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DIVISIBILITY IN THE CHOW GROUP OF ZERO-CYCLES ON A SINGULAR SURFACE

by

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§0. Introduction.

In this paper we study the divisibility of the Chow group $CH^2(X)$ of 0-cycles on a surface X over a field k. When X is smooth this question has been studied by several authors [MSw] [B2] [R] [CT-R], and we extend many of their results to singular surfaces.

The Chow group of a singular surface X is defined as follows. Choose a closed $Y \subset X$ containing the singular locus of X but no irreducible component of X, and let $Z^2(X,Y)$ be the free abelian group on the set of codimension 2 points of $X - Y$. For each closed curve T in X missing Y, and every rational function f on T, the divisor (f) should equal 0 in $CH^2(X)$. If $\dim Y = 0$, $CH^2(X) = CH^2(X,Y)$ is the quotient of $Z^2(X,Y)$ by the subgroup spanned by these divisors; it is independent of Y because by [PW1, 2.2] it is isomorphic to $SK_0(X)$, the subgroup of $K_0(X)$ consisting of elements of rank 0 and determinant 1. If $\dim Y = 1$ we form $CH^2(X) = CH^2(X,Y)$ by adding the extra relations that $(f) = 0$ for every closed curve T on X which is locally cut out by a nonzerodivisor and every $f \in k(T)$ such that the support of (f) misses $T \cap Y$; this group is also independent of Y, because by [LW] we have $CH^2(X,Y) \cong SK_0(X)$.

If X is a surface and $\mathcal{K}_2$ denotes the Zariski sheaf associated to the presheaf $U \mapsto K_2(U)$, there is a well known isomorphism, called "Bloch's Formula" :

$$(0.1) \qquad CH^2(X) \cong SK_0(X) \cong H^2_{\text{zar}}(X, \mathcal{K}_2).$$

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It was discovered by Bloch [B1] for smooth quasiprojective surfaces, extended to all smooth varieties by Quillen [Q], and to singular surfaces by Levine [L1]; see also [PW1, 8.9]. For regular surfaces, (0.1) also follows from the Brown-Gersten spectral sequence [BG]. For general 2-dimensional noetherian schemes, (0.1) follows from Thomason's generalization [TT, 10.3] of the Brown-Gersten spectral sequence.

Our results relate $CH^2(X)$ to the Zariski cohomology of a certain sheaf $\mathcal{H}^2$ on X. To define it, fix an integer n such that $\frac{1}{n} \in k$, let μ_n denote the étale sheaf of n^{th} roots of unity, and set $\mu_n^{\otimes 2} = \mu_n \otimes \mu_n$. By definition, $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$ is the Zariski sheaf associated to the presheaf $U \mapsto H^2_{\text{et}}(U, \mu_n^{\otimes 2})$ of étale cohomology. Since this sheaf has exponent n, it is convenient to adopt the notation that G/n denotes G/nG and ${}_nG$ denotes $\{x \in G : nx = 0\}$ for any abelian group or sheaf G. Here is our first result.

THEOREM A. — *Let X be a quasiprojective surface over a field k containing $\frac{1}{n}$. Then the Chern class $c_{2,2} : K_2(U) \to H^2_{\text{et}}(U, \mu_n^{\otimes 2})$ induces an isomorphism :*

$$CH^2(X)/n \cong H^2_{\text{zar}}(X, \mathcal{K}_2)/n \cong H^2_{\text{zar}}(X, \mathcal{H}^2(\mu_n^{\otimes 2}))$$

This result was originally proven in the smooth case by Bloch and Ogus [BO], and generalized to the case of isolated singularities by Barbieri-Viale [BV1, 3.9]. We give a short proof of Theorem A in §1, using the Nisnevich topology on X, a method suggested to us by R. Thomason.

After submitting this paper, which contained a second more technical proof of Theorem A in §2, we became aware of the following unpublished result of Ray Hoobler [Hoob] which, given Bloch's formula (0.1), immediately implies Theorem A.

HOOBLER'S THEOREM 0.2. — *Let k be a field containing $\frac{1}{n}$.*

1) If A is a semilocal ring, essentially of finite type over k, then the Chern class $c_{22} : K_2(A) \to H^2_{\text{et}}(A, \mu_n^{\otimes 2})$ is an isomorphism.

2) If X is a quasiprojective scheme over k, there is an isomorphism of (Zariski) sheaves

$$c_{2,2} : \mathcal{K}_2/n \to \mathcal{H}^2(\mu_n^{\otimes 2}).$$

When X or A is smooth over k, this theorem is implicit in Merkurjev and Suslin's work [MS, §18]; see [B3, 3.3] [CT-R, p.168] and [PW2, 4.3]. When X is a singular curve, this theorem was proven in [PW2, 5.2].

Our original proof of Theorem A is therefore obsolete. As a favor to the reader, we have deleted it. It was the original §2 of this paper.

The current §2 gives a short survey of the étale Chern classes c_{ij}. We also prove that the isomorphism in Theorem A lifts Grothendieck's Chern class $c_{2,4} : K_0(X) \to H^4_{\mathrm{et}}(X, \mu_n^{\otimes 2})$ to $SK_0(X)$ in the sense that $c_{2,4}$ is the composite

$$SK_0(X) \twoheadrightarrow SK_0(X)/n \cong H^2_{\mathrm{zar}}(X, \mathcal{H}^2) \xrightarrow{\gamma} H^4_{\mathrm{et}}(X, \mu_n^{\otimes 2}),$$

γ being the edge map in the Leray spectral sequence for $X_{\mathrm{et}} \to X_{\mathrm{zar}}$. When X is smooth this proves that the "cycle map" considered in [CT-R] and [Sai,§5] is just $c_{2,4}$.

In §3 we consider the normalization $\pi : \tilde{X} \longrightarrow X$ of X. Using Mayer-Vietoris sequences, we relate $CH^2(X)/n$ to the Chow group $CH^2(\tilde{X})/n$. Let Y denote the singular locus of X, and set $\tilde{Y} = \pi^{-1}(Y)$, so that we have a cartesian square :

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{j} & \widetilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i} & X. \end{array}$$

THEOREM B. — *Assume that k contains μ_n and $\frac{1}{n}$. Then there is an exact sequence for the sheaf $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$:*

$$H^1(\widetilde{X}, \mathcal{H}^2) \oplus H^1(Y, \mathcal{H}^2) \to H^1(\widetilde{Y}, \mathcal{H}^2) \to H^2(X, \mathcal{H}^2) \to H^2(\widetilde{X}, \mathcal{H}^2) \to 0.$$

Using Theorem A and the two isomorphisms $H^1(Y, \mathcal{H}^2) \cong SK_1(Y)/n$ and $H^1(\tilde{Y}, \mathcal{H}^2) \cong SK_1(\tilde{Y})/n$ of [PW2, 5.1], we can restate Theorem B as follows.

COROLLARY C. — *With n as in Theorem B, there is an exact sequence :*

$$H^1(\tilde{X}, \mathcal{H}^2) \oplus SK_1(Y)/n \to SK_1(\tilde{Y})/n \to CH^2(X)/n \to CH^2(\tilde{X})/n \to 0.$$

In the Appendix, we indicate how much of Corollary C can be obtained from pure K-theoretic techniques, i.e., without resorting to $\mathcal{H}^2$.

In §4 we relate the n-torsion in the Chow group of $\tilde{X}$ to the term $H^1(\tilde{X}, \mathcal{H}^2)$ appearing in Corollary C, as well as to the quotient $H^1(\tilde{X}, \mathcal{K}_2)$ of $SK_1(\tilde{X})$. When X is smooth, we know by [B3, 1.12][MS, 8.7.8(e)] that there is an exact sequence :

$$(0.3) \qquad 0 \to H^1(X, \mathcal{K}_2)/n \to H^1(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \to {}_nCH^2(X) \to 0.$$

When X is a surface with isolated singularities, (0.3) needs to be modified because the subsheaf ${}_n\mathcal{K}_2$ of n-torsion elements in $\mathcal{K}_2$ has more complicated cohomology. Indeed, the vanishing of $H^2(X, {}_n\mathcal{K}_2)$ in the smooth case is the basis for the proof of (0.3) in [MS], but if X has isolated singularities we show in 4.2 that

$$H^2(X, {}_n\mathcal{K}_2) \cong H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})).$$

This group is just $H^2(X, \mathcal{O}_X^*)/n$ when $\mu_n \subset k$, and we know that it can be nonzero for normal surfaces; see [PW1, 5.9]. We are able to prove the following generalization of (0.3) in §4. (Again, we have deleted those parts which Hoobler's Theorem makes obsolete.)

THEOREM D. — *Let X be a quasiprojective surface over a field k containing $\frac{1}{n}$. Assume that X is normal, or more generally that* Sing(X) *is finite. Then there is an exact sequence :*

$$H^0(X, \mathcal{K}_2/n) \xrightarrow{\gamma} H^2(X, {}_n\mathcal{K}_2) \to H^1(X, \mathcal{K}_2)/n \to H^1(X, \mathcal{K}_2/n) \to {}_nCH^2(X) \to 0$$

Remark. Presumably the map $H^0(X, \mathcal{H}^2) \xrightarrow{\gamma} H^2(X, \mathcal{H}^1)$ in Theorem D is the differential in the Leray spectral sequence converging to $H^*_{\text{et}}(X, \mu_n^{\otimes 2})$. If so, and we write $NH^3(X)$ for the kernel of $H^3_{\text{et}}(X, \mu_n^{\otimes 2}) \to H^0(X, \mathcal{H}^3)$, then we may restate Theorem D as the following exact sequence, which generalizes part of the sequence of [Suslin, 4.4].

$$(0.4) \qquad 0 \to H^1(X, \mathcal{K}_2)/n \to NH^3(X) \to {}_nCH^2(X) \to 0$$

COROLLARY E (Collino [C]). — *Suppose that k is either an algebraically closed field, or the reals $\mathbb{R}$, or a local field. Let X be a surface having only isolated singularities. Then the n-torsion in $CH^2(X)$ is finite for every n with $\frac{1}{n} \in k$.*

Proof. Fix n and let k be any field such that $H^i_{et}(k, M)$ is finite for constructible n-torsion sheaves M. Then each $H^q_{et}(X, \mu_n^{\otimes i})$ is finite by [SGA4, XVI.5.1]. When X is a surface, the Leray spectral sequence $H^p(X, \mathcal{H}^q) \Rightarrow H^*_{et}(X, \mu_n^{\otimes i})$ degenerates enough to show that the group $H^1(X, \mathcal{H}^2(\mu_n^{\otimes 2})) = H^1(X, \mathcal{K}_2/n)$ is finite. Now apply Theorem D. $\square$

There is a "degree" map $CH^2(X) \to \mathbb{Z}^c$, where c denotes the number of irreducible proper components of X. The image Λ has finite index in $\mathbb{Z}^c$, and $CH^2(X) \cong \Lambda \oplus A_0(X)$, where $A_0(X)$ is the group of zero cycles of relative "degree" zero. Therefore all of our divisibility results are actually statements about the divisibility of the subgroup $A_0(X)$ of $CH^2(X)$.

In §5 we apply our structural results to surfaces over special kinds of fields : algebraically closed fields, number fields and the field $\mathbb{R}$ of real numbers. If $k = \bar{k}$ we show that $A_0(X)$ is n-divisible — and hence that $CH^2(X)/n \cong (\mathbb{Z}/n)^c$ — for every surface X and every n prime to $char(k)$, a well-known result for smooth surfaces.

If k is a number field and X is smooth, Bass has conjectured that $K_0(X)$ and therefore $SK_0(X) \cong CH^2(X)$ is a finitely generated abelian group. This would imply that $CH^2(X)/n$ is finite. By results of Colliot-Thélène and Raskind [CT-R91], and of Salberger, this finiteness is known to hold for every smooth projective surface X such that $H^2(X, \mathcal{O}_X) = 0$ and Bloch's Conjecture holds for X (see 5.3.1); in particular it holds for all surfaces which are not of general type. Bass' conjecture does not carry over to singular surfaces; we give examples of seminormal affine and projective surfaces over any number field k such that both $CH^2(X)/n$ and ${}_nCH^2(X)$ are infinite.

Finally we consider varieties over the real numbers $\mathbb{R}$, relating $CH^2(X)$ to the topological space $X(\mathbb{R})$. If the singular locus of X has codimension ≥ 2 and $d = \dim X$, we show in theorem 5.8 that

$$CH^d(X) \cong \mathbb{Z}^c \oplus (\mathbb{Z}/2)^{t-R} \oplus V, \tag{0.5}$$

where $t = \dim H^d(X(\mathbb{R}), \mathbb{Z}/2)$, R is the number of irreducible proper components of X having a smooth real point and V is a divisible abelian group. This calculation extends results of Colliot-Thélène and Ischebeck [CT-I] for smooth projective varieties. When $d = 2$, the case of a real surface with isolated singularities, (0.5) yields isomorphisms

$$SK_0(X)/2 \cong CH^2(X)/2 \cong (\mathbb{Z}/2)^{t+c-R}, \qquad A_0(X)/2 \cong (\mathbb{Z}/2)^{t-R},$$

where c, R and $t = \dim H^2(X(\mathbb{R}), \mathbb{Z}/2)$ are defined above. Finally, we use Corollary C to extend (0.5) to any real surface in theorem 5.12. (For technical reasons, we need to add a summand $(\mathbb{Z}/2)^\varepsilon$ to (0.5) when $\tilde{X}$ is not smooth, but we suspect that $\varepsilon = 0$ in all cases.) These results may be applied to compute $CH^2(X)/2$ of real surfaces having a one-dimensional singular locus, including the so-called "real umbrellas" (see 5.13).

Notation

We fix an integer n. If G is an abelian group or sheaf, we shall write G/n, $n \cdot G$ and ${}_nG$ respectively for the cokernel, image and kernel of the homomorphism $G \xrightarrow{n} G$.

X will always denote a noetherian scheme over $\mathbb{Z}[\frac{1}{n}]$; by "surface" we will mean a 2-dimensional quasiprojective scheme defined over a field k (with

$\frac{1}{n} \in k$). The étale sheaf of n^{th} roots of unity is μ_n, and we shall also consider $\mu_n^{\otimes 2} = \mu_n \otimes \mu_n$. We write $\mathcal{H}^q$ or $\mathcal{H}^q(\mu_n^{\otimes i})$ for the Zariski sheaf on X associated to the presheaf $U \mapsto H^q_{\text{et}}(U, \mu_n^{\otimes i})$, and $\mathcal{K}_n$ for the Zariski sheaf associated to the presheaf $U \mapsto K_n(U)$.

§1. Proof of Theorem A

Our goal in this section is to give a short proof of Theorem A, without using Hoobler's Theorem (0.2). The only Chern class used in this section is $c_{22} : K_2(X)/n \to H^2_{\text{et}}(X, \mu_n^{\otimes 2})$.

Let k be a field containing $\frac{1}{n}$. If X is any surface over k, not only do we have Bloch's formula (0.1), but we can apply the right exact functor $H^2_{\text{zar}}(X, -)$ to the sequence $\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2 \to \mathcal{K}_2/n \to 0$ to get canonical isomorphisms :

$$CH^2(X)/n \cong H^2_{\text{zar}}(X, \mathcal{K}_2)/n \cong H^2_{\text{zar}}(X, \mathcal{K}_2/n).$$

Therefore, Theorem A just states that there is an isomorphism

$$H^2_{\text{zar}}(X, \mathcal{K}_2/n) \cong H^2_{\text{zar}}(X, \mathcal{H}^2(\mu_n^{\otimes 2})). \tag{1.1}$$

If X is a *smooth* variety over k then, as mentioned in the introduction, Theorem A is well known. Indeed, the sheaf map $c_{22} : \mathcal{K}_2 \to \mathcal{H}^2(\mu_n^{\otimes 2})$ induces an isomorphism of sheaves

$$c_{22} : \mathcal{K}_2/n \cong \mathcal{H}^2(\mu_n^{\otimes 2}). \tag{1.2}$$

The following elementary lemma, whose proof is left to the reader, immediately proves Theorem A — that (1.1) holds — for surfaces with isolated singularities, i.e., surfaces X with $\dim(\text{Sing}(X)) = 0$.

LEMMA 1.3. — *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of Zariski sheaves on a noetherian scheme X. Suppose the kernel and cokernel of f are supported on a union of closed d-dimensional subschemes of X. Then $H^{d+1}(X, \mathcal{F}) \to H^{d+1}(X, \mathcal{G})$ is onto, and*

$$H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{G}) \qquad \textit{for all } i \geq d+2.$$

For general surfaces, our proof of Theorem A uses a result of Y. Nisnevich which was pointed out to us by R. Thomason. Let X_{nis} denote X endowed with the Nisnevich topology introduced in [N1]. This topology is intermediate between the étale and Zariski topologies on X in the sense that there are natural morphisms of sites $X_{\text{et}} \to X_{\text{nis}} \to X_{\text{zar}}$. Let $\mathcal{K}_2^{\text{nis}}$ and $\mathcal{H}^2_{\text{nis}}(\mu_n^{\otimes 2})$ denote the (Nisnevich) sheaves on X_{nis} associated to the presheaves $U \mapsto K_2(U)$ and $U \mapsto H^2_{\text{et}}(X, \mu_n^{\otimes 2})$, respectively. The following result essentially follows from Gabber's theorem [G, Th.1], see also [N2, 8.6].

THEOREM 1.4. — *Let X be a quasiprojective scheme over a field k with $\frac{1}{n} \in k$. Then the Chern class c_{22} induces an isomorphism of sheaves on X_{nis} :*

$$c_{22} : \mathcal{K}_2^{\mathrm{nis}}/n \cong \mathcal{H}^2_{\mathrm{nis}}(\mu_n^{\otimes 2})$$

COROLLARY 1.5 (Nisnevich version of theorem A). — *Let X be a quasiprojective surface over a field k with $\frac{1}{n} \in k$. Then $SK_0(X) \cong H^2_{\mathrm{nis}}(X, \mathcal{K}_2^{\mathrm{nis}})$ and*

$$SK_0(X)/n \cong H^2_{\mathrm{nis}}(X, \mathcal{K}_2^{\mathrm{nis}})/n \cong H^2_{\mathrm{nis}}(X, \mathcal{K}^{\mathrm{nis}}/n) \cong H^2_{\mathrm{nis}}(X, \mathcal{H}^2_{\mathrm{nis}}(\mu_n^{\otimes 2})).$$

Proof. When $\dim X = 2$, X_{nis} has cohomological dimension 2. Therefore $H^2_{\mathrm{nis}}(X, -)$ is right exact hence $H^2_{\mathrm{nis}}(X, \mathcal{K}_2^{\mathrm{nis}})/n \cong H^2_{\mathrm{nis}}(X, \mathcal{K}^{\mathrm{nis}}/n)$. The argument that $SK_0(X) \cong H^2_{\mathrm{nis}}(X, \mathcal{K}_2^{\mathrm{nis}})$ is identical to the argument for the Zariski topology, using the spectral sequence of [TT, 10.8] :

$$E_2^{p,q} = H^p_{\mathrm{nis}}(X, \mathcal{K}_{-q}^{\mathrm{nis}}) \Rightarrow K_{-p-q}(X).$$

In detail, the following terms of E_2 are known to live to E_∞ :

$$\begin{aligned} H^0_{\mathrm{nis}}(X, \mathcal{K}_0^{\mathrm{nis}}) &= H^0_{\mathrm{zar}}(X, \mathcal{K}_0) = H^0_{\mathrm{zar}}(X, \mathbb{Z}) \\ H^0_{\mathrm{nis}}(X, \mathcal{K}_1^{\mathrm{nis}}) &= H^0_{\mathrm{zar}}(X, \mathcal{K}_1) = H^0_{\mathrm{zar}}(X, \mathcal{O}_X^*) \\ H^1_{\mathrm{nis}}(X, \mathcal{K}_1^{\mathrm{nis}}) &\cong H^1_{\mathrm{zar}}(X, \mathcal{K}_1) \cong \mathrm{Pic}(X). \end{aligned}$$

Therefore this spectral sequence yields a filtration on K_0 whose associated graded groups are the same as those associated to the analogous spectral sequence for X_{zar}. □

The morphism $X_{\mathrm{nis}} \to X_{\mathrm{zar}}$ yields a commutative diagram, in which the maps labelled '$\cong$' are isomorphisms by (0.1) and (1.5) :

$$\begin{array}{ccccc} SK_0(X)/n & \cong & H^2_{\mathrm{zar}}(X, \mathcal{K}_2/n) & \xrightarrow{c_{22}} & H^2_{\mathrm{zar}}(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \\ & & \downarrow \cong & & \downarrow \\ & & H^2_{\mathrm{nis}}(X, \mathcal{K}_2^{\mathrm{nis}}/n) & \xrightarrow{\cong} & H^2_{\mathrm{nis}}(X, \mathcal{H}^2_{\mathrm{nis}}(\mu_n^{\otimes 2})). \end{array}$$

We claim that both the kernel and cokernel of the Chern class map c_{22} : $\mathcal{K}_2/n \to \mathcal{H}^2_{\mathrm{zar}}(\mu_n^{\otimes 2})$ are sheaves supported on the subscheme $\mathrm{Sing}(X_{\mathrm{red}})$ of

X_{zar}, which has dim ≤ 1. Lemma 1.3 will then yield the surjectivity of the map $H^2_{\mathrm{zar}}(X, \mathcal{K}_2/n) \to H^2_{\mathrm{zar}}(\mathcal{H}^2(\mu_n^{\otimes 2}))$, and theorem A will follow from a chase of the above diagram.

This claim is immediate from (1.2) when X is reduced, because then X is smooth in codimension 0. When X is not reduced, we argue as follows. Let $\mathcal{K}_2^{\mathrm{red}}$ and $\mathcal{H}^2_{\mathrm{red}}$ denote the Zariski sheaves associated to the presheaves $U \mapsto K_2(U_{\mathrm{red}})$ and $U \mapsto H_{\mathrm{et}}(U_{\mathrm{red}}, \mu_n^{\otimes 2})$, respectively. Since the canonical isomorphism of sites $X_{\mathrm{et}} \cong (X_{\mathrm{red}})_{\mathrm{et}}$ identifies the sheaves μ_n on X and on X_{red}, we have $\mathcal{H}^2(\mu_n^{\otimes 2}) \cong \mathcal{H}^2_{\mathrm{red}}$. Therefore it suffices to show that $\mathcal{K}_2 \cong \mathcal{K}_2^{\mathrm{red}}$; we prove this assertion in 1.7 below.

The ability to ignore nilpotent ideals in theorem A, as well as in the other results in this paper, rests upon the following lemma and its corollary.

LEMMA 1.6. — *If A is a commutative ring containing $\frac{1}{n}$ then $K_2(A)/n \cong K_2(A_{\mathrm{red}})/n$.*

Proof. Let I be the nilradical of A, so that $A_{\mathrm{red}} = A/I$. In the K-theory sequence

$$K_2(A, I) \to K_2(A) \to K_2(A/I) \to SK_1(A, I) \to 0$$

both $K_2(A, I)$ and $SK_1(A, I)$ are uniquely n-divisible by [W1, 1.4]. Hence $K_2(A, I)/n = SK_1(A, I)/n = 0$ and $\mathrm{Tor}(\mathbb{Z}/n, SK_1(A, I)) = 0$. The lemma is now elementary. □

COROLLARY 1.7. — *If X is any scheme over $\mathbb{Z}[\frac{1}{n}]$ then $\mathcal{K}_2/n \cong \mathcal{K}_2^{\mathrm{red}}/n$. In particular,*
(i) If $\dim(X) = 1$ then $SK_1(X)/n \cong SK_1(X_{\mathrm{red}})/n$;
(ii) if $\dim(X) = 2$ then $SK_0(X)/n \cong SK_0(X_{\mathrm{red}})/n$.

Proof. From the Brown-Gersten spectral sequence of [TT], we see that : (i) when $\dim(X) = 1$ then $SK_1(X) \cong H^1(X, \mathcal{K}_2)$, hence $SK_1(X)/n \cong H^1(X, \mathcal{K}_2/n)$, and (ii) when $\dim(X) = 2$ then $SK_0(X) \cong H^2(X, \mathcal{K}_2)$, hence $SK_0(X)/n \cong H^2(X, \mathcal{K}_2/n)$. □

§2 The étale Chern classes c_{ij}

We begin this section with a short summary of étale Chern classes. Then (in 2.3) we show that the isomorphism in Theorem A is a lift of Grothendieck's Chern class $c_{24} : K_0(X) \to H^4_{\mathrm{et}}(X, \mu_n^{\otimes 2})$. Recall that with our fixed notation the integer n is fixed, and all schemes X are defined over $\mathbb{Z}[\frac{1}{n}]$.

Classical étale Chern classes 2.1. The classical étale Chern classes are set maps $c_i = c_{i,2i} : K_0(X) \to H^{2i}_{\mathrm{et}}(X, \mu_n^{\otimes i})$, constructed by Grothendieck to satisfy the following axioms. They are natural in X. By convention, $c_0(x) = 1$

for all $x \in K_0(X)$. The c_i satisfy the Whitney sum formula $c_\ell(x+y) = \sum_{i+j=\ell} c_i(x) \cup c_j(y)$. The first Chern class $c_1 = c_{12}$ is the composition of the determinant map $\det : K_0(X) \to \mathrm{Pic}(X) \cong H^1_{\mathrm{et}}(X, \mathbb{G}_m)$ with the boundary map $\partial : H^1_{\mathrm{et}}(X, \mathbb{G}_m) \to H^2_{\mathrm{et}}(X, \mu_n)$ arising from the Kummer sequence on X_{et}. If L is a line bundle on X, $c_i(L) = 0$ for all $i > 1$.

These axioms determine all the other Chern class maps c_i as follows. A typical element of $K_0(X)$ has the form $[E] - [F]$, where E and F are vector bundles on X. It is possible to replace X by a flag bundle X' over X because $H^{2i}_{\mathrm{et}}(X, \mu_n^{\otimes i})$ injects into $H^{2i}(X', \mu_n^{\otimes i})$ by [J, 2.2.4]. Doing so, we may assume that $[E]$ and $[F]$ are sums of classes of line bundles in $K_0(X)$. The Whitney sum formula then determines $c_i([E])$, $c_i([F])$ and finally (by induction on i) $c_i([E] - [F])$. Note that if F is a trivial vector bundle then $c_i(F) = 0$ for $i \neq 0$ and we have the simple formula $c_i([E] - [F]) = c_i(E)$.

The Product Formula [Gr, I(1.6), II(2.7)] is sometimes listed as an axiom. It expresses $c_i(x \cdot y)$ as a universal polynomial Q_i in the classes $c_1(x), \ldots, c_{i-1}(x)$ and $c_1(y), \ldots, c_{i-1}(y)$ when x and y have rank zero. The polynomial Q_2 is $c_2(x \cdot y) = -c_1(x)c_1(y)$, but Q_3 has 4 terms and Q_4 has 10 terms; see [Gr, I(1.18)] and [W3, 3.6].

Higher étale Chern classes 2.2. Less classical are the higher Chern classes $c_i = c_{ij}$, defined by Quillen and Illusie in 1974 and exposed in the articles [Shek][Soulé][GilRR]. Fixing the indices $i \geq 1$, $0 \leq j < 2i$ and setting $m = 2i - j > 0$ for convenience, c_{ij} is an additive homomorphism from $K_m(X)/n$ to $H^j_{\mathrm{et}}(X, \mu_n^{\otimes i})$. These c_{ij} are natural in the scheme X over $\mathbb{Z}[1/n]$.

The most important higher Chern class is c_{11}; it is defined on $K_1(X)$ as the composite of the natural projection $\det : K_1(X) \to H^0(X, \mathcal{O}_X^*) = H^0_{\mathrm{et}}(X, \mathbb{G}_m)$ with the boundary map $\partial : H^0_{\mathrm{et}}(X, \mathbb{G}_m) \to H^1_{\mathrm{et}}(X, \mu)$ associated to the Kummer sequence. All other Chern classes vanish on the summand $H^0(X, \mathcal{O}_X^*)$ of $K_1(X)$. The Chern class c_{10} vanishes on $K_2(X)$ [Soulé, p.279].

The Product Formula for c_{ij} is simpler for higher K-theory than it is for K_0. If $x \in K_{m_1}(X)$, $y \in K_{m_2}(X)$ with $m_1, m_2 > 0$ then this formula reads

$$\text{(2.2.1)} \qquad c_i(x \cdot y) = \sum_{i_1 + i_2 = i} \frac{-(i-1)!}{(i_1 - 1)!(i_2 - 1)!} c_{i_1}(x) \cup c_{i_2}(y).$$

In particular, $c_{22} : K_2(X) \to H^2_{\mathrm{et}}(X, \mu_n^{\otimes 2})$ coincides up to sign with the classical Galois symbol : if $x, y \in K_1(X)$ then $c_{22}(\{x, y\}) = -c_1(x) \cup c_1(y)$. (Cf. [Shek].)

Mod n variation 2.2.2. We can extend higher Chern classes to K-theory with coefficients, replacing $K_m(X)/n$ with the larger group $K_m(X; \mathbb{Z}/n)$ when $m \geq 2$. These classes were defined by Soulé in [Soulé] for affine schemes;

we can immediately extend the definitions to quasi-projective schemes using Jouanolou's trick. The first new aspect of these mod n Chern classes is that c_{10} is the projection $K_2(X;\mathbb{Z}/n) \to {}_nK_1(X) \to {}_nH^0(X,\mathcal{O}_X^*) = H^0(X,\mu_n)$.

As long as n is odd, these c_i are additive and satisfy the Product Formula (2.2.1). There are technical problems that arise when n is even, starting with the possible lack of additivity of c_{22}; we refer the reader to [W3] for a discussion.

This completes our survey of étale Chern classes.

We now turn to the study of $c_{24} : K_0(X) \to H^4_{\text{et}}(X,\mu_n^{\otimes 2})$. If $\dim(X) = 2$, the Leray spectral sequence for $\varepsilon : X_{\text{et}} \to X_{\text{zar}}$ and the sheaf $\mu_n^{\otimes 2}$ yields an exact sequence :

$$H^3_{\text{et}}(X,\mu_n^{\otimes 2}) \to H^0_{\text{zar}}(X,\mathcal{H}^3(\mu_n^{\otimes 2})) \to H^2_{\text{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes 2})) \xrightarrow{\gamma} H^4_{\text{et}}(X,\mu_n^{\otimes 2}).$$

THEOREM 2.3. — *Let X be a 2-dimensional noetherian scheme over $\mathbb{Z}[\frac{1}{n}]$. Then the following diagram commutes.*

$$\begin{array}{ccc} SK_0(X) & \longrightarrow & K_0(X) \\ {\scriptstyle H^2(c_{22})}\downarrow & & \downarrow{\scriptstyle c_{24}} \\ H^2_{\text{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes 2})) & \xrightarrow{\gamma} & H^4_{\text{et}}(X,\mu_n^{\otimes 2}) \end{array}$$

Here $H^2(c_{22})$ is induced by the map $c_{22} : \mathcal{K}_2 \to \mathcal{H}^2(\mu_n^{\otimes 2})$ via Bloch's formula (0.1).

Proof. Replacing X by a suitable flag bundle X' and using the splitting principle, we may write any element s of $SK_0(X)$ as a product $u_1 \cdot u_2$ in $K_0(X')$, where $u_i = [L_i] - 1$ for appropriate line bundles L_i on X'. Since $H^*_{\text{et}}(X,\mu_n^{\otimes 2})$ injects into $H^*_{\text{et}}(X',\mu_n^{\otimes 2})$ by [J, 2.2.4] we may replace X by X' in computing $c_{24}(s)$. Let λ_i denote the image of u_i in $H^1(X,\mathcal{O}_X^*)$ under the map $\det : K_0(X) \to \text{Pic}(X) \cong H^1(X,\mathcal{K}_1)$. Under Bloch's formula, s corresponds to the product $\lambda_1 \cdot \lambda_2$ in $H^2(X,\mathcal{K}_2)$. Therefore we must show that γ sends the element $H^2(c_{22})(\lambda_1 \cdot \lambda_2)$ of $H^2_{\text{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes 2}))$ to the element $c_{24}(u_1 \cdot u_2)$ of $H^4_{\text{et}}(X,\mu_n^{\otimes 2})$.

The Product Formula for the Chern classes c_{24} and c_{22} (see 2.1 and 2.2) yields

$$c_{24}(u_1 \cdot u_2) = -c_{12}(u_1) \cup c_{12}(u_2)$$
$$H^2(c_{22})(\lambda_1 \cdot \lambda_2) = -H^1(c_{11})(\lambda_1) \cup H^1(c_{11})(\lambda_2)$$

where $H^1(c_{11}) : H^1_{\rm zar}(X,\mathcal{K}_1) \to H^1_{\rm zar}(X,\mathcal{H}^1(\mu_n))$ is induced from $c_{11} : \mathcal{K}_1 \to \mathcal{H}^1(\mu_n)$. Now the Leray filtration on $H^*_{\rm et}(X,\mu_n^{\otimes *})$ is preserved by cup products, such as $H^2_{\rm et}(X,\mu_n) \otimes H^2_{\rm et}(X,\mu_n) \to H^4_{\rm et}(X,\mu_n^{\otimes 2})$, and the induced product on the associated graded groups, such as $H^1(X,\mathcal{H}^1(\mu_n)) \otimes H^1(X,\mathcal{H}^1(\mu_n)) \xrightarrow{\cup} H^2(X,\mathcal{H}^2(\mu_n^{\otimes 2}))$ is the usual product. This compatibility, together with the vanishing of $H^3(X_{\rm zar},-)$ and $H^4(X_{\rm zar},-)$ when $\dim X = 2$, implies that there is an injection $\rho : H^1(X,\mathcal{H}^1) \hookrightarrow H^2_{\rm et}(X,\mu_n)/\varepsilon^* H^2_{\rm zar}(X,\mu_n)$ compatible with γ in the sense that for $a_i \in H^1(X,\mathcal{H}^1)$ we have $\gamma(a_1 \cup a_2) = (\rho a_1) \cup (\rho a_2)$ in $H^4_{\rm et}(X,\mu_n^{\otimes 2})$. Taking $a_i = H^1(c_{11})(\lambda_i)$, it suffices to show that in $H^2_{\rm et}(X,\mu_n)/\varepsilon^* H^2_{\rm zar}(X,\mu_n)$ we have $c_{12}(u_i) = \rho H^1(c_{11})(\lambda_i)$. This assertion about c_{12} is a special case of the more general result 2.4 below. □

Our factorization of c_{12} requires some observations about the filtration on $H^2_{\rm et}(X,\mu_n)$ associated to the Leray spectral sequence for the morphism of sites $\varepsilon : X_{\rm et} \to X_{\rm zar}$. In rows $q = 0$ and $q = 1$ of the spectral sequence we have H^p of the sheaves $\mathcal{H}^0(\mu_n) \cong \mu_n$ and $\mathcal{H}^1(\mu_n) \cong \mathcal{O}^*_X/n$. Therefore the bottom layer of the filtration is the image of $\varepsilon^* : H^2_{\rm zar}(X,\mu_n) \to H^2_{\rm et}(X,\mu_n)$. Assuming for simplicity that $H^3_{\rm zar}(X,\mu_n) = 0$, the next layer of the filtration is given by an injection

$$\rho : H^1_{\rm zar}(X,\mathcal{O}^*_X/n) \hookrightarrow H^2_{\rm et}(X,\mu_n)/\varepsilon^* H^2_{\rm zar}(X,\mu_n).$$

Finally, we define ∂ to be the map

$$H^1_{\rm zar}(X,\mathcal{O}^*_X) \cong H^1_{\rm et}(X,\mathbb{G}_m) \xrightarrow{\partial} H^2_{\rm et}(X,\mu_n)$$

arising from the Kummer sequence on $X_{\rm et}$.

PROPOSITION 2.4. — *Let X be a scheme over $\mathbb{Z}[\frac{1}{n}]$. Then :*
a) The map $H^1(c_{11})$ is induced from the natural quotient map $\mathcal{O}^*_X \to \mathcal{O}^*_X/n$.
b) Assuming for simplicity that $H^3_{\rm zar}(X,\mu_n) = 0$, the following diagram commutes.

$$\begin{array}{ccccc}
K_0(X) & \xrightarrow{c_{12}} & H^2_{\rm et}(X,\mu_n) & \longrightarrow & H^2_{\rm et}(X,\mu_n)/\varepsilon^* H^2_{\rm zar}(X,\mu_n) \\
\downarrow \det & & \uparrow \partial & & \uparrow \rho \\
\mathrm{Pic}(X) & \cong & H^1_{\rm zar}(X,\mathcal{O}^*_X) & \xrightarrow{H^1(c_{11})} & H^1_{\rm zar}(X,\mathcal{O}^*_X/n)
\end{array}$$

Remark 2.4.1 : Part *b*) remains valid if $H^3_{\rm zar}(X,\mu_n) \neq 0$, provided we replace $H^1_{\rm zar}(X,\mathcal{O}^*_X/n)$ by the kernel of the differential

$$d^2 : H^1_{\rm zar}(X,\mathcal{O}^*_X/n) \to H^3_{\rm zar}(X,\mu_n).$$

Proof : By definition, the sheafification $\mathcal{K}_1 \to \mathcal{H}^1(\mu_n)$ of the Chern class c_{11} is the composite of the sheafification det : $\mathcal{K}_1 \xrightarrow{\cong} \mathcal{O}_X^*$ of det : $K_1(X) \to H^0(X, \mathcal{O}_X^*)$ and the natural map $\mathcal{O}_X^* \to \mathcal{O}_X^*/n \cong \mathcal{H}^1(\mu_n)$, which is the sheafification of the Kummer map $H^0(X, \mathcal{O}_X^*) \to H^1_{\text{et}}(X, \mu_n)$. Thus the sheaf version of c_{11} is naturally isomorphic to the map $\mathcal{O}_X^* \to \mathcal{O}_X^*/n$. Applying H^1 yields $H^1(c_{11})$, proving a).

The left-hand square of b) commutes by the definition of c_{12}. For the right-hand square, observe that the Leray spectral sequence for $H^*_{\text{et}}(X, \mu_n)$ is just the hypercohomology spectral sequence for $\mathbb{H}^*(X, \mathbf{R}\varepsilon_*\mu_n)$, where $\mathbf{R}\varepsilon_*$ denotes the total higher direct image functor. We compare this with the hypercohomology spectral sequence for the cochain complex $\mathcal{C} : \mathcal{O}_X^* \xrightarrow{n} \mathcal{O}_X^*$ concentrated in degrees 0 and 1. The natural map $\mathcal{O}_X^* \to \mathbf{R}\varepsilon_*\mathbb{G}_m$ induces a map $\eta : \mathcal{C} \to \mathbf{R}\varepsilon_*\mu_n$ fitting into a morphism of triangles in the derived category of X_{zar} :

$$\begin{array}{ccccccc}
\mathcal{C} & \to & \mathcal{O}_X^* & \xrightarrow{n} & \mathcal{O}_X^* & \to & \mathcal{C}[1] \\
\downarrow \eta & & \downarrow & & \downarrow & & \downarrow \eta \\
\mathbf{R}\varepsilon_*\mu_n & \to & \mathbf{R}\varepsilon_*\mathbb{G}_m & \xrightarrow{n} & \mathbf{R}\varepsilon_*\mathbb{G}_m & \xrightarrow{\partial} & \mathbf{R}\varepsilon_*\mu_n[1]
\end{array}$$

Since $H^*_{\text{et}}(X, \mathcal{F}) = \mathbb{H}^*(X, \mathbf{R}\varepsilon_*\mathcal{F})$ for every étale sheaf $\mathcal{F}$, the bottom row is the Kummer sequence. From the right-hand square, we see that ∂ is the map

(2.5) $H^1(X, \mathcal{O}_X^*) \to \mathbb{H}^1(X, \mathcal{C}[1]) = \mathbb{H}^2(X, \mathcal{C}) \xrightarrow{\eta} \mathbb{H}^2(X, \mathbf{R}\varepsilon_*\mu_n) = H^2_{\text{et}}(X, \mu_n)$.

Since $H^0(\mathcal{C}) = \mu_n$ and $H^1(\mathcal{C}) = \mathcal{O}_X^*/n$, $H^2(X, \mathcal{C})$ is the second level of the Leray filtration of $H^2_{\text{et}}(X, \mu_n)$ and we have a commutative diagram :

$$\begin{array}{ccccc}
H^1(X, \mathcal{O}_X^*) & \to & \mathbb{H}^2(X, \mathcal{C}) & \to & H^1_{\text{zar}}(X, \mathcal{O}_X^*/n) \\
\downarrow \cong & & \downarrow \eta & & \downarrow \rho \\
H^1_{\text{et}}(X, \mathbb{G}_m) & \xrightarrow{\partial} & H^2_{\text{et}}(X, \mu_n) & \to & H^2_{\text{et}}(X, \mu_n)/\varepsilon^* H^2_{\text{zar}}(X, \mu_n)
\end{array}$$

Since the top row is $H^1(c_{11})$, we conclude that $\partial \equiv \rho \cdot H^1(c_{11})$ modulo $\varepsilon^* H^2_{\text{zar}}(X, \mu_n)$. $\square$

§3. A Mayer-Vietoris sequence

Let X be a quasiprojective variety defined over a field k (with $1/n \in k$), having normalization $\pi : \widetilde{X} \to X$. We let Y denote the scheme $\operatorname{Sing} X$ defined by the conductor ideal, and set $\widetilde{Y} = \pi^{-1}(Y)$, forming the cartesian square

$$\begin{array}{ccc} \widetilde{Y} & \overset{j}{\hookrightarrow} & \widetilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \overset{i}{\hookrightarrow} & X. \end{array} \tag{3.1}$$

In [PW2, 2.1] we proved that, for every torsion abelian sheaf T on the big étale site of X, there is a Mayer-Vietoris sequence in étale cohomology

$$\cdots \to H^q_{\text{et}}(X,T) \to H^q_{\text{et}}(\widetilde{X},T) \oplus H^q_{\text{et}}(Y,T) \to H^q_{\text{et}}(\widetilde{Y},T) \to H^{q+1}_{et}(X,T) \to \cdots.$$

This sequence is natural in X, so we may sheafify in the Zariski topology of X. For $T = \mu_n^{\otimes 2}$, the resulting exact sequence of sheaves on X_{zar} is :

$$\cdots \to \mathcal{H}^q_X \to \mathbf{R}^q\pi_*(\mu_n^{\otimes 2}) \oplus i_*\mathcal{H}^q_Y \to \mathbf{R}^q(\pi j)_*(\mu_n^{\otimes 2}) \to \mathcal{H}^{q+1}_X \to \cdots. \tag{3.1.1}$$

Our notation in (3.1.1) is that $\mathcal{H}^q_X$ and $\mathcal{H}^q_Y$ are the sheaves on X_{zar} and Y_{zar} associated to the presheaf $U \mapsto H^q_{\text{et}}(U, \mu_n^{\otimes 2})$, while $\mathbf{R}^q\pi_*(\mu_n^{\otimes 2})$ and $\mathbf{R}^q(\pi j)_*(\mu_n^{\otimes 2})$ are the sheaves on X_{zar} associated to the presheaves $U \mapsto H^q_{\text{et}}(\pi^{-1}(U), \mu_n^{\otimes 2})$ and $U \mapsto H^q_{\text{et}}((\pi j)^{-1}(U), \mu_n^{\otimes 2})$. We shall also need the sheaves $\mathcal{H}^2_{\widetilde{X}}$ on $\widetilde{X}_{\text{zar}}$ and $\mathcal{H}^2_{\widetilde{Y}}$ on $\widetilde{Y}_{\text{zar}}$, which are defined similarly. We propose to break up (3.1.1) into shorter sequences.

PROPOSITION 3.2. — *If* $\dim(Y) \leq 1$ *and* $\mu_n \subset \mathcal{O}_X$ *then* $\mathbf{R}^2(\pi j)_*(\mu_n^{\otimes 2}) \cong (\pi j)_*\mathcal{H}^2_{\widetilde{Y}}$, *and the map*

$$\mathbf{R}^2\pi_*(\mu_n^{\otimes 2}) \to \mathbf{R}^2(\pi j)_*(\mu_n^{\otimes 2})$$

is onto.

Proof. We proceed stalkwise. If $x \notin Y$ then the stalks of $\mathbf{R}^2(\pi j)_*(\mu_n^{\otimes 2})$ and $(\pi j)_*\mathcal{H}^2_{\widetilde{Y}}$ are zero at x, and there is nothing to prove. Fix $y \in Y$ and let A denote the stalk of $\pi_*\mathcal{O}_{\widetilde{X}}$ at y, i.e, A is the semilocal ring such that $\operatorname{Spec} A = \pi^{-1}(\operatorname{Spec} \mathcal{O}_{X,y})$. The stalk of $(\pi j)_*\mathcal{O}_{\widetilde{Y}}$ at y is A/I, where I is the conductor ideal from A to $\mathcal{O}_{X,y}$. As π is finite, A/I is semilocal of dimension ≤ 1.

The stalk of $\mathbf{R}^2(\pi j)_*(\mu_n^{\otimes 2})$ at y is $H^2_{\text{et}}(A/I, \mu_n^{\otimes 2})$ and the stalk of $(\pi j)_*\mathcal{H}^2_{\widetilde{Y}}$ is $H^0(A/I, \mathcal{H}^2)$. We first show that they are isomorphic. Since $\dim(A/I) \leq 1$, the Leray spectral sequence for $\mu_n^{\otimes 2}$ yields an exact sequence

$$0 \to H^1_{\text{zar}}(A/I, \mathcal{H}^1) \to H^2_{\text{et}}(A/I, \mu_n^{\otimes 2}) \to H^0_{\text{zar}}(A/I, \mathcal{H}^2) \to 0.$$

If $\mu_n \subset A/I$, then $\mathcal{H}^1(\mu_n^{\otimes 2}) \cong \mathcal{H}^1(\mu_n) \cong \mathcal{O}^*/n$. Because A/I is semilocal, we calculate that

$$H^1_{\text{zar}}(A/I, \mathcal{O}^*/n) \cong H^1_{\text{zar}}(A/I, \mathcal{O}^*)/n \cong \text{Pic}(A/I)/n = 0.$$

Hence $H^2_{\text{et}}(A/I, \mu_n^{\otimes 2}) \cong H^0_{\text{zar}}(A/I, \mathcal{H}^2)$, proving the first assertion.

Since the stalk of $\mathbf{R}^2\pi_*(\mu_n^{\otimes 2})$ is $H^2_{\text{et}}(A, \mu_n^{\otimes 2})$, the sheaf map

$$\mathbf{R}^2\pi_*(\mu_n^{\otimes 2}) \to \mathbf{R}^2(\pi j)_*(\mu_n^{\otimes 2})$$

is stalkwise the map $H^2_{\text{et}}(A, \mu_n^{\otimes 2}) \to H^2_{\text{et}}(A/I, \mu_n^{\otimes 2})$. Consider the commutative diagram given by cup product on étale cohomology :

$$\begin{array}{ccc}
H^1_{\text{et}}(A, \mu_n) \otimes H^1_{\text{et}}(A, \mu_n) & \xrightarrow{\cup} & H^2_{\text{et}}(A, \mu_n^{\otimes 2}) \\
\downarrow \text{onto} & & \downarrow \\
H^1_{\text{et}}(A/I, \mu_n) \otimes H^1_{\text{et}}(A/I, \mu_n) & \xrightarrow{\cup} & H^2_{\text{et}}(A/I, \mu_n^{\otimes 2}).
\end{array}$$

The left vertical map is onto because A and A/I are semilocal, so that $H^1_{\text{et}}(A, \mu_n) \cong A^*/n$ and $H^1_{\text{et}}(A/I, \mu_n) \cong (A/I)^*/n$. By Hoobler's Theorem (0.2) we have $H^2_{\text{et}}(A/I, \mu_n^{\otimes 2}) \cong K_2(A/I)/n$. Since $K_2(A/I)$ is generated by symbols, the bottom map is onto. $\square$

COROLLARY 3.3. — *If* $\dim(Y) \leq 1$ *and* $\mu_n \subset \mathcal{O}_X$, *then there is a short exact sequence of sheaves on* X_{zar} *:*

$$0 \to \mathcal{H}^2_X \to \mathbf{R}^2\pi_*(\mu_n^{\otimes 2}) \oplus i_*\mathcal{H}^2_Y \to (\pi j)_*\mathcal{H}^2_{\widetilde{Y}} \to 0.$$

Proof. We must show that the map $\mathbf{R}^1\pi_*(\mu_n^{\otimes 2}) \to \mathbf{R}^1(\pi j)_*(\mu_n^{\otimes 2})$ in (3.1.1) is stalkwise onto, hence onto. Given $y \in Y$, let A be the stalk of $\pi_*\mathcal{O}_{\widetilde{X}}$ at y, and $I \subseteq A$ the conductor ideal from A to $\mathcal{O}_{X,y}$, as in the proof of 3.2. The stalk map is then $H^1_{\text{et}}(A, \mu_n^{\otimes 2}) \to H^1_{\text{et}}(A/I, \mu_n^{\otimes 2})$.

If $\mu_n \subset A$, then $H^1_{\text{et}}(A, \mu_n^{\otimes 2}) \cong A^* \otimes \mu_n$ and $H^1_{\text{et}}(A/I, \mu_n^{\otimes 2}) \cong (A/I)^* \otimes \mu_n$ by Kummer Theory. Therefore the stalk map is onto. $\square$

Remark 3.3.1. The restriction that $\mu_n \subset \mathcal{O}_X$ is not necessary in 3.2; we presume that it is not necessary in 3.3 either. If $n = \ell$ is prime and $\mu_n \not\subset \mathcal{O}_X$, the usual transfer argument involving $B = A[\mu_\ell]$ shows that both 3.2 and 3.3 hold in this case as well.

The long exact cohomology sequence attached to the sequence of sheaves in 3.3 is

$$\begin{aligned}(3.4)\qquad \cdots \to H^1(X, \mathcal{H}^2) \to H^1(X, \mathbf{R}^2\pi_*\mu_n^{\otimes 2}) \oplus H^1(Y, \mathcal{H}^2) \to H^1(\widetilde{Y}, \mathcal{H}^2)\\ \to H^2(X, \mathcal{H}^2) \to H^2(X, \mathbf{R}^2\pi_*\mu_n^{\otimes 2}) \to 0.\end{aligned}$$

Our next result compares the cohomology of the sheaf $\mathbf{R}^q\pi_*\mu_n^{\otimes 2}$ to the cohomology groups $H^i(\widetilde{X}, \mathcal{H}^q_{\widetilde{X}}) = H^i(X, \pi_*\mathcal{H}^q_{\widetilde{X}})$.

PROPOSITION 3.5. — *Let X be a surface over a field k containing $1/n$, with normalization $\pi : \widetilde{X} \to X$, and fix $q \geq 0$. Then the natural map*

$$H^i(X, \mathbf{R}^q\pi_*(\mu_n^{\otimes 2})) \to H^i(\widetilde{X}, \mathcal{H}^q_{\widetilde{X}})$$

is an isomorphism for all $i \geq 2$ and onto for $i = 1$. If $\widetilde{X}$ is regular it is an isomorphism for all i.

Proof. First suppose that $\widetilde{X}$ is regular. By the Bloch-Ogus style argument of [PW2, 2.7], $\mathbf{R}^q\pi_*(\mu_n^{\otimes 2}) \cong \pi_*\mathcal{H}^q_{\widetilde{X}}$. In general the normal surface $\widetilde{X}$ has only isolated singularities, so the kernel and cokernel of the natural map $\eta^q : \mathbf{R}^q\pi_*(\mu_n^{\otimes 2}) \to \pi_*\mathcal{H}^q_{\widetilde{X}}$ are supported on the finite set $\pi_* \operatorname{Sing}\widetilde{X}$. Now apply 1.3. $\square$

Remark 3.6. As observed in [PW2, 2.4] the map $\mathbf{R}^q\pi_*(\mu_n^{\otimes 2}) \to \pi_*\mathcal{H}^q_{\widetilde{X}}$ is not an isomorphism in general. For $y \in Y$ let A be the semilocal ring of $\widetilde{X}$ at the finite set $\pi^{-1}(y)$. Then there is a Leray spectral sequence converging to the stalk at y :

$$E_2^{p,q} = H^p_{\text{zar}}(A, \mathcal{H}^q) \Rightarrow H^{p+q}_{\text{et}}(A, \mu_n^{\otimes 2}) = (\mathbf{R}^{p+q}\pi_*\mu_n^{\otimes 2})_y.$$

Since A is semilocal and normal of dimension 2 the sheaves $\mathcal{H}^0$ and $\mathcal{H}^1$ on $\operatorname{Spec} A$ have flasque resolutions of lengths respectively 2 and 3; see (4.1.2). In particular $H^i(X,\mathcal{H}^0) = 0$ for $i \geq 2$, so the spectral sequence degenerates to the exact sequence

$$0 \to H^1(A,\mathcal{H}^1) \to (\mathbf{R}^2\pi_*\mu_n^{\otimes 2})_y \to (\pi_*\mathcal{H}^2_{\widetilde{X}})_y \to H^2(A,\mathcal{H}^1) \to (\mathbf{R}^3\pi_*\mu_n^{\otimes 2})_y.$$

If k contains a primitive n^{th} root of 1, then $\mathcal{H}^1(\mu_n^{\otimes 2}) \cong \mathcal{K}_1/n \cong \mathcal{O}_X^*/n$ and the group $H^1(A,\mathcal{H}^1)$ may be described via the exact sequence in [BV1, 3.4] :

$$0 \to \operatorname{Pic}(A)/n \to H^1(A,\mathcal{H}^1) \to {}_nH^2(A,\mathcal{O}_X^*) \to 0.$$

Since A is semilocal this yields an isomorphism : $H^1(A,\mathcal{H}^1) \cong {}_nH^2(A,\mathcal{K}_1)$. This group may be nonzero for a seminormal local ring A of dimension 2 ; see [PW1, 5.9] and 4.2.1.

THEOREM B. — *Let X be a surface defined over a field k containing $1/n$ and μ_n. Set $Y = \operatorname{Sing} X$ and $\widetilde{Y} = \pi^{-1}(Y)$ where $\pi : \widetilde{X} \to X$ is the normalisation of X. Then there is an exact sequence for the sheaf $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$:*

$$H^1(\widetilde{X},\mathcal{H}^2) \oplus H^1(Y,\mathcal{H}^2) \to H^1(\widetilde{Y},\mathcal{H}^2) \to H^2(X,\mathcal{H}^2) \to H^2(\widetilde{X},\mathcal{H}^2) \to 0.$$

Proof. Use the cohomology exact sequence (3.4) and 3.5, which gives an isomorphism

$$H^2(X,\mathbf{R}^2\pi_*\mu_n^{\otimes 2}) \cong H^2(\widetilde{X},\mathcal{H}^2)$$

and a surjection

$$H^1(X,\mathbf{R}^2\pi_*\mu_n^{\otimes 2}) \to H^1(\widetilde{X},\mathcal{H}^2). \quad \square$$

COROLLARY C. — *With the same hypotheses as in Theorem B there is an exact sequence :*

$$H^1(\widetilde{X},\mathcal{H}^2) \oplus SK_1(Y)/n \to SK_1(\widetilde{Y})/n \to CH^2(X)/n \to CH^2(\widetilde{X})/n \to 0.$$

Proof. Theorem A yields the two isomorphisms $H^2(X,\mathcal{H}^2) \cong CH^2(X)/n$ and $H^2(\widetilde{X},\mathcal{H}^2) \cong CH^2(\widetilde{X})/n$. So we are left to show that $H^1(Y,\mathcal{H}^2)$ is $H^1(Y,\mathcal{K}_2/n) \cong SK_1(Y)/n$ and that $H^1(\widetilde{Y},\mathcal{H}^2)$ is $H^1(\widetilde{Y},\mathcal{K}_2/n) \cong SK_1(\widetilde{Y})/n$. This follows from [PW2, 5.1] when Y and $\widetilde{Y}$ are reduced, and from 1.7 in the general case. (Of course, it also follows from Hoobler's Theorem 0.2.) $\square$

§4. Proof of Theorem D

In order to prove theorem D, we shall need flasque resolutions for each of the sheaves $\mathcal{H}^1$, $\mathcal{H}^2$, $\mathcal{K}_2$, $n\cdot\mathcal{K}_2$, $\mathcal{K}_2/n$ and ${}_n\mathcal{K}_2$ on a surface with isolated singularities. The following standard method is implicit in [PW1, (5.1)]; in our applications, X will always be a scheme with isolated singularities over a field k, and Y will be a finite set containing $\operatorname{Sing} X$.

If X is a scheme, we set $X^i = \{x \in X \text{ of codimension i}\}$. If $x \in X$ we shall write $i_x M$ for the direct image sheaf of an abelian group M under $x \hookrightarrow X$.

Resolutions 4.1. Let Y be a finite set of closed points of a scheme X. Given functors F^p $(p = 0, 1, 2)$ on schemes over k, let $\mathcal{F}^p$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto F^p(U)$. Suppose given a flasque resolution of $\mathcal{F}^0$ on $X - Y$:

$$0 \to \mathcal{F}^0|(X-Y) \xrightarrow{\varepsilon} \coprod_{s\in X^0} i_s F^0(s) \xrightarrow{\partial} \coprod_{t\in X^1} i_t F^1(t) \xrightarrow{\partial} \coprod_{\substack{x\in X^2\\ x\notin Y}} i_x F^2(x) \to 0.$$

If $\dim(X) \geq 2$, the following is a flasque resolution of $\mathcal{F}^0$ on X.

(4.1.1)

$$0 \to \mathcal{F}^0 \to \coprod_{s\in X^0} i_s F^0(s) \oplus \coprod_{y\in Y} i_y(\mathcal{F}^0_y) \xrightarrow{\begin{pmatrix} \partial & 0 \\ -\Delta & \varepsilon \end{pmatrix}}$$

$$\coprod_{t\in X^1} i_t F^1(t) \oplus \coprod_{y\in Y}\coprod_{s<y} i_y F^0(s) \xrightarrow{\begin{pmatrix} \partial & 0 \\ \Delta & \partial \end{pmatrix}} \coprod_{\substack{x\in X^2\\ x\notin Y}} i_x F^2(x) \oplus \coprod_{y\in Y}\coprod_{t<y} i_y F^1(t) \to 0$$

Here the notation $s < y$ (resp. $t < y$) means the set of all points $s \in X^0$ (resp. $t \in X^1$) whose closure contains y, and Δ is the obvious diagonal.

Now suppose that X is a scheme with isolated singularities over a field k, and that Y contains $\operatorname{Sing} X$. Setting $F^0 = H^0(-, \mu_n^{\otimes 2})$ and $F^1 = F^2 = 0$ yields the flasque resolution

(4.1.2)

$$0 \to \mathcal{H}^0 \to \coprod_s i_s H^0(s, \mu_n^{\otimes 2}) \oplus \coprod_y i_y H^0(\mathcal{O}_{X,y}, \mu_n^{\otimes 2}) \to \coprod_{s<y} i_y H^0(s, \mu_n^{\otimes 2}) \to 0.$$

This yields the observation that $H^i(X, \mathcal{H}^0) = 0$ for $i \geq 2$. Similarly, if we set $F^0 = H^1(-, \mu_n^{\otimes 2})$ and $F^1 = \mu_n$, we obtain the flasque resolution

(4.1.3)

$$0 \to \mathcal{H}^1 \to \begin{bmatrix} \coprod_s i_s H^1(s, \mu_n^{\otimes 2}) & \oplus \\ \coprod_y i_y H^1(\mathcal{O}_{X,y}, \mu_n^{\otimes 2}) & \end{bmatrix} \to \begin{bmatrix} \coprod_{t\in X^1} i_t \mu_n(t) & \oplus \\ \coprod_{s<y} i_y H^1(s, \mu_n^{\otimes 2}) & \end{bmatrix} \to \coprod_{t<y} i_y \mu_n(t) \to 0.$$

(Cf. [BV1, (3.6)].) Consequently $H^i(X,\mathcal{H}^1)=0$ for $i\geq 3$. As another example, let ${}_nK_2$ be the kernel of $K_2 \xrightarrow{n} K_2$. On the smooth variety $X-Y$ we have the resolution

$$0\to {}_n\mathcal{K}_2 \xrightarrow{\varepsilon} \coprod_s i_s({}_nK_2(s)) \xrightarrow{\partial} \coprod_t i_t\mu_n(t)\to 0$$

from the universal exactness of the Gersten resolution for K_2 given in [Gray, Cor. 6]; *cf.* [MS, 8.7.8]. Hence we have a resolution for the sheaf ${}_n\mathcal{K}_2$ on X which is identical to (4.1.3) except that $H^1(-,\mu_n^{\otimes 2})$ has been replaced by ${}_nK_2$.

PROPOSITION 4.2. — *Let X be a quasiprojective scheme with isolated singularities, defined over a field k containing $1/n$. Then*

$$H^2(X,{}_n\mathcal{K}_2)\cong H^2(X,\mathcal{H}^1(\mu_n^{\otimes 2})).$$

In particular, if k contains μ_n and $\dim(X)\geq 2$ *then $\mathcal{H}^1(\mu_n^{\otimes 2})\cong \mathcal{O}_X^*/n$ and*

$$H^2(X,\mathcal{H}^1)\cong H^2(X,\mathcal{O}_X^*)/n\cong [\oplus_{y\in Y}Cl(\mathcal{O}_{X,y})/n]/image\ of\ Cl(A),$$

where A is the semilocal ring of X at Y and $Cl(A)$ denotes the divisor class group of A.

Proof. By [Suslin, 3.13], there is a surjection $H^1_{\text{et}}(F;\mu_n^{\otimes 2})\to {}_nK_2(F)$ for any field F. In addition, the following diagram commutes by [Suslin, 3.14].

$$\begin{array}{ccccccc}
\coprod_t \mu_n(t)\oplus\coprod_{s<y} H^1(s,\mu_n^{\otimes 2}) & \xrightarrow{(\Delta,\partial)} & \coprod_{t<y}\mu_n(t) & \longrightarrow & H^2(X,\mathcal{H}^1) & \to & 0\\
\text{onto}\downarrow & & \| & & \downarrow & & \\
\coprod_t \mu_n(t)\oplus\coprod_{s<y} {}_nK_2(s) & \xrightarrow{(\Delta,\partial)} & \coprod_{t<y}\mu_n(t) & \longrightarrow & H^2(X,{}_n\mathcal{K}_2) & \to & 0
\end{array}$$

The right vertical map is an isomorphism by the 5-lemma.

Now suppose that k contains a primitive n^{th} root of unity ξ. The map $k(s)^*/n\to {}_nK_2(s)$ sending f to $\{\xi,f\}$ is onto by [MS]. If we tensor the exact sequence of [PW1, (5.4)]

$$\coprod_t \mathbb{Z}\oplus\coprod_{s<y} k(s)^*\to\coprod_{t<y}\mathbb{Z}\to H^2(X,\mathcal{O}_X^*)\to 0$$

with $\mu_n(k) \cong \mathbb{Z}/n$, we obtain an isomorphism $H^2(X, \mathcal{O}_X^*)/n \cong H^2(X, {}_n\mathcal{K}_2)$. The final interpretation in terms of divisor class groups is just a restatement of [PW1, 5.5]. $\square$

Remark 4.2.1. The group $H^2(X, \mathcal{H}^1)$ can be nonzero if X has more than one singular point. An example of a normal surface with $H^2(X, \mathcal{H}^1) = H^2(X, \mathcal{O}_X^*) = \mathbb{Z}/2$ is given in [PW, 5.9]. That example is easily modified to replace $\mathbb{Z}/2$ by $\mathbb{Z}/n$ when $\mu_n \subset k$.

For Theorem D, we shall need to consider flasque resolutions of $\mathcal{K}_2/n$ and $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$. On $X - Y$ it is well-known (see [B3, 2.3], [MS, §18], [CT-R, p.168] and [PW2, 4.3]) that the flasque resolution

$$(4.3.1) \qquad 0 \to \mathcal{K}_2/n \to \coprod_{s \in X^0} i_s K_2(s)/n \to \coprod_{t \in X^1} i_t k(t)^*/n \to \coprod_{x \in X^2} i_x \mathbb{Z}/n \to 0$$

may be identified with the Bloch-Ogus flasque resolution of $\mathcal{H}^2$, giving an isomorphism $c_{22} : \mathcal{K}_2/n \to \mathcal{H}^2$ of sheaves on $X - Y$. The method of 4.1 gives a morphism of flasque resolutions covering $c_{22} : \mathcal{K}_2/n \to \mathcal{H}^2$. In fact, these resolutions are isomorphic by Hoobler's theorem (0.2).

Here is a useful variation which uses the resolution (4.1.1) to establish acyclicity for $j_*j^*(\mathcal{H}^2)$. Let X_Y denote the semilocal scheme of X at the finite set Y, and let $j : X_Y \hookrightarrow X$ denote the inclusion.

LEMMA 4.4. — *If X is a surface with isolated singularities, the sheaves $j_*j^*(\mathcal{K}_2/n)$ and $j_*j^*(\mathcal{H}^2)$ are isomorphic and acyclic on X.*

Proof. The sheaves are isomorphic by Hoobler's theorem (0.2). The method of 4.1 gives us a flasque resolution
(4.4.1)

$$\begin{aligned} 0 \to j_*j^*(\mathcal{K}_2/n) \to \coprod_{s \in X_Y^0} i_s K_2(s)/n \oplus \coprod_{y} i_y K_2(\mathcal{O}_{X,y})/n \to \\ \coprod_{t \in X_Y^1} i_t k(t)^*/n \oplus \coprod_{s<y} i_y K_2(s)/n \to \coprod_{t<y} i_y k(t)^*/n \to 0. \end{aligned}$$

(*cf.* [PW1, 4.1].) In particular, $H^*(X, j_*j^*\mathcal{K}_2/n) \cong H^*(X_Y, \mathcal{K}_2/n)$.

Now X_Y is semilocal, so $H^2(X_Y, \mathcal{K}_2) \cong SK_0(X_Y) = 0$, and $H^1(X_Y, \mathcal{K}_2) = 0$ as well, because it is a quotient of $SK_1(X_Y) = 0$. That is, $\mathcal{K}_2$ is acyclic on X_Y (*cf.* [PW1, 6.5]). Since $\mathcal{K}_2/n$ and $n \cdot \mathcal{K}_2$ are both quotients of $\mathcal{K}_2$, and $H^2(X_Y, -)$ is right exact, we get $H^2(X_Y, \mathcal{K}_2/n) = H^2(X_Y, n \cdot \mathcal{K}_2) = 0$. From the cohomology sequence associated to the exact sequence of sheaves

$$0 \to n \cdot \mathcal{K}_2 \to \mathcal{K}_2 \to \mathcal{K}_2/n \to 0,$$

we conclude that $H^1(X_Y, \mathcal{K}_2/n) \cong H^2(X_Y, n \cdot \mathcal{K}_2) = 0$. $\square$

From the standard resolution (4.1.1) of $\mathcal{K}_2/n$ on X associated to (4.3.1), it follows that every element of $H^1(X, \mathcal{K}_2/n) \cong H^1(X, \mathcal{H}^2)$ may be represented by a family (f_t, w_{sy}) with $f_t \in k(t)^*/n$ and $w_{sy} \in K_2(s)/n$. We shall need a simpler family of representatives with $w_{sy} = 0$ and $f_t = 1$ whenever $t < y$ for some y. To this end, we give another acyclic resolution of $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$, following [PW1, 3.1]. Recall that for any sheaf $\mathcal{F}$ on X there is a natural adjunction map $\eta : \mathcal{F} \to j_*j^*\mathcal{F}$.

PROPOSITION 4.5. — *Let X be a surface with isolated singularities, and suppose that Y contains* $\operatorname{Sing} X$ *and meets every irreducible component of X. Let X^p_* denote the set of points in X^p whose closure misses Y. Then the following diagram commutes, and its rows are acyclic resolutions of $\mathcal{K}_2/n$ and $\mathcal{H}^2$.*

$$\begin{array}{ccccccccc}
0 \to & \mathcal{K}_2/n & \xrightarrow{\eta} & j_*j^*(\mathcal{K}_2/n) & \to & \coprod_{t\in X^1_*} i_t k(t)^*/n & \to & \coprod_{x\in X^2_*} i_x \mathbb{Z}/n & \to 0 \\
& \cong\downarrow c_{22} & & \cong\downarrow c_{22} & & \| & & \| & \\
0 \to & \mathcal{H}^2 & \xrightarrow{\eta} & j_*j^*\mathcal{H}^2 & \to & \coprod_{t\in X^1_*} i_t k(t)^*/n & \to & \coprod_{x\in X^2_*} i_x \mathbb{Z}/n & \to 0.
\end{array}$$

Proof (*cf.* [PW1, 3.2]). On $X - Y$, we have the flasque resolution (4.3.1) of $\mathcal{K}_2/n$ and $\mathcal{H}^2$. Therefore we get flasque resolutions of the type (4.1.1) on X, and a morphism of resolutions covering $\mathcal{K}_2/n \to \mathcal{H}^2$. The resolution of $\mathcal{K}_2/n$ maps onto the flasque resolution (4.4.1) of $j_*j^*\mathcal{K}_2/n$; since every component of X meets Y, the kernel complex is :

$$0 \to 0 \to \coprod_{X^1_*} i_t k(t)^*/n \to \coprod_{X^2_*} i_x \mathbb{Z}/n \to 0.$$

But this is also the kernel complex of the map from the resolution of $\mathcal{H}^2$ to the resolution of $j_*j^*\mathcal{H}^2$. The result now follows from 4.4 and a diagram chase. $\square$

Porism 4.5.1. Using the resolution of type (4.1.1), we can represent any element of $H^1(X, \mathcal{K}_2/n)$ by an element of $\coprod_{t\in X^1} k(t)^*/n \oplus \coprod_{s<y} K_2(s)/n$.

The proof of 4.5 shows that this element differs by a coboundary from an element of the subgroup $\coprod_{t\in X^1_*} k(t)^*/n$.

Definition 4.6. Let $\mathcal{H}^3_{sm}$ be the Zariski sheaf in X defined by

$$0 \to \mathcal{H}^3_{sm} \to \coprod_{t\in X^1_*} i_t k(t)^*/n \xrightarrow{\partial} \coprod_{x\in X^2_*} i_x \mathbb{Z}/n \to 0.$$

By construction, the acyclic resolution of 4.5 gives an exact sequence :

$$0 \to \mathcal{H}^2(\mu_n^{\otimes 2}) \to j_* j^* \mathcal{H}^2(\mu_n^{\otimes 2}) \to \mathcal{H}^3_{sm} \to 0.$$

This sequence is due to L. Barbieri-Viale [BV2], who identified $\mathcal{H}^3_{sm}$ with the sheaf associated to smooth étale cohomology H^3_{sm}. Taking cohomology, we get an exact sequence

$$H^0(X_Y, \mathcal{H}^2) \to H^0(X, \mathcal{H}^3_{sm}) \to H^1(X, \mathcal{H}^2) \to 0. \tag{4.6.1}$$

Every element of $H^0(X, \mathcal{H}^3_{sm})$ is represented by a family of elements $\bar{f}_t \in k(t)^*/n$ such that $\sum \partial \bar{f}_t = 0$. Since the Bloch-Ogus differential ∂ is the reduction of the divisor map, any choice $f_t \in k(t)^*$ of liftings of the $\bar{f}_t$ determines a divisor D (of codimension 2 on X) such that the divisor $\sum (f_t)$ equals nD.

LEMMA 4.6.2. — *The class of the divisor D in $CH^2(X,Y)$ is independent of the choices made, so $\tilde{\tau}(\{\bar{f}_t\}) = [D]$ determines a homomorphism*

$$\tilde{\tau} : H^0(X, \mathcal{H}^3_{sm}) \to CH^2(X,Y).$$

The image of $\tilde{\tau}$ is the n-torsion subgroup ${}_nCH^2(X,Y)$ of $CH^2(X,Y)$.

Proof. If f'_t is another lifting of $\bar{f}_t$, then there are $g_t \in k(t)^*$ such that $f'_t = f_t g_t^n$. Therefore $\sum (f'_t) = nD + n\sum (g_t)$. Since $[D] = [D + \sum (g_t)]$ in $CH^2(X,Y)$, $[D]$ is independent of the choice, and $\tilde{\tau}$ is well-defined. Since $H^0(X, \mathcal{H}^3_{sm})$ has exponent n, so does the image of $\tilde{\tau}$. Conversely, if D is a divisor with $n[D] = 0$ in $CH^2(X,Y)$, there are $f_t \in k(t)^*$, $t \in X^1_*$, such that $nD = \sum (f_t)$ and therefore $[D] = \tilde{\tau}(\{\bar{f}_t\})$. □

PROPOSITION 4.7. — *The map $\tilde{\tau}$ induces a surjection*

$$\tau : H^1(X, \mathcal{H}^2) \to {}_nCH^2(X,Y).$$

Proof. Combining (4.6.1) with 4.6.2, we see that it suffices to show that the image of $H^0(X_Y, \mathcal{H}^2) \to H^0(X, \mathcal{H}^3_{sm})$ lies in the kernel of $\tilde{\tau}$. Using the resolution (4.4.1) for $\mathcal{H}^2$ on X_Y, we can represent an element of $H^0(X_Y, \mathcal{H}^2)$ by a family of elements $\alpha_y \in H^0(\mathcal{O}_{X,y}, \mu_n^{\otimes 2})$, $y \in Y$, such that : (i) the images $\alpha_s \in H^2(s, \mu_n^{\otimes 2})$ are independent of the pair $s < y$ for each $s \in X^0$, and (ii) $\partial_Y(\sum \alpha_s) = 0$, where ∂_Y is the map

$$\coprod_{s \in X^0} H^2(s, \mu_n^{\otimes 2}) \to \coprod_{t \in X_Y^1} k(t)^*/n.$$

The image β of $\sum \alpha_s$ in $\coprod\{k(t)^*/n \ : \ t \in X_*^1\}$ represents the image of this element under the map $H^0(X_Y, \mathcal{H}^2) \to H^0(X, \mathcal{H}^3_{sm})$. To compute $\tilde{\tau}(\beta)$, we consider the commutative diagram of Gersten-Quillen complexes for $\mathcal{K}_2 \to \mathcal{K}_2/n$ on $X - Y$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \coprod_{s \in X^0} K_2(s) & \xrightarrow{\text{tame}} & \coprod_{t \in X^1} k(t)^* & \xrightarrow{\text{div}} & \coprod_{x \in X_*^2} \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \coprod_{s \in X^0} K_2(s)/n & \xrightarrow{\partial} & \coprod_{t \in X^1} k(t)^*/n & \xrightarrow{\partial} & \coprod_{x \in X_*^2} \mathbb{Z}/n & \longrightarrow & 0
\end{array}$$

As with (4.3.1), we can identify the bottom complex with the Bloch-Ogus complex for $\mathcal{H}^2$. Lifting each α_s to an element $\tilde{\alpha}_s \in K_2(s)$ and setting $\sum f_t = \sum \text{tame}(\tilde{\alpha}_s)$, the associated divisor $\sum (f_t)$ vanishes because $\text{div}(\text{tame}(\tilde{\alpha}_s)) = 0$. Since $\tilde{\tau}(\beta)$ is defined via $\sum (f_t)$, we have $\tilde{\tau}(\beta) = 0$. □

By abuse of notation, we will also use the symbol τ to denote the map from $H^1(X, \mathcal{K}_2/n)$ to $CH^2(X) = H^2(X, \mathcal{K}_2)$ obtained by precomposing τ with $H^1(X, \mathcal{K}_2/n) \to H^1(X, \mathcal{H}^2)$. Let $n \cdot \mathcal{K}_2$ denote the image of $\mathcal{K}_2 \xrightarrow{n} \mathcal{K}_2$, and let π denote the quotient map $\mathcal{K}_2 \to n \cdot \mathcal{K}_2$.

COROLLARY 4.7.1. — *The composite*

$$H^1(X, \mathcal{K}_2/n) \xrightarrow{\tau} H^2(X, \mathcal{K}_2) \xrightarrow{\pi} H^2(X, n \cdot \mathcal{K}_2)$$

is the boundary map in the cohomology sequence associated to the exact sequence

$$0 \to n \cdot \mathcal{K}_2 \to \mathcal{K}_2 \to \mathcal{K}_2/n \to 0.$$

Proof. Since the Gersten-Quillen resolution of $\mathcal{K}_2$ is universally exact [Gray, Cor. 6], we have a short exact sequence of resolutions on $X - Y$:

$$\begin{array}{ccccccccc}
0 & \to & n\cdot\mathcal{K}_2 & \to & \coprod i_s n\cdot K_2(s) & \to & \coprod i_t k(t)^{*n} & \to & \coprod i_x n\cdot\mathbb{Z} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{K}_2 & \to & \coprod i_s K_2(s) & \to & \coprod i_t k(t)^* & \to & \coprod i_x\mathbb{Z} & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \mathcal{K}_2/n & \to & \coprod i_s K_2(s)/n & \to & \coprod i_t k(t)^*/n & \to & \coprod i_x\mathbb{Z}/n & \to 0.
\end{array}$$

The method of 4.1 gives a short exact sequence of resolutions on X of type (4.1.1). By 4.5, an element f of $H^1(X,\mathcal{K}_2/n)$ is represented by a family $\{\bar{f}_t \in k(t)^*/n, t \in X^1_*\}$. Lift $\bar{f}_t$ to $f_t \in k(t)^*$; the boundary map sends f to the element of $H^2(X, n\cdot\mathcal{K}_2)$ represented by the divisor $nD = \sum(f_t)$. But by construction $\pi\tau(f) = \pi([D]) = [nD]$. □

LEMMA 4.8. — *The composite $H^1(X,\mathcal{K}_2) \to H^1(X,\mathcal{K}_2/n) \xrightarrow{\tau} CH^2(X,Y)$ is zero.*

Proof. Using the resolution [PW1, 3.2] of $\mathcal{K}_2$, which is the integral analogue of 4.5, an element $\lambda \in H^1(X,\mathcal{K}_2)$ is represented by a family of units $\{f_t \in k(t)^*, t \in X^1_*\}$ whose total divisor $\sum(f_t)$ is zero. Reducing modulo n, the image $\bar{\lambda}$ of λ in $H^1(X,\mathcal{K}_2/n)$ is represented by $\{\bar{f}_t\}$. Using this choice in the construction of τ shows that $\tau(\bar{\lambda}) = 0$. □

COROLLARY 4.8.1. — *In the cohomology sequence*

$$H^1(X, n\cdot\mathcal{K}_2) \xrightarrow{\partial} H^2(X, {}_n\mathcal{K}_2) \to H^2(X,\mathcal{K}_2) \xrightarrow{\pi} H^2(X, n\cdot\mathcal{K}_2) \to 0,$$

the map ∂ is onto and there is an isomorphism $\pi : H^2(X,\mathcal{K}_2) \cong H^2(X, n\cdot\mathcal{K}_2)$.

Proof. Using 4.7.1, we have constructed a commutative diagram

$$\begin{array}{ccccccc}
H^1(X,\mathcal{K}_2) & \longrightarrow & H^1(X,\mathcal{K}_2/n) & \longrightarrow & H^2(X, n\cdot\mathcal{K}_2) & \longrightarrow & H^2(X,\mathcal{K}_2) \\
 & & \downarrow\tau & & \| & & \\
 & & H^2(X,\mathcal{K}_2) & \xrightarrow{\pi} & H^2(X, n\cdot\mathcal{K}_2) & \longrightarrow & 0
\end{array}$$

with exact rows in which the image of τ is the kernel of the composite $H^2(X,\mathcal{K}_2) \xrightarrow{n} H^2(X,\mathcal{K}_2)$. Hence $\ker(\pi) \subseteq \mathrm{im}(\pi)$. It follows from 4.8 that π is an isomorphism, and hence that ∂ is onto. □

PROPOSITION 4.9 (= Theorem D). — *There is an exact sequence :*

$$H^0(X,\mathcal{K}_2/n)\to H^2(X,\mathcal{H}^1)\to H^1(X,\mathcal{K}_2)/n\to H^1(X,\mathcal{K}_2/n)\xrightarrow{\tau} {}_nCH^2(X)\to 0.$$

Proof. Combine the cohomology exact sequence of 4.7.1,

$$H^0(X,\mathcal{K}_2/n)\to H^1(X,n\cdot\mathcal{K}_2)\to H^1(X,\mathcal{K}_2)\to H^1(X,\mathcal{K}_2/n)\xrightarrow{\pi\tau} H^2(X,n\cdot\mathcal{K}_2),$$

with 4.8.1 and 4.2, which gives the exact sequence

$$H^1(X,\mathcal{K}_2)\to H^1(X,n\cdot\mathcal{K}_2)\xrightarrow{\partial} H^2(X,\mathcal{H}^1)\to 0,$$

together with the observation that multiplication by n equals the composite $H^1(X,\mathcal{K}_2)\to H^1(X,n\cdot\mathcal{K}_2)\to H^1(X,\mathcal{K}_2)$. □

§5. Applications

In this section we apply our structural results to describe $CH^2(X)$ when X is a reduced surface over a field k which is either algebraically closed, a number field or the field $\mathbb{R}$ of real numbers.

If Y is a closed subspace of X containing $\mathrm{Sing}(X)$, the relative Chow group $CH^2(X,Y)$ is defined as the quotient of the free abelian group on the set X^2_* of points of codimension 2 in $X-Y$, modulo the subgroup generated by divisors of functions $f\in k(Z)^*$, where Z is a closed curve on X locally defined by a regular function, having no components in common with Y, and the support of the divisor (f) belongs to $X-Y$. If Y is finite then this agrees with the group $CH^2(X,Y)$ of [PW1] used in the proof of Theorem D. We know by [LW] that $CH^2(X,Y)$ is isomorphic to $SK_0(X)$, and hence that it is independent of Y, so we shall omit Y from the notation and write $CH^2(X)$ for $CH^2(X,Y)$.

There is a natural map from $CH^2(X)$ to the classical group $CH_0(X)$ of cycles of dimension zero modulo rational equivalence. For singular surfaces X we do not always have $CH^2(X)\cong CH_0(X)$, as [PW1, 2.6] and 5.6.1 below show.

It is useful to restrict our attention to a slightly smaller group, the subgroup $A_0(X)$ of 0-cycles of degree 0 in $CH^2(X)$. If X_i is a component of X which is proper, there is a nonzero degree map $CH^2(X_i)\to\mathbb{Z}$; it is onto if $k=\bar{k}$ or if $\tilde{X}_i(k)\neq\emptyset$. If X has c proper components we define $A_0(X)=A_0(X,Y)$ to be the kernel of the resulting degree map

$$\deg : CH^2(X)\to\oplus CH^2(X_i)\to\mathbb{Z}^c.$$

If k is algebraically closed, the degree map is a split surjection and we have $CH^2(X) \cong \mathbb{Z}^c \oplus A_0(X)$. In general the image Λ of deg is a lattice isomorphic to $\mathbb{Z}^c$, and we have a decomposition :

$$(5.0) \qquad CH^2(X) \cong \Lambda \oplus A_0(X), \qquad \Lambda \cong \mathbb{Z}^c.$$

Clearly this yields a direct summand $\Lambda/n \cong (\mathbb{Z}/n)^c$ of $CH^2(X)/n \cong SK_0(X)/n$. If k is algebraically closed, we now show that the other summand vanishes.

THEOREM 5.1. — *Let X be a surface defined over an algebraically closed field k.*
If X has isolated singularities, $A_0(X)$ is divisible and $CH^2(X)/n \cong (\mathbb{Z}/n)^c$ for all n.
If $\mathrm{char}(k) = 0$, *then $A_0(X)$ is divisible and $CH^2(X)/n \cong (\mathbb{Z}/n)^c$ for all n.*
If $\mathrm{char}(k) \neq 0$ *and $1/n \in k$, $A_0(X)$ is n-divisible and $CH^2(X)/n \cong (\mathbb{Z}/n)^c$.*

Proof. First suppose that $Y = \mathrm{Sing}\, X$ is finite. If D is a 0-cycle of degree zero on $X - Y$ then by Bertini's Theorem we can find a curve Z missing Y but containing the support of D. The resulting map $\pi : \widetilde{Z} \to X$ from the normalization $\widetilde{Z}$ of Z induces a map from $\mathrm{Pic}_0(\widetilde{Z})$ to $A_0(X)$, and [D] is in the image of this map. Now $\mathrm{Pic}_0(\widetilde{Z})$ is a divisible abelian group by [Mum, p.62]. Thus [D] is divisible in $A_0(X)$.

In the general case, the proper components of X are in bijective correspondence with the proper components of $\widetilde{X}$, so the above proves that $CH^2(\widetilde{X})/n \cong (\mathbb{Z}/n)^c$. By corollary C, $A_0(X)/n$ is a quotient of $SK_1(\widetilde{Y})/n$, which is zero by lemma 5.2 below. □

Remark 5.1.1. If X has isolated singularities, then $CH^2(X) \cong CH^2(\widetilde{X})$ because $SK_1(\widetilde{Y}) = 0$. (This follows from the exact sequence (A.2) of the appendix; see A.3 or [PW1, 8.6]). It then follows from [C] that ${}_nCH^2(X)$ is finite for every n prime to $\mathrm{char}(k)$, an upper bound being the n-torsion in the Chow group $CH^2(\overline{X})$ of the projective closure $\overline{X}$ of $\widetilde{X}$. If X is affine, then we actually know that ${}_nCH^2(X) = 0$ by [L2, 2.6].

However, when $\mathrm{Sing}(X)$ is 1-dimensional we cannot bound ${}_nCH^2(X)$ unless we know more about $SK_1(\widetilde{Y})$, $\widetilde{Y} = \pi^{-1}(\mathrm{Sing}\, X)$.

LEMMA 5.2. — *Let Y be a 1-dimensional quasiprojective scheme of finite type over an algebraically closed field k. Then $SK_1(Y)$ is n-divisible for every n prime to* $\mathrm{char}(k)$.

Proof. Suppose first that Y is reduced. By [Gil, 1.27], the map

$$\mathrm{Pic}(Y) \otimes k^* \to H^1(Y, \mathcal{K}_2) \cong SK_1(Y)$$

is onto. Since k^* is n-divisible, so is $SK_1(Y)$. If Y is not reduced then by 1.7 we have

$$SK_1(Y)/n = SK_1(Y_{\mathrm{red}})/n = 0. \qquad \square$$

Remark 5.2.1. If Y is affine then $SK_1(Y) = SK_1(Y_{\mathrm{red}})$. In general, however, $SK_1(Y) \to SK_1(Y_{\mathrm{red}})$ is neither injective nor surjective; see [W2, ex.2].

We now briefly consider the situation where k is a number field. If X is smooth, the situation is somewhat conjectural. We cannot hope for the singular case to be any better, and the following example shows that it is worse.

Example 5.3. Let $\widetilde{Y}$ be the coordinate axes ($xy = 0$) in the affine plane $\widetilde{X} = \mathbb{A}^2_k$, and let X be the affine scheme obtained by glueing the two axes together into a line $Y \cong \mathbb{A}^1$, as in [P]. The coordinate ring for X is the subring $\{f \in k[x,y] : f(t,0) = f(0,t)\}$ of $k[x,y]$. It is classical that $SK_1(Y) = SK_1(\widetilde{X}) = SK_0(\widetilde{X}) = 0$; an almost equally classical Mayer-Vietoris argument (in the style of [P]) shows that

$$CH^2(X) \cong SK_0(X) \cong SK_1(\widetilde{Y}) \cong K_2(k).$$

If k is a number field, this group is a direct sum of infinitely many finite cyclic groups, and both $CH^2(X)/n$ and ${}_nCH^2(X)$ are infinite for all n prime to $\mathrm{char}(k)$.

A slightly more sophisticated calculation, left to the reader (using A.3), shows that the projective closure $\overline{X}$ of X satisfies $CH^2(\overline{X}) \cong \mathbb{Z} \oplus K_2(k)$, which has the same qualitative properties as $CH^2(X)$.

In contrast to this example, we would like to point out the following finiteness result of Colliot-Thélène & Raskind and Salberger [CT-R91, thm.C](cf. [Sai]). Note that if k is the ground field of $k(X)$, then X is geometrically connected over k.

THEOREM 5.3.1. — *Let X be a smooth projective variety over a number field. If $H^2(X, \mathcal{O}_X) = 0$ then the torsion subgroup of $CH^2(X)$ is finite.*

In order to summarize the conjectural state of affairs for a smooth surface X over a number field k, let $\mathrm{Alb}_X(k)$ denote the group of k-rational points on the Albanese variety of X. By the Mordell-Weil theorem, this is a finitely generated abelian group. Bloch's conjecture asserts that if $H^2(X, \mathcal{O}_X) = 0$ then the kernel of $A_0(X) \to \mathrm{Alb}_X(k)$ is contained in the torsion subgroup of $A_0(X)$, which is finite (for X projective) by 5.3.1. Bloch's conjecture is known to be true for all surfaces except those of general type [BKL].

PROPOSITION 5.4. — *Consider the following assertions about a smooth surface X over a number field k.*

(1) *(Bloch's conjecture). If $H^2(X, \mathcal{O}_X) = 0$ then the torsion subgroup of $A_0(X)$ contains the kernel of $A_0(X) \to \mathrm{Alb}_X(k)$.*

(2) *$A_0(X)$ is a finitely generated abelian group.*

(3) *(Bass' Conjecture) $K_0(X)$ is a finitely generated abelian group.*

(4) *$A_0(X)/n$ is finite for every n.*

(5) *The kernel of $c_{24} : SK_0(X)/n \to H^4_{\mathrm{et}}(X, \mu_n^{\otimes 2})$ is finite.*

Then (1) implies (2), (2) $\Leftrightarrow$ (3), (3) implies (4) and (4) $\Leftrightarrow$ (5). Moreover, if $H^2(X, \mathcal{O}_X) = 0$ then (1) $\Leftrightarrow$ (2).

Proof. We have seen that (1) $\Rightarrow$ (2). Since $\mathrm{Pic}(X)$ is finitely generated by the Mordell-Weil Theorem, (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) is clear. Finally, S. Saito proved in [Sai] that the image of the "cycle map"

$$CH^2(X)/n \cong SK_0(X)/n \cong H^2(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \to H^4_{\mathrm{et}}(X, \mu_n^{\otimes 4})$$

is finite. By Theorem 2.2, this map is just c_{24}. Since $CH^2(X) \cong \mathbb{Z}^c \oplus A_0(X)$, this gives the equivalence of (4) and (5). $\square$

Finally, we consider varieties over the real numbers $\mathbb{R}$, and interpret our results in terms of topological invariants. By a "real variety" we shall mean a reduced quasiprojective scheme X over $\mathbb{R}$. Let X be an m-dimensional real variety and let Y denote $\mathrm{Sing}(X)$. Let L be the union of all irreducible components of X having dimension strictly less than m.

Since X lies in some $\mathbb{P}^N_{\mathbb{R}}$ we can topologize the set $X(\mathbb{R})$ of real points as a subspace of the real projective space $\mathbb{RP}^N = \mathbb{P}^N_{\mathbb{R}}(\mathbb{R})$. The subspace $M = X(\mathbb{R}) - Y(\mathbb{R}) - L(\mathbb{R})$ is a smooth m-dimensional manifold, and in fact $X(\mathbb{R})$ has the structure of a stratified manifold with top stratum M. Our first result interprets $\dim H^m(X(\mathbb{R}))$ in terms of M.

PROPOSITION 5.5. — *Let t denote the number of path-connected components M_i of M whose closure $\overline{M}_i$ in $X(\mathbb{R})$ is compact. Then*

$$t = \dim H_m(X(\mathbb{R}), Y(\mathbb{R}); \mathbb{Z}/2).$$

If $\dim Y(\mathbb{R}) \leq m - 2$, *then in addition*

$$t = \dim H_m(X(\mathbb{R}); \mathbb{Z}/2) = \dim H^m(X(\mathbb{R}); \mathbb{Z}/2).$$

Proof. The second assertion follows from the first, using the exact sequence

$$0 \to H_m(X(\mathbb{R})) \to H_m(X(\mathbb{R}), Y(\mathbb{R})) \to H_{m-1}(Y(\mathbb{R})).$$

By excision, $H_m(X(\mathbb{R}),Y(\mathbb{R})) \cong \oplus H_m(\overline{M}_i, \overline{M}_i - M_i)$, the sum being taken over all path-connected components of M. Fix i and give $\overline{M}_i$ the structure of a cell complex such that every $(m-1)$-cell in the m-manifold M_i is the boundary of exactly two m-cells; every pair of m-cells in M_i may be connected by a finite sequence of m-cells and $(m-1)$-cells. A cellular m-chain σ is therefore just a finite sum of m-cells in M_i, and if $\sigma \neq 0$ then σ cannot be a cycle unless it is the sum of all the m-cells in M_i. If $\overline{M}_i$ is compact, there are finitely many m-cells and $H_m(\overline{M}_i, \overline{M}_i - M_i) = \mathbb{Z}/2$. However, if $\overline{M}_i$ is not compact then there are infinitely many m-cells, and therefore $H_m(\overline{M}_i, \overline{M}_i - M_i) = 0$. □

We now assume that the singular set of X has codimension ≥ 2, for example that X is a surface with isolated singularities. Exactly as in the case of surfaces, the group $CH^m(X) = CH^m(X,Y)$ is defined to be $Z^m(X,Y)/R^m(X,Y)$, where $Z^m(X,Y)$ is the free abelian group on the set X_*^m and $R^m(X,Y)$ is generated by divisors (f) of rational functions on curves in X missing Y. We are going to relate $CH^m(X)$ to $t = \dim H^m X(\mathbb{R})$.

Let $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification and $\tau : X_{\mathbb{C}} \to X$ the natural map. Pushing forward zero-cycles missing $Y_{\mathbb{C}}$ produces a transfer map $\tau_* : CH^m(X_{\mathbb{C}}) \to CH^m(X)$. The following result was proven in [CT-I, 3.1] for smooth, proper real varieties.

THEOREM 5.6. — *Let X be an m-dimensional real variety such that $Y = \mathrm{Sing}(X)$ has codimension ≥ 2. Then there is an isomorphism*

$$\theta : CH^m(X)/\tau_* CH^m(X_{\mathbb{C}}) \cong H^m(X(\mathbb{R}); \mathbb{Z}/2) \cong (\mathbb{Z}/2)^t,$$

where t is the number of connected components of M having compact closure in $X(\mathbb{R})$.

Remark 5.6.1. For any proper real variety X, [CT-I, 3.1] yields

$$CH_0(X)/\tau_* CH_0(X_{\mathbb{C}}) \cong (\mathbb{Z}/2)^s,$$

where s is the number of connected components of $X(\mathbb{R})$. For singular X we can have $s \neq t$, hence $CH^2(X) \neq CH_0(X)$.

For the proof of 5.6, we shall need some information about the Picard group of a smooth real curve Z. Topologically, $Z(\mathbb{R})$ is a smooth 1-manifold, every component of which is diffeomorphic to either S^1 or $\mathbb{R}$. The following result is proven in [PW2, 1.4].

COMESSATTI–WITT THEOREM 5.7. — *Let Z be an irreducible smooth real curve with $Z(\mathbb{R}) \neq \emptyset$. If $Z(\mathbb{R})$ has λ components Z_i diffeomorphic to S^1, there is an isomorphism*

$$\mathrm{Pic}(Z)/2 \xrightarrow{\cong} (\mathbb{Z}/2)^\lambda$$

whose i^{th} *component* $\mathrm{Pic}(Z)\to\mathbb{Z}/2$ *sends [P] to 1 iff* $P\in Z_i$ $(1\le i\le\lambda)$.

Historical Remark 5.7.1. Comessatti [Com] and Witt [Witt] calculated $\mathrm{Pic}(Z)$ when Z is projective and smooth, and the general case follows easily. Their key step used the 1882 calculations of period matrices by Weichold [Whd]. This transcendental step may be replaced by the algebraic result [Kn, 3.4], which is valid over any real closed field.

Proof of Theorem 5.6 : Let M_i be a path-connected component of M whose closure in $X(\mathbb{R})$ is compact. Define the homomorphism

$$\theta_i : Z^m(X,Y)\to\mathbb{Z}/2$$

to be the characteristic function of $M_i\subset X^m_*$.

We first claim that θ_i induces a map $\bar\theta_i : CH^m(X)/\tau_*CH^m(X_{\mathbb{C}})\to\mathbb{Z}/2$. If $P\in(X_{\mathbb{C}})^m_*$ then $\theta_i\tau_*([P])=0$ because either $\tau(P)\notin M_i$ or else $\tau_*([P])=2[\tau(P)]$. Thus θ_i sends $\tau_*Z^m(X_{\mathbb{C}},Y_{\mathbb{C}})$ to zero. To establish the claim, it suffices to check that θ_i sends the relations $R^m(X,Y)$ for $CH^m(X,Y)$ to zero. For this, choose a curve Z on X missing Y and $f\in k(Z)^*$. Because $\overline{M_i}$ is compact and $Z(\mathbb{R})\subset M$, $Z(\mathbb{R})\cap M_i$ is a compact subspace of $Z(\mathbb{R})$. On the real locus $\widetilde{Z}(\mathbb{R})$ of the normalization $\widetilde{Z}$ of Z, every component Δ lying over $Z(\mathbb{R})\cap M_i$ must be a circle. By the Comessatti-Witt Theorem 5.7, the divisor (f) has even degree on each Δ, so $\theta_i((f))=0$. This establishes the claim.

The map $\theta=(\bar\theta_1,\dots,\bar\theta_t) : CH^m(X)/\tau_*CH^m(X_{\mathbb{C}})\to(\mathbb{Z}/2)^t$ is onto, since a basis of $(\mathbb{Z}/2)^t$ is given by the $\theta([P_i])$ with $P_i\in M_i$. To prove that θ is injective, we argue as in [CT-I]. Since $CH^m(X)\to CH^m(X_{\mathbb{C}})\xrightarrow{\tau}CH^m(X)$ is multiplication by 2, $2\,CH^m(X)\subseteq\tau_*CH^m(X_{\mathbb{C}})$. If $P\notin M$ then P is a smooth complex point of X and $\theta([P])=0$; moreover, $[P]\in\tau_*CH^m(X_{\mathbb{C}})$ because $[P]=\tau_*([P'])$ for either of the two points P' of $X_{\mathbb{C}}$ over P.

Two points P and Q of M are said to be *related* if there is a smooth real curve Z, a proper morphism $\pi : Z\to X$ with $\pi(Z)\cap Y=\emptyset$ and a path $f:[0,1]\to Z(\mathbb{R})$ such that πf is a path in M from P to Q. If P and Q lie in the same component M_i of M, then there is a finite sequence $P=P_0,\dots,P_n=Q$ such that P_j and P_{j+1} are related for every j. It suffices to show : (1) $[P]=[Q]$ in $CH^m(X)/2$ whenever P and Q are related (hence whenever P and Q lie on the same M_i), and (2) if $\overline{M_i}$ is not compact then there exists some point Q of M_i such that $[Q]=0$ in $CH^2(X)/2$.

Suppose first that P and Q are related by $\pi : Z\to X$ and $f:[0,1]\to Z(\mathbb{R})$. Then $f(0)$ and $f(1)$ lie on the same path component Z_i of $Z(\mathbb{R})$, which is diffeomorphic to either S^1 or $\mathbb{R}$. By the Commessatti-Witt Theorem 5.7, $[f(0)]=[f(1)]$ in $\mathrm{Pic}(Z)/2$. Via the map $\mathrm{Pic}(Z)\to CH^m(X)$, we see that $[P]=\pi_*[f(0)]$ equals $[Q]=\pi_*[f(1)]$ in $CH^m(X)/2$.

Now suppose that M_i is a component of M but $\overline{M}_i$ is not compact. Embed X as an open subvariety of a projective variety X' such that $Y' = \mathrm{Sing}(X')$ is the closure of Y, and let M_i' be the component of $X'(\mathbb{R}) - Y'(\mathbb{R})$ containing M_i. Since $\overline{M}_i{}'$ is compact there is a point P at infinity in $\overline{M}_i{}'$ which is related (in X') to some point Q of M_i, say by $\pi : Z' \to X'$ and $f' : [0,1] \to Z'(\mathbb{R})$. The subscheme $Z = \pi^{-1}X$ of Z' is a smooth curve, and $f'(1)$ lies on an unbounded component Δ of $Z(\mathbb{R})$. But the Comessatti-Witt calculation then yields $[f'(1)] = 0$ in $\mathrm{Pic}(Z)/2$, and hence $[Q] = \pi_*[f'(1)] = 0$ in $CH^m(X)/2$. □

In order to relate theorem 5.6 to the Chow group $CH^m(X)$, we need to choose an explicit form of the decomposition $CH^m(X) \cong \mathbb{Z}^c \oplus A_0(X)$ in (5.0). Let E denote the number of proper components X^i of X having no smooth real points; pick a smooth complex point P_i on each such X^i. Pick a smooth real point P_i on each of the remaining $R = c - E$ proper components X^i. These $[P_i]$ form a basis for a summand $\mathbb{Z}^c$ of $CH^m(X)$ which is mapped isomorphically onto the image $\Lambda = (2\mathbb{Z})^E \oplus (\mathbb{Z})^R$ of $\deg : CH^m(X) \to \mathbb{Z}^c$. As the kernel of deg is $A_0(X)$ this gives our explicit decomposition $CH^m(X) \cong \mathbb{Z}^c \oplus A_0(X)$.

Now $X_{\mathbb{C}}$ has $c' \geq c$ proper components ($c' = c$ if every proper component of X is geometrically irreducible). As Q runs over all points of $X_{\mathbb{C}}$ such that $\tau(Q) = P_i$ for some i, the $[Q]$ span a summand $\mathbb{Z}^{c'}$ of $CH^m(X_{\mathbb{C}})$; this yields an explicit decomposition $CH^m(X_{\mathbb{C}}) \cong \mathbb{Z}^{c'} \oplus A_0(X_{\mathbb{C}})$. As in the case $m = 2$, $A_0(X_{\mathbb{C}})$ is a divisible abelian group (the proof of 5.1 goes through). The explicit decompositions are compatible with τ_* in the sense that $\tau_*(\mathbb{Z}^{c'}) \subseteq \mathbb{Z}^c$ and $\tau_* A_0(X_{\mathbb{C}}) \subseteq A_0(X)$. That is,

$$CH^m(X)/\tau_* CH^m(X_{\mathbb{C}}) \cong \mathbb{Z}^c/\tau_*(\mathbb{Z}^{c'}) \oplus A_0(X)/\tau_* A_0(X_{\mathbb{C}}).$$

THEOREM 5.8. — *Let X be an m-dimensional real variety such that $\mathrm{Sing}(X)$ has codimension ≥ 2. Let R (resp. E) denote the number of proper components of X having some (resp. no) smooth real point, and set $t = \dim H^m(X(\mathbb{R}); \mathbb{Z}/2)$.*

The decomposition (5.0) is $CH^m(X) \cong \mathbb{Z}^{R+E} \oplus A_0(X)$, and we have :

$$A_0(X) \cong (\mathbb{Z}/2)^{t-R} \oplus \tau_* A_0(X_{\mathbb{C}}).$$

Since the group $\tau_* A_0(X_{\mathbb{C}})$ is divisible (by 5.1), this yields

$$CH^m(X)/2 \cong (\mathbb{Z}/2)^{t+E}.$$

Proof. Suppose that $Q \in X_{\mathbb{C}}$ has $\tau(Q) = P_i$. If $P_i \in X(\mathbb{R})$ then $\tau_*[Q] = 2[P_i]$; if $P_i \notin X(\mathbb{R})$ then $\tau_*[Q] = [P_i]$. Hence $\mathbb{Z}^c/\tau_*(\mathbb{Z}^{c'}) \cong (\mathbb{Z}/2)^R$. By Theorem 5.6 this yields $A_0(X)/\tau_* A_0(X_{\mathbb{C}}) \cong (\mathbb{Z}/2)^{t-R}$. This must be a summand of $A_0(X)$ since the subgroup $\tau_* A_0(X_{\mathbb{C}})$ is divisible, whence the description of $A_0(X)$. The calculation of $CH^m(X)/2$ is immediate. □

COROLLARY 5.9. — *Let X be a real surface with isolated singularities, and set $t = \dim H^2(X(\mathbb{R});\mathbb{Z}/2)$. Then*

$$SK_0(X)/2 \cong CH^2(X)/2 \cong (\mathbb{Z}/2)^{t+E}$$

and the torsion subgroup of $CH^2(X)$ contains $(\mathbb{Z}/2)^{t-R}$ as a summand.

Example 5.10. Let X be the real affine surface in $\mathbb{A}^3_{\mathbb{R}}$ defined by the equation $y^2 + z^2 = x^2(x-1)$. $X(\mathbb{R})$ is a surface of revolution having two connected components : the isolated point $(0,0,0)$ and a smooth unbounded component. In fact X is normal with $(0,0,0)$ the only singularity. Corollary 5.9 yields $SK_0(X)/2 = 0$.

Here is a direct computation giving the stronger result that $SK_0(X) = 0$. $SK_0(X)$ is generated by all $[P]$ as P ranges over the smooth points of X. If P is real then there exists a real line in $\mathbb{A}^3$ meeting X only at P transversely. If P is complex then a computation similar to that in [W5] shows that $[P] = 0$ in $SK_0(X)$ because P is a complete intersection in X.

(5.11) In order to extend our results to all real surfaces, we need a generalization due to Scheiderer [Sch] [CT-S, 2.3.2] of a result of Colliot-Thélène and Parimala [CT-P]. Let $\mathrm{Spec}_r(X)$ denote the real spectrum of a d-dimensional real variety and let $\varphi : \mathrm{Spec}_r(X) \to X_{\mathrm{zar}}$ denote the support map. The main result of [CT-P] is that if X is smooth, then the natural map $\mathcal{H}^n(\mathbb{Z}/2) \to \varphi_*(\mathbb{Z}/2)$ is an isomorphism when $n > d$. Scheiderer proves that it is a surjection for $n = d$, and that $R^i\varphi_*\mathbb{Z}/2 = 0$ for $i \neq 0$ and all X. As in [PW2, 5.5.3] this yields a natural map from $H^i(X, \mathcal{H}^n)$ to

$$H^i(X, \varphi_*\mathbb{Z}/2) \cong H^i(\mathrm{Spec}_r(X), \mathbb{Z}/2) \cong H^i(X(\mathbb{R});\mathbb{Z}/2).$$

If X is smooth and $n > d$, this map is an isomorphism for all i.

PROPOSITION 5.11.1. — *Let X be a quasiprojective scheme over $\mathbb{R}$ of dimension d. Then the natural map*

$$H^d(X, \mathcal{H}^n(\mathbb{Z}/2)) \to H^d(X(\mathbb{R}), \mathbb{Z}/2)$$

is onto for $n = d$ and an isomorphism for $n > d$.

Proof. If $n > d$ this follows from Lemma 1.3. Let $\mathcal{H}$ denote the image of $\mathcal{H}^d \to \varphi_*\mathbb{Z}/2$. As H^d is right exact, $H^d(X, \mathcal{H}^d(\mathbb{Z}/2)) \to H^d(X, \mathcal{H})$ is onto. As the cokernel of $\mathcal{H} \hookrightarrow \varphi_*\mathbb{Z}/2$ is supported on $\mathrm{Sing}(X_{\mathrm{red}})$, lemma 1.1 gives a surjection from $H^d(X, \mathcal{H})$ to $H^d(X, \varphi_*\mathbb{Z}/2) \cong H^d(X(\mathbb{R});\mathbb{Z}/2)$. □

Example 5.11.2. When X is a real curve the resulting isomorphism $H^1(X, \mathcal{H}^2) \cong H^1(X(\mathbb{R});\mathbb{Z}/2)$ was proven by other methods in [PW2, 5.5].

Example 5.11.3. Suppose that X is a real surface. The composition

$$w_2 : CH^2(X)/2 \cong SK_0(X)/2 \cong H^2(X, \mathcal{H}^2(\mathbb{Z}/2)) \to H^2(X(\mathbb{R}); \mathbb{Z}/2)$$

is a surjection by 5.11.1. If X has isolated singularities we claim that w_2 is exactly the map θ of 5.6 having kernel $(\mathbb{Z}/2)^E$. To see this, we observe that all groups involved are finite and that $(\mathbb{Z}/2)^E \subseteq \ker(w_2)$ as $X_{\mathbb{C}}(\mathbb{R}) = \emptyset$. Hence $\ker(w_2) = (\mathbb{Z}/2)^E$. To see that $w_2 = \theta$ it suffices to choose a component M_i with $\overline{M}_i$ compact and pass to an affine open neighborhood U of $\overline{M}_i$ containing no other relatively compact smooth components, where the two isomorphisms $\mathbb{Z}/2 \cong \mathbb{Z}/2$ must be the same.

Remark 5.11.4. Consider the natural map

$$w_1 : H^1(X, \mathcal{H}^2(\mathbb{Z}/2)) \to H^1(X(\mathbb{R}); \mathbb{Z}/2).$$

If X is a smooth surface, Colliot-Thélène and Scheiderer [CT-S, (3.2)] have shown that w_1 is always a surjection, but need not be an isomorphism. We do not know if w_1 is always a surjection for normal surfaces; our ignorance about w_1 for $\tilde{X}$ forces us to introduce the term $Q = \operatorname{coker}(w_1) \cong (\mathbb{Z}/2)^\varepsilon$ in theorem 5.12 below. As a lower bound, we know that the rank of $H^1(X, \mathcal{H}^2(\mathbb{Z}/2))$ is at least $t - R$. This follows from 5.8, using the surjection $H^1(X, \mathcal{H}^2(\mathbb{Z}/2)) \to {}_2CH^2(X)$ of Theorem D.

From a topological viewpoint, $w_1 : H^1(X, \mathcal{H}^2(\mathbb{Z}/2)) \to H^1(X(\mathbb{R}); \mathbb{Z}/2)$ is a kind of "first Stiefel-Whitney class" in the sense that it is induced from the composition of the algebraic-to-topological K-map $K_1(X) \to KO^{-1}(X(\mathbb{R}))$ and the Stiefel-Whitney class $KO^{-1}(X(\mathbb{R})) \xrightarrow{w_1} H^1(X(\mathbb{R}); \mathbb{Z}/2)$. To see this, note that $H^1(X, \mathcal{H}^2(\mathbb{Z}/2))$ is isomorphic to $H^1(X, \mathcal{K}_2/2)$, which in turn is the quotient of $SK_1(X)$ corresponding to the quotient $H^1(X(\mathbb{R}); \mathbb{Z}/2)$ of $KO^{-1}(X(\mathbb{R}))$ in the Atiyah-Hirzebruch spectral sequence.

THEOREM 5.12. — *Let X be a real surface with normalization $\tilde{X}$. Write R (resp. E) for the number of proper components of X having a (resp. no) smooth real point. Then :*

a) The decomposition (5.0) is $CH^2(X) \cong \mathbb{Z}^{R+E} \oplus A_0(X)$.

b) Set $t = \dim H^2(X(\mathbb{R}); \mathbb{Z}/2)$ and $\varepsilon = \dim(Q)$, where $Q \cong (\mathbb{Z}/2)^\varepsilon$ denotes the cokernel of $w_1 : H^1(\tilde{X}, \mathcal{H}^2(\mathbb{Z}/2)) \to H^1(\tilde{X}(\mathbb{R}); \mathbb{Z}/2)$. (If $\tilde{X}$ is smooth then $\varepsilon = 0$.) Then :

$$A_0(X) \cong (\mathbb{Z}/2)^{t-R+\varepsilon} \oplus \tau_* A_0(X_{\mathbb{C}}).$$

c) The subgroup $\tau_* A_0(X_{\mathbb{C}})$ is divisible, and we have

$$CH^2(X)/2 \cong H^2(X(\mathbb{R}); \mathbb{Z}/2) \oplus (\mathbb{Z}/2)^E \oplus Q \cong (\mathbb{Z}/2)^{t+E+\varepsilon}$$
$$CH^2(X)/\tau_* CH^2(X_{\mathbb{C}}) \cong H^2(X(\mathbb{R}); \mathbb{Z}/2) \oplus Q \cong (\mathbb{Z}/2)^{t+\varepsilon}.$$

Proof. It suffices to calculate $CH^2(X)/2$, since the other calculations all follow from it via the argument of 5.8 and the decomposition

$$CH^2(X)/\tau_*CH^2(X_{\mathbb{C}}) \cong \mathbb{Z}^c/\tau_*(\mathbb{Z}^{c'}) \oplus A_0(X)/\tau_*A_0(X_{\mathbb{C}}).$$

Since $\tilde{X} \to X$ is a birational isomorphism, there is a bijection between the proper components of X and $\tilde{X}$, both with and without smooth real points. Writing H^iM for $H^i_{\text{top}}(M;\mathbb{Z}/2)$, and $\mathfrak{E}$ for $(\mathbb{Z}/2)^E$, the considerations of 5.11 give a commutative diagram

$$\begin{array}{ccccccc}
H^1(\tilde{X},\mathcal{H}^2)\oplus SK_1(Y)/2 & \to & SK_1(\tilde{Y})/2 & \to & CH^2(X)/2 & \longrightarrow & CH^2(\tilde{X})/2 \to 0 \\
\downarrow w_1 & & \downarrow \cong & & \downarrow w_2 & & \cong \downarrow w_2 \\
H^1\tilde{X}(\mathbb{R}) \oplus H^1Y(\mathbb{R}) & \longrightarrow & H^1\tilde{Y}(\mathbb{R}) & \longrightarrow & H^2X(\mathbb{R})\oplus\mathfrak{E} & \to & H^2\tilde{X}(\mathbb{R})\oplus\mathfrak{E} \to 0.
\end{array}$$

The top row is the exact sequence of Theorem B. The bottom row is the Mayer-Vietoris sequence for singular cohomology, resulting from the excision isomorphism $H^*(X(\mathbb{R}),Y(\mathbb{R})) \cong H^*(\tilde{X}(\mathbb{R}),\tilde{Y}(\mathbb{R}))$. The right vertical map is an isomorphism by 5.9 and 5.11.3. By 5.11.2 or [PW2, 5.5] and a diagram chase we get the desired exact sequence :

$$0 \to Q \to CH^2(X)/2 \xrightarrow{w_2} H^2X(\mathbb{R}) \oplus (\mathbb{Z}/2)^E \to 0. \qquad \square$$

Example 5.13. A "real umbrella" is a real surface whose singular locus is a line sticking out of the smooth locus. For example, consider the affine surface $X \subset \mathbb{A}^3_{\mathbb{R}}$ defined by $x^4 + y^4 = x^2(1 - z^2)$. The singular locus Y is the z-axis and $X(\mathbb{R}) - Y(\mathbb{R})$ has two bounded components. From 5.5 we see that $H^2(X(\mathbb{R});\mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$, i.e., $t = 2$. The normalization $\tilde{X}$ is the surface in $\mathbb{A}^4_{\mathbb{R}}$ defined by $y^2 = tx$, $x^2 + z^2 + t^2 = 1$, and $\tilde{X}(\mathbb{R})$ is homeomorphic to S^2 even though $\text{Sing}(\tilde{X})$ has 2 points. Since $H^1(S^2;\mathbb{Z}/2) = 0$ we have $\varepsilon = 0$ in theorem 5.12 and hence

$$CH^2(X) \cong (\mathbb{Z}/2)^2 \oplus \tau_*CH^2(X_{\mathbb{C}}).$$

We claim that the divisible group $\tau_*CH^2(X_{\mathbb{C}})$ is torsionfree. Since the composition $\tau_*CH^2(X_{\mathbb{C}}) \to CH^2(X_{\mathbb{C}}) \xrightarrow{\tau_*} \tau_*CH^2(X_{\mathbb{C}})$ is multiplication by 2, it suffices to show that $CH^2(X_{\mathbb{C}})$ is torsionfree. The Mayer-Vietoris sequence (given as (A.2) in the Appendix) is :

$$SK_1(\tilde{Y}_{\mathbb{C}}) \to CH^2(X_{\mathbb{C}}) \to CH^2(\tilde{X}_{\mathbb{C}}) \to 0.$$

Since $\tilde{Y}$ is defined by $z^2 + t^2 = 1$ and $x = y = 0$, $\tilde{Y}_{\mathbb{C}} \cong \text{Spec}(\mathbb{C}[u, u^{-1}])$ for $u = z + it$. It is well-known that $SK_1(\mathbb{C}[u, u^{-1}]) = 0$, so $CH^2(X_{\mathbb{C}}) \cong CH^2(\tilde{X}_{\mathbb{C}})$. But by [L2, 2.6] (see 5.1.1), $CH^2(\tilde{X}_{\mathbb{C}})$ is torsionfree.

Appendix : K-theoretic excision methods

For purposes of comparison, it is useful to see what can be proven about the groups $SK_0(X)$ using K-theoretical methods.

(A.0) *Standing Assumptions.* We assume that X is a finite-dimensional noetherian scheme with finite normalization $\pi : \tilde{X} \to X$, and that the conductor subscheme Y of X has dimension at most 1. Letting $\tilde{Y}$ denote its preimage $Y \times_X \tilde{X}$, this data fits together into the following cartesian square.

$$\begin{array}{ccc} \tilde{Y} & \overset{j}{\hookrightarrow} & \tilde{X} \\ \downarrow & & \downarrow \pi \\ Y & \overset{i}{\hookrightarrow} & X. \end{array}$$

Since $\dim(Y) \leq 1$ we have $K_0(Y) = H^0(Y, \mathbb{Z}) \oplus \mathrm{Pic}(Y)$, i.e., $SK_0(Y) = 0$. (See [TT, 10.8] when Y is not quasiprojective.) Similarly, $SK_0(\tilde{Y}) = 0$. If X is affine, it is classical that there are exact Mayer-Vietoris sequences

$$(A.1)\ K_1(X) \to K_1(\tilde{X}) \oplus K_1(Y) \to K_1(\tilde{Y}) \to K_0(X) \to K_0(\tilde{X}) \oplus K_0(Y) \to K_0(\tilde{Y})$$

$$(A.2)\ SK_1(X) \to SK_1(\tilde{X}) \oplus SK_1(Y) \to SK_1(\tilde{Y}) \to SK_0(X) \to SK_0(\tilde{X}) \to 0.$$

We want to extend these sequences to non-affine X.

THEOREM A.3. — *Let X be as in (A.0), with* $\dim(Y) \leq 1$. *Then $SK_0(X) \to SK_0(\tilde{X})$ is always onto. Moreover the sequences (A.1) and (A.2) are exact, provided that any one of the following holds :*

- $\dim(Y) = 0$ *or Y is contained in an affine open subset of X ;*
- *the generic points of $\tilde{Y}$ are separable over the generic points of Y ;*
- *we localize all groups by inverting the primes in the set* $\{\mathrm{char}\, k(y), y \in Y\}$.

COROLLARY A.4. — *Let X be as in (A.0) with* $\dim(Y) \leq 1$. *If* $1/n \in \mathcal{O}_X$ *then there is an exact sequence*

$$SK_1(\tilde{Y})/n \to SK_0(X)/n \to SK_0(\tilde{X})/n \to 0.$$

To prove A.3, we show that the "double relative" obstruction $K_{-1}(X, \tilde{X}, Y)$ to excision always vanishes when $\dim(Y) \leq 1$. This easily yields the exactness of the sequence

$$K_0(X) \to K_0(\tilde{X}) \oplus K_0(Y) \to K_0(\tilde{Y})$$

and hence surjectivity of $SK_0(X) \to SK_0(\tilde{X})$. Under the supplementary hypotheses of A.3, we shall prove that the obstruction $K_0(X, \tilde{X}, Y)$ also vanishes, which formally yields the exact sequence (A.1); see [GW, 5.1]. The argument of [PW1, 8.6] shows that exactness of (A.1) implies exactness of (A.2).

If Y is contained in an affine open subset of X, exactness of (A.1) and (A.2) was proven in [PW1, 8.5]; when X is not quasiprojective one needs to replace the space $BQ\mathcal{H}_Z(X)$ in *loc. cit.* by the relative term $K(X \text{ on } Z)$ of [TT, 5.1].

In the other cases we use the following Brown-Gersten spectral sequence to calculate $K_*(X, \tilde{X}, Y)$; the desired vanishing results follow from A.6 below.

PROPOSITION A.5. — *Let $\tilde{F}_q$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto K_q(U, \tilde{X} \times_X U, Y \times_X U)$ of "double relative" K-theory. Then each $\tilde{F}_q$ is supported on Y, $\tilde{F}_q = 0$ for $q \leq 0$ and there is a fourth quadrant Brown-Gersten spectral sequence*

$$E_2^{pq} = H^p_{\mathrm{zar}}(Y, \tilde{F}_{-q}) \Rightarrow K_{-p-q}(X, \tilde{X}, Y).$$

Proof. We say that a presheaf F of (fibrant) spectra on X has the *Mayer-Vietoris property* if for every open U and V the square of spectra

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

is homotopy cartesian. By [TT, 8.1] the presheaf K^B has the Mayer-Vietoris property. Let $K(U, Y \times U)$ denote the homotopy fiber of $K^B(U) \to K^B(Y \times U)$, and let $F(U) = K(U, \tilde{X} \times U, Y \times U)$ denote the homotopy fiber of $K(U, Y \times U) \to K(\tilde{X} \times U, \tilde{Y} \times U)$. By definition, $K_q(U, X \times U, Y \times U) = \pi_q F(U)$. Using the Reduction and Cobase-change Lemmas of [W4], it follows that the presheaves $K(-, Y \times -)$ and F both have the Mayer-Vietoris property. Since X is finite-dimensional and noetherian, this implies the existence of the Brown-Gersten spectral sequence.

The stalk of $\tilde{F}_q$ at $x \in X$ is the usual double relative K-group

$$K_q(\mathcal{O}_{X,x}, \mathcal{O}_{\tilde{X},x}, \mathcal{I}_x),$$

which is classically known to vanish for $q \leq 0$. For $U \subseteq X - Y$ we have $F(U) \simeq *$, hence each $\pi_q F(U) = 0$. This implies that each $\tilde{F}_q = \pi_q \tilde{F}$ is supported on Y. $\square$

COROLLARY A.6. — *If* $\dim(Y) \leq 1$ *then*

$$K_0(X,\tilde{X},Y) \cong H^1(Y,\mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\tilde{Y}/Y})$$

and $K_{-1}(X,\tilde{X},Y)=0$. *In particular, if the generic points of* $\tilde{Y}$ *are separable over the generic points of* Y *then* $K_0(X,\tilde{X},Y)=0$.

Proof. By [GW, (1.1)], $\tilde{F}_1$ is the sheaf $\mathcal{I}/\mathcal{I}^2 \otimes \Omega_{\tilde{Y}/Y}$. The generically separable condition ensures that the support of $\Omega_{\tilde{Y}/Y}$ has dimension 0. □

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