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1 Introduction

Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds, $\mathcal{M}$ a coherent module over the ring $\mathcal{D}_X$ of differential operators on X, F an $\mathbb{R}$-constructible object on X. In this first paper, we give a criterion insuring that the derived direct images of the $\mathcal{D}_X$-module $F \otimes \mathcal{M}$ are coherent $\mathcal{D}_Y$-modules, and we prove related duality and Künneth formulas. Part of these results were announced in [20, 21].

In [22], making full use of these results, we shall associate to $(\mathcal{M}, F)$ a characteristic class and show its compatibility with direct image, thus obtaining an index theorem generalizing (in some sense) the Atiyah-Singer index theorem as well as its relative version [1, 3].

Let us describe our results with more details, beginning with the non-relative case for the sake of simplicity.

An elliptic pair on a complex analytic manifold X is the data of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ and an $\mathbb{R}$-constructible sheaf F on X (more precisely, objects of the derived categories), these data satisfying the transversality condition

$$\operatorname{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X. \tag{1.1}$$

Here $\operatorname{char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$, $SS(F)$ the micro-support of F (see [12]) and $T^*_X X$ the zero section of the cotangent bundle T^*X.

This notion unifies many classical situations. For example, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then the pair $(\mathcal{M}, \mathbb{C}_X)$ is elliptic. If U is an open subset of X with smooth boundary ∂U, the pair $(\mathcal{M}, \mathbb{C}_U)$ is elliptic if and only if ∂U is non characteristic for $\mathcal{M}$. If X is the complexification of a real analytic manifold M, then $(\mathcal{M}, \mathbb{C}_M)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M in the classical sense. If F is $\mathbb{R}$-constructible on X, then $(\mathcal{O}_X, F)$ is an elliptic pair. If $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, we can associate to it the coherent $\mathcal{D}_X$-module $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and the results obtained for the elliptic pair $(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathbb{C}_X)$ will give similar results for $\mathcal{G}$. See §8 for a more detailed discussion.

If $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds, we generalize the preceding definition and introduce the notion of an f-elliptic pair, replacing in (1.1) $\operatorname{char}(\mathcal{M})$ by $\operatorname{char}_f(\mathcal{M})$, the f-characteristic variety of $\mathcal{M}$ (this set was already defined in [19] when f is smooth).

The main results of this paper assert that if the pair $(\mathcal{M}, F)$ is f-elliptic, f is proper on $\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(F)$ and $\mathcal{M}$ is endowed with a good filtration, then:

1) the direct image (in the sense of $\mathcal{D}$-modules) $\underline{f}_!(\mathcal{M} \otimes F)$ has $\mathcal{D}_Y$-coherent cohomology,

2) the duality morphism

$$\underline{f}_!(D'F \otimes \underline{D}_X \mathcal{M}) \longrightarrow \underline{D}_Y \underline{f}_!(\mathcal{M} \otimes F)$$

is an isomorphism (here, $\underline{D}$ denotes the dualizing functor for $\mathcal{D}$-modules and D' is the simple dual for sheaves),

3) there is a Künneth formula for elliptic pairs,

4) direct image commutes with microlocalization.

See Theorem 4.2, Theorem 5.15, Theorem 6.7 and Theorem 7.5 below for more details.

In fact, we obtain these results in a relative situation over a smooth complex manifold S, working with the rings of relative differential operators. This relative setting makes notations a little heavy but it gives us the freedom on the base manifold we need in the proofs. Even if we want the final result over a base manifold reduced to a point, in the proofs, we need to use other bases. So, it is better to work in a relative situation everywhere. Moreover, the base change Theorem 6.5 is a natural way to get the Künneth formula for elliptic pairs.

The idea of the proof of the finiteness result goes as follows.

First, using the graph embedding, we are reduced to prove the theorem for a closed embedding (this one does not offer much difficulty) and for a projection. Then, using the same trick as in [8], we reduce to the case $Y = S$. Then it remains to treat the case where $X = Z \times S$, $f : X \longrightarrow S$ is the second projection, $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module endowed with a good filtration and $F = G \boxtimes \mathbb{C}_S$ where G is an $\mathbb{R}$-constructible sheaf on Z. We call it the projection case and we have to prove that in this case $Rf_!(F \otimes \mathcal{M} \otimes_{\mathcal{D}_{X|S}} \mathcal{O}_X)$ is $\mathcal{O}_S$ coherent and $\mathcal{O}_S$ dual to $Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_X - d_S])$.

For that purpose, we "trivialize" F by replacing it by a bounded complex of sheaves of the form $\oplus_\alpha \mathbb{C}_{U_\alpha}$, the U_α's being relatively compact subanalytic open subsets of X satisfying the regularity condition:

$$D'(\mathbb{C}_{U_\alpha}) = \mathbb{C}_{\overline{U}_\alpha}.$$

This construction is made possible thanks to the triangulation theorem and a result of Kashiwara [11].

Next, we consider the relative realification $\mathcal{M}_{\mathbb{R}|S}$ of $\mathcal{M}$ obtained by adding the relative Cauchy-Riemann system to the $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$ and remark that since $\mathcal{M}$ is assumed to be good we may always find a resolution of $\mathcal{M}_{\mathbb{R}|S}$ by finite free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules near subsets of $Z^{\mathbb{R}} \times S$ of the form $K \times \Delta$ where K is a compact subset of Z and Δ is an open polydisc in S. Moreover, since the solutions of the relative Cauchy-Riemann system are the same in the sheaves of analytic functions, differentiable functions or distributions with holomorphic parameters in S, we can compute the holomorphic solutions of $\mathcal{M}$ as the relative analytic, differentiable or distributional solutions of $\mathcal{M}_{\mathbb{R}|S}$.

Now, the elliptic hypothesis insures the regularity theorem, that is, the isomorphism

$$F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X \xrightarrow{\sim} R\mathcal{H}om(D'F, \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X).$$

Applying $Rf_!$ to this isomorphism, we shall compute both sides using the trivialization of F and a finite free resolution of the relative realification of $\mathcal{M}$ using analytic (resp.

differentiable) solutions for the left (resp. right) hand side. This will give us a continuous $\mathcal{O}_S$-linear quasi-isomorphism

$$\mathcal{R}_1^{\cdot} \xrightarrow{\sim} \mathcal{R}_2^{\cdot} \tag{1.2}$$

where the components of the left (resp. right) hand side are DFN-free (resp. FN-free) topological modules over the Fréchet algebra $\mathcal{O}_S$. The coherence then follows from an extension of Houzel's finiteness theorem [7] due to one of the authors [25]. Note that we found no way of applying the original Houzel's theorem in our situation since it is not obvious to find the requested chain of nuclear quasi-isomorphisms for a given elliptic pair.

The duality result is proved along the same lines once we have a clear construction of the general duality morphism which makes it easy to check its compatibility with the various simplifications and transformations used in the proof.

Note that the hypothesis that the $\mathcal{D}$-module $\mathcal{M}$ is endowed with a good filtration could be relaxed by using cohomological descent techniques as in [24]. However, doing so would have cluttered the proof with unessential technical difficulties. This is why we have preferred to stay to a simpler setting, sufficient for all known applications.

Our theorems provide a wide generalization of many classical results as shown in the last section.

In particular, we obtain Grauert's theorem [6] (in the smooth case) on direct images of coherent $\mathcal{O}$-modules and the corresponding duality result of Ramis-Ruget-Verdier [15, 16]. Since we treat $\mathcal{D}$-modules, we are allowed to "realify" the manifolds by adding the Cauchy-Riemann system to the module, and the rigidity of the complex situation disappears, which makes the proofs much simpler and, may be, more natural than the classical ones.

We also obtain Kashiwara's theorem [9] on direct images of coherent $\mathcal{D}$-modules as well as its extension to the non-proper case of [8] (whose detailed proof had never been published) and the corresponding duality result of [23, 24].

In the absolute case, we regain and generalize many well-known theorems concerning regularity, finiteness or duality for $\mathcal{D}$-modules (in particular those of [2, 13, 14]), see §8 for a more detailed discussion.

2 Elliptic pairs and regularity

2.1 Relative $\mathcal{D}$-modules

In this section, we recall some basic facts about relative $\mathcal{D}$-modules.

In the sequel, by an analytic manifold we mean a complex analytic manifold X of finite dimension d_X. Keeping the notations of [12], we denote by

$$\tau : TX \longrightarrow X \qquad \text{and} \qquad \pi : T^*X \longrightarrow X$$

the tangent and cotangent bundles of X.

To every complex analytic map $f : X \longrightarrow Y$, we associate the natural maps

$$\begin{array}{ccccc} TX & \underset{f'}{\longrightarrow} & X \times_Y TY & \underset{f_\tau}{\longrightarrow} & TY \\ T^*X & \underset{{}^t f'}{\longleftarrow} & X \times_Y T^*Y & \underset{f_\pi}{\longrightarrow} & T^*Y. \end{array}$$

Let S be an analytic manifold. A *relative analytic manifold over* S is an analytic manifold X endowed with a surjective analytic submersion $\epsilon_X : X \longrightarrow S$. We often use the notation $X|S$ for such an object when we want to avoid confusion on the basis and set for short $d_{X|S} = d_X - d_S$.

A *morphism* $f : X|S \longrightarrow Y|S$ *of relative analytic manifolds* is the data of a complex analytic map $f : X \longrightarrow Y$ such that $\epsilon_Y \circ f = \epsilon_X$.

Let $X|S$ be a relative analytic manifold over S.

Since $\epsilon : X \longrightarrow S$ is smooth, the map

$$TX \underset{\epsilon'}{\longrightarrow} X \times_S TS$$

is surjective. Its kernel is thus a sub-bundle of TX. We denote it by $TX|S$ and call it the *relative tangent bundle of* $X|S$. Its holomorphic sections form the sheaf $\Theta_{X|S}$ of vertical holomorphic vector fields on $X|S$. Recall that a holomorphic vector field θ is vertical if and only if

$$\theta(h \circ \epsilon_X) = 0$$

for any section h of $\mathcal{O}_S$. The dual map

$$X \times_S T^*S \underset{{}^t\epsilon'}{\longrightarrow} T^*X$$

is injective. Its cokernel is thus a quotient-bundle of T^*X which is isomorphic to the dual of $TX|S$. This is the *relative cotangent bundle of* $X|S$, we denote it by $T^*X|S$ and denote by

$$p_{X|S} : T^*X \longrightarrow T^*X|S$$

the canonical projection. The holomorphic sections of $\bigwedge^p T^*X|S$ form the sheaf $\Omega^p_{X|S}$ of relative holomorphic differential forms of degree p. To shorten the notations, we set

$$\Omega_{X|S} = \Omega^{d_{X|S}}_{X|S}.$$

To every morphism $f : X|S \longrightarrow Y|S$, we associate the natural maps

$$\begin{array}{ccccc} TX|S & \underset{f'}{\longrightarrow} & X \times_Y TY|S & \underset{f_\tau}{\longrightarrow} & TY|S \\ T^*X|S & \underset{{}^t f'}{\longleftarrow} & X \times_Y T^*Y|S & \underset{f_\pi}{\longrightarrow} & T^*Y|S. \end{array}$$

Note that we use the same notations as in the non-relative case since the context will avoid any confusion.

The subring of $\mathcal{H}om_{\mathbb{C}_X}(\mathcal{O}_X, \mathcal{O}_X)$ generated by the derivatives along vertical holomorphic vector fields and multiplication with holomorphic functions is denoted by $\mathcal{D}_{X|S}$. We call it the *ring of relative differential operators on* $X|S$.

The basic algebraic properties of $\mathcal{D}_{X|S}$ are easily obtained using the usual filtration/graduation techniques. We will not review them here and refer the reader to [18, 19].

As usual, we denote by $\mathrm{Mod}(\mathcal{D}_{X|S})$ the abelian category of left $\mathcal{D}_{X|S}$-modules and by $\mathrm{Coh}(\mathcal{D}_{X|S})$ the full subcategory of coherent modules. The category $\mathrm{Coh}(\mathcal{D}_{X|S})$ is a thick subcategory of $\mathrm{Mod}(\mathcal{D}_{X|S})$ (i.e. it is full and stable by kernel, cokernel and extensions).

A coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$ is *good* if, in a neighborhood of any compact subset of X, $\mathcal{M}$ admits a finite filtration by coherent $\mathcal{D}_{X|S}$-submodules $\mathcal{M}_k$ ($k = 1, \dots, \ell$) such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $\mathrm{Good}(\mathcal{D}_{X|S})$ the full subcategory of $\mathrm{Coh}(\mathcal{D}_{X|S})$ consisting of good $\mathcal{D}_{X|S}$-modules. This definition ensures that $\mathrm{Good}(\mathcal{D}_{X|S})$ is the smallest thick subcategory of $\mathrm{Mod}(\mathcal{D}_{X|S})$ containing the modules which can be endowed with good filtrations on a neighborhood of any compact subset of X.

We denote by $\mathbf{D}(\mathcal{D}_{X|S})$ the derived category of $\mathrm{Mod}(\mathcal{D}_{X|S})$ and by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X|S})$ its full triangulated subcategory consisting of objects with bounded amplitude. The full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{X|S})$ consisting of objects with coherent (resp. good) cohomology modules is denoted by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X|S})$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S})$).

We introduce similar notations with the ring $\mathcal{D}_{X|S}$ replaced by the opposite ring $\mathcal{D}^{\mathrm{op}}_{X|S}$ to deal with right $\mathcal{D}_{X|S}$-modules. Since the categories $\mathrm{Mod}(\mathcal{D}_{X|S})$ and $\mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_{X|S})$ are equivalent, we will work only in the most convenient one depending on the problem at hand.

In the sequel, we will often need to work with bimodule structures. Let k be a field. Recall that if A and B are k-algebras, giving a left (A,B)-bimodule structure on an abelian group M is just giving M a left structure of A-module and a left structure of B-module such that

$$\begin{aligned} a\cdot(b\cdot m) &= b\cdot(a\cdot m)\\ (c\cdot a)\cdot(b\cdot m) &= (c\cdot b)\cdot(a\cdot m)\end{aligned}$$

for any $a \in A$, $b \in B$, $c \in k$ and $m \in M$. Hence, it is equivalent to consider that M is endowed with a structure of $A \otimes_k B$-module. Using this point of view it is easy to extend to bimodules the notions and notations defined usually for modules. For example, we will denote by $\mathrm{Mod}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the category of left $\mathcal{D}_{X|S}$ bimodules and by $\mathbf{D}(\mathcal{D}_{X|S} \otimes \mathcal{D}_{X|S})$ the corresponding derived category.

Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S.

Recall that

$$\mathcal{D}_{X|S\to Y|S} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}$$

has a natural structure of left $\mathcal{D}_{X|S}$-module compatible with its structure of right $f^{-1}\mathcal{D}_{Y|S}$-module. Using this transfer module, we may define the relative proper direct image of an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ by the formula

$$\underline{f}_{|S!}(\mathcal{M}) = Rf_!(\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})$$

It is an object of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{Y|S}^{\mathrm{op}})$.

Recall also that if $\mathcal{M}$ (resp. $\mathcal{N}$) is a right (resp. left) $\mathcal{D}_{X|S}$-module then there is on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ a unique structure of right $\mathcal{D}_{X|S}$-module such that

$$\begin{aligned}(m \otimes n) \cdot \theta &= m \cdot \theta \otimes n - m \otimes \theta \cdot n \\ (m \otimes n) \cdot h &= m \cdot h \otimes n = m \otimes h \cdot n\end{aligned}$$

for any sections m, n, θ and h of $\mathcal{M}$, $\mathcal{N}$, $\Theta_{X|S}$ and $\mathcal{O}_X$ respectively.

In the same way, if $\mathcal{N}$, $\mathcal{P}$ are two left $\mathcal{D}_{X|S}$-modules then there is on $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}$ a unique structure of left $\mathcal{D}_{X|S}$-module such that

$$\begin{aligned}\theta \cdot (n \otimes p) &= \theta \cdot n \otimes p + n \otimes \theta \cdot p \\ h \cdot (n \otimes p) &= h \cdot n \otimes p = n \otimes h \cdot p\end{aligned}$$

for any sections n, p, θ and h of $\mathcal{N}$, $\mathcal{P}$, $\Theta_{X|S}$ and $\mathcal{O}_X$ respectively.

Finally, recall the following exchange lemma which will be useful in the sequel.

Lemma 2.1 *If $\mathcal{M}$ is a right $\mathcal{D}_{X|S}$-module and $\mathcal{N}$, $\mathcal{P}$ are left $\mathcal{D}_{X|S}$-modules then the map*

$$\begin{aligned}\mathcal{M} \otimes_{\mathcal{D}_{X|S}} (\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}) &\longrightarrow (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{D}_{X|S}} \mathcal{P} \\ m \otimes (n \otimes p) &\mapsto (m \otimes n) \otimes p\end{aligned}$$

is a canonical isomorphism.

Let X be a relative analytic manifold over S. Recall that the characteristic variety of a coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$ is a conic analytic subset of $T^*X|S$ denoted by $\mathrm{char}_{X|S}(\mathcal{M})$ and that

$$\mathrm{char}(\mathcal{D}_X \otimes_{\mathcal{D}_{X|S}} \mathcal{M}) = p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}).$$

Hence theorem 11.3.3 of [12] gives the equality

$$SS(R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{O}_X)) = p_{X|S}^{-1}\mathrm{char}_{X|S}(\mathcal{M}).$$

The sheaf $\Omega_{X|S}$ of relative holomorphic differential forms of maximal degree is canonically endowed with a structure of right $\mathcal{D}_{X|S}$-module which is compatible with its structure of $\mathcal{O}_X$-module and characterized by the fact that, for every open subset U of X, one has $\omega.\theta = -L_\theta\omega$ if $\omega \in \Omega_{X|S}(U)$ and θ is a vertical vector field defined on U.

Definition 2.2 The *dualizing complex* for right $\mathcal{D}_{X|S}$-modules is the complex of right $\mathcal{D}_{X|S}$-bimodules defined by setting

$$\mathcal{K}_{X|S} = \Omega_{X|S}[d_{X|S}] \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$$

and using the natural structure of right $\mathcal{D}_{X|S}$-bimodule on the sheaf $\Omega_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$.

The *dual* of an object $\mathcal{M}$ of $\mathbf{D}^-(\mathcal{D}_{X|S}^{\mathrm{op}})$ is

$$R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S})$$

as an object of $\mathbf{D}^+(\mathcal{D}_{X|S}^{\mathrm{op}})$. We denote it by $\underline{D}_{X|S}(\mathcal{M})$.

The functor $\underline{D}_{X|S}$ is the *dualizing functor* for right $\mathcal{D}_{X|S}$-modules.

As in algebraic geometry, the terminology used in the preceding definition is justified by the following biduality result.

Lemma 2.3 *There is a canonical sheaf involution of $\Omega_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ interchanging its two right $\mathcal{D}_{X|S}$-module structures.*

Proof: Let us consider the sheaf $\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ where the tensor product uses the left $\mathcal{O}_X$ module structures of the two copies of $\mathcal{D}_{X|S}$. This sheaf is obviously endowed with one structure of left $\mathcal{D}_{X|S}$-module and two structures of right $\mathcal{D}_{X|S}$-module which are compatible with each other.

The involution

$$\begin{array}{ccc} \mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} & \longrightarrow & \mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} \\ P \otimes Q & \mapsto & Q \otimes P \end{array}$$

exchanges the two right structures and preserves the left one.

Tensoring over $\mathcal{D}_{X|S}$ with $\Omega_{X|S}$ using its right structure and the left structure of $\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}$ and applying the exchange lemma 2.1 gives us the requested involution. □

Proposition 2.4 *For any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$, the canonical arrow*

$$\mathcal{M} \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X|S}}(R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S}), \mathcal{K}_{X|S})$$

deduced from the involution of the preceding lemma is an isomorphism.

Proof: Since $\mathcal{M}$ is locally isomorphic to a bounded complex of finite free right $\mathcal{D}_{X|S}$-modules it is sufficient to prove the result for $\mathcal{M} = \mathcal{D}_{X|S}$ where it is an easy consequence of the preceding lemma and the fact that $\Omega_{X|S}$ is a locally free $\mathcal{O}_X$ module of rank one. □

It follows that the characteristic variety does not change by duality:

Proposition 2.5 *If $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ then one has*

$$\mathrm{char}_{X|S}(\mathcal{M}) = \mathrm{char}_{X|S}(\underline{D}_{X|S}\mathcal{M}).$$

2.2 Relative f-characteristic variety

In this subsection, we consider a morphism $f : X|S \longrightarrow Y|S$ of relative analytic manifolds over S and define the relative characteristic variety $\mathrm{char}_f(\mathcal{M})$ of a coherent $\mathcal{D}^{\mathrm{op}}_{X|S}$-module $\mathcal{M}$. First, we consider the case of a relative submersion where such a variety was already defined in [19] for $S = \{\mathrm{pt}\}$. Next, by using the graph embedding, we extend this definition to the general case. Finally, we show how the relative characteristic variety controls the micro-support of $\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S}$.

Let $f : X|S \longrightarrow Y|S$ be a relative analytic submersion over S. Since f is smooth, we have the following exact sequence of vector bundles on X:

$$0 \longrightarrow X \times_Y T^*Y|S \underset{\psi_f}{\longrightarrow} T^*X|S \underset{\phi_f}{\longrightarrow} T^*X|Y \longrightarrow 0.$$

Working as in paragraph III.1.3 of [19] we get the following lemmas.

Lemma 2.6 *Assume $\mathcal{M}_0$ is a coherent $\mathcal{D}_{X|Y}$-module. Then*

$$\mathrm{char}_{X|S}(\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{M}_0) = \phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{M}_0).$$

Lemma 2.7 *Assume $\mathcal{M}$ is a coherent $\mathcal{D}_{X|S}$-module and assume $\mathcal{M}_0$, $\mathcal{N}_0$ are two coherent $\mathcal{D}_{X|Y}$-submodules of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module then*

$$\mathrm{char}_{X|Y}(\mathcal{M}_0) = \mathrm{char}_{X|Y}(\mathcal{N}_0).$$

Hence, we may introduce the following definition.

Definition 2.8 Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X|S}$-module. One defines the *relative characteristic variety* $\mathrm{char}_{f|S}(\mathcal{M})$ *of* $\mathcal{M}$ *with respect to* f to be the subset of $T^*X|S$ which coincide on $T^*U|S$ with $\phi_f^{-1}\mathrm{char}_{U|Y}(\mathcal{M}_0)$ for any open subset U and any coherent $\mathcal{D}_{U|Y}$-submodule $\mathcal{M}_0$ of $\mathcal{M}_{|U}$ which generates $\mathcal{M}_{|U}$ as a $\mathcal{D}_{U|S}$-module.

It is clear that $\mathrm{char}_{f|S}(\mathcal{M})$ is a closed conic analytic subvariety of $T^*X|S$ and that

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{M}) + \psi_f(X \times_Y T^*Y|S).$$

The functor $\mathrm{char}_{f|S}$ is additive:

Proposition 2.9 *If $f : X \longrightarrow Y$ is a relative analytic submersion over S and if*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

is an exact sequence of coherent $\mathcal{D}_{X|S}$-modules then

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{L}) \cup \mathrm{char}_{f|S}(\mathcal{N}).$$

In the sequel we will need the following lemma essentially due to [8].

Lemma 2.10 *Let $f : X|S \longrightarrow Y|S$ be a relative analytic submersion over S and let K be a compact subset of X. Assume $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module which admits a good filtration in a neighborhood of K. Then, in a neighborhood of K, $\mathcal{M}$ has a left resolution by $\mathcal{D}_{X|S}$-modules of the form*

$$\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}$$

where the $\mathcal{D}_{X|Y}$-module $\mathcal{N}$ admits a good filtration and is such that

$$\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{N}) \subset \mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: In this proof, we always work in some neighborhood of K.

Since $\mathcal{M}$ admits a good filtration, we can find a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module. Set $\mathcal{N}_0 = \mathcal{D}_{X|Y}\mathcal{M}_0$. By construction, $\mathcal{N}_0$ is a $\mathcal{D}_{X|Y}$-submodule of $\mathcal{M}$ which generates it as a $\mathcal{D}_{X|S}$-module. Obviously, $\mathcal{N}_0$ admits a good filtration. Moreover, by definition,

$$\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{N}_0) = \mathrm{char}_{f|S}(\mathcal{M}).$$

The kernel $\mathcal{K}$ of the canonical $\mathcal{D}_{X|S}$-linear epimorphism

$$\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

is a $\mathcal{D}_{X|S}$-module which admits a good filtration and we have

$$\mathrm{char}_{f|S}(\mathcal{K}) \subset \mathrm{char}_{f|S}(\mathcal{D}_{X|S} \otimes_{\mathcal{D}_{X|Y}} \mathcal{N}_0) = \mathrm{char}_{f|S}(\mathcal{M}).$$

We may thus start over the same construction with $\mathcal{M}$ replaced by $\mathcal{K}$ and build the requested resolution by induction. □

Now, by using the graph factorization, we will define the notion of relative characteristic variety for a map which is not necessarily a relative submersion.

Let $f : X|S \longrightarrow Y|S$ be any morphism of relative analytic manifolds.

Denote by

$$X \underset{i}{\longrightarrow} X \times_S Y \underset{q}{\longrightarrow} Y$$

the relative graph factorization of f.

First, we notice:

Lemma 2.11 *Assume f is a relative submersion and $\mathcal{M}$ is a coherent $\mathcal{D}_{X|S}$-module. Then*

$$\mathrm{char}_{f|S}(\mathcal{M}) = {}^t i' i_\pi^{-1} \mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})).$$

Hence, for a general f, the following definition is a natural extension of our previous one.

Definition 2.12 For any coherent $\mathcal{D}_{X|S}$-module $\mathcal{M}$, we set

$$\mathrm{char}_{f|S}(\mathcal{M}) = {}^t i' i_\pi^{-1} \mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})).$$

As usual, for an object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X|S})$, we also set

$$\mathrm{char}_{f|S}(\mathcal{M}) = \bigcup_{j\in\mathbb{Z}} \mathrm{char}_{f|S}(\mathcal{H}^j(\mathcal{M})).$$

We also introduce similar definitions for right $\mathcal{D}_{X|S}$-modules.

Note that $\mathrm{char}_{f|S}(\mathcal{M})$ is a closed conic analytic subset of $T^*X|S$ and that

$$\mathrm{char}_{f|S}(\mathcal{M}) = \mathrm{char}_{f|S}(\mathcal{M}) + \psi_f(X \times_Y T^*Y|S).$$

A link between the relative characteristic variety and the micro-local theory of sheaves is given in the following theorem:

Theorem 2.13 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S and assume $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Then*

$$SS(\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \subset p^{-1}_{X|S}\mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: Consider the graph factorization of f:

$$X \underset{i}{\longrightarrow} X \times_S Y \underset{q}{\longrightarrow} Y.$$

By Proposition 5.4.4 of [12], if F is a sheaf on X, then $SS(i_*F)$ is the natural image of $SS(F)$. Hence, in view of the definition of $\mathrm{char}_{f|S}$, it is enough to prove the inclusion:

$$SS(i_!(\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})) \subset p^{-1}\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M}))$$

where we write p instead of $p_{X\times_S Y|S}$. Since

$$i_!(\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \simeq \underline{i}_{|S!}(\mathcal{M}) \otimes^{L}_{\mathcal{D}_{X\times_S Y|S}} \mathcal{D}_{X\times_S Y|S\to Y|S},$$

we have reduced the proof to the case where f is a relative submersion, what we shall assume now.

Since the problem is local on X and

$$\mathrm{char}_{f|S}(\mathcal{M}) = \bigcup_{j\in\mathbb{Z}} \mathrm{char}_{f|S}(\mathcal{H}^j(\mathcal{M})),$$

we may assume $\mathcal{M}$ is a coherent right $\mathcal{D}_{X|S}$-module. Using Lemma 2.10, we are then reduced to consider the case where $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$, for a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$. Now,

$$\begin{aligned}\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S} &\simeq \mathcal{M}_0 \otimes^{L}_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S\to Y|S} \\ &\simeq (\mathcal{M}_0 \otimes^{L}_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}.\end{aligned}$$

This last sheaf is locally on X a direct sum of an infinite number of copies of $\mathcal{M}_0 \otimes^{L}_{\mathcal{D}_{X|Y}} \mathcal{O}_X$. Applying [12] Exercise V.5(i) (which is an easy consequence of Proposition 5.1.1(3) [loc. cit.]) we get successively

$$\begin{aligned}SS(\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) &= SS(\mathcal{M}_0 \otimes^{L}_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \\ &= SS(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X \otimes^{L}_{\mathcal{D}_X} \mathcal{O}_X) \\ &= \mathrm{char}(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X) \\ &= p^{-1}\phi_f^{-1}\mathrm{char}_{X|Y}(\mathcal{M}_0) \\ &= p^{-1}\mathrm{char}_{f|S}(\mathcal{M})\end{aligned}$$

where the third equality comes from theorem 11.3.3 of [12]. □

2.3 Relative elliptic pairs

We shall now define the main object of study of this paper.

Let $\mathbf{D}(X)$ denote the derived category of the category of sheaves of $\mathbb{C}$-vector spaces on X and let $\mathbf{D}^{\mathrm{b}}(X)$ denote the full triangulated subcategory of complexes with bounded amplitude.

Recall that a sheaf F of $\mathbb{C}$-vector spaces is $\mathbb{R}$-constructible if there is a subanalytic stratification of X along the strata of which $H^j(F)$ is a locally constant sheaf of finite rank for any $j \in \mathbb{Z}$. Following [12], we denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of complexes with $\mathbb{R}$-constructible cohomology sheaves. We say for short that an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ is an $\mathbb{R}$-constructible complex. For such an object, SS(F) is a closed subanalytic Lagrangian subset of $T^*X^{\mathbb{R}}$ where $X^{\mathbb{R}}$ denotes X considered with its underlying real analytic manifold structure. We shall identify $T^*X^{\mathbb{R}}$ with $(T^*X)^{\mathbb{R}}$ as for example in [12] and simply denote it by T^*X. In this paper, we will have to consider most of the time the simple dual $D'F$ of F and not its Poincaré-Verdier dual DF. Recall that since X is an oriented topological manifold of dimension $2d_X$:

$$D'F = R\mathcal{H}om(F, \mathbb{C}_X) \qquad \text{and} \qquad DF = R\mathcal{H}om(F, \omega_X) \simeq R\mathcal{H}om(F, \mathbb{C}_X[2d_X])$$

so the two duals coincide up to shift. Since F is constructible, we have the local biduality isomorphism $F \xrightarrow{\sim} D'D'F$.

Definition 2.14 Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. A pair $(\mathcal{M},F)$ is a *relative f-elliptic pair* if:

- $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,
- F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,
- $p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_XX$.

Such a pair is *good* if moreover $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Its *support* is the set $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$. When f is the canonical map $\epsilon_X : X|S \longrightarrow S|S$ we will say for short that $(\mathcal{M},F)$ is a *(good) relative elliptic pair on* $X|S$. When $S = \{\mathrm{pt}\}$, we drop the word "relative" in the preceding definitions.

Since $\mathrm{char}_{f|S}(\mathcal{M})$ contains $\mathrm{char}_{X|S}(\mathcal{M})$, a relative f-elliptic pair is a relative elliptic pair. Moreover, on a neighborhood of $\mathrm{supp}\,\mathcal{M}$,

$$SS(F) \cap X \times_S T^*S \subset T^*_XX.$$

In particular, an elliptic pair $(\mathcal{M},F)$ on X is the data of a complex of coherent right $\mathcal{D}_X$-modules $\mathcal{M}$ and an $\mathbb{R}$-constructible complex F such that

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_XX.$$

We shall see in §8 below why this notion is a natural generalization of that of an elliptic system on a real manifold. There, we will also explain why Theorem 2.15 below may be considered as a generalization of the classical regularity theorem for elliptic systems.

Theorem 2.15 *Let $(\mathcal{M},F)$ be an f-elliptic pair. Then the canonical morphism*

$$F \otimes (\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S}) \longrightarrow R\mathcal{H}om(D'F, \mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S}),$$

induced by the morphism $F \longrightarrow D'D'F$, is an isomorphism.

Proof: By [12] Proposition 5.4.14, we know that if G belongs to $D^b(X)$ (and F is $\mathbb{R}$-constructible as above), the natural morphism

$$F \otimes G \longrightarrow R\mathcal{H}om(D'F, G)$$

is an isomorphism as soon as

$$SS(F)^a \cap SS(G) \subset T^*_X X.$$

Hence, the conclusion follows from Theorem 2.13 by applying the preceding result to

$$G = \mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S \to Y|S}.$$

□

When $Y = S$, we get:

Corollary 2.16 *Let $\mathcal{M}$ and F be objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ respectively. Assume the transversality condition*

$$p^{-1}\mathrm{char}_{X|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$$

where $p : T^*X \longrightarrow T^*X|S$ is the canonical projection. Then the canonical morphism

$$F \otimes (\mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{X|S}}(D'F, \mathcal{M} \otimes_{\mathcal{D}_{X|S}} \mathcal{O}_X)$$

is an isomorphism.

Definition 2.17 The *dual* of a relative pair $(\mathcal{M},F)$ is the pair $(\underline{D}_{X|S}(\mathcal{M}),D'F)$.

It follows from this definition that a relative pair is f-elliptic if and only if so is its dual pair.

3 Tools

If X is a complex analytic manifold, we have already encountered $X^{\mathbb{R}}$, the real underlying analytic manifold to X. Here, we shall also make use of $\overline{X}$, the complex manifold with $X^{\mathbb{R}}$ as underlying space for which the holomorphic functions are the anti-holomorphic functions on X. Recall that $X \times \overline{X}$ is a natural complexification of $X^{\mathbb{R}}$ via the diagonal embedding.

3.1 Dolbeault complexes with parameters

Let Z and S be complex analytic manifolds and let $q_Z : Z \times S \longrightarrow Z$ be the second projection.

We will denote by $\mathcal{A}_{Z\times S|S}$ (resp. $\mathcal{F}_{Z\times S|S}$, $\mathcal{D}b_{Z\times S|S}$) the sheaf of real analytic functions (resp. infinitely differentiable functions, distributions) on $Z \times S$ which are holomorphic in S.

We will also set

$$\mathcal{D}_{Z^{\mathbb{R}}\times S|S} = (\mathcal{D}_{Z\times\overline{Z}\times S|S})_{|Z^{\mathbb{R}}\times S}.$$

using the diagonal embedding of $Z^{\mathbb{R}}$ in $Z \times \overline{Z}$. Locally, operators in $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$ are of the form

$$\sum_{\alpha,\beta} a_{\alpha,\beta}(z,\overline{z},s) D_z^\alpha D_{\overline{z}}^\beta$$

where $a_{\alpha,\beta}(z,\overline{z},s)$ is a section of $\mathcal{A}_{Z\times S|S}$; $(z : U \longrightarrow \mathbb{C}^{d_Z})$ and $(s : V \longrightarrow \mathbb{C}^{d_S})$ being holomorphic local coordinate systems on Z and S respectively.

For any $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module $\mathcal{M}$ we will consider the parametric Dolbeault complex

$$\mathcal{A}^{\cdot,\cdot}_{Z\times S|S}(\mathcal{M})$$

defined by setting

$$\mathcal{A}^{p,q}_{Z\times S|S}(\mathcal{M}) = q_Z^{-1}\mathcal{A}_Z^{p,q} \otimes_{q_Z^{-1}\mathcal{A}_Z} \mathcal{M}$$

the formulas for the differentials being given locally by

$$\begin{aligned} \partial : \mathcal{A}^{p,q}_{U\times S|S}(\mathcal{M}) &\longrightarrow \mathcal{A}^{p+1,q}_{U\times S|S}(\mathcal{M}) \\ a^{p,q}\otimes m &\mapsto \partial a^{p,q}\otimes m + \sum_{i=1}^{d_Z} dz^i \wedge a^{p,q} \otimes D_{z^i} m \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \overline{\partial} : \mathcal{A}^{p,q}_{U\times S|S}(\mathcal{M}) &\longrightarrow \mathcal{A}^{p,q+1}_{U\times S|S}(\mathcal{M}) \\ a^{p,q}\otimes m &\mapsto \overline{\partial} a^{p,q}\otimes m + \sum_{i=1}^{d_Z} d\overline{z}^i \wedge a^{p,q} \otimes D_{\overline{z}^i} m \end{aligned} \tag{3.2}$$

where $(z : U \longrightarrow \mathbb{C}^{d_Z})$ is a holomorphic local coordinate system on Z. Obviously, this definition is independent on the chosen local coordinate system.

When $\mathcal{M}$ is equal to $\mathcal{A}_{Z\times S|S}$ (resp. $\mathcal{F}_{Z\times S|S}$, $\mathcal{D}b_{Z\times S|S}$) we will denote the corresponding parametric Dolbeault complex simply by $\mathcal{A}^{\cdot,\cdot}_{Z\times S|S}$ (resp. $\mathcal{F}^{\cdot,\cdot}_{Z\times S|S}$, $\mathcal{D}b^{\cdot,\cdot}_{Z\times S|S}$). Of course, the natural maps

$$\Omega^p_{Z\times S|S} \longrightarrow \mathcal{A}^{p,\cdot}_{Z\times S|S} \longrightarrow \mathcal{F}^{p,\cdot}_{Z\times S|S} \longrightarrow \mathcal{D}b^{p,\cdot}_{Z\times S|S}$$

are quasi-isomorphisms.

Let $\mathcal{N}$ be another $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module. Using the natural structure of $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module on the sheaf $\mathcal{N}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}$ we define the parametric Dolbeault complex of $\mathcal{M}$ with coefficients in $\mathcal{N}$ by the formula

$$\mathcal{N}^{\cdot,\cdot}(\mathcal{M}) = \mathcal{A}^{\cdot,\cdot}_{Z\times S|S}(\mathcal{N}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}).$$

The associated simple complex is the parametric de Rham complex of $\mathcal{M}$ with coefficients in $\mathcal{N}$. We denote it by $\mathcal{N}^{\cdot}(\mathcal{M})$.

In this paper, we will only use the preceding notions when $\mathcal{N}$ is $\mathcal{F}_{Z\times S|S}$ or $\mathcal{D}b_{Z\times S|S}$. In this case, we have of course

$$\begin{aligned}\mathcal{F}^{p,q}_{Z\times S|S}(\mathcal{M}) &= \mathcal{F}^{p,q}_{Z\times S|S}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}\\ \mathcal{D}b^{p,q}_{Z\times S|S}(\mathcal{M}) &= \mathcal{D}b^{p,q}_{Z\times S|S}\otimes_{\mathcal{A}_{Z\times S|S}}\mathcal{M}\end{aligned}$$

and the differentials ∂ and $\overline{\partial}$ are given locally by formulas similar to (3.1) and (3.2).

3.2 Realification with parameters

Let Z and S be complex analytic manifolds and set $n = d_Z$. Consider $Z\times S$ as a relative manifold over S through the second projection ϵ.

The *parametric realification* of a left $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$ is the sheaf

$$\mathcal{M}_{\mathbb{R}|S} = \mathcal{A}_{Z\times S|S}\otimes_{\mathcal{O}_{Z\times S}}\mathcal{M}.$$

In this formula, the $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module structure is described locally by the formulas

$$\begin{aligned}D_{z_j}(a\otimes m) &= D_{z_j}a\otimes m + a\otimes D_{z_j}m\\ D_{\overline{z}_j}(a\otimes m) &= D_{\overline{z}_j}a\otimes m\\ f(a\otimes m) &= fa\otimes m\end{aligned}$$

where a,f and m are sections of $\mathcal{A}_{Z\times S|S}$ and $\mathcal{M}$ respectively; $(z : U \longrightarrow \mathbb{C}^n)$ being a local holomorphic coordinate system on Z.

Since $\mathcal{A}_{Z\times S|S}$ is flat over $\mathcal{O}_{Z\times S}$, parametric realification is an exact functor.

Let us consider the map

$$\begin{aligned}\delta : Z\times S &\longrightarrow Z\times\overline{Z}\times S\\ (z,s) &\mapsto (z,z,s).\end{aligned}$$

It is clear that

$$\delta^{-1}(\mathcal{D}_{Z\times\overline{Z}\times S|S}) = \mathcal{D}_{Z^{\mathbb{R}}\times S|S}.$$

Hence the sheaf inverse image by δ of a $\mathcal{D}_{Z\times\overline{Z}\times S|S}$-module is naturally a $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module. Moreover, one checks easily that

$$\mathcal{M}_{\mathbb{R}|S} = \delta^{-1}(\mathcal{O}_{Z\times\overline{Z}\times S}\otimes_{q^{-1}\mathcal{O}_{Z\times S}} q^{-1}\mathcal{M}) = \delta^{-1}(\mathcal{M}\boxtimes\mathcal{O}_{\overline{Z}}).$$

where $q : Z \times \overline{Z} \times S \longrightarrow Z \times S$ is the natural projection and $\underline{\boxtimes}$ denotes the external product of $\mathcal{D}$-modules.

As usual, using "side changing" functors, we may also define the parametric realification of a right $\mathcal{D}_{Z\times S|S}$-module $\mathcal{M}$. We still denote it by $\mathcal{M}_{\mathbb{R}|S}$ and check easily that

$$\mathcal{M}_{\mathbb{R}|S} = \mathcal{A}^{0,n}_{Z\times S|S} \otimes_{\mathcal{O}_{Z\times S}} \mathcal{M} = \delta^{-1}(\mathcal{M} \underline{\boxtimes} \Omega_{\overline{Z}}).$$

Parametric realification is a powerful tool to simplify problems dealing with $\mathcal{D}_{Z\times S|S}$-modules thanks to the following result.

Proposition 3.1 *Let K be a compact subset of Z and let Δ be a closed polydisc of S. Assume $\mathcal{M}$ is a good $\mathcal{D}_{Z\times S|S}$-module. Then, in a neighborhood of $K \times \Delta$, $\mathcal{M}_{\mathbb{R}|S}$ has a left resolution by finite free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules .*

Proof: The assumption insures that $\mathcal{M} \underline{\boxtimes} \mathcal{O}_{\overline{Z}}$ is a good $\mathcal{D}_{Z\times\overline{Z}\times S|S}$-module. Hence it is generated by a coherent $\mathcal{O}_{Z\times\overline{Z}\times S}$-module in a neighborhood of the Stein compact subset $\delta(K \times \Delta)$ of the complex analytic manifold $Z \times \overline{Z} \times S$. By Cartan's Theorem A, it is thus finitely generated in a neighborhood of $\delta(K \times \Delta)$. The conclusion follows easily. □

In order to be able to use effectively the preceding proposition in the sequel, we need to understand the links between parametric realification and the finiteness and duality results. These links are made explicit in the following five lemmas. Since the proofs are just easy computational verifications we leave them to the reader. Recall that $\mathcal{H}om^{\cdot}$ denotes as usual the internal Hom functor of the category of complexes of sheaves.

Lemma 3.2 a) *The sheaf $\mathcal{A}^{0,p}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S})$ is naturally endowed with a structure of left $\mathcal{D}_{Z\times S|S}$-module and a structure of right $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-module and the differential $\overline{\partial}$ is compatible with these two structures.*

b) *As a complex of $(\mathcal{D}_{Z\times S|S}, \mathcal{D}^{\mathrm{op}}_{Z^{\mathbb{R}}\times S|S})$-bimodules $\mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S})$ is quasi-isomorphic to $(\mathcal{D}_{Z\times S|S})_{\mathbb{R}|S}[-n]$ (where the realification uses the right module structure of $\mathcal{D}_{Z\times S|S}$)*

Lemma 3.3 *The map*

$$\begin{array}{rcl}\mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}) \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S} & \longrightarrow & \mathcal{F}^{0,\cdot}_{Z\times S|S} \\ (a^{0,p} \otimes Q) \otimes u & \mapsto & a^{0,p} \wedge Qu\end{array}$$

is an isomorphism of complexes of left $\mathcal{D}_{Z\times S|S}$-modules. Combined with the Dolbeault quasi-isomorphism

$$\mathcal{O}_{Z\times S} \longrightarrow \mathcal{F}^{0,\cdot}_{Z\times S|S},$$

it induces in the derived category the isomorphism

$$\mathcal{M} \otimes^{L}_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S} \xrightarrow{\sim} \mathcal{M}_{\mathbb{R}|S}[-n] \otimes^{L}_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}$$

for any complex of right $\mathcal{D}_{Z\times S|S}$-modules $\mathcal{M}$. We have also similar results with $\mathcal{F}$ replaced by $\mathcal{D}b$ or $\mathcal{A}$.

Lemma 3.4 *The map*

$$\begin{array}{ccc} \mathcal{D}b^{n,\cdot}_{Z\times S|S}[-n] & \longrightarrow & \mathcal{H}om^{\cdot}_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{A}^{0,\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}), \mathcal{D}b^{n,n}_{Z\times S|S}) \\ \varphi^{n,r+n} & \mapsto & [\omega^{0,-r}\otimes Q \mapsto (\varphi^{n,r+n}\wedge\omega^{0,-r})\cdot Q] \end{array}$$

is an isomorphism of complexes of right $\mathcal{D}_{Z\times S|S}$-modules. Combined with the Dolbeault quasi-isomorphism

$$\Omega^n_{Z\times S|S} \longrightarrow \mathcal{D}b^{n,\cdot}_{Z\times S|S},$$

it induces in the derived category the isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n]) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}[-n], \mathcal{D}b^{n,n}_{Z\times S|S})$$

for any complex of right $\mathcal{D}_{Z\times S|S}$-modules $\mathcal{M}$. We have also similar results with $\mathcal{D}b$ replaced by $\mathcal{A}$ or $\mathcal{F}$.

Lemma 3.5 *The natural arrow*

$$\begin{array}{rcl} \Omega^{\cdot}_{Z\times S|S}(\mathcal{D}_{Z\times S|S}) & \longrightarrow & \Omega^n_{Z\times S|S}[-n] \\ (\text{resp. } \mathcal{D}b^{\cdot}_{Z\times S|S}(\mathcal{D}_{Z^{\mathbb{R}}\times S|S}) & \longrightarrow & \mathcal{D}b^{2n}_{Z\times S|S}[-2n] \quad) \end{array}$$

of complexes of right $\mathcal{D}_{Y\times S|S}$ (resp. $\mathcal{D}_{Y^{\mathbb{R}}\times S|S}$) modules is a quasi-isomorphism. Together with the relative de Rham quasi-isomorphism

$$\begin{array}{rcl} \epsilon^{-1}\mathcal{O}_S & \xrightarrow{\sim} & \Omega^{\cdot}_{Z\times S|S} \\ (\text{resp. } \epsilon^{-1}\mathcal{O}_S & \xrightarrow{\sim} & \mathcal{D}b^{\cdot}_{Z\times S|S} \quad) \end{array}$$

it induces the $\epsilon^{-1}\mathcal{O}_S$ linear pairing

$$\begin{array}{rcl} \Omega^n_{Z\times S|S}[n]\otimes^L_{\mathcal{D}_{Z\times S|S}}\mathcal{O}_{Z\times S} & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \\ (\text{resp. } \mathcal{D}b^{2n}_{Z\times S|S}\otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}\mathcal{F}_{Z\times S|S} & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \quad). \end{array}$$

Lemma 3.6 *Assume $\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{Z\times S|S})$. We have the commutative diagram*

$$\begin{array}{ccc} (\mathcal{M}\otimes^L_{\mathcal{D}_{Z\times S|S}}\mathcal{O}_{Z\times S})\otimes^L_{\epsilon^{-1}\mathcal{O}_S} R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M},\Omega^n_{Z\times S}[n]) & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \\ \downarrow & & \downarrow \\ (\mathcal{M}_{\mathbb{R}|S}\otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}\mathcal{F}_{Z\times S|S})\otimes^L_{\epsilon^{-1}\mathcal{O}_S} R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S},\mathcal{D}b^{2n}_{Z\times S|S}) & \longrightarrow & \epsilon^{-1}\mathcal{O}_S[2n] \end{array}$$

where the horizontal arrows are constructed by contraction followed by the pairings of the preceding lemma, the first vertical arrow being the tensor product of the isomorphisms of Lemma 3.3 and 3.4 while the second vertical arrow is the identity.

3.3 Trivialization of $\mathbb{R}$-constructible sheaves

In this section, we follow the notations of [12, Ch. VIII]. As in the classical theory of simplicial complexes, the sets $U(x)$ of [loc. cit.] are called open stars. Let us first point out some basic facts about the topology of polyhedra.

Lemma 3.7 *Let (S, Σ) be a simplicial set and let $x \in |S|$. Then*

(a) $y \in U(x) \quad \Rightarrow \quad [x, y] \subset U(x)$

(b) $y \in \partial U(x) \quad \Rightarrow \quad [x, y[\subset U(x)$

where $U(x)$ denotes the open star of x in $|S|$.

Proof: (a) One knows that

$$U(x) = \bigcup_{\sigma \supset \sigma(x)} |\sigma|.$$

Thus, if $y \in U(x)$, there is a $\sigma \supset \sigma(x)$ such that $y \in |\sigma|$. Since it is clear that $]x, y] \subset |\sigma|$ and that $x \in U(x)$, one gets that $[x, y] \subset U(x)$.

(b) The set $\{\sigma \in \Sigma : \sigma \supset \sigma(x)\}$ being finite, one has the equality

$$\overline{U(x)} = \bigcup_{\sigma \supset \sigma(x)} \overline{|\sigma|}.$$

Hence, since $y \in \partial U(x)$, there is a simplex $\sigma \supset \sigma(x)$ such that $y \in \overline{|\sigma|} \setminus |\sigma|$. Let σ' be a simplex included in σ such that $y \in |\sigma'|$. If $\sigma' \supset \sigma(x)$ then $]x, y[\subset |\sigma'| \subset U(x)$ as requested. If $\sigma' \not\supset \sigma(x)$ then $\sigma'' = \sigma' \cup \sigma(x)$ is a simplex of Σ included in σ and $]x, y[\subset |\sigma''| \subset U(x)$ and the conclusion follows. □

Lemma 3.8 *If (S, Σ) is a simplicial set and if $x \in |S|$ then one has the following commutative diagram*

$$\begin{array}{ccc} \partial U(x) \times]0,1] / \partial U(x) \times \{1\} & \xrightarrow{\sim} & U(x) \\ \downarrow & & \downarrow \\ \partial U(x) \times [0,1] / \partial U(x) \times \{1\} & \xrightarrow{\sim} & \overline{U(x)} \end{array}$$

where the horizontal arrows are homeomorphisms, the vertical arrows being the natural inclusions.

Proof: Let us define the continuous application

$$f : \partial U(x) \times [0, 1] \longrightarrow \overline{U(x)}$$

by setting $f(u, t) = (1 - t)u + tx$. The preceding lemma shows that $f(u, t) = f(u', t')$ if either $t = t' = 1$ or $(u, t) = (u', t')$. Moreover, it is clear that for every $u \in \overline{U(x)}$ there is $v \in \partial U(x)$ such that $u \in [x, v]$. From these facts, one deduces that the continuous map

$$g : \partial U(x) \times [0, 1] / \partial U(x) \times \{1\} \longrightarrow \overline{U(x)}$$

associated to f is bijective. Since $\partial U(x) \times [0, 1]$ is a compact space, g is an homeomorphism. To conclude, it remains to note that $f^{-1}(U(x)) = \partial U(x) \times]0, 1]$. □

Proposition 3.9 *If (S,Σ) is a simplicial set, then for every open star $U(x)$ of $x \in |S|$ one has*

$$D'_{|S|}(\mathbb{C}_{U(x)}) = \mathbb{C}_{\overline{U(x)}}$$

Proof: It is clear that

$$\begin{aligned} D'_{|S|}(\mathbb{C}_{U(x)}) &= R\mathcal{H}om(\mathbb{C}_{U(x)}, \mathbb{C}_{|S|}) \\ &= Rj_{U(x)_*}(\mathbb{C}_{U(x)}). \end{aligned}$$

It remains to prove that the canonical arrow

$$\mathbb{C}_{\overline{U(x)}} \longrightarrow Rj_{U(x)_*}(\mathbb{C}_{U(x)})$$

is a quasi-isomorphism on $\overline{U(x)}$. Thanks to the preceding lemma, there is a neighborhood ω of $\partial U(x)$ in $\overline{U(x)}$ and an homeomorphism

$$\phi : \omega \longrightarrow \partial U(x) \times [0, \epsilon[$$

such that $\phi(\omega \cap U(x)) = \partial U(x) \times]0, \epsilon[$. We are thus reduced to show that the canonical arrows

$$\begin{aligned} \mathbb{C} &\longrightarrow \varinjlim_{V \in \mathcal{V}, \eta > 0} H^0(V \times]0, \eta[; \mathbb{C}) \\ 0 &\longrightarrow \varinjlim_{V \in \mathcal{V}, \eta > 0} H^k(V \times]0, \eta[; \mathbb{C}) \quad (k \geq 1) \end{aligned}$$

are isomorphisms when $\mathcal{V}$ is a fundamental system of neighborhoods of $y \in \partial U(x)$. But, using homotopy, it is clear that

$$H^k(V \times]0, \eta[; \mathbb{C}) \xrightarrow{\sim} H^k(V; \mathbb{C})$$

and the proof is complete. □

The following proposition is the main result of this section and will be used as a basic tool in the sequel.

Proposition 3.10 *An $\mathbb{R}$-constructible sheaf F on a real analytic manifold M is quasi-isomorphic to a bounded complex $T^{\cdot}$ of the form*

$$\cdots 0 \longrightarrow \cdots \bigoplus_{i_a \in I_a} \mathbb{C}_{W_{a,i_a}} \longrightarrow \cdots \bigoplus_{i_k \in I_k} \mathbb{C}_{W_{k,i_k}} \longrightarrow \cdots \bigoplus_{i_b \in I_b} \mathbb{C}_{W_{b,i_b}} \longrightarrow 0 \cdots$$

where each family $(W_{k,i_k})_{i_k \in I_k}$ is locally finite, the open subsets W_{k,i_k} being subanalytic, relatively compact, connected and such that

$$D'_M(\mathbb{C}_{W_{k,i_k}}) \simeq \mathbb{C}_{\overline{W}_{k,i_k}},$$

the differential $d^k_{T^\cdot}$ being such that the induced map

$$(d^k_{T^\cdot})_{ji} : \mathbb{C}_{W_{k,i}} \longrightarrow \mathbb{C}_{W_{k+1,j}}$$

is either 0 *if $W_{k,i} \not\subset W_{k+1,j}$ or a complex multiple $c^k_{ji} I_{W_{k+1,j}W_{k,i}}$ of the canonical inclusion map if $W_{k,i} \subset W_{k+1,j}$.*

Moreover, if F has compact support, we may assume that the set I_k is finite for every $k \in \mathbb{Z}$.

Proof: From the theory of $\mathbb{R}$-constructible sheaves, one knows that there is a simplicial set (S, Σ) and an homeomorphism $i : |S| \longrightarrow M$ such that $i^{-1}F$ is a simplicialy constructible sheaf. From a construction due to M. Kashiwara [11] one knows that such a sheaf is quasi-isomorphic to a bounded complex $T^\cdot$ such that each T^k is a locally finite direct sum of the sheaves $\mathbb{C}_{U(\sigma)}$ associated to the open stars of the simplexes of Σ where F is non zero. Since we have just proven in the preceding lemma that for such a sheaf one has

$$D'(\mathbb{C}_{U(\sigma)}) \simeq \mathbb{C}_{\overline{U(\sigma)}}$$

the first part of the proposition is clear.

Concerning the differential of the complex, we note that if σ, σ' are two simplexes of Σ then

$$\mathrm{Hom}(\mathbb{C}_{U(\sigma)}, \mathbb{C}_{U(\sigma')}) \simeq \Gamma(U(\sigma); \mathbb{C}_{U(\sigma)\cap U(\sigma')})$$

hence the conclusion since $U(\sigma)$ is a connected open set.

In case F has compact support K, the open stars $U(\sigma)$ meeting K are in finite number and since only these stars appear in the components of $T^\cdot$, the sets I_k are finite. □

3.4 Topological $\mathcal{O}_S$-modules

Let S be a complex analytic manifold. Recall that the sheaf $\mathcal{O}_S$ of holomorphic functions on S is a multiplicatively convex sheaf of Fréchet algebras over S (see [7, 25]). Also recall that if V is a relatively compact open subset of a Stein open subset U of X then the restriction map

$$\Gamma(U; \mathcal{O}_S) \longrightarrow \Gamma(V; \mathcal{O}_S)$$

is $\mathbb{C}$-nuclear. From this it follows easily that $\Gamma(U; \mathcal{O}_U)$ is a Fréchet nuclear (FN) space and that $\Gamma(\overline{V}, \mathcal{O}_S)$ is a dual Fréchet nuclear (DFN) space.

As in [7], we will consider $\mathcal{O}_S$ as a sheaf of complete bornological algebras and deal with the category $\mathrm{Born}(\mathcal{O}_S)$ of complete bornological modules over $\mathcal{O}_S$. Recall that Houzel has shown that $\mathrm{Born}(\mathcal{O}_S)$ has a natural internal hom functor denoted by $\mathcal{L}_{\mathcal{O}_S}(\cdot, \cdot)$ and an associated tensor product functor denoted by $\cdot \mathbin{\hat{\otimes}_{\mathcal{O}_S}} \cdot$. They are linked by the adjunction formula

$$\mathrm{Hom}_{\mathrm{Born}(\mathcal{O}_S)}(\mathcal{M} \mathbin{\hat{\otimes}_{\mathcal{O}_S}} \mathcal{N}, \mathcal{P}) = \mathrm{Hom}_{\mathrm{Born}(\mathcal{O}_S)}(\mathcal{M}, \mathcal{L}_{\mathcal{O}_S}(\mathcal{N}, \mathcal{P})).$$

We denote by $\mathrm{L}_{\mathcal{O}_S}(\cdot,\cdot)$ the global sections of $\mathcal{L}_{\mathcal{O}_S}(\cdot,\cdot)$ considered as a bornological vector space. For any $\mathcal{M}$ in $\mathrm{Born}(\mathcal{O}_S)$, the functor $\mathrm{L}_{\mathcal{O}_S}(\mathcal{M},\cdot)$ has a left adjoint. We denote it by $\cdot \hat{\otimes} \mathcal{M}$.

Following [15], an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module is a module isomorphic to $E\hat{\otimes}\mathcal{O}_S$ for some Fréchet nuclear (resp. dual Fréchet nuclear) space E. It is easily shown that the $\mathcal{O}_S$ topological dual $\mathcal{L}_{\mathcal{O}_S}(\mathcal{M},\mathcal{O}_S)$ of an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module $\mathcal{M}$ is DFN-free (resp. FN-free). Moreover both FN-free and DFN-free $\mathcal{O}_S$-modules are $\mathcal{O}_S$ reflexive.

The results needed for the proof of the finiteness, duality and base change theorems for relative elliptic pairs are summarized in the three following propositions. The first one is Corollary 5.1 of [25] and the next two ones are easily deduced from the results in §1–2 of [15] (see also [16]).

Proposition 3.11 *Let $\mathcal{M}^\cdot$ (resp. $\mathcal{N}^\cdot$) be a complex of DFN-free (resp. FN-free) $\mathcal{O}_S$-modules. Assume $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ are bounded from above and*

$$u^\cdot : \mathcal{M}^\cdot \longrightarrow \mathcal{N}^\cdot$$

is a continuous $\mathcal{O}_S$-linear morphism. Assume moreover that $u^\cdot$ is a quasi-isomorphism forgetting the topology. Then $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ have $\mathcal{O}_S$-coherent cohomology.

Proposition 3.12 *Let $\mathcal{M}^\cdot$ be a complex of FN-free $\mathcal{O}_S$-modules and let $\mathcal{N}$ be a DFN $\mathcal{O}_S$-module. Assume $\mathcal{M}^\cdot$ is bounded from above and has $\mathcal{O}_S$-coherent cohomology. Then the natural morphism of $\mathbf{D}^+(\mathcal{O}_S)$*

$$\mathcal{L}_{\mathcal{O}_S}(\mathcal{M}^\cdot,\mathcal{N}) \longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{M}^\cdot,\mathcal{N})$$

is an isomorphism.

Proposition 3.13 *Let $\mathcal{M}^\cdot$ be a complex of FN-free (resp. DFN-free) $\mathcal{O}_S$-modules and let $\mathcal{N}$ be an FN (resp. DFN) $\mathcal{O}_S$-module. Assume $\mathcal{M}^\cdot$ is bounded from above and has $\mathcal{O}_S$-coherent cohomology. Then the natural morphism of $\mathbf{D}^-(\mathcal{O}_S)$*

$$\mathcal{M}^\cdot \otimes^L_{\mathcal{O}_S} \mathcal{N} \longrightarrow \mathcal{M}^\cdot \hat{\otimes}_{\mathcal{O}_S} \mathcal{N}$$

is an isomorphism.

In the sequel, when applying the preceding propositions, we will use the following well-known result.

Proposition 3.14 *Assume Z, S are complex manifolds. Denote by $\epsilon : Z \times S \longrightarrow S$ the second projection. Then, we have the following isomorphisms:*

$$\begin{aligned} \epsilon_*\mathcal{F}_{Z\times S|S} &\simeq \Gamma(Z;\mathcal{F}_Z)\,\hat{\otimes}\,\mathcal{O}_S \\ \epsilon_!\mathcal{D}b^{dz,dz}_{Z\times S|S} &\simeq \Gamma_c(Z;\mathcal{D}b^{dz,dz})\,\hat{\otimes}\,\mathcal{O}_S = \mathcal{L}_{\mathcal{O}_S}(\epsilon_*\mathcal{F}_{Z\times S|S},\mathcal{O}_S). \end{aligned}$$

Hence, $\epsilon_! \mathcal{D}b^{d_Z,d_Z}_{Z\times S|S}$ is a DFN-free $\mathcal{O}_S$-module which is the topological dual over $\mathcal{O}_S$ of the FN-free $\mathcal{O}_S$-module $\epsilon_* \mathcal{F}_{Z\times S|S}$. Moreover,

$$\epsilon_* \mathcal{O}_{Z\times S} = \Gamma(Z;\mathcal{O}_Z) \hat{\otimes} \mathcal{O}_S.$$

Hence, if K is a compact subset of Z, we have

$$\epsilon_*[(\mathcal{A}_{Z\times S|S})_{K\times S}] \simeq \Gamma(K;\mathcal{A}_Z) \hat{\otimes} \mathcal{O}_S$$

and $\epsilon_*[(\mathcal{A}_{Z\times S|S})_{K\times S}]$ is a DFN-free topological $\mathcal{O}_S$-module. Finally, if T is another complex manifold and $p : Z\times T\times S \longrightarrow S$ and $q : T\times S \longrightarrow S$ denotes the canonical projections, we have

$$p_* \mathcal{F}_{Z\times T\times S|T\times S} \simeq \epsilon_* \mathcal{F}_{Z\times S|S} \hat{\otimes}_{\mathcal{O}_S} q_* \mathcal{O}_{T\times S}.$$

4 Finiteness

4.1 The case of a projection

Proposition 4.1 *Let Z, S be complex analytic manifolds. Consider $Z\times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Assume that $(\mathcal{M}, F)$ is a good relative elliptic pair with ϵ-proper support on $Z\times S|S$. Then*

$$R\epsilon_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S})$$

is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$.

Proof: By "dévissage", it is obviously sufficient to prove the result when $\mathcal{M}$ is a $\mathcal{D}_{Z\times S|S}$-module which admits a good filtration on a neighborhood of any compact subset of $Z\times S$.

It follows from the relative regularity theorem (Theorem 2.15) that the canonical map

$$F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S} \longrightarrow R\mathcal{H}om(D'F, \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S})$$

is an isomorphism. Using Lemma 3.3, we get the isomorphism

$$\begin{aligned} &R\epsilon_*(F \otimes \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{A}_{Z\times S|S}) \\ &\quad\longrightarrow R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}). \end{aligned} \tag{4.1}$$

Let V be the interior of a closed polydisc Δ of S. Since $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F) \cap \epsilon^{-1}(\Delta)$ is compact, we can find a compact subset K in Z such that $K\times V$ is a neighborhood of $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F) \cap \epsilon^{-1}(V)$. Replacing S by V and G by G_K shows that we may assume from the beginning that G has compact support. Moreover, by Proposition 3.1 we may also assume that $\mathcal{M}_{\mathbb{R}|S}$ is quasi-isomorphic to complex $\mathcal{L}^\cdot$ whose components are free $\mathcal{D}_{Z^{\mathbb{R}}\times S|S}$-modules of finite rank. Since $\mathcal{M}_{\mathbb{R}|S}$ has bounded amplitude, we may even assume $\mathcal{L}^k = 0$ for $k \gg 0$.

We know by Proposition 3.10 that $D'G$ is isomorphic to a bounded complex $T^\cdot$ of the form

$$\cdots \longrightarrow \bigoplus_{i\in I_k} \mathbb{C}_{W_{k,i}} \longrightarrow \cdots$$

where $W_{k,i}$ is relatively compact subanalytic subset of Z such that $D'(\mathbb{C}_{W_{k,i}}) = \mathbb{C}_{\overline{W}_{k,i}}$. Thus $G \simeq D'D'G$ is quasi-isomorphic to a complex $C^\cdot$ of the form

$$\cdots \longrightarrow \bigoplus_{i\in I_k} \mathbb{C}_{\overline{W}_{k,i}} \longrightarrow \cdots$$

It is clear that the sheaf $(\mathcal{A}_{Z\times S})_{|K\times S}$ (resp. $(\mathcal{F}_{Z\times S})_{|U\times S}$) is acyclic for the the functor $\epsilon_{|K\times S_*}$ (resp. $\epsilon_{|U\times S_*}$) for any compact subset K (resp. any open subset U) of Z. Hence we may view isomorphism (4.1) more explicitly as the morphism

$$\begin{aligned} &\epsilon_*((C^\cdot \boxtimes \mathbb{C}_S) \otimes \mathcal{L}^\cdot \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{A}_{Z\times S|S}) \\ &\qquad \longrightarrow \epsilon_*\mathcal{H}om(T^\cdot \boxtimes \mathbb{C}_S, \mathcal{L}^\cdot \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S|S}) \end{aligned} \tag{4.2}$$

in the category of complexes of $\mathcal{O}_S$-modules (not the derived category). Let us denote by $\mathcal{R}_1^\cdot$ (resp. $\mathcal{R}_2^\cdot$) the source (resp. target) of the preceding arrow.

The components of $\mathcal{R}_1^\cdot$ (resp. $\mathcal{R}_2^\cdot$) are easily seen to be finite sums of the sheaves

$$\epsilon_{|\overline{W}_{k,i}\times S_*}(\mathcal{A}_{Z\times S|S|\overline{W}_{k,i}\times S}) \qquad (\text{resp. } \epsilon_{|W_{k,i}\times S_*}(\mathcal{F}_{Z\times S|S|W_{k,i}\times S}) \ ).$$

Hence, $\mathcal{R}_1^\cdot$ (resp. $\mathcal{R}_2^\cdot$) is naturally a complex of DFN-free (resp. FN-free) topological $\mathcal{O}_S$-modules. For these natural topologies, the regularity quasi-isomorphism is clearly continuous. Applying Proposition 3.11 we conclude that $\mathcal{R}_2^\cdot$ has $\mathcal{O}_S$-coherent cohomology and the proof is complete. □

4.2 The general case

Theorem 4.2 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. Assume $(\mathcal{M}, F)$ is a good relative f-elliptic pair with f-proper support; i.e.*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\phi^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then $\underline{f}_{|S!}(\mathcal{M} \otimes F)$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Y|S})$.

Proof: By "dévissage", it is obviously sufficient to prove the result when $\mathcal{M}$ is a $\mathcal{D}_{X|S}$-module which admits a good filtration on a neighborhood of any compact subset of X.

Decomposing f through its graph embedding

$$i : X \longrightarrow X \times Y$$

shows that it is sufficient to prove the finiteness theorem for the second projection

$$p_2 : X \times Y|S \longrightarrow Y|S$$

and the pair $\underline{i}_{|S!}(\mathcal{M}) \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X\times Y|S}))$, $F \boxtimes \mathbb{C}_Y \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X \times Y))$ since by the projection formula we have

$$\underline{i}_{|S!}(\mathcal{M}) \otimes (F \boxtimes \mathbb{C}_Y) \simeq \underline{i}_{|S!}(\mathcal{M} \otimes F).$$

From the definition of $\mathrm{char}_{f|S}(\mathcal{M})$, it is clear that

$$\mathrm{char}_{p_2|S}(\underline{i}_{|S!}(\mathcal{M})) \cap SS(F \boxtimes \mathbb{C}_Y)$$

is contained in the zero section of $T^*(X \times Y|S)$. So if $Y = S$, the theorem is a consequence of the results obtained in the case of a projection.

To conclude, we will show that if f is a relative submersion and the theorem is true for $f : X|Y \longrightarrow Y|Y$ then it is also true for $f : X|S \longrightarrow Y|S$.

We will use a device introduced in [8] and extended in Lemma 2.10.

Let Δ be a polydisc in Y and denote by K the compact subset of X defined by

$$K = \mathrm{supp}(\mathcal{M}) \cap SS(F) \cap f^{-1}(\overline{\Delta}).$$

Using Lemma 2.10, it is easy to see that, in a neighborhood of K, $\mathcal{M}$ is isomorphic to a complex of right $\mathcal{D}_{X|S}$-modules of the form $\mathcal{R} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$ where $\mathcal{R}$ is a coherent right $\mathcal{D}_{X|Y}$-submodule which admits a good filtration and is such that

$$\phi^{-1}\mathrm{char}_{X|Y}(\mathcal{R}) \subset \mathrm{char}_{f|S}(\mathcal{M}).$$

Moreover, this complex may be assumed to be bounded from above.

Since the functor $\underline{f}_{|S!}$ has finite cohomological dimension, it is thus sufficient to prove the coherence on Δ of the cohomology of $\underline{f}_{|S!}(F \otimes \mathcal{M})$ when $\mathcal{M}$ has the special form $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$ where $\mathcal{M}_0$ is a coherent $\mathcal{D}_{X|Y}$-module which admits a good filtration.

In this case, one knows that the complex $\underline{f}_{|Y!}(F\otimes\mathcal{M}_0)$ has $\mathcal{O}_Y$-coherent cohomology, and the chain of isomorphisms

$$\begin{aligned}
&Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})\\
&\quad= Rf_!(F \otimes (\mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})\\
&\quad\xrightarrow{\sim} Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S\to Y|S})\\
&\quad\xrightarrow{\sim} Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X) \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S}
\end{aligned}$$

shows that $\underline{f}_{|S!}(F \otimes \mathcal{M})$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{Y|S})$. □

Corollary 4.3 *In the situation of the preceding theorem, the well known formula:*

$$\underline{f}_{|S!}(F \otimes \mathcal{M}) \otimes^{L}_{\mathcal{D}_{Y|S}} \mathcal{O}_Y \xrightarrow{\sim} Rf_!(F \otimes \mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{O}_X)$$

gives the inclusion:

$$\mathrm{char}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M})) \subset f_\pi {}^t f'^{-1} \mathrm{char}_{X|S}(\mathcal{M})$$

Proof: By Theorem 2.13 and Proposition 5.4.4 and 5.4.14 of [12], we know that

$$\begin{aligned} p^{-1}_{Y|S}\mathrm{char}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M})) &= SS(\underline{f}_{|S!}(F \otimes \mathcal{M}) \otimes^{L}_{\mathcal{D}_{Y|S}} \mathcal{O}_Y) \\ &= SS(Rf_!(F \otimes \mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{O}_X)) \\ &\subset f_\pi {}^t f'^{-1} SS(F \otimes \mathcal{M} \otimes^{L}_{\mathcal{D}_{X|S}} \mathcal{O}_X) \\ &\subset f_\pi {}^t f'^{-1}(SS(F) + p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M})). \end{aligned}$$

Note that by hypothesis:

$$p^{-1}_{X|S}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X.$$

Moreover, one has:

$$p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M}) + {}^t f'(X \times_Y T^*Y) \subset p^{-1}_{X|S}\mathrm{char}_{f|S}(\mathcal{M}).$$

Hence,

$$\left[p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M}) + SS(F)\right] \cap {}^t f'(X \times_Y T^*Y) \subset p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M}) \cap {}^t f'(X \times_Y T^*Y)$$

so

$${}^t f'^{-1}(SS(F) + p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M})) \subset {}^t f'^{-1}(p^{-1}_{X|S}\mathrm{char}_{X|S}(\mathcal{M}))$$

and the proof is complete. □

5 Duality

Let $f : X|S \longrightarrow Y|S$ be a morphism of relative complex manifolds. Our aim in this section is to prove that, under suitable hypotheses, duality commutes with direct images (see Theorem 5.15 for a precise statement). The proof will use the graph decomposition of f and various "dévissages". Hence, it is necessary to construct first the natural transformation:

$$\underline{f}_{|S!} \circ \underline{D}_{X|S} \longrightarrow \underline{D}_{Y|S} \circ \underline{f}_{|S!}$$

and to check its compatibility with respect to composition in f. This will be a consequence of the explicit construction of the trace morphism for $\mathcal{D}$-modules given in the next section. We follow the lines of [24] (see also [11, 17]).

5.1 The relative duality morphism

Recall that if $f : X \longrightarrow Y$ is a holomorphic map then it induces integration maps

$$f_*^{p,q} : f_! \mathcal{D}b_X^{p+d_X,q+d_X} \longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y}$$

commuting with the Dolbeault operators. We will use this fact and the machinery of distributional Dolbeault complexes of §3.1 to construct canonically the duality map for right $\mathcal{D}$-modules.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. To simplify notations, we will set

$$\mathcal{D}b_X^{\cdot,\cdot}(\mathcal{M}) = \mathcal{D}b_{X\times\{\mathrm{pt}\}|\{\mathrm{pt}\}}^{\cdot,\cdot}(\mathcal{M}_{\mathbb{R}|\{\mathrm{pt}\}}).$$

Hence, the components are

$$\mathcal{D}b_X^{p,q}(\mathcal{M}) = \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$$

and the differentials are given in a local coordinate system $z : U \longrightarrow \mathbb{C}^{d_Z}$ by

$$\begin{aligned} \partial^{p,q} : \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} &\longrightarrow \mathcal{D}b_X^{p+1,q} \otimes_{\mathcal{O}_X} \mathcal{M} \\ u \otimes P &\mapsto \partial^{p,q} u \otimes P + \sum_{i=1}^{d_X} dz_i \wedge u \otimes D_{z_i} P \end{aligned}$$

and

$$\begin{aligned} \overline{\partial}^{p,q} : \mathcal{D}b_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M} &\longrightarrow \mathcal{D}b_X^{p,q+1} \otimes_{\mathcal{O}_X} \mathcal{M} \\ u \otimes P &\longrightarrow \overline{\partial}^{p,q} u \otimes P \end{aligned}$$

respectively. Also recall that we denote by $\mathcal{D}b_X^{\cdot}(\mathcal{M})$ the simple complex associated with $\mathcal{D}b_X^{\cdot,\cdot}(\mathcal{M})$.

Lemma 5.1 *The differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ is compatible with the right $\mathcal{D}_X$-module structure of its components and, in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X^{\mathrm{op}})$, one has a canonical isomorphism:*

$$\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)[d_X] \xrightarrow{\sim} \Omega_X.$$

Proof: The compatibility of the differential of $\mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$ with the right $\mathcal{D}_X$-module structure of its components is a direct consequence of the local forms of ∂ and $\overline{\partial}$ recalled above.

Using the fact that $\mathcal{D}_X$ is flat over $\mathcal{O}_X$ and the Dolbeault resolution of Ω_X^p, we get the quasi-isomorphisms

$$\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot} \otimes_{\mathcal{O}_X} \mathcal{D}_X \xrightarrow{\sim} \mathcal{D}b_X^{p,\cdot}(\mathcal{D}_X).$$

Hence, Weil's lemma shows that the natural morphism

$$DR_X^{\cdot}(\mathcal{D}_X) \longrightarrow \mathcal{D}b_X^{\cdot}(\mathcal{D}_X)$$

from the holomorphic to the distributional de Rham complex of $\mathcal{D}_X$ is a quasi-isomorphism of complexes of right $\mathcal{D}_X$-modules and the conclusion follows from the Spencer quasi-isomorphism

$$DR^{\cdot}_X(\mathcal{D}_X) \simeq \Omega_X[-d_X].$$

□

Lemma 5.2 *To any morphism $f : X \longrightarrow Y$ of analytic manifolds one can associate a canonical integration morphism*

$$f_* : f_! \mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X] \longrightarrow \mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y].$$

in the category of bounded complexes of right $\mathcal{D}_Y$-modules.

Proof: At the level of components the integration morphism is obtained as the following chain of morphisms:

$$\begin{aligned} f_!\mathcal{D}b^{p+d_X,q+d_X}(\mathcal{D}_{X\to Y}) &\xrightarrow{\sim} f_!(\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\ &\xrightarrow{\sim} f_!\mathcal{D}b_X^{p+d_X,q+d_X} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\longrightarrow \mathcal{D}b_Y^{p+d_Y,q+d_Y} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \\ &\xrightarrow{\sim} \mathcal{D}b_Y^{p+d_Y,q+d_Y}(\mathcal{D}_Y). \end{aligned}$$

To get the second morphism one has used the projection formula, the fact that $\mathcal{D}b_X$ is a soft sheaf and the fact that $\mathcal{D}_Y$ is locally free over $\mathcal{O}_Y$. The third arrow is deduced from the integration of distributions along the fibers of f.

To conclude, we need to show that the integration morphism is compatible with the differentials of the complexes involved. Thanks to the local forms of the differentials, this is an easy computational verification and we leave it to the reader. □

Lemma 5.3 *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are morphisms of complex analytic manifolds, one has the following commutative diagram:*

$$\begin{array}{ccc} g_!(f_!\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) & \xrightarrow{1} & g_!(\mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y] \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to Z}) \\ \downarrow 2 & & \downarrow 3 \\ g_!f_!\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Z})[2d_X] & \xrightarrow{4} & \mathcal{D}b^{\cdot}_Z(\mathcal{D}_Z)[2d_Z]. \end{array}$$

In this diagram, arrow (1) is deduced by tensor product and proper direct image from f_*, arrow (2) is an isomorphism deduced from the projection formula, arrow (3) is g_* and arrow (4) is equal to $(g \circ f)_*$.

Proof: Going back to the definition of the various morphisms, one sees easily that the commutativity of the preceding diagram is a consequence of the Fubini theorem for distributions, that is, the formula

$$(g \circ f)_*(u) = g_*(f_*(u))$$

where g_* and f_* denotes the push-forward of distributions along g and f respectively, u being a distribution with $g \circ f$ proper support. □

Proposition 5.4 *If $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds then there is a canonical integration arrow*

$$\int_f : \underline{f}_!(\Omega_X[d_X]) \longrightarrow \Omega_Y[d_Y]$$

in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y^{\mathrm{op}})$. Moreover, if $g : Y \longrightarrow Z$ is a second morphism of complex analytic manifolds then

$$\int_{g \circ f} = \int_g \circ \underline{g}_!(\int_f)$$

Proof: One gets the arrow $\int_f$ by composing the morphisms:

$$\begin{aligned}
\underline{f}_!(\Omega_X[d_X]) &\xrightarrow{\sim} Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \\
&\xrightarrow{\sim} Rf_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\
&\xrightarrow{\sim} f_!(\mathcal{D}b^{\cdot}_X(\mathcal{D}_{X\to Y})[2d_X]) \\
&\longrightarrow \mathcal{D}b^{\cdot}_Y(\mathcal{D}_Y)[2d_Y] \\
&\xrightarrow{\sim} \Omega_Y[d_Y]
\end{aligned}$$

Let us point out that the second and last isomorphisms come from Lemma 5.1, that the third one is deduced from the fact that $\mathcal{D}b^{p,q}_X(\mathcal{D}_{X\to Y})$ is c-soft and that the fourth arrow is given by Lemma 5.2.

The compatibility of integration with composition is then a direct consequence of Lemma 5.3. □

Corollary 5.5 *If $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds over S then there is a canonical arrow*

$$\int_{f|S} : \underline{f}_{|S!}(\Omega_{X|S}[d_{X|S}]) \longrightarrow \Omega_{Y|S}[d_{Y|S}].$$

Moreover, if $g : Y|S \longrightarrow Z|S$ is another morphism of relative analytic manifolds over S then

$$\int_{g \circ f|S} = \int_{g|S} \circ \underline{g}_{|S!}(\int_{f|S})$$

Proof: Using the canonical morphism

$$\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X\to Y} \longrightarrow \Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$$

and the integration morphism

$$\int_f : Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \longrightarrow \Omega_Y[d_Y]$$

we get the morphism

$$Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X\to Y}) \longrightarrow \Omega_Y[d_Y].$$

Since

$$\mathcal{D}_{X\to Y} \simeq \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{D}_{Y|S}} f^{-1}\mathcal{D}_Y$$

as a ($\mathcal{D}_{X|S}$,$f^{-1}\mathcal{D}_Y^{\mathrm{op}}$)-bimodule and Ω_Y is a right $\mathcal{D}_Y$-module, we get a $\mathcal{D}_{Y|S}$-linear morphism:

$$Rf_!(\Omega_X[d_X] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \longrightarrow \Omega_Y[d_Y].$$

Tensoring on both sides by $\epsilon_Y^{-1}\Omega_S^{\otimes -1}[-d_S]$ and using the projection formula, we get the requested relative integration map:

$$Rf_!(\Omega_{X|S}[d_{X|S}] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \longrightarrow \Omega_{Y|S}[d_{Y|S}].$$

The last part of the corollary is then an easy consequence of the similar result for $S = \{\mathrm{pt}\}$. □

Definition 5.6 One defines the direct image of a right ($\mathcal{D}_{X|S}, \mathcal{D}_{X|S}$)-bimodule $\mathcal{M}$ by setting:

$$\underline{\underline{f}}_{|S!}(\mathcal{M}) \quad = \quad Rf_!((\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})$$

Lemma 5.7 *There is a canonical isomorphism*

$$[(\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}$$
$$\xrightarrow{\sim} \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}$$

compatible both with the structure of left $\mathcal{D}_{X|S}$-module and the structure of right ($f^{-1}\mathcal{D}_{Y|S}, f^{-1}\mathcal{D}_{Y|S}$)-bimodule.

Proof: One has the chain of isomorphisms

$$\begin{aligned}
&[(\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}\\
&\xrightarrow{\sim} [\mathcal{D}_{X|S} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S\to Y|S}] \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}\\
&\xrightarrow{\sim} \mathcal{D}_{X|S\to Y|S} \otimes^L_{\mathcal{O}_X} \mathcal{D}_{X|S\to Y|S}\\
&\xrightarrow{\sim} \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S}\\
&\xrightarrow{\sim} \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{D}_{Y|S}} (f^{-1}\mathcal{D}_{Y|S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S})\\
&\xrightarrow{\sim} \mathcal{D}_{X|S\to Y|S} \otimes_{f^{-1}\mathcal{D}_{Y|S}} (f^{-1}\mathcal{D}_{Y|S} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_{Y|S})
\end{aligned}$$

In the second isomorphism we have used the exchange lemma. In the fourth line, the last tensor product uses the structure of left $f^{-1}\mathcal{O}_Y$-module of $f^{-1}\mathcal{D}_{Y|S}$. In the fifth isomorphism the last tensor product uses the structure of right $f^{-1}\mathcal{O}_Y$-module of the first factor and the structure of left $f^{-1}\mathcal{O}_Y$-module of the second one. Finally, in the last line, we have used the exchange lemma again. □

Proposition 5.8 *Let $\mathcal{M}$ be a right $\mathcal{D}_{X|S}$-module and let $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}_{X|S}$ be the associated right $\mathcal{D}_{X|S}$-bimodule. If $f : X \longrightarrow Y$ is a morphism of relative analytic manifolds over S then one has the following canonical isomorphism*

$$\underline{\underline{f}}_{|S!}(\mathcal{M} \otimes^L_{\mathcal{O}_X} \mathcal{D}_{X|S}) \xrightarrow{\sim} \underline{f}_{|S!}(\mathcal{M}) \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S}$$

in the derived category $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S} \otimes \mathcal{D}^{\mathrm{op}}_{Y|S})$.

Proof: This is a direct consequence of the preceding lemma. □

Definition 5.9 The differential trace map associated to a morphism

$$f : X|S \longrightarrow Y|S$$

of relative analytic manifolds over S is defined to be the arrow

$$\mathrm{tr}_f : \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S} \longrightarrow \mathcal{K}_{Y|S}$$

in the derived category $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S} \otimes \mathcal{D}^{\mathrm{op}}_{Y|S})$ obtained by composing the following arrows:

$$\begin{aligned} \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S} &= \underline{\underline{f}}_{|S!}(\Omega_{X|S}[d_{X|S}] \otimes^L_{\mathcal{O}_X} \mathcal{D}_{X|S}) \\ &\xrightarrow{\sim} (\underline{f}_{|S!}\Omega_{X|S}[d_{X|S}]) \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \\ &\xrightarrow{\sim} \Omega_{Y|S}[d_{Y|S}] \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \end{aligned}$$

where the first arrow comes from the definition of $\mathcal{K}_{X|S}$ (see p. 11), the second one being a consequence of the preceding proposition and the third one being constructed by tensor product with the integration arrow of Corollary 5.5. By construction, tr_f is compatible with the composition of maps.

Proposition 5.10 *Assume $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds over S. Then the differential trace map*

$$\mathrm{tr}_f : \underline{\underline{f}}_{|S!}K^{\cdot}_{X|S} \longrightarrow K^{\cdot}_{Y|S}$$

induces a morphism

$$\mathrm{du}_f : \underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow \underline{D}_{Y|S}(\underline{f}_{|S!}\mathcal{M})$$

for any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}^{\mathrm{op}}_{X|S})$. Moreover, this morphism is functorial in $\mathcal{M}$ and compatible with composition in f.

Proof: Since, by definition,

$$\underline{D}_{X|S}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}, \mathcal{K}_{X|S})$$

we have a canonical morphism

$$\underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S})$$

in $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{Y|S})$. Composing this morphism with the morphism

$$R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_{Y|S}}(\underline{f}_{|S!}\mathcal{M}, \mathcal{K}_{Y|S})$$

associated to tr_f gives the requested duality morphism. The construction shows that it is natural in $\mathcal{M}$. The compatibility with composition in f comes from the corresponding property of tr_f. □

To conclude this section, we will show that the differential duality morphism is compatible with the duality morphism of complex analytic geometry.

Recall that since $\mathcal{D}_{X|S}$ is an $\mathcal{O}_X$-module, we have a well defined scalar extension functor

$$\begin{array}{rcl} E_{X|S} : \mathrm{Mod}(\mathcal{O}_X) & \longrightarrow & \mathrm{Mod}(\mathcal{D}^{\mathrm{op}}_{X|S}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}. \end{array}$$

The image by this functor of an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent as a right $\mathcal{D}_{X|S}$-module if and only if $\mathcal{F}$ is coherent as an $\mathcal{O}_X$-module. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, one sets

$$D_{X|S}(\mathcal{F}) = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_{X|S}[d_{X|S}])$$

hence, we have the canonical isomorphism:

$$D_{X|S}(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S} \xrightarrow{\sim} \underline{D}_{X|S}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}).$$

Moreover, if $f : X|S \longrightarrow Y|S$ is a morphism of relative analytic manifolds and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module we have the canonical isomorphism:

$$(Rf_!\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \xrightarrow{\sim} \underline{f}_{|S!}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}).$$

With these facts in mind, we can now state:

Proposition 5.11 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds. Assume $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. Then we have the commutative diagram:*

$$\begin{array}{ccc} Rf_! D_{X|S}(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} & \longrightarrow & D_{Y|S}(Rf_!\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S} \\ \downarrow & & \downarrow \\ \underline{f}_{|S!}\underline{D}_{X|S}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S}) & \longrightarrow & \underline{D}_{Y|S}(\underline{f}_{|S!}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_{X|S})) \end{array}$$

where the first and second horizontal arrows come respectively from the geometric and differential duality morphisms, and the vertical arrows isomorphisms are deduced from the compatibility with direct image and duality of the scalar extensions functors $E_{X|S}$ and $E_{Y|S}$.

Proof: Consider the morphism

$$\begin{array}{rcl} \Omega_{X|S}[d_{X|S}] & \longrightarrow & \mathcal{K}_{X|S} \\ \omega & \mapsto & \omega \otimes 1_{X|S}. \end{array} \tag{5.1}$$

It follows easily from the definition of the differential integration map that we have the following commutative diagram:

$$\begin{array}{ccc} Rf_!\Omega_{X|S}[d_{X|S}] & \longrightarrow & \Omega_{Y|S}[d_{Y|S}] \\ \downarrow & & \downarrow \\ \underline{\underline{f}}_{|S!}\mathcal{K}_{X|S} & \longrightarrow & \mathcal{K}_{Y|S}. \end{array}$$

In the preceding diagram the horizontal arrows are the geometric and differential trace maps, the first vertical arrow is deduced from (5.1) by using the canonical section $1_{X|S\to Y|S}$ of $\mathcal{D}_{X|S\to Y|S}$ and the second vertical arrow is (5.1) with X replaced by Y.

Since the geometric and differential duality morphisms are directly constructed from the corresponding trace maps the result is easily reduced to the commutativity of the preceding diagram. □

5.2 The case of a closed embedding

Proposition 5.12 *Let $i : X|S \longrightarrow Y|S$ be a closed relative embedding. Then, for every coherent right $\mathcal{D}_{X|S}$-module $\mathcal{M}$, the canonical morphism*

$$\underline{i}_{|S!}\underline{D}_{X|S}(\mathcal{M}) \longrightarrow \underline{D}_{Y|S}(\underline{i}_{|S*}\mathcal{M})$$

is an isomorphism.

Proof: Since the problem is local on X, we may assume $\mathcal{M}$ has a bounded resolution by finite free right $\mathcal{D}_{X|S}$-modules. Thus it is sufficient to prove the result for $\mathcal{M} = \mathcal{D}_{X|S}$. Since we have

$$\mathcal{D}_{X|S} = \mathcal{O}_X \otimes_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S}$$

it follows from Proposition 5.11 that the result is a direct consequence of the corresponding result for $\mathcal{O}$-modules. Since we do not have a precise direct reference for this well known result we recall it in the following lemma. □

Lemma 5.13 *If $i : Z \longrightarrow X$ is a closed embedding of analytic manifolds, then for any object $\mathcal{F}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Z)$ the complex $Ri_!\mathcal{F}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ and the geometric duality morphism*

$$Ri_!R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \Omega_Z[d_Z]) \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(Ri_!\mathcal{F}, \Omega_X[d_X])$$

is an isomorphism.

Proof: Since the result is of local nature and the duality morphism is compatible with composition, it sufficient to consider the case when $\mathcal{F} = \mathcal{O}_Z$ and

$$\begin{array}{rcl} i : U' & \longrightarrow & U' \times U'' \\ z' & \mapsto & (z', 0) \end{array}$$

where U' (resp. U'') is an open neighborhood of 0 in $\mathbb{C}^{d_Z}$ (resp. $\mathbb{C}$).

In this case, the arrow

$$i_!\Omega_Z[d_Z] \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \Omega_X[d_X])$$

corresponds up to shift to the arrow

$$i_!\Omega_Z \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \Omega_X[1])$$

deduced from the arrow

$$\begin{array}{rcl} i_!\Omega_Z & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(i_!\mathcal{O}_Z, \mathcal{D}b_X^{d_X,\cdot}[1]) \\ i_!\omega & \mapsto & (i_!h \mapsto i_*(h\omega)) \end{array} \tag{5.2}$$

by using the natural map from $\mathcal{H}om_{\mathcal{O}_X}$ to $R\mathcal{H}om_{\mathcal{O}_X}$ and the Dolbeault resolution $\Omega_X \xrightarrow{\sim} \mathcal{D}b_X^{d_X,\cdot}$. Using the negative Koszul complex $K_\cdot(z'', \mathcal{O}_X)$ as a free resolution of $i_!\mathcal{O}_Z$, the map (5.2) corresponds to the morphism of complexes

$$i_!\Omega_Z \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(K_\cdot(z'', \mathcal{O}_X), \mathcal{D}b_X^{d_X,\cdot}) \tag{5.3}$$

which associates to $i_!\omega$ the morphism of complexes which sends h to $i_*(h_{|Z}\omega)$ in degree zero and is zero in other degrees.

The target of the preceding arrow is the simple complex associated to the double complex $K^{\cdot,\cdot}$ below

$$\begin{array}{ccccccc} \mathcal{D}b_X^{d_X,0} & \longrightarrow & \mathcal{D}b_X^{d_X,1} & \cdots & \longrightarrow & \mathcal{D}b_X^{d_X,d_X} \\ \uparrow z'' & & \uparrow z'' & \cdots & & \uparrow z'' \\ \mathcal{D}b_X^{d_X,0} & \longrightarrow & \mathcal{D}b_X^{d_X,1} & \cdots & \longrightarrow & \mathcal{D}b_X^{d_X,d_X} \end{array}$$

where the horizontal maps are the $\overline{\partial}$ Dolbeault operators, the vertical ones being multiplication by z''. In the preceding diagram the term of bidegree $(0,0)$ is in the upper left corner and the image of $i_!\omega$ by the arrow (5.3) corresponds to the section $i_*\omega$ of $\mathcal{D}b_X^{d_X,1}$ in bidegree $(-1,1)$.

Since the canonical inclusion of $K_\cdot(z'', \Omega_X)$ in the simple complex $sK^{\cdot,\cdot}$ is a quasi-isomorphism, it follows that the cohomology of $sK^{\cdot,\cdot}$ is concentrated in degree zero and that $H^0(sK^{\cdot,\cdot})$ is isomorphic to $\Omega_X/z''\Omega_X$.

Now we have successively

$$\begin{aligned} i_*\omega &= \omega(z') \wedge \delta(z'')dx'' \wedge dy'' \\ &= \omega(z') \wedge \overline{\partial}\left(\frac{dz''}{2i\pi z''}\right) \\ &= \overline{\partial}\left(\omega(z') \wedge \frac{dz''}{2i\pi z''}\right) \end{aligned}$$

and this shows that $i_*\omega$ has the same cohomology class in $H^0(sK^{\cdot,\cdot})$ as the section $\omega(z') \wedge (dz''/2i\pi)$ of $K^{0,0}$. Hence the arrow (**) corresponds at the level of H^0 to the isomorphism

$$\begin{array}{rcl} i_!\Omega_Z & \longrightarrow & \Omega_X/z''\Omega_X \\ i_!\omega & \mapsto & \left[\omega(z') \wedge \dfrac{dz''}{2i\pi}\right]_{z''\Omega_X} \end{array}$$

and the conclusion follows. □

5.3 The case of a projection

As for the finiteness theorem our starting point will be the case of a projection.

Proposition 5.14 *Let Z, S be complex analytic manifolds and denote by n the complex dimension of Z. Consider $Z \times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Assume that $(\mathcal{M},F)$ is a good relative elliptic pair with ϵ-proper support on $Z \times S|S$. Then the natural pairing*

$$R\epsilon_*(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}) \otimes^L_{\mathcal{O}_S} R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n])) \longrightarrow \mathcal{O}_S$$

identifies each complex with the $\mathcal{O}_S$ dual of the other.

Proof: Since the dual of a relative elliptic pair is a relative elliptic pair, we need only to show that the map

$$\begin{aligned} &R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z\times S|S}}(\mathcal{M}, \Omega^n_{Z\times S|S}[n]) \\ &\qquad \longrightarrow \quad R\mathcal{H}om_{\mathcal{O}_S}(R\epsilon_*(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}), \mathcal{O}_S) \end{aligned}$$

deduced from the duality pairing is an isomorphism in the derived category.

Using Lemmas 3.3, 3.4 and 3.6 and the regularity quasi-isomorphism, it is equivalent to prove that the canonical map

$$\begin{aligned} &R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}, \mathcal{D}b^{n,n}_{Z\times S|S})) \\ &\qquad \longrightarrow \quad R\mathcal{H}om_{\mathcal{O}_S}(R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}), \mathcal{O}_S)) \end{aligned} \tag{5.4}$$

is an isomorphism in the derived category.

We will work as in the proof of Proposition 4.1 and use the notations introduced there. Using the resolution $\mathcal{L}^{\cdot}$ of $\mathcal{M}_{\mathbb{R}|S}$ and the resolution $T^{\cdot}$ of $D'G$, we will compute explicitly the preceding morphism. We already know that

$$\begin{aligned} &R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \\ &\qquad \xrightarrow{\sim} \quad \epsilon_* \mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) = \mathcal{R}^{\cdot}_2. \end{aligned}$$

Since $\mathcal{D}b_{W_{k,i}\times S}$ is acyclic for the functor $\epsilon_{|W_{k,i}\times S_!}$, we get

$$\begin{aligned} &R\epsilon_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{M}_{\mathbb{R}|S}, \mathcal{D}b^{n,n}_{Z\times S|S})) \\ &\qquad \xrightarrow{\sim} \quad \epsilon_!((T^{\cdot} \boxtimes \mathbb{C}_S) \otimes \mathcal{H}om_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}}(\mathcal{L}^{\cdot}, \mathcal{D}b^{n,n}_{Z\times S|S})). \end{aligned}$$

We denote by $\mathcal{R}^{\cdot}_3$ this last complex.

The components of $\mathcal{R}^{\cdot}_2$ (resp. $\mathcal{R}_3$) are finite sums of the sheaves

$$\epsilon_{|W_{k,i}\times S_*}(\mathcal{F}_{W_{k,i}\times S}) \qquad (\text{resp. } \epsilon_{|W_{k,i}\times S_!}(\mathcal{D}b^{n,n}_{W_{k,i}\times S})\ )$$

which are FN-free (resp. DFN-free) $\mathcal{O}_S$-modules. Hence, $\mathcal{R}_2^\cdot$ and $\mathcal{R}_3^\cdot$ are naturally complexes of topological $\mathcal{O}_S$-modules.

For any open subset U of Z, we know that

$$\epsilon_{U\times S|S!}\mathcal{D}b^{n,n}_{U\times S|S} = \mathcal{L}_{\mathcal{O}_S}(\epsilon_{|U\times S|S_*}\mathcal{F}_{U\times S|S}, \mathcal{O}_S)$$

where the second member of the preceding equality is the sheaf of continuous $\mathcal{O}_S$-linear homomorphisms between the FN-free $\mathcal{O}_S$-module $\epsilon_{|U\times S|S_*}\mathcal{F}_{U\times S|S}$ and $\mathcal{O}_S$. Hence we get the canonical isomorphism

$$\mathcal{R}_3^k \xrightarrow{\sim} \mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^{-k}, \mathcal{O}_S)$$

for any integer k. One checks easily that these maps define an isomorphism of complexes

$$\mathcal{R}_3^\cdot \xrightarrow{\sim} \mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^\cdot, \mathcal{O}_S).$$

Moreover, the composition of this morphism with the natural morphism

$$\mathcal{L}_{\mathcal{O}_S}(\mathcal{R}_2^\cdot, \mathcal{O}_S) \longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{R}_2^\cdot, \mathcal{O}_S) \tag{5.5}$$

gives the map (5.4).

Since $\mathcal{R}_2^\cdot$ has $\mathcal{O}_S$-coherent cohomology, Proposition 3.12 shows that (5.5) is a quasi-isomorphism and the proof is complete. □

5.4 The general case

Theorem 5.15 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative analytic manifolds over S. Assume ($\mathcal{M}$, F) is a good relative f-elliptic pair with f-proper support; i.e.*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\phi^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then the duality morphism

$$\underline{f}_{|S!}(D'F \otimes \underline{D}_{X|S}(\mathcal{M})) \longrightarrow \underline{D}_{Y|S}(\underline{f}_{|S!}(F \otimes \mathcal{M}))$$

is an isomorphism.

Proof: Using the factorization of f through its graph embedding

$$i : X \longrightarrow X \times Y$$

we deduce from the results obtained for closed embeddings that the theorem will be true if it is true for the second projection

$$q : X \times Y|S \longrightarrow Y|S$$

and the pair

$$(\underline{i}_{|S!}(\mathcal{M}), F \boxtimes \mathbb{C}_Y) \in Ob(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X\times Y|S}) \times \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X \times Y)).$$

From the definition of $\mathrm{char}_{f|S}(\mathcal{M})$, it is clear that

$$\mathrm{char}_{q|S}(\underline{i}_{|S!}(\mathcal{M})) \cap SS(F \boxtimes \mathbb{C}_Y)$$

is in the zero section of $T^*(X \times Y|S)$. So if $Y = S$, the theorem is a consequence of the results obtained in the product case.

To conclude, we will show that if f is a relative submersion and the theorem is true for $f : X|Y \longrightarrow Y|Y$ then it is also true for $f : X|S \longrightarrow Y|S$.

Let us assume first that there is a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$ such that

$$\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}.$$

One get successively:

$$\begin{aligned}
&\underline{D}_{Y|S}(\underline{f}_!(F \otimes \mathcal{M}))\\
&\quad\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{Y|S}}([Rf_!(F \otimes \mathcal{M}_0 \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X)] \otimes^L_{\mathcal{O}_Y} \mathcal{D}_{Y|S}, \mathcal{K}_{Y|S})\\
&\quad\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_Y}(\underline{f}_{|Y!}(F \otimes \mathcal{M}_0), \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{K}_{Y|S}\\
&\quad\xrightarrow{\sim} \underline{f}_{|Y!}(D'F \otimes \underline{D}_{X|Y}(\mathcal{M}_0)) \otimes_{\mathcal{O}_Y} \mathcal{K}_{Y|S}\\
&\quad\xrightarrow{\sim} Rf_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|Y}}(\mathcal{M}_0, \mathcal{K}_{X|Y}) \otimes^L_{\mathcal{D}_{X|Y}} \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{K}_{Y|S})\\
&\quad\xrightarrow{\sim} Rf_!(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|Y}}(\mathcal{M}_0, \mathcal{K}_{X|S}) \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})\\
&\quad\xrightarrow{\sim} \underline{f}_{|S!}(D'F \otimes R\mathcal{H}om_{\mathcal{D}_{X|S}}(\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}, \mathcal{K}_{X|S}))\\
&\quad\xrightarrow{\sim} \underline{f}_{|S!}(D'F \otimes \underline{D}_{X|S}(\mathcal{M}))
\end{aligned}$$

and the theorem is proved.

The general case is reduced to the preceding case by using Lemma 2.10 as in the proof of Theorem 4.2.

□

6 Base change and Künneth formula

6.1 Base change

Recall that to any morphism $b : S_b \longrightarrow S$ of complex manifolds is associated a base change functor

$$\begin{aligned}
(\cdot)_b : \mathrm{Man}(S) &\longrightarrow \mathrm{Man}(S_b)\\
X|S &\longrightarrow X \times_S S_b|S_b
\end{aligned}$$

which transforms relative manifolds over S into relative manifolds over S_b. The aim of this section is to study the behavior of relative elliptic pairs under this functor. The main result is Theorem 6.5.

Let us fix a base change map $b : S_b \longrightarrow S$. For any relative manifold $X|S$, we denote by $X_b|S_b$ its image by the base change functor $(\cdot)_b$ and by b_X the projection from X_b to X. By construction, we have the cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{\epsilon_X} & S \\ \uparrow{\scriptstyle b_X} & \square & \uparrow{\scriptstyle b} \\ X_b & \xrightarrow[\epsilon_{X_b}]{} & S_b. \end{array}$$

Hence, there is a canonical ring morphism

$$b_X^{-1}\mathcal{D}_{X|S} \longrightarrow \mathcal{D}_{X_b|S_b}$$

and we may introduce the following definition.

Definition 6.1 The *base change functor for relative right $\mathcal{D}$-modules* is the functor

$$\begin{array}{ccc} \mathbf{D}(\mathcal{D}_{X|S}^{\mathrm{op}}) & \longrightarrow & \mathbf{D}(\mathcal{D}_{X_b|S_b}^{\mathrm{op}}) \\ \mathcal{M} & \mapsto & b_X^{-1}\mathcal{M} \otimes^L_{b_X^{-1}\mathcal{D}_{X|S}} \mathcal{D}_{X_b|S_b}. \end{array}$$

This functor clearly induces a functor from $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S}^{\mathrm{op}})$ to $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X_b|S_b}^{\mathrm{op}})$.

The *base change functor for sheaves of $\mathbb{C}$-vector spaces* is the functor

$$\begin{array}{ccc} \mathbf{D}(X) & \longrightarrow & \mathbf{D}(X_b) \\ F & \mapsto & b_X^{-1}F. \end{array}$$

This functor clearly induces a functor from $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ to $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X_b)$. Since the context will avoid any possible confusion, we denote all these functors by $(\cdot)_b$.

Let us consider now a morphism $f : X|S \longrightarrow Y|S$ of relative manifolds. We denote by $f_b : X_b|S_b \longrightarrow Y_b|S_b$ the image of f by the base change associated with b. One checks easily that the square:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle b_X} & & \uparrow{\scriptstyle b_Y} \\ X_b & \xrightarrow[f_b]{} & Y_b \end{array} \tag{6.1}$$

is cartesian.

Proposition 6.2 *Using the notations introduced above:*

a) There is a canonical morphism

$$b_X^{-1}\mathcal{D}_{X|S\to Y|S} \otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b} \longrightarrow \mathcal{D}_{X_b|S_b\to Y_b|S_b}$$

in $\mathbf{D}^{\mathrm{b}}(b_X^{-1}\mathcal{D}_{X|S} \otimes f_b^{-1}\mathcal{D}_{Y_b|S_b}^{\mathrm{op}})$.

b) *The preceding morphism induces a natural morphism*

$$(\underline{f}_{|S!}(\mathcal{M}))_b \longrightarrow \underline{f_b}_{|S_b!}(\mathcal{M}_b)$$

for $\mathcal{M}$ *in* $\mathbf{D}(\mathcal{D}_{X|S}^{\mathrm{op}})$.

c) *The morphisms in (a) and (b) are isomorphisms if either f or b is a closed embedding.*

Proof: Since

$$b_X^{-1}\mathcal{D}_{X|S\to Y|S} \otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b} \xrightarrow{\sim} b_X^{-1}\mathcal{O}_X \otimes^L_{b_X^{-1}f^{-1}\mathcal{O}_Y} f_b^{-1}\mathcal{D}_{Y_b|S_b}$$

and

$$\mathcal{D}_{X_b|S_b\to Y_b|S_b} \xrightarrow{\sim} \mathcal{O}_{X_b} \otimes_{f_b^{-1}\mathcal{O}_{Y_b}} f_b^{-1}\mathcal{D}_{Y_b|S_b}$$

the canonical morphism $b_X^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{X_b}$ induces the morphism in (a).

For any $\mathcal{M}$ in $\mathbf{D}(\mathcal{D}_{X|S}^{\mathrm{op}})$, we construct the morphism in (b) as the chain of morphisms:

$$\begin{aligned}
(\underline{f}_{|S!}\mathcal{M})_b &= b_Y^{-1}Rf_!(\mathcal{M}\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})\otimes^L_{b_Y^{-1}\mathcal{D}_{Y|S}}\mathcal{D}_{Y_b|S_b}\\
&\xrightarrow{\sim} Rf_{b!}b_X^{-1}(\mathcal{M}\otimes^L_{\mathcal{D}_{X|S}}\mathcal{D}_{X|S\to Y|S})\otimes^L_{b_Y^{-1}\mathcal{D}_{Y|S}}\mathcal{D}_{Y_b|S_b}\\
&\xrightarrow{\sim} Rf_{b!}(b_X^{-1}\mathcal{M}\otimes^L_{b_X^{-1}\mathcal{D}_{X|S}} b_X^{-1}\mathcal{D}_{X|S\to Y|S}\otimes^L_{f_b^{-1}b_Y^{-1}\mathcal{D}_{Y|S}} f_b^{-1}\mathcal{D}_{Y_b|S_b})\\
&\longrightarrow Rf_{b!}(b_X^{-1}\mathcal{M}\otimes^L_{b_X^{-1}\mathcal{D}_{X|S}}\mathcal{D}_{X_b|S_b\to Y_b|S_b})\\
&\xrightarrow{\sim} \underline{f_b}_{|S_b!}(\mathcal{M}_b).
\end{aligned}$$

This chain of morphisms is obtained using the definition of the base change functor, the fact that the square (6.1) is cartesian, the projection formula, the morphism constructed in (a) and again the definition of the base change functor.

To conclude the proof, it is sufficient to show that if either f or b is a closed embedding then the morphism constructed in (a) is an isomorphism.

Assume f is a closed embedding. The problem being local, we may assume there are open neighborhoods U and V of zero in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively with $X = U \times S$, $Y = U \times V \times S$ and

$$\begin{aligned}
f : U\times S &\longrightarrow U\times V\times S\\
(u,s) &\longrightarrow (u,0,s).
\end{aligned}$$

Then, we get $X_b = U \times S_b$, $Y_b = U \times V \times S_b$ and

$$\begin{aligned}
f_b : U\times S_b &\longrightarrow U\times V\times S_b\\
(u,s_b) &\longrightarrow (u,0,s_b).
\end{aligned}$$

Moreover, $b_X(u, s_b) = (u, b(s_b))$ and $b_Y(u, v, s_b) = (u, v, b(s_b))$. In this simple geometric situation, we have the Koszul quasi-isomorphisms:

$$\begin{aligned}
K.(\mathcal{O}_Y; v_1,\ldots,v_n) &\xrightarrow{\sim} f_*\mathcal{O}_X\\
K.(\mathcal{O}_{Y_b}; v_1\circ b_Y,\ldots,v_n\circ b_Y) &\xrightarrow{\sim} f_{b*}\mathcal{O}_{X_b}
\end{aligned}$$

where $v_1, \dots, v_n$ denotes the functions on Y induced by the standard coordinates on V. Hence, we get the isomorphisms:

$$\begin{aligned} b_X^{-1}\mathcal{O}_X \otimes^L_{b_X^{-1}f^{-1}\mathcal{O}_Y} f_b^{-1}\mathcal{D}_{Y_b|S_b} &\simeq f_b^{-1}K.(\mathcal{D}_{Y_b|S_b}; v_1 \circ b_Y, \dots, v_n \circ b_Y) \\ \mathcal{O}_{X_b} \otimes^L_{f_b^{-1}\mathcal{O}_{Y_b}} f_b^{-1}\mathcal{D}_{Y_b|S_b} &\simeq f_b^{-1}K.(\mathcal{D}_{Y_b|S_b}; v_1 \circ b_Y, \dots, v_n \circ b_Y) \end{aligned}$$

and the conclusion follows.

The case where b is a closed embedding is treated in a similar way. □

The following easy lemma will be useful in the sequel. We leave its proof to the reader.

Lemma 6.3 *Let $f : X|S \longrightarrow Y|S$ be a relative submersion and let $b : S_b \longrightarrow S$ be a base map. Consider X as a relative manifold over Y through the map f and assume $\mathcal{N}$ is an object of $\mathbf{D}(\mathcal{D}^{\mathrm{op}}_{X|Y})$. Then*

$$f_b : X_b|Y_b \longrightarrow Y_b|Y_b$$

is the image of $f : X|Y \longrightarrow Y|Y$ by the base change associated with b_Y and

$$(\mathcal{N} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S})_b \simeq \mathcal{N}_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b}.$$

The behavior of the characteristic variety under base change is given by the following result.

Proposition 6.4 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative manifolds. In the diagram*

$$T^*X_b|S_b \underset{{}^t(b_X)'}{\longleftarrow} X_b \times_X T^*X|S \underset{(b_X)_\pi}{\longrightarrow} T^*X|S$$

the first arrow is an isomorphism and for any object $\mathcal{M}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ we have

$$\mathrm{char}_{f_b|S_b}(\mathcal{M}_b) \subset {}^t(b_X)'(b_X)_\pi^{-1}\mathrm{char}_{f|S}(\mathcal{M}).$$

Proof: Using the graph factorization of f and part (c) of Proposition 6.2 we are reduced to the case where f is a relative submersion. In this case, assume $\mathcal{M}$ is generated as a right $\mathcal{D}_{X|S}$-module by a coherent right $\mathcal{D}_{X|Y}$-module $\mathcal{M}_0$. Thanks to Lemma 6.3 the epimorphism

$$\mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S} \longrightarrow \mathcal{M} \longrightarrow 0$$

induces the epimorphism

$$(\mathcal{M}_0)_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b} \longrightarrow \mathcal{M}_b \longrightarrow 0.$$

Hence,

$$\mathrm{char}_{f_b|S_b}(\mathcal{M}_b) \subset \phi_f^{-1}\mathrm{char}_{X_b|Y_b}((\mathcal{M}_0)_{b_Y})$$

and the result will be true for $f : X|S \longrightarrow Y|S$ and the base change by b if it is true for $f : X|Y \longrightarrow Y|Y$ and the base change by b_Y. In other words, we are reduced to the obvious case where $Y = S$. □

Theorem 6.5 *Let $f : X|S \longrightarrow Y|S$ be a morphism of relative manifolds and let $b : S_b \longrightarrow S$ be a base map. Denote by $f_b : X_b|S_b \longrightarrow Y_b|S_b$ the image of f by the base change associated to b. Assume $(\mathcal{M},F)$ is a good relative f-elliptic pair. Then*

a) $(\mathcal{M}_b,F_b)$ is an f_b-elliptic pair in some neighborhood of $\operatorname{supp}\mathcal{M}_b$,

b) the canonical morphism

$$[\underline{f}_{|S!}(F \otimes \mathcal{M})]_b \longrightarrow \underline{f_b}_{|S_b!}(F_b \otimes \mathcal{M}_b)$$

is an isomorphism.

Proof: Since $(\mathcal{M},F)$ is a relative elliptic pair, F is non characteristic for b_X in a neighborhood of $\operatorname{supp}\mathcal{M}$ and [12, Proposition 5.4.13] gives us an estimate of the micro-support of F_b which together with the preceding proposition gives us (a).

To prove part (b), we will use the graph factorization of f and part (c) of Proposition 6.2 to reduce the problem to the case where $f : X|S \longrightarrow Y|S$ is a relative submersion.

As in the preceding proposition, it is sufficient to treat the case $Y = S$. Assume $\mathcal{M}_0$ is a right $\mathcal{D}_{X|Y}$-module and set $\mathcal{M} = \mathcal{M}_0 \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_{X|S}$. We have successively:

$$\begin{aligned}[\underline{f}_{|S!}(\mathcal{M})]_b &\simeq [\underline{f}_{|Y!}(\mathcal{M}_0) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y|S}]_b \\ &\simeq [\underline{f}_{|Y!}(\mathcal{M}_0)]_{b_Y} \otimes_{\mathcal{O}_{Y_b}} \mathcal{D}_{Y_b|S_b}\end{aligned}$$

and

$$\begin{aligned}\underline{f_b}_{|S_b!}(\mathcal{M}_b) &\simeq \underline{f_b}_{|S_b!}[(\mathcal{M}_0)_{b_Y} \otimes_{\mathcal{D}_{X_b|Y_b}} \mathcal{D}_{X_b|S_b}] \\ &\simeq \underline{f_b}_{|Y_b!}[(\mathcal{M}_0)_{b_Y}] \otimes_{\mathcal{O}_{Y_b}} \mathcal{D}_{Y_b|S_b}.\end{aligned}$$

Hence, using Lemma 2.10, we see that the theorem will be true for $f : X|S \longrightarrow Y|S$ and the base change by b if it is true for $f : X|Y \longrightarrow Y|Y$ and the base change by b_Y.

Finally, factorizing f and b through their graphs and using once more part (c) of Proposition 6.2, we see that it is sufficient to treat the case where $f : Z \times S|S \longrightarrow S|S$ is the second projection. We may also assume that the corresponding f-elliptic pair is of the form $(\mathcal{M}, G \boxtimes \mathbb{C}_S)$ where G and $\mathcal{M}$ are objects of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Z\times S|S})$ respectively, and that $b : T \times S \longrightarrow S$ is the first projection. This product case is treated in Proposition 6.6 below. □

Proposition 6.6 *Let Z, S, T be complex analytic manifolds. Consider $Z \times S$ as a relative analytic manifold over S through the second projection ϵ. Let G be an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$ and set $F = G \boxtimes \mathbb{C}_S$. Consider the cartesian square*

$$\begin{array}{ccc} Z \times S & \xrightarrow{\epsilon} & S \\ \uparrow{\scriptstyle p} & \square & \uparrow{\scriptstyle b} \\ Z \times T \times S & \xrightarrow{\eta} & T \times S \end{array}$$

where the maps are the canonical projections. Assume that $(\mathcal{M},F)$ is a relative elliptic pair with ϵ-proper support on $Z \times S|S$. Then the canonical map

$$\begin{aligned} b^{-1}R\epsilon_*(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times S}) \otimes_{b^{-1}\mathcal{O}_S} \mathcal{O}_{T\times S} \\ \longrightarrow R\eta_*(p^{-1}F \otimes p^{-1}\mathcal{M} \otimes^L_{p^{-1}\mathcal{D}_{Z\times S|S}} \mathcal{O}_{Z\times T\times S}) \end{aligned}$$

is an isomorphism.

Proof: Thanks to the regularity theorem 2.15 and Lemma 3.3, it is equivalent to prove that the canonical morphism:

$$\begin{aligned} b^{-1}R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \otimes_{b^{-1}\mathcal{O}_S} \mathcal{O}_{T\times S} \\ \longrightarrow R\eta_* R\mathcal{H}om(p^{-1}D'F, p^{-1}\mathcal{M}_{\mathbb{R}} \otimes^L_{p^{-1}\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times T\times S|T\times S}) \end{aligned}$$

is an isomorphism. For short, let us denote by $\mathcal{S}_1^{\cdot}$ (resp. $\mathcal{S}_2^{\cdot}$) the source (resp. target) of the preceding arrow. Clearly, it is sufficient to prove that for any open polydisc Δ of T, the induced morphism

$$Rb_*(\mathcal{S}^{\cdot}_{1|\Delta\times S}) \longrightarrow Rb_*(\mathcal{S}^{\cdot}_{2|\Delta\times S}) \tag{6.2}$$

is an isomorphism. We will compute this morphism explicitly as in the proof of Proposition 4.1. Using the notations introduced there, we already know that

$$\begin{aligned} R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}|S} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) \\ \xrightarrow{\sim} \epsilon_*\mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} \mathcal{F}_{Z\times S}) = \mathcal{R}^{\cdot}_2. \end{aligned}$$

Since $\mathcal{R}^{\cdot}_2$ has $\mathcal{O}_S$-coherent cohomology,

$$\begin{aligned} Rb_*(\mathcal{S}^{\cdot}_{1|\Delta\times S}) &\simeq Rb_*(b^{-1}\mathcal{R}^{\cdot}_2 \otimes^L_{b^{-1}\mathcal{O}_S} \mathcal{O}_{\Delta\times S}) \\ &\simeq \mathcal{R}^{\cdot}_2 \otimes^L_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}. \end{aligned}$$

Moreover, we have:

$$\begin{aligned} Rb_*(\mathcal{S}^{\cdot}_{2|\Delta\times S}) &\simeq R\epsilon_* R\mathcal{H}om(D'F, \mathcal{M}_{\mathbb{R}} \otimes^L_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} p_*\mathcal{F}_{Z\times\Delta\times S|\Delta\times S}) \\ &\simeq \epsilon_*\mathcal{H}om(T^{\cdot} \boxtimes \mathbb{C}_S, \mathcal{L}^{\cdot} \otimes_{\mathcal{D}_{Z^{\mathbb{R}}\times S|S}} p_*\mathcal{F}_{Z\times\Delta\times S|\Delta\times S}). \end{aligned}$$

Let us denote $\mathcal{R}^{\cdot}_4$ this last complex. Since we have the isomorphism

$$\epsilon_* p_* i_{W\times\Delta\times S*} i^{-1}_{W\times\Delta\times S} \mathcal{F}_{Z\times\Delta\times S|\Delta\times S} \simeq \Gamma(W;\mathcal{F}_W) \hat{\otimes} \Gamma(\Delta;\mathcal{O}_\Delta) \hat{\otimes} \mathcal{O}_S$$

for any open subset W of Z, a direct computation shows that

$$\mathcal{R}^{\cdot}_4 \xrightarrow{\sim} \mathcal{R}^{\cdot}_2 \hat{\otimes}_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}.$$

Clearly, the morphism (6.2) corresponds to the canonical morphism

$$\mathcal{R}^{\cdot}_2 \otimes^L_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S} \longrightarrow \mathcal{R}^{\cdot}_2 \hat{\otimes}_{\mathcal{O}_S} b_*\mathcal{O}_{\Delta\times S}.$$

Since $\mathcal{R}^{\cdot}_2$ has $\mathcal{O}_S$-coherent cohomology, Proposition 3.13 shows that it is an isomorphism and the conclusion follows. □

6.2 Künneth formula

Theorem 6.7 *Let $f_1 : X_1|S \longrightarrow Y_1|S$ and $f_2 : X_2|S \longrightarrow Y_2|S$ be two morphisms of relative manifolds. Assume*

i) $(\mathcal{M}_1,F_1)$ is a good relative f_1-elliptic pair with f_1-proper support,

ii) $(\mathcal{M}_2,F_2)$ is a good relative f_2-elliptic pair with f_2-proper support.

Then:

a) $(\mathcal{M}_1 \boxtimes_S \mathcal{M}_2, F_1 \boxtimes_S F_2)$ is a good relative $f_1 \times_S f_2$-elliptic pair with $f_1 \times_S f_2$-proper support,

b) the natural morphism

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \boxtimes_S \underline{f_2}_{|S!}(F_2 \otimes \mathcal{M}_2) \longrightarrow \underline{f_1 \times_S f_2}_{|S!}[(F_1 \boxtimes_S F_2) \otimes (\mathcal{M}_1 \boxtimes_S \mathcal{M}_2)]$$

is an isomorphism.

Proof: Part (a) being obvious, we skip directly to part (b). Since

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1)$$

has $\mathcal{D}_{Y_1|S}$-coherent cohomology, the formula

$$f_1 \times_S f_2 = (\mathrm{id}_{X_1} \times_S f_2) \circ (f_1 \times_S \mathrm{id}_{X_2})$$

allows us to restrict to the case $f_2 = \mathrm{id}_{X_2}$. So, we need only to prove that the canonical map

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \boxtimes_S (F_2 \otimes \mathcal{M}_2) \longrightarrow \underline{f_1 \times_S \mathrm{id}_{X_2}}_{|S!}(F_1 \boxtimes_S F_2) \otimes (\mathcal{M}_1 \boxtimes_S \mathcal{M}_2)$$

is an isomorphism. Using the projection formula, we may get rid of F_2. So we assume $F_2 = \mathbb{C}_{X_2}$. The problem being local on $Y_1 \times_S X_2$, we may further assume that $\mathcal{M}_2$ is equal to $\mathcal{D}_{X_2|S}$.

The image of $f_1 : X_1|S \longrightarrow Y_1|S$ under the base change associated with $\epsilon_2 : X_2 \longrightarrow S$ is

$$f_1 \times_S \mathrm{id}_{X_2} : X_1 \times_S X_2|X_2 \longrightarrow Y_1 \times_S X_2|X_2.$$

Hence, by Theorem 6.5, we have the isomorphism:

$$[\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1)]_{\epsilon_2} \xrightarrow{\sim} \underline{(f_1 \times_S \mathrm{id}_{X_2})}_{|X_2!}[(F_1 \otimes \mathcal{M}_1)_{\epsilon_2}].$$

By scalar extension, we get the isomorphism:

$$\underline{f_1}_{|S!}(F_1 \otimes \mathcal{M}_1) \boxtimes_S \mathcal{D}_{X_2|S} \xrightarrow{\sim} \underline{f_1 \times_S \mathrm{id}_{X_2}}_{|S!}[(F_1 \boxtimes_S \mathbb{C}_{X_2}) \otimes (\mathcal{M}_1 \boxtimes_S \mathcal{D}_{X_2|S})]$$

and the conclusion follows. □

7 Microlocalization

Here, we shall prove that direct image commutes with microlocalization. More precisely, denote by $\mathcal{E}_X$ the sheaf of (finite order) microdifferential operators on T^*X (see [18] or [19] for a detailed exposition).

Consider a morphism $f : X \longrightarrow Y$ of complex analytic manifolds and the associated diagram:

$$T^*X \xleftarrow[{}^tf']{} X \times_Y T^*Y \xrightarrow[f_\pi]{} T^*Y$$

and recall that the microlocal proper direct image of a right $\mathcal{E}_X$-module $\mathcal{M}$ is defined through the formula

$$\underline{f}_!(\mathcal{M}) = Rf_{\pi!}({}^tf'^{-1}\mathcal{M} \otimes^L_{{}^tf'^{-1}\mathcal{E}_X} \mathcal{E}_{X\to Y}),$$

where $\mathcal{E}_{X\to Y}$ denotes the micro-differential transfer module associated to f.

Also recall that the microlocalization of a right $\mathcal{D}_X$-module $\mathcal{M}$ is the right $\mathcal{E}_X$-module $\mathcal{M}\mathcal{E}$ defined on T^*X by setting

$$\mathcal{M}\mathcal{E} = \pi_X^{-1}\mathcal{M} \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X.$$

In this section, we prove that, under the hypothesis of the finiteness theorem, we have

$$[\underline{f}_!(\mathcal{M} \otimes F)]\mathcal{E} \simeq \underline{f}_![(\mathcal{M} \otimes F)\mathcal{E}].$$

This result was established by Kashiwara [9] when $F = \mathbb{C}_X$ and f is projective. It was also announced in a non proper case in [8].

7.1 The topology of the sheaf $\mathcal{C}_{Y|X}(0)$

Let us show that the sheaf $\mathcal{C}_{Y|X}(0)$ of [18] is naturally a sheaf of topological vector spaces and that its sections on a compact subset of T^*_YX form a DFN space.

Proposition 7.1 *Let X be a complex analytic manifold. Assume Y is a complex submanifold of X and denote by $\mathcal{C}_{Y|X}(0)$ the sheaf of holomorphic microfunctions of order 0 on T^*_YX. Then, for any compact subset $K \subset T^*_YX$, the space*

$$\Gamma(K; \mathcal{C}_{Y|X}(0))$$

has a canonical DFN topology.

Proof: Locally, we may use a coordinate system $(x_1, \ldots, x_d, y_1, \ldots, y_{n-d})$ where Y is defined by the equations

$$x_1 = 0, \cdots, x_d = 0.$$

Denote by $(y_1, \cdots, y_{n-d}, \xi_1, \ldots, \xi_d)$ the corresponding coordinates on T^*_YX. It follows from [18, Theorem 1.4.5] that, for any open subset U of T^*_YX, the formula

$$\int \delta(p - \langle x, \xi\rangle)u(x,y)dx = \sum_{j=-\infty}^{0} a_j(y,\xi)\delta^{(j)}(p) \tag{7.1}$$

establishes a one to one correspondence between holomorphic microfunctions

$$u(x,y) \in \Gamma(U; \mathcal{C}_{Y|X}(0))$$

and sequences of homogeneous holomorphic functions

$$a_j(x,\xi) \in \Gamma(U; \mathcal{O}_{T^*_Y X}(j)) \qquad (j \leq 0)$$

such that for any compact subset $K \subset U$

$$\sum_{j=-\infty}^{0} |a_j(x,\xi)|_K \frac{\epsilon^{-j}}{(-j)!} < +\infty$$

for some $\epsilon > 0$.

Let us first construct the requested DFN topology in two special cases.

Case a. Assume K is a convex compact subset of $T^*_Y X$ on which $\xi_k \neq 0$. Denote by $p : \dot{T}^*_Y X \longrightarrow P^*_Y X$ the canonical projection. The preceding discussion shows that the map

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(p(K) \times \{0\}; \mathcal{O}_{P^*_Y X \times \mathbb{C}}) \\ u(x,y) & \mapsto & f_k(y,\xi,\tau) = \sum_{j=0}^{+\infty} a_{-j}(y, \xi/\xi_k) \frac{\tau^j}{j!} \end{array}$$

is an isomorphism. Using this isomorphism, we endow $\Gamma(K; \mathcal{C}_{Y|X}(0))$ with the usual DFN topology of $\Gamma(p(K) \times \{0\}; \mathcal{O}_{P^*_Y X \times \mathbb{C}})$. If, moreover, $\xi_\ell \neq 0$ on K, one has

$$f_k(y,\xi,\tau) = f_\ell(y,\xi,\tau\xi_k/\xi_\ell).$$

Hence, the DFN topology of $\Gamma(K; \mathcal{C}_{Y|X}(0))$ does not depend on k.

Case b. Let π denote the canonical projection of the bundle $T^*_Y X$ on its base Y identified to the zero section. Assume K is a convex compact subset of $T^*_Y X$ such that $\pi(K) \subset K$. It follows from (7.1) that

$$\begin{array}{rcl} \Gamma(K; \mathcal{C}_{Y|X}(0)) & \longrightarrow & \Gamma(\pi(K); \mathcal{O}_Y) \\ u(x,y) & \mapsto & a_0(y,0) \end{array}$$

is an isomorphism. We use this isomorphism to transport on $\Gamma(K; \mathcal{C}_{Y|X}(0))$ the usual DFN topology of $\Gamma(\pi(K); \mathcal{O}_Y)$.

One checks easily that, if $K_1 \subset K_2$ are two compact subsets of $T^*_Y X$ of the kind treated in case (a) or (b) above, then the restriction map

$$\Gamma(K_2; \mathcal{C}_{Y|X}(0)) \longrightarrow \Gamma(K_1; \mathcal{C}_{Y|X}(0))$$

is continuous.

Let K be an arbitrary compact subset of $T^*_Y X$. The preceding discussion shows that we can find a finite covering $(K_i)_{i\in I}$ of K by compact subsets such that $\Gamma(K_i; \mathcal{C}_{Y|X}(0))$ and $\Gamma(K_i \cap K_j; \mathcal{C}_{Y|X}(0))$ are DFN spaces. Thanks to the exact sequence

$$0 \longrightarrow \Gamma(K; \mathcal{C}_{Y|X}(0)) \xrightarrow{\alpha} \prod_{i\in I} \Gamma(K_i; \mathcal{C}_{Y|X}(0)) \xrightarrow{\beta} \prod_{i,j\in I} \Gamma(K_i \cap K_j; \mathcal{C}_{Y|X}(0)),$$

we may use α to transport on $\Gamma(K; \mathcal{C}_{Y|X}(0))$ the DFN topology of $\ker \beta$. To show that this topology is independent of the chosen covering, it is sufficient to show that it is equivalent to the topology induced by a finer covering. Since such a topology is obviously weaker, the conclusion follows from the closed graph theorem.

Since a direct computation shows that the above defined topology is independent of the chosen coordinate systems, the conclusion follows easily. □

Corollary 7.2 *Let X be a complex analytic manifold. Assume K is a compact subset of T^*X. Then*

$$\Gamma(K; \mathcal{E}_X(0))$$

has a canonical DFN topology.

Proof: Apply the preceding proposition to $\mathcal{C}_{\Delta_X|X\times X}(0)$. □

Proposition 7.3 *Let X, Z be complex analytic manifolds and let Y be a complex submanifold of X. We identify $T^*_{(Z\times Y)}(Z \times X)$ and $Z \times T^*_Y X$. We denote by $q : Z \times T^*_Y X \longrightarrow T^*_Y X$ the second projection. Then, for any Stein compact subset $K \subset Z$, one has*

$$Rq_![(\mathcal{C}_{Z\times Y|Z\times X}(0))_{K\times T^*_Y X}] \simeq \Gamma(K; \mathcal{O}_Z) \mathbin{\hat{\otimes}} \mathcal{C}_{Y|X}.$$

Proof: Let S be a complex manifold. Denote by $p_S : Z \times S \longrightarrow S$ the second projection. By classical results of analytic geometry, we know that

$$Rp_{S!}[(\mathcal{O}_{Z\times S})_{K\times S}] \simeq \Gamma(K; \mathcal{O}_Z) \mathbin{\hat{\otimes}} \mathcal{O}_S.$$

Using the explicit isomorphisms constructed in the proof of the preceding proposition, the conclusion follows easily. □

Corollary 7.4 *Let Z, Y be complex analytic manifolds and denote by*

$$f : Z \times Y \longrightarrow Y$$

the second projection. Assume K is a Stein compact subset of Z. Then

$$Rf_{\pi!}[(\mathcal{E}_{Z\times Y\to Y}(0))_{K\times T^*Y}] \simeq \Gamma(K; \mathcal{O}_Z) \mathbin{\hat{\otimes}} \mathcal{E}_Y(0).$$

Proof: Apply the preceding proposition to $\mathcal{C}_{Z\times\Delta_Y|Z\times(Y\times Y)}(0)$. □

7.2 Direct image and microlocalization

Theorem 7.5 *Assume $f : X \longrightarrow Y$ is a morphism of complex analytic manifolds and $(\mathcal{M}, F)$ is an f-elliptic pair on X with f-proper support. Then the canonical map*

$$[\underline{f}_!(\mathcal{M} \otimes F)]\mathcal{E} \longrightarrow \underline{f}_!([\mathcal{M} \otimes F]\mathcal{E})$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_Y)$.

Proof: Recall that we have the commutative diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{{}^t f'} & X \times_Y T^*Y & \xrightarrow{f_\pi} & T^*Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xleftarrow{\sim} & X & \xrightarrow[f]{} & Y \end{array}$$

Hence, we have successively

$$\begin{aligned} &\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\ &\quad= Rf_{\pi!}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\ &\quad= Rf_{\pi!}[\pi^{-1}(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y] \\ &\quad= Rf_{\pi!}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)]. \end{aligned}$$

Note that there is a canonical map

$$\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y \longrightarrow \mathcal{E}_{X\to Y}. \tag{7.2}$$

Hence, we get a canonical morphism

$$\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \longrightarrow \underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X]. \tag{7.3}$$

When f is a closed embedding, (7.2) is an isomorphism. Hence (7.3) is an isomorphism for any $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ and any $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$.

In the general case, consider the graph embedding

$$i : X \longrightarrow X \times Y$$

and the projection

$$p : X \times Y \longrightarrow Y.$$

Since $(\mathcal{M}, F)$ is an f-elliptic pair, the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ is p-elliptic. Since our result holds for closed embeddings and

$$\underline{i}_!\mathcal{M} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \underline{i}_!(\mathcal{M} \otimes F),$$

we are reduced to prove the theorem for the pair $(\underline{i}_!\mathcal{M}, F \boxtimes \mathbb{C}_Y)$ and the map p.

We may thus assume that f is the second projection from $X = Z \times Y$ to Y and that $F = G \boxtimes \mathbb{C}_Y$ where G is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(Z)$. Moreover, working as in §4, we may also assume that $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{D}_{X|Y}} \mathcal{D}_X$ where $\mathcal{N}$ is a coherent $\mathcal{D}_{X|Y}$-module. In this case,

$$\begin{aligned}
&\pi_Y^{-1}[\underline{f}_!(\mathcal{M} \otimes F)] \otimes_{\pi_Y^{-1}\mathcal{D}_Y} \mathcal{E}_Y \\
&\quad = Rf_{\pi!}[\pi^{-1}(\mathcal{M} \otimes F) \otimes^L_{\pi^{-1}\mathcal{D}_X} (\pi^{-1}\mathcal{D}_{X\to Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{D}_Y} f_\pi^{-1}\mathcal{E}_Y)] \\
&\quad = Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y)]
\end{aligned}$$

and

$$\underline{f}_![\pi_X^{-1}(\mathcal{M} \otimes F) \otimes_{\pi_X^{-1}\mathcal{D}_X} \mathcal{E}_X] = Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}].$$

Hence, we are reduced to show that the canonical arrow

$$\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0) \longrightarrow \mathcal{E}_{X\to Y}(0)$$

induces an isomorphism

$$\begin{aligned}
&Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \\
&\quad \xrightarrow{\sim} Rf_{\pi!}[\pi^{-1}(\mathcal{N} \otimes (G \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)]
\end{aligned} \tag{7.4}$$

As a matter of fact, $\mathcal{E}_{X\to Y} \simeq \mathcal{E}_{X\to Y}(0) \otimes_{f_\pi^{-1}\mathcal{E}_Y(0)} f_\pi^{-1}\mathcal{E}_Y$ as a $(\mathcal{D}_{X|Y},\mathcal{E}_Y)$-bimodule and a scalar extension of (7.4) gives the theorem.

Using the realification process as in §4, we may assume from the beginning that Z is a complexification of a real analytic manifold M and that G is supported by M.

Since the result is local on T^*Y (hence on Y), we may assume also that $\mathcal{N}$ has a projective resolution $\mathcal{L}^{\cdot}$ by finite free $\mathcal{D}_{X|Y}$-modules (see Proposition 3.1).

As for G, we may assume it is isomorphic to a bounded complex $T^{\cdot}$ of the type

$$0 \longrightarrow \cdots \bigoplus_{i_a \in I_a} \mathbb{C}_{K_{a,i_a}} \longrightarrow \cdots \bigoplus_{i_k \in I_k} \mathbb{C}_{K_{k,i_k}} \longrightarrow \cdots \bigoplus_{i_b \in I_b} \mathbb{C}_{K_{b,i_b}} \longrightarrow 0$$

where the sets I_k are finite and K_{k,i_k} is a subanalytic compact subset of M (see Proposition 3.10).

Hence,

$$\mathcal{N} \otimes (F \boxtimes \mathbb{C}_Y) \simeq \mathcal{L}^{\cdot} \otimes (T^{\cdot} \boxtimes \mathbb{C}_Y)$$

and the components of this last complex are finite direct sums of sheaves of the type

$$\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}$$

where K is a subanalytic compact subset of M.

Note that

$$\begin{aligned}
&\pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes^L_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0)) \\
&\quad \xrightarrow{\sim} \pi^{-1}(\mathcal{O}_X)_{K\times Y} \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0)
\end{aligned} \tag{7.5}$$

$$\begin{aligned}
&\pi^{-1}(\mathcal{D}_{X|Y} \otimes \mathbb{C}_{K\times Y}) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0) \\
&\quad \xrightarrow{\sim} (\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}
\end{aligned} \tag{7.6}$$

The right hand side of (7.5) is acyclic for $f_{\pi !}$ thanks to usual properties of Stein compact subsets. Moreover, Corollary 7.4 shows that the right hand side of (7.6) is also acyclic for $f_{\pi !}$. Hence, the morphism (7.4) of $\mathbf{D}^b(\mathcal{E}_Y(0))$ is represented in $C^b(\mathcal{E}_Y(0))$ by the morphism

$$f_{\pi !}[\pi^{-1}(\mathcal{L}^{\cdot} \otimes (T^{\cdot} \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} (\pi^{-1}\mathcal{O}_X \otimes_{f_\pi^{-1}\pi_Y^{-1}\mathcal{O}_Y} f_\pi^{-1}\mathcal{E}_Y(0))] \qquad (7.7)$$
$$\longrightarrow f_{\pi !}[\pi^{-1}(\mathcal{L}^{\cdot} \otimes (T^{\cdot} \boxtimes \mathbb{C}_Y)) \otimes_{\pi^{-1}\mathcal{D}_{X|Y}} \mathcal{E}_{X\to Y}(0)]$$

Let us denote by $R^{\cdot}$ the complex

$$f_![\mathcal{L}^{\cdot} \otimes (T^{\cdot} \otimes \mathbb{C}_Y) \otimes_{\mathcal{D}_{X|Y}} \mathcal{O}_X].$$

Its components are direct sums of sheaves of the type

$$f_![(\mathcal{O}_X)_{K\times Y}] \simeq \Gamma(K;\mathcal{O}_Z) \hat{\otimes} \mathcal{O}_Y$$

which are DFN-free $\mathcal{O}_Y$-modules. It is easy to check that the $\mathcal{O}_Y$-linear differential of $R^{\cdot}$ is continuous with respect to the these natural topologies. Hence, we may consider $R^{\cdot}$ as a topological complex of DFN-free $\mathcal{O}_Y$-modules. Using Corollary 7.4, we have successively

$$\begin{aligned} Rf_{\pi !}[(\mathcal{E}_{X\to Y}(0))_{K\times T^*Y}] &\simeq \Gamma(K;\mathcal{O}_Z) \hat{\otimes} \mathcal{E}_Y(0) \\ &\simeq [\Gamma(K;\mathcal{O}_Z) \hat{\otimes} \pi_Y^{-1}\mathcal{O}_Y] \hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \\ &\simeq \pi_Y^{-1} f_![(\mathcal{O}_X)_{K\times Y}] \hat{\otimes}_{\pi_Y^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \end{aligned}$$

and (7.7) is represented as the canonical morphism

$$\pi^{-1}R^{\cdot} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0) \longrightarrow \pi^{-1}R^{\cdot} \hat{\otimes}_{\pi^{-1}\mathcal{O}_Y} \mathcal{E}_Y(0).$$

Since $R^{\cdot}$ has $\mathcal{O}_Y$-coherent cohomology, Proposition 3.13 allows us to conclude the proof. □

Corollary 7.6 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module endowed with a good filtration. Assume:*

(i) f is proper on $\operatorname{supp}\mathcal{M}$,

(ii) f_π is finite on ${}^tf'^{-1}(\operatorname{char}\mathcal{M}) \cap (X \times_Y \dot{T}^*Y)$, where $\dot{T}^*Y = T^*Y \setminus T^*_YY$.

Then, for $j \neq 0$, $H^j(\underline{f}_!\mathcal{M})$ is a flat connection (i.e. its characteristic variety is contained in the zero section).

Proof: The second hypothesis implies that $\underline{f}_!(\mathcal{M}\mathcal{E})$ is concentrated in degree zero on $\dot{T}^*Y$. The first hypothesis and Theorem 7.5 imply that

$$(\underline{f}_!\mathcal{M})\mathcal{E} \simeq \underline{f}_!(\mathcal{M}\mathcal{E}).$$

Hence, for $j \neq 0$, $\operatorname{supp} H^j[(\underline{f}_!\mathcal{M})\mathcal{E}]$ is contained in the zero section. Since $\mathcal{E}$ is flat over $\pi^{-1}\mathcal{D}$, the conclusion follows easily. □

This Corollary has important applications when studying correspondences of $\mathcal{D}$-modules, such as, for example, the Penrose correspondence. We refer the interested reader to [5] for more details.

8 Main corollaries

8.1 Extension to the non proper case

In this subsection, we shall generalize Theorems 4.2 and 5.15 to a non proper situation, using the techniques of [8, 12].

Let $f : X|S \longrightarrow Y|S$ be a morphism of complex manifolds over S and let $\varphi : X \longrightarrow \mathbb{R}$ be a real analytic function. Set

$$\Lambda_\varphi = \{(x, d\varphi(x)) : x \in X\}.$$

This is a Lagrangian submanifold of T^*X (which is not conic for a non locally constant φ). We also associate to φ the following subsets of X:

$$\begin{aligned} Z_t &= \{x \in X : \varphi(x) \leq t\}, \\ U_t &= \{x \in X : \varphi(x) < t\}, \end{aligned}$$

and denote by $j_t : U_t \longrightarrow X$ the open embedding. Recall finally that the image of a subset S of T^*X by the antipodal map is denoted by S^a.

Corollary 8.1 *Let $\mathcal{M}$ and F be objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{X|S})$ and $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$ respectively and assume:*

i) for each $t \in \mathbb{R}$, f is proper on $\operatorname{supp}\mathcal{M} \cap \operatorname{supp} F \cap Z_t$,

ii) $p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,

iii) there is $t_0 \in \mathbb{R}$ such that

$$\Lambda_\varphi \cap (p^{-1}\mathrm{char}_{f|S}(\mathcal{M}) + SS(F)^a) \subset \pi^{-1}(Z_{t_0}).$$

Then:

a) setting

$$F_t = j_{t!}j_t^{-1}F \simeq F_{U_t},$$

the canonical morphisms:

$$\begin{aligned} \underline{f}_{|S!}(F_t \otimes \mathcal{M}) &\longrightarrow \underline{f}_{|S!}(F \otimes \mathcal{M}) \\ \underline{f}_{|S*}(D'F_t \otimes \underline{D}_{X|S}(\mathcal{M})) &\longleftarrow \underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}\mathcal{M}) \end{aligned}$$

are isomorphisms for $t > t_0$,

b) both

$$\underline{f}_{|S!}(F \otimes \mathcal{M}) \qquad \text{and} \qquad \underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}(\mathcal{M}))$$

are objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}^{\mathrm{op}}_{Y|S})$, *c) the natural duality morphism:*

$$\underline{f}_{|S*}(D'F \otimes \underline{D}_{X|S}(\mathcal{M})) \longrightarrow \underline{D}_{Y|S}\underline{f}_{|S!}(F \otimes \mathcal{M})$$

is an isomorphism.

Note that replacing $SS(F)$ by $SS(F)^a$ in hypothesis (iii), we get a similar conclusion after interchanging $\underline{f}_{|S!}$ and $\underline{f}_{|S*}$. Also note that it would be possible to generalize to a non proper situation the results of §6 but for the sake of brevity, we leave it to the reader.

Proof: Let $x \in X$. If $x \in \operatorname{supp} \mathcal{M} \cap \operatorname{supp} F$ and $x \notin Z_{t_0}$, then $d\varphi(x) \notin \operatorname{char}(\mathcal{M}) + SS(F)^a$ by hypothesis (iii) and in particular $d\varphi(x) \neq 0$. Applying Proposition 5.4.8 of [12], we find for $t > t_0$:

$$SS(F_t) \subset SS(F) + \mathbb{R}^+\Lambda_\varphi$$

where $\mathbb{R}^+\Lambda_\varphi = \{(x; \lambda d\varphi(x)) : x \in X, \lambda > 0\}$. Since:

$$p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) \cap (SS(F) + \mathbb{R}^+\Lambda_\varphi) \subset T^*_X X \cup \pi^{-1}(Z_{t_0}),$$

again by hypothesis (iii), we obtain that $(\mathcal{M}, F_t)$ satisfies the hypothesis of Theorems 4.2 and 5.15 for $t > t_0$. Hence, the conclusions of these theorems apply to the pair $(\mathcal{M}, F_t)$ and part (b) and (c) are consequences of part (a) which we shall now prove.

First, we consider the morphism

$$\underline{f}_{|S!}(F_t \otimes \mathcal{M}) \longrightarrow \underline{f}_{|S!}(F \otimes \mathcal{M}). \tag{8.1}$$

Set $G = F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}$. By Theorem 2.15, hypothesis (ii) and Proposition 5.4.14 of [12] we have:

$$SS(G) \subset SS(F) + p^{-1}\operatorname{char}_{f|S}(\mathcal{M}).$$

Since

$$p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) = p^{-1}\operatorname{char}_{f|S}(\mathcal{M}) + {}^t f'(X \times_Y T^*Y),$$

the above morphism (8.1) is an isomorphism by Proposition 5.4.17 of [12].

To prove the second isomorphism in (a), consider the chain of isomorphisms which follows from the regularity theorem applied first to F_t, then to F:

$$\begin{aligned}
&Rf_*(D'F_t \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\quad\simeq Rf_*R\mathcal{H}om(F_t, \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\quad\simeq Rf_*Rj_{t*}j_t^{-1}R\mathcal{H}om(F, \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}) \\
&\quad\simeq Rf_*Rj_{t*}j_t^{-1}(D'F \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}).
\end{aligned}$$

Set

$$G = D'F \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S}.$$

Isomorphism (8.1) applied to $(D'F_t,\ \underline{D}_{X|S}\mathcal{M})$ tells us in particular that the projective system

$$Rf_*(D'F_t \otimes \underline{D}_{X|S}\mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{D}_{X|S\to Y|S})$$

is essentially constant for $t > t_0$. Hence, the projective system $Rf_*Rj_{t*}j_t^{-1}G$ is also essentially constant for $t > t_0$ and using the Mittag-Leffler theorem we get the isomorphism

$$Rf_*G \xrightarrow{\sim} Rf_*Rj_{t*}j_t^{-1}G$$

which completes the proof. □

8.2 Special cases and examples

In this subsection, we will consider various special situations and give the corresponding form of Theorem 4.2 and 5.15 leaving the reader do the same thing for Theorem 6.7.

First, let us specialize our results to the non relative case taking $S = \{\mathrm{pt}\}$.

Corollary 8.2 *Let $f : X \longrightarrow Y$ be a morphism of complex analytic manifolds. Assume $(\mathcal{M},F)$ is a good f-elliptic pair with f-proper support i.e.:*

- *$\mathcal{M}$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$,*
- *F is an object of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,*
- *$\mathrm{char}_f(\mathcal{M}) \cap SS(F) \subset T^*_XX$,*
- *$\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is f-proper.*

Then

- *$\underline{f}_!(\mathcal{M} \otimes F)$ is an object of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Y^{\mathrm{op}})$,*
- *$\underline{f}_![\underline{D}_X(\mathcal{M}) \otimes D'F] \xrightarrow{\sim} \underline{D}_Y[\underline{f}_!(\mathcal{M} \otimes F)]$.*

When we take $F = \mathbb{C}_X$ in the preceding corollary we recover the coherence theorem for $\mathcal{D}$-modules of Kashiwara [9] (who treated only projective morphisms). Moreover, using Corollary 8.1, we also recover the finiteness theorem for non proper morphisms of [8] and the corresponding duality result of [24].

It is well known that an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if and only if the induced $\mathcal{D}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is itself coherent. Moreover, this scalar extension process is compatible with direct images and duality (see Proposition 5.11). Applying the preceding corollary to the pair $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathbb{C}_X)$ we recover Grauert's coherence theorem [6] and Ramis-Ruget-Verdier's relative duality theorem [15, 16] in the important special case of analytic manifolds.

Taking $Y = \{\mathrm{pt}\}$ in the preceding corollary, we get the following absolute result:

Corollary 8.3 *Let X be a complex analytic manifold. Assume $(\mathcal{M},F)$ is a good elliptic pair with compact support i.e.:*

- $\mathcal{M}$ *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X^{\mathrm{op}})$,
- F *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,
- $\mathrm{char}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,
- $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is compact.*

Then the complexes

$$\mathrm{R}\Gamma(X; \mathcal{M} \otimes F \otimes^L_{\mathcal{D}_X} \mathcal{O}_X) \quad \text{and} \quad \mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \Omega_X[d_X]))$$

have finite dimensional cohomology and are dual one to each other.

In the special case where $F = \mathbb{C}_X$, we get an absolute finiteness and duality result for good $\mathcal{D}_X$-modules which was considered by Mebkhout in [14]. For coherent analytic sheaves, the preceding corollary corresponds to the very classical Cartan-Serre [4] and Serre [26]'s theorems.

In the case $Y = S$, Theorem 4.2 and 5.15 give information on analytic families of absolute elliptic pairs.

Corollary 8.4 *Let $X|S$ be a relative analytic manifold and let $(\mathcal{M}, F)$ be a relative elliptic pair on $X|S$ i.e.:*

- $\mathcal{M}$ *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_{X|S}^{\mathrm{op}})$,
- F *is an object of* $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X)$,
- $p^{-1}\mathrm{char}_{X|S}(\mathcal{M}) \cap SS(F) \subset T^*_X X$,
- $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is ϵ_X-proper,*

where $p : T^*X \longrightarrow T^*X|S$ is the canonical projection. Then

$$Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X) \quad \text{and} \quad Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_{X|S}])$$

are objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$, dual one to each other, i.e. the canonical morphism:

$$Rf_! R\mathcal{H}om_{\mathcal{D}_{X|S}}(F \otimes \mathcal{M}, \Omega_{X|S}[d_{X|S}]) \longrightarrow R\mathcal{H}om_{\mathcal{O}_S}(Rf_!(F \otimes \mathcal{M} \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X), \mathcal{O}_S)$$

is an isomorphism.

Combining the preceding corollary with the base change formula, we get:

Corollary 8.5 *Let $X|S$ be a relative analytic manifold. For any $s \in S$, denote by b_s the canonical inclusion of $\{s\}$ in S. Assume $(\mathcal{M},F)$ is a good relative elliptic pair with ϵ-proper support on $X|S$. Then for any $s \in S$, $(\mathcal{M}_{b_s},F_{b_s})$ is a good elliptic pair with compact support (on a neighborhood of $\mathrm{supp}\,\mathcal{M}_{b_s}$ in X_{b_s}, the fiber of X over s) and the Euler-Poincaré index:*

$$\chi(\mathrm{R}\Gamma(X_{b_s}; \mathcal{M}_{b_s} \otimes F_{b_s} \otimes^L_{\mathcal{D}_{X_{b_s}}} \mathcal{O}_{X_{b_s}}))$$

is a locally constant function on S.

Proof: Let us denote by $\mathcal{I}$ the ideal of holomorphic functions vanishing at s. The base change Theorem 6.5 tells us that:

$$\mathrm{R}\Gamma(X_{b_s}; \mathcal{M}_{b_s} \otimes F_{b_s} \otimes^L_{\mathcal{D}_{X_{b_s}}} \mathcal{O}_{X_{b_s}}) = [\mathcal{O}_S/\mathcal{I} \otimes^L_{\mathcal{O}_S} R\epsilon_*(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X)]_s.$$

We know by the finiteness theorem that

$$R\epsilon_*(\mathcal{M} \otimes F \otimes^L_{\mathcal{D}_{X|S}} \mathcal{O}_X)$$

has $\mathcal{O}_S$ coherent cohomology. Hence, it is locally quasi-isomorphic to a bounded complex of finite free $\mathcal{O}_S$-modules. The conclusion follows easily since $[\mathcal{O}_S/\mathcal{I}]_s = \mathbb{C}$. □

Remark 8.6 Let $P_0 : E \longrightarrow F$, $P_1 : E \longrightarrow F$ be two complex analytic linear differential operators between holomorphic vector bundles on X. Assume that their principal symbols induce the same morphism of fiber bundles

$$\sigma : \pi^{-1}E \longrightarrow \pi^{-1}F.$$

Then, $P_\lambda = (1-\lambda)P_0 + \lambda P_1$ is a one parameter analytic family of operators with principal symbols equal to σ. Combining this remark with the preceding corollary, we recover, for example, the fact that the index of an elliptic operator on a compact real analytic manifold depends only on its principal symbol.

Let us now consider a few explicit examples. For the sake of brevity, we only consider non-relative situations.

Example 8.7 Let M be a real analytic manifold with X as a complexification and $\mathcal{M}$ a good $\mathcal{D}_X$-module. Then, as we have already noticed in the introduction, $\mathcal{M}$ is elliptic on M in the classical sense if and only if $(\mathcal{M}, \mathbb{C}_M)$ is elliptic. In fact, $SS(\mathbb{C}_M) = T^*_M X$. Since $\mathbb{C}_M \otimes \mathcal{O}_X = \mathcal{A}_M$, the sheaf of real analytic functions on M and

$$R\mathcal{H}om(D'\mathbb{C}_M, \mathcal{O}_X) = \mathcal{B}_M$$

the sheaf of Sato's hyperfunctions, the regularity theorem 2.15 entails the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M). \tag{8.2}$$

This is the Petrowski theorem for $\mathcal{D}$-modules which is often proved using micro-differential equations as in [18]. Moreover, if M is compact and $\mathcal{M}$ is good, Corollary 8.3 asserts that the spaces

$$H^j(\mathrm{R}\Gamma(M; R\mathcal{H}om_{\mathcal{D}_M}(\mathcal{M}, \mathcal{B}_M))) = \mathrm{Ext}^j_{\mathcal{D}_M}(M; \mathcal{M}, \mathcal{B}_M)$$

and

$$H^{n-j}(\mathrm{R}\Gamma(M; \Omega_M \otimes^L_{\mathcal{D}_M} \mathcal{M})) = \mathrm{Tor}^{\mathcal{D}_M}_{j-n}(M; \Omega_M, \mathcal{M}),$$

are finite dimensional and dual to each other. Note that for solutions of elliptic operators the duality and finiteness theorems are well-known results.

Example 8.8 Let X be a complex manifold, U an open subset with real analytic boundary. Then $(\mathcal{M}, \mathbb{C}_U)$ is an elliptic pair if and only if the boundary ∂U is non characteristic for $\mathcal{M}$, that is, $\mathrm{char}(\mathcal{M}) \cap T^*_{\partial U}X \subset T^*_X X$. The regularity theorem yields the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (\mathcal{O}_X)_{\overline{U}}) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_U(\mathcal{O}_X)). \tag{8.3}$$

In other words, the holomorphic solutions on U of the system $\mathcal{M}$ extend holomorphically through the boundary. If U is relatively compact, and $\mathcal{M}$ is good, we get that the spaces $\mathrm{Ext}^j_{\mathcal{D}_X}(U, \mathcal{M}, \mathcal{O}_X)$ and $\mathrm{Tor}^{\mathcal{D}_X}_{j-n}(\overline{U}; \Omega_X, \mathcal{M})$ are finite dimensional and dual to each other.

Note that the regularity theorem is due to Zerner [27] (for the 0-th cohomology) and [2], both in case of one equation with one unknown, then to Kashiwara [10] for systems. The finiteness theorem is due to [2], this last result being extended in various directions by Kawai [13].

Example 8.9 One can generalize both preceding examples as follows. Let M be a real analytic manifold, X being a complexification of M and let U be an open subset of M with real analytic boundary. Then $(\mathcal{M}, \mathbb{C}_U)$ is an elliptic pair if and only if $\mathcal{M}$ is elliptic on M on a neighborhood of $\overline{U}$ and moreover the conormal vectors to ∂U in M are hyperbolic with respect to $\mathcal{M}$. Then we get the isomorphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, (\mathcal{A}_M)_{\overline{U}}) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_U(\mathcal{B}_M)).$$

(i.e.: the hyperfunction solutions of $\mathcal{M}$ on U are real analytic and extend analytically through the boundary), and we also get finiteness and duality results that we do not develop here.

Example 8.10 A general situation including the preceding examples is the following.

Let $X = \bigsqcup_\alpha X_\alpha$ be a subanalytic μ-stratification (cf. [12, Chap. VIII]) and assume:

$$\begin{cases} SS(F) & \subset & \bigsqcup_\alpha T^*_{X_\alpha}X, \\ \mathrm{char}(\mathcal{M}) \cap T^*_{X_\alpha}X & \subset & T^*_X X \quad \forall \alpha. \end{cases} \tag{8.4}$$

(In other words, F is locally constant on the strata X_α and these strata are non characteristic for $\mathcal{M}$.)

Then of course, the pair $(\mathcal{M}, F)$ is elliptic. If, moreover, $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is compact we may apply Theorem 4.2 and Theorem 5.15 and we obtain new finiteness and duality results.

Example 8.11 For any $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X))$, the pair $(\mathcal{O}_X, F)$ is elliptic. Since $F \simeq \Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{O}_X \otimes F[-n]$ and $D'F \simeq R\mathcal{H}om_{\mathcal{D}_X}(F \otimes \mathcal{O}_X, \mathcal{O}_X)$, one recovers the classical finiteness and duality theorem on constructible sheaves. In fact if M is a real analytic manifold and $i : M \hookrightarrow X$ denote a complexification of M, to $G \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(M))$ one associates the elliptic pair $(\mathcal{O}_X, i_*G)$.

Example 8.12 Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and let $x_0 \in X$. Let $B(x_0, \varepsilon)$ denote the open ball with center x_0 and radius $\varepsilon > 0$ in some local chart at x_0. By a result of Kashiwara [10], the pair $(\mathcal{M}, \mathbb{C}_{B(x_0,\varepsilon)})$ is elliptic for $0 < \varepsilon \ll 1$. If X is open in $\mathbb{C}^n$ and $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(X))$ has compact support, one proves similarly that $(\mathcal{M}, F * \mathbb{C}_{B(0,\varepsilon)})$ is elliptic for $0 < \varepsilon \ll 1$. (Here "$*$" denotes the convolution of sheaves; cf. [12] Exercise 2.20.)

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