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Exposé VI : Semi-stable reduction and p-adic etale cohomology

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Exposé VI

SEMI–STABLE REDUCTION AND P–ADIC ETALE COHOMOLOGY

by **Kazuya KATO**

1. — Introduction

This paper is a result of joint study with J.–M. Fontaine. I learned from him the main ideas in the study in this paper.

Let A be a complete discrete valuation ring with field of fractions K and with residue field k, such that $char(K) = 0$, $char(k) = p > 0$ and k is perfect. Let X be a proper scheme over A with semi–stable reduction (that is, X is regular and $X \otimes_A k$ is a reduced divisor with normal crossings on X). The purpose of this paper is to give a partial solution to a conjecture of Fontaine–Jannsen on the p–adic etale cohomology

$$H^m_{et}(X_{\overline{K}}, \mathbf{Q}_p)$$

($X_{\overline{K}} = X \otimes_A \overline{K}$ with $\overline{K}$ an algebraic closure of K).

We recall the conjecture (cf. Jannsen [J] § 5 for the first form of the conjecture; the final precise form of the conjecture introduced here was formulated in Fontaine [Fo3]). Let K_0 be the field of fractions of the ring $W(k)$ of Witt vectors. Let D be the "m–th crystalline cohomology with logarithmic poles in the semi–stable situation" defined in [H2] [HK], which is a K_0–vector space endowed with a frobenius $\varphi : D \to D$, a monodromy operator $\mathcal{N} : D \to D$, and an isomorphism with the de Rham cohomology

$$\rho_\pi : K \otimes_{K_0} D \xrightarrow{\sim} H^m_{DR}(X_K/K)$$

(ρ_π is defined canonically once one fixes a prime element π of A). The conjecture says that the $\mathbf{Q}_p$–vector space $V = H^m_{et}(X_{\overline{K}}, \mathbf{Q}_p)$ endowed with the action of $\mathrm{Gal}(\overline{K}/K)$, and the K_0–vector space D endowed with φ, $\mathcal{N}$, and ρ_π, can be reconstructed from each other. To state the precise form of the conjecture, one needs a ring B_{st} of Fontaine ([Fo3]), which is defined also by fixing a prime element of A. Cf. (2.2) for the review of the definition of B_{st}. It is related to his older rings B_{crys} and B_{DR} (cf. [Fo1]; we review also these rings in (2.2)); B_{st} is a subring of B_{DR} containing B_{crys}. Properties of B_{st} used in the statement of the conjecture are that B_{st} is endowed with the frobenius $\varphi : B_{st} \to B_{st}$, a monodromy operator $\mathcal{N} : B_{st} \to B_{st}$, and a natural action of $\mathrm{Gal}(\overline{K}/K)$. We also have to recall that B_{DR} is a complete discrete valuation field with valuation ring B^+_{dR} and hence filtered by the valuation.

The conjecture is the following.

CONJECTURE (1.1). — *There exists a canonical isomorphism*

$$B_{st} \otimes_{\mathbf{Q}_p} V \cong B_{st} \otimes_{K_0} D$$

with preserves φ, $\mathcal{N}$, the action of $\mathrm{Gal}(\overline{K}/K)$, *and the filtration after taking* $B_{DR} \otimes_{B_{st}}$.

Here φ on the left (resp. right) hand side in (1.1) is $\varphi \otimes 1$ (resp. $\varphi \otimes \varphi$), $\mathcal{N}$ on the left (resp. right) hand side is $\mathcal{N} \otimes 1$ (resp. $N \otimes 1 + 1 \otimes \mathcal{N}$), the action of $\sigma \in \mathrm{Gal}(\overline{K}/K)$ on the left (resp. right) hand side is $\sigma \otimes \sigma$ (resp. $\sigma \otimes 1$), the filtration on $B_{DR} \otimes_{\mathbf{Q}_p} V$ is $fil^\cdot B_{DR} \otimes_{\mathbf{Q}_p} V$, and the filtration on

$$B_{DR} \otimes_{K_0} D = B_{DR} \otimes_K H^m_{DR}(X_K/K) \quad (via\ \rho_\pi),$$

where we use the same prime element π in the definitions of B_{st} and ρ_π, is the tensor product of the filtrations on B_{DR} and the Hodge filtration on $H^m_{DR}(X_K/K)$. Since the $\mathrm{Gal}(\overline{K}/K)$–invariant part of B_{st} is K_0 and $\{x \in B_{st};\ \varphi(x) = x,\ \mathcal{N}(x) = 0,\ x \in B^+_{DR}\} = \mathbf{Q}_p$ ([Fo3]), one will have as a consequence of the conjecture,

$$D \cong \{x \in B_{st} \otimes_{\mathbf{Q}_p} V; \sigma(x) = x \quad for\ all\ \sigma \in \mathrm{Gal}(\overline{K}/K)\}$$

$$V \cong \{x \in B_{st} \otimes_{K_0} D; \varphi(x) = x\,,\ \mathcal{N}(x) = 0\,,\ x \in fil^0(B_{DR} \otimes_{K_0} D)\}\,.$$

This conjecture is the "semi–stable reduction version" of the crystalline conjecture of Fontaine [Fo1] on $H^m_{et}(X_{\overline{K}}, \mathbf{Q}_p)$ for the case X is smooth over A, which was studied and proved by Fontaine, Messing and Faltings ([FM], [Fa2]). A new phenomenon in the semi–stable reduction case is that the monodromy operator is involved. In [Fo3], Conjecture (1.1) was proved in the case of abelian varieties.

Our result is the following

THEOREM (1.2). — *The conjecture* (1.1) *is true if* $p > 2\dim(X_K) + 1$.

In the "ordinary semi–stable reduction" case, $H^m_{et}(X_{\overline{K}}, \mathbf{Q}_p)$ was studied by Hyodo [H1] without any assumption on p (cf. also [P], Appendix by L. Illusie).

In § 2 we give preliminaries. In § 3 and § 4, we give interpretations of the ring B_{st} and the space $\{x \in B_{st} \otimes D; \mathcal{N} = 0\}$ by the theory of crystalline cohomology with logarithmic poles, respectively. In § 5, we state a result, whose proof will be given elsewhere, on the relation between p–adic vanishing cycles and a certain complex $s_n^{\log}(r)$ related to crystalline cohomology with log poles (in the good reduction case, this result is proved in Kurihara [Ku]). By combining § 3 – § 5, we prove Thm. (1.2) in § 6.

I thank Professor J.–M. Fontaine without whose help, I could do nothing about the subject of this paper. I also thank Professors L. Illusie and W. Messing for advice. The method in § 6 is a modification of the method in the joint paper [KM] of W. Messing and the author which treated the good reduction case.

2. — Preliminaries

In this § 2, we fix notations, review the definitions of the rings B_{crys}, B_{DR} and B_{st}, and give some comments on the crystalline cohomology with logarithmic poles.

(2.1). — We use the following notations. Let A, K, k and $\overline{K}$ be as in § 1. Let $\overline{A}$ be the integral closure of A in $\overline{K}$, and let

$$A_n = A \otimes_{\mathbf{Z}} \mathbf{Z}/p^n\mathbf{Z}\,, \quad \overline{A}_n = \overline{A} \otimes_{\mathbf{Z}} \mathbf{Z}/p^n\mathbf{Z}\,,$$

$$S = \mathrm{Spec}(A)\,, \quad S_n = \mathrm{Spec}(A_n)\,, \quad \overline{S} = \mathrm{Spec}(\overline{A})\,, \quad \overline{S}_n = \mathrm{Spec}(\overline{A}_n)\,.$$

(2.2). — We review the definitions of the rings B_{crys}, B_{DR} and B_{st} (cf. [Fo1], [Fo3], [Fo4], [FM]). For $n \geq 1$, let

$$B_n = \Gamma((\overline{S}_n/W_n)_{crys}, \mathcal{O}_{\overline{S}_n/W_n})$$

i.e. the ring of global sections of the structure sheaf of the crystalline site of $\overline{S}_n$ over $W_n = \mathrm{Spec}(W_n(k))$. Then the canonical homomorphism $B_n \to \overline{A}_n$ is surjective. We denote by J_n the kernel of this surjection, and by $J_n^{[r]}$ the r–th divided power of J_n. For a sequence $s = (s_n)_{n \geq 0}$ of elements of $\overline{A}$ such that $(s_{n+1})^p = s_n$ for all $n \geq 0$, let $\varepsilon(s) = ((\widetilde{s}_n)^{p^n})_n \in \varprojlim_n B_n$ where for $a \in \overline{A}$, we denoted by $\widetilde{a}$ any element of B_n whose image in $\overline{A}_n$ coincides with that of a (then $\widetilde{a}^{p^n}$ depends only on a, and is independent of the choice of $\widetilde{a}$). This map ε defines an injective homomorphism

$$\log \varepsilon : \mathbb{Z}_p(1) \longrightarrow \varprojlim_n B_n .$$

Let

$$B^+_{crys} = \mathbb{Q} \otimes \varprojlim_n B_n , \quad B_{crys} = B^+_{crys}[t^{-1}]$$

$$B^+_{DR} = \varprojlim_r (\mathbb{Q} \otimes \varprojlim_n B_n/J_n^{[r]}) , \quad B_{DR} = B^+_{DR}[t^{-1}]$$

where t is any non–zero element in the image of $\mathbb{Q} \otimes \varepsilon : \mathbb{Q}_p(1) \to B^+_{crys}$. Then B_{crys} has a frobenius endomorphism by the crystalline cohomology theory, B_{DR} has a filtration by the fact that it is a complete discrete valuation field with valuation ring B^+_{DR} (the residue field is $\mathbb{C}_p = \mathbb{Q} \otimes \varprojlim_n \overline{A}_n$, and the t as above is a prime element), and B_{crys} and B_{DR} are endowed with natural actions of $\mathrm{Gal}(\overline{K}/K)$.

Now fix a prime element π of A.

For a sequence $s = (s_n)_{n \geq 0}$ of elements of $\overline{A}$ such that

$$s_0 = \pi , \quad (s_{n+1})^p = s_n \quad \text{for} \quad n \geq 0 ,$$

we have $\varepsilon(s)\pi^{-1} \in \mathrm{Ker}((B_{DR}^{+})^{\times} \to \mathbf{C}_p^{\times})$ and hence

$$u_s = \log(\varepsilon(s)\pi^{-1}) \in B_{DR}^{+}$$

is defined. Fontaine defines B_{st}^{+} and B_{st} by

$$B_{st}^{+} = B_{crys}^{+}[u_s]\,, \quad B_{st} = B_{crys}[u_s]$$

as subrings of B_{DR}. It was shown by Fontaine [Fo3] that u_s is transcendental over B_{crys} and hence B_{st}^{+} (resp. B_{st}) is a polynomial ring in one variable over B_{crys}^{+} (resp. B_{crys}). The rings B_{st}^{+} and B_{st} depend on the prime element π, but do not depend on the choice of s. The frobenius $\varphi : B_{st} \to B_{st}$ is defined by extending the φ of B_{crys} by $\varphi(u_s) = pu_s$, and the monodromy operator $\mathcal{N} : B_{st} \to B_{st}$ is defined to be the unique B_{crys}-derivation such that $\mathcal{N}(u_s) = 1$. These operators φ and $\mathcal{N}$ are also independent of the choice of s. Finally B_{st} is $\mathrm{Gal}(\overline{K}/K)$-stable in B_{DR} and hence $\mathrm{Gal}(\overline{K}/K)$ acts on B_{st}.

(2.3). — In this paper we use freely the terminologies concerning log structures introduced in [HK] and [Ka2], without explaining the definitions of them. We just mention here that a logarithmic structure on a scheme X in the sense of Fontaine–Illusie is, by definition, a sheaf of commutative monoids with a unit on the etale site X_{et} which is endowed with a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$ satisfying $\alpha^{-1}(\mathcal{O}_X^{\times}) \xrightarrow{\sim} \mathcal{O}_X^{\times}$ via α (cf. [Fa3] for another formulation of logarithmic structures).

We make the following conventions.

(2.3.1) A scheme X with a log structure M is denoted as (X, M). If M is the trivial log structure (that is, $M = \mathcal{O}_X^{\times}$ with the inclusion map $\mathcal{O}_X^{\times} \to \mathcal{O}_X$), we abbreviate (X, M) as X.

(2.3.2) If X is a scheme and M is a log structure on X, and if the inverse image of $a \in \Gamma(X, \mathcal{O}_X)$ in $\Gamma(X, M)$ consists of a single element b, we sometimes identify b with a.

(2.3.3). — For a scheme X and $a \in \Gamma(X, \mathcal{O}_X)$, we denote by $\mathcal{L}(a)$ the fine log structure on X associated ([HK] (2.2)) to $\mathbf{N} \to \mathcal{O}_X\,;\, 1 \longmapsto a$. For example, the "canonical log structure" on $\mathrm{Spec}(A)$ defined in (2.6) below coincides with $\mathcal{L}(\pi)$ for any prime element π of A.

(2.4). — In the paper [HK], we discussed the crystalline cohomology theory only for schemes with fine log structures. But the log structure on $\mathrm{Spec}(\overline{A})$ which we discuss in this paper is not fine though it is a filtered inductive limit of fine log structures. We say a log structure M on a scheme X is *integral* if "$ac = bc$ implies $a = b$" holds in M (a fine log structure is integral, and the log structure on $\mathrm{Spec}(\overline{A})$ discussed later is integral). By replacing the category of schemes with fine log structures in [HK] by the category of schemes with integral log structures, we obtain the definition of the crystalline sites for schemes with integral log structures. For schemes with fine log structures, this does not change their crystalline sites.

With this definition of the crystalline site :

(2.4.1) Let (T, L) be a scheme with a fine log structure such that $\mathcal{O}_T$ is killed by some non-zero integer, and assume T is endowed with a PD (= divided power) ideal. For a scheme with an integral log structure (X, M) over (T, L), we denote by

$$H^q((X,M)/(T,L))$$

the q-th cohomology of the structure sheaf $\mathcal{O}_{X/T}$ on the crystalline site $((X,M)/(T,L))_{crys}$.

(2.4.2). — With (T, L) as in (2.4.1), let $f : (X, M) \to (Y, N)$ be a morphism of schemes with integral log structures over (T, L). Then, if N is fine, a morphism between the crystalline topoi

$$f_{crys} : ((X,M)/(T,L))^{\sim}_{crys} \longrightarrow ((Y,N)/(T,L))^{\sim}_{crys}$$

is associated to f. The construction of f_{crys} is the same as in the fine case.

(2.4.3). — Let (T, L) be as in (2.4.1) and let $\{(X_\lambda, M_\lambda)\}_\lambda$ be a filtered inductive system of schemes with fine log structures over (T, L) such that all transition morphism $X_\lambda \to X_\mu$ are affine. Let X be the projective limit of $\{X_\lambda\}_\lambda$ and let M be the inductive limit on X of the inverse images of M_λ. Then,

$$H^q((X,M)/(T,L)) \cong \varinjlim H^q((X_\lambda, M_\lambda)/(T,L)) .$$

If (Y, N) is a scheme with a fine log structure over (T, L) and the $(X_\lambda, M_\lambda) \to (T, L)$ factor through morphisms $f_\lambda : (X_\lambda, M_\lambda) \to (Y, N)$ which are compatible with the transition morphisms, we have

$$R^q f_{crys*}(\mathcal{O}_{X/S}) \cong \varinjlim R^q (f_\lambda)_{crys*}(\mathcal{O}_{X_\lambda/S})$$

where f is the limit of f_λ. These facts follow from [SGA4] (tome 2) Exposé VI.

(2.5). — In [FM], a morphism of schemes which is flat and locally of complete intersection is called syntomic and syntomic morphisms behave well in their theory. We shall use the logarithmic version of this notion.

We say a morphism $f : (X, M) \to (Y, N)$ of schemes with fine log structures is *syntomic* if f is an integral morphism [HK] (2.10), the underlying morphism $X \to Y$ is flat and locally of finite presentation, and if etale locally on X there is a factorization of $f : (X, M) \xrightarrow{i} (Z, L) \xrightarrow{h} (Y, N)$ with (Z, L) a scheme with a fine log structure satisfying the following conditions : i is an exact closed immersion [HK] (2.8), h is smooth [HK] (2.9), and the ideal of X in Z is generated at each point of X by a regular sequence.

Just as in the case of the original definition, we can show

(2.5.1). — If f is syntomic and we have another factorization $(X, M) \xrightarrow{i'} (Z', L') \xrightarrow{h'} (Y, N)$ of f with L' fine, i' an exact closed immersion and h' smooth, then the ideal of X in Z' is defined at each point of X by a regular sequence. Furthermore, if we have such a factorization of a syntomic morphism and if we are given a quasi-coherent ideal I of $\mathcal{O}_Y$ and a divided power structure on I, the divided power envelope of X in Z' is flat over Y (cf. [FM] for the case without log structures).

(2.5.2). — We have the base change theorem of crystalline cohomology for syntomic morphisms (cf. [BBM] (2.3.5) and [B] V 3.5.1 for the case without log structures) :

Assume we have a commutative diagram of shemes with fine log structures

$$\begin{tikzcd}
(X', M') \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & (X, M) \arrow[d, "f"] \\
(Y', N') \arrow[r, "g"] \arrow[d] & (Y, N) \arrow[d] \\
(T', L') \arrow[r, "v"] & (T, L)
\end{tikzcd}$$

such that the upper square is cartesian, f is syntomic, the underlying morphism $f : X \to Y$ is quasi-compact and quasi-separated, and $\mathcal{O}_T$ is annihilated by some non-zero integer. Assume T and T' are endowed with quasi-coherent PD-ideals and $v : T' \to T$ is a PD-morphism. Then, for any quasi-coherent flat crystal of $\mathcal{O}_{X/T}$-modules on $((X, M)/(T, L))_{crys}$, we have a canonical isomorphism

$$Lg^*_{crys} Rf_{crys*}(\mathcal{F}) \cong Rf'_{crys*} g'^*_{crys}(\mathcal{F})\,.$$

(2.6). — For a scheme over the discrete valuation ring A, we define the *canonical log structure* as the sheaf of regular functions which are invertible on the generic fiber.

In what follows, we denote the canonical log structure on $S = \mathrm{Spec}(A)$ (resp. $\overline{S} = \mathrm{Spec}(\overline{A})$) by N (resp. $\overline{N}$) and the inverse image of N on $S_n = \mathrm{Spec}(A_n)$ by N_n (resp. of $\overline{N}$ on $\overline{S}_n = \mathrm{Spec}(\overline{A}_n)$ by $\overline{N}_n$). Then $\overline{N}$ is the inductive limit of inverse images of the canonical log structures on $\mathrm{Spec}(A')$, where A' ranges over all discrete valuation rings in $\overline{A}$ which are finite over A, and $\overline{N}_n$ is the inductive limit of the inverse images of the log structures on $\mathrm{Spec}(A'/p^n A')$ defined in the way above.

We shall denote the inverse image of N on $\mathrm{Spec}(k)$ by L.

For $a \in A - \{0\}$, the images of a in any of the log structures introduced here are denoted by class(a).

3. — A crystalline interpretation of B_{st}

We give an interpretation (3.7) of the ring B_{st} by the theory of crystals with logarithmic poles. Let

$$h : (\overline{S}_n, \overline{N}_n) \longrightarrow (S_n, N_n)$$

be the canonical morphism (cf. (2.6) for the notation), and let h_{crys} : $((\overline{S}_n, \overline{N}_n)/W_n)^{\sim}_{crys} \to ((S_n, N_n)/W_n)^{\sim}_{crys}$ be the induced morphism (2.4.2).

In this section we compute the higher direct images of the structure sheaf for this morphism

$$R^q h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})$$

and show that $h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})$ is closely related to B_{st}.

PROPOSITION (3.1). — *$R^q h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n}) = 0$ for $q \neq 0$ and $h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})$ is a quasi–coherent flat crystal of $\mathcal{O}_{S_n/W_n}$–modules on $((S_n, N_n)/W_n)_{crys}$.*

(3.2). — We give a description (3.3) of the crystal $h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})$. To describe a crystal by a connection just as in the usual theory (without log structures) of crystals, we embed (S_n, N_n) in a smooth object. Fix a prime element π of A, let $Z_n = \mathrm{Spec}(W_n[t])$ with t an indeterminate, let $E_n = \mathrm{Spec}(R_n)$ be the PD–envelope of S_n in Z_n with respect to the closed immersion $S_n \to Z_n\,;\, t \longmapsto \pi$, and endow Z_n (resp. E_n) with the log structure $\mathcal{L}(t)$ (cf. (2.3.3)). Since $(Z_n, \mathcal{L}(t))$ is smooth ([HK] (2.9)) over W_n, a quasi–coherent crystal of $\mathcal{O}_{S_n/W_n}$–modules $\mathcal{F}$ on $((S_n, N_n)/W_n)_{crys}$ is characterized by the R_n–module $\mathcal{F}(E_n)$ and the connection with log poles

$$\nabla : \mathcal{F}(E_n) \longrightarrow \mathcal{F}(E_n)\mathrm{d}\log(t)$$

([HK] (2.17)). Here $\mathcal{F}(E_n)\mathrm{d}\log(t)$ means $\mathcal{F}(E_n) \otimes_{W_n[t]} \Gamma(Z_n, \omega^1_{Z_n/W_n})$ with ω^1 the differential module with log poles [HK] (2.5) (then $\Gamma(Z_n, \omega^1_{Z_n/W_n})$ is a free $W_n[t]$–module of rank one with base $\mathrm{d}\log(t)$). Let

$$P_n = \mathcal{F}(E_n) \quad \text{with } \mathcal{F} = h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})\,. \tag{3.2.1}$$

Note the homomorphism $B_n \to \overline{A}_n$ factors canonically as $B_n \to P_n \to \overline{A}_n$ and the kernel of $P_n \to \overline{A}_n$ has a natural PD–structure.

PROPOSITION (3.3). — (1) *To each p^n-th root β of π in $\overline{A}$, there exists a canonically defined element v_β of* $\mathrm{Ker}(P_n^\times \to \overline{A}_n^\times)$ *such that we have a PD-isomorphism*

$$B_n < V > \xrightarrow{\sim} P_n\,; \quad V \longmapsto v_\beta - 1$$

where $B_n < V >$ denotes the PD-polynomial ring over B_n in one variable V. If $\zeta \in \overline{A}$ and $\zeta^{p^n} = 1$, then $v_{\zeta\beta} = \tilde{\zeta}^{p^n} v_\beta$ where $\tilde{\zeta}$ is any element of B_n whose image in $\overline{A}_n$ coincides with that of ζ.

(2) *The connection $\nabla : P_n \to P_n \mathrm{d}\log(t)$ is the unique B_n-linear map satisfying*

$$\nabla((v_\beta - 1)^{[i]}) = (v_\beta - 1)^{[i-1]} v_\beta \mathrm{d}\log(t) \quad \textit{for all } i\,.$$

(in particular, $\nabla(v_\beta) = v_\beta \mathrm{d}\log(t)$).

(3) *Let $\varphi : P_n \to P_n$ be the frobenius, which is induced by the frobenius $(Z_n, \mathcal{L}(t)) \to (Z_n, \mathcal{L}(t))$ defined by the usual frobenius of W_n and by $t \longmapsto t^p$. Then, φ is the unique PD-homomorphism which extends the frobenius of B_n and satisfies $\varphi(v_\beta) = (v_\beta)^p$.*

(4) *The natural action of* $\mathrm{Gal}(\overline{K}/K)$ *on P_n is characterized by the following properties. It extends the natural action of* $\mathrm{Gal}(\overline{K}/K)$ *on B_n, it preserves the PD-structure, and satisfies $\sigma(v_\beta) = v_{\sigma(\beta)}$ $(\sigma \in \mathrm{Gal}(\overline{K}/K))$.*

DEFINITION (3.4). — *Define $\mathcal{N} : P_n \to P_n$ by*

$$\nabla(a) = \mathcal{N}(a)\mathrm{d}\log(t) \quad \textit{for } a \in P_n\,.$$

Then $\mathcal{N}$ is a B_n-derivation.

DEFINITION (3.5). — *For a primitive p^n-th root β of π in $\overline{A}$, define*

$$u_\beta = \log(v_\beta) \in P_n$$

where log *is defined by the PD-structure on* $\mathrm{Ker}(P_n \to \overline{A}_n)$.

COROLLARY (3.6). — *The map $\mathcal{N} : P_n \to P_n$ is surjective,*

$$\{a \in P_n\,;\ \mathcal{N}^i(a) = 0\} = \bigoplus_{0 \le j < i} B_n(u_\beta)^{[j]}\,,$$

$$\{a \in P_n\,;\ \mathcal{N}^i(a) = 0 \quad \text{for some } i\} = B_n < u_\beta >\,,$$

and u_β is transcendental over B_n.

From (3.3) and (3.6), we have

THEOREM (3.7). — *There exists a canonical B^+_{crys}–isomorphism between the ring B^+_{st} of Fontaine and*

$$\{a \in \mathbf{Q} \otimes \varprojlim_{n} P_n\,;\ \mathcal{N}^i(a) = 0 \quad \text{for some } i \ge 0\}$$

where B^+_{st} and P_n are defined using the same prime element π, which preserves φ, $\mathcal{N}$ and the action of $\mathrm{Gal}(\overline{K}/K)$.

Indeed, the isomorphism is given by sending $u_s \in B^+_{st}$ for $s = (s_n)_n$ ($s_n \in \overline{A}$, $s_0 = \pi$, $(s_{n+1})^p = s_n$) (cf. (2.2)) to $(u_{s_n})_n \in \varprojlim_{n} P_n$. The inverse map is induced from $\varprojlim_{n} P_n \to B_{DR}\,;\ ((v_{s_n} - 1)^{[i]})_n \longmapsto (i!)^{-1}(\varepsilon(s)\pi^{-1} - 1)^i$.

(3.8). — We prove (3.1). We follow the argument of Fontaine [Fo1] § 3. Let F be any object of $((S_n, N_n)/W_n)crys$ and let N_F be the log structure of F. By (2.4.3) we have

$$R^q h_{crys*}(\mathcal{O}_{\overline{S}_n/W_n})(F) = \varinjlim_{A'} H^q((S'_n, N'_n)/(F, N_F)) \tag{3.8.1}$$

where A' ranges over all discrete valuation ring in $\overline{A}$ which are finite over A, $S'_n = \mathrm{Spec}(A'/p^n A')$ and N'_n is the inverse image of the canonical log structure on $\mathrm{Spec}(A')$ (2.6). For such A', take a prime element π' of A' and write $\pi = (\pi')^i a$ for $i \ge 1$ and $a \in (A')^\times$. Let $\widetilde{\pi}$ be a section of N_F whose image in N_n is class(π) (2.6). Let

$$Z' = \mathrm{Spec}(\mathcal{O}_F[t', u^{\pm 1}]/((t')^i u - \alpha(\widetilde{\pi}))) \quad (\alpha : N_F \to \mathcal{O}_F)$$

where t' and u are indeterminates. Then the morphism $(S'_n, N'_n) \to (F, N_F)$ factors as $(S'_n, N'_n) \to (Z', \mathcal{L}(t')) \to (F, N_F)$ where the first arrow is an exact closed immersion defined by $\mathcal{O}_{Z'} \to \mathcal{O}_{S'_n}$; $t' \longmapsto \pi'$, $u \longmapsto a$ and by $\mathcal{L}(t') \to N'_n$; $t' \longmapsto \text{class}(\pi')$, and the second is a smooth morphism defined by $N_F \to \mathcal{L}(t')$; $\widetilde{\pi} \longmapsto (t')^i u$. Let F' be the PD–envelope of S'_n in Z'. Then ([HK] (2.20))

$$H^q((S'_n, N'_n)/(F, N_F)) \cong H^q(F', \mathcal{O}_{F'} \otimes_{\mathcal{O}_{Z'}} \omega^{\cdot}_{Z'/F}) . \tag{3.8.2}$$

Note $\omega^1_{Z'/F}$ is a free $\mathcal{O}_{Z'}$–module with basis $\mathrm{d}\log(t')$. To pass to $\varinjlim_{A'}$, let $A'' = A'[\pi'']$ where π'' is a p^n–th root of π', $S'' = \mathrm{Spec}(A''/p^n A'')$, N''_n on S''_n the inverse image of the canonical log structure of $\mathrm{Spec}(A'')$, $Z'' = \mathrm{Spec}(\mathcal{O}_{Z'}[t'']/((t'')^{p^n} - t'))$, and form the commutative diagram

$$\begin{array}{ccc} (S''_n, N''_n) & \longrightarrow & (Z'', \mathcal{L}(t'')) \\ \downarrow & & \downarrow \\ (S', N'_n) & \longrightarrow & (Z', \mathcal{L}(t')) \end{array}$$

with $\mathcal{O}_{Z''} \to \mathcal{O}_{S''}$; $t'' \longmapsto \pi''$, $\mathcal{L}(t'') \to N''_n$; $t'' \longmapsto \text{class}(\pi'')$, $\mathcal{L}(t') \to \mathcal{L}(t'')$; $t' \longmapsto (t'')^{p^n}$.

Then, $\omega^1_{Z'/F} \to \omega^1_{Z''/F}$ annihilates $\mathrm{d}\log(t')$ and hence is the zero map. This shows that

$$\varinjlim_{A'} H^q((S'_n, N'_n)/(F, N_F)) = 0 \quad \text{for } q \neq 0 .$$

We have shown $Rh_{crys^*}(\mathcal{O}_{\overline{S}_n/W_n}) = h_{crys^*}(\mathcal{O}_{\overline{S}_n/W_n})$. This and the base change theorem (2.5.2) shows that $h_{crys^*}(\mathcal{O}_{\overline{S}_n/W_n})$ is flat. The fact that $h_{crys^*}(\mathcal{O}_{\overline{S}_n/W_n})$ is quasi–coherent is shown easily.

(3.9). — We prove (3.3). The following proof is due to the suggestion of W. Messing (my original proof was a direct computation using (3.8.1) and (3.8.2) and was long). Fix a p^n–th root β of π in $\overline{A}$, and regard $\mathrm{Spec}(B_n < V >)$ (V an indeterminate) as an object of

$((\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n}))_{crys}$ as follows : $B_n < V > \longrightarrow \overline{A}_n$ is $V \longmapsto 0$, the PD-structure on $\mathrm{Ker}(B_n < V > \to \overline{A}_n)$ is the usual one, the log structure of $\mathrm{Spec}(B_n < V >)$ which we denote by $\mathcal{L}$ is associated to $\overline{A} - \{0\} \to B_n < V >$; $a \longmapsto \widetilde{a}^{p^n}$ where $\widetilde{a}$ denotes any lifting of a to B_n (we will denote by η the induced map $\overline{A} - \{0\} \to \mathcal{L}$) and $(\mathrm{Spec}(B_n < V >), \mathcal{L}) \to (E_n, N_{E_n})$ is given by the PD-homomorphism $R_n \to B_n < V >$: $t \longmapsto (1+V)^{-1}\widetilde{\beta}^{p^n}$ and by $N_{E_n} \to \mathcal{L}$; $t \longmapsto (1+V)^{-1}\eta(\beta)$. Then, $\mathrm{Spec}(B_n < V >)$ is a terminal object in $((\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n}))_{crys}$, and this fact implies $B_n < V > \xrightarrow{\sim} H^0((\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n})) = P_n$.

Indeed, for any object F of $(\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n}))_{crys}$ with the log structure N_F, the unique morphism $F \to \mathrm{Spec}(B_n < V >)$ in $((\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n}))_{crys}$ is given as follows. Let $\widetilde{\beta}$ be any section of N_F whose image in $\overline{N}_n$ is the class of β. Then $\widetilde{\beta}^{p^n}$ is independent of the choice of $\widetilde{\beta}$. Since the images of $\widetilde{\beta}^{p^n}$ and t under $N_F \to \overline{N}_n$ coincide, there exists a unique section v_β of $\mathcal{O}_F^\times$ such that $\widetilde{\beta}^{p^n} = tv_\beta$ in N_F. Define the PD-homomorphism $B_n < V > \to \mathcal{O}(F)$ by $V \longmapsto v_\beta - 1$, and extend this morphism to the log structures by $\eta(a) \longmapsto \widetilde{a}^{p^n}$ ($a \in \overline{A} - \{0\}$) where $\widetilde{a}$ is any lifting of $\mathrm{class}(a) \in \overline{N}_n$ to N_F (then $\widetilde{a}^{p^n}$ is independent of the choice of $\widetilde{a}$). It is easily checked that this construction yields a unique morphism in $((\overline{S}_n, \overline{N}_n)/(E_n, N_{E_n}))_{crys}$. The properties of P_n in (3.3)(1)–(4) (with v_β defined as in the above argument) are checked easily.

4. — Crystalline interpretation of $(B_{st} \otimes D)^{\mathcal{N}=0}$

In this section, let $S = \mathrm{Spec}(A)$, N and L be as in (2.6), and let (X, M) be a scheme with a log structure over (S, N) satisfying the following conditions (i), (ii), (iii) :

(i) (X, M) is smooth over (S, N).

(ii) The underlying morphism $X \to S$ is proper.

(iii) Let $Y = X \otimes_A k$ and let M_Y be the inverse image of M on Y. Then the induced morphism $(Y, M_Y) \to (\mathrm{Spec}(k), L)$ is of Cartier type [HK] (2.12).

For example, if (X, M) is a fiber product over (S, N) of a finite family of proper A-schemes with semistable reduction with the canonical log structures (2.6), then (X, M) satisfies the conditions above. Moreover the conditions above are stable under base changes $(S', N') \to (S, N)$ where S' is the spectrum of a complete discrete valuation ring with perfect residue field dominating S and N' is the canonical log structure of S'.

Fix $m \geq 0$ and let

$$D_n = H^m((Y, M_Y)/(W_n, W_n(L))) \quad (n \geq 1)$$

$$D_\infty = \varprojlim_n D_n , \quad D = \mathbf{Q} \otimes D_\infty ,$$

where $W_n(L)$ is the canonical lifting of L to W_n defined in [HK] (3.1). Then as in [HK], D_n is a $W_n(k)$–module of finite length, D_∞ is a $W(k)$–module of finite type, and we have a frobenius–linear operator $\varphi : D_n \to D_n$ called the *frobenius* and a linear operator $\mathcal{N} : D_n \to D_n$ called the *monodromy operator* which induce $D_\infty \to D_\infty$ and $D \to D$ denoted by the same letters φ and $\mathcal{N}$, respectively.

Fix a prime element π of A to define B_{st}^+. The aim of this section is to prove

THEOREM (4.1). — *The kernel of*

$$\mathcal{N} = \mathcal{N} \otimes 1 + 1 \otimes \mathcal{N} : B_{st}^+ \otimes_{K_0} D \longrightarrow B_{st}^+ \otimes_{K_0} D$$

is canonically isomorphic to

$$\mathbf{Q} \otimes \varprojlim_n H^m((\overline{X}_n, \overline{M}_n)/W_n)$$

where $(\overline{X}_n, \overline{M}_n) = (X, M) \times_{(S,N)} (\overline{S}_n, \overline{N}_n)$ *for* $n \geq 1$.

For $n \geq 1$, let $X_n = X \otimes \mathbf{Z}/p^n\mathbf{Z}$, let M_n be the inverse image of M on X_n, and let the notations be as

$$\begin{array}{ccccc} (\overline{X}_n, \overline{M}_n) & \xrightarrow{\bar{f}_n} & (\overline{S}_n, \overline{N}_n) & & \\ {\scriptstyle g_n}\Big\downarrow & \square & \Big\downarrow{\scriptstyle h_n} & & \text{(cartesian diagram)} \\ (X_n, M_n) & \xrightarrow{f_n} & (S_n, N_n) . & & \end{array}$$

Let $Z_n = \mathrm{Spec}(W_n[t])$ and $E_n = \mathrm{Spec}(R_n)$ be the PD–envelope of S_n in Z_n with the log structure N_{E_n} as in (3.2), where we use the same prime element π to define $S_n \hookrightarrow Z_n$; $t \longmapsto \pi$.

LEMMA (4.2). — *There exists a long exact sequence*

$$\cdots \longrightarrow H^m((\overline{X}_n, \overline{M}_n)/W_n) \longrightarrow$$
$$P_n \otimes_{R_n} (R^m(f_n)_{crys*}(\mathcal{O}_{X_n/W_n}))(E_n) \xrightarrow{\mathcal{N}} P_n \otimes_{R_n} (R^m(f_n)_{crys}(\mathcal{O}_{X_n/W_n}))(E_n)$$
$$\longrightarrow H^{m+1}((\overline{X}_n, \overline{M}_n)/W_n) \longrightarrow \cdots .$$

PROOF. Let $((X^{\cdot}, M^{\cdot}), (Z^{\cdot}, M_{Z^{\cdot}}))$ be an embedding system [HK] (2.18) for $(X, M) \to (\mathrm{Spec}(W[t]), \mathcal{L}(t))$, $t \longmapsto \pi$. Let $(F_n^i, M_{F_n^i})$ be the PD–envelope [HK](2.16) of (X_n^i, M_n^i) in $(Z_n^i, M_{Z_n^i})$. Then, for any crystal $\mathcal{F}$ on $((X_n, M_n)/W_n)_{crys}$, we have an exact sequence of complexes in the topos $(X_n^{\cdot})_{et}^{\sim}$ ([HK] (2.18))

$$0 \longrightarrow C'([-1] \xrightarrow{\mathrm{d}\log(t)} C \longrightarrow C' \longrightarrow 0 \tag{4.2.1}$$

where C (resp. C') is defined on each X_n^i as the complex

$$\mathcal{F}_{F_n^i} \otimes_{\mathcal{O}_{Z_n^i}} \omega^{\cdot}_{Z_n^i/W_n} \quad (\text{resp.}\, \mathcal{F}_{F_n^i} \otimes_{\mathcal{O}_{Z_n^i}} \omega^{\cdot}_{Z_n^i/\mathrm{Spec}(W_n[t])}).$$

Consider the case $\mathcal{F} = R(g_n)_{crys^*}(\mathcal{O}_{\overline{X}_n/W_n})$. By taking the inductive limit of the base change theorem (2.5.2) and by (3.1), we see $\mathcal{F} = (f_n)^*_{crys}(h_n)_{crys^*}(\mathcal{O}_{\overline{S}_n/W_n})$. We have

$$H^m((X_n^{\cdot})_{et}, C) = H^m((\overline{X}_n, \overline{M}_n)/W_n),$$
$$H^m((X_n^{\cdot})_{et}, C') = H^m((X_n^{\cdot})_{et}^{\sim}, P_n \otimes_{\mathcal{O}(E_n)} \mathcal{O}_{F_n^{\cdot}} \otimes_{\mathcal{O}_{Z_n^{\cdot}}} \omega^{\cdot}_{Z_n^{\cdot}/\mathrm{Spec}(W_n[t])})$$
$$= P_n \otimes_{\mathcal{O}(E_n)} H^m((X_n^{\cdot})_{et}, \mathcal{O}_{F_n^{\cdot}} \otimes_{\mathcal{O}_{Z_n^{\cdot}}} \omega^{\cdot}_{Z_n^{\cdot}/\mathrm{Spec}(W_n[t])})$$

where the last equation follows from the flatness of P_n over $\mathcal{O}(E_n)$. Hence (4.2) is obtained by taking the long exact sequence of cohomology groups associated to the exact sequence (4.2.1).

LEMMA (4.3). — *For any $W_n(k)$-module T having a nilpotent $W_n(k)$-linear operator $\mathcal{N} : T \to T$, the map*

$$\mathcal{N} \otimes 1 + 1 \otimes \mathcal{N} : P_n \otimes T \longrightarrow P_n \otimes T$$

is surjective.

PROOF. This is reduced to the case $T = W_n(k)$ and $\mathcal{N} : T \to T$ is the zero map, i.e. to (3.6).

DEFINITION (4.4). — (1) *For a category $\mathcal{C}$, let $ps(\mathcal{C})$ be the category of projective systems in $\mathcal{C}$ with index set $\mathbb{N}$.*

(2) *For an additive category, let $\mathbb{Q} \otimes \mathcal{C}$ be the category with the same objects as $\mathcal{C}$ but with morphisms $\mathrm{Hom}_{\mathbb{Q}\otimes\mathcal{C}} = \mathbb{Q} \otimes \mathrm{Hom}_{\mathcal{C}}$. An object T of $\mathcal{C}$ is denoted by $\mathbb{Q} \otimes T$ when it is regarded as an object of $\mathbb{Q} \otimes \mathcal{C}$.*

(4.5). — Now we prove (4.1). By [HK] (5.2), we have an isomorphism in $\mathbb{Q} \otimes ps(Ab)$ (Ab denotes the category of abelian groups)

$$\mathbb{Q} \otimes \{(R^m(f_n)_{crys*}(\mathcal{O}_{X_n/W_n})(E_n)\}_n \cong \mathbb{Q} \otimes \{R_n \otimes_{W_n} D_n\}_n .$$

By this and (4.2) (4.3), we have an exact sequence in $\mathbb{Q} \otimes ps(Ab)$

$$0 \longrightarrow \mathbb{Q} \otimes \{H^m((\overline{X}_n, \overline{M}_n)/W_n)\}_n \longrightarrow$$

$$\mathbb{Q} \otimes \{P_n \otimes_{W_n} D_n\}_n \xrightarrow{\mathcal{N}} \mathbb{Q} \otimes \{P_n \otimes_{W_n} D_n\}_n \longrightarrow 0 .$$

Furthermore this map $\mathcal{N}$ is $\mathcal{N} \otimes 1 + 1 \otimes \mathcal{N}$ with the first $\mathcal{N} : P_n \to P_n$ and the second $\mathcal{N} : D_n \to D_n$ the monodromy operator. Hence we have

$$\mathbb{Q} \otimes \varprojlim_n H^m((\overline{X}_n, \overline{M}_n)/W_n) \cong$$

$$\mathrm{Ker}(\mathcal{N} : (\mathbb{Q} \otimes \varprojlim_n P_n) \otimes_{K_0} D \longrightarrow (\mathbb{Q} \otimes \varprojlim_n P_n) \otimes_{K_0} D).$$

Since $\mathcal{N}$ is nilpotent on D, we can replace $\mathbb{Q} \otimes \varprojlim_n P_n$ by the part of it on which $\mathcal{N}$ is nilpotent, i.e. by B_{st}^+ (3.7).

5. — The complex $s_n^{\log}(t)$

Let (X, M) be a scheme with a fine log structure which is syntomic over W. For $0 \leq r < p$, we define an object $s_{n,X}^{\log}(r)$ of the derived category $D(X_{et}, \mathbb{Z}/p^n\mathbb{Z})$ supported on $(X \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})_{et}$. We state a result (5.4) on the relation between $s_{n,X}^{\log}(r)$ and p–adic vanishing cycles whose proof will be given elsewhere. This $s_{n,X}^{\log}(r)$ is the log pole version of the complex $R\nu_*(S_{n,X}^r)$ studied in [Ka1] and [Ku] where $S_{n,X}^r$ is the sheaf on the syntomic site defined in [FM] and ν is the canonical morphism from the syntomic site to the etale site.

(5.1). — Take an embedding system $((X^{\cdot}, M^{\cdot}), (Z^{\cdot}, N^{\cdot}))$ for $(X, M) \to W$ with a lifting of frobenius $F : (Z^{\cdot}, N^{\cdot}) \to (Z^{\cdot}, N^{\cdot})$. Let $X_n^i = X^i \otimes \mathbb{Z}/p^n\mathbb{Z}$, $Z_n^i = Z^i \otimes \mathbb{Z}/p^n\mathbb{Z}$, M_n^i (resp. N_n^i) the inverse image of M^i on X_n^i (resp. N^i on Z_n^i). Here F is a lifting of frobenius means that the morphisms on (Z_1^i, N_1^i) induced by F are the absolute frobeniuses [HK] (2.12) and F commutes with the canonical frobenius of W. Let $(F_n^i, M_{F_n^i})$ be the PD–envelope [HK] (2.16) of (X_n^i, M_n^i) in (Z_n^i, N_n^i), and let $J_{F_n^i}^{[r]} \subset \mathcal{O}_{F_n^i}$ be the r–th divided power of $J_{F_n^i} = \mathrm{Ker}(\mathcal{O}_{F_n^i} \to \mathcal{O}_{X_n^i})$.

For $0 \leq r < p$, we define a complex $s_{n,(X^{\cdot},M^{\cdot});(Z^{\cdot},N^{\cdot})}^{\log}(r)$ in $(X^{\cdot})_{et}^{\sim}$ (denoted simply by $s_{n,X^{\cdot}}^{\log}(r)$) as follows.

First, denote by $j_{n,X}^{\log}(r)$ the complex in $(X^{\cdot})_{et}^{\sim}$ which gives on each X^i the familiar complex

$$J_{F_n^i}^{[r]} \xrightarrow{d} J_{F_n^i}^{[r-1]} \otimes_{\mathcal{O}_{Z^i}} \omega_{Z^i/W}^1 \xrightarrow{d} \cdots \quad \cdots \xrightarrow{d} J_{F_n^i}^{[r-q]} \otimes_{\mathcal{O}_{Z^i}} \omega_{Z^i/W}^q \to \cdots .$$

Let $\varphi : \mathcal{O}_{F_n^i} \to \mathcal{O}_{F_n^i}$ be the homomorphism induced by $F : (Z^{\cdot}, M_{Z^{\cdot}}) \to (Z^{\cdot}, M_{Z^{\cdot}})$. Assume $0 \leq r < p$. Then $\varphi(J_{F_n^i}^{[r]}) \subset p^r\mathcal{O}_{F_n^i}$. We define the map $p^{-r}\varphi : J_{F_n^i}^{[r]} \to \mathcal{O}_{F_n^i}$ by the law $(p^{-r}\varphi)(a \bmod p^n) = b \bmod p^n$ for $a \in J_{F_{n+r}^i}^{[r]}$ and $b \in \mathcal{O}_{F_{n+r}^i}$ such that $\varphi(a) = p^r b$. This map is well defined by the flatness of F_n^i over W_n (2.5.1). We have a homomorphism of complexes

$$p^{-r}\varphi : j_{n,X^{\cdot}}^{\log}(r) \longrightarrow j_{n,X^{\cdot}}^{\log}(0)$$

whose degree q part is $((p^{q-r})\varphi$ on $J_{F_n^i}^{[r-q]}) \otimes (p^{-q}\varphi$ on $\omega^q_{Z^i/W})$. Finally we define $s^{\log}_{n,X^\cdot}(r)$ as the mapping fiber of

$$1 - p^{-r}\varphi : j^{\log}_{n,X^\cdot}(r) \longrightarrow j^{\log}_{n,X^\cdot}(0)\,.$$

Here for a homomorphism $h : C \to C'$ of complexes, by the mapping fiber of h we mean the complex whose degree q part is $C^q \oplus (C')^{q-1}$ and whose differential is given by

$$C^q \oplus (C')^{q-1} \longrightarrow C^{q+1} \oplus (C')^q\,;\ (a,b) \longmapsto (dx, h(x) - dy)\,.$$

Let $\theta : (X^\cdot)^\sim_{et} \to X^\sim_{et}$ be the canonical morphism, and define

$$s^{\log}_{n,X}(r) = R\theta_*(s^{\log}_{n,X^\cdot}(r))\,.$$

Note

$$R\theta_*(j^{\log}_{n,X^\cdot}(r)) = Ru_{(X_n,M_n)/W_n*}(J^{[r]}_{X/W_n})$$

where u is the canonical morphism $((X_n, M_n)/W_n)^\sim_{crys} \to X^\sim_{et}$ and $J^{[r]}_{X/W_n}$ is the r–th divided power of $J_{X/W_n} = \mathrm{Ker}(\mathcal{O}_{X_n/W_n} \to \mathcal{O}_{X_n})$. Thus we have a distinguished triangle

$$s^{\log}_{n,X}(r) \longrightarrow Ru_{(X_n,M_n)/W_n*}(J^{[r]}_{X_n/W_n}) \xrightarrow{1-p^{-r}\varphi} Ru_{(X_n,M_n)/W_n*}(\mathcal{O}_{X_n/W_n}) \to .$$

This shows that the object $s^{\log}_{n,X}(r)$ in $D(X_{et}, \mathbb{Z}/p^n\mathbb{Z})$ is independent of the choice of an embedding system with a lifting of frobenius.

This definition of $s^{\log}_{n,X}(r)$ is just to add log poles to the complex $R\nu_*(S^r_{n,X})$ (cf. [Ka1]). Hence, if (X, M) and X are syntomic over W, "to add log poles" defines a canonical morphism

$$R\nu_*(S^r_{n,X}) \longrightarrow s^{\log}_{n,X}(r)\,.$$

The same method as in [Ka1] (which treated the case without log structures) defines a product structure

$$s^{\log}_{n,X}(r) \otimes^L s^{\log}_{n,X}(r') \longrightarrow s^{\log}_{n,X}(r + r') \quad (0 \le r,\ r',\ r + r' < p)\,.$$

THEOREM (5.4). — *Let A be as before and let X be a scheme over A with semi-stable reduction endowed with the canonical log structure. Let*

$$X \otimes_A \overline{k} \xrightarrow{\overline{i}} \overline{X} \xleftarrow{\overline{j}} X_{\overline{K}}$$

be the inclusion maps and consider the sheaf of p-adic vanishing cycles $\overline{i}_*\overline{i}^* R\overline{j}_*(\mathbf{Z}/p^n\mathbf{Z}(r))$, where (r) means the Tate twist. Then for $0 \leq r < p-1$, we have a canonical isomorphism

$$s_{n,X}^{\log}(r) \cong \tau_{\leq r}\overline{i}_*\overline{i}^* R\overline{j}_*(\mathbf{Z}/p^n\mathbf{Z}(r))\,.$$

Here $s_{n,X}^{\log}(r)$ is defined to be the inductive limit of the inverse images of $s_{n,X\otimes_A A'}^{\log}(r)$ where A' ranges over all discrete valuation rings in $\overline{A}$ which are finite over A and $X \otimes_A A'$ is endowed with the log structures as fiber products where X, $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$ are endowed with the canonical log structures (2.6).

The "without log pole" version of (5.4) was proved in [Ku] (cf. also [Ka1]). The method of the proof of (5.4) is similar to that in [Ku]. The key point is that, in the place where the result of [BK] on p-adic vanishing cycles in the good reduction case is used in [Ku], we can use the generalization by Hyodo [H1] of the result of [BK] to the semi-stable reduction case.

COROLLARY (5.5). — *Let X be a proper scheme over A with semi-stable reduction. Then if $m \leq r < p-1$ or if $\dim(X_K) \leq r < p-1$, there exists a canonical isomorphism*

$$H^m(\overline{X}, s_n^{\log}(r)) \xrightarrow{\sim} H_{et}^m(X_{\overline{K}}, \mathbf{Z}/p^n\mathbf{Z}(r))\,.$$

This follows from (5.4) and the proper base change theorem for the etale cohomology $H_{et}^m(X_{\overline{K}}, \mathbf{Z}/p^n\mathbf{Z}) \cong H_{et}^m(\overline{X}, \overline{i}_*\overline{i}^* R\overline{j}_*\mathbf{Z}/p^n\mathbf{Z})$.

The isomorphism in (5.5) has the following properties as will be shown elsewhere, which we shall use in §6.

(5.6.1). — It is compatible with the action of $\mathrm{Gal}(\overline{K}/K)$.

(5.6.2). — The isomorphism $H^0(\overline{X}, s_n^{\log}(1)) \xrightarrow{\sim} H^0(\overline{X}, \mathbb{Z}/p^n\mathbb{Z}(1))$ is inverse to the map $\mathbb{Z}/p^n\mathbb{Z}(1) \to \{x \in J_n^{[1]}\,;\, p^{-1}\varphi(x) = x\} \subset B_n$ induced by ε in (2.2).

(5.6.3). — When m and r vary satisfying $m \le r < p-1$, the isomorphisms of (5.5) are compatible with the product structures.

(5.6.4). — For a line bundle $\mathcal{L}$ on $\overline{X}$, the Chern class of $\mathcal{L}$ in the syntomic cohomology $H^2(\overline{X}_{syn}, S_n^1)$ [FM] is sent to the Chern class of $\mathcal{L}$ in $H^2_{et}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(1))$ by the composite map $H^2(\overline{X}_{syn}, S_n^r) \to H^2(\overline{X}, s_n^{\log}(r)) \to H^2_{et}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}(1))$.

6. — Conjecture of Fontaine–Jannsen

In this section we prove Thm. (1.2). The following is the logarithmic version of the method in [KM] in which the good reduction case was considered. Let X be a proper scheme over A with semi-stable reduction. Let

$$V^m = H^m_{et}(X_{\overline{K}}, \mathbb{Q}_p)\,, \quad D^m = \mathbb{Q} \otimes \varprojlim_n H^m((Y, M_Y)/(W_n, W_n(L)))\,,$$

where M_Y is the inverse image on Y of the canonical log structure M on X (2.6).

(6.1). — We define a canonical B_{st}-linear homomorphism

$$B_{st} \otimes_{\mathbb{Q}_p} V^m \longrightarrow B_{st} \otimes_{K_0} D^m \tag{6.1.1}$$

for $m < p-1$, which is compatible with the action of $\mathrm{Gal}(\overline{K}/K)$ and with the frobenius φ and the monodromy operator $\mathcal{N}$. The canonical homomorphism $s_n^{\log}(r) \to j_n^{\log}(r) \subset j_n^{\log}(0)$ induces

$$H^m(\overline{X}, s_n^{\log}(r)) \longrightarrow H^m((\overline{X}_n, \overline{M}_n)/W_n)^{\varphi = p^r}$$

where $\varphi = p^r$ means the part on which the frobenius acts by p^r. By (4.1) and (5.5), we obtain by taking $\mathbb{Q} \otimes \varprojlim_n$,

$$V^m(r) \longrightarrow (B^+_{st} \otimes D^m)^{\mathcal{N}=0,\, \varphi = p^r} \quad \textit{for } m \le r < p-1\,.$$

By tensoring with $\mathbf{Q}_p(-r)$ and using the canonical map $B_{st}^+ \otimes \mathbf{Q}_p(-r) \to B_{st}$, we obtain

$$V^m \longrightarrow (B_{st} \otimes D^m)^{\mathcal{N}=0,\,\varphi=1}.$$

For a fixed m such that $m < p-1$, this map is independent of the choice of r such that $m \leq r < p-1$. This defines the desired homomorphism (6.1.1).

(6.2). — Assume X_K is geometrically connected and of dimension d. Then we have trace maps

$$V^{2d} \xrightarrow{\sim} \mathbf{Q}_p(-d),$$

$$K \otimes_{K_0} D^{2d} \cong H^{2d}_{DR}(X_K/K) \xrightarrow{\sim} K.$$

In the latter isomorphism, by replacing K by a finite extension which is Galois over K_0 and by taking the $\mathrm{Gal}(K/K_0)$–invariant part, we obtain the trace map

$$D^{2d} \xrightarrow{\sim} K_0$$

(this isomorphism also follows from the Poincaré duality of the de Rham–Witt complex $W_n\omega^{\cdot}_Y$ proved in Hyodo [H2] and the isomorphism $D^m = \mathbf{Q} \otimes \varprojlim_n H^m(Y, W_n\omega^{\cdot}_Y)$).

Assume $2d < p-1$. Then the following diagram is commutative.

$$\begin{array}{ccc} B_{st} \otimes_{\mathbf{Q}_p} V^{2d} & \longrightarrow & B_{st}(-d) \cong B_{st} \\ {\scriptstyle \text{by (6.1.1)}}\big\downarrow & & \big\downarrow{\scriptstyle id} \\ B_{st} \otimes_{K_0} D^{2d} & \longrightarrow & B_{st}\,. \end{array}$$

Indeed, this follows from the compatibility (5.6.4) with the Chern class of line bundles (cf. [FM] § 6.3).

(6.3). — We prove that the homomorphism (6.1.1) is an isomorphism if $2\dim(X_K) < p-1$. We may assume X is geometrically connected and $m \leq 2d$

where $d = \dim(X_K)$. Consider the commutative diagram

$$\begin{array}{ccc} B_{st} \otimes V^m \times B_{st} \otimes V^{2d-m} & \longrightarrow & B_{st} \otimes V^{2d} \cong B_{st} \\ \downarrow & & \downarrow \scriptstyle{id.} \\ B_{st} \otimes D^m \times B_{st} \otimes D^{2d-m} & \longrightarrow & B_{st} \otimes D^{2d} \cong B_{st} \end{array}$$

induced by cup products. The Poincaré duality shows that the horizontal pairings are perfect pairings of free B_{st}–modules of finite ranks. From this, we see that the map (6.1.1) is an injection and its image is a B_{st}–direct summand. Since

$$\dim_{\mathbb{Q}_p}(V^m) = \dim_K(H^m_{DR}(X_K/K)) = \dim_{K_0}(D^m)\,,$$

we have the surjectivity of (6.1.1).

(6.4). — We show that the isomorphism

$$B_{DR} \otimes_{\mathbb{Q}_p} V^m \xrightarrow{\sim} B_{DR} \otimes_K H^m_{DR}(X_K/K) \tag{6.4.1}$$

induced by (6.1.1) preserves the filtrations. We prove first :

(6.4.2). — The isomorphism (6.4.1) sends fil^i of the left hand side into fil^i of the right hand side for any $i \in \mathbb{Z}$.

It suffices to prove this for one choice of i, and so take $i = r$ with $m \leq r < p-1$. Our task is to show that the image of $\varprojlim_n H^m(\overline{X}_n, s_n^{\log}(r)) \to B_{DR} \otimes H^m_{DR}$ is contained in fil^r. This will follow if we show that the map

$$\varprojlim_n H^m((\overline{X}_n, \overline{M}_n)/W_n)) \longrightarrow B_{DR} \otimes H^m_{DR} \tag{6.4.3}$$

sends the image of $\varprojlim_n H^m(((\overline{X}_n, \overline{M}_n)/W_n)_{crys}, J^{[r]}_{\overline{X}_n/W_n})$ into fil^r. In [KM], it is proved that for any proper syntomic scheme X over A with smooth generic fiber, we have a canonical isomorphism

$$\mathbb{Q} \otimes \varprojlim_n H^m((\overline{X}_n/W_n)_{crys}\,,\ \mathcal{O}_{\overline{X}_n/W_n}/J^{[r]}_{\overline{X}_n/W_n}) \cong (B^+_{DR} \otimes H^m_{DR}(X_K/K))/fil^r$$

(here all things are without log structures). In the situation of this section, the same method shows that there is an isomorphism

$$\mathbb{Q} \otimes \varprojlim_n H^m(((\overline{X}_n, \overline{M}_n)/W_n)_{crys}, \mathcal{O}_{\overline{X}_n/W_n}/J^{[r]}_{\overline{X}_n/W_n})$$

$$\cong (B^+_{DR} \otimes H^m_{DR}(X_K/K))/fil^r$$

which is compatible with (6.4.3). Thus we obtain (6.4.2).

Once we have (6.4.2), the fact that (6.4.1) is an isomorphism of filtrations is reduced to the injectivity of the maps

$$\text{(6.4.4)} \qquad \mathrm{gr}^i(B_{DR} \otimes V^m) \longrightarrow \mathrm{gr}^i(B_{DR} \otimes D^m)$$

induced by (6.4.1). Since $\mathrm{gr}^i(B_{DR}) \cong \mathbb{C}_p(i)$, this map is rewritten as

$$\text{(6.4.5)} \qquad \mathbb{C}_p(i) \otimes V^m \longrightarrow \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_p(i-j) \otimes H^{m-j}(X_K, \Omega^j)\,.$$

The bijectivity of (6.4.4) is proved by using the Poincaré duality by the argument as in (6.3).

(6.5). — By (6.4), we have proved the de Rham conjecture of Fontaine [Fo1] in the semi-stable reduction case under the assumption $2\dim(X_K) < p-1$. However this conjecture is already proved by Faltings [Fa2] with no such assumption by a different method. We obtained in (6.4) (the bijectivity of (6.4.4) with $i = 0$) a new proof of the existence of the Hodge–Tate decomposition ([Fa1]) under the assumption of Thm. (1.2).

REFERENCES

[B] P. Berthelot. — *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Math. 407, Springer–Verlag, 1974.

[BBM] P. Berthelot, L. Breen, W. Messing. — *Théorie de Dieudonné cristalline II*, Lecture Notes in Math. 930, Springer–Verlag, 1982.

[Fa1] G. Faltings. — *p–adic Hodge theory*, Journal of the AMS 1 (1988), 255–299.

[Fa2] G. Faltings. — *Crystalline cohomology and p–adic Galois–representations*, in Algebraic Analysis, Geometry and Number theory, The Johns Hopkins Univ. Press. (1989), 25–80.

[Fa3] G. Faltings. — *F–isocrystals on open varieties, results and conjectures*, The Grothendieck Festschrift, Vol. 2, 219–248, Progress in Math. 87, Birkhäuser, 1990.

[Fo1] J.-M. Fontaine. — *Sur certains types de représentations p–adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti–Tate*, Ann. of Math. 115 (1982), 529–577.

[Fo2] J.-M. Fontaine. — *Cohomologie de de Rham, cohomologie cristalline et représentations p–adiques*, in Algebraic geometry, Lecture Notes in Math. 1016, Springer–Verlag (1983), 86–108.

[Fo3] J.-M. Fontaine. — *Letter to U. Jannsen*, Nov. 26, (1987).

[Fo4] J.-M. Fontaine. — *Le corps des périodes p–adiques*, this volume.

[FM] J.-M. Fontaine and W. Messing. — *p–adic periods and p–adic etale cohomology*, Contemporary Math. 67 (1987), 179–207.

[H1] O. Hyodo. — *A note on p–adic etale cohomology in the semi–stable reduction case*, Invent. Math. 91 (1988), 543–557.

[H2] O. Hyodo. — *On the de Rham–Witt complex attached to semi–stable family*, Comp. Math. 78 (1991), 241–260.

[HK] O. Hyodo and K. Kato. — *Semi–stable reduction and crystalline cohomology with logarithmic poles*, this volume.

[J] U. Jannsen. — *On the ℓ–adic cohomology of varieties over number fields and its Galois cohomology*, in Galois groups over $\mathbb{Q}$, Springer–Verlag (1989), 315–360.

[Ka1] K. KATO. — *On p–adic vanishing cycles (Application of ideas of Fontaine–Messing*, Advanced studies in Pure Math. 10 (1987), 207–251.

[Ka2] K. KATO. — *Logarithmic structures of Fontaine–Illusie*, in Algebraic analysis, geometry and number theory, The Johns Hopkins Univ. Press (1989), 191–224.

[KM] K. KATO and W. MESSING. — *Syntomic cohomology and p–adic etale cohomology*, Tôhoku Math. J. 44 (1992), 1–9.

[Ku] M. KURIHARA. — *A note on p–adic etale cohomology*, Proc. Japan Academy 63 (1987), 275–278.

[P] B. PERRIN–RIOU. — *Représentations p–adiques ordinaires*, this volume.

[SGA4] M. ARTIN, A. GROTHENDIECK, J.–L. VERDIER. — *Théorie des topos et cohomologie étale des schémas*, tome 2, Lecture Notes in Math. 270, Springer–Verlag, 1972.

Added in proof : Recently, generalizations of the results of this paper were obtained in the following papers by Takeshi Tsuji.

- "Syntomic complexes and p–adic vanishing cycles"
- "Log crystalline cohomology and log syntomic cohomology".

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