Astérisque

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Astérisque, tome 218 (1993), p. 177-186
<http://www.numdam.org/item?id=AST_1993__218__177_0>

SESHADRI CONSTANTS ON SMOOTH SURFACES

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Introduction

Let X be a smooth complex projective variety of dimension n, and let L be a numerically effective line bundle on X. Following Demailly [**De2**], one defines the *Seshadri constant* of L at a point $x \in X$ to be the real number

$$\epsilon(L,x) = \inf_{C \ni x} \frac{L \cdot C}{m_x(C)} \quad ,$$

where the infimum is taken over all irreducible curves C passing through x, and $m_x(C)$ is the multiplicity of C at x. It is profitable to view $\epsilon(L,x)$ as a local measure of how positive L is at x. For example if L is very ample, then $\epsilon(L,x) \geq 1$; on a surface X the same is true more generally if $L = \mathcal{O}_X(D)$ for an ample effective divisor $D \ni x$ which is smooth at x. In general, if $f : Bl_x(X) \longrightarrow X$ denotes the blowing up of X at x and $E = f^{-1}(x)$ is the exceptional divisor, then for $\epsilon > 0$ the $\mathbb{R}$-divisor $f^*L - \epsilon \cdot E$ is nef if and only if $\epsilon \leq \epsilon(L,x)$. (Consult [**De2**, §6] for other interpretations.) Similarly, one defines the global Seshadri constant

$$\epsilon(L) = \inf_{x \in X} \epsilon(L,x).$$

Thus Seshadri's criterion for ampleness states that $\epsilon(L) > 0$ if and only if L is ample.

Recent interest in Seshadri constants stems from the fact that they govern a simple method for producing sections of adjoint bundles $K_X + kL$ (c.f. [**De2**, (6.8)]). In brief, by means of vanishing theorems on the blow-up $Bl_x(X)$, a lower bound on $\epsilon(L,x)$ yields an explicit value of k such that $K_X + kL$ has a section which is non-zero at x (see (3.4) below). We shall see in §3 that Seshadri

*Partially supported by NSF Grant DMS 91-05183
**Partially supported by NSF Grant DMS 89-02551

constants alone cannot account for the known results on global generation and very ampleness of adjoint bundles ([**Rdr**], [**Del**], [**EL**]). However they remain very interesting in their own right as measures of local positivity. The subtlety of these invariants is reflected in the fact, pointed out by Demailly, that they are already rather difficult to compute on surfaces.

The purpose of this note is to study Seshadri constants in this first non-trivial case, when X is a smooth projective surface. One might anticipate that in general $\epsilon(L,x)$ could become small on fairly arbitrary algebraic subsets of X. Somewhat surprisingly, our main result shows that this is not the case:

THEOREM. *Let L be an ample line bundle on a smooth complex projective surface X. Then $\epsilon(L,x) \geq 1$ for all except perhaps countably many points $x \in X$, and moreover if $c_1(L)^2 > 1$, then the set of exceptional points is in fact finite. More generally, given an integer $e > 1$, suppose that*

$$c_1(L)^2 \geq 2e^2 - 2e + 1 \quad \textit{and} \quad c_1(L)\cdot\Gamma \geq e \ \textit{ for every irreducible curve } \Gamma \subset X.$$

Then $\epsilon(L,x) \geq e$ for all but finitely many $x \in X$.

On the other hand, simple examples (constructed by Miranda) show that $\epsilon(L,x)$ can take on arbitrarily small values at isolated points. We hope that this gives some sense of the kind of picture one might hope for in higher dimensions.

The proof of the theorem is completely elementary, the essential point being simply to view the question variationally. Specifically, suppose that L is an ample line bundle, and $C = C_0 \subset X$ is a curve with $m = m_x(C) > C \cdot L$ for some point $x = x_0 \in C$. By combining a simple computation in deformation theory (§1) with the Hodge index theorem, we show that (C,x) cannot move in a non-trivial one-parameter family (C_t, x_t) with $m_{x_t}(C_t) \geq m$ for all t. In other words, pairs (C,x) forcing $\epsilon(L,x) < 1$ are rigid, and the first statement of the Theorem follows at once. We were inspired in this argument by work of G. Xu [**Xu**], who uses related but much more elaborate calculations to study geometric genera of subvarieties of general hypersurfaces in projective space. We present some examples and open questions in §3.

We have benefitted from discussions with J. Kollár, W. Lang, R. Miranda, Y.-T. Siu, H. Tsuji, E. Viehweg, G. Xiao, and G. Xu.

§1. Deformations of Singular Curves on a Surface

This section is devoted to a proof, in the spirit of [**Xu**], of an elementary lemma concerning the deformation theory of singular curves on a surface. While the result in question is certainly well known in the folklore, we include an argument here for lack of a suitable reference and for the convenience of the reader.

We consider the following situation. X is a smooth complex projective surface, and we suppose given a one-parameter family

$$\{ C_t \ni x_t \}_{t \in \Delta}$$

consisting of curves $C_t \subset X$ plus a point $x_t \in C_t$, parametrized by a smooth curve or small disk Δ. Setting $C = C_0$ and $x = x_0$ for $0 \in \Delta$, the deformation determines a Kodaira-Spencer map

$$\rho : T_0\Delta \longrightarrow H^0(C, N),$$

where $N = \mathcal{O}_C(C)$ is the normal bundle to C in X.

LEMMA 1.1. *Assume that $m_{x_t}(C_t) \geq m$ for all $t \in \Delta$. Then $\rho(\frac{d}{dt}) \in H^0(C, N)$ vanishes to order $\geq (m-1)$ at x.*

REMARK. We say that a section $s \in H^0(C, N)$ vanishes to order $\geq k$ at a (possibly singular) point $y \in C$ if s is actually a section of the subsheaf $N \otimes \mathfrak{m}_y^k \subset N$, where $\mathfrak{m}_y$ is the maximal ideal sheaf of y.

PROOF OF LEMMA 1.1: We simply make an explicit computation. Specifically, the assertion is local on C and Δ, so we can assume that Δ is a small disk with coordinate t, and that C lies in an open subset U of $\mathbb{C}^2$ with coordinates (z, w), and $x = (0, 0)$. The total space $\mathcal{C} \subset U \times \Delta$ of the deformation is then defined by a power series $F(z, w, t) = f_t(z, w)$ where $C_t = \{f_t = 0\}$. We may suppose that $x_t = (a(t), b(t))$ for suitable power series $a(t), b(t)$. Then the curve defined by

$$\phi_t(z, w) =_{\text{def}} F(z + a(t), w + b(t), t)$$

has multiplicity $\geq m$ at $(0, 0)$ for all $t \in \Delta$. Expanding $\phi_t(z, w) = \sum \phi_i(z, w) t^i$ as a power series in t, it follows that $\phi_i \in (z, w)^m$ for all i. On the other hand,

$$\phi_1(z, w) = \frac{\partial f_0}{\partial z}(z, w) \cdot a'(0) + \frac{\partial f_0}{\partial w}(z, w) \cdot b'(0) + \frac{\partial F}{\partial t}(z, w, 0),$$

and since $\frac{\partial f_0}{\partial z}(z, w), \frac{\partial f_0}{\partial w}(z, w) \in (z, w)^{m-1}$, we find that

$$\frac{\partial F}{\partial t}(z, w, 0) \in (z, w)^{m-1}.$$

But $\frac{\partial F}{\partial t} | C$ is the local expression for $\rho(\frac{d}{dt}) \in H^0(C, N)$, and the lemma follows. ∎

COROLLARY 1.2. *In the situation of the Lemma, assume in addition that C is reduced and irreducible, and that the Kodaira-Spencer deformation class $\rho(\frac{d}{dt}) \in H^0(C, N)$ is non-zero. Then $C \cdot C \geq m(m-1)$.*

PROOF: This follows from the Lemma plus the fact that $c_1(N)$ represents $C \cdot C$. In more detail, let $f : Y \longrightarrow X$ be the blowing-up of X at x, with exceptional divisor $E \subset Y$. Then $f^*C = C' + kE$, where $C' \subset Y$ is the proper transform of C, and $k = m_x(C) \geq m$. Note that C' is the blowing-up of C at x. Put $s = \rho(\frac{d}{dt})$, so that $0 \neq s \in H^0(C, \mathfrak{m}_x^{m-1} \otimes \mathcal{O}_C(C))$. Then s induces a non-zero section

$$s' \in H^0(C', f^*(\mathcal{O}_C(C)) \otimes \mathcal{O}_Y((1-m)E)|_{C'}).$$

This implies that $\deg f^*(\mathcal{O}_C(C))|_{C'} \geq (m-1)E \cdot C' = k(m-1)$. It follows that

$$C \cdot C = \deg \mathcal{O}_C(C) = \deg f^*(\mathcal{O}_C(C))|_{C'} \geq k(m-1) \geq m(m-1),$$

as claimed. ∎

§2. Proof of the Theorem

We now give the proof of the theorem stated in the Introduction.

As in the statement, let L be an ample line bundle on the smooth surface X. Then there are only finitely many algebraic families of reduced irreducible (i.e. integral) curves on X of bounded degree with respect to L. Therefore for fixed $d > 0$ the set

$$S_d = \left\{ (C, x) \;\middle|\; x \in C \subset X \text{ an integral curve },\ m_x(C) > C \cdot L,\ C \cdot L \leq d \right\}$$

is parametrized by a finite union of irreducible quasi-projective varieties. Consequently

$$S = \left\{ (C, x) \;\middle|\; x \in C \subset X \text{ a reduced irreducible curve },\ m_x(C) > C \cdot L \right\}$$

consists of at most countably many algebraic families. The first statement of the theorem will follow if we prove that each of these families is discrete.

Suppose to the contrary that there exists a non-trivial continuous family $\{ (C_t, x_t) \}_{t \in \Delta}$ of reduced irreducible curves $C_t \subset X$, plus points $x_t \in C_t$, with

$$(*) \qquad m_t =_{\text{def}} mult_{x_t}(C_t) > C_t \cdot L \qquad \text{for all } t \in \Delta.$$

Without loss of generality we may assume here that Δ is a smooth irreducible curve (or a disk). Since each C_t is reduced, we have $m_y(C_t) = 1$ for all but finitely many $y \in C_t$. So it follows from (*) that the curves $\{C_t\}$ must themselves move in a non-trivial family. Hence for general $t^* \in \Delta$ the corresponding Kodaira-Spencer map

$$T_{t^*}\Delta \longrightarrow H^0(C_{t^*}, N_{C_{t^*}/X})$$

is non-zero. Let $C = C_{t^*}$ and $m = m_{t^*}$ for such a point $t^* \in \Delta$. Corollary 1.2 then implies that $C \cdot C \geq m(m-1)$. On the other hand, $(C^2)(L^2) \leq (C \cdot L)^2$ thanks to the Hodge index theorem, and since $C \cdot L \leq m - 1$ by assumption, we find:

$$m(m-1) \leq (C^2)(L^2) \leq (C \cdot L)^2 \leq (m-1)^2.$$

This is a contradiction when $m > 1$, which proves the first statement of the Theorem.

Suppose next that $L^2 \geq 2$. To prove the finiteness of the exceptional points, it is enough to show that $S = S_d$ for some d, i.e. that any reduced irreducible curve C with $m = m_x(C) > C \cdot L$ for some $x \in X$ has bounded L-degree. To this end observe first that there exists a large integer N with the property that for any point $y \in X$ there is a divisor $D_y \in |N \cdot L|$ with $m_y(D_y) \geq N$. Indeed, it follows from Riemann-Roch that for $n \gg 0$:

$$h^0(X, nL) \sim \frac{n^2 L^2}{2} \geq n^2,$$

whereas it is only $\binom{n+1}{2} \sim \frac{n^2}{2}$ conditions to impose an n-fold point at $y \in X$. Suppose now that C is a reduced irreducible curve with $m = m_x(C) > C \cdot L$ for some $x \in X$. Setting $D = D_x$, we claim next that C must appear as a component of D. In fact, if C were to meet D properly, then

$$m \cdot N \leq m_x(C) \cdot m_x(D) \leq C \cdot D = N(C \cdot L),$$

whence $m \leq C \cdot L$, a contradiction. But once we know that C appears as a component of $D_x \in |N \cdot L|$, we find that

$$C \cdot L \leq D_x \cdot L = N \cdot L^2,$$

which gives the required bound.

Finally, fix $e \geq 2$, and assume that $L^2 \geq 2e^2 - 2e + 1$ and that $\Gamma \cdot L \geq e$ for all curves $\Gamma \subset X$. Suppose that $C \subset X$ is an integral curve such that $m = m_x(C) > \frac{C \cdot L}{e}$. If (C, x) moves in a non-trivial family satisfying this same condition, then the lower bound on $C \cdot L$ shows that C itself must move. Then one argues as above that

$$(2e^2 - 2e + 1)m(m-1) \leq (L^2)(C^2) \leq (C \cdot L)^2 \leq (em-1)^2.$$

But we claim this is a contradiction when $m \geq 2$. In fact the function

$$f(m) = (2e^2 - 2e + 1)m(m-1) - (em-1)^2$$

is increasing for $m \geq 1$, and $f(2) > 0$. Hence pairs (C, x) with $m_x(C) > \frac{C \cdot L}{e}$ are rigid. The finiteness of the exceptional points is similarly proved much as before.

This completes the proof of the Theorem.

§3. Complements, Examples and Open Problems

We collect in this section some applications, examples and open questions.

We begin with an example, given by Miranda, to show that $\epsilon(L, x)$ can take on arbitrarily small values at isolated points. Miranda's construction improves and simplifies a more cumbersome example we had produced where $\epsilon(L, x) \leq \frac{1}{2}$.

EXAMPLE 3.1. Let $D \subset \mathbf{P}^2$ be an irreducible plane curve of degree d with a point $x \in D$ of multiplicity m. Let D' be a second irreducible curve of degree d, meeting D transversely. Choosing D' generally, we may suppose that all the curves in the pencil spanned by D and D' are irreducible. Blow up the base-points of the pencil to obtain a surface X, admitting a map $f : X \longrightarrow \mathbf{P}^1$ with irreducible fibres, among them $D \subset X$. Observe that f has a section $S \subset X$ meeting D transversely at one point. Fix an integer $a \geq 2$. It follows from the Nakai criterion that the divisor $L = aD + S$ on X is ample. But $L \cdot D = 1$ whereas $m_x(D) = m$, so $\epsilon(L, x) \leq \frac{1}{m}$. Note that by taking suitable a we can make L^2 arbitrary large, and by taking L to be a multiple of $aD + S$ we can arrange that $L \cdot \Gamma$ be bounded below by any preassigned integer. ∎

As Viehweg points out, once one has an example of a surface where $\epsilon(L, x)$ is small at isolated points, one gets examples of higher dimensional varieties where the Seshadri constant becomes small on a codimension two subset:

EXAMPLE 3.2. Let (X,L) be as in Example (3.1), and for $n \geq 3$ let $Y = X \times \mathbf{P}^{n-2}$ and put $N = p_1^*(L) \otimes p_2^*(\mathcal{O}_{\mathbf{P}}(1))$. By taking curves in $X \times \{z\}$, one sees that

$$\epsilon(N, (x, z)) \leq \epsilon(L, x) \quad \text{for all } z \in \mathbf{P}^{n-2}.$$

In particular $\epsilon(N, y)$ can be arbitrarily small in codimension two. ∎

It would be very interesting to understand whether Seshadri constants are otherwise well-behaved:

PROBLEM 3.3. Let L be an ample line bundle on a smooth projective variety X. Does there always exist a point $x \in X$ at which $\epsilon(L, x) \geq 1$? If $L^n \gg 0$ is $\epsilon(L, x) \geq 1$ off a subset of codimension two?

Unfortunately the elementary methods of the present paper do not seem to shed much light on this question.

As noted in the Introduction, bounds on Seshadri constants lead to statements on the existence of sections of adjoint bundles. On surfaces, adjoint bundles are well understood thanks to the celebrated theorem of Reider [**Rdr**]. It is interesting to compare Reider's results with the statements obtained from our main Theorem. To this end recall first the well-known:

PROPOSITION 3.4. *Let X be a smooth complex projective variety of dimension n, and let L be an ample (or nef and big) line bundle on X. Fix a point $x \in X$ and a positive integer $k \geq \frac{n}{\epsilon(L,x)}$. If $L^n > (\frac{n}{k})^n$, then $K_X + kL$ has a section which does not vanish at x.*

Sketch of Proof: Let $f : Y \longrightarrow X$ be the blowing up of X at x, and denote by $E \subset Y$ the exceptional divisor. Setting $\epsilon = \epsilon(L, x)$ we have the linear equivalence of $\mathbb{R}$-divisors:

$$k \cdot f^*L - nE \equiv \frac{n}{\epsilon}(f^*L - \epsilon E) + (k - \frac{n}{\epsilon})f^*L,$$

and therefore $k \cdot f^*L - nE$ is nef and big. On the other hand $K_Y = f^*K_X + (n-1)E$, whence $f^*(K_X + kL) - E \equiv K_Y + (k \cdot f^*L - nE)$. Kawamata-Viehweg vanishing then gives

$$H^1(Y, \mathcal{O}(f^*(K_X + kL) - E) = 0$$

which in turn implies the existence of the required section. ∎

In particular, taking $e = 2$ in the main theorem implies:

COROLLARY 3.5. *Let X be a smooth complex projective surface and let L be an ample line bundle on X such that $L^2 \geq 5$ and $\Gamma \cdot L \geq 2$ for all irreducible curves $\Gamma \subset X$. Then at all but finitely many points $x \in X$, $K_X + L$ has a section which is non-vanishing at x.*

On the other hand, it is a consequence of Reider's theorem that under the hypotheses of (3.5), $K_X + L$ is in fact globally generated. Hence we may view our main theorem here as a sort of local Reider-type result, which however holds only off a finite set. A proof of the global generation of $K_X + L$ using vanishing theorems for $\mathbf{Q}$-divisors appears in [**EL**, §1].

While the results of the present paper give a fairly complete picture of the behavior of the Seshadri constants $\epsilon(L,x)$ for a given line bundle L on a smooth surface X, it is less clear what happens as L varies. The essential question here, which is in effect posed by Demailly [**De2**, (6.9)], is the following:

PROBLEM 3.6. Let X be a smooth projective variety, and for an ample line bundle L consider the global Sheshadri constant $\epsilon(L)$ defined in the Introduction. As L varies are these constants bounded away from zero? In other words, setting

$$\epsilon(X) =_{\text{def}} \inf\{\epsilon(L) \mid L \text{ ample on } X\},$$

is it always the case that $\epsilon(X) > 0$?

Our sense is that there may well exist surfaces where $\epsilon(X) = 0$, although we have been unable to construct any. This ties in with the following considerations.

Given an ample line bundle L on a smooth projective variety X, define $\nu(L)$ to be the least integer ν such that νL is very ample. Note that if X is a curve of genus g, then $\nu(L) \leq 2g+1$ for all ample L. In general, if there is a fixed ν such that $\nu(L) \leq \nu$ for every ample line bundle L on X, then $\epsilon(X) \geq \frac{1}{\nu}$. On the other hand, the following example, due to Kollár, shows that it need not be the case in general that $\nu(L)$ is bounded from above.

EXAMPLE 3.7. [Kollár]. We give an example of a surface X carrying a family of ample line bundles L_n such that $\nu(L_n) \to \infty$ with n.

We start with an elliptic curve E, and put $Y = E \times E$. Fix a point $P \in E$, and define on Y the divisors:

$$h = pr_1^*(P), \quad v = pr_2^*(P), \quad \delta = \text{ diagonal } \subset E \times E.$$

Next, given a positive integer $n \geq 2$ consider the divisor

$$M_n = n \cdot h + (n^2 - n + 1) \cdot v - (n-1)\delta.$$

Then $M_n^2 = 2$ and $M_n \cdot v > 0$, and consequently M_n is ample. [Proof: The inequalities imply by Riemann Roch that M_n has a section, and since Y is homogeneous it follows that M_n is in any event nef. If $M_n \cdot C = 0$ for some effective curve C, then the Hodge index theorem shows that $C^2 < 0$, which is absurd. Hence the Nakai criterion applies.] Finally, let $R = v + h$, and let $B \in |2R|$ be a smooth divisor.

For our surface X we take the double cover $f : X \longrightarrow Y$ of Y branched along B. Let $L_n = f^*(M_n)$. Then L_n is ample and we claim that the natural inclusion

$$(*) \qquad H^0(Y, \mathcal{O}_Y(n \cdot M_n)) \longrightarrow H^0(X, \mathcal{O}_X(n \cdot L_n))$$

is an isomorphism. It follows that $n \cdot L_n$ cannnot very ample, and hence $\nu(L_n) > n$. For the claim, observe that $f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-R)$, and therefore

$$f_*(\mathcal{O}_X(n \cdot L_n)) = \mathcal{O}_Y(n \cdot M_n) \oplus \mathcal{O}_Y(n \cdot M_n - R).$$

So to verify that the map in (*) is bijective, it suffices to prove that $H^0(Y, \mathcal{O}_Y(n \cdot M_n - R)) = 0$. But this follows from the computation that $(n \cdot M_n - R)^2 < 0$. [Note that the specific choices we have made are relatively unimportant; the essential point is simply that $M_n \cdot R$ grows much more quickly than $M_n \cdot M_n$.] ∎

Finally, we note that the definition of the Seshadri constant of a line bundle at a point can be generalized to measure positivity along a subvariety. Let X be a smooth projective variety, and let $V \subset X$ be a subvariety, say smooth to fix ideas. Let $f : Bl_V(X) \longrightarrow X$ be the blowing up of X along V, with exceptional divisor $E \subset Bl_V(X)$. Given an ample line bundle L on X, define the Seshadri constant of L along V to be

$$\epsilon(L, V) = \sup\{\epsilon \mid f^*L - \epsilon \cdot E \text{ is nef}\}.$$

Paoletti [P] has investigated these invariants when V is a curve in $\mathbf{P}^3$ (or a general smooth threefold X), and $L = \mathcal{O}_{\mathbf{P}^3}(1)$. In this case $\epsilon(L, V)$ detects such classical information as the presence of multisecant lines, but it seems to be a more delicate invariant. Paoletti proves the striking result that under suitable numerical hypotheses, $\epsilon(L, V)$ governs the gonality of the curve V. It would be interesting to see what other concrete geometric properties are influenced by these invariants. It would also be useful to develop some techniques for computing or estimating $\epsilon(L, V)$; some first steps in this direction appear in [P].

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