17) is less direct than here (cf. his comment at the end of p. 171 in op. cit.). It also makes use of special properties of the class of blow-ups of projective spaces he considers.

We shall in this paper apart from the generalization to arbitrary toric varieties improve the treatment of error terms under Möbius inversion in **[52]**. One of the tools to do this will be the following result about the Möbius function $\mu : \mathbb{Z}^n_{>0} \to Z$ introduced in (11.9).

Lemma 11.19. — *Let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Let ε be any posititve real number. Then the following holds.*

(a) $\sum_{\Pi(\mathbf{d})\leq b} |\mu(\mathbf{d})| = O(b^{1/f+\varepsilon})$
(b) $\sum_{\Pi(\mathbf{d})\geq b}(|\mu(\mathbf{d})|/\Pi(\mathbf{d})) = O(b^{1/f-1+\varepsilon})$

Proof

(a) This follows from the fact (cf. (11.15)(e)) that the sum $\sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d})^s$ over all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ converges for $s > 1/f$ (cf. e.g. [**68**, p. 5]).

(b) Let $y(m) = \sum_{\Pi(\mathbf{d})=m} |\mu(\mathbf{d})|$ where $\mathbf{d} \in \mathbb{Z}^n_{>0}$. Then we have to prove that

$$\sum_{y\geq b} y(m)/m = O(b^{1/f-1+\varepsilon})$$

for any $\varepsilon > 0$.

To show this, put $y(0) = 0$ and $Y(m) = y(0) + \cdots + y(m)$ for $m \geq 0$. Then, by partial summation we obtain:

$$\sum_{m\geq b} y(m)/m = -Y(b-1)/b + \sum_{m\geq b} Y(m)/m(m+1)$$

for any positive integer b.

But $Y(m) = O(m^{1/f+\varepsilon})$ by (a). Hence

$$Y(b-1)/b = O(b^{1/f-1+\varepsilon})$$

and

$$\sum_{m\geq b} Y(m)/m(m+1) \leq \sum_{m\geq b} Y(m)/m^2 = O(\sum_{m\geq b} m^{1/f-2+\varepsilon}) = O(b^{1/f-1+\varepsilon}).$$

This completes the proof. □

Proposition 11.20. — *Let $D_0 = \sum_{\rho\in\Delta(1)} D_\rho$ and suppose that $\mathcal{O}(D_0)$ is generated by its global sections. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in U_0(\mathbb{R})$ be an n-tuple of real numbers different from* 0*. Then,*

$$|\Pi(\alpha)| \leq \sup_{\sigma\in\Delta_{\max}} |\alpha^{D(\sigma)}|.$$

Proof. — Let $P \in U_0(\mathbb{R})$ be the image of $\alpha \in U_0(\mathbb{R})$ and let σ be a maximal cone such that $P \in C_\sigma(\mathbb{R})$ and let $\chi^{m(\sigma)}$ be the character which on U_σ represents the Cartier divisor with Weil divisor D_0. Then $|\chi^{m(\sigma)}(P)| \leq 1$ (cf. (9.2)). Now note that $\chi^{m(\sigma)}(P) = \alpha^{D_0}/\alpha^{D(\sigma)}$ by the definition of $D(\sigma)$. This completes the proof. □

Note that the argument is valid for arbitrary locally compact local fields.

Notation 11.21

(a) Let $\pi_{\mathbb{R},\mathrm{an}} : X_0(\mathbb{R}) \to X(\mathbb{R})$ be the analytic map defined by $\pi : X_0 \to X$ and let $C_\sigma(\mathbb{R}) \subset X(\mathbb{R})$ be the compact subset defined in (9.1). Then, $C_{0,\sigma}(\mathbb{R})$ is the inverse image of $C_\sigma(\mathbb{R})$ under $\pi_{\mathbb{R},\mathrm{an}}$.

(b) Let ρ_i, $1 \leq i \leq n$ be the rays of Δ, let D_i, $1 \leq i \leq n$ be U-invariant irreducible Weil divisors corresponding to these rays and let x_i, $1 \leq i \leq n$ be indeterminates indexed by the rays. Let

$$D = l_1 D_1 + \cdots + l_n D_n, \quad l_1, \dots, l_n \in \mathbb{Z}$$

be a U-invariant Weil divisor on X_Δ. Then $\mathbf{x}^D$ denotes the Laurent monomial $\prod_{i=1}^n x_i^{l_i}$.

Proposition 11.22. — *Let $\sigma \in \Delta$ be a maximal cone of the complete regular fan Δ and let*

$$(\rho_1, \dots, \rho_{r+d})$$

$d = \dim\Delta$, $r + d = \mathrm{Card}\,\Delta(1)$ *be an ordering of the rays of Δ such that*

$$(\rho_{r+1}, \dots, \rho_{r+d})$$

are the rays of σ. Let

$$\{n^{(j)},\ 1 \leq j \leq d\}$$

be the $\mathbb{Z}$-basis of N, consisting of the generators of the rays ρ_{r+j} of σ and let

$$\{m^{(j)},\ 1 \leq j \leq d\}$$

be the dual $\mathbb{Z}$-basis of M. Finally, let

$$D(j) = \sum_{\rho\in\Delta(1)} \langle m^{(j)}, n_\rho\rangle D_\rho$$

for $1 \le j \le d$.

Then $C_{0,\sigma}(\mathbb{R})$ is the subset of $X_0(\mathbb{R})$ of real n-tuples $\mathbf{x} = (x_1, \ldots, x_{r+d})$ for which

$$(*) \qquad |x^{D(j)}| \le 1, \quad j \in \{1, \ldots, d\}.$$

Proof. — It suffices by continuity to show that an $\mathbb{R}$-point P_0 on U_0 satisfies

$$|\mathbf{x}^{D(j)}(P_0)| \le 1 \quad \text{for} \quad 1 \le j \le d$$

if and only if $P = \pi(P_0) \in C_\sigma(\mathbb{R})$. To see this, let

$$-L(P) = \sum_{j=1}^{d} \lambda_j n^{(j)}, \quad \lambda_j \in \mathbb{R}.$$

Then $P \in C_\sigma(\mathbb{R})$ if and only if

$$\langle m^{(j)}, L(P)\rangle \ge 0$$

for $1 \le j \le d$ (cf. (9.1)). Now note that

$$|\mathbf{x}^{D(j)}(P_0)| = \exp\langle m^{(j)}, L(P)\rangle$$

by the definitions of $m^{(j)}$, $D(j)$ and L. This completes the proof. □

Remark 11.23. — Let $\tau = \sigma^{(j)}$ be the (unique) adjacent maximal cone with

$$\tau \cap \rho_{r+j} = \{0\},$$

and let b_j be the multiplicity of D_{r+j} in $D(\tau)$. Then,

$$D(\tau) - D(\sigma) = b_j D(j)$$

by (8.9) so that

$$|\mathbf{x}^{D(\tau)-D(\sigma)}| = |\mathbf{x}^{D(j)}|^{b_j}.$$

Also, if $\mathbf{x} \in C_{0,\sigma}(\mathbb{R})$, then $|\mathbf{x}^{D(\tau)-D(\sigma)}| \le 1$ for all maximal cones $\tau \in \Delta$ by (9.8).

If $\sum_{\rho\in\Delta(1)} D_\rho$ is ample, then $b_j > 0$ for $1 \le j \le d$ (cf. (8.9)). We may therefore in this case (cf. also (9.10)) replace $(*)$ by the condition:

$$(*') \qquad |\mathbf{x}^{D(\tau)-D(\sigma)}| \le 1, \quad \tau \in \Delta_{\max}.$$

Sublemma 11.24. — *Let* $(e_1, \dots, e_r)$ *be non-negative integers and let* e *be a positive integer. Let*

$$S(B, e; e_1, \dots, e_r) = \sum (\prod_{i=1}^{r} g_i^{e_i/e-1})$$

be the sum over all r*-tuples* $(g_1, \dots, g_r)$ *of positive integers satisfying*

$$\prod_{i=1}^{r} g_i^{e_i} \leq B$$

and

$$\max(g_1, \dots, g_r) \leq B.$$

Then,

$$S(B, e; e_1, \dots, e_r) = O(B^{1/e}(1 + \log B)^{r-1}).$$

Proof. — Suppose that $e_r = 0$. Then,

$$S(B, e; e_1, \dots, e_r) \leq \begin{cases} S(B, e; e_1, \dots, e_{r-1}) \left(\sum_{i=1}^{[B]} 1/i\right) & \text{if } r > 1 \\ \left(\sum_{i=1}^{[B]} 1/i\right) & \text{if } r = 1. \end{cases}$$

We may and shall therefore assume that $(e_1, \dots, e_r)$ is a positive r-tuple. The statement is easy to verify for $r = 1$ by means of a comparison with an integral. Suppose $r > 1$ and put $m = g_r$, $a = e_r$. Then

$$S(B, e; e_1, \dots, a) = \sum_{m=1}^{[\sqrt[a]{B}]} m^{a/e-1} S([B/m^a], e; e_1, \dots, e_{r-1}).$$

The induction assumption for $S([B/m^a], e; e_1, \dots, e_{r-1})$, $1 \leq m \leq [\sqrt[a]{B}]$ gives

$$S(B, e; e_1, \dots, a) = O\left(\sum_{m=1}^{[\sqrt[a]{B}]} m^{a/e-1}(B/m^a)^{1/e}(1 + \log(B/m^a)^{r-2})\right),$$

$$S(B, e; e_1, \dots, a) = O\left(B^{1/e} \sum_{m=1}^{[\sqrt[a]{B}]} m^{-1}(1 + \log(B/m^a)^{r-2})\right)$$

and the desired result follows. □

Lemma 11.25

(a) $A(B) = O(B(1 + \log B)^{r-1})$.

(b) *Let*

$$u_k = (0, \dots, 0, 1, 0, \dots, 0), \quad 1 \le k \le n$$

be the k-th unit vector in the standard basis for $\mathbb{R}^n$ and let $\delta_k(B)$ be the set of n-tuples $\mathbf{g}$ *of positive integers such that*

(i) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \le B$,

(ii) $\sup_{\sigma \in \Delta_{\max}} |(\mathbf{g} + u_k)^{D(\sigma)}| > B$.

Suppose that $\operatorname{Card} \Delta(1) - \dim \Delta > 1$. *Then,*

$$\operatorname{Card} \delta(B) = O(B(1 + \log B)^{r-2}).$$

Proof

(a) Let $\sigma \in \Delta$ be a maximal cone and let $A_\sigma(B)$ be the number of positive n-tuples $\mathbf{g}$ of integers such that

$$\mathbf{g} = (g_1, \dots, g_n) \in C_{0,\sigma}(\mathbb{R})$$

and

$$\sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \le B.$$

It suffices to prove that

$$A_\sigma(B) = O(B(1 + \log B)^{r-1})$$

for each maximal cone σ of Δ.

Fix $\sigma \in \Delta_{\max}$ and let $(\rho_1, \dots, \rho_n)$ be an ordering of the rays of Δ such that the last $d = n - r$ rays are the one-dimensional faces of σ. Then

$$D(\sigma) = \sum_{i=1}^{r} e_i D_i$$

for some non-negative integers $e_1, \dots, e_r$. (We use here the assumption that $\mathcal{O}(D_0)$ is generated by its global sections.)

Let $\{m^{(j)} \in M,\ 1 \le j \le d\}$ be the $\mathbb{Z}$-basis of M in (11.22) and let

$$E(j) = D_{r+j} - \sum_{\rho \in \Delta(1)} \langle m^{(j)}, n_\rho \rangle D_\rho, \quad j \in \{1, \dots, d\}.$$

Then each $\mathbf{x}^{E(j)}$ is a Laurent monomial in $(x_1, \dots, x_r)$.

Moreover, by (11.22)-(11.23) it follows that $A_\sigma(B)$ is the cardinality of the set of $r + d$-tuples $\mathbf{g} = (g_1, \dots, g_{r+d})$ of positive integers such that

$$(\#) \qquad \mathbf{g}^{D(\sigma)} = \prod_{i=1}^{r} g_i^{e_i} \le B$$

$$(\#\#) \qquad g_{r+j} \le \mathbf{g}^{E(j)}, \quad j \in \{1, \dots, d\}$$

Now note that by (8.9)(iii):

$$D(\sigma) - (D_1 + \cdots + D_r) = \sum_{j=1}^{d} E(j)$$

and hence that

$$\prod_{j=1}^{d} \mathbf{g}^{E(j)} = \prod_{i=1}^{r} g_i^{e_i - 1}.$$

There is, therefore, for each r-tuple $(g_1, \ldots, g_r) \in \mathbb{Z}^r_{>0}$ at most $\prod_{i=1}^r g_i^{e_i-1}$ d-tuples satisfying (##). Hence,

$$A_\sigma(B) \leq \sum \left(\prod_{i=1}^{r} g_i^{e_i-1}\right)$$

where in the sum $(g_1, \ldots, g_r)$ runs over all r-tuples of positive integers satisfying $\prod_{i=1}^r g_i^{e_i} \leq B$. Finally, note that

$$\max(g_1, \ldots, g_r) \leq |\Pi(\mathbf{g})| \leq \sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$$

by (11.20). Hence $A_\sigma(B) = O(B(1 + \log B)^{r-2})$ by the sublemma above.

(b) It suffices to count the set $\delta_\sigma(B)$ of n-tuples $\mathbf{g} = (g_1, \ldots, g_n)$ satisfying (i) and (ii) and the additional condition

(iii) $$\mathbf{g} + u_k \in C_{0,\sigma}(\mathbb{R})$$

for some maximal cone $\sigma \in \Delta$.

Now order the rays as in (a) with

$$\mathbf{g}^{D(\sigma)} = \prod_{i=1}^{r} g_i^{e_i}$$

and recall that

$$\mathbf{g}^{D(\sigma)} < (\mathbf{g} + u_k)^{D(\sigma)}$$

by (i)-(iii). Then $k \in \{1, \ldots, r\}$ and $e_k > 0$. After a permutation of the first r rays we may thus assume that $k = 1$. Moreover, from the arguments in (a), it follows that $\delta_\sigma(B)$ is the number of n-tuples $\mathbf{g}$ of positive integers satisfying:

(I) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{g}^{D(\sigma)}| \leq B$
(II) $g_{r+j} \leq (\mathbf{g} + u_1)^{E(j)}$, $j \in \{1, \ldots, d\}$
(III) $(g_1 + 1)^{e_1} \prod_{i=2}^r g_i^{e_i} > B$

Also, by (8.9)(iii) we have just as in (a) that

$$\prod_{j=1}^{d}(\mathbf{g}+u_1)^{E(j)} = (g_1+1)^{e_1-1}\prod_{i=2}^{r} g_i^{e_i-1} \le 2^{e_1-1}\prod_{i=1}^{r} g_i^{e_i-1}.$$

There are therefore for each r-tuple $(g_1,\dots,g_r)\in\mathbb{Z}_{>0}^r$ at most $2^{e_1-1}\prod_{i=1}^{r} g_i^{e_i-1}$ d-tuples satisfying (II). Moreover, from (I) and (III) we conclude that

$$g_1 = [(B/\prod_{j=2}^{r} g_i^{e_i})^{1/e_1}] \le (B/\prod_{i=2}^{r} g_i^{e_i})^{1/e_1}.$$

Hence,

$$\begin{aligned}\operatorname{Card}\delta_\sigma(B) &\le 2^{e_1-1}\sum\left((B/\prod_{i=2}^{r} g_i^{e_i})^{1-1/e_1}\prod_{i=2}^{r} g_i^{e_i-1}\right)\\ &= 2^{e_1-1}B^{1-1/e_1}\sum\left(\prod_{i=2}^{r} g_i^{e_i/e_1-1}\right)\end{aligned}$$

where in the sums $(g_2,\dots,g_r)$ runs over all r-tuples of positive integers satisfying

$$\prod_{i=1}^{r} g_i^{e_i}\le B.$$

We therefore obtain from the sublemma above that

$$\operatorname{Card}\delta_\sigma(B) = O(B(1+\log B)^{r-2})$$

as was to be proved. □

Lemma 11.26. — *Suppose that $r>1$. Then there exists for each $\varepsilon>0$ a positive constant $C=C(\varepsilon)>0$ depending only on ε such that*

$$0\le A(B)-A_{\mathbf{d}}(B)\Pi(\mathbf{d})\le C\Pi(\mathbf{d})B(1+\log B)^{r-2}$$

for all $\mathbf{d}\in\mathbb{Z}_{>0}^n$ and all $B\ge 1$.

Proof. — Fix $\mathbf{d}\in\mathbb{Z}_{>0}^n$ and $B\ge 1$. Then (cf. the proof of (11.7)) $A(B)-A_{\mathbf{d}}(B)\Pi(\mathbf{d})$ is equal to the cardinality of the set $\Omega(B)$ of $\mathbf{q}\in\mathbb{Z}_{>0}^n$ such that

(i) $\sup_{\sigma\in\Delta_{\max}}|(\mathbf{q}+\mathbf{d}-(1,\dots,1))^{D(\sigma)}|>B$
(ii) $\sup_{\sigma\in\Delta_{\max}}|\mathbf{q}^{D(\sigma)}|\le B$

But there exists for each $\mathbf{q}\in\Omega(B)$ an n-tuple $\mathbf{e}$ of positive integers with

$$e_i\le d_i,\quad i\in\{1,\dots,n\}$$

and an integer $k \in \{1, \dots, n\}$ such that $\mathbf{q} + \mathbf{e}$ belongs to the set $\delta_k(B)$ in (11.25). Hence

$$\operatorname{Card} \Omega(B) \leq \Pi(\mathbf{d}) \sum_{k=1}^{n} \operatorname{Card} \delta_k(B)$$

so that the desired assertion follows from the previous lemma. □

Main lemma 11.27. — *Let Δ be a complete regular d-dimensional fan with n rays such that $r := n - d > 1$ and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then the Euler product*

$$\prod_p \kappa_p = \prod_p \operatorname{Card} \widetilde{X}_0(\mathbb{Z}/p\mathbb{Z})/p^n$$

converges absolutely to a positive constant κ and

$$\mathcal{C}_0(B)^+ - \kappa A(B) = O(B(1 + \log B)^{r-2+1/f+\varepsilon})$$

for each $\varepsilon > 0$.

Proof. — The first assertion about κ is just a restatement of (11.15) since

$$\kappa_p = \sum_d^{(p)} (\mu(\mathbf{d})/\Pi(\mathbf{d}))$$

by (11.14). To prove the last assertion, we fix $\varepsilon > 0$ and use the identity (cf. (11.11))

$$\mathcal{C}_0^+(B) = \sum_{\mathbf{d}} \mu(\mathbf{d}) A_{\mathbf{d}}(B), \quad \mathbf{d} \in \mathbb{Z}^n_{>0}.$$

Then,

$$\mathcal{C}_0^+(B) - \kappa A(B) = \sum_{\mathbf{d}} \mu(\mathbf{d}) A_{\mathbf{d}}(B) - \sum_{\mathbf{d}} \mu(\mathbf{d}) A(B)/\Pi(\mathbf{d}), \quad \mathbf{d} \in \mathbb{Z}^n_{>0}.$$

where the last sum is absolute convergent by (11.15).

It is therefore sufficient to prove that

$$\sum_{\mathbf{d}} |\mu(\mathbf{d})| (A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) = O(B(1 + \log B)^{r-2+1/f+\varepsilon}).$$

But there exists by (11.26) a constant C such that

$$0 \leq A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B) \leq CB(1 + \log B)^{r-2}$$

for all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ and all $B \in \mathbb{Z}_{>0}$.

Hence by (11.19)(a) we get that

$$\begin{aligned} \sum_{\mathbf{d}} |\mu(\mathbf{d})| (A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) &\leq CB(1 + \log B)^{r-2}) \sum_{\mathbf{d}} |\mu(\mathbf{d})| \\ &= O(B(1 + \log B)^{r-2+1/f+\varepsilon}) \end{aligned}$$

for the partial sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ such that $\Pi(\mathbf{d}) \leq (1 + \log B)$.

For $\mathbf{d} \in \mathbb{Z}^n_{>0}$ with $\Pi(\mathbf{d}) > (1 + \log B)$, we use the fact that (cf. (11.25))

$$0 \leq A(B) - A_{\mathbf{d}}(B)\Pi(\mathbf{d}) \leq A(B) = O(B(1 + \log B)^{r-1}).$$

Hence, by (11.19)(b) we get that

$$\begin{aligned}\sum_{\mathbf{d}} |\mu(\mathbf{d})|(A(B)/\Pi(\mathbf{d}) - A_{\mathbf{d}}(B)) &= O(B(1 + \log B)^{r-1}) \sum_{\mathbf{d}} |\mu(\mathbf{d})|/\Pi(\mathbf{d}) \\ &= O(B(1 + \log B)^{r-2+1/f+\varepsilon})\end{aligned}$$

for the partial sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ such that $\Pi(\mathbf{d}) > (1+\log B)$. We therefore obtain the desired upper bound for the sum of all $\mathbf{d} \in \mathbb{Z}^n_{>0}$ by combing the two partial sums. This completes the proof. □

Let m_∞ be the Borel measure associated to the toric norm on $X(\mathbb{R})$ (cf. (9.11)(b)) and let n_∞ be the Borel measure on $X_0(\mathbb{R})$ determined by the induced norm of the toric norm (cf. (3.30) and (9.12)).

Notation 11.28. — Let $B \geq 1$ be a real number. Then,

(a) $D(B) \subset U_0(\mathbb{R})$ is the set of n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ such that

(i) $\inf(x_1, \ldots, x_n) \geq 1$,

(ii) $\sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| \leq B$.

(b) $I(B) = \int_{D(B)} \sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| dn_\infty$.

Lemma 11.29. — *We have*

$$A(B) = I(B) + O(B(1 + \log B)^{r-2}).$$

Proof. — Let $\widetilde{X}_1 = \operatorname{Spec} \mathbb{Z}[x_\rho]$, $\rho \in \Delta(1)$. Then $\widetilde{X}_1(\mathbb{Z})$ is a $\mathbb{Z}$-lattice in $\widetilde{X}_1(\mathbb{R}) = X_1(\mathbb{R}) = \mathbb{R}^n$. Let

$$d\mathbf{x} = dx_1 \cdots dx_n$$

be the Haar measure on $X_1(\mathbb{R})$ such that the volume of a fundamental domain of $\widetilde{X}_1(\mathbb{Z}) \subset X_1(\mathbb{R})$ is equal to 1. Then (cf. (9.12)) the following equality holds on $X_0(\mathbb{R}) \subset X_1(\mathbb{R})$

$$\int_{X_0(\mathbb{R})} f dn_\infty = \int_{X_0(\mathbb{R})} \left(f(\mathbf{x}) / \sup_{\sigma \in \Delta_{\max}} |\mathbf{x}^{D(\sigma)}| \right) d\mathbf{x}$$

for any continuous function f on $X_0(\mathbb{R})$ with compact support. It is therefore also true for the characteristic function g of $D(B) \subset X_0(\mathbb{R})$ since there exists an L^1-Cauchy sequence $(f_i)_{i=1}^\infty$ in $C_c(X_1(\mathbb{R}))$ converging to g (cf. [**38**, Ch IX, §3]). Hence,

$$I(B) = \int_{D(B)} d\mathbf{x}.$$

while $A(B)$ is the number of lattice points in $D(B)$. Let $\gamma(B)$ be the set of n-tuples $\mathbf{g}=(g_1,\dots,g_n)$ of positive integers such that

(i) $\sup_{\sigma\in\Delta_{\max}}|\mathbf{g}^{D(\sigma)}|\leq B$,

(ii) $\sup_{\sigma\in\Delta_{\max}}|(\mathbf{g}+\mathbf{1})^{D(\sigma)}|>B$ for $\mathbf{1}=(1,\dots,1)$.

Then, $D(B)$ contains a union of of $A(B)-\operatorname{Card}\gamma(B)$ disjoint n-dimensional unit cubes and is contained in a union of $A(B)$ n-dimensional unit cubes. Hence, by an argument due to Archimedes one has

$$0\leq A(B)-I(B)\leq \operatorname{Card}\gamma(B).$$

There is for each $\mathbf{g}\in\gamma(B)$ a binary n-tuple $\mathbf{e}$ such that $\mathbf{g}+\mathbf{e}$ belongs to one of the sets $\delta_k(B)$ described in (11.25). Therefore,

$$\operatorname{Card}\gamma(B)\leq 2^n\sum_{k=1}^{n}\operatorname{Card}\delta_k(B)$$

so that the desired assertion follows from (op. cit.). □

Let n_p be the measure on $X_0(\mathbb{Q}_p)$ determined by the induced norm of the toric norm on $X(\mathbb{Q}_p)$. Then,

$$\operatorname{Card}\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z})/p^n=n_p(\widetilde{X}_0(\mathbb{Z}_p))$$

by (9.14) and (2.15). We may thus summarize the results obtained so far as follows.

***Theorem 11.30* (preliminary form).** — *Let Δ be a complete regular d-dimensional fan with n rays such that $r:=n-d>1$ and let $X=X_\Delta$ be the toric $\mathbb{Q}$-variety defined by Δ. Suppose that the anticanonical sheaf of X is generated by its global sections. Let H be the height function on $X(\mathbb{Q})$ defined by the toric adelic norm $\|\ \|$ for X and let $C(B)$ be the number of $\mathbb{Q}$-points on U of toric height at most B.*

Let $\pi: X_0\to X$ be the principal universal torsor of X and let n_p and n_∞ be the Borel measures on $X_0(\mathbb{Q}_p)$ resp. $X_0(\mathbb{R})$ determined by the induced norm of the toric norms. Finally, let $D(B)\subset U_0(\mathbb{R})$ be as in (11.28) and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then,

$$\prod_p n_p(\widetilde{X}_0(\mathbb{Z}_p))$$

converges absolutely to a positive constant κ and

$$C(B)-\kappa\frac{\operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R}^+))}{\operatorname{Card}\widetilde{T}(\mathbb{Z})}\int_{D(B)}\sup_{\sigma\in\Delta_{\max}}|\mathbf{x}^{D(\sigma)}|dn_\infty$$
$$=O(B(1+\log B)^{r-2+1/f+\varepsilon})$$

for any $\varepsilon>0$.

Proof. — We have

$$\begin{aligned} c(m) &= c_0(m)/\operatorname{Card}(\widetilde{T}(\mathbb{Z})) \\ &= c_0(m)^+ \operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)/\operatorname{Card}(\widetilde{T}(\mathbb{Z})) \end{aligned}$$

(cf. (11.4)) and hence

$$\mathcal{C}(B) = \operatorname{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\operatorname{Card}(\widetilde{T}(\mathbb{Z}))^{-1}\mathcal{C}_0(B)^+.$$

The theorem is therefore a consequence of (11.27) and (11.29). □

It remains to study the integral:

$$\int_{D(B)} \sup_{\sigma\in\Delta_{\max}} |\mathbf{x}^{D(\sigma)}| dn_\infty = \int_{D(B)} d\mathbf{x}.$$

We shall do this by means of the toric canonical splitting over $\mathbb{R}$.

Notation 11.31. — Let $B \geq 1$ be a real number, $\sigma \in \Delta$ be a cone and $D(B)$ be as in (11.28). Then,

$$D(B,\sigma) = D(B) \cap C_\sigma(\mathbb{R}).$$

It follows immediately from the definition of $C_\sigma(\mathbb{R})$ that

$$D(B,\sigma) \cap D(B,\tau) = D(B,\sigma\cap\tau)$$

for any two cones $\sigma, \tau \in \Delta$.

Proposition 11.32. — *We have*

$$\int_{D(B)} \mathbf{x} = \sum_{\sigma\in\Delta_{\max}} \left(\int_{D(B,\sigma} \mathbf{dx} \right)$$

where $\mathbf{dx}$ *is the Lebesgue measure on* $\widetilde{X}_1(\mathbb{R}) = \mathbb{R}^{r+d}$ *normalized by* $\widetilde{X}_1(\mathbb{Z}) = \mathbb{Z}^{r+d}$.

Proof. — It follows from the definition that the sets $D(B,\sigma)$, $\sigma \in \Delta_{\max}$ form a covering of $D(B)$. It is therefore sufficient to prove that

$$\int_{D(B,\sigma\cap\tau)} \mathbf{dx} = 0$$

for each pair (σ,τ) of maximal cones in Δ. But this is done exactly as in the proof of $m(C_\sigma(\mathbb{R}) \cap C_\tau(\mathbb{R})) = 0$ in (9.16). □

Now fix a maximal d-dimensional cone $\sigma \in \Delta$ and choose an ordering $(\rho_1, \ldots, \rho_n)$ of the rays in Δ such that the last d rays are the one-dimensional faces of σ and such that (cf. (8.7))

$$D(\sigma) = \sum_{i=1}^{r} e_i D_i$$

for some non-negative integers $e_1, \ldots, e_r$, $r = n - d$.

Let $\{m^{(j)},\ 1 \leq j \leq d\}$ be the $\mathbb{Z}$-basis of M which is dual to the $\mathbb{Z}$-basis $\{n^{(j)},\ 1 \leq j \leq d\}$ of N, consisting of generators of the rays ρ_{r+j}, $1 \leq j \leq d$ of σ. Also, let

$$E(j) = D_{r+j} - \sum_{\rho \in \Delta(1)} \langle m^{(j)}, n_\rho \rangle D_\rho, \quad j \in \{1, \ldots, d\}$$

as in the proof of (11.25). Recall that $\mathbf{x}^{E(j)}$, $1 \leq i \leq d$ is a Laurent monomial in $(x_1, \ldots, x_r)$ and that (cf. (8.9)(iii))

$$\prod_{j=1}^{d} \mathbf{x}^{E(j)} = \mathbf{x}^{D(\sigma)} / \prod_{i=1}^{r} x_i \tag{11.33}$$

$D(B, \sigma)$ is then by (11.22)-(11.23) the set of $\mathbf{x} = (x_1, \ldots, x_{r+d})$ such that

(i) $\min(x_1, \ldots, x_{r+d}) \geq 1$
(ii) $\prod_{j=1}^{r} x_j^{e_j} \leq B$
(iii) $x_{r+j} \leq \mathbf{x}^{E(j)}$, $j \in \{1, \ldots, d\}$

Notations 11.34. — $\Omega(B, \sigma)$ is the set of all real r-tuples $(x_1, \ldots, x_r)$ such that

(i) $\min(x_1, \ldots, x_r) \geq 1$,
(ii) $\mathbf{x}^{D(\sigma)} \leq B$,
(iii) $\mathbf{x}^{E(j)} \geq 1$, $j \in \{1, \ldots, d\}$.

We now first integrate with respect to $(x_{r+1}, \ldots, x_{r+d})$ and then with respect to $(x_1, \ldots, x_r)$. By Fubini's theorem it follows that:

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{\Omega(B,\sigma)} \left(\prod_{j=1}^{d} (\mathbf{x}^{E(j)} - 1 \right) dx_1 \cdots dx_r \tag{11.35}$$

and by (11.33) that

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{\Omega(B,\sigma)} \mathbf{x}^{D(\sigma)} \left(\prod_{j=1}^{d} (1 - \mathbf{x}^{-E(j)} \right) \frac{dx_1}{x_1} \cdots \frac{dx_r}{x_r}. \tag{11.36}$$

We may regard $\omega = \dfrac{dx_1}{x_1} \cdots \dfrac{dx_r}{x_r}$ as a global T-invariant differential form on $T(\mathbb{R}) \subset U_0(\mathbb{R})$. Moreover, since $D(\sigma) - D_0$ and $D_{r+j} - E(j)$ are principal divisors one has

$$\mathbf{x}^{D(\sigma)} = \mathbf{x}^{D_0}$$

and

$$\mathbf{x}^{E_j} = x_{r+j}, \quad j \in \{1, \ldots, d\}$$

on $T(\mathbb{R})$.

Let $F(B) \subset T(\mathbb{R})$ be the subset of $T(\mathbb{R}) \subset U_0(\mathbb{R})$ satisfying

$$\min(x_1, \ldots, x_{r+d}) \geq 1, \tag{$*$}$$

$$(**) \qquad \mathbf{x}^{D_0} \leq B.$$

Then the projection onto the r first coordinates defines an analytic isomorphism between $\Omega(B,\sigma)$ and $F(B,\sigma)$. Hence from (11.36) we deduce that

$$(11.37) \qquad \int_{D(B,\sigma)} \mathbf{dx} = \int_{F(B,\sigma)} \mathbf{x}^{D_0} \left(\prod_{j=1}^{d}(1-1/x_{r+j})\right) \frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r}$$

Lemma 11.38. — *We have*

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{F(B)} \mathbf{x}^{D_0}\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r} + O(B(1+\log B)^{r-2})$$

for $B \geq 1$.

Proof. — It suffices by (11.37) to show that

$$\int_{F(B)} (\mathbf{x}^{D_0}/x_{r+j})\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r} = O(B(1+\log B)^{r-2})$$

for $j = 1,\dots,d$. But the measure $|\omega|$ on $T(\mathbb{R})$ defined by $\omega = \dfrac{dx_1}{x_1}\cdots\dfrac{dx_r}{x_r}$ (cf. (1.13)) does not depend on the maximal cone σ used to define $\{x_1,\dots,x_r\}$. We may thus in the description of $|\omega|$ replace the coordinates $(x_1,\dots,x_r)$ by any set of r coordinates corresponding to the rays outside a given maximal cone τ of Δ. We shall choose τ such that $-n_{r+j} \in \tau$. Then D_{r+j} has multiplicity ≥ 2 in $D(\tau)$ (cf. (8.7)). After a permutation of the rays we may change the index of ρ_{r+j} to 1 and assume that $(\rho_1,\dots,\rho_r)$ are the rays not in τ. Then it suffices to show that

$$\int_{F(B)} (\mathbf{x}^{D(\tau)}/x_1)\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r} = O(B(1+\log B)^{r-2}).$$

We shall in fact consider the integral over the larger subset $G(B)$ defined by

$$\begin{aligned} x_1 &\geq 0 \\ x_i &\geq 1, \quad i \in \{2,\dots,r\} \\ \mathbf{x}^{D(\tau)} &\leq B. \end{aligned}$$

We shall also assume that $r \geq 2$ and leave the trivial case $r = 1$ to the reader.

Now note that

$$D(\tau) = f_1D_1 + \cdots + f_rD_r$$

for some non-negative integers $e = f_1 \geq 2$, $f_2,\dots,f_r$ (cf. (8.7)). Let $E(\tau) = D(\tau) - f_1D_1$ and let $H(B)$ be the subset of $\mathbb{R}^{r-1}$ defined by

$$\mathbf{x}^{E(\tau)} \leq B \quad \text{and} \quad \min(x_2,\dots,x_r) \geq 1.$$

Then by integrating with respect to $x_1 \in [0, (B/\mathbf{x}^{E(t)})^{1/e}]$ we obtain

$$\begin{aligned}\int_{G(B)} (\mathbf{x}^{D(\tau)}/x_1)\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r} &= \frac{B^{1-1/e}}{e-1}\int_{H(B)} \mathbf{x}^{E(\tau)/e}\frac{dx_2}{x_2}\cdots\frac{dx_r}{x_r} \\ &= e^{r-1}\frac{B^{1-1/e}}{e-1}\int_{H(B^{1/e})} \mathbf{x}^{E(\tau)}\frac{dx_2}{x_2}\cdots\frac{dx_r}{x_r}.\end{aligned}$$

The latter integral can be calculated by means of the variable substitution $y_i = f_i \log x_i$, $2 \leq i \leq n$. This gives (cf. (11.39)-(11.40) below):

$$\int_{H(B^{1/e})} \mathbf{x}^{E(\tau)}\frac{dx_2}{x_2}\cdots\frac{dx_r}{x_r} = O(B^{1/e}(1+\log\frac{B}{e})^{r-2})$$

thereby completing the proof. □

Now let $b = \log B \geq 0$ and $y_i = \log x_i$ for $i = 1, \ldots, n$. This gives an analytic isomorphism between $T(\mathbb{R})^+$ and $V = \mathrm{Hom}(\mathrm{Pic}\, X, \mathbb{R})$ which sends the subset $T(\mathbb{R})_{\geq 1} \subset T(\mathbb{R})^+$ with all coordinates $x_1, \ldots, x_{r+d} \geq 1$ onto the cone $\sigma_{\mathrm{eff}}(X, U) \subset V$ described in (10.18).

Let E_b be the set of all $\varphi \in \sigma_{\mathrm{eff}}(X, U)$ with $0 \leq \varphi(D) \leq b$ and let $d\nu$ be the unique Haar measure on V such that $\mathrm{Vol}(V/L) = 1$ for the $\mathbb{Z}$-lattice $L = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{Z})$ in V. Then, by the variable substitution $y_i = \log x_i$, we get

$$\int_{D(B,\sigma)} \mathbf{dx} = \int_{E_b} \exp(y_1 + \cdots + y_n)\prod_{j=1}^{d}(1-\exp(-y_{r+j}))d\nu$$

(11.39) $$\int_{F(B)} \mathbf{x}^D\frac{dx_1}{x_1}\cdots\frac{dx_r}{x_r} = \int_{E_b} \exp(y_1+\cdots+y_n)d\nu.$$

Let $\lambda : V \to \mathbb{R}$ be the linear form $\lambda(y_1, \ldots, y_n) = y_1 + \cdots + y_n$ obtained by evaluating at the anticanonical class. By integrating along the fibres of λ (cf. (10.18)-(10.19)), we obtain

(11.40) $$\int_{E(b)} \exp(y_1+\cdots+y_n)d\nu = \alpha(X)\int_0^b \exp(y)y^{r-1}dy$$

where $\alpha(X)$ is Peyre's constant. The integral on the right hand side is equal to

$$B\sum_{k=0}^{r-1}(-1)^k\frac{(r-1)!}{(r-1-k)!}(\log B)^{r-1-k}.$$

We therefore conclude from (11.38)-(11.40) that

(11.41) $$\int_{D(B,\sigma)} \mathbf{dx} = \alpha(X)B(\log B)^{r-1} + O(B(1+\log B)^{r-2})$$

for any maximal cone σ of Δ and hence by (11.32) that

$$\int_{D(B)} \mathbf{dx} = \alpha(X)\,\mathrm{Card}(\Delta_{\max})B(\log B)^{r-1} + O(B(1+\log B)^{r-2}). \tag{11.42}$$

If we combine this with (11.30) then we obtain the asymptotic formula

$$\mathcal{C}(B) = CB(\log B)^{r-1} + O(B(1+\log B)^{r-2+1/f+\varepsilon}) \tag{11.43}$$

for any $\varepsilon > 0$ with the constant

$$C = (\mathrm{Card}\,\widetilde{T}(\mathbb{Z}))^{-1}\,\mathrm{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\alpha(X)\,\mathrm{Card}(\Delta_{\max})\Pi_p n_p(\widetilde{X}_0(\mathbb{Z}_p)).$$

We now verify that this result is compatible with Peyre's conjecture (7.7)

$$\mathcal{C}(B) = \alpha_{\mathrm{Peyre}}(X)\tau(X, \|\ \|)B(\log B)^{r-1}(1+o(1)) \tag{11.44}$$

This is by theorem (6.19) (cf. also (5.21)-(5.22)) true if the following identity holds:

$$\begin{aligned} \overline{\Theta}^1_{\{\infty\}}(T^1(A_\mathbb{Q})/T(\mathbb{Q}))m_\infty(X(\mathbb{R})) \\ = (\mathrm{Card}\,\widetilde{T}(\mathbb{Z}))^{-1}\,\mathrm{Card}(U_0(\mathbb{R})/U_0(\mathbb{R})^+)\,\mathrm{Card}(\Delta_{\max}). \end{aligned} \tag{11.45}$$

Here $\overline{\Theta}^1_{\{\infty\}}$ is the Haar measure on $T^1(A_\mathbb{Q})/T(\mathbb{Q})$ corresponding to the Haar measure $\Theta_{\{\infty\}}$ on $T(A_\mathbb{Q})$ given by the adelic order norm (5.9)(b) and the convergence factors $\beta_p = \mu_p(\widetilde{T}(\mathbb{Z}_p))^{-1}$ and $\beta_\infty = 1$ under the bijection described after (5.19). Hence if $\mathbf{K} \subset T(\mathbb{R})$ is a Borel set, and $\overline{\mathbf{K}}$ the image of $\prod_p \widetilde{T}(\mathbb{Z}_p) \times \mathbf{K}$ in $T(A_\mathbb{Q})/T^1(A_\mathbb{Q}) = \mathrm{Hom}(\mathrm{Pic}\,X, \mathbb{R})$, then

$$\Theta_{\{\infty\}}\left(\prod_p \widetilde{T}(\mathbb{Z}_p) \times \mathbf{K}\right) = \int_{\overline{K}} d\nu$$

for the Haar measure $d\nu$ on $T(A_\mathbb{Q})/T^1(A_\mathbb{Q}) = \mathrm{Hom}(\mathrm{Pic}\,X, \mathbb{R})$ used in (11.38)-(11.40).

Now recall that the bijection betwen measures on $T^1(A_\mathbb{Q})/T(\mathbb{Q})$ and $T(A_\mathbb{Q})$ is established by applying (5.19) twice to the chain of normal closed subgroups

$$T(\mathbb{Q}) \subseteq T^1(A_\mathbb{Q}) \subseteq T(A_\mathbb{Q})$$

with the counting measure on $T(\mathbb{Q})$ and the measure $d\nu$ on $T(A_\mathbb{Q})/T^1(A_\mathbb{Q})$. Note also that $T(\mathbb{R})^+ \times \prod_p \widetilde{T}(\mathbb{Z}_p)$ is a fundamental domain for the $T(\mathbb{Q})$-action on $T(A_\mathbb{Q})$. It is straightforward from this description that:

$$\overline{\Theta}^1_{\{\infty\}}(T^1(A_\mathbb{Q})/T(\mathbb{Q})) = \mathrm{Card}(T(\mathbb{R})/T(\mathbb{R})^+)(\mathrm{Card}\,\widetilde{T}(\mathbb{Z}))^{-1} \tag{11.46}$$

which is equal to 1. It is easy to verify that there is a commutative diagram with exact sequences

$$\begin{tikzcd} 1 \arrow[r] & T(\mathbb{R})^+ \arrow[r] \arrow[d] & U_0(\mathbb{R})^+ \arrow[r] \arrow[d] & U(\mathbb{R})^+ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & T(\mathbb{R}) \arrow[r] & U_0(\mathbb{R}) \arrow[r] & U(\mathbb{R}) \arrow[r] & 1 \end{tikzcd} \tag{11.47}$$

where the upper $+$-index means the subgroup given by the real connected component containing 1. One can e.g. make use of the fact that the subgroups in the first row are generated by squares of elements in the groups below.)

Finally, by (9.16) we have:

$$m_\infty(X(\mathbb{R})) = \mathrm{Card}(U(\mathbb{R})/U(\mathbb{R})^+)\,\mathrm{Card}(\Delta_{\max}) \tag{11.48}$$

Therefore, (11.45) follows from (11.46)-(11.48). We have thus given a new very different proof of the following result of Batyrev-Tschinkel [**4**].

Theorem 11.49. — *Let Δ be a complete regular d-dimensional fan with n rays and let $X = X_\Delta$ be the toric $\mathbb{Q}$-variety defined by Δ. Suppose that the anticanonical sheaf of X is generated by its global sections. Let H be the toric anticanonical height function on $X(\mathbb{Q})$ defined by the toric adelic norm $\|\ \|$ for X (cf. (11.2)) and let $\mathcal{C}(B)$ be the number of $\mathbb{Q}$-points on U of toric anticanonical height at most B. Let $\alpha_{\mathrm{Peyre}}(X)$ and $\tau(X, \|\ \|)$ be the numbers in (7.7) and let f be the smallest integer such that there exist f rays of Δ not contained in a cone of Δ. Then,*

$$\mathcal{C}(B) = \alpha_{\mathrm{Peyre}}(X)\tau(X, \|\ \|)B(\log B)^{r-1} + O(B(1+\log B)^{r-2+1/f+\varepsilon})$$

for any $\varepsilon > 0$.

The following toric surface has been discussed in talks by Batyrev and Tschinkel:

Example 11.50. — Let $N = \mathbb{Z}^2$, $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (-1,2)$, $n_4 = (-1,1)$, $n_5 = (-1,0)$, $n_6 = (-1,-1)$, $n_7 = (0,-1)$, $n_8 = (1,-1)$, $n_9 = (2,-1)$. Then there exists a unique complete regular fan (N, Δ) with 9 two-dimensional cones σ_i, $1 \le i \le 9$ and 9 one-dimensional cones ρ_i, $1 \le i \le 9$ where σ_i, $i \in \mathbb{Z}/9\mathbb{Z}$ is generated by n_{i+4} and n_{i+5} and ρ_i is generated by n_i.

Let

$$D_0 = \sum_{\rho \in \Delta(1)} D_\rho$$

and let U_i, $1 \le i \le 9$ be the affine toric plane defined by σ_i. Let χ^{m_i}, $m_i = m(\sigma_i) \in M$ be the unique character of $U \subset X_\Delta$ which on U_i represents the Cartier divisor with Weil divisor D_0 and let

$$D(\sigma_i) := D_0 + \sum_{\rho\in\Delta(1)} \langle -m(\sigma_i), n_\rho\rangle D_\rho.$$

Finally, let D_i be the prime divisor which corresponds to ρ_i. Then,

$$D(\sigma_{i-1}) = D(\sigma_i) = D(\sigma_{i+1}) = D_{i-2} + 2D_{i-1} + 3D_i + 2D_{i+1} + D_{i+2}$$

for all $i \in \mathbb{Z}/9\mathbb{Z}$ with $i \equiv 0 \bmod 3$.

There exists thus for each closed point P on X_Δ a divisor $D(\sigma)$, $\sigma \in \Delta_{\max}$ with $P \notin \operatorname{Supp} D(\sigma)$ so that $\mathcal{O}(D_0)$ is generated by its global sections. Also, it is easy to see that $m = 0, m_3, m_6, m_9$ are the only elements $m \in M$ such that

$$D_0 + \sum_{\rho\in\Delta(1)} \langle -m, n_\rho\rangle D_\rho$$

is effective. Therefore, by [**24**, p. 66] it follows that:

$$H^0(X_\Delta, \mathcal{O}(D_0)) = \mathbb{Q} \oplus \mathbb{Q}\chi^{-m_3} \oplus \mathbb{Q}\chi^{-m_6} \oplus \mathbb{Q}\chi^{-m_9}.$$

This means that the morphism $g : X_\Delta \to \mathbb{P}^3_{\mathbb{Q}}$ defined by the linear sytem $|D_0|$ restricts to the open immersion restriction

$$g_U : U \to \mathbb{P}^3_{\mathbb{Q}} \quad \text{defined by} \quad (1, \chi^{-m_3}, \chi^{-m_6}, \chi^{-m_9}).$$

If we introduce projective coordinates (z_0, z_3, z_6, z_9) for $\mathbb{P}^3_{\mathbb{Q}}$, then the scheme-theoretic image X' of g is given by the closed subscheme with equation $z_0^3 = z_3z_6z_9$. The morphism $g : X \to X'$ is a toric morphism corresponding to a morphism of fans $(N, \Delta) \to (N, \Delta')$. The fan Δ' is the complete non-regular fan with three two-dimensional cones τ_{3i}, $i = 1, 2, 3$ generated by n_{3i+3} and n_{3i+6}.

Therefore, the number $\mathcal{C}(B)$ of $\mathbb{Q}$-points on $U \subset X_\Delta$ of toric height at most B is equal to the number of integral points $(z_0, z_3, z_6, z_9) \in \mathbb{P}^3(\mathbb{Z})$ with $z_0^3 = z_3z_6z_9 \neq 0$ of height:

$$\max(|z_0|, |z_3|, |z_6|, |z_9|) \leq B.$$

The Picard group of X_Δ is of rank $r = \operatorname{Card}\Delta(1) - \dim\Delta = 9-2 = 7$ (cf. (8.2)). Further, if p is a prime number, then we can determine the cardinality of $\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z})$ by means of the bijection between $\widetilde{U}(\mathbb{Z}/p\mathbb{Z})$-orbits under the action of $\widetilde{U}(\mathbb{Z}/p\mathbb{Z})$ on $\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z})$ and the cones in Δ (see [**24**, p. 94]). This gives

$$\operatorname{Card}\widetilde{X}_\Delta(\mathbb{Z}/p\mathbb{Z}) = 1(p-1)^2 + 9(p-1) + 9 = p^2 + 7p + 1.$$

Therefore, the principal universal torsor $\widetilde{X}_0$ over $\widetilde{X} = \widetilde{X}_\Delta$ satisfies

$$\operatorname{Card}\widetilde{X}_0(\mathbb{Z}/p\mathbb{Z}) = \operatorname{Card}\widetilde{X}(\mathbb{Z}/p\mathbb{Z})\operatorname{Card}\widetilde{T}(\mathbb{Z}/p\mathbb{Z}) = (p^2 + 7p + 1)(p-1)^7,$$

and (cf. (2.15))

$$n_p(\widetilde{X}_0(\mathbb{Z}_p)) = (1 + 7/p + 1/p^2)(1 - 1/p)^7.$$

Also, Peyre's constant $\alpha(X)$ is easily seen to be equal to the $\mathrm{Vol}(P)/9$, where P is the polytope in $\mathbb{R}^6_{\geq 0}$ with coordinates $(y_1, y_2, y_4, y_5, y_7, y_8)$ defined by

$$\max(y_1 + 2y_2 + 2y_4 + y_5, y_4 + 2y_5 + 2y_7 + y_8, y_7 + 2y_8 + 2y_1 + y_2) \leq 1$$

and a further calculation gives $\alpha(X) = 1/25920$.

Hence by (11.43) we get that

$$\mathcal{C}(B) = CB(\log B)^6 + O(B(1 + \log B)^{6-1/2+\varepsilon}) \tag{11.51}$$

for any $\varepsilon > 0$ with

$$C = \frac{1}{720} \prod_p (1 + 7/p + 1/p^2)(1 - 1/p)^7.$$

This asymptotic formula was first established by Batyrev and Tschinkel as a corollary of [**4**] (cf. [**6**] which we received after completing this paper). Another treatment of this asymptotic formula, but without a discussion of the value for C, can be found in the paper of Fouvry [**22**]. We have also after the completion of this paper received the paper [**13**] of de la Bretèche on the counting function for $z_0^3 = z_3 z_6 z_9 \neq 0$. He obtains a more precise asymptotic formula than in (11.51). He has also just before this book goes to press improved the estimates of the error terms for other toric varieties.

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