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Speculations about the Topology of Rational Points: An up-date.

B. Mazur

In my talk, over a year ago,[1] I tried a conjecture out on my audience which at first glance may have seemed unreasonable, but which surprisingly still eludes attempts at its demolition. Since an account of my talk has been subsequently published **[Mz 1]** there is no need for me to repeat much of it here. Nevertheless I want to present here a brief "up-date", to slip in some commentary on **[Mz 1]**,[2] and to mention (why not?) a few more questions.

First, the conjecture. A *variety* will mean a reduced scheme of finite type over a field (usually $\mathbf{Q}$).

Conjecture 1. *Let V be a smooth variety over $\mathbf{Q}$ such that $V(\mathbf{Q})$ is Zariski-dense in V. Then the topological closure of $V(\mathbf{Q})$ in $V(\mathbf{R})$ consists of a (finite) union of connected components of $V(\mathbf{R})$.*

In **[Mz 1]** the following consequences of Conjecture 1 (and the triangulability theorems of Hironaka **[Hi]** and Lojasiewicz **[Lo]**) were mentioned:

Conjecture 2. *The topological closure of the set of rational points of any variety over $\mathbf{Q}$ in its real locus is homeomorphic to the complement of a finite subcomplex in a finite simplicial complex; it has the homotopy type of a finite simplicial complex.*

and:

Conjecture 3. *The topological closure of the set of rational points of any variety over $\mathbf{Q}$ has at most a finite number of connected components.*

[1] in the series of lectures at Columbia University whose write-ups are contained in this *Astérisque* volume

[2] and to correct an erroneous assertion there: specifically in §5 of **[Mz 1]**, replace the phrase "the same result holds" by the phrase "the Brauer-Manin obstruction is the only obstruction to weak approximation"

My original motivation to consider these conjectures in their full generality came from thinking about the work of Matijasevic. Conjecture 3, for example, would imply that $\mathbb{Z}$ is *not* **Diophantinely Definable** in $\mathbb{Q}$. For further discussion of this, see **[Mz 1,2]**.

The condition of **smoothness** in Conjecture 1 is necessary. For a counterexample, in the nonsmooth case, consider the affine cone $X : x^2+y^2-3z^2=0$ over $\mathbb{Q}$. Since the corresponding conic has no points rational over $\mathbb{Q}_2$ (or over $\mathbb{Q}_3$) the only $\mathbb{Q}$-rational point of X is its vertex. But its $\mathbb{R}$-rational locus is connected and Zariski-dense in X.[3]

§1. Obstructions to the existence of rational points.

It may be that conjecture 1 has remained undemolished for so long simply because we have very few "obstructions" to the existence of rational points. We have, of course, the Brauer-Manin obstruction, first studied in **[Mn 1]**; let us recall its definition.

Let $\mathbb{A}_k$ denote the adele ring of a global field k. Let X be a smooth projective variety over k. Let $Br(X)$ denote the Brauer group of the scheme $X_{/k}$. Local Class Field Theory enables one to define a natural (continuous) right-linear pairing

$$\gamma : X(\mathbb{A}_k) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$$

(cf. 3.1 of **[CT - Sa 2]**) and by Global Class Field Theory, the restriction of the pairing γ to $X(k) \times Br(X)$ is trivial. Let

$$X(\mathbb{A}_k)^{Br} = \{x \in X(\mathbb{A}_k) \mid \gamma(x,b) = 0 \quad \text{for all} \quad b \in Br(X)\};$$

i.e., $X(\mathbb{A}_k)^{Br}$ is the "left-kernel" of γ, given the topology it inherits as a closed subspace of $X(\mathbb{A}_k)$. The image of the natural injection $X(k) \hookrightarrow X(\mathbb{A}_k)$ is then contained in $X(\mathbb{A}_k)^{Br}$.

Let $\overline{X(k)} \subset X(\mathbb{A}_k)$ denote the topological closure of $X(k)$ in $X(\mathbb{A}_k)$; so $\overline{X(k)} \subset X(\mathbb{A}_k)^{Br}$.

[3] I thank the referee for suggesting this simple example

Definition. *Say that* **"the Brauer-Manin obstruction is the** ***only*** **obstruction to weak approximation"** *to X over k if we have equality*

$$\overline{X(k)} = X(\mathbb{A}_k)^{Br}.$$

To simplify the terminology in this article, call a variety $X_{/k}$ such that $\overline{X(k)} = X(\mathbb{A}_k)^{Br}$ a **Brauer-Manin variety**. Conjecture 1 holds for all Brauer-Manin varieties. Although the condition of being "Brauer-Manin" is expected to be a very stringent condition we have no example, at present, of a variety $V_{/\mathbb{Q}}$ which has been actually proven to *fail to be* Brauer-Manin. It might be interesting to rectify this situation.

The Brauer-Manin invariant has a certain limited sensitivity in that it factors through the Chow group of zero-cycles mod rational equivalence, in the following sense. For L any field extension of $\mathbb{Q}$, let $\mathcal{C}X(L)$ denote the set of classes of *zero-cycles of degree one* taken up to rational equivalence, rational over L. We have the natural mapping $X(L) \to \mathcal{C}X(L)$. Put $\mathcal{C}X(\mathbb{A}_k) = \prod_v \mathcal{C}X(k_v)$. The right-linear pairing γ induces a pairing on the product,

$$\mathcal{C}\gamma : \mathcal{C}X(\mathbb{A}_k) \times Br(X) \longrightarrow \mathbb{Q}/\mathbb{Z} \qquad (\text{cf.}[CT - SD]).$$

If $\mathcal{C}X(\mathbb{A}_k)^{Br}$ denotes the subset of elements $c \in \mathcal{C}X(\mathbb{A}_k)$ such that $\mathcal{C}\gamma(c, b) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $b \in Br(X)$, i.e., the "left-kernel" of this pairing, we have a commutative diagram

$$\begin{array}{ccccc} X(k) & \subset & X(\mathbb{A}_k)^{Br} & \subset & X(\mathbb{A}_k) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}X(k) & \subset & \mathcal{C}X(\mathbb{A}_k)^{Br} & \subset & \mathcal{C}X(\mathbb{A}_k) \end{array}$$

(with the right-hand square Cartesian).

For a detailed discussion of the Brauer-Manin (and related) conditions, see **[CT-SD]**, **[CT 1,2]**, and **[CT-Sa 1,2]**. In some recent work "Schinzel's Hypothesis[4] (concerning the conditions on a finite set of polynomials $P_1(x), \ldots, P_\nu(x) \in \mathbb{Z}[x]$ sufficient to guarantee that there are infinitely many integers n for which $P_1(n), \ldots, P_\nu(n)$ are all prime numbers) is shown to imply

[4]sometimes called "Hypothesis (H)"; for a precise statement, see loc. cit.

the assertion that various classes $\mathcal{V}$ of algebraic varieties are Brauer-Manin. In particular, modulo Schinzel's Hypothesis, the following class $\mathcal{V}_0$ is proved in **[CT-SD]** to be Brauer-Manin, continuing earlier work of Colliot-Thélène-Sansuc, and of Serre.

$\mathcal{V}_0$ = *the class of smooth, projective, geometrically integral varieties* $X_{/\mathbf{Q}}$ *endowed with a dominant morphism* $\varphi : X \to P^1$ *defined over* $\mathbf{Q}$, *such that the generic fiber of* φ *is a "generalized Severi-Brauer variety" (see* **[CT-SD]***) over the function field* $\mathbf{Q}(t)$ *of* $P^1_{/\mathbf{Q}}$.

This class of varieties $x_{/\mathbf{Q}}$ includes, for example, the class of (smooth, projective) pencils of conics defined over $\mathbf{Q}$ possessing $\mathbf{Q}$-rational points.

§2. Abelian varieties.

As a consequence of the local and global duality theorems of Tate, if $A_{/\mathbf{Q}}$ is an abelian variety with the property that $A(\mathbf{Q})$ is Zariski-dense in A, and if the Tate-Shafarevich group $(A_{/\mathbf{Q}}$ is finite, then $A_{/\mathbf{Q}}$ is Brauer-Manin *if and only if* Conjecture 1 holds for $A_{/\mathbf{Q}}$ (this was explained to me by Lan Wang; unpublished note).

For an explicit example of an abelian variety which is, using the above result, actually *proven* to be Brauer-Manin, one can cite the elliptic curve $y^2 = 4x^3 - 4x + 1$ which is a modular elliptic curve of conductor 37 isomorphic to $X_0(37)^+$, whose Mordell-Weil group is infinite cyclic generated by the point $(x, y) = (0, 1)$, and which is proved to have vanishing Tate-Shafarevich group in **[K]**; since all elliptic curves over $\mathbf{Q}$ satisfy Conjecture 1, by the result of L. Wang quoted above, this elliptic curve is then Brauer-Manin.

Conjecture 1 implies that, for a simple abelian variety $A_{/\mathbf{Q}}$ with Mordell-Weil group of positive rank, the topological closure of the Mordell-Weil group in $A(\mathbf{R})$ is open (of finite index).

The analogous statement is *false* if $A(\mathbf{R})$ is replaced with $A(\mathbf{Q}_p)$; i.e., there are simple abelian varieties such that the topological closure of $A(\mathbf{Q})$ in $A(\mathbf{Q}_p)$ is a p-adic Lie subgroup of $A(\mathbf{Q}_p)$ which is of dimension greater than 0, and strictly less than the dimension of $A(\mathbf{Q}_p) = \dim A$.

The natural way to get such a simple abelian variety A is to find one with the rank of its Mordell-Weil group greater than 0 and strictly less than $d = \dim A$. For $d = 2$ the only worked-out example of this that I know is the abelian surface studied in **[G-G]**. Factors of the jacobians of Fermat curves, however, provide a source of lots of d-dimensional abelian varieties $A_{/\mathbf{Q}}$ for larger d, with Mordell-Weil rank r in the required range $0 < r < d$. I am thankful to W. McCallum for communicating to me the following explicit examples of these.

Let $\ell > 7$ be a prime number, and s an integer in the range $1 < s < \ell-2$, with $s = (\ell-1)/2$. Let $J(\ell, s)_{/\mathbf{Q}}$ denote the jacobian, defined over $\mathbf{Q}$, of the curve $C(\ell, s) : y^\ell = x^s(x-1)$. Then the dimension of $J(\ell, s)$, i.e., the genus of $C(\ell, s)$, is $(\ell-1)/2$.

Theorem. *With the above restrictions on ℓ and s, the abelian variety $J(\ell, s)_{/\mathbf{Q}}$ is simple and its Mordell-Weil group is of rank r in the range $0 < r < (\ell-1)/2$, if the prime number ℓ is a regular prime number congruent to* -1 mod 3.

Remark. The same conclusion, i.e., simplicity of $J(\ell, s)_{/\mathbf{Q}}$ and $0 < \operatorname{rank} A(\mathbf{Q}) < (\ell-1)/2$, holds under the following more general hypothesis:
(i) $\ell > 7, s \neq (\ell-1)/2$ as before, and if $\ell \equiv 1 \bmod 3$, then s does not have order 3 modulo ℓ,

and

(ii) If $\mathbf{Q}(\mu_\ell)$ is the cyclotomic field of ℓ-th roots of unity, and if V_ℓ denotes the $\mathbf{F}_\ell$-vector space of ℓ-torsion in the ideal class group of $\mathbf{Q}(\mu_\ell)$, then $\dim_{\mathbf{F}_\ell} V_\ell < (\ell+5)/8$.

Briefly, the reason for the running conditions $\ell > 7$ and $1 < s < \ell-2$, with $s \neq (\ell-1)/2$ is to guarantee that the Gross-Rohrlich point (**[G-R]**) in $J(\ell, s)$ have infinite order. The requirement that if $\ell \equiv 1 \bmod 3$ then s not be of order 3 modulo ℓ guarantees that $J(\ell, s)$ be simple (**[K-R]**). The condition that ℓ be regular (resp. the more general condition (ii)) guarantees that the Mordell-Weil rank is smaller than the dimension; see **[F]** (resp. **[Mc]**).

Question. *Under the above conditions on ℓ and s, is the topological closure of the Gross-Rohrlich point P in $J(\ell, s)(\mathbf{R})$ open?*

§3. Families of quadratic twists of elliptic curves

Let $G(x) \in \mathbb{Q}(x)$ be a monic cubic polynomial with distinct roots and let E be the elliptic curve over $\mathbb{Q}$ defined by the equation $y^2 = G(x)$. If d is a nonzero rational number, let E_d denote the elliptic curve $d \cdot y^2 = G(x)$, which we will refer to as the **twist** of E by d. If $D(t)$ is a polynomial in $\mathbb{Q}(t)$ not identically zero, consider the pencil of elliptic curves, all twists of E, defined by the equation

$$E_{D(t)} : \quad D(t) \cdot y^2 = g_3(x). \tag{*}$$

Conjecture 1 implies that either the set $\mathcal{T}$ of $t \in \mathbb{R}$ for which $E_{D(t)}$ has Mordell-Weil rank > 0 is *finite*, or else it is *dense* in $\mathbb{R}$. If the Birch Swinnerton-Dyer Conjecture is true, and if $D(t)$ takes both positive and negative values, then Rohrlich has shown that the set $\mathcal{T}$ is *not* finite (this follows from **[R]** Theorem 2; see also the discussion in §6 of **[Mz 1]**).

For what polynomials $D(t)$ is the set $\mathcal{T}$ dense in $\mathbb{R}$?

I am thankful to the referee for pointing out to me that if $D(t)$ is separable and of degree 2 then the surface (*) is unirational over $\mathbb{Q}$ and that there are two approaches to seeing this, the cubic surface approach (Lemma 1.2, p. 33 of **[C-T]**) and the Châtelet surface approach (Prop. 8.3 and Remark 8.3.1 of **[CT-S-SwD]**). Consequently one can show, if $D(t)$ is of degree 2, that $\mathcal{T}$ is dense in $\mathbb{R}$.

Kuwata and Wang **[K-W]** have more recently extended this result (again, independent of any conjectures) to polynomials $D(t)$ of degree ≤ 3. Specifically, the case of degrees ≤ 2 having been treated, there is no loss of generality in supposing that $D(t)$ is a monic polynomial of degree 3, with no multiple roots, so that we have two elliptic curves $E : y^2 = G(x)$, and $F : y^2 = D(t)$, and *excluding the special cases* $j(E) = j(F) = 0$ and $j(E) = j(F) = 1728$, Kuwata and Wang show that $\mathcal{T}$ is dense in $\mathbb{R}$, and, in fact, they show more: they prove that there are infinitely many square-free positive integers (also square-free *negative* integers) d for which the quadratic twists of E and F by d *both* have positive Mordell-Weil rank.

As Kuwata and Wang ask in their article, does a similar result hold for three elliptic curves over $\mathbb{Q}$? Or for N of them? Or for N simple abelian varieties over $\mathbf{Q}$?

Question. *Are pencils of elliptic curves Brauer-Manin?*

A positive answer to this question in the case where the polynomial $D(t)$ of (*) above is separable and of degree 2 may be found in **[CT-S-SwD]**.

§4. Heights, measures, and dynamics

Alongside the question of the topological placement of rational points in real loci is the question of how rational points are distributed, if they are ordered with respect to a chosen height function. In various guises such questions have been (explicitly and implicitly) studied in special cases via the classical circle method and its subsequent developments: by Siegel in the context of quadratic forms; by Weil and many others in the generalization of Siegel's work to adelic studies of the arithmetic of semi-simple groups; for homogeneous forms of higher degree in large numbers of variables, e.g., as in Birch's **[B 1,2]**, or **[D-L]** or in **[Sch]**, in which Schmidt axiomatizes a class (of systems) of homogeneous forms for which the circle method applies. Relevant to distributional questions, as well, are the studies of pre-homogeneous vector spaces continuing work of Sato and Shintani, cf., e.g., **[D-W]**, **[Wr]**, **[W-Y]** and the vast literature related to the analytic continuation of zeta-functions of quite general homogeneous forms of high degree: see Igusa's treatise **[Ig]**, and, e.g., **[Ca-N]**, **[Ho]**, **[Licht]**, **[Sa]**.

That such questions are topical is clear from the recent work of Batyrev, Duke, Fomenko, Franke, Golubeva, Heath-Brown, Hooley, Iwaniec, Manin, Moroz, Rudnik, Sarnak, Schulze-Pillot, Silverman, Tschinkel, Vaserstein, and others (cf. Silverman's survey of some of this work in his article **[Si 2]** appearing in this *Astérisque* volume; also **[B-M]**, **[C-S]**, **[D]**, **[D-R-S]**, **[D-S-P]**, **[F-M-T]**, **[F-G]**, **[Mn 2]**, **[M-Tsch]**, **[Tsch]**).

Measures coming from metric height functions. Fix a $\mathbf{Z}$-lattice Ω (i.e., free abelian group of finite rank) and a nondegenerate, positive-definite symmetric $\mathbf{R}$-valued bilinear form, denoted $\langle\ ,\ \rangle$ on Ω. Let V be the $\mathbf{Q}$-vector space $\Omega \otimes_{\mathbf{Z}} \mathbf{Q}$, and define a non-negative real-valued function $v \longmapsto \|v\|$ for all vectors $v \in V$ by the rule

$$\|v\|^2 := \min_{w} \langle w, w \rangle$$

where w ranges through all vectors in Ω which are nonzero scalar multiples of v. This norm function $\| * \| : V \to \mathbf{R}^+$ admits a natural extension to $V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ and its extension is invariant under multiplication by nonzero scalars. It induces, therefore, an $\mathbf{R}^+$-valued function H (which could be called the **metric height function**) on the $\overline{\mathbf{Q}}$-valued points of $\mathbf{P}_V$ the projective space of all lines through the origin in V. More precisely, if x is a $\overline{\mathbf{Q}}$-valued points of $\mathbf{P}_V$, put $H(x) := \|v\|$, where v is any nonzero vector on $V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which generates the line determined by the point x.[5]

The metric on the vector space V also determines a Riemannian metric on the real locus of $\mathbf{P}_V$. Namely, the real locus $\mathbf{P}_V(\mathbf{R})$ admits a 2:1 covering by the unit sphere in the (Euclidean) inner product space $V(\mathbf{R})$ and the induced Riemannian metric on the unit sphere (being invariant under multiplication by -1) descends to yield a Riemannian metric on $\mathbf{P}_V(\mathbf{R})$. If $X \subset \mathbf{P}_V$ is any smooth projective subvariety over $\mathbf{Q}$, restrict the metric height H to the $\overline{\mathbf{Q}}$-valued points of X, and the Riemannian metric on $\mathbf{P}_V(\mathbf{R})$ to the real locus $X(\mathbf{R})$. The restricted Riemannian metric defines a volume form on $X(\mathbf{R})$; let us denote by μ the volume form obtained from this by multiplying by the appropriate scalar to obtain the normalization

$$\int_{X(\mathbf{R})} \mu = 1.$$

Given an open set U in the real locus $X(\mathbf{R})$ which is "nice" in the sense, say, that its boundary in $X(\mathbf{R})$ is a piece-wise differentiable (C^∞) subcomplex,

[5]More generally, if G is a reductive algebraic group over $\mathbf{Q}$ admitting an algebraic linear representation on a finite dimensional $\mathbf{Q}$-vector space V, and if we endow V with a lattice and metric (compatible with the representation in a certain sense) then J.-F. Burnol **[Bu]** has defined an explicit "metric" height function on the quasi-projective variety $\mathcal{M}_{/\mathbf{Q}}$ defined as the quotient of the semi-stable points of V under the action of G. If G is the trivial group, then $\mathcal{M} = \mathbf{P}_V$ and Burnol's height function is equal to the classical metric height function defined above.

one can try to form the limit as $r \longmapsto \infty$ of the ratio

$$\{x \in X(\mathbb{Q}) \cap U | H(x) \leq r\} / \{x \in X(\mathbb{Q}) | H(x) \leq r\}.$$

If this limit exists, call it $\nu(U)$. If $\nu(U)$ exists for all "nice" open subsets, then it is visibly finitely additive on this class of subsets; call it a **Diophantine measure** attached to H. If ν exists, it is natural to try to compare it to μ. At one extreme, we can ask if the rational points of X are equidistributed in the sense that ν (is defined, and) is equal to μ on "nice" open subsets $U \subset X(\mathbf{R})$. For projective space itself, such equidistribution results are classical and straightforward: If V is a metrized finite-dimensional vector space over $\mathbb{Q}$, and X is either $\mathbf{P}_V$, or, more generally, a subvariety isomorphic to projective space of some dimension imbedded in $\mathbf{P}_V$ by a Veronese imbedding (over $\mathbb{Q}$) then $\nu = \mu$ on "nice" open subsets $U \subset X(\mathbb{R})$.[6] In particular, we have the equidistribution of rational points for plane conics $X_{/\mathbb{Q}}$ possessing a rational point, since these curves can be realized as the image of the quadratic Veronese imbedding of $\mathbf{P}^1_{/\mathbb{Q}}$. There are other methods for studying distributional questions in more general contexts, e.g., Hardy and Littlewood's "circle method" and its modern elaborations. Also, for quadrics of any dimension, and more generally, for homogeneous spaces of the for G/H where G is a reductive group, and H is either a reductive (see **[D-R-S]**) or parabolic subgroup of G (see **[F-M-T]**). See also **[Th]** for an explicit count of rational points of flag varieties. A delicate type of "equidistributional question" originally posed by Linnik **[Linn]** (and towards which he had substantial results) is the following: fix a positive-definite ternary quadratic form $g(x, y, z)$. For a **given** integer, consider the set $\mathcal{S}_n$ of integral solutions $g(x, y, z) = n$ scaled back to the unit ellipsoid $\mathcal{E}$, i.e.,

$$\mathcal{S}_n = \{(x, y, z)/\sqrt{n} \mid (x, y, z) \in \mathbb{Z}^3, g(x, y, z) = n\}.$$

[6] but consider the projective plane with one point blown-up, imbedded in projective space via a complete linear system: the precise relationship between ν and μ, here, does not seem to have been settled, even though the corresponding asymptotic count of rational points has long been known (cf. [Se] 2.12)

Linnik's Problem. Letting n run through any finite sequence of integers for which $\mathcal{S}_n$ is nonempty, is it the case that for all C^∞ functions f on $\mathcal{E}$, that the sequence of averages of f on the sets $\mathcal{S}_n$ tends to the integral of f on $\mathcal{E}$, computed with respect to the normalized measure on $\mathcal{E}$ coming from its induced Riemannian metric?

The *complete solution* of Linnik's problem (see **[D]**, **[F-G]**,**[D,S-P]**) uses Iwaniec's delicate bound (**[Iw]** on the Fourier coefficients of modular forms of half-integral weights (see also the prior work **[A-J]** on the diagonal form $x^2+y^2+z^2$, and the even earlier work on the analogous problems for positive-definite forms of rank ≥ 4 **[Mal]**, **[P]**).

A good "test" of the power of the methods we have at hand for the treatment of quadrics might be to try to apply them to equidistribution questions for certain *pencils* of quadrics. But, generally, if the circle method doesn't apply to a form of higher degree, we seem to have little other handle on distributional questions for it; e.g., consider the special cubic surface

$$X_3 : x^3 + y^3 + z^3 - 3t^3 = 0.$$

For a discussion of this surface, see **[H-B]** and **[SD]**. Some extensive numerical experiments being done by Vaserstein[7] on this surface is consistent with Manin's general "linear growth conjecture" which, for this example, predicts that the number of integral points (x, y, z, t) on X_3, with g.c.d. $= 1$, and $x^2 + y^2 + z^2 + t^2 \leq R^2$ grows approximately linearly in R). For a discussion of the asymptotics for other cubic surfaces, see **[Tsch]**.

Following the well-known construction of Tate, in some recent work (**[Si 1,2]**, **[C-S]**) Silverman and Call study the **canonical height function** $H_{L,\varphi}$ (see also **[Se]**) associated to a self-morphism $\varphi : X \to X$ defined over $\mathbf{Q}$ together with a "line bundle" $L \in \mathrm{Pic}(X) \otimes \mathbf{R}$ which is an eigenvector for φ with eigenvalue of absolute value $\alpha > 1$. This is defined by taking any Weil height function $\mathcal{H}$ for L and setting

$$H_{L,\varphi}(x) = \lim_{\nu \to \infty} \mathcal{H}(\varphi^\nu x)^{\alpha^{-\nu}}.$$

[7]Vaserstein tabulates this for points with $\max(|x|, |y|, |z|, |t|) \leq 10,000$. Also, for a study of integral points on this variety, see [C-V].

If X also has *trivial canonical bundle*, then a nowhere vanishing section σ of the canonical bundle (over $\mathbf{R}$) is uniquely determined up to multiplication by a nonzero real number. Restricting σ to the real locus $X(\mathbf{R})$ gives us a nowhere vanishing volume form, and multiplying by the appropriate scalar so that the total integral over $X(\mathbf{R})$ is 1 gives us a **canonical volume form** on $X(\mathbf{R})$. The canonical volume form is preserved by all (biregular) automorphisms of X defined over $\mathbf{R}$.

In view of Silverman's work on Wehler $K3$ surfaces,[8] it would be interesting to understand what (equidistributional) relationship, if any, the canonical heights of Wehler $K3$ surfaces have with respect to the canonical volume form. The dynamics of these $K3$ examples seem interesting. In **[Mz 1]** we reproduced a computer-drawn picture provided to us by Curt McMullen of a (piece of an) orbit of an automorphism of infinite order on a Wehler $K3$ surface.

Here is another example of an automorphism of infinite order on a Wehler $K3$ surface given to me by McMullen. Consider the intersection of the zeroes of the bi-homogeneous (1,1)-form φ and the (1,1)-form ψ in $\mathbf{P}^2 \times \mathbf{P}^2$ given in affine coordinates (a, b) and (x, y) by the equations:

$$\varphi = y + b - 2ax$$
$$\psi = x^2 + y^2 + a^2(1 + y^2) + b^2(1 + x^2 + y^2) - 7.$$

This picture (diagram 1 below) is of the projection of seven orbits (14,000 iterates) of $\mathbf{Q}$-rational points onto the (a, b)-plane. There is a pair of elliptic fixed points lying over $(a, b) = (0, 0)$, and so appropriate neighborhoods of these points should have "Kolmogorov-Arnold-Moser" dynamics. In fact, reading from the projection of these elliptic fixed points "outwards", the seven orbits depicted below *seem to be* of the following types:

1, 2: part of an elliptic island around (0,0)

3: lies in an elliptic island of period 6

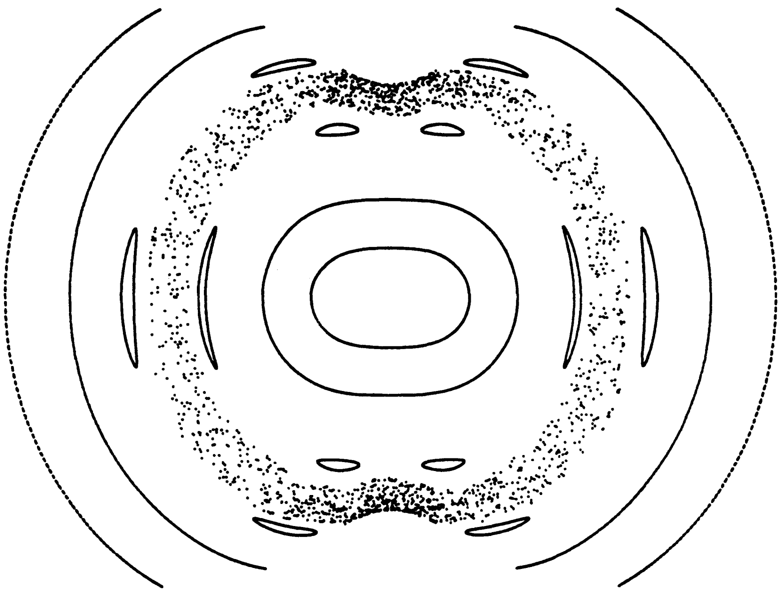
[8] Recall that a Wehler $K3$ surface X is a smooth subvariety in $P^2 \times P^2$ defined as the complete intersection of two bihomogeneous forms, one of bidegree (1,1) and the other of bidegree (2,2). The two projections of X to the factors P^2 are each of degree two, and therefore determine two involutions on the surface X which generate an infinite dihedral group of automorphisms of X; (see [We]).

4: is chaotic

5" lies in an elliptic island of period 6

6, 7: ?? these, as McMullen describes them, seem to "want to be" nice elliptic orbits but they "run into" the discriminant locus.

In contrast to the picture in **[Mz 1]** (which is suggestive of something close to hyperbolic dynamics where, conceivably, there might be dense orbits) these KAM dynamics *preclude*, of course the possibility of dense orbits. For this type of Wehler K3 surface, then, the dynamics alone will not be sufficient to check Conjecture 1.



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