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The Function $\exp[-p\,\mathrm{Trace}\sqrt{2A}]$ as a Laplace Transform on Symmetric Matrices

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Abstract. — This note shows that if $p > 0$ and if S_+ is the set of symmetric positive definite matrices, then the function on S_+ defined by $A \mapsto \exp(-\mathrm{Trace}\, p\sqrt{2A})$ is the Laplace transform of a non positive function concentrated on S_+ if $n \geq 2$. This function is explicitely computed for $n = 2$. This computation is generalized to a Lorentz cone. The link of this question with the inverse Gaussian distributions in probability theory is also discussed, as well as the general problem of considering $\det L(A)$ as a Laplace transform on symmetric matrices when $L(\lambda)$ is a Laplace transform on the real line.

§1. Introduction. For $p > 0$, define the stable probability distribution of order 1/2 on $\mathbb{R}^+$:

$$\mu_p(dx) = \frac{p}{\sqrt{2\pi}}\, x^{-\frac{3}{2}} \exp\left(-\frac{p^2}{2x}\right) \mathbb{1}_{(0,+\infty)}(x)\, dx \tag{1.1}$$

Then its Laplace transform, evaluated at $\lambda > 0$, is :

$$\int_0^\infty \exp(-\lambda x)\, \mu_p(dx) = \exp(-p\sqrt{2\lambda}) \tag{1.2}$$

(See e.g. Feller 1970, p. 436 (3.4)).

Probability distributions (1.1) can be imbedded in the three parameter family of the so called "generalized inverse Gaussian distributions" defined for (a, b, λ) in $(0, +\infty) \times [0, +\infty) \times \mathbb{R}$ by

$$\mu_{\lambda,a,b}(dx) = \left(K_\lambda(\sqrt{ab})\right)^{-1} a^{\frac{\lambda}{2}} b^{-\frac{\lambda}{2}} x^{\lambda-1} \exp -\tfrac{1}{2}(ax + bx^{-1})\, \mathbb{1}_{(0,+\infty)}(x)\, dx \qquad (1.3),$$

where K_λ is a Bessel function (Watson 1966, p. 91).

Probability distributions (1.3) have a natural extension to the space of symmetric (n, n) real matrices, which extends nicely the fact that (1.3) is the distribution of a random continued fraction whose coefficients are independent and gamma distributed (see Letac and Seshadri 1983). This extension has been performed by E. Bernadac (1992) and even been made on general symmetric cones (see Bernadac 1993 and 1995). In this extension, the gamma distributions are replaced by the Wishart distributions on symmetric real matrices or on symmetric cones.

However, in this extension, the particular role played by $\lambda = -1/2$ when specializing (1.3) to (1.1) disappears, and although the extension of (1.3) to matrices is natural, extension of (1.1) is not. So one can look for an other path, and instead of trying to generalize (1.3), through for instance continued fractions, one can try to generalize (1.1) to symmetric matrices through (1.2). To describe what we have in mind, it is better to introduce a few definitions now.

Let E be a Euclidean space with dimension n, and let S be the space of symmetric endomorphisms of E. We equip S also with a Euclidean structure through the scalar product on S

$$(a, b) \mapsto \frac{1}{n} \operatorname{Trace} ab \,.$$

If $I \subset \mathbb{R}$, one denotes by $S(I)$ the set of a in S with eigenvalues in I; $S(I)$ is convex if I is an interval. For simplicity we write $S_+ = S((0, +\infty))$ (resp. $\bar{S}_+ = S([0, +\infty)))$ the cone of symmetric positive-definite (resp. positive) endomorphisms. Also, if e is a basis in E and a is in S, we write $[a]_e$ as its matrix in base e.

Let $f : I \to \mathbb{R}$ be any function. Suppose that a is in $S(I)$ and that e is an orthonormal basis which diagonalizes a, with $[a]_e = \operatorname{Diag}(\lambda_1, \dots, \lambda_n)$. Then it is a standard exercise to show that $\tilde{f}(a)$ in S defined by

$$[\tilde{f}(a)]_e = \operatorname{Diag}\left(f(\lambda_1), \dots, f(\lambda_n)\right) \tag{1.4}$$

actually does not depend on e. Thus $\tilde{f} : S(I) \mapsto S$ is a well defined function. Furthermore, if I is an interval and if the derivative f' exists on I, then $\tilde{f}$ is differentiable, and its differential $(\tilde{f})'(a)$ on a, evaluated at the point h of S, is computed as follows : defining $g : I \times I \to \mathbb{R}$ by :

$$g(\lambda, \lambda) = f'(\lambda) \qquad \text{and} \qquad g(\lambda, \mu) = \big(g(\lambda) - g(\mu)\big)/(\lambda - \mu) \quad \text{if} \quad \lambda \neq \mu$$

then, if e is an orthonormal basis with $[a]_e = \operatorname{Diag}(\lambda_1, \dots, \lambda_n)$, we have

$$\left[(\tilde{f})'(a)(h)\right]_e = \left(g(\lambda_i, \lambda_j) h_{ij}\right), \qquad \text{for} \quad [h]_e = (h_{ij}) \,. \tag{1.5}$$

The proof of (1.5) is a not so easy exercise in advanced calculus.

From (1.5), one deduces two facts. Asume that I is an open interval, and consider the function

$$a \longmapsto \operatorname{Trace} \tilde{f}(a) \qquad S(I) \longrightarrow \mathbb{R} \tag{1.6}$$

Then if f' exists, the differential of (1.6) in a is $(\tilde{f'})(a)$, from (1.5) (Note that we identify S with its dual through the Euclidean structure of S, and the differential of a real function on S can then be called a gradient). Furthermore, assume that f is convex on I. Then (1.6) will be convex on $S(I)$: to see this point, assume that f'' exists. Then, for arbitrary u in S and a in $S(I)$ (which is an open convex subset of S), there exists $\alpha > 0$ such that the function

$$(-\alpha, \alpha) \longrightarrow \mathbb{R} \qquad t \longmapsto F(t) = \operatorname{Trace} \tilde{f}(a+tu)$$

is well defined. With the help of (1.5) we compute

$$F''(0) = \operatorname{Trace} \tilde{f}''(a) u^2 .$$

Since $f'' \geq 0$, then $\tilde{f}''(a)$ is in $\bar{S}_+$, as well as $u\tilde{f}''(a)u$. Thus $F''(0) \geq 0$. This implies that (1.6) is convex. The case where f'' does not exist is then treated by approximation.

To come back to our initial problem, i.e. a suitable generalization of (1.1) through (1.2), we consider (1.6) when f is the logarithm of the Laplace transform L of some positive measure μ on $\mathbb{R}$. Let us assume that for all λ in the open interval I

$$L(\lambda) = \exp f(\lambda) = \int_{\mathbb{R}} \exp(-\lambda x)\, \mu(dx) < \infty \tag{1.7}$$

It is well known that f is convex on I. Thus, as we have seen, (1.6) is convex, and one can wonder if there exists a positive measure $\tilde{\mu}$ on S such that for all a in $S(I)$ one has

$$\operatorname{Det} \tilde{L}(a) = \exp \operatorname{Trace} \tilde{f}(a) = \int_S \exp\left(-\operatorname{Trace}(ax)\right) \tilde{\mu}(dx) \tag{1.8}$$

An instance for which it is true is the case $I = \mathbb{R}$ and $f(\lambda) = \sigma^2\lambda^2/2$: clearly $\tilde{\mu}$ is a suitable Gaussian distribution on S. An other instance for which it is almost true is the case where $I = (0, +\infty)$ and $f(\lambda) = -p \operatorname{Log} \lambda$, where $p > 0$. Here (1.7) holds with

$$\mu(dx) = \frac{x^{p-1}}{\Gamma(p)} \mathbb{1}_{(0,+\infty)}(x)\, dx .$$

However $\tilde{\mu}$ defined by (1.8) will be positive if and only if

$$p \in \left\{\tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \tfrac{n-1}{2}\right\} \cup \left(\tfrac{n-1}{2}, +\infty\right) \tag{1.9}$$

This result (1.9) is due to Gindikin (1975). It has been rediscovered again and again : see Casalis and Letac (1994) for references, and a short proof.

We are now able to state the aim of this note : to study the existence of a positive $\tilde{\mu}$ in (1.8) when $I = (0, +\infty)$ and $f(\lambda) = -p\sqrt{2\lambda}$ (compare (1.2) and (1.7)). As we shall see (section 5) the answer is negative for $n \geq 2$, and we shall prove this by computing explicitely a signed measure $\tilde{\mu}$ such that (1.8) holds when $n = 2$. Explicit calculations for $n \geq 3$ seem hopeless. Section 2 is devoted to a general study of (1.8). Section 3 specializes to $n = 2$. Section 4 studies an integral equation that we meet by considering the case $f(\lambda) = -p\sqrt{2\lambda}$ and a slight extension of the problem to the Lorentz cone, which appears in section 5.

§2. Properties of $\tilde{\mu}$ for general n. We keep the notations of the introduction; furthermore we denote by $\mathbb{O}(E)$ and $\mathbb{O}(S)$ the orthogonal groups of the Euclidean spaces E and S. There is a natural representation of $\mathbb{O}(E)$ in $\mathbb{O}(S)$ defined as follows : if u is in $\mathbb{O}(E)$, then for all a in S, $g_u(a) = uau^{-1}$ is in S.

Furthermore $\operatorname{Trace}\left(g_u(a)\right)^2 = \operatorname{Trace} a^2$, thus g_u is in $\mathbb{O}(S)$. An argument of convexity shows easily that if u is in the subgroup $\mathbb{O}_+(E)$ of rotations of $\mathbb{O}(E)$, then g_u is in $\mathbb{O}_+(S)$ too. Clearly $g_{u_1}g_u = g_{u_1u}$, and $u \mapsto g_u$ defines an homomorphism from $\mathbb{O}(E)$ to $\mathbb{O}(S)$ and from $\mathbb{O}_+(E)$ to $\mathbb{O}_+(S)$. Note also that

$$uau^{-1} = a \quad \text{for all } u \text{ in } \mathbb{O}_+(E) \qquad \Longleftrightarrow \qquad a \in \mathbb{R}.\mathrm{id}_E \tag{2.1}$$
$$uau^{-1} = a \quad \text{for all } a \text{ in } S \qquad \Longleftrightarrow \qquad u = \pm\,\mathrm{id}_E \tag{2.2}$$

Denote by G and G_+ the respective images of $\mathbb{O}(E)$ and $\mathbb{O}_+(E)$ in $\mathbb{O}(S)$ by $u \mapsto g_u$. It is easy to see that a and b in S are in the same G_+ orbit —thus in the same G orbit— if and only if their spectrum coincide. More precisely if $\lambda_1(a) \leq \lambda_2(a) \leq \ldots \leq \lambda_n(a)$ is the sequence of not necessarily distinct eigenvalues of a, then there exists u in $\mathbb{O}_+(E)$ such that $b = uau^{-1}$ if and only if $\lambda_j(a) = \lambda_j(b)$ $j = 1, \ldots, n$. The necessary condition is clear; to prove the sufficient condition, if e and f are orthonormal basis of E such $a(\vec{e}_j) = \lambda_j(a)\vec{e}_j$ and $b(\vec{f}_j) = \lambda_j(b)\vec{f}_j$ then one takes u in $\mathbb{O}_+(E)$ such that $u(\vec{f}_j) = \vec{e}_j$. However, if such a u has determinant -1, one has to replace $\vec{f}_1$ by $-\vec{f}_1$, still an eigenvector of b.

Assume now that I and μ are as in (1.7) and suppose that (1.8) holds with a signed measure $\tilde{\mu}$. For u in $\mathbb{O}(E)$ we have :

$$\operatorname{Trace} \tilde{f}(a) = \operatorname{Trace} \tilde{f}\left(g_u(a)\right) .$$

Thus (1.8) becomes

$$\int_S \exp\left(-\operatorname{Trace}(ax)\right) \tilde{\mu}(dx) = \exp \operatorname{Trace} \tilde{f}(a) = \exp \operatorname{Trace} \tilde{f}\left(g_u(a)\right)$$
$$= \int_S \exp\left(-\operatorname{Trace}\left(a g_{u^{-1}}(x)\right)\right) \tilde{\mu}(dx) = \int_S \exp\left(-\operatorname{Trace}(ay)\right) \tilde{\mu}_1(dy)$$

where $\tilde{\mu}_1(dy)$ is the image of $\tilde{\mu}$ by $x \mapsto y = g_{u^{-1}}(x)$.

Thus $\tilde{\mu}$ is invariant by G and G_+. Now S is split by G_+ in orbits and the set of these orbits is parametrized by the increasing sequence of the eigenvalues of any element of the orbit, i.e. by

$$H = \left\{h \in \mathbb{R}^n \; ; \; h_1 \leq h_2 \leq \ldots \leq h_n\right\} .$$

Choosing an arbitrary orthonormal basis e of E, one can say that, since $\tilde{\mu}$ is invariant by G_+, there exists a signed measure ν on H such that if dU denotes the Haar measure of mass 1 on the group $\mathbb{O}_+(n)$ of rotation matrices of order n, then $\tilde{\mu}(dx)$ is the image of $\nu(dh)\,dU$ by the map

$$(h,U) \longmapsto x \qquad \text{with} \qquad [x]_e = U \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} U^{-1}.$$

In (1.8), if $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of a, we get :

$$\int_{\mathbb{R}^n} \exp -(\lambda_1 t_1 + \ldots + \lambda_n t_n)\,\mu(dt_1)\ldots\mu(dt_n) = \exp\left(f(\lambda_1) + \ldots + f(\lambda_n)\right)$$

$$= \int_H \nu(dh) \int_{\mathbb{O}_+(n)} dU \exp - \operatorname{Trace} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} U \begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} U^{-1} \qquad (2.3)$$

and (2.3) shows that the image of $\nu(dh)\,dU$ by $(h,U) \mapsto (t_1,\ldots,t_n) =$ diagonal of $U\left(\begin{smallmatrix} h_1 & 0 \\ 0 & h_n \end{smallmatrix}\right)U^{-1}$ is $\mu(dt_1)\ldots\mu(dt_n)$.

The task of extracting ν from this information seems rather difficult for $n \geq 3$. For $n = 2$, however, things are feasible : we have to find $\nu(dh_1, dh_2)$ such that the image of

$$\nu(dh_1, dh_2)\,\frac{d\theta}{2\pi} \qquad \text{on} \qquad \{h \in \mathbb{R}^2 \ ; \ h_1 \leq h_2\} \times [0, 2\pi[$$

by $(h,\theta) \mapsto (t_1,t_2) = (h_1 \cos^2\theta + h_2 \sin^2\theta, h_1 \sin^2\theta + h_2 \cos^2\theta)$ is $\mu(dt_1)\mu(dt_2)$. We solve this problem in the next section.

§3. How to compute $\tilde{\mu}$ for $n = 2$.

To have a clear geometrical picture of the case $n = 2$, we adapt the notations. The Euclidean plane E is now identified with $\mathbb{R}^2$ and S is identified to $\mathbb{R}^3$ by the parametrization

$$\mathbb{R}^3 \longrightarrow S \ : \quad (a,b,c) \longmapsto M(a,b,c) = \begin{bmatrix} a+b & c \\ c & a-b \end{bmatrix}.$$

Thus the scalar product in S is

$$\tfrac{1}{2} \operatorname{Trace}\left(M(a,b,c)M(a',b',c')\right) = aa' + bb' + cc'$$

which is the canonical scalar product in $\mathbb{R}^3$. We consider the measure $\tilde{\mu}$ that we look for as a measure $\tilde{\mu}(da,db,dc)$ on $\mathbb{R}^3$. Thus (1.8) can be written

$$\int_{\mathbb{R}^2} \exp -(\lambda_1 t_1 + \lambda_2 t_2)\,\mu(dt_1)\mu(dt_2) = \int_{\mathbb{R}^3} \exp -\left(\lambda_1(a+b) + \lambda_2(a-b)\right)\tilde{\mu}(da,db,dc) \qquad (3.1)$$

The images of $\tilde{\mu}$ by $(a,b,c) \mapsto (a,b)$ and of $\mu(dt_1)\mu(dt_2)$ by $(t_1,t_2) \mapsto (a,b) = (\frac{t_1+t_2}{2}, \frac{t_1-t_2}{2})$ coincide. We denote it by $\pi(da,db)$.

Doing $\lambda_1 = \lambda_2$ in (3.1) shows that the image of $\tilde{\mu}$ by $(a,b,c) \mapsto a$ is the positive measure $\pi(da)$ defined as the image of the convolution $\mu * \mu$ by the homothety $t \mapsto \frac{t}{2} = a$.

Therefore we write :

$$\pi(da,db) = \pi(da)\, Q_a(db)\ , \quad \tilde{\mu}(da,db,dc) = \pi(da)\, \nu_a(db,dc)\ . \tag{3.2}$$

Recall that in (3.2), $Q_a(db)$ is a known positive measure and that $\nu_a(db,dc)$ has to be found. Before giving two examples, we make the following remark :

Proposition 3.1 : *Let μ be a positive measure on $\mathbb{R}$ such that $\int e^{-\lambda t}\mu(dt) < \infty$ for all λ in the open interval I. For λ_0 in $\mathbb{R}$, define $\mu^0(dt) = e^{-\lambda_0 t}\mu(dt)$ and consider the $\pi^0(da)$, $\pi^0(da,db)$ and $Q^0_a(db)$ similarly associated to μ^0 as in (3.2). Then $\pi^0(da) = \exp(-2\lambda_0 a)\,\pi(da)$, $\pi^0(da,db) = \exp(-4\lambda_0 a)\,\pi(da,db)$ and $Q^0_a(db) = \exp(-2\lambda_0 a)\, Q_a(db)$.*

In particular, $Q_a(db)$ is a bounded measure π almost every where.

Proof : $(\mu^0 * \mu^0)(dt) = \exp(-\lambda_0 t)\,(\mu * \mu)(dt)$ imply the three identities. Finally, since there exists λ_0 such that μ^0 is bounded, this implies that $Q^0_a(db)$ is bounded, as well as Q_a, π almost every where. ∎

Example A : Let p be > 0 and take $\mu(dt) = \dfrac{t^{p-1}}{\Gamma(p)}\mathbb{1}_{(0,+\infty)}(t)\, dt$. Then :

$$(\mu * \mu)(dt) = \frac{t^{2p-1}}{\Gamma(2p)}\mathbb{1}_{(0,+\infty)}(t)\, dt$$

$$\pi(da) = \frac{2^{2p}}{\Gamma(2p)}a^{2p-1}\mathbb{1}_{(0,+\infty)}(a)\, da$$

$$\pi(da,db) = \frac{4}{(\Gamma(p))^2}(a^2-b^2)^{p-1}\mathbb{1}_{|b|<a}(a,b)\, da\, db$$

$$Q_a(db) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}\left(1-\left(\frac{b}{a}\right)^2\right)^{p-1}\mathbb{1}_{(-a,a)}(b)\,\frac{db}{a}\ . \tag{3.3}$$

The constant in (3.3) has been simplified with the duplication formula of the gamma function : see Whittaker and Watson (1927), bottom of p. 240).

Example B : For $p > 0$ we take $\mu = \mu_p$ as in (1.1). Hence $\mu_p * \mu_p = \mu_{2p}$ and

$$\pi(da) = \frac{p}{\sqrt{\pi}}a^{-\frac{3}{2}}\exp\left(-\frac{p^2}{a}\right)\mathbb{1}_{(0,+\infty)}(a)\, da$$

$$\pi(da,db) = \frac{2p^2}{\pi}\,(a^2-b^2)^{-\frac{3}{2}}\exp\left(-\frac{p^2 a}{a^2-b^2}\right)\mathbb{1}_{|b|<a}(a,b)\, da\, db$$

$$Q_a(db) = \frac{2p}{\sqrt{\pi}}\left(1-\left(\frac{b}{a}\right)^2\right)^{-\frac{3}{2}}\exp\left(\frac{p^2 b^2}{a(a^2-b^2)}\right)\mathbb{1}_{|b|<a}(b)\, db\ . \tag{3.4}$$

We now state a theorem.

Theorem 3.2 : *Assume that μ has no atoms. Then a signed measure $\tilde{\mu}$ satisfying* (1.8) *for $n = 2$ exists if and only if for π almost all a, $Q_a(db)$ as defined by* (3.2) *is absolutely continuous, with density $q_a(b)$, and there exists a signed measure $K_a(dr)$ with bounded variation on* $(0, +\infty)$ *such that*

$$q_a(b) = \frac{1}{\pi}\int_b^\infty \frac{K_a(dr)}{\sqrt{r^2-b^2}} \tag{3.5}$$

Under these circumstances, $\nu_a(db, dc)$, as defined by (3.2), *is the image of $K_a(dr)\,\frac{d\theta}{2\pi}$ on* $(0, +\infty) \times [0, 2\pi[$ *by* $(r, \theta) \mapsto (b, c) = (r\cos\theta, r\sin\theta)$.

Furthermore if μ is concentrated on $(0, +\infty)$, *then K_a is concentrated on* $(0, a]$.

Proof : Suppose that $\tilde{\mu}$ exists. Thus $\nu_a(db, dc)$ is invariant by rotation in the (b, c) plane, i.e. ν_a is the image of a measure $K_a(dr)\frac{d\theta}{2\pi}$ on $(0, +\infty) \times [0, 2\pi[$ by $(r, \theta) \mapsto (b, c) = (r\cos\theta, r\sin\theta)$.

Let us observe that $Q_a(\{0\}) = 0$ for π-almost all a. If not, there exists $A \subset \mathbb{R}$ such that

$$0 < \int_A \pi(da)\, Q_a(\{0\}) \leq \mu\otimes\mu(\{(t,t)\,;\; t \in \mathbb{R}\}) ,$$

and since μ has no atoms, the right hand term of the above inequality is 0 : a contradiction.

Denote by $\alpha(dz)$ the image of $\frac{d\theta}{2\pi}$ on $[0, 2\pi[$ by $\theta \mapsto z = \cos\theta$. We have

$$\alpha(dz) = \frac{1}{\pi}(1-z^2)^{-\frac{1}{2}}\,\mathbb{1}_{(-1,1)}(z)\, dz .$$

Thus $Q_a(db)$, defined by (3.2), is the image of $K_a(dr)\,\alpha(dz)$ by $(r, z) \mapsto b = rz$. Hence Q_a is simply the convolution of K_a and α in the *multiplicative* group $\mathbb{R}^*$. Furthermore, because of the invariance by rotation of ν_a in (3.2), Q_a, as well as α, is a symmetric measure. Thus, taking their restrictions to $(0, +\infty)$, Q_α is the convolution of K_a and α in the multiplicative group $\mathbb{R}_+^*$. Since α has a density, necessarily Q_a must have one too, denoted by $q_a(b)$. The densities of Q_a and α with respect to the Haar measure $\frac{db}{b}$ of $\mathbb{R}_+^*$ are respectively $bq_a(b)$ and $\frac{b}{\pi}(1-b^2)^{-\frac{1}{2}}\mathbb{1}_{(0,1)}(b)$. Thus, for $b > 0$:

$$bq_a(b) = \int_0^\infty \frac{b}{\pi r}\left(1 - \frac{b^2}{r^2}\right)^{-\frac{1}{2}} \mathbb{1}_{(0,1)}\left(\frac{b}{r}\right) K_a(dr) ,$$

which gives (3.5).

The converse part is plain. Eventually, to see that $K_a(dr)$ is concentrated on $(0, a]$ if μ is on $(0, +\infty)$ one observes that $\pi(da, db)$ is on $\{(a, b)\,;\; |b| < a\}$, $Q_a(db)$ is on $(-a, a)$. Finally one uses the Titchmarsh theorem (see Donoghüe (1969) p. 224) to get the result. ∎

One can test this theorem on Example A; (3.3) and (3.5) give for $0 < b < a$ and for a constant C :

$$C\left(1-\left(\frac{b}{a}\right)^2\right)^{p-1}\frac{1}{a}=\int_b^a \frac{K_a(dr)}{\sqrt{r^2-b^2}}\,.$$

Denoting $y = 1-\left(\frac{b}{a}\right)^2$ and making the change of variable $x = 1-\left(\frac{r}{a}\right)^2$ in the integral, we get

$$C\,y^{p-1}=\int_0^y \frac{\tilde{K}(dx)}{\sqrt{y-x}} \tag{3.6}$$

where $\tilde{K}$ is the image of K_a. If $p > \frac{1}{2}$ the solution of (3.6) is $\tilde{K}(dx) = C_1 x^{p-\frac{3}{2}}\, 1\!\!1_{(0,1)(x)}\, dx$. If $p = \frac{1}{2}$, it is $C_1\delta_0(dx)$. If $0 < p < \frac{1}{2}$, there are no signed measure $\tilde{K}$ satisfying (3.6) : we get back the Gindikin result (1.9) for $n = 2$.

The remainder of the paper is essentially devoted to the solution of (3.5) in the case of Example B, i.e. with $g_a(b)$ given by (3.4). We write it for $0 < b < a$:

$$\frac{2p}{\sqrt{\pi}}\left(1-\left(\frac{b}{a}\right)^2\right)^{-\frac{3}{2}}\exp\frac{-p^2b^2}{a(a^2-b^2)}=\frac{1}{\pi}\int_a^b \frac{K_a(dr)}{\sqrt{r^2-b^2}} \tag{3.7}$$

Denoting $y = \dfrac{a^2}{a^2-b^2}$ and making the change of variable $x = \dfrac{a^2}{a^2-r^2}$, we get

$$2p\,y\,\exp-\frac{p^2}{a}(y-1)=\frac{1}{a\sqrt{\pi}}\int_y^\infty \frac{\tilde{K}(dx)}{\sqrt{x-y}}\,, \tag{3.8}$$

where $\tilde{K}(dx)$ is the image, multiplied by $\sqrt{x}$, of $K_a(dr)$ by $r \mapsto x$. The next section solves integral equation (3.7) and an extension of it.

§4. An integral equation

Theorem 4.1 : *Let q be > 0 and n be an integer ≥ 2. Let μ_n be a signed Radon measure on $[0,\infty)$ such that*

$$\int_0^\infty \exp(-\lambda x)\,|\mu_n|(dx)$$

is finite for all $\lambda > 0$ and such that, for all $y > 0$:

$$y^{\frac{n}{2}}e^{-qy}=\frac{q^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_y^\infty (x-y)^{\frac{n-3}{2}}\exp(-qx)\,\mu_n(dx) \tag{4.1}$$

Then μ_n is unique, absolutely continuous and its density f_n is as follows :

(i) If $n = 2p$ is even, then f_n is a polynomial $x^p + g(x)$, with degree of $g < p$, and defined by

$$\int_0^\infty x^{p+k-\frac{3}{2}} \exp(-qx)\, f_n^{(k)}(x)\, dx = 0\,, \qquad k = 0,1,\dots(p-1)\,. \tag{4.2}$$

(ii) If $n = 2p+1$ is odd, then

$$f_n(x) = (-q)^{-p} \exp(qy) \left(\frac{d}{dy}\right)^p \left(y^{p+\frac{1}{2}} \exp(-qy)\right) \tag{4.3}$$

Examples : $\quad f_2(x) = x - \frac{2}{q}\,, \qquad f_3(x) = x^{3/2} - \frac{3}{2q}\, x^{1/2}$

$$f_4(x) = x^2 - \frac{10}{3q}\, x + \frac{5}{4q^2}\,, \qquad f_5(x) = x^{5/2} - \frac{5}{q}\, x^{3/2} + \frac{15}{4q^2}\, x^{1/2} \tag{4.4}$$

Proof : We prove the uniqueness. If μ_n and μ'_n are solutions of (4.1), then $\beta = \mu_n - \mu'_n$ satisfies for all $y > 0$

$$\int_y^\infty (x-y)^{\frac{n-3}{2}} \exp(-qx)\, \beta(dx) = 0 \tag{4.5}$$

Multiplying (4.5) by y^{s-1}, with $s > 0$, integrating with respect to y on $(0, +\infty)$, and applying Fubini (since $\int_0^\infty \exp(-\lambda x)\, |\beta|(dx) < \infty$), we get for all $s > 0$

$$0 = \int_0^\infty e^{-qx} \beta(x) \int_0^x y^{s-1} (x-y)^{\frac{n-3}{2}} dy = \frac{\Gamma(s)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(s + \frac{n-1}{2}\right)} \int_0^\infty x^{s-1} e^{-qx} x^p \beta(dx)\,,$$

i.e. the Mellin transform of $\exp(-qx)\, x^p \beta(dx)$ is 0. This implies $\beta = 0$.

We now show the existence of a solution μ_n of (4.1) with $\mu_n(dx) = f_n(x)\, dx$, with f_n of $C^\infty(0, +\infty)$ class such that all its derivatives are slowly increasing, i.e. for all $\lambda > 0$, $\lim_{x \to +\infty} \exp(-\lambda x) f^{(k)}(x) = 0$.
Now, changing x in $s = x - y$ in (4.1), we get

$$y^{\frac{n}{2}} = \frac{q^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty s^{\frac{n-3}{2}} \exp(-qs)\, f_n(y+s)\, ds\,. \tag{4.6}$$

Because of the postulated regularity of f_n, (4.6) can be derivated under the integral sign an arbitrary number of times.

(i) If $n=2p$, p derivations of (4.6) yield

$$p! = \frac{q^{p-\frac12}}{\Gamma\left(p-\frac12\right)} \int_0^\infty s^{p-\frac32} \exp(-qs)\, f_n^{(p)}(y+s)\, ds\ .$$

Since $f_n^{(n)}(x) = p!$ satisfies this relation we can take $f_n = x^p + g$ where g has degree $< p$. Finally, writing $f_n(y+s) = \sum_{k=0}^p \frac{y^k}{k!} f_n^{(k)}(s)$, and identifying the polynomials in y leads to (4.2).

(ii) If $n = 2p+1$, induction on $k = 0, 1, \dots, p-1$ shows from (4.1) that

$$(-q)^p \left(\frac{d}{dy}\right)^k \left(y^{p+\frac12} \exp -qy\right) = \frac{(-1)^k}{(p-k-1)!} \int_y^\infty (x-y)^{p-1-k} \exp(-qx)\, f_n(x)\, dx\ .$$

Derivating this formula for $k = p-1$ once more gives (4.3). ∎

§5. exp $\left(-\operatorname{Trace} p\sqrt{2A}\right)$ **for (2, 2) symmetric matrices.**

We now apply the previous theory to find $\tilde\mu(da, db, dc)$ such that if A is a positive $(2,2)$ symmetric matrix we have

$$\exp\left(-\operatorname{Trace} p\sqrt{2A}\right) = \int_{\mathbf{R}^3} \exp\left(-\operatorname{Trace} A \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix}\right) \tilde\mu(da, db, dc)\ . \qquad (5.1)$$

In (4.1), do $n = 2$ and $q = p^2/a$ $(a > 0)$. From (4.4) we get

$$y \exp\left(-\frac{p^2}{a} y\right) = \frac{p}{\sqrt{\pi a}} \int_y^\infty e^{-\frac{p^2}{a}x} \left(x - \frac{2a}{p^2}\right) \frac{dx}{\sqrt{x-y}}$$

and from the uniqueness in Th. 4.1 we get that $\tilde K$ in (3.8) is

$$\tilde K(dx) = 2p^2\sqrt{a}\, \exp\left(-\frac{p^2}{a}(x-1)\right) \left(x - \frac{2a}{p^2}\right) \mathbb{1}_{(0,+\infty)}(x)\, dx\ . \qquad (5.2)$$

Thus $K_a(dr)$ in (3.7) is

$$K_a(dr) = 4p^2 a^{\frac52} \exp\left(-\frac{p^2 r^2}{a(a^2-r^2)}\right) \left[\frac{a}{(a^2-r^2)^{\frac52}} - \frac{2}{p^2(a^2-r^2)^{\frac32}}\right] r\, \mathbb{1}_{(0,a)}(r)\, dr\ . \quad (5.3)$$

$\nu_a(db, dc)$, as defined by (3.2), is the image of $K_a(dr)\frac{d\theta}{2\pi}$ by $(r,\theta) \mapsto (b,c) = (r\cos\theta, r\sin\theta)$, i.e.

$$\begin{aligned}\nu_a(db, dc) = &\frac{2p^2 a^{\frac52}}{\pi} \exp\left(-\frac{p^2(b^2+c^2)}{a(a^2-b^2-c^2)}\right) \\ &\times \left[\frac{a}{(a^2-b^2-c^2)^{\frac52}} - \frac{2}{p^2(a^2-b^2-c^2)^{\frac32}}\right] \mathbb{1}_{b^2+c^2<a^2}(b,c)\, db\, dc\end{aligned} \qquad (5.4)$$

and since $\tilde{\mu}(da, db, dc) = \pi(da)\, \nu_a(db, dc)$, with $\pi(da)$ given by (3.4), we get at last

$$\tilde{\mu}(da, db, dc) = \frac{2p^3 a}{\pi^{\frac{3}{2}}} \exp\left(-\frac{p^2 a}{a^2-b^2-c^2}\right) \times \left[\frac{a}{(a^2-b^2-c^2)^{\frac{5}{2}}} - \frac{2}{p^2(a^2-b^2-c^2)^{\frac{3}{2}}}\right] 1\!\!1_{\sqrt{b^2+c^2}<a}(a, b, c)\, da\, db\, dc$$

as satisfying (5.1).

One can observe that $\tilde{\mu}$ is never a positive measure. It is concentrated on the cone of revolution $\{(a, b, c); \sqrt{b^2+c^2} < a\}$ which is nothing but, with the parametrization introduced in §3, the cone of positive definite symmetric endomorphisms. The positive part of $\tilde{\mu}$ is concentrated inside the convex hull of one sheet of the hyperboloid :

$$\left\{(a, b, c)\,;\quad b^2 + c^2 - \left(a^2 - \frac{p^2}{4}\right)^2 + \frac{p^4}{16} = 0\right\}.$$

Note the difference with Example A where, from (1.9), $\tilde{\mu}$ is positive if p is big enough.

Finally, the above computation of $\tilde{\mu}$ shows that if $n \geq 3$, there is no positive measure $\nu(dx)$ on the space S of symmetric (n, n) matrices such that

$$\int_S \exp(-\operatorname{Trace} Ax)\, \nu(dx) = \exp -p \operatorname{Trace} \sqrt{2A}$$

for all A in the cone of symmetric positive definite matrices. To see it, observe that this formula would be true for A only positive thus for

$$A = \begin{pmatrix} a+b & c & 0 \\ c & a-b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and this would imply that the positive measure ν is linearly projected on the non positive measure $\tilde{\mu}$.

§6. Extension to the Jordan algebra of the Lorentz cone.

Because of the complexity of the calculation, we have not been able to solve the problem of the title for $n \geq 3$. In this section we sketch a generalization of the problem, and we solve a significative specialization of it, extending section 5 and using calculations made in section 4.

The idea is to consider the space of symmetric endomorphisms as a particular instance of an Euclidean Jordan algebra. An excellent reference on the subject is the new book by Faraut and Koranyi (1994). This object is a Euclidean space S with scalar product $<a, b>$ and a bilinear symmetric product

$$S \times S \longrightarrow S \qquad\qquad (a, b) \longmapsto a \circ b,$$

such that there exists a neutral element e (i.e. $a \circ e = a$ for all a) and such that for all a, b, c, d in S, the following properties hold

$$(1) \qquad \langle a, b\circ c\rangle = \langle a\circ b, c\rangle$$

$$(2) \quad (a\circ b)\circ(c\circ d)+(a\circ d)\circ(b\circ c)+(a\circ c)\circ(b\circ d) = \big(a\circ(c\circ d)\big)\circ b+\big(a\circ(b\circ c)\big)\circ d+\big(a\circ(b\circ d)\big)\circ c.$$

In the case of the space S of symmetric endomorphisms of a Euclidean space E the product $a \circ b$ is $\frac{1}{2}(ab+ba)$. Replacing the real numbers by complex, quaternions and octonions give other instances of these algebras; we shall describe a fifth instance with the Jordan algebra of the Lorentz cone in a moment. One can prove that these five instance are essentially the only ones.

If S is such an algebra, one defines similarly two real functions called "determinant" and "trace" on S. Attached to S is a positive integer r called the "rank" of S. If S is the space of symmetric endomorphisms of E, then $r = \dim E$. In general we normalize such that

$$\frac{1}{r}\,\mathrm{Trace}\,(a \circ b) = \langle a, b\rangle\ . \tag{6.1}$$

Again, if $I \subset \mathbb{R}$, one defines a suitable subset $S(I)$ of S and, for $f : I \to \mathbb{R}$, a map $\tilde{f} : S(I) \to S$. The problem of extending the Laplace transform of μ on $\mathbb{R}$ as in (1.7) to a $\tilde{\mu}$ on S such that an extension of (1.8) holds :

$$\exp \mathrm{Trace}\,\tilde{f}(a) = \int_S \exp\left(-\,\mathrm{Trace}\,(a \circ x)\right) \tilde{\mu}(dx) \tag{6.2}$$

can be raised. However, we shall be content here to consider only the Jordan algebra of the Lorentz cone which is the only one with rank $r = 2$ and the case $f(\lambda) = -p\sqrt{2\lambda}$, with $p > 0$.

We define now the Jordan algebra of the Lorentz cone by taking first a Euclidean space E with dimension $n \geq 2$, where the scalar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and the squared norm $\|\vec{a}\|^2 = \vec{a}^{\,2}$. On $S = \mathbb{R}\times E$ the scalar and the bilinear symmetric products of $a = (a_0, \vec{a})$ and $b = (b_0, \vec{b})$ are defined by

$$\langle a, b\rangle = a_0 b_0 + \vec{a}.\vec{b} \qquad a \circ b = \left(\langle a, b\rangle\ ,\ a_0\vec{b} + b_0\vec{a}\,\right) \tag{6.3}$$

and we call the following quantities $2a_0$ and $a_0^2 - \vec{a}^{\,2}$ the trace and the determinant of $a = (a_0, \vec{a})$. The set $C = \{a \in S\ ;\ a_0 > \|\vec{a}\|\}$ is called the Lorentz cone. If $E = \mathbb{R}^2$ with its canonical Euclidean structure, $S = \mathbb{R} \times \mathbb{R}^2$ is isomorphic to symmetric $(2,2)$ real matrices by

$$(a_0\ ,\ (a_1, a_2)) \longmapsto \begin{bmatrix} a_0{+}a_1 & a_2 \\ a_2 & a_0{-}a_1 \end{bmatrix}\ .$$

Now if $a \in \overline{C} = \{a \in S\ ;\ a_0 \geq \|\vec{a}\|\}$, there exists a unique $u = u(a)$ in $\overline{C}$ such that $u \circ u = a$ (or $u = \sqrt{a}\,$), which is given by

$$u_0 = u_0(a) = \left[\tfrac{1}{2}(a_0{+}\sqrt{\det a})\right]^{1/2} \qquad \vec{u} = \vec{u}(a) = \frac{\vec{a}}{2u_0}\ . \tag{6.4}$$

Therefore the aim of this section is to compute the signed measure $\tilde{\mu}$ on $S = \mathbb{R} \times E$ such that (6.2) holds when $f = -p\sqrt{2\lambda}$, i.e. for $(a_0, \vec{a})$ in $\overline{C}$:

$$\exp\left(-2p\left(a_0 + \sqrt{a_0^2 - \vec{a}^2}\right)^{1/2}\right) = \int_S \exp\left(-2(a_0 x_0 + \vec{a}.\vec{x})\right) \tilde{\mu}(dx) \qquad (6.5)$$

Note that no special knowledge of Jordan algebras is required to understand and solve the problem (6.5) : previous explanations just gave the motivation and the background of it.

We now imitate the previous sections : doing $\vec{a} = \vec{0}$ in (6.5) gives the Laplace transform of the image $\pi(dx_0)$ of $\tilde{\mu}$ by $(x_0, \vec{x}) \mapsto x_0$. We can also equip E with an orthonormal basis e and identify E with $\mathbb{R}^n$. Then doing $\vec{a} = (1, 0, \dots 0)$ in (6.5) gives the Laplace transform of the image $\pi(dx_0, dx_1)$ of $\tilde{\mu}$ by $(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1)$, i.e.

$$\exp -2p\left(a_0 + \sqrt{a_0^2 - a_1^2}\right)^{1/2} = \int_{\mathbb{R}^2} \exp\left(-2a_0 x_0 - 2a_1 x_1\right) \pi(dx_0, dx_1)$$

Writing $\lambda_1 = a_0 + a_1$ and $\lambda_2 = a_0 - a_1$ shows

$$\exp -p\left(\sqrt{2\lambda_1} + \sqrt{2\lambda_2}\right) = \int \exp\left(-x_0(\lambda_1 + \lambda_2) - x_1(\lambda_1 - \lambda_2)\right) \pi(dx_0, dx_1) .$$

Finally if $\pi(dx_0, dx_1) = \pi(dx_0) Q_{x_0}(dx_1)$ one sees that $\pi(dx_0)$, $\pi(dx_0, dx_1)$ and $Q_{x_0}(dx_1)$ are given by formulas (3.4), where (x_0, x_1) replaces (a, b).

Now $\tilde{\mu}$ is invariant by the transformations g_u of S defined by $g_u(a_0, \vec{a}) = \left(a_0, u(\vec{a})\right)$ when u varies in $\mathbb{O}(E)$. Thus

$$\tilde{\mu}(dx_0, d\vec{x}) = \pi(dx_0)\, \nu_{x_0}(d\vec{x}) ,$$

where $\nu_{x_0}(d\vec{x})$ is invariant by $\mathbb{O}(E)$.

There exists a signed measure $K_{x_0}(dr)$ on $(0, +\infty)$ such that ν_{x_0} is the image of $K_{x_0}(dr)\, \sigma(d\theta)$ (where σ is the uniform probability measure on the unit sphere $S(E)$ of E) by $(r, \theta) \mapsto \vec{x} = r\theta$.

Coming back to the basis, and writing $\theta = (\theta_1, \dots, \theta_n)$, we see that $Q_{x_0}(dx_1)$ is the image of $K_{x_0}(dr)\, \sigma(d\theta)$ by $(r, \theta) \mapsto \theta_1$. Denoting by $\alpha(d\theta_1)$ the image of $\sigma(d\theta)$ by $\theta \mapsto \theta_1$, the known Q_{x_0} is the convolution in the multiplicative group $\mathbb{R}^*$ of the unknown $K_{x_0}(dr)$ with the known $\alpha(d\theta_1)$. Actually, the computation of $\alpha(d\theta_1)$ is quite standard; the fastest way to proceed is to observe that $\sigma(d\theta)$ is the distribution of $\vec{X}/\|\vec{X}\|$, where $\vec{X}$ is Gaussian distributed in E with mean $\vec{0}$ and covariance identity. Thus the distributions of θ_1^2 and of $X_1^2(X_1^2 + X_2^2 + \dots + X_n^2)^{-1}$ are

$$\beta^{(1)}_{\frac{1}{2}, \frac{n-1}{2}} = t^{-1/2}(1-t)^{(n-3)/2} \mathbb{1}_{(0,1)}(t) \frac{dt}{B\left(\frac{1}{2}, \frac{n-1}{2}\right)} .$$

Since the random variable θ_1 is symmetric, then :

$$\alpha(d\theta_1) = (1 - \theta_1^2)^{(n-3)/2} \mathbb{1}_{(-1,1)}(\theta_1) \frac{d\theta_1}{B\left(\frac{1}{2}, \frac{n-1}{2}\right)} .$$

Working, as in Theorem 3.2, in $\mathbb{R}_+^*$ rather than $\mathbb{R}^*$, the analogue of the integral equations (3.5) and (3.7) is :

$$\frac{2p}{\sqrt{\pi}}\left(1-\left(\frac{x_1}{x_0}\right)^2\right)^{-\frac{3}{2}} \exp-\frac{p^2x_1^2}{x_0(x_0^2-x_1^2)} = \frac{1}{B(\frac{1}{2},\frac{n-1}{2})}\int_{x_1}^{x_0}(r^2-x_1^2)^{\frac{n-3}{2}}\frac{K_{x_0}(dr)}{r^{n-2}} .$$

As in (3.7) we make the change of variables :

$$x=\frac{x_0^2}{x_0^2-r^2} \qquad\qquad y=\frac{x_0^2}{x_0^2-x_1^2}$$

and we get the generalization of (3.8) :

$$2p\,y^{\frac{n}{2}}\exp-\frac{p^2}{x_0}(y-1)=\frac{x_0^{n-3}\sqrt{\pi}}{B(\frac{1}{2},\frac{n-1}{2})}\int_y^{\infty}(x-y)^{\frac{n-3}{2}}\tilde{K}(dx)\,, \tag{6.6}$$

where $\tilde{K}(dx)$ is the image, multiplied by $x^{(3-n)/2}$, of $r^{2-n}K_{x_0}(dr)$ by $r\mapsto x$. Equation (6.6) is essentially solved by Theorem 4.1, and keeping the notation f_n used there we get :

$$\tilde{K}(dx)=\frac{2p^n}{\Gamma(\frac{n}{2})}\,x_0^{\frac{7-3n}{2}}f_n(x)\exp\left(-\frac{p^2}{x_0}(x-1)\right)\mathbb{1}_{(0,+\infty)}(x)\,dx .$$

Taking the image of $x^{(n-3)/2}\,\tilde{K}(dx)$ by $x\mapsto r$, we get :

$$K_{x_0}(dr)=\frac{4p^nx_0^{\frac{5-n}{2}}}{\Gamma(\frac{n}{2})}(x_0^2-r^2)^{-\frac{n-3}{2}}f_n\Big(\frac{x_0^2}{x_0^2-r^2}\Big)\exp\Big(-\frac{p^2r^2}{x_0(x_0^2-r^2)}\Big)r\mathbb{1}_{(0,+\infty)}(r)\,dr .$$

Now, $\nu_{x_0}(d\vec{x})$ is the image of $K_{x_0}(dr)\,\sigma(d\theta)$ by $(r,\theta)\mapsto\vec{x}=r\theta$. Recall that the image of $r^{n-1}dr\,\sigma(d\theta)$ by $(r,\theta)\mapsto\vec{x}=r\theta$ is "a" Lebesgue measure of E, i.e. is invariant by translation. However, to get "the" Lebesgue measure of E, i.e. the only one which gives mass 1 to any unit cube built on an orthonormal basis, we have to introduce a factor obtained by the computation of the volume of the unit ball. We skip this standard computation and obtain that $d\vec{x}$ is the image of

$$\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,r^{n-1}\,dr\;\sigma(d\theta)\,,$$

by $(r,\theta)\mapsto r\theta=\vec{x}$. Thus we get :

$$\nu_{x_0}(d\vec{x})=\frac{2p^nx_0^{\frac{5-n}{2}}}{\pi^{\frac{n}{2}}}(x_0^2-\vec{x}^2)^{\frac{3-n}{2}}r^{2-n}f_n\Big(\frac{x_0^2}{x_0^2-\vec{x}^2}\Big)\exp-\frac{p^2\vec{x}^2}{x_0^2-\vec{x}^2}\mathbb{1}_{\|\vec{x}\|<x_0}(\vec{x})\,d\vec{x} .$$

And the final solution of (6.5) is :

$$\tilde{\mu}(dx_0, d\vec{x}) = \frac{2p^{n+1}}{\pi^{\frac{n+1}{2}}} x_0^{\frac{2-n}{2}} \|\vec{x}\|^{2-n} (x_0^2 - \vec{x}^2)^{\frac{3-n}{2}} f_n\Big(\frac{x_0^2}{x_0^2 - \vec{x}^2}\Big) \times \exp - \frac{p^2}{x_0(x_0^2 - \vec{x}^2)} \mathbb{1}_{\|\vec{x}\| < x_0}(x_0, \vec{x})\, dx_0\, d\vec{x}\,,$$

where f_n is defined in Theorem 4.1.

References

Bernadac, E. (1992), "Fractions continues sur les matrices symétriques réelles et la loi gaussienne inverse", *C. R. Acad. Sc. Paris 315,* Série I, 329–332.

Bernadac, E. (1993), "Fractions continues aléatoires sur un cône symétrique", *C. R. Acad. Sc. Paris 316,* Série I, 859–864.

Bernadac, E. (1995), "Random continued fractions and inverse Gaussian distribution on a symmetric cone", *J. Theor. Prob. 8,* 221–260.

Casalis, M. and Letac, G. (1994), "Characterization of the Jorgensen set in the generalized linear model", *Test 3,* 145–162.

Donoghüe, W. (1969), *"Distributions and Fourier transforms"*, Academic Press, New York.

Faraut, J. and Koranyi, A. (1994), *"Analysis on symmetric cones"*, Oxford University Press, Cambridge.

Feller, W. (1970), *An introduction to probability theory and its applications,* Vol. II, 2d ed. Wiley, New York.

Gindikin, S. (1975), "Invariant generalized functions in homogeneous spaces", *J. of Funct. Anal. and Applications 9,* 50–52.

Letac, G. and Seshadri, V. (1983), "A characterization of the inverse Gaussian distribution by continued fractions", *Z. Wahrsch. Verw. Gebiete 62,* 485–489.

Watson, G. N. (1966), *"A treatise on the theory of Bessel functions"*, 4th ed. Cambridge Univ. Press, Cambridge.

Whittaker, E. T. and Watson, G. N. (1927), *"Modern Analysis"*, Cambridge Univ. Press.

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