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# On a specialization map in $K_2$-cohomology

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## Introduction

Let $K$ be an algebraic number field, $\overline{K}$ its algebraic closure. Let $K_\wp$ be the completion of $K$ at a prime $\wp$ lying above $p$, $p > 2$. Denote by $o_\wp$ the integers in $K_\wp$ and let $k$ be its residue field. A central observation in the Iwasawa theory of abelian varieties is the following:

Let $A/K$ be an abelian variety with good reduction at all primes above $p$ and let

$$0 \longrightarrow \mathcal{A}^0(p) \longrightarrow \mathcal{A}(p) \longrightarrow \mathcal{A}(p)^{et} \longrightarrow 0$$

be the étale-local decomposition of its $p$-divisible group over $o_\wp$. Then we have a specialization map $A(\overline{K}_\wp) \to A(\overline{k})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}(p)(\overline{K}_\wp) & \hookrightarrow & A(\overline{K}_\wp) \\ \downarrow & & \downarrow \\ \mathcal{A}(p)^{et}(\overline{k}) & \hookrightarrow & A(\overline{k}) \end{array} .$$

Here the groups on the left hand side coincide with the $p$-primary torsion group of the corresponding groups on the right hand side. This fact can be used to show that the $p$-primary Selmer group over the cyclotomic $\mathbb{Z}_p$-extension $K_\infty$, denoted by $S_{p^\infty}(K_\infty, A)$ only depends on the Tate-module $T_p(A)$ and coincides with the flat cohomology $H^1_{fl}(o_\infty, \mathcal{A}(p))$, where $o_\infty$ is the ring of integers in $K_\infty$. The result is originally due to Greenberg [Gr].

Now let $X$ be a smooth projective variety over $K_\wp$ with good reduction, $\overline{X} = X \underset{K_\wp}{\times} \overline{K}_\wp$, and $Y_k$ be the closed fiber of a smooth proper model of $X$ over $o_\wp$ and let us consider its motive $H^2(X)(2)$. It turns out that, similarly to the Kummer sequence of an abelian variety we can associate a Kummer sequence to the motive $H^2(X)(2)$

$$0 \to H^2_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^1_{Zar}(\overline{X}, \mathcal{K}_2) \to H^1_{Zar}(\overline{X}, \mathcal{K}_2) \otimes \mathbb{Z}[\frac{1}{p}] \to 0$$

where $H^1_{Zar}(\overline{X}, \mathcal{K}_2)$ denotes the Zariski-$K$-cohomology of the Quillen $K$-sheaf $\mathcal{K}_2$ (compare [CT-R], [S]).

In his paper [S] P. Schneider gave two possible candidates for Iwasawa-Modules of an arbitrary motive $H^i(X)(n)$, one of which generalizes the flat

Mazur cohomology $H^1_{fl}(o_\infty, \mathcal{A}(p))$, the other one the Selmer group in case of $H^1(\hat{A})(1)$ for an abelian variety $A$ and its dual $\hat{A}$. In order to give more sense to this new theory there should be a close link between the two candidates. A similar analysis as in the case of an abelian variety, which, however, we are not going to specify here, leads, for the motive $H^2(X)(2)$, to the following question, which is also interesting within the study of motivic cohomology:

Is there a specialization map

$$H^1(\overline{X}, \mathcal{K}_2) \longrightarrow H^1(Y_{\overline{k}}, \mathcal{K}_2)$$

such that the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & H^2_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \longrightarrow & H^1(\overline{X}, \mathcal{K}_2) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(Y_{\overline{k}}, \mathcal{K}_2)(p) & \longrightarrow & H^1(Y_{\overline{k}}, \mathcal{K}_2) \end{array}$$

where the left vertical map should be surjective.

In this paper we give an affirmative answer under a slightly stronger hypothesis.

**Theorem 1**

*Assume $K_\wp$ is unramified over $\mathbb{Q}_p$. Let $X/K_\wp$ be a smooth projective variety with ordinary good reduction, such that* $\dim X < p-1$. *Assume that* $\mathrm{Pic}(Y_{\overline{k}})(p) \equiv 0$, *where $Y_k$ is defined as above. Then we have a specialization map*

$$f : H^1(\overline{X}, \mathcal{K}_2) \longrightarrow H^1(Y_{\overline{k}}, \mathcal{K}_2)$$

*which is surjective on the $p$-primary torsion groups.*

The main idea of the proof is to show that the map from the étale cohomology group $H^2_{et}(\overline{X}, \mathbb{Z}/p^n(2))$ onto the highest quotient of the filtration induced by the spectral sequence of $p$-adic vanishing cycles (which degenerates under the assumptions of the theorem) factorizes through the $E_2^{0,2}$-term of the Bloch-Ogus spectral sequence of coniveau. This provides an interesting link between results of Bloch-Kato on $p$-adic étale cohomology in the case of ordinary reduction and assertions of Suslin, Lichtenbaum, Colliot-Thélène and Raskind with respect to the sheaf $\mathcal{K}_2$, resp. $\mathcal{K}_2/p^n$ and its Zariski-cohomology, in particular in characteristic $p$.

The second part of this paper has been taken from the author's thesis [La]. He wishes to express his gratitude to P. Schneider for his constant support and valuable advice. Furthermore he thanks C. Beckmann, W. Raskind and P. Salberger for useful discussions and U. Jannsen for proposing the question preceding Prop. 1.1. Finally he appreciates the referee's suggestions to improve the paper.

## Contents



## §1 Construction of the specialization map

We will construct the specialization map in a more general context. Let $S$ be the spectrum of a discrete valuation ring with generic point $\eta$ and closed point $s$. Denote by $K = k(\eta)$, $k = k(s)$ the corresponding residue fields. Let $\mathcal{X}$ be a smooth $S$-scheme with generic fiber $X_\eta$ and closed fiber $X_s$. Fix a uniformizing element $\pi$ of $S$ and regard $\pi$ as a section in $H^0(X_\eta, G_m)$. Let $O_{\mathcal{X},x}$ denote the local ring at a point $x \in X$. Bloch has proven the Gersten-Conjecture for $\mathcal{K}_2/\operatorname{Spec} O_{\mathcal{X},x}$ ([Bl], Corollary A2).
Let $i_x : \operatorname{Spec} k(x) \to \mathcal{X}$ denote the inclusion of a point $x \in \mathcal{X}$. In particular, let $i_{\mathcal{X}} : \operatorname{Spec} k(\mathcal{X}) \to \mathcal{X}$, $i_{X_\eta} : \operatorname{Spec} k(X_\eta) \to \mathcal{X}$ and $i_{X_s} : \operatorname{Spec} k(X_s) \to \mathcal{X}$ be the inclusion of the generic points of $\mathcal{X}$, $X_\eta$ and $X_s$. The Gersten-Quillen-resolutions of $\mathcal{K}_1/X_s$, $\mathcal{K}_2/\mathcal{X}$ and $\mathcal{K}_2/X_\eta$ give rise to a commutative diagram with exact rows and columns:

$$
(*) \qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & i_{X_s*}k(X_s)^* \arrow[r] \arrow[d] & \coprod_{y\in X_s^1} i_{y*}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
i_{\mathcal{X}*}\mathcal{K}_2(k(\mathcal{X})) \arrow[r] \arrow[d, "="'] & \coprod_{y\in\mathcal{X}^1} i_{y*}k(y)^* \arrow[r, "\varphi"] \arrow[d] & \coprod_{y\in\mathcal{X}^2} i_{y*}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
i_{X_\eta*}\mathcal{K}_2(k(X_\eta)) \arrow[r] & \coprod_{y\in X_\eta^1} i_{y*}k(y)^* \arrow[r] \arrow[d] & \coprod_{y\in X_\eta^2} i_{y*}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

Passing to global sections we derive, by using homological algebra, the following exact sequence

$$(**)\qquad H^0(X_\eta,\mathcal{K}_2)\longrightarrow H^0(X_s,\mathcal{K}_1)\longrightarrow H^1(\mathcal{X},\mathcal{K}_2)\longrightarrow H^1(X_\eta,\mathcal{K}_2)\xrightarrow{\partial}\mathrm{Pic}(X_s)$$

Consider now the following commutative diagram with exact vertical sequences

$$(***)\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{K}_2(k(X_s)) & \longrightarrow & \coprod_{y\in X_s^1} k(y)^* & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
\mathcal{K}_3(k(\mathcal{X})) & \to & \coprod_{y\in\mathcal{X}^1}\mathcal{K}_2(k(y)) & \xrightarrow{\Gamma} & \coprod_{y\in\mathcal{X}^2} k(y)^* & \longrightarrow & \dots \\
{=}\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{K}_3(k(X_\eta)) & \to & \coprod_{y\in X_\eta^1}\mathcal{K}_2(k(y)) & \xrightarrow{\Gamma_\eta} & \coprod_{y\in X_\eta^2} k(y)^* & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Note that the upper and the lower horizontal sequence compute Zariski-$K$-cohomology. By applying the snake lemma (we don't need the Gersten Conjecture for $\mathcal{K}_3/\mathcal{X}$ — this was pointed out to me by P. Salberger) we get from the diagram $(***)$ a boundary map

$$H^1(X_\eta,\mathcal{K}_3)\xrightarrow{\partial'}H^1(X_s,\mathcal{K}_2)\ .$$

By composing $\partial'$ with the map

$$\begin{aligned}H^1(X_\eta,\mathcal{K}_2)&\longrightarrow H^1(X_\eta,\mathcal{K}_3)\\ \alpha&\longmapsto\pi\sqcup\alpha\end{aligned}$$

we have defined our specialization map

$$f: H^1(X_\eta,\mathcal{K}_2)\longrightarrow H^1(X_s,\mathcal{K}_2)$$

(compare with 12.16.5 and the discussion thereafter in [J1] and §8 in [G]).

If $i : X_s \hookrightarrow \mathcal{X}$, $j : X_\eta \hookrightarrow \mathcal{X}$ denote the natural inclusions of the closed and open fiber, there are natural maps

$$\begin{aligned} i^* &: H^1(\mathcal{X}, \mathcal{K}_2) \longrightarrow H^1(\mathcal{X}_s, \mathcal{K}_2) \\ j^* &: H^1(\mathcal{X}, \mathcal{K}_2) \longrightarrow H^1(X_\eta, \mathcal{K}_2) \quad . \end{aligned}$$

Now the following question is quite natural:
Are the maps $i^*$ and $f$ compatible with each other?

The affirmative answer is given in the next proposition.

**Proposition 1.1.**

*Assume that $k$ is infinite or that the relative dimension of $\mathcal{X}$ over $S$ is positive. Then the diagram*

$$\begin{array}{ccc} H^1(\mathcal{X}, \mathcal{K}_2) & \xrightarrow{j^*} & H^1(X_\eta, \mathcal{K}_2) \\ & {\scriptstyle i^*} \searrow & \downarrow f \\ & & H^1(X_s, \mathcal{K}_2) \end{array}$$

*commutes.*

**Remark:**

For technical reasons we cannot avoid the disturbing assumption, the necessity of which will become clear in the proof.

Before proving Proposition 1.1 we show an analogous statement for "Pic", namely that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{X}) = \mathrm{Pic}(X_\eta) & \xrightarrow{\sqcup \pi} & H^1(X_\eta, \mathcal{K}_2) \\ -\downarrow & & \downarrow \partial \\ \mathrm{Pic}(X_s) & \xrightarrow{=} & \mathrm{Pic}(X_s) \end{array} \quad .$$

(The left vertical map is the reduction map for "Pic" on the closed fiber.)
The claim for "Pic" follows immediately from the following lemma.

**Lemma 1.2.**

*Let $v$ denote the discrete valuation on $K^*$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Pic}(X_\eta)\otimes K^* & \stackrel{\mathrm{id}\otimes v}{\longrightarrow} & \mathrm{Pic}(X_\eta)\otimes \mathbb{Z} = \mathrm{Pic}(\mathcal{X}) \\ {\scriptstyle \text{cup-product}}\big\downarrow & & \big\downarrow{\scriptstyle -} \\ H^1(X_\eta,\mathcal{K}_2) & \stackrel{\partial}{\longrightarrow} & \mathrm{Pic}(X_s) \end{array} .$$

**Remark:**

The statement of this lemma is used throughout the proof of Proposition 1.3 in [R1], but not proven there.

**Proof of the lemma:**

Consider again the diagram $(*)$.
Using homological algebra we see that the map $\partial$ is induced by the restriction $\hat{\varphi}$ of $\varphi$:

$$\hat{\varphi}: \coprod_{y\in X^1_\eta\subset X^1} i_{y*}k(y)^* \longrightarrow \coprod_{y\in X^1_s} i_{y*}\mathbb{Z} \quad .$$

Let $y\in X^1_\eta$ and let $x^y_1,\ldots,x^y_r$ be the points in $X^1_s$ which lie in the closure $\overline{\{y\}}$ of $y$ in $\mathcal{X}$. These are points of codimension 1 in $\overline{\{y\}}$. For $\alpha\in k(y)^*$ we then have

$$\hat{\varphi}(\alpha) = \sum_{i=1}^r ord_{x_i,y}(\alpha) \in \coprod_{x^y_1,\ldots,x^y_r} \mathbb{Z} \quad .$$

Note that $O_{\overline{\{y\}},x^y_i}$ has dimension 1 and therefore $ord_{x^y_i}$ is defined on Quot $O_{\overline{\{y\}},x^y_i} = k(y)$. The surjection $\coprod_{y\in X^1_\eta}\mathbb{Z} \twoheadrightarrow \mathrm{Pic}(X_\eta)$ defines a surjection

$$\coprod_{y\in X^1_\eta} K^* \twoheadrightarrow \mathrm{Pic}(X_\eta)\otimes K^* \quad .$$

Regard an element $z\otimes\beta\in \mathrm{Pic}(X_\eta)\otimes K^*$ in this way as an element

$$\sum_i \beta^{n_i} \in \coprod_{y\in X^1_\eta} K^* \subset \coprod_{y\in X^1_\eta} k(y)^* \quad .$$

Then we have

$$^-\circ(\mathrm{id}\otimes v)(z\otimes\beta) = v(\beta)\cdot\overline{z}\in \mathrm{Pic}(X_s) \quad .$$

Here $\overline{z}$ is defined as follows: Let $(n_1, \dots, n_r) \in \coprod_{x \in X^1_\eta} \mathbb{Z}$ be a cycle, which is defined in the points $z_1, \dots, z_r$ and which represents $z$. Under $^- : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_s)$ let $^-(z_i) = \sum_{j=1}^{k_i} n^i_j x^i_j$. Then $\overline{z}$ is the image of $\sum_{i=1}^{r} n_i \sum_{j=1}^{k_i} n^i_j$ under the map $\coprod_{y \in X^1_s} \mathbb{Z} \to \mathrm{Pic}(X_s)$. On the other hand $z \otimes \beta$ is mapped to

$$\delta(\sum_{i=1}^{r} \beta^{n_i}) = \sum_{i=1}^{r} \hat{\varphi}(\beta^{n_i}) = \sum_{i=1}^{r} n_i \sum_{j=1}^{k_i} ord_{x^i_j}(\beta) \quad .$$

It remains to show: $v(\beta) \cdot n^i_j = ord_{x^i_j}(\beta)$. Let $\beta = \pi$ w.l.o.g. $ord_{x^i_j}(\pi) = \ell(O_{\overline{\{z_i\}}, x^i_j}/\pi)$, where $\ell(A)$ denotes the length of a local ring $A$.

Since $\mathcal{X}/\operatorname{Spec} S$ is smooth and $S$ is regular, $\mathcal{X}$ is regular, i.e. all local rings of the structure sheaf of $O_{\mathcal{X}}$ are UFDs. Therefore Weil-divisors correspond uniquely to Cartier-divisors. The effective Cartier divisor, which belongs to $\overline{\{z_i\}}$, induces an effective Cartier divisor $(U, f)$ in a neighbourhood $U$ of $\operatorname{Spec} O_{\mathcal{X}, x^i_j}$ with $f \in O_{\mathcal{X}, x^i_j}$ such that $U \cap \overline{\{z_i\}} = (f)$ as divisors on $U$. Then: $O_{\overline{\{z_i\}}, x^i_j} = O_{\mathcal{X}, x^i_j}/f$ and therefore

$$O_{\overline{\{z_i\}}, x^i_j}/\pi = O_{\mathcal{X}, x^i_j}/(\pi, f) = O_{X_s, x^i_j}/(\overline{f}) \quad ,$$

where $\overline{f}$ denotes the restriction of $f$ on $U \cap X_s$. By definition we have

$$n^i_j = v_{x^i_j}(\overline{f}) = \ell(O_{X_s, x^i_j}/(\overline{f})) \quad .$$

This finishes the proof of lemma 1.2.

**Proof of Proposition 1.1.**

Throughout the proof we will use the following well-known facts about algebraic $K$-groups:

— $K_2(F) = K_2^{Mil}(F)$ for any field $F$, i.e., $K_2(F)$ is generated by symbols.

— The Gersten-Quillen spectral sequence as given in ([Q], §5) is compatible with $\lambda$-operations ([So], Theorem 4). Since there are canonical maps $\alpha_i : K_i^{Mil}(F) \to K_i(F)$ for any field $F$ and all $i$ such that $Im(\alpha_i) \subset F^i_\gamma K_i(F)$ ($F^i_\gamma$ means $\gamma$-filtration, compare [So]), we can conclude by the compatibility of the boundary maps (as proved in [Gra]) that there is a map between the complex denoted by $(S_n)$ in ([K1], §1) using Milnor $K$-groups and the Gersten-Quillen complex ([Q], §5). This implies that the boundary in the Gersten-Quillen resolution, when restricted to Milnor $K$-groups via the maps $\alpha_i$, coincides with the tame symbol as given in ([K1], §1).

Consider now the diagram $(***)$ which was used to construct the specialization map $f$. The maps $\Gamma$ and $\Gamma_\eta$ are induced by the "tame symbol". Let $[\alpha] \in H^1(\mathcal{X}, \mathcal{K}_2)$ be represented by the cycle $\alpha \in \coprod_{y \in X^1} k(y)^*$ with image $\alpha_\eta \in \coprod_{y \in X^1_\eta} k(y)^*$. $\alpha_\eta = \sum_{i=1}^{r} \alpha^i_\eta$ supported in in the points $y_1, \ldots, y_r$. Similarly as for the situation for "Pic"above, the boundary map $\delta$ is given by the restriction $\hat{\Gamma}$ of $\Gamma$:

$$\hat{\Gamma} : \coprod_{y \in X^1_\eta} \mathcal{K}_2(k(y)) \longrightarrow \coprod_{y \in X^1_s} k(y)^* \quad .$$

$\pi \sqcup [\alpha_\eta]$ is represented by the "symbol"

$$\{\pi, \alpha_\eta\} := \sum_{i=1}^{r} \{\pi, \alpha^i_\eta\} \in \coprod_{y \in X^1_\eta} \mathcal{K}_2(k(y)) \quad .$$

Then we have

$$\hat{\Gamma}(\{\pi, \alpha_\eta\}) = -\hat{\Gamma}(\{\alpha_\eta, \pi\}) = \sum_{y \in X^1_s} \prod_{i=1}^{r} tame_{y_{i,y}}(\{\alpha^i_\eta, \pi\})^{-1} \quad ,$$

where $tame_{y_{i,y}}(\{\alpha^i_\eta, \pi\})$ denotes the $y$-component of the image under $\hat{\Gamma}|_{\mathcal{K}_2(k(y_i))}$, considered as an element in $\coprod_{y \in X^1_s} k(y)^*$. It is clear that $\hat{\Gamma}(\{\pi, \alpha_\eta\}) = f([\alpha_\eta])$.

In order to be able to compare $f([\alpha_\eta])$ with $i^*([\alpha])$, we have to study the map $i^*$ more precisely on "symbols": We will construct a map between the Gersten-Quillen-resolutions of the sheaves $\mathcal{K}_2/\mathcal{X}$ and $i_*\mathcal{K}_2/X_s$ which extends the canonical map $\mathcal{K}_2/\mathcal{X} \to i_*\mathcal{K}_2/X_s$. Let $tame_{X_s}$ be the tame symbol on $\mathcal{K}^M_3(k(\mathcal{X}))$ with respect to the discrete valuation $v_{X_s}$, associated to the generic point of $X_s$. Here $\mathcal{K}^M_3$ is Milnor $K$-group.

**Claim 1:**

*The diagram*

$$\begin{array}{ccc} \mathcal{K}_2/\mathcal{X} & \longrightarrow & i_{X_*}\mathcal{K}_2(k(\mathcal{X})) \\ \downarrow & & \downarrow f^0 \\ i_*\mathcal{K}_2/X_s & \longrightarrow & i_{X_s*}\mathcal{K}_2(k(X_s)) \end{array}$$

*commutes, where $f^0$ is defined as follows:*
*If $U \subset X_\eta$, then $f^0(i_{\mathcal{X}_*}\mathcal{K}_2(k(\mathcal{X}))(U)) = 0$. If $U \subset \mathcal{X}$, $U \cap X_s \neq \emptyset$, then $f^0$ is given by*

$$f^0(\{\alpha, \beta\}) := tame_{X_s}(\{\alpha, \beta, \pi\}) \quad .$$

**Proof:**

We show the claim in each stalk. If $x \in X_\eta$, the claim follows. Let $x \in X_s$. For the stalk of $\mathcal{K}_2/\mathcal{X}$ in $x$ we have $\varinjlim_{U \ni x} \mathcal{K}_2(U) = \mathcal{K}_2(O_{\mathcal{X},x})$ (compare [Q] (5.3)). The map $\mathcal{K}_2/\mathcal{X} \to i_*\mathcal{K}_2/X_s$ is given in $x$ as follows:

$$\mathcal{K}_2(O_{\mathcal{X},x}) \to \varinjlim_{U \ni x} \mathcal{K}_2(X_s \cap U) = \mathcal{K}_2(O_{X_s,x}) \quad .$$

Observe that $(X_s \cap U)_U$ is a system of neighbourhoods of $x$ in $X_s$. Now let $O_{\mathcal{X},\pi}$ be the local ring at the generic point of $X_s$. The commutative diagram

$$\begin{array}{ccccccc} & & O_{\mathcal{X},x} & \hookrightarrow & O_{\mathcal{X},\pi} & & \\ & & \downarrow & & \downarrow & & \\ O_{X_s,x} & = & O_{\mathcal{X},x}/\pi & \hookrightarrow & O_{\mathcal{X},\pi/\pi} & = & k(X_s) \end{array}$$

gives rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{K}_2(O_{\mathcal{X},x}) & \longrightarrow & \mathcal{K}_2(O_{\mathcal{X},\pi}) \\ \downarrow & & \downarrow \\ \mathcal{K}_2(O_{X_s,x}) & \longrightarrow & \mathcal{K}_2(k(X_s)) \end{array} \quad .$$

It therefore suffices to show that the diagram

$$\begin{array}{ccc} \mathcal{K}_2(O_{\mathcal{X},\pi}) & \longrightarrow & \mathcal{K}_2(k(\mathcal{X})) \\ & \searrow & \downarrow f^0 \\ & & \mathcal{K}_2(k(X_s)) \end{array}$$

commutes.

Since $O_{\mathcal{X},\pi}$ is a local ring with infinite residue field $k(X_s)$ (here we need the assumption in the proposition) we know that $\mathcal{K}_2^M(O_{\mathcal{X},\pi}) = \mathcal{K}(O_{\mathcal{X},\pi})$, i.e. that $\mathcal{K}_2(O_{\mathcal{X},\pi})$ is generated by symbols $\{a,b\}$, $a,b \in O^*_{\mathcal{X},\pi}$ (compare [Guin], pages 22, 95 and [v.d.K.]). So the map $\mathcal{K}_2(O_{\mathcal{X},\pi}) \to \mathcal{K}_2(k(X_s))$ is just induced by the projection $O_{\mathcal{X},\pi} \to k(X_s)$. For $\{a,b\} \in \mathcal{K}_2(O_{\mathcal{X},\pi})$ we obviously have $v_{X_s}(a) = v_{X_s}(b) = 0$. This means that $tame_{X_s}(\{a,b,\pi\}) = \{\overline{a},\overline{b}\}$ (compare [K1] §1). Therefore the diagram also commutes in the stalks in $x \in X_s$.

**Claim 2:**

*The diagram*

$$\begin{array}{ccc} i_{\mathcal{X}_*}\mathcal{K}_2(k(\mathcal{X})) & \xrightarrow{d_1} & \coprod\limits_{\substack{x \text{ codim } 1\\ \text{in } \mathcal{X}}} i_{\mathcal{X}_*}k(\mathcal{X})^* \\ f^0\downarrow & & \downarrow f^1 \\ i_{X_{s*}}\mathcal{K}_2(k(X_s)) & \xrightarrow{d_1'} & \coprod\limits_{\substack{y \text{ codim } 1\\ \text{in } X_s}} i_{y_*}k(y)^* \end{array}$$

*commutes, where $f^1$ is defined as follows:*
*It is zero in a stalk in $x_0 \in X_\eta$. In a neighbourhood $U$ of $x_0 \in X_s$ it is given as follows:*
*Let $\alpha = \sum_{i=1}^{r} \alpha^i + z$, where $\alpha^i \in k(x_i)^*$ for a point $x_i \in U$ of codimension 1 in $X_\eta$ and $z \in k(X_s)^*$. Consider now the following complex of Milnor $K$-groups:*

$$\mathcal{K}_3^M(k(\mathcal{X})) \xrightarrow{(d_x)_x} \coprod_{\substack{x \text{ codim } 1\\ \text{in } U}} \mathcal{K}_2(k(x)) \xrightarrow{(tame_{x,y})} \coprod_{\substack{y \text{ codim } 2\\ \text{in } U}} k(y)^* .$$

*It is shown in Prop. 1 in [K1] that this is in fact a complex. Then define $f^1(\alpha)$ by*

$$f^1(\alpha) := \sum_{y \in X_s^1} (\prod_{i=1}^{r} tame_{x_i,y}(\{\alpha^i,\pi\}))^{-1} .$$

**Proof of claim 2:**

From the exactness of the Gersten-Quillen-resolution it follows that

$$(tame_{X_s,y} \circ tame_{X_s}(\{\alpha,\beta,\pi\}) \cdot (\prod_{\substack{x\in U \\ x\in X_\eta^1}} tame_{x,y} \circ (\sum_{\substack{x \text{ codim } 1 \\ in\ U \\ x\in X_\eta}} tame_x(\{\alpha,\beta,\pi\}))) = 1$$

($tame_{X_s}$ denotes the tame symbol on $\mathcal{K}_3^M(k(\mathcal{X}))$, associated to the generic point of $X_s$ and $tame_{X_{s,y}}$ the tame symbol on $\mathcal{K}_2(k(X_s))$ associated to the point $y \in X_s^1$).

We also know that $tame_x(\{\alpha,\beta,\pi\}) = \{tame_x(\alpha,\beta),\pi\}$, because $\pi$ is a unit in $O_{\mathcal{X},x}, x \in X_\eta^1$. From these considerations, together with the definition of $f^0$ and $f^1$ above, Claim 2 follows.

Since the sheaf $\coprod_{\substack{x \text{ codim } 2 \\ in\ \mathcal{X}}} i_{x_*}\mathbb{Z}$ of cycles of codimension 2 is equal to the cokernel of $d_1$, the map

$$\overline{f^1} : \coprod_{\substack{x \text{ codim } 2 \\ in\ \mathcal{X}}} i_{x_*}\mathbb{Z} \longrightarrow \coprod_{\substack{y \text{ codim } 2 \\ in\ X_s}} i_{y_*}\mathbb{Z}$$

together with the maps $f^0$ and $f^1$ give rise to a morphism between the Gersten-Quillen-resolution of $\mathcal{K}_2/X$ and $i_*$ (more precisely, the Gersten-Quillen-resolution of $\mathcal{K}_2/X_s$, and $i : X_s \hookrightarrow X$). Since $i_*$ transforms acyclic sheaves in acyclic sheaves, we can conclude for abstract reasons of homological algebra that the map $f^1$ induces the map

$$i^* : H^1(\mathcal{X},\mathcal{K}_2) \longrightarrow H^1(X_s,\mathcal{K}_2)$$

on cohomology. If $[\alpha]$ is a cohomology class in $H^1(\mathcal{X},\mathcal{K}_2)$, the comparison immediately implies that

$$i^*([\alpha]) = f^1([\alpha]) = f([\alpha_\eta]) \ .$$

This finishes the proof of Proposition 1.

**Remarks:**

— The above maps $f^0$ and $f^1$ reflect the operation "intersection of algebraic cycles on $\mathcal{X}$ with the closed fiber $X_s$" on $K$-theoretic symbols.

— The referee pointed out that it is possible to remove the hypothesis in Proposition 1.1, i.e., to prove it for any smooth $S$-scheme $\mathcal{X}$.

## §2 Proof of Theorem 1

We now consider the following situation.
Let $K_\wp/\mathbb{Q}_p$ be a local field of *char* 0. $X/K_\wp$ projective, smooth, geometrically connected with good reduction. If $o_\wp$ are the integers in $K_\wp$, let now $R$ be a discrete valuation ring with Quot $R = L$ and residue field $\overline{k}$, where $k$ denotes the residue field of $o_\wp$. Obviously we have $\overline{o}_\wp = \varinjlim_R R$, where $R$ is as above.
Let $\mathcal{X}$ be a smooth proper model of $X$ over $o_\wp$, $\mathcal{X}_R := \mathcal{X} \underset{o_\wp}{\times} R$ and $Y_k$, $Y_{\overline{k}}$ the closed fiber of $\mathcal{X}$ and $\mathcal{X}_R$. Let $X_L = X \underset{K_\wp}{\times} L$. The exact sequence (**), applied to our present situation, looks as follows:

$$\begin{array}{ccccccccc} H^0(X_L,\mathcal{K}_2) & \longrightarrow & H^0(Y_{\overline{k}},\mathcal{K}_1) & \longrightarrow & H^1(\mathcal{X}_R,\mathcal{K}_2) & \longrightarrow & H^1(X_L,\mathcal{K}_2) & \xrightarrow{\partial} & \mathrm{Pic}(Y_{\overline{k}}). \\ & & \| & & & & & & \\ & & \overline{k}^* & & & & & & \end{array}$$

The composition of the first map with the canonical map from $K_2(X_L)$ to $H^0(X_L,\mathcal{K}_2)$ yields the tame symbol map $K_2(X_L) \to K_1(Y_{\overline{k}}) \cong \overline{k}^*$. It is surjective.

The Gersten-Quillen spectral sequence is contravariant for flat morphisms of noetherian schemes. In particular the spectral sequence for $\overline{X} = \varprojlim X_L$ (over $L$ as above) is the inductive limit of the spectral sequences for $X_L$.

This implies:

$$H^1(\overline{X},\mathcal{K}_2) = \varinjlim H^1(X_L,\mathcal{K}_2) \quad .$$

For a finite extension of discretely valued fields $L'/L$ with valuations $v_L, v_{L'}$ and with ramification index $e$ the diagram

$$\begin{array}{ccc} L & \xrightarrow{v_L} & \mathbb{Z} \\ \downarrow & & \downarrow\cdot e \\ L' & \xrightarrow{v_{L'}} & \mathbb{Z} \end{array}$$

commutes. Therefore Lemma 1.2 implies that the following diagram commutes:

$$\begin{array}{ccc} H^1(X_L,\mathcal{K}_2) & \xrightarrow{\partial} & \mathrm{Pic}(Y_{\overline{k}}) \\ \downarrow & & \downarrow\cdot e \\ H^1(X_{L'},\mathcal{K}_2) & \xrightarrow{\partial} & \mathrm{Pic}(Y_{\overline{k}}) \end{array}$$

Passing to the limit $R \to \overline{o}_{\wp}$, $L = \operatorname{Quot} R \to \overline{K}$ we get the following exact sequence

$$0 \longrightarrow \varinjlim_R H^1(\mathcal{X}_R, \mathcal{K}_2) \longrightarrow H^1(\overline{X}, \mathcal{K}_2) \xrightarrow{\partial} \operatorname{Pic}(Y_{\overline{k}}) \otimes \mathbb{Q} \quad .$$

In particular, we can conclude for the $p$-primary torsion:

$$\varinjlim_R H^1(\mathcal{X}_R, \mathcal{K}_2)(p) \cong H^1(\overline{X}, \mathcal{K}_2)(p) \quad .$$

Since for $R \subset R'$ the diagram

$$\begin{array}{ccc} & & \mathcal{X}_{R'} \\ & \nearrow & \\ Y_{\overline{k}} & & \downarrow \\ & \searrow & \\ & & \mathcal{X}_R \end{array}$$

commutes, we can consider the natural reduction map $i^*$ on the limit

$$i^* \colon \varinjlim_R H^1(\mathcal{X}_R, \mathcal{K}_2) \longrightarrow H^1(Y_{\overline{k}}, \mathcal{K}_2) \quad .$$

Using Proposition 1.1, it is now clear that Theorem 1 follows from the following proposition.

**Proposition 2.1:**

*The notations are as above. In addition let $Y_k$ be ordinary in the sense of ([BK], §7) and* $\operatorname{Pic}(Y_{\overline{k}})(p) = 0$, $\dim X < p - 1$, $K/\mathbb{Q}_p$ *unramified. Then the map*

$$i^* \colon \varinjlim_R H^1(\mathcal{X}_R, \mathcal{K}_2) \longrightarrow H^1(Y_{\overline{k}}, \mathcal{K}_2)$$

*is surjective on the $p$-primary torsion groups.*

**Remark:**

Proposition 2.1 completes a result of Raskind, asserting that the $\ell$-primary torsion groups for $\ell \nmid p$ of $H^1(\mathcal{X}_R, \mathcal{K}_2)$ and $H^1(Y_{\overline{k}}, \mathcal{K}_2)$ under the map $i^*$ are isomorphic ([R1], Theorem 2.5 b).

**Proof of Proposition 2.1:**

We first note that

$$H^1(\overline{X}, \mathcal{K}_2)(p) \cong H^2_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

by ([CT-R], Theorem 2.1). The exact sequence

$$0 \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{K}_2/p^n \longrightarrow 0 \quad (\text{in char } p)$$

together with the $p$-divisibility of $H^0(Y_{\overline{k}}, \mathcal{K}_2)$ ([R2], Theorem 2.3) induces an isomorphism

$$H^1(Y_{\overline{k}}, \mathcal{K}_2)(p) \cong \varinjlim_n H^0(Y_{\overline{k}}, \mathcal{K}_2/p^n) \quad .$$

Throughout the proof let $i$ denote the closed immersions $Y_{\overline{k}} \hookrightarrow \mathcal{X}_R$ resp. $Y_{\overline{k}} \hookrightarrow \mathcal{X}_{\overline{o}_\wp}$ and $f$ the open immersions $\overline{X} \hookrightarrow \mathcal{X}_{\overline{o}_\wp}$ resp. $X_L \hookrightarrow \mathcal{X}_R$. For a scheme $Z$ let $\pi\colon Z_{et} \to Z_{Zar}$ be the canonical morphism of sites. The idea of the proof can be described as follows:

We have seen that $i^*$ induces a map

$$H^2_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \varinjlim_n H^0_{et}(Y_{\overline{k}}, \mathcal{K}_2/p^n)$$

on the $p$-primary torsion groups. Since (with the assumptions in Proposition 2.1) the spectral sequence of $p$-adic vanishing cycles

$$H^i(Y_{\overline{k}}, i^* R^j f_* \mathbb{Z}/p^m(n)) \Longrightarrow H^{i+j}(\overline{X}, \mathbb{Z}/p^m(n))$$

degenerates ([K2], Theorem 4.3), we have a surjection

$$\begin{aligned} H^2_{et}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) &\twoheadrightarrow \varinjlim_n H^0_{et}(Y_{\overline{k}}, i^* R^2 f_* \mathbb{Z}/p^n(2)) \\ &\cong \varinjlim_n H^0_{et}(Y_{\overline{k}}, W_n\Omega^2_{log}) \end{aligned}$$

where the last isomorphism is proven in ([BK] §6), using the assumption that $Y_k$ is ordinary. $H^i_{et}(Y_{\overline{k}}, W_n\Omega^j_{log})$ denotes logarithmic Hodge-Witt cohomology, as introduced in ([I], §3.3). New results of Lichtenbaum (compare Lemma 2.2 in the proof) imply that $\mathcal{K}_2/p^n \cong \pi_* W_n\Omega^2_{log}$. Therefore the proof of the proposition boils down to the compatibility of the above maps under this isomorphism, more precisely to the following

**Claim:** The diagram on the following page is commutative.

That the left vertical arrow is an isomorphism, follows from the proper base change theorem. The maps (1), (2) and (3) are canonical maps of sections of presheaves into sections of sheaves. In (3) we have to observe that there is an isomorphism $\mathcal{H}^2 \cong \mathcal{K}_2/p^n$, where $\mathcal{H}^2$ is the sheaf (in the Zariski topology) associated to the presheaf $U \to H^2_{et}(U, \mathbb{Z}/p^n(2))$. This follows from a comparison of the Quillen resolution of the sheaf $\mathcal{K}_2$ (and tensoring with $\mathbb{Z}/p^n$) and the Bloch-Ogus resolution of the Zariski sheaf $\mathcal{H}^2$ using the fact that the Galois symbol $K_2(F)/p^n \to H^2(F, \mu_{p^n}^{\otimes 2})$ is an ismorphism for a field $F$, *char* $F \neq p$ (Merkurjev-Suslin), see ([CT-R], section 1). Then (3) is just the natural map from the $E^2$- to the $E_2^{0,2}$-term arising from the Bloch-Ogus spectral sequence of coniveau. $\hat{R}^2 f_* \mathbb{Z}/p^n(2)$ is the Zariski sheaf associated to the presheaf $U \to H^2_{et}(f^{-1}(U), \mathbb{Z}/p^n(2))$ on $\mathcal{X}_{\overline{o}_\wp}$. (7) is induced by the universal property of the sheafification functor

$$H^2_{et}(f^{-1}(U), \mathbb{Z}/p^n(2)) \xrightarrow{id} H^2_{et}(f^{-1}(U), \mathbb{Z}/p^n(2)) \longrightarrow \mathcal{H}^2(f^{-1}(U))$$

The diagrams (A) and (B) commute for trivial reasons.
We have a commutative diagram of isomorphisms:

$$\begin{array}{ccc} H^0(\overline{X}, \mathcal{K}_2/p^n) & \cong & H^0(\mathcal{X}_{\overline{o}_\wp}, f_*\mathcal{K}_2/p^n) \\ \downarrow\cong & & \downarrow\cong \\ \varinjlim_{L \text{ as above}} H^0(X_L, \mathcal{K}_2/p^n) & \cong & \varinjlim_{R \text{ as above}} H^0(\mathcal{X}_R, f_*\mathcal{K}_2/p^n) \end{array}$$

**Lemma 2.2:**

$$\mathcal{K}_2/p^n \cong \pi_* W_n \Omega^2_{log} \quad in\ (Y_{\overline{k}})_{Zar} \quad .$$

**Proof:**

Following ([L], Lemma 2.7) we have the exact triangle (in *char* $p$):

$$\begin{array}{ccccc} & & W_n\Omega^2_{log}[-2] & & \\ & {}^{+1}\swarrow & & \nwarrow & \\ \Gamma(2) & & \xrightarrow{p^n} & & \Gamma(2) \end{array}$$

This is one of the axioms for the Lichtenbaum-complexes $\Gamma(n)$ (compare [L]), which are known to exist for $n \leq 2$ and regular schemes over fields. Apply the

$$
\begin{tikzcd}[column sep=small]
 & H^2_{et}(\overline{X},\mathbb{Z}/p^n(2)) \arrow[dl, "(1)"'] \arrow[ddd, "(2)"] \arrow[dr, "(3)"] \arrow[drr, "{[Su]}"] \arrow[ddddl, bend right=60, "\simeq"'] \arrow[l, phantom, "\text{©}"] \arrow[drr, phantom, "\text{Ⓕ}"] & & & \\
H^0_{et}(\mathcal{X}_{\bar{\mathfrak{o}}_{\mathfrak{p}}},R^2f_*\mathbb{Z}/p^n(2)) \arrow[d, equal] \arrow[dr, phantom, "\text{Ⓐ}"] & & H^0_{Zar}(\overline{X},\mathcal{K}_2/p^n) \arrow[r, "(12)"] \arrow[d, equal] \arrow[dl, phantom, "\text{Ⓑ}"] & H^1(\overline{X},\mathcal{K}_2)/\ker(3) & \\
H^0_{Zar}(\mathcal{X}_{\bar{\mathfrak{o}}_{\mathfrak{p}}},\pi_*R^2f_*\mathbb{Z}/p^n(2)) \arrow[d] & H^0_{Zar}(\mathcal{X}_{\bar{\mathfrak{o}}_{\mathfrak{p}}},\hat{R}^2f_*\mathbb{Z}/p^n(2)) \arrow[l] \arrow[r, "(7)"] \arrow[d] & \varinjlim\limits_R H^0_{Zar}(\mathcal{X}_R,f_*\mathcal{K}_2/p^n) \arrow[r, "\simeq", "(13)"'] \arrow[d, phantom, "\text{Ⓓ}"] & \varinjlim\limits_R H^0_{Zar}(\mathcal{X}_R,\mathcal{K}_2/p^n) \arrow[r] \arrow[d] & \varinjlim\limits_R H^1(\mathcal{X}_R,\mathcal{K}_2)/\ker(3) \arrow[d] \arrow[ul, bend right] \\
H^0_{Zar}(Y_{\bar{k}},i^*\pi_*R^2f_*\mathbb{Z}/p^n(2)) \arrow[d, "(4)"] \arrow[dr, phantom, "\text{Ⓔ}"] & H^0_{Zar}(Y_{\bar{k}},i^*\hat{R}^2f_*\mathbb{Z}/p^n(2)) \arrow[l] \arrow[d, "(5)"] & & H^0_{Zar}(Y_{\bar{k}},i^*\mathcal{K}_2/p^n) \arrow[r] \arrow[d] & H^1(Y_{\bar{k}},i^*\mathcal{K}_2)/\mathrm{Im}(\ker(3)) \arrow[d] \\
H^0_{Zar}(Y_{\bar{k}},\pi_*i^*R^2f_*\mathbb{Z}/p^n(2)) \arrow[r, "\simeq", "(6)"'] \arrow[d, "\wr"] & H^0_{Zar}(Y_{\bar{k}},\pi_*W_n\Omega^2_{log}) \arrow[rr, "\simeq", "(11)"'] & & H^0_{Zar}(Y_{\bar{k}},\mathcal{K}_2/p^n) \arrow[r, hook] & H^1(Y_{\bar{k}},\mathcal{K}_2) \\
H^0_{et}(Y_{\bar{k}},i^*R^2f_*\mathbb{Z}/p^n(2)) & & & &
\end{tikzcd}
$$

functor $R\pi_*$ to the above triangle. Then the fact that $R^3\pi_*\Gamma(2) = 0$ (axiom "Hilbert 90") yields the exact triangle

$$\begin{array}{ccc} & \pi_* W_n\Omega^2_{log}[-2] & \\ {}^{+1}\swarrow & & \nwarrow \\ \tau_{\leq 2}R\pi_*\Gamma(2) & \longrightarrow & \tau_{\leq 2}R\pi_*\Gamma(2) \end{array} \quad \text{in } (Y_{\overline{k}})_{Zar} \ ,$$

where $\tau_{\leq 2}\mathcal{K}$ is the truncated complex

$$\to \mathcal{K}^{-1} \to \mathcal{K}^0 \to \mathcal{K}^1 \to Ker\, d^2 \to 0 \to \dots$$

in degree 2. The statement

$$R^2\pi_*\Gamma(2) \cong (\mathcal{K}_2)_{Zar} \quad ([L], \text{ Theorem 2.10.})$$

finishes the proof of the Lemma.
The isomorphism $(\mathcal{K}_2/p^n)_{et} \cong W_n\Omega^2_{log}$ in $(Y_{\overline{k}})_{et}$ implies in particular

$$H^0_{Zar}(Y_{\overline{k}}, \mathcal{K}_2/p^n) \cong H^0_{et}(Y_{\overline{k}}, \mathcal{K}_2/p^n) \ .$$

Therefore (11) is an isomorphism.
The map (4) is induced by the canonical natural transformation

$$i^*\pi_* \longrightarrow \pi_* i^*$$

between the functors of topoi

$$i^*\pi_* : S(\mathcal{X}_{\overline{\mathfrak{o}}_{\wp}})_{et} \longrightarrow S(Y_{\overline{k}})_{Zar}$$

and

$$\pi_* i^* : S(\mathcal{X}_{\overline{\mathfrak{o}}_{\wp}})_{et} \longrightarrow S(Y_{\overline{k}})_{Zar}$$

One can easily see that under the map (4) the diagram (C) commutes. (5) is defined by composing the other arrows in diagram (E). (6) is an isomorphism by ([BK], Theorem 9.2).

**Remark:**
The map

$$i^*R^2 f_*\mathbb{Z}/p^n(2) \longrightarrow \pi_* W_n\Omega^2_{log} \ ,$$

which induces (5) factorizes through the Zariski sheaf

$$R^2\pi_* i^* Rf_*\mathbb{Z}/p^n(2) \ ,$$

which was studied by O. Gabber (compare ([BK], section 6.6)). It is the Zariski analog of the étale sheaf $i^*R^2 f_*\mathbb{Z}/p^n(2)$ of $p$-adic vanishing cycles. But it is not needed here.

**Lemma 2.3:**

*Let* $\mathrm{Pic}(Y_{\overline{k}})(p)$ *be torsion-free,* $R$ *as above,* $L = \mathrm{Quot}\, R$.

*Then:*

$$\varinjlim_{R} H^0(\mathcal{X}_R, \mathcal{K}_2/p^n) \cong H^0(\overline{X}, \mathcal{K}_2/p^n) \quad .$$

**Proof:**

We show:

$$H^0(\mathcal{X}_R, \mathcal{K}_2/p^n) \cong H^0(X_L, \mathcal{K}_2/p^n) \quad .$$

As in the case $H^1(\overline{X}, \mathcal{K}_2)$ the claim then follows from

$$\varinjlim_{L} H^0(X_L, \mathcal{K}_2/p^n) \cong H^0(\overline{X}, \mathcal{K}_2/p^n) \quad .$$

Following Raskind (see [R1], Proposition 1.1) the Gersten-Quillen complex $\mathrm{mod}\, p^n$ is a flabby resolution of the sheaf $\mathcal{K}_2/p^n$ over $\mathcal{X}_R$. Consider now the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \mathcal{K}_2(k(X_L))/p^n & \to & \coprod_{x \in X_L^1} k(x)^*/p^n & \to & \coprod_{x \in X_L^2} \mathbb{Z}/p^n \\
 & & \| & & \uparrow & & \uparrow \\
0 & \to & \mathcal{K}_2(k(\mathcal{X}_R))/p^n & \to & \coprod_{x \in \mathcal{X}_R^1} k(x)^*/p^n & \to & \coprod_{x \in \mathcal{X}_R^2} \mathbb{Z}/p^n \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0 & \to & k(Y_{\overline{k}})^*/p^n & \to & \coprod_{x \in X_{\overline{k}}^1} \mathbb{Z}/p^n \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
\end{array}$$

Compare this with the diagram (*) in §1. This yields the exact sequence:

$$0 \longrightarrow H^0(\mathcal{X}_R, \mathcal{K}_2/p^n) \longrightarrow H^0(X_L, \mathcal{K}_2/p^n) \longrightarrow H^0(Y_{\overline{k}}, \mathcal{K}_1/p^n) \quad .$$

In *char* $p$ we have the exact sequence

$$0 \longrightarrow \mathcal{K}_1 \xrightarrow{p^n} \mathcal{K}_1 \longrightarrow \mathcal{K}_1/p^n \longrightarrow 0 \ .$$

Since $H^0(Y_{\overline{k}}, \mathcal{K}_1) \cong \overline{k}^* \xrightarrow{p^n} H^0(Y_{\overline{k}}, \mathcal{K}_1)$ is an isomorphism, we get:

$$H^0(Y_{\overline{k}}, \mathcal{K}_1/p^n) \cong \mathrm{Pic}(Y_{\overline{k}})_{p^n} = 0 \text{ by assumption, therefore:}$$
$$H^0(\mathcal{X}_R, \mathcal{K}_2/p^n) \cong H^0(X_L, \mathcal{K}_2/p^n)$$

Lemma 2.3 implies that (13) is an isomorphism. The right vertical arrows in diagram II are induced by the map $i^*$. The commutativity of (F) follows from Suslin ([Su], §4). (12) is induced from the short exact sequences of sheaves

$$0 \longrightarrow {}_{p^n}\mathcal{K}_2 \longrightarrow \mathcal{K}_2 \longrightarrow I \longrightarrow 0$$
$$0 \longrightarrow I \longrightarrow \mathcal{K}_2 \longrightarrow \mathcal{K}_2/p^n \longrightarrow 0$$

and the fact that $H^2(\overline{X}, {}_{p^n}\mathcal{K}_2) = 0$. In particular we have:

$$ker\,(3) \cong H^1(\overline{X}, {}_{p^n}\mathcal{K}_2) \ .$$

The unnamed arrows are canonical maps and the unnamed diagrams commute for trivial reasons. It remains to show that the diagram (D) commutes.

**Lemma 2.4**

*The canonical map*

$$R^2 f_* \mathbb{Z}/p^n(2) \longrightarrow f_*\mathcal{H}^2$$

*is an isomorphism.*

**Proof:**

We prove this stalkwise. Let $x \in \overline{X}$. Then it is clear that

$$R^2 f_* \mathbb{Z}/p^n(2)_x = \varinjlim_{\substack{U \ni x \\ U \subset \overline{X}}} H^2_{et}(U, \mathbb{Z}/p^n(2)) = \varinjlim \mathcal{H}^2(U) \ .$$

Let now $x \in Y_{\overline{k}}$. Then we need to show that the canonical map

$$\varinjlim_{U \ni x} H^2(f^{-1}(U), \mathbb{Z}/p^n(2)) \longrightarrow \varinjlim_{U \ni x} \mathcal{H}^2(f^{-1}(U))$$

is an isomorphism. Consider the following commutative diagram, which is taken from the proof of Lemma 1.1 in [CT-R] ($\mathcal{H}^i(\mu_{p^n})$ denotes the Zariski sheaf associated to the presheaf $U \to H^i_{et}(U, \mu_{p^n})$ on $\overline{X}$)

$$\begin{array}{ccccccccc} 0 \to & H^1(\overline{X}, \mathcal{H}^1(\mu_{p^n})) & \to & H^2(\overline{X}, \mu_{p^n}) & \to & H^0(\overline{X}, \mathcal{H}^2(\mu_{p^n})) & \to 0 \\ & \cong\uparrow & & \| & & \downarrow\cong & \\ 0 \to & \mathrm{Pic}(\overline{X})/p^n & \to & H^2(\overline{X}, \mu_{p^n}) & \to & {}_{p^n}Br(\overline{X}) & \to 0. \end{array}$$

The exact upper row arises from the Bloch-Ogus spectral sequence and uses results from [BO]. The lower exact sequence is part of the étale cohomology sequence associated to the Kummer sequence. The isomorphism on the left hand side is also established in [BO]. Now consider the same commutative diagram for $f^{-1}(U)$ instead of $\overline{X}$ for any $U, x \in U$. One gets the following exact sequence using the left and right vertical isomorphisms:

$$0 \to \varinjlim_U \mathrm{Pic}(f^{-1}(U))/p^n \to \varinjlim_U H^2(f^{-1}(U), \mu_{p^n}) \to \varinjlim_U H^0(f^{-1}(U), \mathcal{H}^2(\mu_{p^n})) \to 0$$

One knows that $\varprojlim_U f^{-1}(U) = \mathrm{Spec}\, O_{\mathcal{X}_{\bar{o}_\wp},x}[\frac{1}{p}]$. Therefore we conclude:
$\varinjlim_U \mathrm{Pic}(f^{-1}(U))/p^n = \mathrm{Pic}(\mathrm{Spec}\, O_{\mathcal{X}_{\bar{o}_\wp},x}[\frac{1}{p}])/p^n$. Let $x_R$ be the image sheaf of $x$ under $\mathcal{X}_{\bar{o}_\wp} \to \mathcal{X}_R$. Then we have $\mathrm{Spec}\, O_{\mathcal{X}_{\bar{o}_\wp},x}[\frac{1}{p}] = \varprojlim_R \mathrm{Spec}\, O_{\mathcal{X}_R,x_R}[\frac{1}{p}]$. Since $O_{\mathcal{X}_R,x_R}[\frac{1}{p}]$ is the localization of a UFD it is a UFD itself. This implies $\mathrm{Pic}(\mathrm{Spec}\, O_{\mathcal{X}_R,x_R}[\frac{1}{p}]) = 0$ and also
$\mathrm{Pic}(\mathrm{Spec}\, O_{\mathcal{X}_{\bar{o}_\wp},x}[\frac{1}{p}]) = \varinjlim_R \mathrm{Pic}(\mathrm{Spec}\, O_{\mathcal{X}_R,x_R}[\frac{1}{p}]) = 0$. This means that there is an isomorphism

$$\varinjlim_U H^2_{et}(f^{-1}(U), \mu_{p^n}) \xrightarrow{\cong} \varinjlim_U H^0(f^{-1}(U), \mathcal{H}^2(\mu_{p^n})) \quad .$$

By tensoring with $\mu_{p^n}$ the assertion follows.

Lemma 2.4 implies that the arrow (7) is in fact an isomorphism. Now the commutativity of the diagram (D) follows from the subsequent observations:

a) The sheaf $\mathcal{K}_2$ is locally generated by symbols in the Zariski topology and as well in the étale topology.

b) The following diagrams of Zariski sheaves commute:

$$\begin{tikzcd}
\hat{R}^2 f_* \mathbb{Z}/p^n(2) \arrow[d] & \mathcal{K}_2/p^n \arrow[l] \arrow[d] \\
\pi_* i_* i^* R^2 f_* \mathbb{Z}/p^n(2) & \pi_* i_* i^* (\mathcal{K}_2/p^n)_{et} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
i^*(\mathcal{K}_2/p^n)_{\mathcal{X}_R} \arrow[r] \arrow[dd] & i^* \pi_* (\mathcal{K}_2/p^n)_{\mathcal{X}_R, et} \arrow[d] \\
 & \pi_* i^* (\mathcal{K}_2/p^n)_{\mathcal{X}_R, et} \arrow[d] \\
(\mathcal{K}_2/p^n)_{Zar, Y_{\overline{k}}} \arrow[r, "\cong"'] & \pi_* (\mathcal{K}_2/p^n)_{Y_{\overline{k}}, et}
\end{tikzcd}$$

For the lower isomorphism see Lemma 2.2.

c) The sheaf $i^* R^2 f_* \mathbb{Z}/p^n(2)$ is étale-locally generated by "symbols" of the form $i^* x_1 \otimes i^* x_2$, where $x_i \in R^1 f_* \mathbb{Z}/p^n(1)$, $x_i = \partial(y_i)$ with $y_i \in f_* O^*_{\overline{X}}$ and $\partial$ denotes the boundary operator in the following exact sequence

$$0 \to f_* \mu_{p^n} \to f_* O^*_{\overline{X}} \to f_* O^*_{\overline{X}} \to R^1 f_* \mathbb{Z}/p^n(1) \to 0$$

(compare [BK] §6). Furthermore we may assume that $y_i \in O^*_{\mathcal{X}_{\overline{o}_\wp}}$ for $i = 1, 2$ or that $y_1 \in O^*_{\mathcal{X}_{\overline{o}_\wp}}$ and $y_2 = \pi$, the uniformizing element in $O_K$.

d) The map $\lambda : i^* R^2 f_* \mathbb{Z}/p^n(2) \to W_n \Omega^2_{log}$ which induces the isomorphism (6) is given on "symbols" as follows: Let $y_1, y_2 \in O^*_{\mathcal{X}_{\overline{o}_\wp}}$, $\lambda(i^* \partial(y_1) \otimes i^* \partial(y_2)) = \frac{d\overline{y}_1}{y_1} \wedge \frac{d\overline{y}_2}{\overline{y}_2}$, where $\overline{y}_i$ is the image of $y_i$ under the map $i^* O^*_{\mathcal{X}_{\overline{o}_\wp}} \to O^*_{Y_{\overline{k}}}$. Otherwise $\lambda(i^* \partial(y_1) \otimes \pi) = 0$.

This finishes the proof of Proposition 2.1 and of Theorem 1.

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