Astérisque

JEAN FRANÇOIS MATTEI
MARCEL NICOLAU

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Astérisque, tome 222 (1994), p. 285-302
<http://www.numdam.org/item?id=AST_1994__222__285_0>

EQUISINGULAR UNFOLDINGS OF FOLIATIONS BY CURVES

Jean François MATTEI and Marcel NICOLAU*

0. Introduction. Let F denote a holomorphic foliation by curves with isolated singularities on a complex surface M. The first author constructed in [M] a versal equisingular unfolding, parametrized by a smooth space of parameters K_i^{loc}, of the germ of the foliation F at one of its singular points q_i. The aim of this paper is to show the existence of a versal equisingular unfolding of the global foliation F when M is compact. In this case the parameter space K_e of the versal unfolding can be singular. The problem of finding conditions on F assuring the triviality of any unfolding has been considered by X. Gomez-Mont in [G-M].

An equisingular unfolding of F is an unfolding admiting a reduction of the singularities "with parameters". It is claimed in [M] that there is a one-to-one correspondence between equisingular unfoldings of F and locally trivial unfoldings (cf. Definition 1.6) of the reduction $\widetilde{F}$ of F preserving the divisor which comes from the singular points of F. So we are led to construct a versal locally trivial unfolding of a (possibly non saturated) foliation by curves. The construction of the versal space is carried out in the first two sections. The key point is the identification of locally trivial unfoldings with a certain type of deformations of the complex structure of the underlying manifold. Then we consider the relationship between the global versal space K_e and the local versal spaces K_i^{loc}. We show that under some cohomological assumptions K_e is smooth and naturally identified with the product $\prod K_i^{\text{loc}}$. Finally we apply the above results to show that any equisingular unfolding of a germ of algebraic foliation is still algebraic.

1. Locally trivial unfoldings of foliations by curves.

Let M be a n-dimensional compact complex manifold and let TM be its holomorphic tangent bundle. Given a holomorphic vector bundle E over M we denote by $\mathcal{O}(E)$ the sheaf of germs of holomorphic sections of E. In

*Partially supported by grant PB90-0686 from DGICYT.

particular $\mathcal{O}_M = \mathcal{O}(\mathbb{C})$ and $\Theta_M = \mathcal{O}(TM)$ are respectively the sheaves of germs of holomorphic functions and holomorphic vector fields on M.

By a (singular) holomorphic foliation on M we mean a locally free $\mathcal{O}_M$-submodule F of Θ_M which is closed under the Lie bracket of vector fields. The *singular locus* $S(F)$ of F is the analytic subset whose complementary $M - S(F)$ is the maximal open set on which F defines a foliation in the usual sense, i.e. without singularities. The foliation F is called *saturated* if one has

$$\Gamma(U, \Theta_M) \cap \Gamma(U - S(F), F) = \Gamma(U, F)$$

for any open subset $U \subset M$. It is well known that a foliation F is saturated if and only if $\text{codim} S(F) \geq 2$. In fact it follows from Hartogs' extension theorem that if there is given an analytic subset $\Sigma \subset M$ of codimension greater than one and a non singular foliation F' on $M - \Sigma$ then there is a uniquely defined saturated foliation F on M which coincides with F' on $M - \Sigma$. In particular $S(F) \subset \Sigma$.

A *foliation by curves* is a locally free subsheaf F of Θ_M of rank one. Therefore a foliation by curves is determined by a pair (L, κ) where L is a line bundle over M and $\kappa : L \to TM$ is a non identically zero bundle morphism. The bundle morphism κ induces an injective morphism of sheaves $\mathcal{O}(L) \to \Theta_M$ and we identify $\mathcal{O}(L)$ with its image in Θ_M. Then $S(F)$ is the set of those points $z \in M$ for which $\kappa : L_z \to T_z M$ is the zero map.

In an equivalent way a holomorphic foliation by curves can be defined by a collection of local holomorphic vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ such that $\mathcal{U} = \{U_i\}$ is an open cover of M and $\xi_j = u_{ji} \cdot \xi_i$ on $U_i \cap U_j$ for suitable non vanishing holomorphic functions u_{ji}. Then L is the line bundle associated to the 1-cocycle $\{u_{ji}\}$. Moreover, if $\sigma_i : U_i \to L$ are non vanishing sections with $\sigma_j = u_{ji} \cdot \sigma_i$ then $\kappa : L \to TM$ is the bundle morphism determined by the condition $\kappa \circ \sigma_i = \xi_i$. The singular locus $S(F)$ is just the union $\bigcup \text{Sing}(\xi_i)$ where $\text{Sing}(\xi_i)$ is the subset of U_i where ξ_i vanishes.

To any foliation F there is naturally associated a saturated foliation ${}^sF$ which coincides with F outside $S(F)$. In the case of a foliation by curves ${}^sF$ can be described as follows. Assume that F is defined by local vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ where each U_i is holomorphically equivalent to an open polidisc Δ of $\mathbb{C}^n$. Let $z^1, \ldots, z^n$ be the coordinates on U_i induced by the identification $U_i \equiv \Delta$ and set $\xi_i = \sum \xi_i^\alpha \partial / \partial z^\alpha$. Let v_i be a m.c.d. of the functions $\xi_i^1, \ldots, \xi_i^n$ and define $\hat{\xi}_i = \xi_i / v_i$. Then $\text{codim}(\text{Sing}(\hat{\xi}_i)) \geq 2$. Since the foliation on $U_i \cap U_j$ defined by $\hat{\xi}_j$ is saturated there is a holomorphic function $\hat{u}_{ij}$ such that $\hat{\xi}_i = \hat{u}_{ij} \cdot \hat{\xi}_j$ on $U_i \cap U_j$. Furthermore the functions $\hat{u}_{ij}$ do not vanish. If not $\text{Sing}(\hat{\xi}_j)$ would be of codimension one. Hence the local vector fields $\hat{\xi}_i$ define a saturated foliation ${}^sF$ called the *saturation* of F.

1.1. Example. Let a saturated foliation F and an analytic hypersurface D on M be given. One can find an open cover $\{U_i\}$ of M with the property that there exist $\xi_i \in \Gamma(U_i, \Theta_M)$ and $h_i \in \Gamma(U_i, \mathcal{O}_M)$ such that the collection $\{\xi_i\}$ defines F and $h_i = 0$ defines D on U_i (i.e. if $f \in \Gamma(W, \mathcal{O}_M)$ vanishes on D then $f = \lambda \cdot h_i$ on $W \cap U_i$). Set $\eta_i = h_i \cdot \xi_i$. The collection of vector fields $\{\eta_i\}$ defines a non saturated foliation by curves F^D with singular locus $S(F^D) = D \cup S(F)$ and whose saturation is just F. With more generality and for any given positive integer $k \in \mathbb{N}^*$ one can define the foliation $F^{k \cdot D}$ as the foliation by curves defined by the local vector fields $\eta_i^{(k)} = (h_i)^k \cdot \xi_i$. In section 3 we will consider locally trivial unfoldings of foliations by curves obtained in this way.

From now on F will be a fixed foliation by curves on a compact manifold M defined by a pair (L, κ). Let Ω be an open neighbourhood of 0 in $\mathbb{C}^m$. The product $\Omega \times M$ is endowed with a holomorphic foliation F_Ω of the same codimension as F obtained as the product of F by the foliation on Ω consisting of a single leaf; i.e. F_Ω is the foliation defined by the subsheaf $\mathrm{pr}_1^*\Theta_\Omega \oplus \mathrm{pr}_2^*F$ of $\Theta_{\Omega \times M}$. Here pr_1 and pr_2 denote respectively the natural projections from $\Omega \times M$ onto the first and second factors. We call F_Ω the *trivial unfolding* of F parametrized by Ω.

1.2. Definition. *Let W be an open subset of $\Omega \times M$ and let $\Psi : W \to \Psi(W) \subset \Omega \times M$ be an $\mathbb{C}$-analytic diffeomorphism over the identity of Ω such that the restriction of Ψ to $W \cap (\{0\} \times M)$ is the identity. We say that Ψ is a relative automorphism of the trivial unfolding F_Ω if there is a bundle morphism $\Psi^\sharp : (T\Omega \times L)|W \to (T\Omega \times L)|\Psi(W)$ over Ψ such that the diagram*

$$\begin{array}{ccc} (T\Omega \times L)|W & \xrightarrow{id \times \kappa} & (T\Omega \times TM)|W \\ \Psi^\sharp \downarrow & & \downarrow \Psi_* \\ (T\Omega \times L)|\Psi(W) & \xrightarrow{id \times \kappa} & (T\Omega \times TM)|\Psi(W) \end{array}$$

is commutative, where Ψ_* denotes the tangent map of Ψ.

1.3. Remark. Let Ψ be a local biholomorphism over the identity of Ω and inducing the identity on $\{0\} \times M$. Suppose that Ψ maps the singular locus $S(F_\Omega)$ of F_Ω identically into itself and preserves the foliation F_Ω outside $S(F_\Omega)$. If the foliation F is saturated then Ψ_* induces a bundle morphism $\Psi^\sharp$ fulfiling the conditions required in the above definition. This is no longer true in general if the foliation is not saturated and in this case Ψ need not to be an automorphism of the trivial unfolding. Nevertheless there is a particular type of non saturated foliations for which this property still holds. It is considered in Proposition 1.5.

Assume $\Psi : W \to \Psi(W)$ is a holomorphic diffeomorphism such that W, $\Psi(W)$ are subsets of $\Omega \times U$ where U is a coordinate open subset of M, with coordinates $z = (z^1, \ldots, z^n)$, on which F is defined by a holomorphic vector field ξ. Let $s = (s^1, \ldots, s^m)$ denote the linear coordinates on Ω. Then Ψ is a relative automorphism of F_Ω if and only if it is of the form $\Psi(s,z) = (s, \psi(s,z))$ where ψ is a holomorphic map such that $\psi(0,z) = z$ and fulfiling

$$\psi_* \xi = (u \circ \Psi^{-1}) \cdot \xi, \tag{1}$$

$$\psi_* \frac{\partial}{\partial s^\mu} = (v_\mu \circ \Psi^{-1}) \cdot \xi \qquad \text{for } \mu = 1, \ldots, m, \tag{2}$$

for suitable functions u, v_μ on W. For example, if $\varphi = \varphi(t,z)$ denotes the local flow of ξ and $\sigma = \sigma(s,z)$ is a holomorphic function with $\sigma(0,z) = 0$ then $\Psi(s,z) = (s, \varphi(\sigma(s,z), z))$ is a relative automorphism of the trivial unfolding. The following proposition states that any relative automorphism of F_Ω is locally of this form.

1.4. Proposition. *Let $\Psi(s,z) = (s, \psi(s,z))$ be a relative automorphism of F_Ω with domain $W \subset \Omega \times M$. Assume $U = W \cap (\{0\} \times M)$ is an open subset of M holomorphically equivalent to an open polidisc on which F is defined by a holomorphic vector field ξ. Let $\varphi = \varphi(t,z)$ be the local flow associated to ξ. Then there is a holomorphic function $\sigma = \sigma(s,z)$ defined in a neighbourhood W' of U in W with $\sigma(0,z) = 0$ and such that $\psi(s,z) = \varphi(\sigma(s,z), z)$ on W'.*

Proof. Because of (2) a function $\sigma = \sigma(s,z)$ fulfils the required conditions if and only if σ is a solution of the total differential equation (with parameter $z \in U$)

$$\begin{cases} \dfrac{\partial \sigma}{\partial s^\mu}(s,z) = v_\mu(s,z) \\ \sigma(0,z) = 0. \end{cases} \tag{3}$$

The existence of such a function σ outside the singular locus $S(F_\Omega)$ of F_Ω can be easily seen by using local coordinates $w^1, \ldots, w^n$ such that $\xi = \partial/\partial w^n$. In particular the integrability condition of the equation (3), i.e.

$$\frac{\partial v_\mu}{\partial s^\nu} = \frac{\partial v_\nu}{\partial s^\mu} \qquad \text{for } \mu, \nu = 1, \ldots, m, \tag{4}$$

is fulfiled outside $S(F_\Omega)$. But equalities (4) must then also be verified on the singular set by continuity. So the complex Frobenius theorem with holomorphic parameters applies showing that (3) has a unique solution defined in a neighbourhood of $\{0\} \times U$ in W. ∎

Let $F^{k \cdot D}$ denote a foliation of the type defined in Example 1.1. A relative automorphism of $(F^{k \cdot D})_\Omega$ is also a relative automorphism of F_Ω. The converse is not true in general. In the following proposition we consider a particular situation in which the converse still holds. It will be used in section 3.

1.5. Proposition. *Assume F is saturated and let D be a hypersurface of M. Let Ψ be a given relative automorphism of F_Ω. If Ψ is the identity on $\Omega \times D$ then it is also a relative automorphism of the trivial unfolding $(F^D)_\Omega$.*

Proof. Using the notation in the above proposition we can write Ψ locally as $\Psi = \varphi(\sigma(s,z),z)$ for a given function σ. Suppose D is defined by the equation $h(z) = 0$, set $\hat{\xi} = h \cdot \xi$ and denote by $\hat{\varphi} = \hat{\varphi}(t,z)$ the local flow of $\hat{\xi}$. Then $\hat{\varphi}(t,z) = \hat{\varphi}(a(t,z),z)$ where $a = a(t,z)$ is the holomorphic function determined by the equation

$$\begin{cases} \dfrac{\partial a}{\partial t}(t,z) = h(\hat{\varphi}(t,z)) \\ a(0,z) = 0. \end{cases}$$

Since $z = \hat{\varphi}(t,z)$ for any $z \in D$ we have $a(t,z) = 0$ for $z \in D - S(F)$. The foliation F being saturated $D - S(F)$ is dense in D. So $a(t,\cdot)$ must vanish on D. From this fact we deduce that $a(t,z) = h(z) \cdot b(t,z)$ with $b(t,z) = t + t^2 b_1(t,z)$.

The maps $\Psi(s,)$ are also the identity on D by hypothesis. The above argument also shows that $\sigma(s,z) = h(z) \cdot \tau(s,z)$. The implicit function theorem implies the existence of a function $\hat{\sigma} = \hat{\sigma}(s,z)$ defined for small s such that $\tau(s,z) = b(\hat{\sigma}(s,z),z)$. So we can write $\psi(s,z) = \hat{\varphi}(\hat{\sigma}(s,z),z)$ showing that Ψ is a relative automorphism of $(F^D)_\Omega$. ∎

Let $U \subset M$ be open. Two relative automorphisms Ψ and Ψ' of F_Ω whose domains contain $\{0\} \times U$ are identified if they coincide in a neighbourhood of $\{0\} \times U$ in $\Omega \times M$. The set $\mathcal{G}_\Omega(U)$ of these equivalence classes is a group under the composition of automorphisms. When U runs over the open subsets of M the family $\{\mathcal{G}_\Omega(U)\}$ defines a sheaf $\mathcal{G}_\Omega$ of non abelian groups over M.

In order to obtain a versality theorem for locally trivial unfoldings we are led to consider (germs of) analytic spaces as spaces of parameters. So the general space of parameters we will be the germ $(S,0)$ at $0 \in \mathbb{C}^m$ of a (possibly non reduced) analytic space S defined by $S = \mathrm{supp}(\mathcal{O}_\Omega/\mathcal{I})$ where $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{O}_\Omega$. The restriction F_S of F_Ω to $S \times M$ will be called the *trivial unfolding of F parametrized by S*. The restrictions to $S \times M$ of the elements of $\mathcal{G}_\Omega$ form a sheaf of non abelian groups over M denoted by $\mathcal{G}_S$.

1.6. Definition. *A (germ of) locally trivial unfolding $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ (sometimes simply denoted by $\mathcal{F}$) of F parametrized by $(S,0)$ is given by an analytic space $\mathcal{M}$, a proper $\mathbb{C}$-analytic morphism $\pi : \mathcal{M} \to S$ and a holomorphic isomorphism $\iota : M \to M_0 := \pi^{-1}(0)$ in such a way that there exists an atlas $\{(W_i, \phi_i)\}$ of $\mathcal{M}$, where $\phi_i : W_i \to \phi(W_i) \subset S \times M$ are $\mathbb{C}$-analytic diffeomorphisms, fulfiling*

(i) $pr_1 \circ \phi_i = \pi$,

(ii) $\phi_i|M_0 = \iota^{-1}$,
(iii) $\Psi_{ij} = \phi_i \circ \phi_j^{-1}$ *is a section of* $\mathcal{G}_S$.

For s close to zero $M_s := \pi^{-1}(s)$ is a compact complex manifold and we can think of $\pi : \mathcal{M} \to S$ as a family of deformations of the complex manifold $M \cong M_0$. The restriction F_s of $\mathcal{F}$ to M_s is a foliation by curves locally isomorphic to $F = \iota^* F_0$.

1.7. Remark. Let $\mathcal{M}$ be a complex manifold, $\pi : \mathcal{M} \to \Omega$ a proper holomorphic map and $\iota : M \to M_0 := \pi^{-1}(0)$ a biholomorphism. Let a saturated foliation $\mathcal{F}$ on $\mathcal{M}$ which is transverse to the fibres of π (outside $S(\mathcal{F})$) be given. Then ι induces a well defined saturated foliation $\iota^*\mathcal{F}$ on M. Therefore in the case of saturated foliations one can define the (general) notion of *unfolding* of F parametrized by the smooth space Ω as a 5-tuple $(\mathcal{F}, \mathcal{M}, \pi, \Omega, \iota)$ where the foliation $\mathcal{F}$ is saturated and transverse to the fibres of π and such that $\iota^*\mathcal{F} = F$.

Two locally trivial unfoldings $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ and $(\mathcal{F}', \mathcal{M}', \pi', S, \iota')$ of F parametrized by $(S, 0)$ and defined respectively by the atlas $\{(W_i, \phi_i)\}$ and $\{(W'_k, \phi'_k)\}$ are said to be isomorphic if there is a $\mathbb{C}$-analytic isomorphism over the identity of S, $\Phi : \mathcal{M} \to \mathcal{M}'$, such that: (i) $\Phi \circ \iota = \iota'$ and (ii) $\phi'_k \circ \Phi \circ \phi_i^{-1}$ are sections of $\mathcal{G}_S$.

The underlying analytic space $\mathcal{M}$ of a locally trivial unfolding $\mathcal{F}$ is obtained by glueing together open subsets of $S \times M$ by means of relative automorphisms $\Psi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{G}_S)$. Assume that ξ_i are local vector fields defining F on U_i and let $\varphi_i = \varphi_i(t, z)$ denote the corresponding local flows. By virtue of Proposition 1.4, $\Psi_{ij} = (\mathrm{pr}_1, \psi_{ij})$ where ψ_{ij} is of the form

$$\psi_{ij}(s, z) = \varphi_i(\sigma_{ij}(s, z), z)$$

for suitable functions σ_{ij}. Given $\partial/\partial s \in T_0 S$, let $\varrho_{\mathcal{F}}(\partial/\partial s)$ denote the cohomology class of the cocycle associating to $U_i \cap U_j$ the section of $\mathcal{O}(L)$ given by

$$\theta_{ij} = \left.\frac{\partial \psi_{ij}}{\partial s}\right|_{s=0} = \left(\left.\frac{\partial \sigma_{ij}}{\partial s}\right|_{s=0}\right) \cdot \xi_i.$$

The $\mathbb{C}$-linear map $\varrho_{\mathcal{F}} : T_0 S \to H^1(M, \mathcal{O}(L))$ defined in this way only depends on the isomorphism class of the unfolding and is called the *Kodaira-Spencer map* of $\mathcal{F}$.

Given a morphism of germs of analytic spaces $f : (T, 0) \to (S, 0)$ the fibered product $T \times_S \mathcal{M}$ is constructed by means of sections $\Psi^f_{ij} = (\mathrm{pr}_1, \psi^f_{ij})$ of $\mathcal{G}_T$ where $\psi^f_{ij} = \psi_{ij} \circ (f \times 1)$. In this way $T \times_S \mathcal{M}$ is endowed with a locally trivial unfolding $f^*\mathcal{F}$ of F parametrized by $(T, 0)$ called the pull-back of $\mathcal{F}$

by f. A locally trivial unfolding $(\mathcal{F},\mathcal{M},\pi,S,\iota)$ of F is called ***versal*** if for any other locally trivial unfolding $(\mathcal{F}',\mathcal{M}',\pi',S',\iota')$ of F there is a morphism of germs of analytic spaces $f:(S',0)\to(S,0)$ such that: (i) $\mathcal{F}'$ and $f^*\mathcal{F}$ are isomorphic and (ii) $d_0 f$ is unique. In this case the Kodaira-Spencer map of $\mathcal{F}$ is an isomorphism. If the morphism f itself and not only its linear part is unique then the unfolding $\mathcal{F}$ is called *universal*.

We end this paragraph by showing that any locally trivial unfolding is globally differentiably trivial. This is a consequence of the existence of the "exponential map" stated in the proposition below. This exponential map is also used in the proof of the versality theorem 2.5. Let $(\mathcal{F},\mathcal{M},\pi,S,\iota)$ be a locally trivial unfolding of F defined by an atlas $\{(W_i,\phi_i)\}$ of $\mathcal{M}$ as in Definition 1.6. Assume that F is defined on $U_i=\iota^{-1}(W_i\cap M_0)$ by a local vector field ξ_i and let $\varphi_i=\varphi_i(t,z)$ be the associated flow. Let us consider $S\times L$ as a vector bundle over $S\times M$ and let $\gamma: S\times M\to S\times L$ denote the zero section. In this situation we have

1.8. Proposition. *By shrinking S if necessary we can find an open neighbourhood $\mathcal{V}$ of $\gamma(S\times M)$ in $S\times L$ and a C^∞ map $\tilde{g}_{\mathcal{F}}:\mathcal{V}\to\mathcal{M}$ over the identity of S and holomorphic with respect to S such that the restriction of $\phi_i\circ\tilde{g}_{\mathcal{F}}$ to $\mathcal{V}\cap p^{-1}(U_i)$ can be written in the form*

$$\tilde{g}_{\mathcal{F}}(s,z,t)=(s,\varphi_i(a_i(s,z,t),z)) \tag{5}$$

where t denotes the linear coordinate on $L|U_i$ induced by the choice of ξ_i and $a_i=a_i(s,z,t)$ is a C^∞ function depending holomorphically on s and t and fulfiling

$$a_i(0,z,0)=0 \tag{6}$$

$$\frac{\partial a_i}{\partial t}(0,z,0)=1. \tag{7}$$

Proof. A straightforward computation shows that the condition for the mapping $\tilde{g}_{\mathcal{F}}$ to admit a local writing like (5) does not depend on the choice of ξ_i nor on the choice of ϕ_i. Thus there is a globally defined sheaf $\mathcal{S}$ over $S\times L$ whose elements are the germs of such mappings. The restriction $\mathcal{S}_0$ of $\mathcal{S}$ to $M\equiv\gamma(\{0\}\times M)$ is locally isomorphic to the sheaf $\mathcal{S}_0'$ over $\{0\}\times\mathbb{C}^n\times\{0\}$ which is the restriction of the sheaf over $S\times\mathbb{C}^n\times\mathbb{C}$ whose elements are the germs of C^∞ functions $a=a(s,z,t)$ depending holomorphically on s and t and fulfuling conditions (6) and (7).

By means of partitions of unity one sees that $\mathcal{S}_0'$ is locally soft ("mou"). This implies that $\mathcal{S}_0$ is globally soft (cf. [G, th. 3.4.1]) and it admits a section on $\gamma(\{0\}\times M)$. Since $\gamma(\{0\}\times M)$ has a fundamental system of paracompact neighbourhoods in $S \times L$ the above section can be extended to a section of $\mathcal{S}$ on a neighbourhood $\mathcal{V}$ of $\gamma(\{0\} \times M)$ (cf. [G, th. 3.3.1]). Because of the compacity of M the domain $\mathcal{V}$ can be taken as a neighbourhood of $\gamma(S' \times M)$ in $S' \times L$ for a suitable neighbourhood S' of 0 in S. ∎

1.9. Corollary. *By shrinking S if necessary we can find a C^∞ diffeomorphism $g_{\mathcal{F}} : S \times M \to \mathcal{M}$ over the identity of S and holomorphic with respect to S which takes the leaves of the foliation F in $\{s\} \times M$ onto the leaves of F_s in M_s.*

Proof. Set $g_{\mathcal{F}} = \tilde{g}_{\mathcal{F}} \circ \gamma$. ∎

2. Locally trivial unfoldings of F as families of complex structures on M.

For a given holomorphic vector bundle E over M let $A^{0,k}(E)$ denote the space of differential forms on M of type $(0, k)$ with values in E, i.e. $A^{0,k}(E) = C^\infty(M, \Lambda^k \overline{TM}^* \otimes E)$. Set $\mathcal{T} = TM \oplus \overline{TM}$. We recall that a complex structure τ on M close to the original one is given by an involutive subbundle $T^{0,1}$ of $\mathcal{T}$ such that $\mathcal{T} = T^{1,0} \oplus T^{0,1}$ and $T^{0,1} \cap TM = 0$. Here $T^{1,0} = \overline{T^{0,1}}$. In this situation there is a unique element $\omega \in A^{0,1}(TM)$ such that $T^{0,1} = T^{0,1}_\omega := (\mathrm{id} - \omega)(\overline{TM})$. Involutiveness of $T^{0,1}_\omega$ is equivalent to the integrability of ω, i.e. to the condition $\bar{\partial}\omega - 1/2[\omega, \omega] = 0$. Thus complex structures on M close to the original one are parametrized by a neighbourhood of zero in the space J defined as

$$J = \{\omega \in A^{0,1}(TM) | \bar{\partial}\omega - \frac{1}{2}[\omega, \omega] = 0\}.$$

Given a family of deformations $\pi : \mathcal{M} \to S$ of the complex manifold $M \equiv M_0 = \pi^{-1}(0)$ and a differentiable trivialization $g : S \times M \to \mathcal{M}$ (i.e. a diffeomorphism over the identity of S, depending holomorphically on S and fulfiling $g|\{0\} \times M = \mathrm{id}$) there is a family of complex structures on M for which g is holomorphic. This family of complex structures is parametrized by a family $\{\omega_s\}$ of elements of J depending holomorphically on S and such that $\omega_0 = 0$. We will say that $\{\omega_s\}$ is the family of elements of J associated to $\mathcal{M}$ and the differentiable trivialization g. Conversely, given a holomorphic family $\{\omega_s\}$ of *real analytic* elements of J with $\omega_0 = 0$ there is a (uniquely determined) family of complex structures on M which is parametrized by $\{\omega_s\}$ (cf. [W, Prop. II.3.2] and [D]).

Given $\omega \in J$ close to zero and a diffeomorphism h of M which is C^1-close to the identity we will denote by $\psi = \omega \circ h$ the unique element of J such that $h : M_\psi \to M_\omega$ is holomorphic.

Let us consider now the pair (L, κ) defining the foliation by curves F on M. The bundle morphism κ induces injective $\mathbb{C}$-linear maps $\kappa : A^{0,k}(L) \to A^{0,k}(TM)$. We will identify $A^{0,k}(L)$ with its image in $A^{0,k}(TM)$. In this way the elements of $J_F = J \cap A^{0,1}(L)$ close to zero can be thought as a certain type of complex structures on M. In this paragraph we will see that locally trivial unfoldings of F are naturally parametrized by families of elements of J_F.

A diffeomorphism h of M which is C^1-close to the identity will be called *tangent to* F if, in any local chart $(U, z = (z^1, \dots, z^n))$ of M on which F is defined by a vector field ξ, it can be written

$$h(z) = \varphi(b(z), z) \tag{8}$$

where $b = b(z)$ is a certain differentiable function close to zero and $\varphi = \varphi(t, z)$ is the local flow associated to ξ.

2.1. Proposition. *Let $\omega \in J_F$ be close to zero and let h be a diffeomorphism of M tangent to F. Then $\psi = \omega \circ h$ belongs to J_F.*

Proof. The statement being purely local we can assume that M is just the domain of the local chart $(U, z = (z^1, \dots, z^n))$ in which h is given by (8). Clearly $\psi = \omega \circ h$ is integrable so we only have to see that ψ belongs to $A^{0,1}(L)$. In order to prove this we need the following identities

$$\bar\partial h^\beta = \xi^\beta \cdot \bar\partial b \qquad \text{for } \beta = 1, \dots, n, \tag{9}$$

$$h_*\xi = (1 + \xi(b)) \cdot \xi + \xi(\bar b) \cdot \xi. \tag{10}$$

The first one follows directely from the chain rule. The second equality can be easily checked on $U-\mathrm{Sing}(\xi)$ by taking local coordinates $\{w^1, \dots, w^n\}$ on which $\xi = \partial/\partial w^n$. Then (10) must also be verified on $\mathrm{Sing}(\xi)$ by continuity. From (10) one gets the identities

$$\xi(h^\beta) = (1 + \xi(b)) \cdot \xi^\beta, \tag{11}$$

$$\xi(\bar h^\beta) = \xi(\bar b) \cdot \bar\beta^\beta. \tag{12}$$

Let us write $\omega = \sum_\lambda \omega_\lambda d\bar z^\lambda \otimes \xi = \sum_{\beta,\lambda} \omega_\lambda \xi^\beta d\bar z^\lambda \otimes \partial/\partial z^\beta$ and let $\zeta^1, \dots, \zeta^n$ be ω-holomorphic local coordinates. They must verify $\bar\partial\zeta^\alpha = \omega(\zeta^\alpha)$, that is

$$\frac{\partial\zeta^\alpha}{\partial\bar z^\lambda} = \sum_\beta \omega_\lambda \xi^\beta \frac{\partial\zeta^\alpha}{\partial z^\beta}. \tag{13}$$

The vector 1-form ψ is determined by the condition that the functions $\zeta^\alpha \circ h$ are ψ-holomorphic, i.e.

$$\bar\partial(\zeta^\alpha \circ h) = \psi(\zeta^\alpha \circ h). \tag{14}$$

We want to prove that ψ can be written $\psi = \sum_\delta \psi_\delta d\bar{z}^\delta \otimes \xi$ where ψ_δ are C^∞ functions. If this were the case then equality (14) could be written, using (13),

$$(15)\quad \sum_\beta \frac{\partial \zeta^\alpha}{\partial h^\beta}(\bar{\partial} h^\beta + \sum_\lambda \omega_\lambda \xi^\beta \bar{\partial}\bar{h}^\lambda) = \sum_\beta \frac{\partial \zeta^\alpha}{\partial h^\beta}(\sum_\gamma \xi^\gamma \frac{\partial h^\beta}{\partial z^\gamma} + \sum_{\gamma,\lambda} \xi^\gamma \omega_\lambda \xi^\beta \frac{\partial \bar{h}^\lambda}{\partial z^\gamma}) \cdot \psi^o$$

where $\psi^o = \sum_\delta \psi_\delta d\bar{z}^\delta$. Since $\left(\frac{\partial \zeta^\alpha}{\partial h^\beta}\right)$ is an invertible matrix we see, making use of (9) and (11), that (15) is equivalent to

$$\xi^\beta(\bar{\partial} b + \sum_\lambda \omega_\lambda \bar{\partial}\bar{h}^\lambda) = \xi^\beta(1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda)) \cdot \psi^o.$$

But the expression

$$\bar{\partial} b + \sum_\lambda \omega_\lambda \bar{\partial}\bar{h}^\lambda = (1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda)) \cdot \psi^o$$

defines ψ^o as a differential form of type (0, 1) because the function $(1 + \xi(b) + \sum_\lambda \omega_\lambda \xi(\bar{h}^\lambda))$ does not vanish by hipothesis. Thus $\psi = \psi^o \otimes \xi$ is an element of J_F fulfiling (14). This concludes the proof. ∎

2.2. Remark. Let $\{\omega_s\}$ be a family of elements of J_F with $\omega_0 = 0$ and depending holomorphically on $s \in S$ and let h_s be a holomorphic family of diffeomorphisms of M tangent to F such that $h_0 = \text{id}$. Then the same argument shows that $\psi_s = \omega_s \circ h_s$ is a holomorphic family of elements of J_F.

Let $\tilde{g}_{\mathcal{F}}$ be the exponential map for the unfolding $\mathcal{F}$ constructed in Proposition 1.8. The restriction $g_{\mathcal{F}}$ of $\tilde{g}_{\mathcal{F}}$ to the zero section is a differentiable trivialization of $\mathcal{M}$. In this situation we have

2.3. Proposition. *The family $\{\omega_s\}$ of elements of J associated to $\mathcal{M}$ and $g_{\mathcal{F}}$ is in fact a family of elements of J_F.*

Proof. By construction the composition $\phi_i \circ g_{\mathcal{F}}$, where $\phi_i : W_i \to S \times M$ is a local chart of $\mathcal{M}$ defining the locally trivial unfolding $\mathcal{F}$, is a diffeomorphism tangent to F. The restriction of $\{\omega_s\}$ to $g_{\mathcal{F}}^{-1}(W_i)$ coincides with $0_s \circ (\phi_i \circ g_{\mathcal{F}})$ which, by virtue of Remark 2.2, is a family of integrable local sections of $\overline{TM}^* \otimes L$. Here 0_s denotes the family of J_F which is constantly equal to zero. ∎

2.4. Proposition. *Let $\{\omega_s\}$ be a given holomorphic family of real analytic elements of J_F with $\omega_0 = 0$. Then there exist a locally trivial unfolding $(\mathcal{F}, \mathcal{M}, \pi, S, \iota)$ of F and a real analytic trivialization $g : S \times M \to \mathcal{M}$ such that $\{\omega_s\}$ is the family of vector 1-forms induced by $\mathcal{M}$ and g.*

Proof. For a given point $x \in S \times M$ let $W'_x \subset S \times M$ be a coordinate neighbourhood of x in which F_S is defined by a vector field ξ with associated local flow $\varphi = \varphi(t, z)$. We will construct a structure of $\mathbb{C}$-analytic space on the topological space $S \times M$ by giving, for any point $x \in S \times M$, a local chart $\phi_x : W_x \to S \times M$, where $W_x \subset W'_x$ is a neighbourhood of x, with the properties: (i) ϕ_x is holomorphic when we consider on W_x the complex structure defined by $\{\omega_s\}$, and (ii) ϕ_x can be written in the form

$$\phi_x(s, z) = (s, \varphi(b(s, z), z))$$

where $b = b(s, z)$ is a real analytic function depending holomorphically on $s \in S$ such that $b(0, z) = 0$. Then the compositions $\phi_x \circ \phi_{x'}^{-1}$ will be sections of $\mathcal{G}_S$ and we will obtain an analytic space $\mathcal{M}$ endowed with a trivial unfolding $\mathcal{F}$ of F. The differentiable trivialization will then be given by the map which identifies $S \times M$ with the topological space which underlies $\mathcal{M}$.

Assume $x \in S \times M$ does not belong to the singular locus $S(F_S)$ of F_S. In a neighbourhood of x we can find local coordinates $(s, w^1, \ldots, w^n)$ with $\xi = \partial/\partial w^n$. Then $\varphi(t, w^1, \ldots, w^n) = (w^1, \ldots, w^n + t)$. Since the elements of $\{\omega_s\}$ are real analytic one can prove as in [W, Prop. II.3.2] that there is a real analytic function $\tilde{w}^n = \tilde{w}^n(s, w^1, \ldots, w^n)$ depending holomorphically on S with $\tilde{w}^n(0, w^1, \ldots, w^n) = w^n$ such that $(s, w^1, \ldots, w^{n-1}, \tilde{w}^n)$ is a system of local coordinates holomorphic with respect to $\{\omega_s\}$ on a certain neighbourhood of x. In this case we set

$$b(s, w^1, \ldots, w^n) = \tilde{w}^n(s, w^1, \ldots, w^n) - w^n$$

getting $\phi_x(s, w^1, \ldots, w^n) = (s, w^1, \ldots, w^{n-1}, \tilde{w}^n)$.

Let now $x \in S(F_S)$. On W'_x set $\xi = \sum_\beta \xi^\beta \partial/\partial z^\beta$ and $\omega_s = \sum_\lambda \omega_\lambda d\bar{z}^\lambda \otimes \xi$ where $\omega_\lambda = \omega_\lambda(s, z)$. Let us consider on $W'_x \times \mathbb{C}$ the non-singular vector field $\tilde{\xi} = \xi + \partial/\partial t$. The map $\tilde{\varphi} : W'_x \times \mathbb{C} \to W'_x \times \mathbb{C}$ given by $\tilde{\varphi}(s, z, t) = (s, \varphi(t, z), t)$ is a holomorphic diffeomorphism fulfiling $\tilde{\varphi}_*(\partial/\partial t) = \tilde{\xi}$. The family $\{\tilde{\omega}_s\}$ of real analytic vector 1-forms on $W'_x \times \mathbb{C}$ defined as

$$\tilde{\omega}_s = \sum_\lambda \omega_\lambda d\bar{z}^\lambda \otimes \tilde{\xi}$$

is holomorphic with respect to S and integrable. Let $\widetilde{\mathcal{W}'_x}$ and $\mathcal{W}'_x$ stand for $W'_x \times \mathbb{C}$ and W'_x endowed with the complex structure defined by $\{\tilde{\omega}_s\}$ and $\{\omega_s\}$

respectively. The canonical projection $\pi : \widetilde{\mathcal{W}'_x} \to \widetilde{\mathcal{W}'_x}$ is holomorphic. As in the non-singular case one can see that, if we consider on $W'_x \times \mathbb{C}$ the only complex structure for which $\tilde{\varphi} : W'_x \times \mathbb{C} \to \widetilde{\mathcal{W}'_x}$ is holomorphic, then there is a real analytic function $z^{n+1} = z^{n+1}(s, z^1, \ldots, z^n, t)$ holomorphic with respect to S such that $(s, z^1, \ldots, z^n, z^{n+1})$ is a system of holomorphic local coordinates. Let $b = b(s, z^1, \ldots, z^n)$ be the function defined by the condition

$$z^{n+1}(s, z^1, \ldots, z^n, b(s, z^1, \ldots, z^n)) = 0$$

and set $\psi(s, z^1, \ldots, z^n) = (s, z^1, \ldots, z^n, b(s, z^1, \ldots, z^n))$. Then the composition $\phi_x = \pi \circ \tilde{\varphi} \circ \psi$ fulfils the required conditions. ∎

The above propositions illustrate the way in what locally trivial unfoldings of F can be thought as a certain type of deformations of the complex structure of M. Namely those parametrized by holomorphic families of elements of J_F. Kuranishi's theorem (cf. [K]) states the existence of a versal space for the deformations of a complex structure on a compact manifold. The proof of this theorem given by Douady in [D] can be adapted here to obtain the following

2.5. Theorem. *Let F be a foliation by curves on a complex compact manifold M. There is a germ of analytic space $(K_{tr}, 0)$ parametrizing a versal locally trivial unfolding $\mathcal{F}$ of F.*

2.6. Remarks. (i) As in the proof of Kuranishi's theorem one can see that there is an open neighbourhood V of 0 in $H^1(M, \mathcal{O}(L))$ and an analytic map $\Phi : V \to H^2(M, \mathcal{O}(L))$ such that $(K_{tr}, 0)$ is identified with the germ at 0 of $\Phi^{-1}(0)$. Moreover the jet of order two of Φ at 0 is the quadratic map $v \to [v, v]$. This implies in particular that K_{tr} is smooth in the case $H^2(M, \mathcal{O}(L)) = 0$. Here the braket $[\ ,\ ]$ refers to the structure of graded Lie algebra on $H^*(M, \mathcal{O}(L))$ induced by the inclusion of sheaves $\mathcal{O}(L) \hookrightarrow \Theta_M$.

(ii) Through the proof of the theorem one sees that if $H^0(M, \mathcal{O}(L)) = 0$ then the unfolding $\mathcal{F}$ is universal.

Sketch of the proof. Let us fix a real analytic Hermitian metric on L and a real analytic Riemannian metric on M. From these metrics one constructs an Hermitian product on $A^{0,*}(L)$ in a standard way. The differential complex

$$A^{0,0}(L) \xrightarrow{\bar\partial} A^{0,1}(L) \xrightarrow{\bar\partial} A^{0,2}(L) \longrightarrow \ldots \longrightarrow A^{0,n}(L)$$

is elliptic and if we denote by ϑ the adjoint operator of $\bar\partial$ with respect to the above Hermitian product then the Laplacian $\Delta = \bar\partial\vartheta + \vartheta\bar\partial$ is a real analytic elliptic operator. The space

$$N = \{\omega \in A^{0,1}(L) \mid \vartheta(\bar\partial\omega - \frac{1}{2}[\omega, \omega]) + \bar\partial\vartheta\omega = 0\}$$

is a finite dimensional submanifold whose tangent space at 0 is the space $\mathbb{H}^1 \cong H^1(M, \mathcal{O}(L))$ of harmonic elements of $A^{0,1}(L)$. The elements of N are real analytic because they are solutions of an elliptic equation with real analytic coefficients.

Let $(K_{tr}, 0)$ be the germ at 0 of the intersection $K_{tr} = N \cap J_F$. The analytic space K_{tr} can also be defined as (cf. [K, p. 83])

$$K_{tr} = \{\omega \in N \mid H[\omega, \omega] = 0\}$$

where H denotes the orthogonal projection of $A^{0,2}(L)$ onto the space $\mathbb{H}^2$ of harmonic elements. A neighbourhood of 0 in K_{tr} can be thought as a holomorphic family of real analytic elements of J_F. Proposition 2.4 implies that this family defines a locally trivial unfolding $\mathcal{F}$ of F parametrized by the space K_{tr} itself.

Let $\tilde{g}_{\mathcal{F}}$ denote the exponential map for the unfolding $\mathcal{F}$ constructed in Proposition 1.8. Given $\eta \in A^{0,0}(L)$ close to zero then $h_\eta = \tilde{g}_{\mathcal{F}} \circ \eta$ is a diffeomorphism of M tangent to F. By virtue of Proposition 2.1 there is a map

$$\rho : A^{0,0}(L) \times K_{tr} \longrightarrow J_F$$

defined in a neighbourhood of $(0,0)$ such that $\rho(\eta, \omega) = \omega \circ h_\eta$. One can adapt here Douady's proof of Kuranishi's theorem to see that ρ is an isomorphism when restricted to $E \times K_{tr}$ where E is a subspace of $A^{0,0}(L)$ complementary to $\ker\{\bar{\partial} : A^{0,0}(L) \to A^{0,1}(L)\} \cong H^0(M, \mathcal{O}(L))$. From this fact and Proposition 2.3 it follows the versality of the unfolding $\mathcal{F}$. ∎

3. Equisingular unfoldings.

In this paragraph we assume that M has dimension 2 and that the foliation by curves F is saturated. In this case the singular locus $S(F)$ is a finite set $\{q_1, \ldots, q_k\} \subset M$. We recall that F is called *reduced* at a singular point q_i if there are local coordinates w^1, w^2 centered at q_i such that F is defined in a neighbourhood of q_i by a vector field with a 1-jet of the form

$$w^1 \frac{\partial}{\partial w^1} + \lambda w^2 \frac{\partial}{\partial w^2}$$

where $\lambda \in \mathbb{C}$ is not a strictly positive rational number.

There is a procedure of reduction of the singular foliation F similar to the reduction of singular plane curves (cf. [S]). More precisely there exist a compact manifold $\widetilde{M}$, a divisor $D \subset \widetilde{M}$, a saturated foliation $\widetilde{F}$ on $\widetilde{M}$ and a holomorphic map $\varpi : \widetilde{M} \to M$ with the following properties: (i) ϖ maps $\widetilde{M} - D$ biholomorphically onto $M - \{q_1, \ldots, q_k\}$ identifying $\widetilde{F}|\widetilde{M} - D$ with $F|M - \{q_1, \ldots, q_k\}$, (ii) the singularities of $\widetilde{F}$ are reduced and (iii) if

$x \in D - S(\widetilde{F})$ then the leaf of $\widetilde{F}$ through x is transverse to D or contained in D if x is a regular point of D and is a local irreducible component of D if x is a singular point of D. Furtheremore the 4-tuple $(\widetilde{F}, \widetilde{M}, D, \varpi)$ which is called the *reduction* of F is unique up to isomorphism. We denote by D_0 the union of all the irreducible components of D which are *dicritical*, i.e. those components which are generically transverse to $\widetilde{F}$.

An unfolding $(\mathcal{F}, \mathcal{M}, \pi, \Omega, \iota)$ of F parametrized by an open neighbourhood Ω of 0 in $\mathbb{C}^m$ (cf. Remark 1.7) is said to be *equisingular* if there exist a complex manifold $\widetilde{\mathcal{M}}$, a hypersurface $\mathcal{D} \subset \widetilde{\mathcal{M}}$, a saturated foliation $\widetilde{\mathcal{F}}$ on $\widetilde{\mathcal{M}}$ and a holomorphic map $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ such that: (i) Π maps $\widetilde{\mathcal{M}} - \mathcal{D}$ biholomorphically onto $\mathcal{M} - S(\mathcal{F})$ identifying $\widetilde{\mathcal{F}}|\widetilde{\mathcal{M}} - \mathcal{D}$ with $\mathcal{F}|\mathcal{M} - S(\mathcal{F})$, (ii) $\widetilde{\mathcal{F}}$ is transverse to the fibres of $\pi \circ \Pi$ outside its singular locus and (iii) the restriction of $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \mathcal{D}, \Pi)$ to $\pi^{-1}(0) = \iota(M)$ is the reduction of F_0. Then the 4-tuple $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \mathcal{D}, \Pi)$ is unique up to isomorphism and is called the reduction of $\mathcal{F}$. Because of the unicity of the reduction of a foliation there is a biholomorphism Υ from $\widetilde{M}$ onto $(\pi \circ \Pi)^{-1}(0)$ making commutative the diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\Upsilon} & \widetilde{\mathcal{M}} \\ \varpi \downarrow & & \downarrow \Pi \\ M & \xrightarrow{\iota} & \mathcal{M} \end{array}$$

and such that $\Upsilon^* \widetilde{\mathcal{F}} = \widetilde{F}$. We can thus think of the 5-tuple $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{M}}, \widetilde{\pi}, \Omega, \Upsilon)$ as an unfolding of the reduction $\widetilde{F}$ of F. Here $\widetilde{\pi} = \pi \circ \Pi$. As above $\mathcal{D}_0$ will denote the union of all the dicritical irreducible components of $\mathcal{D}$.

The first author proved in [M] that any unfolding of a germ of foliation at a reduced singularity is trivial. From this fact he deduces that for any equisingular unfolding $\mathcal{F}$ of F the corresponding reduction $\widetilde{\mathcal{F}}$ is a locally trivial unfolding of $\widetilde{F}$. Furthermore, if $\widetilde{\phi}_i : \widetilde{\mathcal{M}} \to \Omega \times \widetilde{M}$ are the local charts trivializing $\widetilde{\mathcal{F}}$ then the compositions $\widetilde{\phi}_j \circ \widetilde{\phi}_i^{-1}$ are the identity on $\Omega \times D_0$. This implies using Proposition 1.5 that $\widetilde{\mathcal{F}}^{\mathcal{D}_0}$ is in fact a locally trivial unfolding of $\widetilde{F}^{D_0}$. It is also shown in [M, Lemme 1.3.1] that, conversely, given a locally trivial unfolding $\widetilde{\mathcal{F}}'$ of $\widetilde{F}^{D_0}$ parametrized by Ω there is an equisingular unfolding $\mathcal{F}$ of F also parametrized by Ω such that its reduction $\widetilde{\mathcal{F}}$ is the saturation ${}^s(\widetilde{\mathcal{F}}')$ of $\widetilde{\mathcal{F}}'$. These facts suggest the following

3.1. Definition. *An equisingular unfolding of F parametrized by a germ of analytic space $(S,0)$ is a locally trivial unfolding $(\widetilde{\mathcal{F}}^{\mathcal{D}_0}, \widetilde{\mathcal{M}}, \widetilde{\pi}, S, \Upsilon)$ of $\widetilde{F}^{D_0}$ where $(\widetilde{F}, \widetilde{M}, D, \varpi)$ is the reduction of F and $\widetilde{F}^{D_0}$ has the meaning of example 1.1.*

Because of the above remarks in the case S is smooth the above definition of equisingular unfolding is equivalent to the first one. The notion of versal equisingular unfolding has an evident meaning and its existence is a corollary of Theorem 2.5.

3.2. Theorem. *Let F be a foliation by curves with isolated singularities on a compact surface M. There is a germ of analytic space $(K_e, 0)$ which parametrizes a versal equisingular unfolding of F.*

3.3. Remarks. (i) The space K_e is nothing but the parameter space $\widetilde{K}_{tr}$ of the versal locally trivial unfolding $(\widetilde{\mathcal{F}}^{\mathcal{D}_0}, \widetilde{\mathcal{M}}, \widetilde{\pi}, \widetilde{K}_{tr}, \Upsilon)$ of $\widetilde{F}^{D_0}$. So the tangent space of K_e at 0 is naturally identified with $H^1(\widetilde{M}, \mathcal{O}(\widetilde{L}))$ where $\widetilde{L}$ is the line bundle associated to the foliation by curves $\widetilde{F}^{D_0}$. Furthermore if $H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) = 0$ then K_e is smooth.

(ii) In the case K_e is smooth there is an equisingular unfolding in the first sense $(\mathcal{F}, \mathcal{M}, \pi, K_e, \iota)$ of F whose reduction is just ${}^s(\widetilde{\mathcal{F}}^{\mathcal{D}_0})$. Then any other equisingular unfolding of F parametrized by a smooth space is a pull-back of $\mathcal{F}$.

For a given germ of singular foliation by curves on $(\mathbb{C}^2, 0)$ there is a versal equisingular unfolding of it parametrized by a (germ of) smooth space. This local versal space is constructed by the first author in [M]. In our situation and using the above notation the parameter space K_i^{loc} of the versal equisingular unfolding of the germ of F at a singular point q_i turns out to be isomorphic to $H^1(D_i, \mathcal{O}(\widetilde{L}))$ where $D_i = \varpi^{-1}(q_i)$. The versality property of the local versal spaces K_i^{loc} induces a "localization" map

$$\chi : K_e \longrightarrow K_1^{\text{loc}} \times \ldots \times K_k^{\text{loc}}. \tag{16}$$

In certain cases the map χ is in fact a biholomorphism. More precisely

3.4. Theorem. *Let F be a foliation by curves with isolated singularities on a compact surface M defined by a pair (L, κ). Assume that $H^1(M, \mathcal{O}(L)) = 0$. Then the differential map $d_0\chi$ of χ at 0 is an isomorphism. Moreover if $H^2(M, \mathcal{O}(L))$ also vanishes then K_e is smooth and χ is an isomorphism.*

Proof. For each $i = 1, \ldots, k$ let B_i and B_i' be open neighbourhoods of q_i identified by a suitable local chart to the polydiscs $\Delta(1)$ and $\Delta(1/2)$ of $\mathbb{C}^2$ of polyradius 1 and 1/2 respectively. Assume that the open sets B_i are disjoint. Set $U = \cup B_i$, $V = M - \cup \overline{B}_i'$ and define $\widetilde{B}_i = \varpi^{-1}(B_i)$, $\widetilde{U} = \varpi^{-1}(U)$ and $\widetilde{V} = \varpi^{-1}(V)$. Notice that any section of $\widetilde{L}$ on $\widetilde{U} \cap \widetilde{V}$ can be extended because of Hartogs' theorem to a section on $\widetilde{U}$. Using this fact and the identifications induced by ϖ the Mayer-Vietoris sequence gives

$$0 \to H^1(\widetilde{M}, \mathcal{O}(\widetilde{L})) \xrightarrow{\alpha^1} H^1(\widetilde{U}, \mathcal{O}(\widetilde{L})) \oplus H^1(V, \mathcal{O}(L)) \xrightarrow{\beta^1} H^1(U \cap V, \mathcal{O}(L)) \to$$
$$\to H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) \xrightarrow{\alpha^2} H^2(\widetilde{U}, \mathcal{O}(\widetilde{L})) \oplus H^2(V, \mathcal{O}(L)) \xrightarrow{\beta^2} H^2(U \cap V, \mathcal{O}(L)) \to 0.$$

The intersection $U \cap V$ is the union of two Stein open subsets. Thus $H^2(U \cap V, \mathcal{O}(L)) = 0$. A theorem by Andreotti and Grauert [A,G] states that $H^*(\widetilde{U}, \mathcal{O}(\widetilde{L})) = \bigoplus_{i=1}^k H^*(\widetilde{B}_i, \mathcal{O}(\widetilde{L}))$ is isomorphic to $\bigoplus_{i=1}^k H^*(D_i, \mathcal{O}(\widetilde{L}))$ and also implies that the restriction of β^* to $H^*(\widetilde{U}, \mathcal{O}(\widetilde{L}))$ is the zero map. It is also proved in [M] that $H^2(D_i, \mathcal{O}(\widetilde{L})) = 0$. Using now the Mayer-Vietoris sequence which computes $H^*(M, \mathcal{O}(L))$ by means of the decomposition $M = U \cup V$ one can see that $H^1(M, \mathcal{O}(L)) = 0$ implies that the restriction of β^1 to $H^1(V, \mathcal{O}(L))$ is injective and therefore α^1 maps $H^1(\widetilde{M}, \mathcal{O}(\widetilde{L}))$ isomorphically onto $\bigoplus_{i=1}^k H^1(D_i, \mathcal{O}(\widetilde{L}))$ but this map is just the differential $d_0\chi$ of χ at 0. The same exact sequence also shows that in the case $H^2(M, \mathcal{O}(L)) = 0$ then the restriction of β^1 to $H^1(V, \mathcal{O}(L))$ is surjective and $H^2(V, \mathcal{O}(L)) = 0$. This implies $H^2(\widetilde{M}, \mathcal{O}(\widetilde{L})) = 0$ concluding the proof. ∎

Let us apply the above result to the case of a foliation by curves on $\mathbb{P}^2$. Recall that line bundles L_d on $\mathbb{P}^2$ are classified by its Chern class $d \in \mathbb{Z} \cong H^2(\mathbb{P}^2, \mathbb{Z})$ and that there are non identically zero morphisms $L_d \to T\mathbb{P}^2$ if and only if $d \leq 1$ (cf. [G-M,O-B]). It is also known from Serre's computations that $H^1(\mathbb{P}^2, \mathcal{O}(L)) = 0$ for any $d \in \mathbb{Z}$ and that $H^2(\mathbb{P}^2, \mathcal{O}(L)) = 0$ for $d > -3$. So we obtain

3.5. Corollary. *Let $F = (L_d, \kappa)$ be a foliation by curves on $\mathbb{P}^2$ with isolated singularities $q_1, \ldots, q_k$. Let K_e (resp. K_i^{loc}) denote the parameter space of the versal equisingular unfolding of F (resp. of the germ of F at q_i). Then the tangent map at 0 of the localization morphism $\chi : K_e \to K_1^{\text{loc}} \times \cdots \times K_k^{\text{loc}}$ is an isomorphism. The map χ itself is an isomorphism if $d = 1, 0, -1, -2$.*

3.6. Remarks. (i) As it is pointed out in [G-M,O-B] the foliations by curves on $\mathbb{P}^2$ having an associated line bundle L_d with Chern class $d = 1, 0, -1, -2$ are those obtained respectively by projectivization of the vector fields

$$X = X_1 \frac{\partial}{\partial z^1} + X_2 \frac{\partial}{\partial z^2} + X_3 \frac{\partial}{\partial z^3}$$

on $\mathbb{C}^3$ such that X_1, X_2 and X_3 are homogeneous polinomials of degree $0, 1, 2$ or 3.

(ii) The space K_e is not smooth in general as it is shown by the examples studied by I. Luengo in [L].

Finally we come back from global to local foliations and using the above methods we obtain

3.7. Theorem. *Let ξ be a polynomial vector field on $\mathbb{C}^2$ having an isolated singularity at the origin. Let F_ξ denote the germ of foliation by curves at 0 defined by ξ. Then any equisingular unfolding of F_ξ is algebraic. More precisely the versal equisingular unfolding of F_ξ is given by an integrable holomorphic differential form*

$$\omega = a(z,t)dz_1 + b(z,t)dz_2 + \sum_{i=1}^{p} c_i(z,t)dt_i$$

on an open subset $\mathbb{C}^2 \times U$ of $\mathbb{C}^2 \times \mathbb{C}^p$ where p is the dimension of the versal space and a, b, c_i are functions which are polynomials with respect to the coordinates $z = (z_1, z_2)$ of $\mathbb{C}^2$.

Proof. The foliation F_ξ extends to a foliation by curves F on $\mathbb{P}^2$ having only isolated singularities at $q_1 = 0, q_2, \ldots, q_k$. Let $(\widetilde{F}, \widetilde{M}, D, \varpi)$ denote the reduction of F. Let us chose a straight line C in $\mathbb{P}^2$ which do not meet any singular point of F. There is a positive number $m \in \mathbb{N}^*$ such that the line bundle associated to the foliation $F^{m \cdot C}$ (cf. Example 1.1) is L_0, i.e. its Chern class is zero. Set $\widetilde{C} = \varpi^{-1}(C)$. Then the argument used to prove the above theorem can be repeated to show that the parameter space K' of the versal locally trivial unfolding $(\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}, \widetilde{\mathcal{M}}, \widetilde{\pi}, K', \Upsilon)$ of $\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}$ is smooth and can be identified with the product $K_1^{\mathrm{loc}} \times \cdots \times K_k^{\mathrm{loc}}$. Moreover one can prove as in [M, Lemme 1.3.1] that there is an equisingular unfolding (in the first sense) $(\mathcal{F}, \mathcal{M}, \pi, K', \iota)$ of F whose reduction is just the saturation of $\widetilde{\mathcal{F}}^{m \cdot \widetilde{C}}$. Then $\mathcal{F}$ is algebraic and the germ at zero of the restriction of $\mathcal{F}$ to the subspace $K_1^{\mathrm{loc}} \times \{0\} \times \cdots \times \{0\}$ of K' is the versal equisingular unfolding of F_ξ. ∎

References

[A,G] A. Andreotti et H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. math. France, **90**, 193-259, 1962.

[D] A. Douady, *Le probléme des modules pour les variétés analytiques complexes*, Séminaire Bourbaqui, 17ᵉ année. No. **277**, 1964-65.

[G] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, **1964**.

[G-M] X. Gomez-Mont, *Unfoldings of holomorphic foliations*, Publicacions Matemàtiques U.A.B., **33**, 501-515, 1989.

[G-M,O-B] X. Gomez-Mont y L. Ortiz-Bobadilla, *Sistemas dinamicos holomorfos en superficies*, Aportaciones Matematicas, No. **3**, Soc. Matem. Mexicana, 1989.

[K] K. Kodaira, *Deformations of compact complex manifolds*, Montréal, 1971.

[L] I. Luengo, *On the existence of complete families of projective plane curves, which are obstructed*, J. London Math. Soc., **36**, 33-43, 1987.

[M] J.F. Mattei, *Modules de feuilletages holomorphes singuliers: I équisingularité*, Invent. Math., **103**, 297-325, 1991.

[S] A. Seidenberg, *Reduction of singularities of the differentiable equation* $AdY = BdX$, Amer. J. Math., **90**, 248-269, 1948.

[W] A. Weil, *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.

Jean François Mattei
Laboratoire de Topologie et Géométrie
URA au CNRS 1408
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse Cedex (France)

Marcel Nicolau
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Spain)