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DENSITIES FOR CERTAIN LEAVES OF REAL ANALYTIC FOLIATIONS

C. ROCHE[1]

I. Introduction.

Let suppose an n dimensional real analytic manifold M be given. We will supose M to be paracompact connected and oriented. A real analytic $n-1$ *foliation with singularities* $\mathcal{F}$ on M is determined by giving an open covering (U_i) of M together with real analytic integrable 1-forms $\omega_i \in \Omega^1(U_i)$ such that on the overlapping charts, $U_i \cap U_j \neq \emptyset$, there exists a non vanishing function $g_{i,j} : U_i \cap U_j \to \mathbf{R}^*$ such that $\omega_i = g_{i,j}\omega_j$. Leaves of $\mathcal{F}$ on U_i are unions of the integral manifolds of the pfaffian equation $\omega_i = 0$.

The singular set of the foliation Sing($\mathcal{F}$) is the analytic subspace of M defined by the annulation of the forms ω_i. In local coordinates of M, each ω_i can be written as

$$\omega_i(x) = \sum_{l=1}^{n} a_l^i(x) dx^l$$

and locally Sing($\mathcal{F}$) is determined by the equations

$$a_1^i(x) = 0, \ldots, a_n^i(x) = 0 \qquad x \in U_i.$$

The hypothesis that the $g_{i,j}$ be non vanishing allowes to suppose that the singular set is of codimension at least 2. Such $\mathcal{F}$ defines on $M \setminus \mathrm{Sing}(\mathcal{F})$ an $n-1$ dimensional analytic foliation: $\mathcal{F}_{reg}$. Leaves of $\mathcal{F}_{reg}$ are called regular leaves of $\mathcal{F}$.

Morover if we suppose $\mathcal{F}$ to be transversally orientable, as will be done in this paper, Theorem A and B of Cartan in the real case [3] show that we can glue the 1-forms in order to suppose that the foliation $\mathcal{F}$ is given by a globally defined real analytic differential form ω, that is $\omega_i = \omega_{|U_i}$.

Consider now a union Γ of regular leaves of such a foliation $\mathcal{F}$, Γ is an immersed $n-1$ real analytic submanifold of M. Γ is called a separating solution

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by Khovanskii if there are two disjoint open sets, L_1 ans L_2 of M such that $M\setminus$ Sing$(\mathcal{F}) \setminus\Gamma = L_1 \cup L_2$, $\Gamma = \bar{L_1} \setminus L_1\setminus$Sing$(\mathcal{F})$ and finally ω points inside L_1 all along Γ.

In [17] we generalize this notion introducing Rollian pfaffian hypersurfaces. A regular leaf V of $\mathcal{F}$ is so called if for each analytic path $\gamma : [0,1] \to M$ intersecting the set V twice, say $\gamma(0) \in V$ and $\gamma(1) \in V$ there is an intermediate point, say $\gamma(t)$, $t \in [0,1]$ where the path is tangent to $\mathcal{F}$. At this point, if $\mathcal{F}$ is determined by the pfaff equation $\omega = 0$

$$\omega(\gamma(t)).\gamma'(t) = 0.$$

Such a Rollian pfaffian hypersurface (Rollian leaf or Rollian ph for short) will be denoted $\{V, \mathcal{F}, M\}$ to emphasize the pfaffian equation verified by V.

Khovanskii's Rolle theorem asserts that every separating solution of $\omega = 0$ is a union of Rollian ph. Separating solutions are not easy to find but, as it was shown in [17], an argument of Haefliger proves that if $M\setminus$Sing$(\mathcal{F})$ is simply connected, each regular leaf of $\mathcal{F}$ is a Rollian ph.

In [17] we used this generalisation to prove the following general finiteness theorem.

Theorem on uniform finiteness. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_q$ be transversally oriented singular foliations on M. If X is a semianalytic subset of M for each compact set K of M there is a constant $b \in \mathbf{R}$ such that for any set of Rollian pfaffian hypersurfaces $\{V_i, \mathcal{F}_i, M\}, i = 1, \ldots, q$ the number of connected components of $X \cap V_1 \cap \cdots \cap V_q$ meeting K is bounded by b.*

A carefull reading of the proof of this theorem in [17] shows that a separating manifold is in fact a locally finite union of Rollian ph as was shown by Khovanskii [5].

As an easy consequence of this result we can mention that a Rollian ph $\{V, \mathcal{F}, M\}$ is a real analytic submanifold of M closed in $M\setminus$Sing$(\mathcal{F})$.

In developping the ideas sketched in Khovanskii's work [5] [6] in joint work with R. Moussu, J.-M. Lion and J.-Ph. Rolin (started in [16]) we tried to consider Rollian ph just as building blocks for a theory similar to that of semianalytic sets. By different methods the same goal is pursued by Tougeron [19]. This idea leads to the problem of the behaviour of the boundary of a Rollian ph. At present time it is not known if the closure of a Rollian ph $\{V, \mathcal{F}, M\}, \bar{V}$ can be stratified with some regularity condition. In a forthcomming paper of F. Cano, J.-M. Lion and R. Moussu an important result on the regularity of the boundary $\bar{V} \setminus V$ of such a Rollian ph will be described. [2].

The study of the boundary of a sole Rollian ph $\{V, \mathcal{F}, M\}$ is most usefull for further research if we describe the structure of the boundary of an intersection $X \cap V$ where X is a semianalytic subset of M. If X is open connected and relatively compact in M, $X \cap V$ is a finite union of leaves of the restricted foliation $\mathcal{F}_{|X}$ each of them is a Rollian ph in X. The behabiour of V at the ends of M is so permited in the case the foliation can be regularly continued.

Let's define a pfaffian subset of M as a finite intersection $W = X \cap V_1 \cap \cdots \cap V_q$ where X is any semianalytic subset of M and the $V_i's$ are Rollian ph of foliations $\mathcal{F}_i$.

The following properties are known for the set $\partial W = \bar{W} \setminus W$. See [8][10].

Theorem on finiteness of the boundary. *The set ∂W is locally arc connected. Morover if $B_a(\rho)$ is the euclidean open ball of center a and radius ρ for $a \in \bar{W}$ the number of connected components of $\partial W \cap B_a(\rho)$ can be bounded by a constant depending only on the foliations $\mathcal{F}_i$ but not on the particular Rollian ph chosen.*

Let $C_y(A)$ be the tangent cone of $A \subset M$ at $y \in M$.

Curve selection lemma. *Let $a \in \partial W$, $u \in C_a(W)$, with $||u|| = 1$ be given, there is a semianalytic subset Y of M such that $W \cap Y$ is a union of paths $\gamma_i((0,1))$ one of them, say γ_0, can be extended in a C^1 way at 0 by $\gamma_0(0) = a$ and $\gamma_0'(0) = u$.*

These curves are pfaffian curves.

In this paper we show that Rollian ph have local volume properties similar to those of semianalytic and subanalytic sets.

A subset Y of $\mathbf{R}^n$ has a k-dimensional density at $y \in \mathbf{R}^n$ if the k-dimensional volume of $B_y(\epsilon) \cap Y$, $vol_k(B_y(\epsilon) \cap Y)$ is finite for small enough $\epsilon > 0$ and the following limit exists

$$\Theta_k(Y, y) = \lim_{\epsilon \to 0+} \frac{vol_k(B_y(\epsilon) \cap Y)}{\epsilon^k}.$$

This quantity is called density of Y at y. If these conditions are not fullfilled we can always consider the corresponding superior limit and inferior limit, which are denoted by $\bar{\Theta}_k(Y, y)$ and $\underline{\Theta}_k(Y, y) \in \bar{\mathbf{R}}_+$ respectively.

In a recent paper [7] Kurdyka and Raby show that subanalytic subsets have a density at every point. Our result is similar, but restricted to the case of Rollian ph as we cannot, at present time, obtain a general decomposition into graphs theorem for pfaffian sets.

Precisely, let M be an open semianalytic subset of $\mathbf{R}^n$

Theorem 1. *Let $\{V, \mathcal{F}, M\}$ be a Rollian pfaffian hypersurface then V has a density at each point of $\bar{V}$.*

The proof of this result uses the same idea of Kurdyka ans Raby and needs a new result on decomposition of Rollian ph into graphs. This decomposition gives a precision to a similar result of Lion [8], [9] and is obtained in a more elementary way. Namely

Proposition 1. *Let ω be an integrable real analytic 1-form, in a neighborhood of $0 \in \mathbf{R}^n$ and a small enough $\epsilon > 0$ be given. Then there is a finite number of hyperplans (H_i) and a subanalytic stratification $\mathcal{N}$ of a ball $B_0(\rho)$ such that: if $\{V, \omega, B_0(\rho)\}$ is a Rollian ph and $N \in \mathcal{N}$ then either*
$V \cap N$ is included in a smooth submanifold of dimension less than $n-1$,
or $V \cap N \subset H_i \oplus H_i^{\perp} \subset \mathbf{R}^n$ is the graph of a locally ϵ–lipschitzian analytic function on an open subset of H_i.

That is, up to a smaller dimensional set, each Rollian ph is a graph of an analytic funtion. This function can be supposed to have a very small derivative.

It is known that strong regularity conditions for stratified objects doesn't imply the existence of densities. Theorem 1 gives an interesting information on the good behaviour of the boundary of a Rollian ph even in case a theorem of regular stratification happens to be obtained.

The generalisation of theorem 1 to all pfaffian sets would be not difficult provided a result similar to Proposition 1 for several pfaffian equations can be proved.

II. Tangents to semianalytic sets and pfaffian equations.

Here we discuss a general stratification procedure preparing a graph decomposition of Rollian ph. In the first two paragraphs the discusion is fairly general and we restrict to the case of a single pfaffian equation in the third paragraph in order to get the proof of Proposition 1. We will use freely the theory of semianalytic sets [1] and stratifications [15]. A stratification is said to be adapted to a set if this set is a union of strata.

The proofs being local we will suppose from now on that M is an open semianalytic subset of $\mathbf{R}^n$.

1. Strongly analytic submanifolds. A subset X of M is a strongly analytic submanifold of M if it is semianalytic in M and a submanifold of M. That is locally at each point *of* X , X is given by the level set of an analytic submersion and at each point *of* M, X is determined locally by a finite number of analytic equalities and inequalities.

Let U be an open semianalytic subset of M, a semianalytic subset X of M is *normal* in U if $X \cap U$ can be described by a finite number of functionally independent analytic functions defined on U. That is, there exist $n-k+1$ analytic functions on U, $f_0, f_1, \ldots, f_{n-k}$ such that

$$X \cap U = \{x \in U / f_0(x) > 0, f_1(x) = 0, \cdots, f_{n-k}(x) = 0 \text{ and}$$
$$df_1(x) \wedge df_2(x) \wedge \cdots \wedge df_{n-k}(x) \neq 0\}.$$

In this case, $X \cap U$ is a strongly analytic submanifold of dimension k.

We recall a fundamental result of the local theory of semianalytic sets.

Lojasiewicz's stratification theorem. *Let X be a semianalytic subset of M, there is a strongly analytic stratification of M adapted to X. Morover for each $x \in M$ there is a semianalytic open neighborhood U of x in M and a strongly analytic stratification of U adapted to $X \cap U$ such that each stratum is normal in U.*

The modern proof of this result is obtained as in Corollary 2.11 of [1] if we observe that a stratum described by all the elements of a separating familly in an open set U is normal in U.

This local decomposition in normal strata will be used to obtain properties of the map $x \mapsto T_xX$ for a strongly analytic submanifold X.

Let U be an open set of M and N a natural number, a subset X of $U \times \mathbf{R}^N$ is called relatively semialgebraic over U if it is of the form

$$X = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s} X_{i,j}$$

where each $X_{i,j}$ is either $\{(x,T) \in U \times \mathbf{R}^N / f_{i,j}(x,T) = 0\}$ or $\{(x,T) \in U \times \mathbf{R}^N / f_{i,j}(x,T) > 0\}$ for $f_{i,j}$ polynomial in T with coefficients analytic in U.

A recent result of Lojasiewicz asserts that if $X \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U the closure of X in $U \times \mathbf{R}^N$ is also relatively semialgebraic over U [12]. We have also

Tarski-Seidenberg-Lojasiewicz Theorem. *If $X \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U and $\pi : (x,T) \mapsto x$ is the natural projection, $\pi(X)$ is a semianalytic subset of U.*

Consider now the Grassmann manifold $G_p, p < n$ of p-dimensional vector subspaces of $\mathbf{R}^n$ naturally embedded as a smooth semialgebraic subset of some $\mathbf{R}^N$.

We will need later the following remark. If T is a p-plane, $T \in G_p$ and $0 \leq s \leq p$ the set of all the s-planes contained in T is an algebraic set in G_s. We can denote this set by $G_s(T)$. So if $X \subset U \times G_p \subset U \times \mathbf{R}^N$ is relatively semialgebraic over U the subset $\{(x, L) \in U \times G_s / \exists T \in G_p, L \subset T, (x,T) \in X\}$ denoted $G_s(X)$ is also relatively semialgebraic over U.

For each p-dimensional strongly analytic submanifold X of M, the map $\gamma_X : X \to M \times G_p : x \mapsto (x, T_x X)$ has been considered by Lojasiewicz[13] and Verdier [20] in the subanalytic setting.

Proposition 2. *For each $x \in \bar{X}$ there is a neighborhood U of x in M such that the image of $\gamma_{X \cap U}$ is a relatively semialgebraic set over U.*

Proof. Let be given a strongly analytic stratification $\mathcal{N}$ of a neighborhood U of x adapted to X so that each stratum is normal in U according to Lojasiewicz's stratification theorem. Let's suppose that S is a stratum of $\mathcal{N}$ of dimension $p = \dim X$ included in X. We have that if $y \in S$ $T_y S = T_y X$. It's easy to show that the map γ_S has a relatively semialgebraic image, as if the normal semianalytic subset S is described by the functionnally independent equations $f_1 = 0, \ldots, f_{n-p} = 0$ and $f_0 > 0$ the image of γ_S is

$$\{(y,T) \in U \times \mathbf{R}^N / df_1(y) \wedge T = 0, \ldots, df_{n-p} \wedge T = 0, f_0(y) > 0\}$$

considering naturally each element of G_p as an $n-p$-linear form.

As finite unions, intersections and closure of relatively semialgebraic sets over U are relatively semialgebraic over U. It's enough now to prove that the image of $\gamma_{X \cap U}$ is the union of the closures of the images of the γ_S for all the strata S of $\mathcal{N}$ of dimension p included in X. That is elementary as at each point of X one can take analytic local coordinates because X is a submanifold of M.

For later reference for each $\epsilon > 0$ let's fix in each dimension $p = 1, 2, \ldots, n$ a finite familly $\mathcal{H}_{p,\epsilon} = \{H_i^p(\epsilon), i = 1, \ldots l(p,\epsilon)\}$, $H_i^p(\epsilon) \in G_p$ such that the balls in $\mathbf{R}^N$ of center $H_i^p(\epsilon)$ and radius ϵ give an open covering of G_p. Denote finally $\mathcal{H}(\epsilon)$ the union of these families for every dimension. We will denote d_G the euclidean distance of $\mathbf{R}^N$ restricted to the Grassmann manifold.

2.Stratification adapted to a system of pfaffian equations. Let $\omega_1, \ldots, \omega_q$ be integrable analytic differential 1-forms on M. For each strongly analytic submanifold X of M we will obtain information on the solutions of the pfaffian system

$$\omega_1 = 0, \cdots, \omega_q = 0$$

restricted to X. This will be done by local stratification of M such that the kernel of the restricted pfaffian system is nearly parallel to a given vectorial space along each stratum.

Denote Ω the familly $\{\omega_i / i = 1, 2, \ldots, q\}$ and if

$$J : \{1, 2, \ldots, s\} \to \{1, 2, \ldots, q\}$$

is a map put $|J| = s$ and $\Omega_J = \{\omega_{J(j)} / j = 1, 2, \ldots, s\}$.

Let N be a strongly analytic submanifold of M, we will say that the familly Ω_J is transverse to N if

$$\dim(T_x N \cap \bigcap_{j=1}^{s} \ker \omega_{J(j)}(x)) = \dim N - s.$$

at every point $x \in N$. This is denoted by $\Omega_J \pitchfork N$.

The familly Ω_J is a basis of Ω along N if $\Omega_J \pitchfork N$ and

$$(*) \qquad T_x N \cap \bigcap_{j=1}^{s} \ker \omega_{J(j)}(x) = T_x N \cap \bigcap_{i=1}^{q} \ker \omega_i(x)$$

at every point $x \in N$.

In [16] we obtained for each semianalytic subset of M a local stratification adapted to it, such that a basis of Ω can be chosen along each stratum.

The following statement says that we can obtain a better precision. Such a stratification can be obtained so that along each stratum N the tangent space of the restricted foliation, $(*)$, is ϵ–parallel to a fixed vector space.

Proposition 3. *Let $\epsilon > 0$ be fixed. If X is a p-dimensional strongly analytic submanifold of M, $\Omega \pitchfork X$ and $x \in \bar{X}$ then there is a semianalytic subset Y of dimension at most $p-1$ an $\rho > 0$ such that if c is a connected component of $B_x(\rho) \cap (X \setminus Y)$ we can choose a $p-q$-plane $H_c = H^{p-q}_{i(c)}(\epsilon)$ so that the distance in the grasmaniann G_{p-q} from H_c to $\cap_{\omega \in \Omega} \ker \omega_{|X}(y)$ is less than ϵ for every $y \in c$.*

Corollary. *Let Ω and $\epsilon > 0$ be given. For each semianalytic subset X of M and $x \in \bar{X}$ there is a strongly analytic stratification of a neighborhood of x, $\mathcal{N}$ adapted to X such that we can choose for each stratum N of $\mathcal{N}$ a map J_N from a natural interval $\{1, 2, \ldots, s(N)\}$ to $\{1, 2, \ldots, q\}$ and a vector space $H_N = H^{\dim N - s(N)}_{i(N)}(\epsilon)$ of the familly $\mathcal{H}(\epsilon)$ verifying:*

1)Ω_{J_N} is a basis of Ω along N;

2)$d_G(H_N, \cap_{j=1}^{s(N)} \ker \omega_{J(j)|_N}) < \epsilon$.

The proof of the corollary is a straightforward consequence of the proof of proposition 1 in [16], applying Proposition 3 at each step of the induction and will not be reproduced here.

Proof of Proposition 3. Let $\rho > 0$ be such that if $U = B_0(\rho)$ the image of $\gamma_{X\cap U}$ is relatively semialgebraic over U.

It's clear that the assumption that $\Omega \not\pitchfork X$ implies that if for $x \in X$ we denote by $\Omega(x) = \cap_{\omega\in\Omega}\ker \omega(x)$ the set $\{(x, L)/x \in X, L = \Omega(x)\}$ is a relatively semialgebraic subset of $U \times G_{n-q}$.

Next observe that if Q is any relatively semialgebraic set over U, $Q \subset U \times G_{p-q}$ then

$$\{x \in X \cap U \quad / \quad T_x X \cap \bigcap_{\omega\in\Omega} \ker \omega(x) \in Q\}$$

is a semianalytic subset of M. We will denote this set by $Q(X, \Omega)$. It suffices to write this set as the natural projection of the set $G_{p-q}(\gamma_{X\cap U}(X \cap U)) \cap G_{p-q}(\Omega(X \cap U)) \cap Q$. This later set being relatively semialgebraic according to Proposition 2. The assertion follows from the Tarski-Seidenberg-Lojasiewicz Theorem.

Let's construct Y and choose the $p-q$-planes by induction. Define $Y_0 = \emptyset$, $X_0 = \emptyset$ and $Q_0 = \emptyset$. Consider the first element H_1 of the familly $\mathcal{H}_{p-q,\epsilon}$. If Q_1' is the open ball in G_{p-q} with center H_1 and radius ϵ put $Q_1 = Q_1' \setminus Q_0$, the set $Q_1(X, \Omega)$ is a semianalytic subset of X. If X_1' is the interior of this set *in* X put $X_1 = X_1' \cup X_0$. For each connected component c of X_1 we know by the definition of $Q_1(X, \Omega)$ that a proper choice for H_c is clearly H_1. That is at each point of X_1 the intersections of the kernels of the forms in Ω restricted to X are ϵ-nearly parallel to the plane H_1.

To proceed with the next element of the familly $\mathcal{H}$ we can put if Y_1' is the complementary to X of the closure *in* X of $Q_1(X, \Omega)$, $Y_1 = Y_1' \cup Y_0$. So Y_1 is a semianalytic subset of X of codimension at least 1.

Suppose that the sets X_l and Y_l have been constructed in such a way to have: X_l is an open semianalytic subset of X, for each connected component c of which a $p-q$-plane H_c in the familly $\{H_j^{p-q}(\epsilon), j < l+1\}$ can be chosen verifying the claim of proposition 3; Y_l is a semianalytic subset of X of codimension at least 1, $Y_l \cap X_l = \emptyset$. It's clear that if we define Q_{l+1}' as the open ball in G_{p-q} with center H_{l+1} and radius ϵ and we put $Q_{l+1} = Q_{l+1}' \setminus Q_l$, the set $Q_{l+1}(X, \Omega)$ is a semianalytic subset of X. If X_{l+1}' is the interior of this set *in* X put $X_{l+1} = X_{l+1}' \cup X_l$. For each connected component c of X_{l+1} we know by the definition of $Q_{l+1}(X, \Omega)$ that a proper choice for H_c is clearly H_{l+1}.

It is now clear that $X_{l(p-q,\epsilon)}$ is a open and dense semianalytic subset of X and Y can be chosen to be it's complement to X. On each connected component of $X \setminus Y$ the choice of $p-q$-plane is done.

3.Graph decomposition of Rollian pfaffian hypersurfaces. In this paragraph we will state the consequences of Proposition 3 for pfaffian sets and with the help of a Theorem of Hardt we will deduce the decomposition into graphs of Rollian ph.

Proposition 4. *Let Ω and $\epsilon > 0$ be given as before. For each semianalytic subset X of M and $x \in \bar{X}$ there is a strongly analytic stratification of a neighborhood of x, $\mathcal{N}$ adapted to X such that we can choose for each stratum N of $\mathcal{N}$ a vector space in $\mathcal{H}(\epsilon)$, H_N such that: for any familly $\{V_i, \omega_i, M\}, i = 1, \ldots, q$ of Rollian pfaffian hypersurfaces the intersection $N \cap X \cap V_1 \cap \cdots \cap V_q$ if non void is a union of a finite number of analytic submanifolds of M : W_m. Morover the restriction to each of the W_m of the orthogonal projection onto H_N, π_N, is a local isomorphism with finite fiber and the norm of the derivative of a sufficiently small local section can be bounded by ϵ.*

This proposition is a direct consequence of the corollary of proposition 3 using the theorem on uniform finiteness.

Let W be an analytic submanifold of M. Suppose that there is an open set U of some vectorial subspace E of $\mathbf{R}^n$ such that W is the graph in $E \oplus E^{\perp}$ of a map φ from U to the orthogonal space of E, $E^{\perp}$. Then we say [7] that W is an ϵ-analytic piece if the norm of the derivative of φ is everywhere bounded by ϵ.

An interesting goal is to decompose pfaffian sets in ϵ-analytic pieces in a finite way. Proposition 4 approches this goal, but we cannot, at present time achieve a further decomposition of M to get pieces where the restriction of the orthogonal projection becomes one to one. The case of codimension 1 pfaffian sets can be treated by an specific argument which seems not to be general enough. We will need for this argument the following Theorem of Hardt as it's proved in [14]. Compare also [4], [18].

Let O be a real analytic manifold.

Hardt's theorem on stratification of maps. *Let $f : E \to O$ be a proper continuous subanalytic map from the subanalytic closed set E of M, and let $\mathcal{G}$ and $\mathcal{T}$ be locally finite families of subanalytic subsets of M and O respectively. There exist a subanalytic stratification $\mathcal{A}$ of M adapted to each element of $\mathcal{G}$ and E and a subanalytic stratification $\mathcal{B}$ of O adapted to each element of $\mathcal{T}$ such that if $\Gamma \in \mathcal{A}$ and $\Gamma \subset E$, we have $f(\Gamma) \in \mathcal{B}$ and the restriction*

$f_{|\Gamma} : \Gamma \to f(\Gamma)$ is analytically equivalent to the projection $f(\Gamma) \times \Delta \to f(\Gamma)$ where Δ is a simplex.

We are now ready to prove the decomposition in ϵ-analytic pieces announced in Proposition 1.

Let ω and ϵ be as in the statement of proposition 1. Let $\mathcal{N}$ be the stratification obtained in Proposition 4. We are going to use Hardt's theorem for each orthogonal projection $\pi_N, N \in \mathcal{N}$ if $\dim H_N = n-1$ and $\dim N = n$ forgetting the frontier condition for a stratification at strata where the intersection with any regular solution of $\omega = 0$ is either the whole stratum of dimension $n-1$ and so a semianalytic subset of M or is of dimension less than $n-1$.

Denote $\mathcal{G}$ the set of elements N of $\mathcal{N}$ such that $\dim H_N = n-1$ and $\dim N = n$. For each $N \in \mathcal{G}$ consider the stratification $\mathcal{A}$ trivialising π_N. Let's show that in each stratum Γ of this new stratification restricted to N any Rollian ph associated to ω is an ϵ-analytic piece. That is, if $\{V, \omega, M\}$ is a Rollian ph meeting Γ, $\pi_{\Gamma|_{V\cap\Gamma}}$ is necessarily one to one.

Suppose there exist two points x_1, x_2 in $V \cap \Gamma$ such that $\pi_N(x_1) = \pi_N(x_2)$. The fiber of π_N restricted to Γ being isomorphic to a simplex, is connected. It's an analytic path meeting the Rollian ph in two points. The fundamental hypothesis proves that this fiber is tangent somewhere inbetween, to the distribution $\omega = 0$. This is not possible as Γ is included in N and the distribution $\omega = 0$ is ϵ-parallel to the orthogonal to this fiber everywhere in N..

The number of elements of $\mathcal{G}$ being finite the proof of proposition 1 is complete.

III. The density of a Rollian leaf at a boundary point

In this short chapter we will give the proof of theorem 1. The whole plan of the proof and the arguments are taken from the beautiful paper [7] of C. Kurdyka and G. Raby. We need only to verify that Rollian ph don't behave more wildly than subanalytic sets. This proof uses the decomposition into ϵ-analytic pieces of Proposition 1 and seems not to work with the result obtained in Proposition 4 alone.

1. Limit values of an ϵ-analytic pfaffian piece. Consider a pfaffian set W verifying the conditions of pieces found in Proposition 4.

That is,

–W is an analytic submanifold of M;

–we can associate to W a vector subspace $H \subset \mathbf{R}^n$ and $\epsilon > 0$ such that the orthogonal projection $\pi_W : H \oplus H^\perp \to H$ is a local isomorphism when

restricted to W;
–for every point $w \in W, d_G(T_w W, H) < \epsilon$.

We will call such a pfaffian set, an ϵ-horizontal piece.

Let's denote by U_W the open set $\pi_W(W)$.

We can think of W as being the graph of some kind of multivalued analytic mapping φ_W. The first problem is to understand the behaviour of W when its argument goes to the boundary of U_W.

Following [7] we state

Lemma 1. *Let W be a pfaffian set which is an ϵ-horizontal piece. If $w \in \bar{W}$ then the tangent cone*

$$C_w(W) \subset \{(x, y) \in H \times H^\perp / ||y|| \leq \epsilon ||x||\}.$$

The proof is the same of that of [7] page 757. You need only to know that by the Curve selection lemma you can reach the point w from inside W following a C^1 curve γ_u chosen to have a given u of the tangent cone as limit direction at w.

As this curve is a pfaffian set it can be chosen short enough so that it has no double points. If γ_1 is the projection $\pi_W \circ \gamma_u$, γ_1 behaves as a curve in an ϵ-analytic piece over γ_1 and the argument in [7] applies.

Corollary. *If W is a pfaffian set which is an ϵ-horizontal relatively compact piece, for small enough ϵ. Then the fiber of the projection π_W restricted to $\bar{W}$ is a finite set.*

The uniform finiteness theorem shows that $\bar{W} \cap (\{x\} \times H^\perp)$ has a finite number of connected components for each x in H. If one of those fibers had an accumulation point z we could not have at that point

$$C_z(\bar{W} \cap (\{x\} \times H^\perp)) \subset C_z(W) \cap H^\perp \subset \{z\}.$$

As Lemma 1 proves.

For pfaffian ϵ-analytic pieces this shows that the underlying analytic map has a finite number of limit values at each point in the closure of its domain.

2.Density of an open subpfaffian set. The subpfaffian sets are defined here in a naïve way. The whole theory of this class is not, at present time, clear enough. Some further properties of these sets can be found in [2]. Precisely let be given a vector subspace H of $\mathbf{R}^n$ a subset Z of H is called subpfaffian in H if there is a relatively compact pfaffian set W in $\mathbf{R}^n$ such that if π is the orthogonal projection onto H we have $Z = \pi(W)$. If Y is a relatively compact subanalytic subset of $\mathbf{R}^n$, Y is a finite union of subanalytic sets Y_i linear projections of relatively compact semianalytic sets L_i of some $\mathbf{R}^{n_i}$: $\pi_i : \mathbf{R}^{n_i} \to \mathbf{R}^n$, $\quad Y_i = \pi_i(L_i)$. As $\pi_i^{-1}(W)$ is a a pfaffian set in $\mathbf{R}^{n_i}$, $W \cap Y_i = \pi_i(\pi_i^{-1}(W) \cap L_i)$ is a subpfaffian set. So $W \cap Y$ is a finite union of subpfaffian sets. It is so, also, for $\pi(W \cap Y)$.

The following property of our subpfaffian sets is not difficult.

Lemma 2. *If Z is a subpfaffian set in some $\mathbf{R}^k$ and if X is a semianalytic subset of $\mathbf{R}^k$ then $Z \cap X$ has a finite number of connected components.*

We can suppose that there is an l and a relatively compact pfaffian set $W \in \mathbf{R}^l$ such that if π is the first k coordinates projection $\mathbf{R}^l \to \mathbf{R}^k$, $\pi(W) = Z$. As $\pi^{-1}(X)$ is semianalytic in $\mathbf{R}^l$ the set $W \cap \pi^{-1}(X)$ has a finite number of connected components. In this particular case we have $\pi(W \cap \pi^{-1}(X)) = Z \cap X$. So $Z \cap X$ has a finite number of connected components.

This result implies, in particular, that if Z is a finite union of subpfaffian sets and X is a line segment the set $Z \cap X$ is a finite union of segments.

We will say that a set Z is *line-finite* if it has this property for any line segment X. The set $Z \cap X$ is so, necessarilly, semianalytic.

We state next a minor generalisation of an argument of [7] in a form most usefull.

Proposition 5. *An open line-finite subset Z of $\mathbf{R}^k$ has a density at every point $z \in \mathbf{R}^k$. Morover*

$$\Theta_k(Z, z) \leq vol_k(B_0(1) \cap C_z(Z)).$$

So open subpfaffian sets have densities everywhere.

We will not reproduce here Kurdyka-Raby's proof as it's one page long. We can remark that in their case the density of Z subanalytic is exactly the upper bound given here in proposition 5. In the subpfaffian case this fact is not known.

All the arguments are ready for the proof of Theorem 1.

3. Densities for Rollian leaves. We will give some details of the final argument of [7] in order to stress the difficulty in the general pfaffian set case. The following lemma is the point where the use of analytic pieces cannot be avoided.

lemma 3. *Let $\varphi : U \to \mathbf{R}^{n-k}$ be a locally ϵ-lipschitzian map defined on an open set $U \subset \mathbf{R}^k$. Let $a \in \bar{U}$ such that φ has a limit b at a. Then for any*

$r > 0$

$$vol_k(U' \cap B_a(\frac{r}{1+\epsilon})) \leq vol_k(\Gamma \cap B_{(a,b)}(r)) \leq (1+\epsilon)^k vol_k(U \cap B_a(r))$$

where Γ *is the graph of* φ *and* $U' = \{x \in U / ||\varphi(x) - b|| \leq \epsilon ||x - a||\}$.

The proof is not difficult if we consider the locally $1+\epsilon$-lipschitzian map $q : x \mapsto (x, \varphi(x))$ and the 1-lipschitzian projection $p : \mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^k$. Just observe that $\Gamma \cap B_{(a,b)}(r) \subset q(U \cap B_a(r))$ and

$$U' \cap B_a(\frac{r}{1+\epsilon}) \subset p(\Gamma \cap B_{(a,b)}(r)).$$

This is a good estimate, as it is shown in [7] that the set U' has the same density as U at point a.

You cannot get such an estimate if you dont have a map φ at hand. In the case of ϵ-horizontal pieces W we can obtain a similar estimate but the upper term must be multiplied by the maximun number of points in the fiber of the projection π_W restricted to W. All the information is lost in that way.

The proof can be concluded. Let $\{V, \mathcal{F}, M\}$ be like in Theorem 1 and $x \in \bar{V}$. Fix $\epsilon > 0$ small enough and apply Proposition 1. Let X be an open semianalytic neighborhood of x such that there is a finite collection of submanifolds $\{\Sigma_i\}$ of X of dimensions less or equal to $n-2$ such that $V \cap X \setminus \cup\Sigma_i$ is a finite union of $N(\epsilon)$ ϵ-analytic pieces : W_j^ϵ =graph(φ_j^ϵ) $j = 1, 2, \ldots, N(\epsilon)$. Denote U_j^ϵ the domain of φ_j^ϵ. It's a finite union of subpfaffian open sets in an $n-1$ plane of $\mathbf{R}^n$.

If $r > 0$ we have

$$vol_{n-1}(V \cap B_x(r)) = \sum_{j=1}^{N(\epsilon)} vol_{n-1}(W_j^\epsilon).$$

We can suppose, for short that $x = 0$, and for r small enough that only one limit value for each function φ_j^ϵ remains in the ball, namelly 0.

Applying the majorations of the last lemma, and the remark about the density of the domain $U_j'^\epsilon$ of φ_j^ϵ we get: if we denote $\lambda(\epsilon) = \sum_{j=1}^{N(\epsilon)} \Theta_{n-1}(U_j^\epsilon, 0)$,

$$\frac{\lambda(\epsilon)}{(1+\epsilon)^{n-1}} \leq \liminf_{r - 0} \frac{vol_{n-1}(V \cap B_0(r))}{r^{n-1}}$$

and

$$\limsup_{r \to 0} \frac{vol_{n-1}(V \cap B_0(r))}{r^{n-1}} \leq (1+\epsilon')^{n-1}\lambda(\epsilon').$$

For any two real positive ϵ and ϵ'.

That is $\lambda(\epsilon)(1+\epsilon)^{1-n} \leq \underline{\Theta}(V,0)$ and $\bar{\Theta}(V,0) \leq (1+\epsilon')^{n-1}\lambda(\epsilon')$.

Kurdyka and Raby conclude by showing that these two lateral densities coincide. That is so because λ is bounded as $\epsilon \to 0$ (just use above inequalities to write $(1+\epsilon)^{1-n}\lambda(\epsilon) \leq \lambda(1)(1+1)^{n-1} \leq \lambda(1)$.)

It's clear from this proof that the point at stake in controlling densities of pfaffian sets is that of decomposing them into graphs.

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