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ERIC AMAR

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A PROBLEM OF IDEALS

Eric Amar

Recently U. Cegrell [2] proved the following result:

Theorem. *Let* $\mathbb{B}$ *be the unit ball of* $\mathbb{C}^n$ *and* $f_1 \in A(\mathbb{B})$, $f_2 \in H^\infty(\mathbb{B})$*such that* $\forall z \in \mathbb{B}$, $|f_1(z)| + |f_2(z)| \geq \delta$ *then there are two functions* g_1, g_2 *in* $H^\infty(\mathbb{B})$ *such that:* $f_1 g_1 + f_2 g_2 = 1$ *in* $\mathbb{B}$.

where, as usual, if Ω is a domain in $\mathbb{C}^n$, $A(\Omega)$ is the algebra of all holomorphic functions in Ω continuous up to the boundary, $A^k(\Omega)$ is the algebra of all holomorphic functions in Ω, $\mathcal{C}^k$ up to $\partial\Omega$ and $H^\infty(\Omega)$ is the algebra of all holomorphic and bounded functions in Ω.

This means, in this special case, that the Corona is true. He uses a nice analysis of pic functions and representing measures in the ball, of independant interest.

The aim of this note is to give a very simple proof of this theorem which is also more general.

In order to state the result, let me give the following definition:

Definition. *We say that the bounded pseudo–convex domain in* $\mathbb{C}^n$ *has the* L^∞_q *property if: for any* $(0,q)$ *form* ω *in* $\mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$, *there is a* $(0, q-1)$ *form* u *in* $\mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$ *such that:* $\overline{\partial} u = \omega$.

As usual, a (0,0) form is just a function.

There are many examples of such domains: the strictly pseudo-convex ones [6], the polydiscs [8], the ellipsoïds [4], [11], the domains of finite type in $\mathbb{C}^2$ [3], [5].

We shall prove the following:

Theorem 1. *Let Ω be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the L_1^∞ condition and let $f_1 \in A(\Omega)$, $f_2 \in H^\infty(\Omega)$ such that $\forall z \in \Omega$, $|f_1(z)|+|f_2(z)| \geq \delta$ then there are two functions g_1, g_2 in $H^\infty(\Omega)$ such that: $f_1g_1 + f_2g_2 = 1$ in Ω.*

Proof:
because f_1 is continuous up to $\partial\Omega$, it is easy to make a function $\chi \in \mathcal{C}^\infty(\overline{\Omega})$ such that:

$$\chi = \begin{cases} 1 & in\ \{|f_1| > \delta/2\} \\ 0 & in\ \{|f_1| < \delta/4\} \end{cases}$$

Now let $\omega := \frac{\overline{\partial}\chi}{f_1f_2}$, then $\omega \in \mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$ because on the set where $\overline{\partial}\chi \neq 0$, $|f_1f_2| > \delta^2/16$. Moreover, $\overline{\partial}\omega = 0$ in Ω, hence, by the L_1^∞ condition, there is a $u \in L^\infty(\Omega)$ such that: $\overline{\partial}u = \omega$.

Let us define

$g_1 := \frac{\chi}{f_1} - uf_2$ and $g_2 := \frac{1-\chi}{f_2} + uf_1$;

then we get:

$\overline{\partial}g_1 = 0$, $\overline{\partial}g_2 = 0$

hence these functions are holomorphic in Ω and:

$f_1g_1 + f_2g_2 = 1.$

Moreover the g_i's are easily seen to be bounded in Ω, hence the theorem. ∎

Now using the Koszul's Complex method as in [9], it is easy to prove, usingexactly the same lines the:

Theorem 2. *Let Ω be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the L_q^∞ condition for $q \leq p-1$ and let $f_1, .., f_{p-1} \in A(\Omega)$, $f_p \in H^\infty(\Omega)$ such that $\forall z \in \mathbb{B}$ $\sum_{i=1}^p |f_i(z)| \geq \delta$; then there are p functions $g_1, ..., g_p$ in $H^\infty(\Omega)$ such that: $\sum_i f_ig_i = 1$ in Ω.*

Now let us define the $\mathcal{C}_p^k$ property for a pseudo-convex bounded domain in an analogous way:

Definition. *We say that the bounded pseudo-convex domain in* $\mathbb{C}^n$ *has the* $\mathcal{C}^k_q$ *property if: for any* $(0,q)$ *form* ω *in* $\mathcal{C}^k(\overline{\Omega})$, *there is a* $(0,q-1)$ *form* u *in* $\mathcal{C}^k(\overline{\Omega})$ *such that:* $\overline{\partial} u = \omega$.

For k finite, the domains listed above with the L^∞_q property have the $\mathcal{C}^k_q$ property too. For $k=\infty$ a very famous theorem by J.J. Kohn [10] says that all pseudo-convex bounded domains with smooth boundary has the $\mathcal{C}^\infty_q$ property.

The same way has above, we can show:

Theorem 3. *Let* Ω *be a pseudo-convex bounded domain in* $\mathbb{C}^n$ *verifying the* $\mathcal{C}^k_q$ *condition for* $q \leq p-1$ *and let* $f_1,..,f_p \in A^k(\Omega)$, *such that* $\forall z \in \Omega\ \sum_{i=1}^p |f_i(z)| \geq \delta$; *then there are* p *functions* $g_1,...,\ g_p$ *in* $A^k(\Omega)$ *such that:* $\sum_i f_i g_i = 1$ *in* Ω.

As a classical corollary we get:

Corollary. *Let* Ω *be a pseudo-convex bounded domain in* $\mathbb{C}^n$ *verifying the* $\mathcal{C}^k_q$ *condition for* $q \leq n$, *then the spectrum of the algebra* $A^k(\Omega)$ *is* $\overline{\Omega}$.

In the case $k=\infty$, M. Catlin [1] and M. Hakim and N. Sibony [7] already proved this result, the method they used is also a division method but slightly different and their method cannot give theorem 1 and 2 here.

References

[1] M. Catlin: *Boundary behavior of holomorphic functions on weakly pseudoconvex domains.*
Thesis, Princeton University

[2] U. Cegrell: *Representing measures in the Spectrum of* $H^\infty(\Omega)$
Proceedings of the International Workshop, Wuppertal 1990
Aspects of Mathematics

[3] M. Christ: *Regularity properties of the* $\overline{\partial}_b$*-equation on weakly pseudoconvex CR manifolds of dimension three.*
J. Am. Math. Soc. **1**, 587-646 (1988)

[4] K. Diederich, J.E. Fornaess, M. Wiegerinck: *Sharp Hölder estimates for $\overline{\partial}$ on ellipsoids.*
Manuscr. Math. **56**, 399–413 (1986)

[5] C. Fefferman, J.J. Kohn: *Hölder estimates on domains in two complex dimensions and on three dimensional CR manifolds.*
Adv. Math. **69**, 233–303 (1988)

[6] H. Grauert, I. Lieb: *Das Ramirezsche Integral und die Lösung der Gleichung $\overline{\partial} f = \alpha$ im Bereich der beschränkten Formen.*
Rice Univ. Studies **56**, 26–50 (1970)

[7] M. Hakim, N. Sibony: *Spectre de $A(\overline{\Omega})$ pour des domaines bornés faiblement pseudoconvexes régulier.*
J. Functional Analysis **37**, 127–135 (1980)

[8] G. Henkin: *Boundary properties of holomorphic functions of several variables.*
J. Soviet Math. **5**, 612–687 (1976)

[9] L. Hörmander: *Generators for some rings of analytic functions.*
Bull. Amer. Math. Soc. **73**, 943–949 (1967)

[10] J.J. Kohn: *Global regularity for $\overline{\partial}$ on weakly pseudoconvex manifolds.*
Trans. Amer. Math. Soc. **181**, 273–292 (1973)

[11] M. Range: *Integral kernels and Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$*
Math. Ann. 288, 63-74 (1990)

Amar Eric
Université Bordeaux I
351 Cours de la Libération
33405 Talence, France