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Density of Integral and Rational Points on Varieties

J. Pila*

Suppose that V is an irreducible affine or projective variety over $\mathbb{R}$. In this note we give upper bounds for the number of integral points of V of height $\leq H$ in the case that V is affine, and for the number of rational points of V of height $\leq H$ if V is projective.

Our results follow easily from a result in a joint paper with Bombieri [1]. That result is the plane curve case of theorem A below. Indeed it was remarked in the introduction to [1] that our results could be generalized to higher dimensions by simple slicing arguments. We explicate such arguments here and compare with related results in the literature.

We define the height $H(P)$ of a point $P = (X_1, \ldots, X_N)$ of affine $N-$ space with rational integral coordinates X_i to be $\max(|X_i|)$. For a point $P = (X_0, \ldots, X_N)$ of projective N space with integral X_i and $\gcd(X_0, \ldots, X_N) = 1$ we set $H(P) = \max(|X_i|)$.

A feature of our estimates is that they are uniform for all varieties of given dimension, degree, and ambient space.

Theorem A. *Let V be an irreducible affine variety, of dimension n and degree d, embedded in N dimensional affine space. Let $H \geq 1$. There is a constant $c(n,d,N)$ such that the number of integral points of V of height $\leq H$ is bounded by*

$$c(n,d,N)H^{n-1+\frac{1}{d}} \exp(12\sqrt{d \log H \log\log H}).$$

The constant $c(n,d,N)$ is effectively computable. The example $V : X_2 = X_1^d$ shows that the exponent $n - 1 + \frac{1}{d}$ is best possible. The case of theorem A for plane curves is theorem 5 of [1], in which the constant is completely explicit. Indeed if one assumes that $H \geq \exp(d^6)$ then one may take $c(1,d,2) = 4$.

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Theorem B. *Let V be an irreducible projective variety, of dimension n and degree d, embedded in N dimensional projective space. Let $H \geq 1$. There is a constant $c'(n,d,N)$ such that the number of rational points of V of height $\leq H$ is bounded by*

$$c'(n,d,N)H^{n+\frac{1}{d}+\varepsilon}\exp(12\sqrt{d\log H\log\log H}).$$

The uniformity of the bound for curves allows one to treat higher dimensional varieties by projections and slicing. A little care is needed to ensure that the slices and projections are sufficiently generic, involving considerations much like those of Lang–Weil [5], especially their lemma 2. The only difficulty in explicating the constants in the theorems is connected with choosing slices and projections with appropriate properties (compare with the remark following lemma 2 of [5]).

For theorem B, we view the projective variety as an affine variety of dimension $n+1$.

Our crude approach yields a correspondingly general result. For discussion and results concerning asymptotics for the number of integral points in various cases, see for example Duke–Rudnick–Sarnak [3].

Let us compare our results with those given in Serre [9, Chapter 13]. Those estimates are obtained using the large sieve, and improve somewhat the estimates obtained by S. D. Cohen [2] along the same lines. Estimates are given there for points integral or rational over number fields. Suppose that K is a number field of degree k, with ring of integers O_K. For the height, we now take the maximum of the absolute values of the coordinates over all embeddings. Suppose that V is defined over K.

Corresponding to our theorem A is an estimate $c''(V,K)H^{(n-\frac{1}{2})k}(\log H)^{\gamma}$, with $\gamma < 1$, and corresponding to theorem B a similar estimate with exponent $(n+\frac{1}{2})k$.

For the situation we consider, that is for the rational field, our estimates are stronger than those in [9], provided $d \geq 3$, both in the exponent, and in the uniformity of the constant. For $d = 2$, the results of [9] are stronger for a fixed V, since the "ε" exponent in our result is not a power of $\log H$. On the other hand, our constant is independent of the height of the variety. Note that while we do not need to assume that our variety is defined over the algebraic numbers, it is easy to see that there is no real gain in generality in this.

The exponent in theorem B does not seem to be best possible. For the case $n = 1$, as remarked in [6], our method gives the correct exponent $\frac{2}{d}$, but not with a uniform constant. Of course for curves much more precise (nonuniform) bounds are available. Nevertheless, one might hope that our method might yield results of sufficient uniformity to serve to get better bounds in higher dimensions.

For $n > 1$, one can ask for bounds with better exponents if one excludes cylinders. Schmidt [7] obtains such a bound for affine surfaces assumed to be non-cylinders and then gets bounds for points on (non-cylindrical) hypersurfaces. Those estimates are also uniform, and the exponent is $n - \frac{1}{2}$, $(n \geq 2)$, so that they are better than our estimate if $d = 2$, and than the estimates in [9] under those additional hypotheses.

In [8] Schmidt goes on to consider hypersurfaces in affine 4-space that are not cylinders. For these he gets the better exponent $2 + \frac{4}{9}$, with constant depending only on $d > 1$. There is a corresponding result for hypersurfaces in N dimensional spaces that are not cylinders on an $N - 3$ dimensional subvariety. Schmidt's method is of an elementary geometric nature.

Under rather restrictive hypotheses, Fujiwara [4] gives bounds with better exponents in some regimes for varieties defined by forms. In particular for non-singular affine hypersurfaces in space of $N \geq 4$ dimensions, and H large (depending on V), a bound with exponent $N - 2 + \frac{2}{N}$, and constant depending only on N, d is given. The non-singularity requirements become considerably more onerous with increasing codimension. Fujiwara also considers distribution of rational points on varieties over finite fields, and indeed this is the approach to integral points. The methods are highly non-elementary, using Deligne's theorem.

Shparlinskii and Skorobogatov [10] improve these results along the same general lines. They eliminate the more stringent hypotheses of Fujiwara. For non-singular complete intersections they give a bound with exponent $N - 2s + \frac{2s^2}{N+s-1}$, where s is the codimension. The constant in the bound depends on V. Further refinements are given by Skorobogatov in [11] using a Lefschetz hyperplane theorem for singular varieties. A measure of the singularity of hyperplane sections is employed to give a generalization of the above result.

Among the methods discussed above, ours is unique in responding to the degree of the variety; indeed, our method is sensitive only to the degree. Thus it seems, for example, that we cannot address Schmidt's conjecture that for plane curves of positive genus, an estimate of the form $c(d, \varepsilon)H^{\varepsilon}$ obtains for the number of integral points of height $\leq H$.

Proof of Theorem A. The proof will be by induction on n, so let us begin with the case of curves.

We will reduce to the case $N = 2$, in which case the conclusion is Theorem 5 of [1], with all constants explicit. For larger N we will find a projection from $\mathbf{A}^N$ to $\mathbf{A}^2$ preserving the irreducibility and degree of V, taking integral points to integral points, and of small height.

Consider projections given by

$$Y = \sum y_i X_i, \quad Z = \sum z_i X_i.$$

The collection of y_i, z_i for which this map is non-degenerate, and for which the image of V is of the same degree as V forms an open set in $\mathbf{A}^{2N}$. The complement B is algebraic, of dimension at most $2N-1$ and degree b depending only on N, d. Indeed, one can get uniform equations for this exceptional locus for all cycles of dimension n and degree d in N space in terms of the Chow coordinates of V.

The number of integral points of such a set of height $\leq T$ is bounded by $c_1(b)T^{2N-1}$, so we can find an integral point in the complement of B of height at most $c_2(b)$. This projection carries V to a plane curve of degree d, and the number of possible preimages of a given (singular) point in the image is uniformly bounded interms of d. Integer points of height $\leq H$ are carried to integer points of height $\leq Nc_2(b)H$, and the estimate for plane curves now completes the proof for space curves.

Let us now assume the validity of the theorem for dimension $n-1$, where $n \geq 2$, and prove the result for dimension n.

Arguing as above with projections we can reduce to the case of hypersurfaces.

Let us now consider hyperplane sections of V. These are generically irreducible and of degree d: the exceptional locus E of hyperplanes has dimension at most $N-1$, and degree at most e depending only on n, N, d. (Again, this is because we can specify this locus uniformly in terms of the Chow point of V as a cycle.)

We can thus find a hyperplane with integral equation

$$\sum a_i X_i = m$$

with $|a_i|, |m| \leq c_3(e, N)$ that intersects V in an irreducible hyperplane of degree d. If we now consider m to be variable, we generate a pencil of hyperplanes. Since this pencil is not contained in E, its intersection with E consists of at most e points m_j.

For these exceptional m_j the intersection of V with the hyperplane is of dimension at most $n-1$ and bounded degree, so they contribute at most $c_4(n, N, d)H^{n-1}$ integral points. For the other non-exceptional hyperplane section we can apply the induction hypothesis. Finally, we observe that any integral point of V of height $\leq H$ lies on a hyperplane of the above pencil for some m with $|m| \leq Nc_3(e, N)H$. This completes the proof. □

Proof of Theorem B. Suppose V is an irreducible projective variety, of dimension n and degree d in projective space of dimension N. Let A be the affine cone over V. Then A is an irreducible affine variety of dimension $n+1$ and degree d in affine space of $N+1$ dimensions. There is an injection of rational points of V of height $\leq H$ into integral points of A of height $\leq H$, so the conclusion follows from theorem A. □

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