VICTOR V. BATYREV
YURI TSCHINKEL

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TAMAGAWA NUMBERS OF POLARIZED ALGEBRAIC VARIETIES

by

Victor V. Batyrev and Yuri Tschinkel

Dedicated to Professor Yu. I. Manin on his 60th birthday

Abstract. — Let $\mathcal{L} = (L, \|\cdot\|_v)$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety V over a number field F. Denote by $N(V, \mathcal{L}, B)$ the number of rational points in V having $\mathcal{L}$-height $\leqslant B$. In this paper we consider the problem of a geometric and arithmetic interpretation of the asymptotic for $N(V, \mathcal{L}, B)$ as $B \to \infty$ in connection with recent conjectures of Fujita concerning the Minimal Model Program for polarized algebraic varieties.

We introduce the notions of *$\mathcal{L}$-primitive varieties* and *$\mathcal{L}$-primitive fibrations*. For $\mathcal{L}$-primitive varieties V over F we propose a method to define an adelic Tamagawa number $\tau_{\mathcal{L}}(V)$ which is a generalization of the Tamagawa number $\tau(V)$ introduced by Peyre for smooth Fano varieties. Our method allows us to construct Tamagawa numbers for $\mathbf{Q}$-Fano varieties with at worst canonical singularities.

In a series of examples of smooth polarized varieties and singular Fano varieties we show that our Tamagawa numbers express the dependence of the asymptotic of $N(V, \mathcal{L}, B)$ on the choice of v-adic metrics on $\mathcal{L}$.

1. Introduction

Let F be a number field (a finite extension of $\mathbf{Q}$), $\mathrm{Val}(F)$ the set of all valuations of F, F_v the v-adic completion of F with respect to $v \in \mathrm{Val}(F)$, and $|\cdot|_v : F_v \to \mathbf{R}$ the v-adic norm on F_v normalized by the conditions $|x|_v = |N_{F_v/\mathbf{Q}_p}(x)|_p$ for p-adic valuations $v \in \mathrm{Val}(F)$.

Consider a projective space $\mathbf{P}^m$ with homogeneous coordinates $(z_0, ..., z_m)$ and a locally closed quasi-projective subvariety $V \subset \mathbf{P}^m$ defined over F (we want to stress that V is not assumed to be projective). Let $V(F)$ be the set of points in V

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with coordinates in F. A **standard height function** $H : \mathbf{P}^m(F) \to \mathbf{R}_{>0}$ is defined as follows

$$H(x) := \prod_{v \in \mathrm{Val}(F)} \max_{j=0,\ldots,m} \{|z_j(x)|_v\}.$$

A basic fact about the standard height function H claims that the set

$$\{x \in \mathbf{P}^m(F) \, : \, H(x) \leq B\}$$

is finite for any real number B [**29**]. We set

$$N(V, B) = \#\{x \in V(F) \, : \, H(x) \leq B\}.$$

It is an experimental fact that whenever one succeeds in proving an asymptotic formula for the function $N(V, B)$ as $B \to \infty$, one obtains the asymptotic

$$N(V, B) = c(V) B^{a(V)} (\log B)^{b(V)-1} (1 + o(1)) \tag{1}$$

with some constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$, and $c(V) \in \mathbf{R}_{>0}$. We want to use this observation as our starting point. It seems natural to ask the following:

Question A. — *For which quasi-projective subvarieties $V \subset \mathbf{P}^m$ defined over F do there exist constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$ and $c(V) \in \mathbf{R}_{>0}$ such that the asymptotic formula* (1) *holds?*

Question B. — *Does there exist a quasi-projective variety V over F with an asymptotic which is different from* (1)*?*

In this paper we will be interested not in Questions A and B themselves but in a related to them another natural question:

Question C. — *Assume that V is an irreducible quasi-projective variety over a number field F such that the asymptotic formula* (1) *holds. How to compute the constants $a(V), b(V)$ and $c(V)$ in this formula via some arithmetical properties of V over F and geometrical properties of V over $\mathbf{C}$?*

To simplify our terminology, it will be convenient for us to postulate:

Assumption. — For all quasi-projective V', V with $V' \subset V \subset \mathbf{P}^m$ and with $|V(F)| = \infty$ there exists the limit

$$\lim_{B \to \infty} \frac{N(V', B)}{N(V, B)}.$$

The following definitions have been useful to us:

Definition S_1. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ over a number field F is called **weakly saturated**, if $|V(F)| = \infty$ and if for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$ one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} < 1.$$

It is important to remark that Question C really makes sense *only for weakly saturated* varieties. Indeed, if there were a locally closed subvariety $W \subset V$ with $\dim W < \dim V$ and

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 1,$$

then it would be enough to answer Question C for each irreducible component of W and for all possible intersections of these components (i.e., one could forget about the existence of V and reduce the situation to a lower-dimensional case). In general, it is not easy to decide whether or not a given locally closed subvariety $V \subset \mathbf{P}^m$ is weakly saturated. We expect (and our assumption implies this) that the orbits of connected subgroups $G \subset PGL(m+1)$ are examples of weakly saturated varieties $V \subset \mathbf{P}^m$ (see 3.2.8).

Definition S_2. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ with $|N(V,B)| = \infty$ is called **strongly saturated**, if for all dense Zariski open subsets $U \subset V$, one has

$$\lim_{B\to\infty} \frac{N(U,B)}{N(V,B)} = 1.$$

First of all, if $V \subset \mathbf{P}^m$ is a strongly saturated subvariety, then for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$, one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 0,$$

i.e., V is weakly saturated.

On the other hand, if $V \subset \mathbf{P}^m$ is weakly saturated, but not strongly saturated, then there must be an infinite sequence $W_1, W_2, \ldots$ of pairwise different locally closed irreducible subvarieties $W_i \subset V$ with $\dim W_i < \dim V$ and $|W_i(F)| = \infty$ such that for an arbitrary positive integer k one has

$$0 < \lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k, B)}{N(V,B)} < 1.$$

Moreover, in this situation one can always choose the varieties W_i to be strongly saturated (otherwise one could find $W_i' \subset W_i$ with $\dim W_i' < \dim W_i$ with the same properties as W_i etc.). The strong saturatedness of each W_i implies that

$$\lim_{B\to\infty} \frac{N(W_{i_1} \cap \cdots \cap W_{i_l}, B)}{N(V,B)} = 0$$

for all pairwise different $i_1, \ldots, i_l$ and $l \geqslant 2$. In particular, one has

$$\sum_{i=1}^{k} \lim_{B\to\infty} \frac{N(W_i, B)}{N(V, B)} = \lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k, B)}{N(V, B)} < 1 \ \forall k > 0.$$

Definition F. — Let V be a weakly saturated quasi-projective variety in $\mathbf{P}^m$ and $W_1, W_2, \ldots$ an infinite sequence of strongly saturated irreducible subvarieties W_i having the property

$$0 < \theta_i := \lim_{B\to\infty} \frac{N(W_i, B)}{N(V, B)} < 1 \ \forall i > 0.$$

We say that the set $\{W_1, W_2, \ldots\}$ forms an **asymptotic arithmetic fibration** on V, if the following equality holds

$$\sum_{i=1}^{\infty} \theta_i = 1.$$

The main purpose of this paper is to explain some geometric and arithmetic ideas concerning weakly saturated varieties and their asymptotic arithmetic fibrations by strongly saturated subvarieties. It seems that the cubic bundles considered in [**9**] are examples of such a fibration. We want to remark that most of the above terminology grew out of our attempts to restore a conjectural picture of the interplay between the geometry of algebraic varieties and the arithmetic of the distribution of rational points on them after we have found in [**9**] an example which contradicted general expectations formulated in [**4**].

In section 2 we consider smooth quasi-projective varieties V over $\mathbf{C}$ together with a polarization $\mathcal{L} = (L, \|\cdot\|_h)$ consisting of an ample line bundle L on V equipped with a positive hermitian metric $\|\cdot\|_h$. Our main interest in this section is a discussion of geometric properties of V in connection with the Minimal Model Program [**27**] and its version for polarized algebraic varieties suggested by Fujita [**20, 23, 24**]. We introduce our main geometric invariants $\alpha_{\mathcal{L}}(V)$, $\beta_{\mathcal{L}}(V)$, and $\delta_{\mathcal{L}}(V)$ for an arbitrary $\mathcal{L}$-polarized variety V. It is important to remark that we will be only interested in the case $\alpha_{\mathcal{L}}(V) > 0$. The number $\alpha_{\mathcal{L}}(V)$ was first introduced in [**4, 5**], it equals to the opposite of the so called *Kodaira energy* (investigated by Fujita in [**20, 23, 24**]). Our basic geometric notion in the study of $\mathcal{L}$-polarized varieties V with $\alpha_{\mathcal{L}}(V) > 0$ is the notion of an *$\mathcal{L}$-primitive* variety. In Fujita's program for polarized varieties with negative Kodaira energy $\mathcal{L}$-primitive varieties play the same role as $\mathbf{Q}$-Fano varieties in Mori's program for algebraic varieties with negative Kodaira dimension. In particular, one expects the existence of so called *$\mathcal{L}$-primitive fibrations*, which are analogous to $\mathbf{Q}$-Fano fibrations in Mori's program. We show that on $\mathcal{L}$-primitive varieties there exists a canonical volume measure. Moreover, this measure allows us to construct a descent of hermitian metrics to the base of $\mathcal{L}$-primitive fibrations. Many geometric ideas of this section are inspired by [**4, 5**].

In section 3 we introduce our main arithmetic notions of *weakly* and *strongly $\mathcal{L}$-saturated* varieties. Our first main diophantine conjecture claims that if an adelic $\mathcal{L}$-polarized quasi-projective algebraic variety V over a number field F is strongly $\mathcal{L}$-saturated, then the corresponding $\mathcal{L}$-polarized complex algebraic variety $V(\mathbf{C})$ is $\mathcal{L}$-primitive. Moreover, we conjecture that if an adelic $\mathcal{L}$-polarized quasi-projective algebraic variety V over a number field F is weakly $\mathcal{L}$-saturated, then the corresponding $\mathcal{L}$-polarized complex algebraic variety $V(\mathbf{C})$ admits an $\mathcal{L}$-primitive fibration having infinitely many fibers W defined over F which form an asymptotic arithmetic fibration. These conjectures allow us to establish a connection between the geometry of $V(\mathbf{C})$ and the arithmetic of V. Following this idea, we explain a construction of an adelic measure on an arbitrary $\mathcal{L}$-primitive variety V with $\alpha_{\mathcal{L}}(V) > 0$ and of the corresponding Tamagawa number $\tau_{\mathcal{L}}(V)$ as a regularized adelic integral of this measure. Our construction generalizes the definition of Tamagawa measures associated with a metrization of the canonical line bundle due to Peyre [**30**]. We expect that for strongly $\mathcal{L}$-saturated varieties V the number $\tau_{\mathcal{L}}(V)$ reflects the dependence of the constant $c(V)$ in the asymptotic formula (1) on the adelic metrization of the ample line bundle L. We discuss the natural question about the behavior of the adelic constant $\tau_{\mathcal{L}}(W)$ for fibers W in $\mathcal{L}$-primitive fibrations on weakly $\mathcal{L}$-saturated varieties.

In section 4 we show that our diophantine conjectures agree with already known examples of asymptotic formulas established for polarized algebraic varieties through the study of analytic properties of height zeta functions.

In Section 5 we illustrate our expectations for the constants $a(V), b(V)$ and $c(V)$ in the asymptotics of $N(V, B)$ on some examples of smooth Zariski dense subsets V in Fano varieties with singularities.

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2. Geometry of $\mathcal{L}$-polarized varieties

2.1. $\mathcal{L}$-closure. — Let V be a smooth irreducible quasi-projective algebraic variety over $\mathbf{C}$, $V(\mathbf{C})$ the set of closed points of V, L an ample invertible sheaf on V, i.e., $L^{\otimes k} = i^*\mathcal{O}_{\mathbf{P}^m}(1)$ for some $k > 0$ and some embedding $i : V \hookrightarrow \mathbf{P}^m$. Since we don't assume V to be compact, the invertible sheaf L on V itself contains too little information about the embedding $i : V \hookrightarrow \mathbf{P}^m$. For instance, let V be an affine variety of positive dimension. Then the space of global sections of L is infinite dimensional and we don't know anything about the projective closure of V in $\mathbf{P}^m$ even though we know that the invertible sheaf $L^{\otimes k}$ is isomorphic to $i^*\mathcal{O}_{\mathbf{P}^m}(1)$. This situation changes if one considers L together with a positive hermitian metric, i.e., an ample metrized invertible sheaf $\mathcal{L}$ associated with L. Let us choose a positive hermitian metric h on $\mathcal{O}_{\mathbf{P}^m}(1)$ (e.g. Fubini-Study metric) and denote by $\|\cdot\|_h$ the induced metric on $L^{\otimes k}$. Thus we obtain a metric $\|\cdot\|$ on L by putting $\|s(x)\| := \|s^k(x)\|_h^{1/k}$ for any $x \in V(\mathbf{C})$ and any section $s \in \mathrm{H}^0(U, L)$ over an open subset $U \subset V$.

Definition 2.1.1. — We call a pair $\mathcal{L} = (L, \|\cdot\|)$ **an ample metrized invertible sheaf** associated with L. We denote by $\mathcal{L}^{\otimes\nu}$ the pair $(L^{\otimes\nu}, \|\cdot\|^{\nu})$.

Our next goal is to show that an ample metrized invertible sheaf contains almost complete information about the projective closure of V in $\mathbf{P}^m$.

Definition 2.1.2. — Let $\mathcal{L} = (L, \|\cdot\|)$ be an ample metrized invertible sheaf on a complex irreducible quasi-projective variety V. We denote by

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L})$$

the subspace of $\mathrm{H}^0(V, L)$ consisting of those global sections s of L over V such that the corresponding continuous function $x \mapsto \|s(x)\|$ $(x \in V(\mathbf{C}))$ is globally bounded on $V(\mathbf{C})$ from above by a positive constant $C(s)$ depending only on s. We call $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L})$ the **space of globally bounded sections** of $\mathcal{L}$.

Proposition 2.1.3. — *Let $\overline{V}$ be the normalization of the projective closure of V with respect to the embedding $i : V \hookrightarrow \mathbf{P}^m$ with $L^{\otimes k} = i^*\mathcal{O}_{\mathbf{P}^m}(1)$. Denote by $e : \overline{V} \to \mathbf{P}^m$ the corresponding finite projective morphism. Then one has a natural isomorphism*

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k}) \cong \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1)).$$

Proof. — Since $\overline{V}(\mathbf{C})$ is compact, the continuous function $x \mapsto \|s(x)\|$ is globally bounded on $\overline{V}(\mathbf{C})$ for any $s \in \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$. Therefore, we obtain that $\mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$ is a subspace of $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$.

Now let $f \in \mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$ be a globally bounded on $V(\mathbf{C})$ section of $i^*\mathcal{O}_{\mathbf{P}^m}(1)$. Since $\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k})$ is a subspace of $\mathrm{H}^0(V, L^{\otimes k})$, the section f uniquely extends to a global meromorphic section $\overline{f} \in \mathrm{H}^0(\overline{V}, e^*\mathcal{O}_{\mathbf{P}^m}(1))$. Since a bounded meromorphic function is holomorphic, $\overline{f}$ is a global regular section of $e^*\mathcal{O}_{\mathbf{P}^m}(1)$ (we apply the theorem of Riemann to some resolution of singularities $\rho : X \to \overline{V}$ and use the fact that $\rho_*\mathcal{O}_X = \mathcal{O}_{\overline{V}}$). Thus we have

$$\mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes k}) \subset \mathrm{H}^0(\overline{V}, e^*\mathcal{O}(1)).$$

□

Definition 2.1.4. — We define the graded $\mathbf{C}$-algebra

$$\mathrm{A}(V, \mathcal{L}) = \bigoplus_{\nu \geqslant 0} \mathrm{H}^0_{\mathrm{bd}}(V, \mathcal{L}^{\otimes\nu}).$$

Using 2.1.3, one immediately obtains:

Corollary 2.1.5. — *The graded algebra* $\mathrm{A}(V,\mathcal{L})$ *is finitely generated.*

Definition 2.1.6. — We call the normal projective variety

$$\overline{V}^{\mathcal{L}} = \operatorname{Proj} \mathrm{A}(V,\mathcal{L}).$$

the $\mathcal{L}$-**closure** of V with respect to an ample metrized invertible sheaf $\mathcal{L}$.

Remark 2.1.7. — By 2.1.3, $\overline{V}^{\mathcal{L}}$ is isomorphic to $\overline{V}$. Therefore, we have obtained a way to define the normalization of the projective closure of V with respect to an $L^{\otimes k}$-embedding via a notion of an ample metrized invertible sheaf $\mathcal{L}$ on V.

2.2. Kodaira energy and $\alpha_{\mathcal{L}}(V)$. — Let X be a normal irreducible algebraic variety of dimension n. We denote by $\mathrm{Div}(X)$ (resp. by $\mathrm{Z}_{n-1}(X)$) the group of Cartier divisors (resp. Weil divisors) on X. An element of $\mathrm{Div}(X)\otimes\mathbf{Q}$ (resp.$\mathrm{Z}_{n-1}(X)\otimes\mathbf{Q}$) is called a $\mathbf{Q}$-Cartier divisor (resp. a $\mathbf{Q}$-divisor). By K_X we denote a divisor of a meromorphic differential n-form on X, where K_X is considered as an element of $\mathrm{Z}_{n-1}(X)$.

Definition 2.2.1. — Let X be a projective variety and L be an invertible sheaf on X. The **Iitaka-dimension** $\kappa(L)$ is defined as

$$\kappa(L) = \begin{cases} -\infty & \text{if } \mathrm{H}^0(X, L^{\otimes\nu}) = 0 \text{ for all } \nu > 0 \\ \operatorname{Max} \dim \phi_{L^{\otimes\nu}}(X) : & \mathrm{H}^0(X, L^{\otimes\nu}) \neq 0 \end{cases}$$

where $\phi_{L^{\otimes\nu}}(X)$ is the closure of the image of X under the rational map

$$\phi_{L^{\otimes\nu}} : X \to \mathbf{P}(\mathrm{H}^0(X, L^{\otimes\nu})).$$

A Cartier divisor L is called **semi-ample** (resp. **effective**), if $L^{\otimes\nu}$ is generated by global sections for some $\nu > 0$ (resp. $\kappa(L) \geqslant 0$).

Remark 2.2.2. — The notions of Iitaka-dimension, ampleness and semi-ampleness obviously extend to $\mathbf{Q}$-Cartier divisors. Let L be a Cartier divisor. Then for all $\kappa_1, k_2 \in \mathbf{N}$ we set

$$\kappa(L^{\otimes k_1/k_2}) := \kappa(L),$$

$$L^{\otimes k_1/k_2} \text{ is ample} \Leftrightarrow L \text{ is ample},$$

and

$$L^{\otimes k_1/k_2} \text{ is semi-ample} \Leftrightarrow L \text{ is semi-ample}.$$

Definition 2.2.3. — Let X be a smooth projective variety. We denote by $\mathrm{NS}(X)$ the group of divisors on X modulo numerical equivalence and set

$$\mathrm{NS}(X)_{\mathbf{R}} = \mathrm{NS}(X) \otimes \mathbf{R}.$$

By $[L]$ we denote the class of a divisor L in $\mathrm{NS}(X)$. The **cone of effective divisors** $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{NS}(X)_{\mathbf{R}}$ is defined as the closure of the subset

$$\bigcup_{\kappa(L) \geqslant 0} \mathbf{R}_{\geqslant 0}[L] \subset \mathrm{NS}(X)_{\mathbf{R}}.$$

Definition 2.2.4. — Let V be a smooth quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$, $\overline{V}^{\mathcal{L}}$ the $\mathcal{L}$-closure of V and ρ some resolution of singularities

$$\rho : X \to \overline{V}^{\mathcal{L}}.$$

We define the number

$$\alpha_{\mathcal{L}}(V) = \inf\{t \in \mathbf{Q} : t[\rho^* L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\}.$$

and call it the **$\mathcal{L}$-index** of V.

Remark 2.2.5. — It is easy to see that the $\mathcal{L}$-index does not depend on the choice of the resolution ρ.

Remark 2.2.6. — The $\mathcal{L}$-index $\alpha_{\mathcal{L}}(V)$ for smooth projective varieties V was first introduced in **[4]** and **[5]**. We remark that the opposite number $-\alpha_{\mathcal{L}}(V)$ coincides with the notion of **Kodaira energy** introduced and investigated by Fujita in **[20, 23, 24]**:

$$\kappa\epsilon(V, L) = -\alpha_{\mathcal{L}}(V) = -\inf\{t \in \mathbf{Q} : \kappa((L)^{\otimes t} \otimes K_V) \geqslant 0\}.$$

From the viewpoint of our diophantine applications it is much more natural to consider $\alpha_{\mathcal{L}}(V)$ instead of its opposite $-\alpha_{\mathcal{L}}(V)$. The only reason that we could see for introducing the number $-\alpha_{\mathcal{L}}(V)$ instead of $\alpha_{\mathcal{L}}(V)$ is some kind of compatibility between the notions of *Kodaira energy* and *Kodaira dimension*, e.g. Kodaira energy must be positive (resp. negative) iff the Kodaira dimension is positive (resp. negative).

The following statement was conjectured in **[4]** (see also **[5, 21, 22]**):

***Conjecture 2.2.7* (Rationality).** — *Assume that $\alpha_{\mathcal{L}}(V) > 0$. Then $\alpha_{\mathcal{L}}(V)$ is rational.*

Remark 2.2.8. — It was shown in **[5]** that this conjecture follows from the Minimal Model Program. In particular, it holds for $\dim V \leqslant 3$. If $\dim V = 1$, then the only possible values of $\alpha_{\mathcal{L}}(V)$ are numbers $2/k$ with $(k \in \mathbf{N})$. If $\dim V = 2$, then $\alpha_{\mathcal{L}}(V) \in \{2/k, 3/l\}$ with $(k, l \in \mathbf{N})$.

Definition 2.2.9. — A normal irreducible algebraic variety W is said to have at worst **canonical** (resp. **terminal**) singularities if K_W is a $\mathbf{Q}$-Cartier divisor and if for some (or every) resolution of singularities

$$\rho : X \to W$$

one has

$$K_X = \rho^*(K_W) \otimes \mathcal{O}(D)$$

where D is an effective $\mathbf{Q}$-Cartier divisor (resp. the support of the effective divisor D coincides with the exceptional locus of ρ). Irreducible components of the exceptional locus of ρ which are not contained in the support of D are called **crepant divisors** of the resolution ρ.

Definition 2.2.10. — A normal irreducible algebraic variety W is called a **canonical $\mathbf{Q}$-Fano** variety, if W has at worst canonical singularities and K_W^{-1} is an ample $\mathbf{Q}$-Cartier divisor. A maximal positive rational number $r(W)$ such that $K_W^{-1} = L^{\otimes r(W)}$ for some Cartier divisor L is called the **index** of a canonical $\mathbf{Q}$-Fano variety W (obviously, one has $r(W) = \alpha_{\mathcal{L}}(W)$ for some positive metric on L).

The following conjecture is due to Fujita [**20**]:

***Conjecture 2.2.11* (Spectrum Conjecture).** — *Let $S(n)$ be the set all possible values of $\alpha_{\mathcal{L}}(V)$ for smooth quasi-projective algebraic varieties V of dimension $\leqslant n$ with an ample metrized invertible sheaf $\mathcal{L}$. Then for any $\varepsilon > 0$ the set*

$$\{\alpha_{\mathcal{L}}(V) \in S(n) \ : \ \alpha_{\mathcal{L}}(V) > \varepsilon\}$$

is finite.

This conjecture follows from the Minimal Model Program [**27**] and from the following conjecture on the boundedness of index for Fano varieties with canonical singularities:

***Conjecture 2.2.12* (Boundedness of Index).** — *The set of possible values of index $r(W)$ for canonical $\mathbf{Q}$-Fano varieties W of dimension n is finite.*

In particular, both conjectures are true for $\mathbf{Q}$-Fano varieties of dimension $n \leqslant 3$ [**2, 20, 23, 24, 28, 32**].

2.3. $\mathcal{L}$-primitive varieties

Definition 2.3.1. — Let X be a projective algebraic variety. We call an effective $\mathbf{Q}$-divisor D **rigid**, if $\kappa(D) = 0$.

Proposition 2.3.2. — An effective $\mathbf{Q}$-divisor D on X is rigid if and only if there exist finitely many irreducible subvarieties $D_1, \ldots, D_l \subset X$ $(l \geqslant 0)$ of codimension 1 such that $D = r_1 D_1 + \cdots + r_l D_l$ with $r_1, \ldots, r_l \in \mathbf{Q}_{>0}$ and

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) = 1 \ \ \forall\ (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}.$$

Proof. — Let D be rigid. Take a positive integer m_0 such that $m_0 D$ is a Cartier divisor and $\dim \mathrm{H}^0(X, \mathcal{O}(m_0 D)) = 1$. Denote by $D_1, \ldots, D_l$ the irreducible components of the divisor (s) of a non-zero section $s \in \mathrm{H}^0(X, \mathcal{O}(mD))$. One has

$$(s) = m_1 D_1 + \cdots + m_l D_l \quad m_1, \ldots, m_l \in \mathbf{N}.$$

Since $\mathcal{O}(D_i)$ admits at least one global non-zero section we obtain that

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1' D_1 + \cdots + n_l' D_l)) \geqslant \dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)),$$

whenever $n_1 \geqslant n_1', \ldots, n_l \geqslant n_l'$ for $(n_1', \ldots, n_l'),\ (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}$. This implies that

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) \geqslant 1 \ \ \forall\, (n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}.$$

On the other hand, for any $(n_1, \ldots, n_l) \in \mathbf{Z}^l_{\geqslant 0}$ there exists a positive integer n_0 such that $n_0 m_1 \geqslant n_1, \ldots, n_0 m_l \geqslant n_l$. Therefore,

$$\dim \mathrm{H}^0(X, \mathcal{O}(n_1 D_1 + \cdots + n_l D_l)) \leqslant \dim \mathrm{H}^0(X, \mathcal{O}(n_0 m_0 D)) = 1,$$

since $\kappa(n_0 m_0 D) = \kappa(D) = 0$. □

Corollary 2.3.3. — *Let $D_1, \ldots, D_l \subset X$ be all irreducible components of the support of a rigid $\mathbf{Q}$-Cartier divisor D. Then a linear combination*

$$n_1 D_1 + \cdots + n_l D_l, \ \ n_1, \ldots, n_l \in \mathbf{Z}$$

is a principal divisor, iff $n_1 = \cdots = n_l = 0$.

Proof. — Assume that $n_1 D_1 + \cdots + n_l D_l$ is linearly equivalent to 0. Then the effective Cartier divisor $D_0 = \sum_{n_i \geqslant 0} n_i D_i$ is linearly equivalent to the effective Cartier divisor $D_0' = \sum_{n_j < 0} (-n_j) D_j$. Since D_0 and D_0' have different supports we have $\dim \mathrm{H}^0(X, \mathcal{O}(D_0)) \geqslant 2$. Contradiction to 2.3.2. □

Definition 2.3.4. — Let V be a smooth quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$ and $\overline{V}^{\mathcal{L}}$ the projective $\mathcal{L}$-closure of V. The variety V is called **$\mathcal{L}$-primitive**, if the number $\alpha_{\mathcal{L}}(V)$ is rational and if for some resolution of singularities

$$\rho \,:\, X \to \overline{V}^{\mathcal{L}}$$

one has $\rho^*(L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X = \mathcal{O}(D)$, where D is a rigid effective $\mathbf{Q}$-Cartier divisor on X.

Remark 2.3.5. — It is easy to see that the notion of an $\mathcal{L}$-primitive variety doesn't depend on the choice of a resolution of singularities ρ. Since V is smooth, we can always assume that the natural mapping

$$\rho \,:\, \rho^{-1}(V) \to V$$

is an isomorphism.

Example 2.3.6. — Let V_1 and V_2 be two smooth quasi-projective varieties with ample metrized invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ (resp. on V_1 and V_2). Assume that V_1 (resp. V_2) is $\mathcal{L}_1$-primitive (resp. $\mathcal{L}_1$-primitive). Then the product $V = V_1 \times V_2$ is $\mathcal{L}$-primitive, where $\mathcal{L} = \pi_1^*\mathcal{L}_1 \otimes \pi_2^*\mathcal{L}_2$.

Our main list of examples of $\mathcal{L}$-primitive varieties is obtained from canonical $\mathbf{Q}$-Fano varieties:

Example 2.3.7. — Let V be the set of nonsingular points of a canonical $\mathbf{Q}$-Fano variety W with an ample metrized invertible sheaf $\mathcal{L} = (L, \|\cdot\|)$ such that $K_W^{-1} = L^{\otimes r(W)}$ ($W = \overline{V}^{\mathcal{L}}$). Then V is an $\mathcal{L}$-primitive variety with $\mathcal{L}$-index $r(W)$. Indeed, let $\rho : X \to W$ be a resolution of singularities. By 2.2.10, we have

$$\rho^*(L)^{\otimes r(W)} \otimes K_X = \mathcal{O}(D),$$

where D is an effective $\mathbf{Q}$-Cartier divisor. Since the support of D consists of exceptional divisors with respect to ρ, D is rigid (see 2.3.2).

We expect that the above examples cover all $\mathcal{L}$-primitive varieties:

Conjecture 2.3.8 **(Canonical Q-Fano contraction).** — *Let V be an $\mathcal{L}$-primitive variety with $\alpha_{\mathcal{L}}(V) > 0$. Then there exists a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ and a birational projective morphism $\pi : X \to W$ to a canonical $\mathbf{Q}$-Fano variety W such that $\pi^*K_W^{-1} \cong \rho^*(L)^{\alpha_{\mathcal{L}}(V)}$ (i.e., $\alpha_{\mathcal{L}}(V) = r(W)$) and the support of D ($\rho^*(L)^{\otimes r(W)} \otimes K_X = \mathcal{O}(D)$) is contained in the exceptional locus of π.*

The above conjecture is expected to follow from the Minimal Model Program using the existence and termination of flips (in particular, it holds for toric varieties).

The following statement will be important in our construction of Tamagawa numbers for $\mathcal{L}$-primitive varieties defined over a number field:

Conjecture 2.3.9 **(Vanishing).** — *For V an $\mathcal{L}$-primitive variety such that $\alpha_{\mathcal{L}}(V) > 0$ we have*

$$\mathrm{h}^i(X, \mathcal{O}_X) = 0 \ \ \forall\, i > 0$$

for any resolution of singularities

$$\rho : X \to \overline{V}^{\mathcal{L}}$$

such that the support of the $\mathbf{Q}$-Cartier divisor $\rho^*(L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X$ is a $\mathbf{Q}$-Cartier divisor with normal crossings. In particular, $\mathrm{Pic}(X)$ *is a finitely generated abelian group and one has a canonical isomorphism*

$$\mathrm{Pic}(X) \otimes \mathbf{Q} \cong \mathrm{NS}(X) \otimes \mathbf{Q}.$$

Remark 2.3.10. — Theorem 1-2-5 in **[27]** implies the vanishing for the structure sheaf for $\mathbf{Q}$-Fano varieties with canonical singularities (even with log-terminal singularities). All canonical (and log-terminal singularities) are rational and it follows that the higher cohomology of the structure sheaf on any desingularization of a canonical or a log-terminal Fano variety must also vanish (by Leray spectral sequence). Therefore, we would obtain the vanishing 2.3.9 for all $\mathcal{L}$-primitive varieties which are birationally equivalent to a Fano variety with at worst log-terminal singularities. The existence of a canonical $\mathbf{Q}$-contraction 2.3.8 would insure this.

Definition 2.3.11. — Let V be an $\mathcal{L}$-primitive variety with $\alpha_{\mathcal{L}}(V) > 0$, $\rho : X \to \overline{V}^{\mathcal{L}}$ any resolution of singularities, $D_1, \ldots, D_l$ irreducible components of the support of the rigid effective $\mathbf{Q}$-Cartier divisor D with $\mathcal{O}(D) = \rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X$. We shall call

$$\mathrm{Pic}(V, \mathcal{L}) := \mathrm{Pic}(X \setminus \bigcup_{i=1}^{l} D_i)$$

the $\mathcal{L}$-**Picard group** of V. The number

$$\beta_{\mathcal{L}}(V) := \mathrm{rk}\,\mathrm{Pic}(V, \mathcal{L})$$

will be called the $\mathcal{L}$-**rank** of V. We define the $\mathcal{L}$-**cone of effective divisors**

$$\Lambda_{\mathrm{eff}}(V, \mathcal{L}) \subset \mathrm{Pic}(V, \mathcal{L}) \otimes \mathbf{R}$$

as the image of $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{NS}(X)_{\mathbf{R}} = \mathrm{Pic}(X) \otimes \mathbf{R}$ under the natural surjective $\mathbf{R}$-linear mapping

$$\tilde{\rho} : \mathrm{Pic}(X) \otimes \mathbf{R} \to \mathrm{Pic}(V, \mathcal{L}) \otimes \mathbf{R}.$$

Remark 2.3.12. — By 2.3.3, one obtains the exact sequence

$$0 \to \mathbf{Z}[D_1] \oplus \cdots \oplus \mathbf{Z}[D_l] \to \mathrm{Pic}(X) \xrightarrow{\tilde{\rho}} \mathrm{Pic}(V, \mathcal{L}) \to 0 \tag{2}$$

and therefore

$$\beta_{\mathcal{L}}(V) = \mathrm{rk}\,\mathrm{Pic}(X) - l.$$

Using these facts, it is easy to show that the group $\mathrm{Pic}(V, \mathcal{L})$ and the cone $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ do not depend on the choice of a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$.

Conjecture 2.2.7 holds in dimension $n \leqslant 3$ as a consequence of the Minimal Model Program. More precisely, it is a consequence of Conjecture 2.3.8 and the following weaker statement:

Conjecture 2.3.13 **(Polyhedrality).** — *Let V be an $\mathcal{L}$-primitive variety with*

$$\alpha_{\mathcal{L}}(V) > 0.$$

Then $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ is a rational finitely generated polyhedral cone.

Definition 2.3.14. — Let $(A, A_{\mathbf{R}}, \Lambda)$ be a triple consisting of a finitely generated abelian group A of rank k, a k-dimensional real vector space $A_{\mathbf{R}} = A \otimes \mathbf{R}$ and a convex k-dimensional finitely generated polyhedral cone $\Lambda \in A_{\mathbf{R}}$ such that $\Lambda \cap -\Lambda = 0 \in A_{\mathbf{R}}$. For $\mathrm{Re}(\mathbf{s})$ contained in the interior of the cone Λ we define the $\mathcal{X}$-function of Λ by the integral

$$\mathcal{X}_{\Lambda}(\mathbf{s}) := \int_{\Lambda^*} e^{-\langle \mathbf{s}, \mathbf{y} \rangle} \mathbf{dy}$$

where $\Lambda^* \in A^*_{\mathbf{R}}$ is the dual cone to Λ and $\mathbf{dy}$ is the Lebesgue measure on $A^*_{\mathbf{R}}$ normalized by the dual lattice $A^* \subset A^*_{\mathbf{R}}$ where $A^* := \mathrm{Hom}(A, \mathbf{Z})$.

Remark 2.3.15. — If Λ is a finitely generated rational polyhedral cone the function $\mathcal{X}_{\Lambda}(\mathbf{s})$ is a rational function in $\mathbf{s}$. However, the explicit determination of this function might pose serious computational problems.

Definition 2.3.16. — Let V be an $\mathcal{L}$-primitive smooth quasi-projective algebraic variety with a metrized invertible sheaf $\mathcal{L}$ and $\alpha_{\mathcal{L}}(V) > 0$. Let X be any resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$. We consider the triple

$$(\mathrm{Pic}(V, \mathcal{L}), \mathrm{Pic}(V, \mathcal{L})_{\mathbf{R}}, \Lambda_{\mathrm{eff}}(V, \mathcal{L}))$$

and the corresponding $\mathcal{X}$-function. Assuming that $\mathrm{Pic}(V, \mathcal{L})$ is a finitely generated abelian group (cf. 2.3.9) and that $\Lambda_{\mathrm{eff}}(V, \mathcal{L})$ is a polyhedral cone (cf. 2.3.13), we define the constant $\gamma_{\mathcal{L}}(V) \in \mathbf{Q}$ by

$$\gamma_{\mathcal{L}}(V) := \mathcal{X}_{\Lambda_{\mathrm{eff}}(V,\mathcal{L})}(\tilde{\rho}(-[K_X])).$$

2.4. $\mathcal{L}$-primitive fibrations and descent of metrics. — Let V be an $\mathcal{L}$-primitive variety of dimension n. We show that there exists a canonical measure on $V(\mathbf{C})$ which is uniquely defined up to a positive constant.

In order to construct this measure we choose a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ and a positive integer k_2 such that $k_2 D$ is a Cartier divisor, where $\mathcal{O}(D) \cong (\rho^* L)^{\otimes \alpha_{\mathcal{L}}(V)} \otimes K_X$. Then $k_1 = k_2 \alpha_{\mathcal{L}}(V)$ is a positive integer. Let

$$g \in \mathrm{H}^0(X, \mathcal{O}(k_2 D))$$

be a non-zero global section (by 2.3.2, it is uniquely defined up to a non-zero constant). We define a measure $\omega_{\mathcal{L}}(g)$ on $X(\mathbf{C})$ as follows. Choose local complex analytic coordinates $z_1, \ldots, z_n$ in some open neighborhood $U_x \subset X(\mathbf{C})$ of a point $x \in V(\mathbf{C})$. We write the restriction of the global section g to U_x as

$$g = s^{k_2 \alpha_{\mathcal{L}}(V)} (dz_1 \wedge \cdots \wedge dz_n)^{\otimes k_2} = s^{k_1} (dz_1 \wedge \cdots \wedge dz_n)^{\otimes k_2},$$

where s is a local section of L. Then we set

$$\omega_{\mathcal{L}}(g) := \left(\frac{\sqrt{-1}}{2} \right)^n \|s\|^{\alpha_{\mathcal{L}}(V)} (dz_1 \wedge d\overline{z}_1) \wedge \cdots \wedge (dz_n \wedge d\overline{z}_n).$$

By a standard argument, one obtains that $\omega_{\mathcal{L}}(g)$ doesn't depend on the choice of local coordinates in U_x and that it extends to the whole complex space $X(\mathbf{C})$. It remains to notice that the restriction of the measure $\omega_{\mathcal{L}}(g)$ to $V(\mathbf{C}) \subset X(\mathbf{C})$ does not depend on the choice of ρ. So we obtain a well-defined measure on $V(\mathbf{C})$.

Remark 2.4.1. — We note that the measure $\omega_{\mathcal{L}}(g)$ depends on the choice of $g \in \mathrm{H}^0(X, \mathcal{O}(k_2 D))$. More precisely, it multiplies by $|c|^{1/k_2}$ if we multiply g by some non-zero complex number c. Thus we obtain that the mapping

$$\|\cdot\|_{\mathcal{L}} \; : \; \mathrm{H}^0(X, \mathcal{O}(k_2 D)) \to \mathbf{R}_{\geqslant 0}$$

$$g \mapsto \int_{X(\mathbf{C})} \omega_{\mathcal{L}}(g) = \int_{V(\mathbf{C})} \omega_{\mathcal{L}}(g)$$

satisfies the property

$$\|cg\|_{\mathcal{L}} = |c|^{1/k_2} \|g\|_{\mathcal{L}} \;\; \forall c \in \mathbf{C}^*.$$

Definition 2.4.2. — Let V be a smooth quasi-projective variety with an ample metrized invertible sheaf $\mathcal{L}$, $\rho : X \to \overline{V}^{\mathcal{L}}$ a resolution of singularities. A regular projective morphism $\pi : X \to Y$ to a projective variety Y $(\dim Y < \dim X)$ is called an **$\mathcal{L}$-primitive fibration on** V if there exists a Zariski dense open subset $U \subset Y$ such that the following conditions are satisfied:

(i) for any point $y \in U(\mathbf{C})$ the fiber $V_y = \pi^{-1}(y) \cap V$ is a smooth quasi-projective $\mathcal{L}$-primitive subvariety;

(ii) $\alpha_{\mathcal{L}}(V) = \alpha_{\mathcal{L}}(V_y) > 0$ for all $y \in U(\mathbf{C})$;

(iii) for any $k \in \mathbf{N}$ such that $k\alpha_{\mathcal{L}}(V) \in \mathbf{Z}$,

$$L_k := \mathrm{R}^0\pi_* \left(\rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X\right)^{\otimes k}$$

is an ample invertible sheaf on Y.

We propose the following version of the Fibration Conjecture of Fujita (see **[20]**):

***Conjecture 2.4.3* (Existence of Fibrations).** — *Let V be an arbitrary smooth quasi-projective variety with an ample metrized invertible sheaf $\mathcal{L}$ and $\alpha_{\mathcal{L}}(V) > 0$. Then there exists a resolution of singularities $\rho : X \to \overline{V}^{\mathcal{L}}$ such that X admits an $\mathcal{L}$-primitive fibration $\pi : X \to Y$ on some dense Zariski open subset $V' \subset V$.*

Remark 2.4.4. — From the viewpoint of the Minimal Model Program, Conjecture 2.4.3 is equivalent to the statement about the conjectured existence of $\mathbf{Q}$-Fano fibrations for algebraic varieties of negative Kodaira-dimension (cf. **[27]**). The existence of an $\mathcal{L}$-primitive fibration is equivalent to the fact that the graded algebra

$$\mathrm{R}(V, \mathcal{L}) = \oplus_{\nu \geqslant 0} \mathrm{H}^0(X, M^{k\nu}), \;\; M := \rho^*(L)^{\otimes \alpha_L(V)} \otimes K_X$$

is finitely generated (cf. 2.4 in **[4]**). One can define Y as $\operatorname{Proj} \mathrm{R}(V, \mathcal{L})$ and X as a common resolution of singularities of $\overline{V}^{\mathcal{L}}$ and of the indeterminacy locus and generic fiber of the natural rational map $\overline{V}^{\mathcal{L}} \to Y$ (cf. **[26]**).

It is important to observe that a metric $\|\cdot\|$ on L induces natural metrics on all ample invertible sheaves L_k on Y:

Definition 2.4.5. — Let L_k be an ample invertible sheaf on Y as above. We define a metric $\|\cdot\|_{\mathcal{L},k}$ on L_k as follows. Let $y \in Y(\mathbf{C})$ be a closed point, $U \subset Y$ be a Zariski open subset containing y, and $s \in \mathrm{H}^0(U, L_k)$ be a section with $s(y) \neq 0$. Then we set

$$\|s(y)\|_{\mathcal{L},k} := \left(\int_{V_y(\mathbf{C})} \omega_{\mathcal{L}}(\pi^* s) \right)^k ,$$

where $V_y(\mathbf{C})$ is the fiber over y of the $\mathcal{L}$-primitive fibration π^*, $\pi^* s$ the π-pullback of s restricted to $\mathcal{L}$-primitive variety $V_y(\mathbf{C})$, and $\omega_{\mathcal{L}}(\pi^* s)$ the corresponding to $\pi^* s$ volume measure on $V_y(\mathbf{C})$. We call $\|\cdot\|_{\mathcal{L},k}$ a k-**adjoint descent** to Y of a metric $\|\cdot\|$ on L.

3. Heights and asymptotic formulas

3.1. Basic terminology and notations. — Let F be a number field, $\mathcal{O}_F \subset F$ the ring of integers in F, $\mathrm{Val}(F)$ the set of all valuations of F, F_v the completion of F with respect to a valuation $v \in \mathrm{Val}(F)$, $\mathrm{Val}(F)_\infty = \{v_1, \dots, v_r\}$ the set of all archimedean valuations of F. For any algebraic variety X over a field F we denote by $X(F)$ the set of its K-rational points.

Definition 3.1.1. — Let E be a vector space of dimension $m+1$ over F, $\mathcal{O}_E \subset E$ a projective $\mathcal{O}_F$-module of rank $m+1$ and $\|\cdot\|_{v_1}, \dots, \|\cdot\|_{v_r}$ the set of Banach norms on the real or complex vector spaces $E_{v_i} = E \times_F F_{v_i}$ corresponding to elements of $\mathrm{Val}(F)_\infty = \{v_1, \dots, v_r\}$. It is well-known that the above data for E define a family $\{\|\cdot\|_v, \ v \in \mathrm{Val}(F)\}$ of v-adic metrics for a standard invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(E)$. If $x \in X(F)$ is a point and $s \in \mathrm{H}^0(U, \mathcal{O}(1))$ is a section over an open subset $U \subset \mathbf{P}(E)$ containing x, then we denote by $\|s(x)\|_v$ the corresponding v-adic norm of s at x. We set $\tilde{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|_v)$ to be the standard invertible sheaf $\mathcal{O}(1)$ on $\mathbf{P}(E)$ together with a family v-adic metrics $\|\cdot\|_v$ defined by the above data and we call $\tilde{\mathcal{O}}(1)$ the **standard ample metrized invertible sheaf** on $\mathbf{P}(E)$.

Definition 3.1.2. — Let X be an algebraic variety over F. For any point $x \in X(F)$ and any regular function $f \in \mathrm{H}^0(U, \mathcal{O}_X)$ on an open subset $U \subset \mathbf{P}(E)$ containing x, we define the v-adic norm $\|f(x)\|_v := |f(x)|_v$. We call this family of v-adic metrics on $\mathcal{O}_X$ the **canonical metrization** of the structure sheaf $\mathcal{O}_X$.

Definition 3.1.3. — Let X be a quasi-projective algebraic variety, L a very ample invertible sheaf on X, $i : X \hookrightarrow \mathbf{P}(E)$ an embedding with $L = i^*\mathcal{O}(1)$. We denote by $\mathcal{L} = (L, \|\cdot\|_v)$ the sheaf L together with v-adic metrics induced from a family of v-adic metrics on the standard ample metrized invertible sheaf $\tilde{\mathcal{O}}(1)$ on $\mathbf{P}(E)$. In this situation we call $\mathcal{L}$ **a very ample metrized sheaf** on X and write $\mathcal{L} = i^*\tilde{\mathcal{O}}(1)$.

Definition 3.1.4. — Let $\mathcal{L} = (L, \|\cdot\|_v)$ be a very ample metrized invertible sheaf on X. Then for any point $x \in X(F)$, the $\mathcal{L}$-**height** of x is defined as

$$H_{\mathcal{L}}(x) = \prod_{v \in \mathrm{Val}(F)} \|s(x)\|_v^{-1},$$

where $s \in \Gamma(U, L)$ is a nonvanishing at x section of L over some open subset $U \subset X$.

Remark 3.1.5. — Using a canonical metrization of the structure sheaf $\mathcal{O}_X$, the linear mapping

$$S^k(\Gamma(U, L)) \to \Gamma(U, L^{\otimes k}) \ \ (k > 0),$$

and the F-bilinear mapping

$$\Gamma(U, L^{\otimes k}) \times \Gamma(U, L^{\otimes -k}) \to \Gamma(U, \mathcal{O}_X),$$

one immediately sees that a family of v-adic metrics on an invertible sheaf L allows to define a family of v-adic metrics on $L^{\otimes k}$ and on any invertible sheaf M such that there exist integers k_1, k_2 $(k_2 \neq 0)$ with $L^{\otimes k_1} = M^{\otimes k_2}$. In this situation we write

$$\mathcal{M} = (M, \|\cdot\|_v) := (L^{\otimes k_1/k_2}, \|\cdot\|_v^{k_1/k_2}),$$

or simply $\mathcal{M} = \mathcal{L}^{k_1/k_2}$. Obviously, one obtains

$$H_{\mathcal{M}}(x) = (H_{\mathcal{L}}(x))^{k_1/k_2} \quad \forall\, x \in X(F).$$

Definition 3.1.6. — Let L be an ample invertible sheaf on a quasi-projective variety X and k a positive integer such that $L^{\otimes k}$ is very ample. We define an **ample metrized invertible sheaf** $\mathcal{L} = (L, \|\cdot\|_v)$ on X associated with L by considering $\mathcal{L}^{\otimes k} := (L^{\otimes k}, \|\cdot\|_v^k)$ as a very ample metrized invertible sheaf on X.

3.2. Weakly and strongly $\mathcal{L}$-saturated varieties. — Let V be an arbitrary quasi-projective algebraic variety over F with an ample metrized invertible sheaf $\mathcal{L}$. We always assume that $V(F)$ is infinite and set

$$N(V, \mathcal{L}, B) := \#\{x \in V(F) \ : \ H_{\mathcal{L}}(x) \leq B\}.$$

Here and in 3.4 we will work under the following

Assumption 3.2.1. — *For all quasi-projective V', V and with $V' \subset V$ and $|V(F)| = \infty$ there exists the limit*

$$\lim_{B\to\infty} \frac{N(V',\mathcal{L},B)}{N(V,\mathcal{L},B)}.$$

Definition 3.2.2. — We call an irreducible quasi-projective algebraic variety V with an ample metrized invertible sheaf $\mathcal{L}$ **weakly $\mathcal{L}$-saturated** if for any Zariski locally closed subset $W \subset V$ with $\dim W < \dim V$, one has

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} < 1.$$

Definition 3.2.3. — We call an irreducible quasi-projective algebraic variety V with an ample metrized invertible sheaf $\mathcal{L}$ **strongly $\mathcal{L}$-saturated** if for any dense Zariski open subset $U \subset V$, one has

$$\lim_{B\to\infty} \frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

Definition 3.2.4. — Let V be a weakly $\mathcal{L}$-saturated variety, $W \subset V$ a locally closed strongly saturated subvariety of smaller dimension. Then we call W an **$\mathcal{L}$-target** of V, if

$$0 < \lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} < 1.$$

Theorem 3.2.5. — *Let V be an arbitrary quasi-projective algebraic variety with an ample metrized invertible sheaf $\mathcal{L}$. Assume that $|V(F)| = \infty$ and that 3.2.1 holds. Then we have:*

(i) *if V is strongly $\mathcal{L}$-saturated then V is weakly $\mathcal{L}$-saturated;*

(ii) *V contains finitely many weakly $\mathcal{L}$-saturated subvarieties $W_1, \ldots, W_k$ with*

$$\lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

(iii) *V contains a strongly $\mathcal{L}$-saturated subvariety W having the property*

$$0 < \lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} < 1.$$

(iv) *if V is weakly saturated and if it doesn't contain a dense Zariski open subset $U \subset V$ which is strongly saturated then V contains infinitely many $\mathcal{L}$-targets.*

Proof. — (i) Let $W \subset V$ be a Zariski closed subset with $\dim W < \dim V$ and $U = V \setminus W$. Then

$$\lim_{B\to\infty} \frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)} + \lim_{B\to\infty} \frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)} = \lim_{B\to\infty} \frac{N(V,\mathcal{L},B)}{N(V,\mathcal{L},B)} = 1.$$

Since V is strongly $\mathcal{L}$-saturated, we have

$$\lim_{B\to\infty}\frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)}=1$$

and therefore

$$\lim_{B\to\infty}\frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)}=0<1.$$

(ii) Let $W\subset V$ be a minimal Zariski closed subset such that

$$\lim_{B\to\infty}\frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)}=1$$

and $W_1,\dots,W_k$ irreducible components of W. It immediately follows from the minimality of W that each W_i is weakly saturated.

(iii) Let $W\subset V$ be an irreducible Zariski closed subset of minimal dimension such that

$$\lim_{B\to\infty}\frac{N(W,\mathcal{L},B)}{N(V,\mathcal{L},B)}=1.$$

The minimality of W implies that W is weakly saturated.

(iv) By (iii) the set of $\mathcal{L}$-targets is nonempty. Assume that the set of all $\mathcal{L}$-targets is finite: $\{W_1,\dots,W_k\}$. The strong saturatedness of each W_i implies that

$$\lim_{B\to\infty}\frac{N(W_{i_1}\cap\cdots\cap W_{i_l},\mathcal{L},B)}{N(V,\mathcal{L},B)}=0$$

for all pairwise different $i_1,\dots,i_l\in\{1,\dots,k\}$ and $l\geqslant 2$. In particular, one has

$$\sum_{i=1}^{k}\lim_{B\to\infty}\frac{N(W_i,\mathcal{L},B)}{N(V,\mathcal{L},B)}=\lim_{B\to\infty}\frac{N(W_1\cup\cdots\cup W_k,\mathcal{L},B)}{N(V,\mathcal{L},B)}<1.$$

We set $U:=V\setminus(W_1\cup\cdots\cup W_k)$. Then

$$\lim_{B\to\infty}\frac{N(U,\mathcal{L},B)}{N(V,\mathcal{L},B)}>0.$$

Since U is not strongly saturated, there exists an irreducible Zariski closed subset $W_0\subset V$ of minimal dimension $<\dim V$ such that

$$\lim_{B\to\infty}\frac{N(W_0\cap U,\mathcal{L},B)}{N(U,\mathcal{L},B)}>0.$$

It follows from the minimality of W_0 that W_0 is strongly saturated. On the other hand, one has

$$\lim_{B\to\infty}\frac{N(W_0,\mathcal{L},B)}{N(V,\mathcal{L},B)}\geqslant\lim_{B\to\infty}\frac{N(W_0\cap U,\mathcal{L},B)}{N(V,\mathcal{L},B)}>0,$$

i.e., W_0 is an $\mathcal{L}$-target and $W_0\notin\{W_1,\dots,W_k\}$. Contradiction. □

Definition 3.2.6. — Let V be a weakly $\mathcal{L}$-saturated variety and $W_1, W_2, \dots$ an infinite sequence of strongly saturated irreducible subvarieties W_i having the property

$$0 < \theta_i := \lim_{B\to\infty} \frac{N(W_i, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} < 1 \;\; \forall i > 0.$$

We say that the set $\{W_1, W_2, \dots\}$ forms an **asymptotic arithmetic $\mathcal{L}$-fibration** on V, if the following equality holds

$$\sum_{i=1}^{\infty} \theta_i = 1.$$

We expect that the main source of examples of weakly and strongly saturated varieties should come from the following situation:

Proposition 3.2.7. — *Assume 3.2.1 and let $G \subset PGL(n+1)$ be a connected linear algebraic group acting on $\mathbf{P}^n$ and $V := Gx \subset \mathbf{P}^n$ a G-orbit of a point $x \in \mathbf{P}^n(F)$. Then V is weakly $\tilde{O}(1)$-saturated.*

Proof. — Let $W \subset V$ be an arbitrary locally closed subset with $\dim W < \dim V$, $\overline{W} \subset V$ its Zariski closure in V and $U := V \setminus \overline{W} \subset V$ the corresponding dense Zariski open subset of V. Then V is covered by the open subsets gU, where g runs over all elements in $G(F)$ (this follows from the fact that G is unirational and that $G(F)$ is Zariski dense in G [**10**]). Therefore, the orbit of $x \in V(F)$ under $G(F)$ is Zariski dense in V. Since the Zariski topology is noetherian we can choose a finite subcovering: $V = \bigcup_{i=1}^{k} g_i U$ (g_i in $G(F)$). Considering $g_i \in G(F)$ as matrices in $PGL(n+1)$ and using standard properties of heights [**29**], one obtains positive constants c_i such that

$$H_{\mathcal{L}}(g_i(x)) \leq c_i H_{\mathcal{L}}(x)$$

for all $x \in \mathbf{P}^n(F)$. It is clear that for $c_0 := \sum_{i=1}^{k} c_i$ we have

$$N(U, \mathcal{L}, B) \leq N(V, \mathcal{L}, B) \leq c_0 N(U, \mathcal{L}, B).$$

It follows that

$$\frac{1}{c_0} \leqslant \lim_{B\to\infty} \frac{N(U, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant 1.$$

Hence

$$\lim_{B\to\infty} \frac{N(W, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant \lim_{B\to\infty} \frac{N(\overline{W}, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} \leqslant 1 - \frac{1}{c_0} < 1.$$

□

We can reformulate the statement of 3.2.7 as follows:

Proposition 3.2.8. — *Let G be a connected linear algebraic group, $H \subset G$ a closed subgroup and $V := G/H$. If 3.2.1 holds then V is weakly saturated with respect to any G-equivariant projective embedding of V.*

It is easy to see that $V = G/H$ is not necessarily weakly saturated with respect to projective embeddings which are not G-equivariant:

Example 3.2.9. — Let $S \subset \mathbf{P}^8$ be the anticanonically embedded Del Pezzo surface which is a blow up of a rational point in $\mathbf{P}^2$. Denote by $\mathcal{L}$ the metrized anticanonical sheaf on S. The unique exceptional curve $C \subset S$ is contained in the union of two open subsets $U_0, U_1 \subset S$ where $U_0 \cong U_1 \cong \mathbf{A}^2$. Therefore, S can be considered as a projective compactification of the algebraic group $\mathbf{G}_a^2$ (after an identification of $\mathbf{G}_a^2$ with U_0 or U_1). This compactification is not $\mathbf{G}_a^2$-equivariant. One has

$$a_{\mathcal{L}}(U_0) = a_{\mathcal{L}}(U_1) = a_{\mathcal{L}}(C) = 2,$$

but

$$a_{\mathcal{L}}(U_0 \setminus C) = a_{\mathcal{L}}(U_1 \setminus C) = 1.$$

Hence, U_0 and U_1 are not weakly $\mathcal{L}$-saturated.

It is easy to show that an equivariant compactification of G/H is not necessarily strongly $\mathcal{L}$-saturated:

Example 3.2.10. — Let $V = \mathbf{P}^1 \times \mathbf{P}^1$. Then V is a G-homogeneous variety with $G = GL(2) \times GL(2)$. However, V is not strongly $\mathcal{L}$-saturated for $L := \pi_1^*\mathcal{O}(k_1) \otimes \pi_2^*\mathcal{O}(k_2)$ ($k_1, k_2 \in \mathbf{N}$), if $k_1 \neq k_2$.

3.3. Adelic $\mathcal{L}$-measure and $\tau_{\mathcal{L}}(V)$. — Now we define an adelic measure $\omega_{\mathcal{L}}$ corresponding to an ample metrized invertible sheaf $\mathcal{L}$ on an $\mathcal{L}$-primitive variety V with $\alpha_{\mathcal{L}}(V) > 0$ which satisfies the assumption 2.3.9. This is a generalization of a construction due to Peyre ([**30**]) for V being a smooth projective variety and $\mathcal{L}$ the metrized canonical line bundle, which in its turn is a generalization of the classical construction of Tamagawa measures on the adelic points of algebraic groups.

Let V be an $\mathcal{L}$-primitive variety of dimension n, $\rho : X \to \overline{V}^{\mathcal{L}}$ a resolution of singularities, k_2 a positive integer such that $k_1 = k_2\alpha_{\mathcal{L}}(V) \in \mathbf{Z}$ and

$$(\rho^*(L)^{\alpha_{\mathcal{L}}(V)} \otimes K_X)^{\otimes k_2} \cong \mathcal{O}(D),$$

where D is a rigid effective Cartier divisor on X.

Definition 3.3.1. — Let g be a non-zero element of the 1-dimensional F-vector space $\mathrm{H}^0(X, \mathcal{O}(D))$ (g is defined uniquely up to an element of F^*). Let $v \in \mathrm{Val}(F)$. We define a measure $\omega_{\mathcal{L},v}(g)$ on $V(F_v)$ as follows. Choose local v-analytic coordinates $x_{1,v}, \ldots, x_{n,v}$ in some open neighborhood $U_x \subset X(F_v)$ of a point $x \in X(F_v)$. We write the restriction of the global section g to U_x as

$$g = s^{k_1}(dx_{1,v} \wedge \cdots \wedge dx_{n,v})^{k_2}$$

where s is a local section of L. Define a v-adic measure on U_x as

$$\omega_{\mathcal{L},v}(g) := \|s\|_v^{k_1/k_2} dx_{1,v} \cdots dx_{n,v} = \|s\|_v^{\alpha_{\mathcal{L}}(V)} dx_{1,v} \cdots dx_{n,v},$$

where $dx_{1,v} \cdots dx_{n,v}$ is the usual normalized Haar measure on F_v^n. By a standard argument, one obtains that $\omega_{\mathcal{L},v}(g)$ doesn't depend on the choice of local coordinates in U_x and that it extends to the whole v-adic space $X(F_v)$. The restriction of $\omega_{\mathcal{L},v}(g)$ doesn't depend on the choice of ρ. So we obtain a well-defined v-adic measure on $V(F_v)$.

Remark 3.3.2. — We remark that $\omega_{\mathcal{L},v}(g)$ depends on the choice of a global section $g \in \mathrm{H}^0(X, \mathcal{O}(D))$: if $g' = cg$ ($c \in F^*$) is another global section, then

$$\omega_{\mathcal{L},v}(g') = |c|_v^{1/k} \omega_{\mathcal{L},v}(g).$$

Our next goal is to obtain an explicit formula for the integral

$$d_v(V) := \int_{V(F_v)} \omega_{\mathcal{L},v}(g) = \int_{X(F_v)} \omega_{\mathcal{L},v}(g)$$

for almost all $v \in \mathrm{Val}(F)$. We use the p-adic integral formula of Denef ([**16**] Th. 3.1) in the same way as the p-adic formula of A. Weil ([**35**] Th. 2.2.3) was used by Peyre in [**30**].

Remark 3.3.3. — By [**1, 26**], we can choose ρ in such a way that ρ is defined over F and all irreducible components $D_1, \ldots, D_l$ ($l \geqslant 0$) of the support of $D = \sum_{i=1}^{l} m_i D_i$ are smooth divisors with normal crossings over the algebraic closure $\overline{F}$.

Definition 3.3.4. — Let G be the image of $\mathrm{Gal}(\overline{F}/F)$ in the symmetric group S_l that acts by permutations on $D_1, \ldots, D_l$. We set $I := \{1, \ldots, l\}$ and denote by I/G the set of all G-orbits in I. For any $J \subset I$ we set

$$D_J := \begin{cases} \bigcap_{j \in J} D_j, & \text{if } J \neq \varnothing \\ X \setminus \bigcup_{j \in I} D_j, & \text{if } J = \varnothing \end{cases},$$

$$D_J^{\circ} = D_J \setminus \bigcup_{j \notin J} D_j.$$

(D_J is defined over F if and only if J is a union of some of G-orbits in I/G.).

We can extend X to a projective scheme $\mathcal{X}$ of finite type over $\mathcal{O}_F$ and divisors $D_1, \ldots, D_l$ to codimension-1 subschemes $\mathcal{D}_1, \ldots, \mathcal{D}_l$ in $\mathcal{X}$ such that for almost all non-archimedean $v \in \mathrm{Val}(F)$ the reductions of X and $\mathcal{D}_1, \ldots, \mathcal{D}_l$ modulo $\wp_v \subset \mathcal{O}_F$ are smooth projective varieties $\mathcal{X}_v$ and $\mathcal{D}_{v,1}, \ldots, \mathcal{D}_{v,l}$ over the algebraic closure $\overline{k_v}$ of the residue field k_v with $\mathcal{D}_{v,i} \neq \mathcal{D}_{v,j}$ for $i \neq j$. Moreover, we can assume that

$$\mathcal{D}_{v,J} := \begin{cases} \bigcap_{j \in J} \mathcal{D}_{v,j}, & \text{if } J \neq \varnothing \\ X \setminus \bigcup_{j \in I} \mathcal{D}_{v,j}, & \text{if } J = \varnothing \end{cases}$$

are also smooth over $\overline{k_v}$.

Definition 3.3.5. — A non-archimedean valuation $v \in \mathrm{Val}(F)$ which satisfies all the above assumptions will be called a **good valuation** for the pair $(\mathcal{X}, \{\mathcal{D}_i\}_{i\in I})$.

Definition 3.3.6. — Let $G_v \subset G$ be a cyclic subgroup generated by a representative of the Frobenius element in $\mathrm{Gal}(\overline{k_v}/k_v)$. We denote by I/G_v the set of all G_v-orbits in I. If $j \in I/G_v$, then we set b_j to be the length of the corresponding G_v-orbit and put $r_j = m_j/k_2$, where m_j is the multiplicity of irreducible components of D corresponding to the G_v-orbit j.

The following theorem is a slightly generalized version of Th. 3.1 in **[16]**:

Theorem 3.3.7. — *Let $v \in \mathrm{Val}(F)$ be a good non-archimedean valuation for*

$$(\mathcal{X}, \{\mathcal{D}_i\}_{i\in I}).$$

Then

$$d_v(V) = \int_{X(F_v)} \omega_{\mathcal{L},v}(g) = \frac{c_\varnothing}{q_v^n} + \frac{1}{q_v^n} \sum_{\varnothing\neq J\subset I/G_v} c_J \prod_{j\in J} \left(\frac{q_v^{b_j} - 1}{q_v^{b_j(r_j+1)} - 1} \right),$$

where q_v is the cardinality of k_v, and c_J is the cardinality of the set of k_v-rational points in $\mathcal{D}_J^\circ$.

Let us consider an exact sequence of Galois $\mathrm{Gal}(\overline{F}/F)$-modules:

$$0 \to \mathbf{Z}[\mathcal{D}_1] \oplus \cdots \oplus \mathbf{Z}[\mathcal{D}_l] \to \mathrm{Pic}(\mathcal{X}) \xrightarrow{\tilde{\rho}} \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \to 0 \tag{3}$$

Theorem 3.3.8. — *Assume that X has the property $\mathrm{h}^1(X, \mathcal{O}_X) = 0$. Then*

$$d_v(V) = 1 + \frac{1}{q_v}\mathrm{Tr}(\Phi_v|\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l) + O\left(\frac{1}{q_v^{1+\varepsilon}}\right),$$

where

$$\varepsilon = \min\{1/2, r_1, \ldots, r_l\}$$

and Φ_v is the Frobenius morphism.

Proof. — By conjectures of Weil proved by Deligne **[15]**, one has

$$\frac{c_J}{q_v^n} = O\left(\frac{1}{q_v}\right) \quad J \neq \varnothing$$

(since $\dim D_J \leqslant n-1$ for $J \neq \varnothing$) and

$$\frac{c_\varnothing}{q_v^n} = \sum_{k=0}^{2n} (-1)^k \mathrm{Tr}(\Phi_v|\mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l)),$$

where $\mathrm{H}_c^k(\cdot, \mathbf{Q}_l)$ denotes the étale cohomology group with compact supports. Using long cohomology sequence of the pair $(\mathcal{X}, \bigcup_i \mathcal{D}_i)$, one obtains isomorphisms

$$\mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) = \mathrm{H}_c^k(\mathcal{X}, \mathbf{Q}_l) \text{ for } k = 2n, 2n-1$$

and the short exact sequence

$$0 \to \mathrm{H}_c^{2n-2}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) \to \mathrm{H}_c^{2n-2}(\mathcal{X}, \mathbf{Q}_l) \to \bigoplus_{i=1}^{l} \mathrm{H}_c^{2n-2}(\mathcal{D}_i, \mathbf{Q}_l) \to 0.$$

Using isomorphisms

$$\mathrm{H}_c^{2n-2}(\mathcal{D}_i, \mathbf{Q}_l(n-1)) \cong \mathbf{Q}_l, \mathrm{H}_c^{2n}(\mathcal{X}, \mathbf{Q}_l(n)) \cong \mathbf{Q}_l,$$

Poincaré duality

$$\mathrm{H}_c^{2n-2}(\mathcal{X}, \mathbf{Q}_l(n-1)) \times \mathrm{Pic}(\mathcal{X}) \xrightarrow{\sim} \mathbf{Q}_l$$

and the vanishing property $\mathrm{h}^1(X, \mathcal{O}_X) = 0$, we obtain

$$\mathrm{H}_c^{2n-1}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l) = 0$$

and

$$\frac{c_\varnothing}{q_v^n} = 1 + \frac{1}{q_v}\mathrm{Tr}(\Phi_v | \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l)$$

$$+ \sum_{k=0}^{2n-3} \frac{(-1)^k}{q_v^n} \mathrm{Tr}(\Phi_v | \mathrm{H}_c^k(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i, \mathbf{Q}_l))$$

$$= 1 + \frac{1}{q_v}\mathrm{Tr}(\Phi_v | \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i) \otimes \mathbf{Q}_l) + O\left(\frac{1}{q_v^{3/2}}\right).$$

On the other hand, for $J \neq \varnothing$ one has

$$\prod_{j \in J} \left(\frac{q_v^{b_j} - 1}{q_v^{b_j(r_j+1)} - 1} \right) = O\left(\frac{1}{q_v^{1+\varepsilon_0}}\right),$$

where $\varepsilon_0 = \min\{r_1, \ldots, r_l\}$. □

Definition 3.3.9. — We define the convergency factors

$$\lambda_v := \begin{cases} L_v(1, \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)), & \text{if } v \text{ is good} \\ 1 & \text{otherwise} \end{cases}$$

where L_v is the local factor of the Artin L-function corresponding to the G-module $\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)$ and we set

$$\omega_{\mathcal{L},S} := \sqrt{|\mathrm{disc}(F)|}^{-n} \prod_{v \in \mathrm{Val}(F)} \lambda_v^{-1} \omega_{\mathcal{L},v}(g),$$

where $\mathrm{disc}(F)$ is the absolute discriminant of $\mathcal{O}_F$ and S is the set of bad valuations.

By the product formula, $\omega_{\mathcal{L},S}$ doesn't depend on the choice of g.

Definition 3.3.10. — Denote by $\mathbf{A}_F$ the adele ring of F. Let $\overline{X(F)}$ be the closure of $X(F)$ in $X(\mathbf{A}_F)$ (in direct product topology). Under the vanishing assumption $\mathrm{h}^1(X, \mathcal{O}_X) = 0$, we define the constant

$$\tau_{\mathcal{L}}(V) = \lim_{s \to 1} (s-1)^{\beta_{\mathcal{L}}(V)} L_S(s, \mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)) \int_{\overline{X(F)}} \omega_{\mathcal{L},S},$$

where $\beta_{\mathcal{L}}(V)$ is the rank of the submodule of $\mathrm{Gal}(\overline{F}/F)$-invariants of the module $\mathrm{Pic}(\mathcal{X} \setminus \bigcup_{i=1}^{l} \mathcal{D}_i)$.

3.4. Main strategy. — Now we proceed to discuss our main strategy in understanding the asymptotic for the number $N(V, \mathcal{L}, B)$ as $B \to \infty$ for an arbitrary $\mathcal{L}$-polarized quasi-projective variety. Again, we shall make the assumption 3.2.1. Our approach consists in 4 steps including 3 subsequent simplifications of the situation:

Step 1 (reduction to weakly $\mathcal{L}$-saturated varieties): By 3.2.5 (ii), every quasi-projective $\mathcal{L}$-polarized variety V contains a finite number of weakly $\mathcal{L}$-saturated varieties $W_1, \dots W_k$ such that

$$\lim_{B \to \infty} \frac{N(W, \mathcal{L}, B)}{N(V, \mathcal{L}, B)} = 1.$$

Therefore, it would be enough to understand separately the asymptotics of

$$N(W_i, \mathcal{L}, B), i \in \{1, \dots, k\}$$

modulo the asymptotics

$$N(W_{i_1} \cap \dots \cap W_{i_l}, \mathcal{L}, B)$$

for low-dimensional subvarieties $W_{i_1} \cap \dots \cap W_{i_l}$, where $\{i_1, \dots, i_l\} \subset \{1, \dots, k\}$ are subsets of pairwise different elements with $l \geqslant 2$.

For our next reduction step, we need:

Conjecture 3.4.1. — *Let V be a weakly $\mathcal{L}$-saturated variety which doesn't contain an open Zariski dense and strongly $\mathcal{L}$-saturated subset $U \subset V$. Then the set of $\mathcal{L}$-targets of V forms an asymptotic arithmetic $\mathcal{L}$-fibration.*

Step 2 (reduction to strongly $\mathcal{L}$-saturated varieties): Let V be an arbitrary weakly $\mathcal{L}$-saturated variety. Then either V contains a strongly $\mathcal{L}$-saturated Zariski open subset or, according to 3.4.1, we obtain an asymptotic arithmetic $\mathcal{L}$-fibration of V by $\mathcal{L}$-targets. In the first situation, it is enough to understand the asymptotic of $N(U, \mathcal{L}, B)$ for the strongly $\mathcal{L}$-saturated variety U (we note that the complement $V \setminus U$ consists of low-dimensional irreducible components). In the second situation, it is enough to understand the asymptotic of $N(W_i, \mathcal{L}, B)$ for each of the $\mathcal{L}$-targets $W_i \subset V$.

For our next reduction step, we need:

Conjecture 3.4.2. — *Let V be a smooth strongly $\mathcal{L}$-saturated quasi-projective variety. Then the complex analytic variety $V(\mathbf{C})$ is $\mathcal{L}$-primitive.*

Step 3 (reduction to $\mathcal{L}$-primitive varieties): Every quasi-projective algebraic variety V is a disjoint union of finitely many locally closed smooth subvarieties V_i. Therefore, if one knows the asymptotic for each $N(V_i, \mathcal{L}, B)$ then one immediately obtains the asymptotic for $N(V, \mathcal{L}, B)$.

Definition 3.4.3. — Let V be an $\mathcal{L}$-primitive algebraic variety over a number field F, $\rho : X \to \overline{V}^{\mathcal{L}}$ a desingularization over F of the closure of V with the exceptional locus consisting of smooth irreducible divisors $D_1, \ldots, D_l$. We consider $\mathrm{Pic}(X)$ and $\mathrm{Pic}(V, \mathcal{L})$ as $\mathrm{Gal}(\overline{F}/F)$-modules and we denote by $\beta_{\mathcal{L}}(V)$ the rank of $\mathrm{Gal}(\overline{F}/F)$-invariants in $\mathrm{Pic}(V, \mathcal{L})$ and by $\delta_{\mathcal{L}}(V)$ the cardinality of the cohomology group

$$\mathrm{H}^1(\mathrm{Gal}(\overline{F}/F), \mathrm{Pic}(V, \mathcal{L})).$$

Remark 3.4.4. — Using the long exact cohomology sequence associated with (3), one immediately obtains that $\beta_{\mathcal{L}}(V)$ and $\delta_{\mathcal{L}}(V)$ do not depend on the choice of the resolution ρ.

Step 4 (expected asymptotic formula): Let V be a strongly $\mathcal{L}$-saturated (and $\mathcal{L}$-primitive) smooth quasi-projective variety. Assume that $a_{\mathcal{L}}(V) > 0$. Then we expect that the following asymptotic formula holds:

$$N(V, \mathcal{L}, B) = c_{\mathcal{L}}(V) B^{\alpha_{\mathcal{L}}(V)} (\log B)^{\beta_{\mathcal{L}}(V)-1} (1 + o(1)),$$

where

$$c_{\mathcal{L}}(V) := \frac{\gamma_{\mathcal{L}}(V)}{\alpha_{\mathcal{L}}(V)(\beta_{\mathcal{L}}(V) - 1)!} \delta_{\mathcal{L}}(V) \tau_{\mathcal{L}}(V),$$

$\gamma_{\mathcal{L}}(V)$ is an invariant of the triple

$$(\mathrm{Pic}(V, \mathcal{L}), \mathrm{Pic}(V, \mathcal{L})_{\mathbf{R}}, \Lambda_{\mathrm{eff}}(V, \mathcal{L}))$$

(2.3.16), $\delta_{\mathcal{L}}(V)$ is a cohomological invariant of the $\mathrm{Gal}(\overline{F}/F)$-module $\mathrm{Pic}(V, \mathcal{L})$ (3.4.3) and $\tau_{\mathcal{L}}(V)$ is an adelic invariant of a family of v-adic metrics $\{\| \cdot \|_v\}$ on L (3.3.10).

In sections 4 and 5 we discuss some examples which show how the constants

$$\alpha_{\mathcal{L}}(V), \beta_{\mathcal{L}}(V), \delta_{\mathcal{L}}(V), \gamma_{\mathcal{L}}(V), \tau_{\mathcal{L}}(V)$$

appear in asymptotic formulas for the number of rational points of bounded $\mathcal{L}$-height on algebraic varieties. Naturally, we expect that the exhibited behavior is typical. However, we also feel that one should collect more examples which could help to clarify the general situation.

3.5. $\mathcal{L}$-primitive fibrations and $\tau_{\mathcal{L}}(V)$. — We proceed to discuss our observations concerning the arithmetic conjecture 3.4.1 and its relation to the geometric conjecture 2.4.3.

Let V be a weakly $\mathcal{L}$-saturated smooth quasi-projective variety with $a_{\mathcal{L}}(V) > 0$ which is not strongly saturated and which doesn't contain Zariski open dense strongly saturated subvarieties. We distinguish the following two cases:

Case 1. V is not $\mathcal{L}$-primitive. In this case we expect that some Zariski open dense subset $U \subset V$ admits an $\mathcal{L}$-primitive fibration which is defined by a projective regular morphism $\pi : X \to Y$ over F to a low-dimensional normal irreducible projective variety Y satisfying the conditions (i)-(iii) in 2.4.2, for an appropriate smooth projective compactification X of U (see 2.4.3). It seems natural to expect that all fibers satisfy the vanishing assumption 2.3.9. Thus we see that for any $y \in Y(F)$ such that $V_y = \pi^{-1}(y) \cap V$ is $\mathcal{L}$-primitive we can define the adelic number $\tau_{\mathcal{L}}(V_y)$. Furthermore, we expect that every $\mathcal{L}$-target W is contained in an appropriate $\mathcal{L}$-primitive subvariety V_y which is a fiber of the $\mathcal{L}$-primitive fibration $\pi : V \to U$ on V. In particular, Step 4 of our main strategy implies that if every $\mathcal{L}$-target W *coincides* with a suitable $\mathcal{L}$-primitive fiber V_y then one should expect the asymptotic

$$N(V, \mathcal{L}, B) = c_{\mathcal{L}}(V) B^{\alpha_{\mathcal{L}}(V)} (\log B)^{\beta_{\mathcal{L}}(V)-1} (1 + o(1)),$$

where the numbers $\alpha_{\mathcal{L}}(V)$ (resp. $\beta_{\mathcal{L}}(V)$) coincide with the numbers $\alpha_{\mathcal{L}}(V_y)$ (resp. $\beta_{\mathcal{L}}(V_y)$) for the corresponding $\mathcal{L}$-targets V_y and the constant $c_{\mathcal{L}}(V)$ is equal to the sum

$$\sum_y c_{\mathcal{L}}(V_y) = \sum_y \frac{\gamma_{\mathcal{L}}(V_y)}{\alpha_{\mathcal{L}}(V_y)(\beta_{\mathcal{L}}(V_y) - 1)!} \delta_{\mathcal{L}}(V_y) \tau_{\mathcal{L}}(V_y),$$

where y runs over all points in $Y(F)$ such that V_y is an $\mathcal{L}$-target of V. It is natural to try to understand the dependence of $\tau_{\mathcal{L}}(V_y)$ on the choice of a point $y \in Y(F)$. We expect that the number $\tau_{\mathcal{L}}(V_y)$ can be interpreted as a "height" of y. More precisely, the examples we considered suggest the following:

Conjecture 3.5.1. — *There exist a family of v-adic metrics on K_Y and two positive constants $c_2 > c_1 > 0$ such that*

$$c_1 H_{\mathcal{F}}(y) \leqslant \tau_{\mathcal{L}}(V_y) \leqslant c_2 H_{\mathcal{F}}(y) \ \ \forall y \in Y(F) \cap U,$$

where $U \subset Y$ is some dense Zariski open subset and $\mathcal{F}$ is a metrized $\mathbf{Q}$-invertible sheaf associated with the $\mathbf{Q}$-Cartier divisor $L_1^{-1} \otimes K_Y$ (recall that L_1 is the tautological ample $\mathbf{Q}$-Cartier divisor on Y defined by the graded ring $\mathrm{R}(V, \mathcal{L})$ *(see 2.4.4)).*

Case 2. V is $\mathcal{L}$-primitive (but not strongly saturated!). We don't know examples of a precise asymptotic formula in this situation.

Example 3.5.2. — Let V be a Fano diagonal cubic bundle over $\mathbf{P}^3$ with the homogeneous coordinates $(X_0 : X_1 : X_2 : X_3)$ defined as a hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$ by the equation

$$X_0Y_0^3 + X_1Y_1^3 + X_2Y_2^3 + X_3Y_3^3 = 0$$

in $\mathbf{P}^3 \times \mathbf{P}^3$ (see [**9**]). We expect that V is not strongly saturated with respect to a metrized anticanonical sheaf $L := \mathcal{O}(3, 1)$ and that the corresponding $\mathcal{L}$-targets are the splitting diagonal cubics in fibers of the natural projection $\pi : V \to \mathbf{P}^3$ (this leads to the failure of the expected asymptotic formula in Step 4 for this example).

The next example was suggested to us by Colliot-Thélène:

Example 3.5.3. — Let V be an analogous diagonal quadric bundle over $\mathbf{P}^3$ defined as a hypersurface in $\mathbf{P}^3 \times \mathbf{P}^3$ by the equation

$$X_0Y_0^2 + X_1Y_1^2 + X_2Y_2^2 + X_3Y_3^2 = 0.$$

For infinitely many fibers $V_x = \pi_1^{-1}(x)$ $(x \in \mathbf{P}^3(F))$ we have $\operatorname{rk}\operatorname{Pic}(V_x) = 2$. At the same time, we have also $\operatorname{rk}\operatorname{Pic}(V) = 2$. We consider the height function associated to some metrization of the line bundle $L := \mathcal{O}(3, 2)$. On the one hand, we think that the asymptotic on the whole variety is $c(V)B\log B(1+o(1))$ for $B \to \infty$ with some $c(V) > 0$. On the other hand, if $X_0X_1X_2X_3$ is a square in F we get already about $B\log B$ solutions. Another important observation is the *expected* convergency of the series

$$\sum_{x\in\mathbf{P}^3(F):V_x\cong\mathbf{P}^1\times\mathbf{P}^1} c(V_x).$$

The latter would be a consequence of the following two facts. First, the condition $V_x \cong \mathbf{P}^1 \times \mathbf{P}^1$ $(x = (X_0 : X_1 : X_2 : X_3))$ is equivalent to the conditions that V_x contains an F-rational point and that the product $X_0X_1X_2X_3$ is a square in F. The number of F-rational points $x' = (X_0 : X_1 : X_2 : X_3 : Z)$ with $H_{\mathcal{O}(1)}(x') \leqslant B$ lying on the hypersurface with the equation $X_0X_1X_2X_3 = Z^2$ in the weighted projective space $\mathbf{P}^4(1,1,1,1,2)$ can be estimated from above by $B^2(\log B)^3(1 + o(1))$.

Secondly, we expect

$$c(V_x) = H_{\mathcal{O}(4)}^{-1}(x)(1 + o(1)),$$

uniformly over the base $\mathbf{P}^3(F)$. This would imply the claimed convergency.

4. Height zeta-functions

4.1. Tauberian theorem. — One of the main techniques in the proofs of asymptotic formulas for the counting function

$$N(V,\mathcal{L},B) := \#\{x \in V(F) \ : \ H_{\mathcal{L}}(x) \leq B\}$$

has been the use of *height zeta functions*. Let $\mathcal{L}$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety X. We define the height zeta function by the series

$$Z(X,\mathcal{L},s) := \sum_{x\in X(F)} H_{\mathcal{L}}(x)^{-s}$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$. After establishing the analytic properties of $Z(X,\mathcal{L},s)$ one uses the following version of a Tauberian theorem:

Theorem 4.1.1. — *([**14**]) Suppose that there exist an $\varepsilon > 0$ and a real number $\Theta(\mathcal{L}) > 0$ such that*

$$Z(X,\mathcal{L},s) = \frac{\Theta(\mathcal{L})}{(s-a)^b} + \frac{f(s)}{(s-a)^{b-1}}$$

for some $a > 0$, $b \in \mathbf{N}$ and some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) >$ *$a - \varepsilon$. Then we have the following asymptotic formula*

$$N(X,\mathcal{L},B) = \frac{\Theta(\mathcal{L})}{a\cdot(b-1)!} B^a(\log B)^{b-1}(1+o(1))$$

for $B \to \infty$.

4.2. Products. — Let X_1 and X_2 be two smooth quasi-projective varieties with ample metrized invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ (resp. on X_1 and X_2). Denote by $X = X_1 \times X_2$ the product and by $\mathcal{L}$ the product of $\mathcal{L}_1$ and $\mathcal{L}_2$ (with the obvious metrization). Clearly,

$$Z(X,\mathcal{L},s) = Z(X_1,\mathcal{L}_1,s)\cdot Z(X_2,\mathcal{L}_2,s).$$

Assume that for $i = 1,2$ we have

$$Z(X_i,\mathcal{L}_i,s) = \frac{\Theta(\mathcal{L}_i)}{(s-\alpha_{\mathcal{L}_i}(X_i))^{\beta_{\mathcal{L}_i}(X_i)}} + \frac{f_i(s)}{(s-\alpha_{\mathcal{L}_i}(X_i))^{\beta_{\mathcal{L}_i}(X_i)-1}}$$

with some functions $f_i(s)$ which are holomorphic in the domains

$$\mathrm{Re}(s_i) > \alpha_{\mathcal{L}_i}(X_i) - \varepsilon$$

for some $\varepsilon > 0$. There are two possibilities:

Case 1: $\alpha_{\mathcal{L}_1}(X) = \alpha_{\mathcal{L}_2}(X)$. In this situation the constant $\Theta(\mathcal{L})$ at the pole of highest order $\beta_{\mathcal{L}_1}(X_1) + \beta_{\mathcal{L}_2}(X_2)$ is given by $\Theta(\mathcal{L}) = \Theta(\mathcal{L}_1)\Theta(\mathcal{L}_2)$.

Case 2: $\alpha_{\mathcal{L}_1}(X) < \alpha_{\mathcal{L}_2}(X)$. In this situation the constant is a sum

$$\Theta(\mathcal{L}) = \sum_{x \in X_1(F)} H_{\mathcal{L}_1}^{-\alpha_{\mathcal{L}_2}}(x)\Theta(\mathcal{L}_2).$$

Consider the projection $X \to X_1$ and denote by V_x the fiber over $x \in X_1(F)$. We notice

$$\tau_{\mathcal{L}}(V_x) = H_{\mathcal{L}_1}^{-\alpha_{\mathcal{L}_2}}(x)\tau_{\mathcal{L}_2}(X_2).$$

We denote by M the $\mathbf{Q}$-Cartier divisor $\pi_1^*\mathcal{L}_1^{\alpha_{\mathcal{L}_2}} \otimes K_{V_1}$. We obtain that $\pi_1^*\mathcal{L}_1^{\alpha_{\mathcal{L}_2}} = M \otimes K_{V_1}^{-1}$. So we have $\tau_{\mathcal{L}}(V_x) \sim H^{-1}_{M\otimes K_{V_1}^{-1}}$. We observe that Tamagawa numbers of fibers depend on the height of the points on the base.

4.3. Symmetric product of a curve. — Let C be a smooth irreducible curve of genus $g \geq 2$ over F. We denote by $X = C^{(m)}$ the m-th symmetric product of C and by $Y := \mathrm{Jac}(C)$ the Jacobian of C. We fix an $m > 2g-2$. We have a fibration

$$\pi : C^{(m)} \to Y,$$

with $\mathbf{P}^{m-g}$ as fibers. We denote by V_y a fiber over $y \in Y(F)$.

Let $\tilde{C} \to \mathrm{Spec}(\mathcal{O}_F)$ be a smooth model of C over the integers and $\mathcal{L}$ an ample hermitian line bundle on $\tilde{C}$. It defines a height function

$$H_{\mathcal{L}} : C(F) \to \mathbf{R}_{>0}$$

which extends to $X(F)$. Observe that $\alpha_{\mathcal{L}}(X) = (m+1-g)/d$, where $d := \deg_{\mathbf{Q}}(\mathcal{L})$. Consider the height zeta function

$$Z(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

This function was introduced by Arakelov in [**3**].

Theorem 4.3.1 ([**17**]). — *Let $\mathcal{L}$ be an ample hermitian line bundle on $\tilde{C}$. There exist an $\varepsilon(\mathcal{L}) > 0$ and a real number $\Theta(\mathcal{L}) \neq 0$ such that the height zeta function has the following representation*

$$Z(X, \mathcal{L}, s) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))} + f(s)$$

with some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

This is Theorem 8 in ([**17**], p. 422). Arakelov gives an explicit expression for the constant $\Theta(\mathcal{L})$ ([**3**]). We are very grateful to J.-B. Bost for pointing out to us that Arakelov's formula is not correct and for allowing us to use his notes on the Arakelov zeta function.

Theorem 4.3.2. — *With the notations above we have*

$$\Theta(\mathcal{L}) = \sum_{y\in Y(F)} \tau_{\mathcal{L}}(V_y).$$

Proof. — We outline the proof for $F = \mathbf{Q}$. For $\mathrm{Re}(s) \gg 0$ one can rearrange the order of summation and one obtains

$$Z(X, \mathcal{L}, s) := \sum_{y\in Y(F)} \sum_{x\in V_y(F)} H_{\mathcal{L}}(x)^{-s}.$$

It is proved in ([**17**], p. 420-422) that the sums

$$Z(V_y, \mathcal{L}, s) := \sum_{x\in V_y(F)} H_{\mathcal{L}}(x)^{-s}$$

have simple poles at $s = \alpha_{\mathcal{L}}(X)$ with non-zero residues and that one can "sum" these expressions to obtain a function with a simple pole at $s = \alpha_{\mathcal{L}}(X)$ and meromorphic continuation to $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$ for some $\varepsilon(\mathcal{L}) > 0$. Moreover, the residue at this pole is obtained as a sum over $y \in Y(F)$ of the residues of $Z(V_y, \mathcal{L}, s)$.

Choosing an element $z(y)$ in the class of $y \in \mathrm{Jac}(C)(F)$, and denoting by

$$E(y) := \Gamma(C, \mathcal{O}(z(y)))$$

one can identify the fiber as $V_y = \mathbf{P}(E(y))$, where $f \in E(y) = \Gamma(C, \mathcal{O}(z(y)))$ is mapped to $z(y) + \mathrm{div}(f)$. The height is given by the formula

$$H_{\mathcal{L}}(z(y) + \mathrm{div}(f)) := H_{\mathcal{L}}(z(y)) \exp(\int_{C(\mathbf{C})} \log|f| c_1(\mathcal{L})).$$

This defines a metrization of the anticanonical line bundle on $V_y = \mathbf{P}(E(y))$. Assuming the Tamagawa number conjecture for $\mathbf{P}^n$ (for suitable metrizations of the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^n$) we obtain

$$\lim_{s\to\alpha_{\mathcal{L}}(X)} (s - \alpha_{\mathcal{L}}(X)) Z(V_y, \mathcal{L}, s) = \tau_{\mathcal{L}}(V_y).$$

□

One can write down an explicit formula for $\tau_{\mathcal{L}}(V_y)$. For $f \in E(y)_{\mathbf{C}} \backslash \{0\}$ we define

$$\Phi(f) := \exp(\frac{1}{d} \int_{C(\mathbf{C})} \log|f| c_1(\mathcal{L}))$$

and we put $\Phi(0) = 0$. It follows that

$$\tau_{\mathcal{L}}(V_y) = \frac{1}{2} \alpha_{\mathcal{L}}(X) H_{\mathcal{L}}(y)^{-\alpha_{\mathcal{L}}(X)} \cdot \frac{\mathrm{vol}(\{f \in E(y)_{\mathbf{R}} \mid \Phi(f) \le 1\})}{\mathrm{vol}(E(y)_{\mathbf{R}}/E(y))},$$

where the volumes are calculated with respect to some Lebesgue measure on the space $E(z(y))_{\mathbf{R}}$. Arakelov relates this last expression to the Neron-Tate height of

$y \in \mathrm{Jac}(C)$. A detailed calculation due to Bost indicates that Arakelov's formula is correct only up-to $O(1)$.

4.4. Homogeneous spaces G/P. — Let G be a split semisimple linear algebraic group defined over a number field F. It contains a Borel subgroup P_0 defined over F and a maximal torus which is split over F. Let P be a standard parabolic. Denote by $Y_P = P\backslash G$ (resp. $X = P_0\backslash G$) the corresponding flag variety.

A choice of a maximal compact subgroup $\mathbf{K}$ such that $G(\mathbf{A}_F) = P_0(\mathbf{A}_F)\mathbf{K}$ defines a metrization on every line bundle L on the flag varieties Y_P ([**19**], p. 426). We will denote by

$$H_{\mathcal{L}} : P(F)\backslash G(F) \to \mathbf{R}_{>0}$$

the associated height. We consider the height zeta function

$$Z(X, \mathcal{L}, s) := \sum_{x \in X(F)} H_{\mathcal{L}}(x)^{-s}.$$

Theorem 4.4.1. — *Let $\mathcal{L}$ be an ample metrized line bundle on X. There exist an $\varepsilon(\mathcal{L}) > 0$ and a real number $\Theta(\mathcal{L}) \neq 0$ such that the height zeta function has the following representation*

$$Z(X, \mathcal{L}, s) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)}} + \frac{f(s)}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)-1}}$$

with some function $f(s)$ which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

This theorem follows from the identification of the height zeta function with an Eisenstein series and from the work of Langlands. The formula (2.10) in ([**19**], p. 431) provides an expression for $\Theta(\mathcal{L})$ which we will now analyze.

There is a canonical way to identify the faces of the closed cone of effective divisors $\Lambda_{\mathrm{eff}}(X) \subset \mathrm{Pic}(X)_{\mathbf{R}}$ with $\Lambda_{\mathrm{eff}}(Y_P)$ as P runs through the set of standard parabolics. A line bundle L such that its class is contained in the interior of the cone $\in \Lambda_{\mathrm{eff}}(X)$ defines a line bundle

$$[L_Y] := \alpha_{\mathcal{L}}(X)[L] + [K_X]$$

which is contained in the interior of the face $\Lambda_{\mathrm{eff}}(Y) \subset \Lambda_{\mathrm{eff}}(X)$ for some $Y = Y_P$. We have a fibration $\pi_{\mathcal{L}} : X \to Y$ with fibers isomorphic to the flag variety $V := P_0\backslash P$. A fiber over $y \in Y(F)$ will be denoted by V_y. Denote by $\mathcal{K}_Y$ the canonical line bundle on Y with the metrization defined above.

Theorem 4.4.2. — *We have*

$$\Theta(\mathcal{L}) = \sum_{y \in Y(F)} \gamma_{\mathcal{L}}(X)\tau_{\mathcal{L}}(V_y).$$

Proof. — In the domains of absolute and uniform convergence we can rearrange the order of summation and we obtain

$$Z(X, \mathcal{L}, s) = \sum_{y \in Y(F)} \sum_{x \in P_0(F) \backslash P(F)} H_{\mathcal{L}}(yx)^{-s}.$$

One can check that the sums

$$Z(V_y, \mathcal{L}, s) := \sum_{x \in V_y(F)} H_{\mathcal{L}}(x)^{-s} = \sum_{x \in P_0(F) \backslash P(F)} H_{\mathcal{L}}(yx)^{-s}$$

have poles at $s = \alpha_{\mathcal{L}}(X)$ of order $\beta_{\mathcal{L}}(X)$ with non-zero residues, and that they admit meromorphic continuation to $\alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$ for some $\varepsilon(\mathcal{L}) > 0$. Moreover, the constant $\Theta(\mathcal{L})$ is obtained as a sum over $y \in Y(F)$ of the residues of $Z(V_y, \mathcal{L}, s)$. From the Tamagawa number conjecture for $P_0 \backslash P$ (with varying metrizations of the anticanonical line bundle) we obtain

$$\lim_{s \to \alpha_{\mathcal{L}}(X)} (s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} Z(V_y, \mathcal{L}, s) = \gamma_{\mathcal{L}}(X) \tau_{\mathcal{L}}(V_y).$$

The sum

$$\sum_{y \in Y(F)} \gamma_{\mathcal{L}}(X) \tau_{\mathcal{L}}(V_y)$$

converges for $[L_Y]$ contained in the interior of $\Lambda_{\mathrm{eff}}(Y)$. □

Let us recall the explicit formula for $\tau_{\mathcal{L}}(V_y)$ (see (2.9) in [**19**], p. 431):

$$\lim_{s \to \alpha_{\mathcal{L}}(X)} (s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} \sum_{x \in P_0(F) \backslash P(F)} H_{\mathcal{L}}(xy)^{-s} = \gamma_1 c_P H_{\mathcal{L}_Y^{-1} \otimes \mathcal{K}_Y}(y)$$

where $\gamma_1 \in \mathbf{Q}$ is an explicit constant and the constant c_P is defined in ([**19**], p. 430). It follows that

$$\Theta(\mathcal{L}) = \gamma_1 c_P \sum_{y \in Y(F)} H_{\mathcal{L}_Y^{-1} \otimes \mathcal{K}_Y}(y).$$

Next we observe that there is an explicit constant $\gamma_2 \in \mathbf{Q}$ such that we have

$$\gamma_2 \tau_{\mathcal{L}}(V_y) = c_P \cdot H_{\mathcal{L}_Y^{-1} \otimes \mathcal{K}_Y}(y)$$

for all $y \in Y(F)$. To see this, we first identify $c_P = \gamma_2 \tau_{\mathcal{L}}(V)$, this is done by a computation of local factors of intertwining operators ([**30**], p.160-161). The next step involves the comparison of Tamagawa measures on V_y for varying $y \in Y(F)$. Finally, we have $\gamma_{\mathcal{L}}(X) = \gamma_1 \gamma_2$.

4.5. Toric varieties. — There are many equivalent ways to describe a toric variety X over a number field F together with some projective embedding (see, for example, [**12, 25, 13**]). For us, it will be useful to view $X = X_\Sigma$ as a collection of the following data ([**6**]):

1. A splitting field E of the algebraic torus T and the group $G = \mathrm{Gal}(E/F)$.
2. The lattice of E-rational characters of T, which we denote by M and its dual lattice N.
3. A G-invariant complete fan Σ in $N_{\mathbf{R}} = N \otimes \mathbf{R}$.

There is an isomorphism between the group of G-invariant integral piecewise linear functions $\varphi \in PL(\Sigma)^G$ and classes of T-linearized line bundles on X_Σ. For $\varphi \in PL(\Sigma)^G$ we denote the corresponding line bundle by $L(\varphi)$.

We define metrizations of line bundles as follows. Let $G_v \subset G$ be the decomposition group at v. We put $N_v = N^{G_v}$ for the lattice of G_v-invariants of N for non-archimedean valuations v and $N_v = N_{\mathbf{R}}^{G_v}$ for archimedean v. We have the logarithmic map

$$T(F_v)/T(\mathcal{O}_v) \to N_v$$

which is an embedding of finite index for all non-archimedean v, an isomorphism of lattices for almost all non-archimedean valuations and an isomorphism of real vector spaces for archimedean valuations. We denote by $\bar{t}_v$ the image of $t_v \in T(F_v)$ under this map.

Definition 4.5.1 ([**6**, p. 607]). — For every $\varphi \in PL(\Sigma)^G$ and $t_v \in T(F_v)$ we define the local height function

$$H_{\Sigma,v}(t_v, \varphi) := e^{\varphi(\bar{t}_v)\log q_v}$$

where q_v is the cardinality of the residue field of F_v for non-archimedean valuations and $\log q_v = 1$ for archimedean valuations. For $t \in T(\mathbf{A}_F)$ we define the global height function as

$$H_\Sigma(t, \varphi) := \prod_{v \in \mathrm{Val}(F)} H_{\Sigma,v}(t_v, \varphi).$$

We proved in ([**6**], p. 608) that this pairing can be extended to a pairing

$$H_\Sigma : T(\mathbf{A}_F) \times PL(\Sigma)^G_{\mathbf{C}} \to \mathbf{C}$$

and that it defines a simultaneous metrization of T-linearized line bundles on X. We will denote such metrized line bundles by $\mathcal{L} = \mathcal{L}(\varphi)$. We consider the height zeta function

$$Z(T, \mathcal{L}, s) = \sum_{t \in T(F)} H_{\mathcal{L}}(t)^{-s}.$$

Theorem 4.5.2 ([**8**]). — *Let $\mathcal{L}$ be an invertible sheaf on X (with the metrization introduced above) such that its class $[L]$ is contained in the interior of $\Lambda_{\mathrm{eff}}(X)$. There* *exist an $\varepsilon(\mathcal{L}) > 0$ and a $\Theta(\mathcal{L}) > 0$ such that the height zeta function has the following representation*

$$Z(\mathcal{L}, T, S) = \frac{\Theta(\mathcal{L})}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)}} + \frac{f(s)}{(s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)-1}}$$

where $f(s)$ is a function which is holomorphic for $\mathrm{Re}(s) > \alpha_{\mathcal{L}}(X) - \varepsilon(\mathcal{L})$.

Remark 4.5.3. — The computation of the constants $\alpha_{\mathcal{L}}(X)$ and $\beta_{\mathcal{L}}(X)$ in specific examples is a problem in linear programming. For the anticanonical line bundle on a smooth toric variety X we have $\alpha_{\mathcal{K}^{-1}}(X) = 1$ and $\beta_{\mathcal{K}^{-1}}(X) = \dim PL(\Sigma)^G_{\mathbf{R}} - \dim M^G_{\mathbf{R}}$.

Our goal is to identify the constant $\Theta(\mathcal{L})$. Let us recall some properties of toric varieties and introduce more notations (see **[6]**). The cone of effective divisors $\Lambda_{\text{eff}}(X)$ is generated by the classes of irreducible components of $X \backslash T$ which we denote by $[D_1], ..., [D_r]$. These divisors correspond to Galois orbits $\Sigma_1(1), ..., \Sigma_r(1)$ on the set of 1-dimensional cones in Σ. The line bundle $\mathcal{L}$ defines a face $\Lambda(\mathcal{L})$ of $\Lambda_{\text{eff}}(X)$. We denote by $J = J(\mathcal{L})$ the maximal set of indices $J \subset [1, ..., r]$ such that we have

$$\alpha_{\mathcal{L}}(X)[L] + [K_X] = \sum_{j \in J} r_j [D_j]$$

with $[D_j] \in \Lambda(\mathcal{L})$ and some $r_j \in \mathbf{Q}_{>0}$. We denote by $I = I(\mathcal{L})$ the set of indices $i \notin J(\mathcal{L})$. We denote by M_J the lattice given by

$$M_J := \{m \in M \mid < e, m >= 0 \quad \mathbf{R}_{\geq 0} e \in \cup_{i \in I} \Sigma_i(1)\},$$

We denote by $M_I := M/M_J$ and by N_* the corresponding dual lattices. We have an exact sequence of algebraic tori

$$1 \to T_I \to T \to T_J \to 1$$

which induces a map $\pi_{\mathcal{L}} : T(F) \to T_J(F)$ with finite cokernel and an exact sequence of lattices

$$0 \to N_I \to N \to N_J \to 0.$$

The restriction of the fan $\Sigma \subset N_{\mathbf{R}}$ to $N_{I,\mathbf{R}}$ will be denoted by Σ_I. It is again a G-invariant fan and it will define an equivariant compactification X_I of T_I. The class of the piecewise linear function $\varphi_I \in PL(\Sigma_I)^G$ with $\varphi_I(e) = 1$ for $e \in \cup_{i \in I} \Sigma_i$ corresponds to the class of the anticanonical line bundle $[-K_I] \in \mathrm{Pic}(X_I)$.

The line bundle $\mathcal{L}$ defines a fibration of varieties $\pi_{\mathcal{L}} : X_{\Sigma} \to Y$ with fibers isomorphic to X_I, which, when restricted to T, gives rise to the exact sequence of tori above. We denote the fiber over $y \in T_J(F)$ by $X_{I,y}$.

Theorem 4.5.4. — *We have*

$$\Theta(\mathcal{L}) = \gamma_{\mathcal{L}}(X)\delta_{\mathcal{L}}(X) \sum_{y \in \pi_{\mathcal{L}}(T(F))} \tau_{\mathcal{L}}(X_{I,y}).$$

Proof. — Let $\mathcal{L} = \mathcal{L}(\varphi)$ be an invertible sheaf on T with the metrization introduced above. In the domain of absolute and uniform convergence we can rearrange the order of summation and we obtain

$$Z(T, \mathcal{L}, s) = \sum_{y \in \pi_{\mathcal{L}}(T(F))} \sum_{x \in T_{I,y}(F)} H_\Sigma(yx, \varphi)^{-s}.$$

From the proof of our main theorem in **[8]** it follows that the sums

$$Z(T_{I,y}, \mathcal{L}, s) = \sum_{x \in T_I(F)} H_\Sigma(yx, \varphi)^{-s}$$

have a pole at $\alpha_{\mathcal{L}}(X)$ of order $\beta_{\mathcal{L}}(X)$. Moreover, the constant $\Theta(\mathcal{L})$ is obtained as a sum over $y \in \pi_{\mathcal{L}}(T(F))$ of residues of $Z(T_{I,y}, \mathcal{L}, s)$. Now we want to use the Tamagawa number conjecture for the anticanonical line bundle (with varying metrizations) on the toric variety X_I to conclude the proof.

In **[7]** we proved this conjecture for a specific metrization and under the assumption that the fan Σ is *regular*. We want to demonstrate that our proof goes through in the general case needed above.

Our main idea was to use the Poisson summation formula on the adelic group $T(\mathbf{A}_F)$ and to obtain an integral representations for the height zeta function. We denote by $\mathcal{A}_I = (T_I(\mathbf{A}_F)/\mathbf{K}T_I(F))^*$ the group of unitary characters of $T_I(\mathbf{A}_F)$ which are trivial on $T_I(F)$ and on the maximal compact subgroup $\mathbf{K} \subset T_I(\mathbf{A}_F)$. Using the the adelic definition of the height function we obtain

$$Z(T_{I,y}, \mathcal{L}, s) = \sum_{t \in T_I(F)} H_{\mathcal{L}}(yt)^{-s} = \int_{\mathcal{A}_I} d\chi \int_{T_I(\mathbf{A}_F)} H_{\mathcal{L}}(yt)^{-s} \chi(t) d\mu,$$

where $d\mu$ is a Haar measure on $T(\mathbf{A}_F)$ and $d\chi$ is the orthogonal Haar measure on $\mathcal{A}_I$. To apply our technical theorem in **[7]** about the analytic continuation and the residues of such integrals we need to know that

$$\int_{T_I(\mathbf{A}_F)} H_{\mathcal{L}}(yt)^{-s} \chi(t) d\mu = \prod_{i \in I} L_i(\chi_i, s) \cdot \zeta_{\Sigma_I}(\chi, s) \cdot \zeta_\infty(\chi, s)$$

where

$$\zeta_{\Sigma_I}(\chi, s) = \prod_{v \in \mathrm{Val}(F)} \zeta_{\Sigma_I, v}(\chi, s)$$

is an absolutely convergent Euler product for $\mathrm{Re}(s) > \alpha_{\mathcal{L}(X)} - \varepsilon(\mathcal{L})$, $L_i(\chi_i, s)$ are Hecke L-functions (with some induced characters χ_i) and $\zeta_\infty(\chi, s)$ satisfies certain growth conditions.

First we observe that it is unnecessary to assume that the fan Σ_I is regular. Using our definition of the height function we see that the calculation of the Fourier transform (see **[6]**) reduces to summations of the function $q_v^{\varphi(\overline{t_v}) + im_v}$ ($m_v \in M_{I,\mathbf{R}}$) over the lattice $N_{I,v}$ (resp. to integrations in cones for archimedean valuation). A piecewise linear function φ induces a piecewise linear function on any subdivision of the fan. Clearly, the result of such summations and integrations does not depend on any subdivisions.

Next we see that for a fixed $y \in T_J(F)$ we have $H_{\mathcal{L},v}(yt) = H_{\mathcal{L},v}(t)$ for almost all v and all $t \in T_I(\mathbf{A}_F)$. Now we can refer to lemma 5.10 in [**8**] which proves the required statement. The local integrals for the remaining finitely many non-archimedean valuations will be absorbed into $\zeta_{\Sigma_I}(\chi, s)$. And finally, we need to check that the estimates of the Fourier transform of $H_{\mathcal{L},v}(yt)$ at archimedean valuations are still satisfied for any $y \in T_J(F)$. This is straightforward.

We can now apply the main technical theorem of [**7**] and obtain

$$\lim_{s \to \alpha_{\mathcal{L}}} (s - \alpha_{\mathcal{L}}(X))^{\beta_{\mathcal{L}}(X)} Z(T_{I,y}, \mathcal{L}, s) = \gamma_{\mathcal{L}}(X)\delta_{\mathcal{L}}(X)\tau_{\mathcal{L}}(X_{I,y}).$$

□

Remark 4.5.5. — It is possible to compute $\tau_{\mathcal{L}}(X_{I,y})$ and to observe that it is related to the height of the point $y \in T_J(F)$.

Remark 4.5.6. — Similar statements hold for equivariant compactifications of homogeneous spaces G/U where G is a split reductive group and U is its maximal unipotent subgroup [**33**]. We hope that these results can be extended to equivariant compactifications of other homogeneous spaces, in particular, to equivariant compactifications of reductive and non-reductive groups.

5. Singular Fano varieties

5.1. Weighted projective spaces. — Let $W := \mathbf{P}(w) = \mathbf{P}(w_0, \ldots, w_n)$ be a weighted projective space of dimension n with weights $w = (w_0, \ldots, w_n)$. We remark that W is a rational variety over $\mathbf{Q}$ with $\mathrm{Pic}(W) \cong \mathbf{Z}$. Moreover, the anticanonical class K_W^{-1} is an ample $\mathbf{Q}$-Cartier divisor. So W is a (singular) Fano variety of index

$$r = \frac{w_0 + \cdots + w_n}{\mathrm{l.c.m.}\{w_0, \ldots, w_n\}}.$$

One could try to generalize the method of Schanuel [**31**] for counting $\mathbf{Q}$-rational points of bounded height on usual projective spaces to the case of weighted projective spaces. Let $z_0, z_1, \ldots, z_n$ be homogeneous coordinates on Y. Then a first approximation to counting points of bounded height would be a counting of all $(n+1)$-tuples $(x_0, x_1, \ldots, x_n) \in \mathbf{Z}^n \setminus \{0\}$ satisfying the conditions

$$|x_i| \leqslant B^{\frac{w_i}{w_0+w_1+\cdots+w_n}} \quad i = 0, \ldots, n.$$

Since the volume of the domain restricted by these inequalities is

$$B = \prod_{i=0}^{n} B^{\frac{w_i}{w_0+w_1+\cdots+w_n}},$$

one could expect that the asymptotic number of solutions of these inequalities agrees with "expected" linear growth for the anticanonical height. However, this "intuition" turns out to be wrong, in general, because the singularities of Y could be even worse than canonical. A typical class of singularities that appear on Y are so called log-terminal singularities introduced by Kawamata [**28**]. We give below a simple example of a Del Pezzo surface with a log-terminal singularity and we show that for every dense Zariski open subsets $U \subset W$ the number $N(U, B)$ of F-rational points of anticanonical height $\leqslant B$ in U has more than linear growth:

$$N(U, B) = c(U)B^{2-\frac{4}{m+2}}(1 + o(B)).$$

Moreover, there are no dense Zariski open subsets $U' \subset X$ such that the adelic term in the constant $c(U)$ in the asymptotic formula for $N(U, B)$ would be independent of U for $U \subset U'$.

Example 5.1.1 **(Del-Pezzo surface with a log-terminal singularity)**
Let $W = \mathbf{P}(1, 1, m)$ be a singular weighted projective plane with weights $(1, 1, m)$, $m \geqslant 2$. Then the anticanonical class of W is an ample $\mathbf{Q}$-Cartier divisor (i.e., W is Del Pezzo surface) and $p = (0 : 0 : 1)$ is the unique singularity of W. Let $X \to W$ be the minimal resolution of the singularity at $p \in W$. Then X is isomorphic to a ruled surface $\mathbf{F}_m = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(m))$ and the exceptional divisor $E = f^{-1}(p)$ is a smooth rational curve which is a section of the $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ and $\langle E, E\rangle = -m$. Then we have

$$K_X = f^*K_W + \frac{2-m}{m}E.$$

Therefore, p is canonical $\Leftrightarrow m = 2$ and p is log-terminal $\Leftrightarrow m \geqslant 3$.

The group $\mathrm{Pic}(X)$ is isomorphic to $\mathbf{Z}^2$ where $\{[E], [C]\}$ is a $\mathbf{Z}$-basis. Moreover, $[E], [C]$ are generators for the cone $\Lambda_{\mathrm{eff}}(X)$ of effective divisors in $\mathrm{Pic}(X)_{\mathbf{R}}$. We have

$$K_X = -2E - (m+2)C,$$

$$L := f^*(-K_X) = \frac{m+2}{m}E + (m+2)C$$

and

$$\alpha_L(W) = \inf\{t \in \mathbf{Q} \; : \; t[L] + [K_X] \in \Lambda_{\mathrm{eff}}(X)\} = \frac{2m}{m+2}.$$

Since X is a smooth toric variety, we can apply our main result in [**8**] and obtain the following:

Theorem 5.1.2. — *Let $\pi : W \setminus p \to \mathbf{P}^1$ be the natural projection, C_x the fiber of π over $x \in \mathbf{P}^1(\mathbf{Q})$. Then for any dense Zariski open subset in $W \setminus p$, one has*

$$N(U,B) = c(U)B^{2-\frac{4}{m+2}}(1+o(B))$$

Moreover,

$$c(U) = \sum_{x \in \mathbf{P}^1(\mathbf{Q}) \cap \pi(U)} c(C_x).$$

5.2. Vaughan-Wooley cubic

Example 5.2.1. — Let $Y \subset \mathbf{P}^5$ be a singular cubic defined by the equation $z_0z_1z_2 - z_3z_4z_5 = 0$, X the intersection of Y with the linear subspace in $\mathbf{P}^5$ with the equation:

$$z_0 + z_1 + z_2 - z_3 - z_4 - z_5 = 0.$$

Vaughan and Wooley proved [**34**]:

Theorem 5.2.2. — *Let $U \subset X$ be the complement in X to the following 15 divisors $D_{i_1i_2i_3}, D_{ij} \subset X$ $(\{i_1,i_2,i_3\} = \{0,1,2\},\ i \in \{0,1,2\},\ j \in \{3,4,5\})$, where*

$$D_{i_1i_2i_3} = \{(z_0 : \ldots : z_5) \in \mathbf{P}^5 \ : \ z_{i_1} = z_3,\ z_{i_2} = z_4,\ z_{i_3} = z_5\}$$

$$D_{ij} = \{(z_0 : \ldots : z_5) \in X \ : \ z_i = z_j = 0\}.$$

If $N(U,B)$ is the number of $\mathbf{Q}$-rational points in U of the anticanonical height $\leqslant B$, then there exist some constants $c_1 > c_2 > 0$ such that

$$c_2B^2(\log B)^5 \leqslant N(U,B) \leqslant c_1B^2(\log B)^5.$$

We want to show that this result is compatible with predictions in [**4**]. First of all we note that Y is a 4-dimensional toric Fano variety: an equivariant compactification of a 4-dimensional algebraic torus T with respect to a 4-dimensional polyhedron Δ with 6 lattice vertices

$$v_0 = (0,0,0,0),\ v_1 = (1,0,0,0),\ v_2 = (0,1,0,0),$$

$$v_3 = (0,0,1,0),\ v_4 = (0,0,0,1),\ v_5 = (1,1,-1,-1)$$

(Δ is the support of global sections of a very ample divisor Y corresponding to the embedding $Y \hookrightarrow \mathbf{P}^4$). The polyhedron Δ has 9 faces Θ_{ij} of codimension 1:

$$\Theta_{ij} = \mathrm{Conv}(\{v_0,v_1,v_2,v_3,v_4,v_5\} \setminus \{v_i,v_j\}).$$

Each face Θ_{ij} defines an torus invariant divisor $Y_{ij} \subset Y \setminus T$ such that $D_{ij} = Y_{ij} \cap X$. It is easy to check that all singularities of Y are at worst terminal and that the hypersurface $X \subset Y$ intersects all strata Y_{ij} transversally. From these facts we obtain that the only exceptional divisors with the discrepancy 0 that appear in a resolution of singularities of X come from singularities in $X \cap T$. We write down the affine equation of $X \cap T$ as

$$1 + x + y - z - t - \frac{xy}{zt},$$

where $T = \operatorname{Spec} \mathbf{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}, t^{\pm 1}]$. From this equation one immediately sees that the only singularities in $X \cap T$ are the A_1-double points lying on the curve $C : x = y = z = t$. Therefore, we obtain $\operatorname{rk} \operatorname{Cl}(X) = \operatorname{rk} \operatorname{Cl}(Y) = 9 - \dim T = 5$. Moreover, there exists exactly one crepant divisor (over C) in a resolution of singularities of X. So the predicted power of $\log B$ in the asymptotic formula for $N(U, B)$ is $(\operatorname{rk} \operatorname{Cl}(X) - 1) + 1 = 5$.

5.3. Cubic $xyz = u^3$. — We consider the singular cubic surface $X \subset \mathbf{P}^3$ over $\mathbf{Q}$ given by the homogeneous equation $xyz = u^3$. This is a toric variety, an equivariant compactification of the torus

$$T = \operatorname{Spec} \mathbf{Q}[x, y, z]/(xyz - 1)$$

given by the condition $u \neq 0$. We can fix an isomorphism $T \cong \mathbf{G}_m^2$ by choosing $\{x, y\}$ as a basis of the group of algebraic characters of T. Consider the problem of the computation of the asymptotic of

$$N(T, B) = \operatorname{Card}\{(x, y) \in (\mathbf{Q}^*)^2 \ : \ H(x, y) \leq B\ \}$$

for $B \to \infty$, where

$$H(x, y) = \prod_{v \in \operatorname{Val}(\mathbf{Q})} \max\{\|x\|_v, \|y\|_v, \|(xy)^{-1}\|_v, \|1\|_v\}$$

This problem is addressed in **[18]**. We would like to use this problem as a down-to-earth illustration of our general theory of height zeta functions of toric varieties. First of all we note that the relation $\|x\|_v \|y\|_v \|(xy)^{-1}\|_v = \|1\|_v = 1$ implies that

$$H_v(x, y) := \max\{\|x\|_v, \|y\|_v, \|(xy)^{-1}\|_v\}.$$

Since $l_1 = \log \|x\|_v$, $l_2 = \log \|y\|_v$, $l_3 = \log \|(xy)^{-1}\|_v$ are linear functions on the logarithmic space $N_{\mathbf{R},v} \cong \mathbf{R}^2$:

$$N_{\mathbf{R},v} = \begin{cases} T(\mathbf{Q}_v)/T(\mathcal{O}_v) \otimes_{\mathbf{Z}} \mathbf{R}, & \text{if } v = p \in \operatorname{Spec} \mathbf{Z} \\ T(\mathbf{Q}_v)/T(\mathcal{O}_v), & \text{if } v = \infty, \end{cases}$$

we can consider $\log h_v(x, y)$ as a piecewise linear function on $N_{\mathbf{R},v}$. Let $e_1 = (-2, 1)$, $e_2 = (1, -2)$ and $e_3 = (1, 1)$ be lattice vectors in $\mathbf{Z}^2 \subset \mathbf{R}^2$. We define the following 3 convex cones in $\mathbf{R}^2$:

$$\sigma_1 = \mathbf{R}_{\geqslant 0} e_2 + \mathbf{R}_{\geqslant 0} e_3,$$

$$\sigma_2 = \mathbf{R}_{\geqslant 0} e_1 + \mathbf{R}_{\geqslant 0} e_3,$$

$$\sigma_3 = \mathbf{R}_{\geqslant 0} e_1 + \mathbf{R}_{\geqslant 0} e_2.$$

Then $N_{\mathbf{R},v} = \bigcup_{i=1}^{3} \sigma_i$ and the restriction of $\log H_v(x,y)$ to σ_i coincides with the linear function l_i. Let

$$A := \bigoplus_{v \in \mathrm{Val}(\mathbf{Q})} T(\mathbf{Q}_v)/T(\mathcal{O}_v)$$

be the logarithmic adelic group. In order to compute the height zeta function

$$Z(s) = \sum_{(x,y) \in (\mathbf{Q}^*)^2 = T(\mathbf{Q})} H(x,y)^{-s}$$

we use the natural homomorphism Log of $T(\mathbf{Q})$ to A. Denote by B the subgroup $\mathrm{Log}(T(\mathbf{Q})) \subset A$. We remark that the kernel of Log consists of 4 elements of finite order in $(\mathbf{Q}^*)^2$ and the quotient A/B is isomorphic to $\mathbf{R}^2$. Moreover the functions $H_v(x,y)^{-s}$ on each $T(\mathbf{Q}_v)/T(\mathcal{O}_v)$ define a natural extension of $h(x,y)^{-s}$ to a function on A. So we obtain:

$$Z(s) = 4 \sum_{b \in B} \prod_{v \in \mathrm{Val}(\mathbf{Q})} H_v(b_v)^{-s} \tag{4}$$

The main idea of our proof in **[6]** is to apply the Poisson summation formula on the group A and to express the height zeta function $Z(s)$ as an integral

$$Z(s) = \frac{4}{(2\pi)^2} \int_{\mathbf{R}^2} \left(\prod_p Q_p(s, i\mathbf{m}) \cdot Q_\infty(s, i\mathbf{m}) \right) \mathbf{dm},$$

where $\mathbf{m} = (m_1, m_2) \in \mathbf{R}^2$, $\mathbf{dm} = dm_1 dm_2$,

$$Q_p(s, i\mathbf{m}) = \sum_{(b_{1,p}, b_{2,p}) \in \mathbf{Z}^2} h_p(b_p)^{-s} p^{i<b,\mathbf{m}>},$$

and

$$Q_\infty(s, i\mathbf{m}) = \int_{\mathbf{R}^2} H_\infty(b)^{-s} \exp(i < b, \mathbf{m} >).$$

An exact computation of $Q_p(s, i\mathbf{m})$ and $Q_\infty(s, i\mathbf{m})$ can be obtained by a subdivision of each of the cones $\sigma_1, \sigma_2, \sigma_3$ into a union of 3 subcones generated by a basis of the lattice $\mathbf{Z}^2 \subset \mathbf{R}^2$. (From the viewpoint of toric geometry this means that we reduce the counting problem for rational points in a torus with respect to a singular compactification to a counting problem for rational points in a torus with respect to the minimal resolution of singularities of this compactification). This calculation is done in **[6]**, Section 2 for arbitrary smooth toric varieties. What remains is the analytic continuation of the integral and the identification of the constant at the leading pole. For this it is necessary to work on the whole complexified space $PL(\Sigma)_{\mathbf{C}}$ and to invoke the technical theorems in Section 6 in **[7]**. Applying the main theorem of **[7]**, we obtain

$$N(T,B) = \frac{\gamma_{\mathcal{K}^{-1}}(X) \delta_{\mathcal{K}^{-1}}(X) \tau_{\mathcal{K}^{-1}}(X)}{6!} B(\log B)^6 (1 + o(1))$$

for $B \to \infty$. The constants are as follows: $\gamma_{\mathcal{K}^{-1}}(X) = 1/36$, $\delta(X) = 1$ and $\tau_{\mathcal{K}^{-1}}(X) = \tau_{\mathcal{K}^{-1}}(X)_\infty \prod_p \tau_{\mathcal{K}^{-1}}(X)_p$ where

$$\tau_{\mathcal{K}^{-1}}(X)_p = \left(1 + \frac{7}{p} + \frac{1}{p^2}\right) \cdot (1 - 1/p)^7$$

for all primes p and $\tau_{\mathcal{K}^{-1}}(X)_\infty = 9 \cdot 4$. Similar statements hold over any number field. One can compute the constant $\gamma_{\mathcal{K}^{-1}}(X)$ by observing that $\Lambda^*_{\mathrm{eff}}(X)$ (the dual cone to the cone of effective divisors) is a union of two simplicial cones.

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VICTOR V. BATYREV, Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany • *E-mail* : `batyrev@bastau.mathematik.uni-tuebingen.de`

YURI TSCHINKEL, Dept. of Mathematics, U.I.C., Chicago (IL) 60607-7045, U.S.A. *E-mail* : `yuri@math.uic.edu`