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Geometric structures and characteristic forms

Masahide Kato

On complex manifolds, we consider various holomorphic geometric structures such as affine structures, projective structures and conformal structures. When they admit such geometric structures, then their Chern classes satisfy certain formulae. For compact Kähler manifolds, these formulae were first found in the affine case by Atiyah [A], in the projective case by Gunning [G], and in the conformal case by Kobayashi-Ochiai [KO]. In the following, we shall introduce new characteristic forms defined by projective Weyl curvature tensors and conformal Weyl curvature tensors, and study their relations with the Chern forms. As byproducts, we obtain generalizations and refinements of the formulae quoted above. Details of this note will be found in [K1] and [K2].

Notation

$\Omega^p(E)$	:	the sheaf of germs of holomorphic p-forms with values in a holomorphic vector bundle E,
$\mathcal{O}(E) \simeq \Omega^0(E)$	:	the sheaf of germs of holomorphic sections of a holomorphic vector bundle E,
Θ	:	the sheaf of germs of holomorphic vector fields,
T	:	the sheaf of germs of differentiable vector fields,
$\mathcal{A}^r(G)$	:	the sheaf of germs of differentiable r-forms with values in a differentiable vector bundle G,
$\mathcal{A}^{p,q}(G)$	:	the sheaf of germs of differentiable (p,q)-forms with values in a differentiable vector bundle G.

1 Affine structures

Let X be a complex manifold of dimension $n \geq 1$. Take a locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of X such that on each U_α, there is a system of local coordinates $z_\alpha = (z_\alpha^1, z_\alpha^2, \ldots, z_\alpha^n)$. Put

$$\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$$

and

$$\tau_{\alpha\beta} = \text{the Jacobian matrix of } \varphi_{\alpha\beta}.$$

On $U_\alpha \cap U_\beta$, we consider an $n \times n$-matrix-valued holomorphic 1-form

$$a_{\alpha\beta} = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}.$$

It is well-known and easy to check that the set $\{a_{\alpha\beta}\}$ define an element of $H^1(X, \Omega^1(\mathrm{End}\Theta))$.

Definition 1.1 *The cohomology class*

$$a_X = \{a_{\alpha\beta}\} \in H^1(X, \Omega^1(\mathrm{End}\Theta))$$

is called the obstruction to the existence of holomorphic affine connections of X.

Definition 1.2 *For a complex manifold X with $a_X = 0$ there exists a (holomorphic) 0-cochain $\{a_\alpha\}$ such that $\delta\{a_\alpha\} = \{a_{\alpha\beta}\}$, which is called a holomorphic affine connection of X. If X has a holomorphic affine connection, we also say that X admits an affine structure. There always exists a C^∞ 0-cochain such that $\delta\{a_\alpha\} = \{a_{\alpha\beta}\}$ in the natural sense, where the a_α is an element of $\Gamma(U_\alpha, \mathcal{A}^{1,0}(\mathrm{End}T))$. The 0-cochain is called a C^∞ affine connection.*

Let $\theta = \{a_\alpha\}$ be a C^∞ affine connection. Then we have

$$a_{\alpha\beta} = a_\beta - \tau_{\alpha\beta}^{-1} a_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

The curvature form of the C^∞ affine connection

$$\Theta_\alpha = da_\alpha + a_\alpha \wedge a_\alpha$$

satisfies the equation

$$\Theta_\beta = \tau_{\alpha\beta}^{-1} \Theta_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Let t be an indeterminate and A be an $n \times n$ matrix. Define polynomials φ_0, $\varphi_1, \ldots, \varphi_n$ by

$$\det(I - \frac{1}{2\pi i} tA) = \Sigma_{k=0}^{n} \varphi_k(A) t^k.$$

The Chern forms are defined by

$$c_q(\theta) = \varphi_q(\Theta_\alpha).$$

The following is a well-known fundamental fact.

Theorem 1.1 *For any $q = 0, 1, \ldots, n$, the Chern form $c_q(\theta)$ is a d-closed C^∞ $2q$-form. The corresponding de Rham cohomology class $[c_q(\theta)]$ is real and independent of the choice of the connection θ.*

Corollary 1.1 [A] *If a compact complex manifold with dimension $n \geq 1$ admits a holomorphic affine connection then all q-th Chern forms with $2q \geq n$ vanish. If , further, the manifold is of Kähler then all q-th Chern classes, $q \geq 1$, are zero.*

Proof. Note that the Chern forms defined by a holomorphic affine connection are holomorphic. Therefore if $2q > n$ then the q-th Chern form vanishes. Note that d-closed holomorphic n-form represents a real de Rham cohomology class only if it represents a zero class. Hence the n-th Chern class vanishes. If the manifold is of Kähler then any holomorphic form is harmonic. Since the Chern classes are real, they vanish by Hodge theory. ∎

2 Projective structures

In this section, we assume that $n = \dim X \geq 2$. We use the notation of section 1. On $U_\alpha \cap U_\beta$, we define a scalar-valued holomorphic 1-form

$$\sigma_{\alpha\beta} = (n+1)^{-1} d\log(\det \tau_{\alpha\beta}) = (n+1)^{-1}\mathrm{Trace}(\tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta})$$

and an $n \times n$ matrix-valued holomorphic 1-form $\rho_{\alpha\beta}$ by

$$(\rho_{\alpha\beta})^j_k = \sigma_{\alpha\beta k} dz^j_\beta,$$

where

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta j} dz^j_\beta,$$

and $(A)^j_k$ indicates the (j,k)-component of the matrix A. Put

$$p_{\alpha\beta} = a_{\alpha\beta} - \rho_{\alpha\beta} - I \cdot \sigma_{\alpha\beta}, \tag{1}$$

where I is the identity matrix of size n. The 1-cochain $\{\sigma_{\alpha\beta}\}$ is a cocycle of $H^1(X, \Omega^1)$, and $\{\rho_{\alpha\beta}\}$ and $\{p_{\alpha\beta}\}$ are cocycles of $H^1(X, \Omega^1(\mathrm{End}\Theta))$.

Definition 2.1 *The cohomology class*

$$p_X = \{p_{\alpha\beta}\} \in H^1(X, \Omega^1(\mathrm{End}\Theta))$$

is called the obstruction to the existence of holomorphic projective connections of X.

Definition 2.2 *For a complex manifold X with $p_X = 0$ there exists a (holomorphic) 0-cochain $\{p_\alpha\}$ such that $\delta\{p_\alpha\} = \{p_{\alpha\beta}\}$, which is called a holomorphic projective connection of X. If X has a holomorphic projective connection, we also say that X admits a projective structure. There always exists a C^∞ 0-cochain $\{p_\alpha\}$ such that $\delta\{p_\alpha\} = \{p_{\alpha\beta}\}$ in a natural sense, where p_α is an element of $\Gamma(U_\alpha, \mathcal{A}^{1,0}(\mathrm{End}\Theta))$. The 0-cochain $\{p_\alpha\}$ is called a C^∞ projective connection.*

Let $\pi = \{p_\alpha\}$ be a C^∞ projective connection on X, that is, on each U_α there is an $n \times n$ matrix-valued C^∞ $(1,0)$-form p_α such that

$$p_\beta = p_{\alpha\beta} + \tau_{\alpha\beta}^{-1} p_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{2}$$

We write the (j,k)-component of p_α and $p_{\alpha\beta}$ as

$$\begin{aligned} (p_\alpha)^j_k &= p^j_{\alpha ik} dz^i_\alpha, \\ (p_{\alpha\beta})^j_k &= p^j_{\alpha\beta ik} dz^i_\beta. \end{aligned}$$

By the definition (1), we have $p^j_{\alpha\beta kl} = p^j_{\alpha\beta lk}$. Therefore it is easy to see that the $n \times n$ matrix-valued 1-form q_α defined by

$$(q_\alpha)^j_k = p^j_{\alpha ki} dz^i_\alpha$$

is also a projective connection. Hence $\{2^{-1}(p_\alpha + q_\alpha)\}$ is also a projective connection. Therefore we may assume that

$$p^j_{\alpha kl} = p^j_{\alpha lk} \tag{3}$$

holds. Since $\mathrm{Trace}(p_\beta) = 0$, it follows from (2) that

$$\mathrm{Trace}(p_\beta) = \mathrm{Trace}(p_\alpha).$$

Since $\{p_\alpha - n^{-1}\mathrm{Trace}(p_\alpha) I\}$ is also a projective connection, we may assume that

$$p^j_{\alpha ij} = 0 \tag{4}$$

holds. The projective connection satisfying (3) is said to be *normal.* The projective connection satisfying (4) is said to be *reduced.* Thus any complex manifold admits a normal reduced C^∞ projective connection. As we see easily from the above argument, *if a complex manifold admits a holomorphic projective connection, the manifold admits also a normal reduced holomorphic projective connection.* In the following in this section, we consider only normal reduced (holomorphic or C^∞) projective connections.

The projective Weyl curvature tensor $\{W_\alpha\}$ associated with a (normal reduced) C^∞ projective connection $\pi = \{p_\alpha\}$ is defined by

$$W_\alpha = dp_\alpha + p_\alpha \wedge p_\alpha + \frac{1}{n-1} X_\alpha$$

where

$$\begin{aligned}(X_\alpha)^j_k &= X_{\alpha kl} dz^j_\alpha \wedge dz^l_\alpha, \\ X_{\alpha kl} &= -\frac{\partial p^r_{\alpha kl}}{\partial z^r_\alpha} + p^r_{\alpha sl} p^s_{\alpha kr}.\end{aligned}$$

The projective Weyl curvature tensor satisfies the equation

$$W_\beta = \tau^{-1}_{\alpha\beta} W_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{5}$$

Thus we have

Proposition 2.1 *The projective Weyl curvature tensor $\{W_\alpha\}$ is an element of $\Gamma(X, \mathcal{A}^2(\mathrm{End}T))$. If the projective connection is holomorphic, then the associated projective Weyl curvature tensor is an element of $\Gamma(X, \Omega^2(\mathrm{End}\Theta))$.*

We define C^∞ $2q$-forms, $P_q(\pi)$, $q = 0, 1, \ldots, n$, associated with the normal reduced projective connection π by

$$P_q(\pi) = \varphi_q(W_\alpha).$$

In view of (5), the $P_q(\pi)$ are indeed defined on the whole X.

Theorem 2.1 [K1] *i) $P_q(\pi)$ is a d-closed smooth $2q$-form.*
ii) The de Rham cohomology class $[P_q(\pi)]$ is a real cohomology class and is independent of the choice of the normal reduced C^∞ projective connection π.

Definition 2.3 [K1] *The d-closed smooth $2k$-form $P_q = P_q(\pi)$ is called the q-th projective Weyl form associated with the normal reduced C^∞ projective connection π.*

Since π is reduced, we have

$$P_1(\pi) = 0.$$

Theorem 2.1 follows immediately from the property of the Chern forms and the following formula.

Theorem 2.2 [K1] *Let X be a complex manifold of dimension $n \geq 2$. Let π be any normal reduced C^∞ projective connection. Then there is a C^∞ affine connection θ on X which satisfies the following equality;*

$$\sum_{q=0}^{n} c_q(\theta)t^q = (1+at)^{n+1} \sum_{q=0}^{n} P_q(\pi)\left(\frac{t}{1+at}\right)^q,$$

or, equivalently

$$\sum_{q=0}^{n} P_q(\pi)t^q = (1-at)^{n+1} \sum_{q=0}^{n} c_q(\theta)\left(\frac{t}{1-at}\right)^q,$$

where

$c_q(\theta)$ = *the q-th Chern form associated with θ*

$a = \dfrac{1}{n+1}c_1(\theta)$,

$P_q(\pi)$ = *the q-th projective Weyl form associated with π.*

The projective Weyl forms are holomorphic for holomorphic projective connections. Therefore by the same reasoning as the proof of Corollary 1.1, we have

Corollary 2.1 [K1] *If a compact (not necessarily Kähler) complex manifold with dimension $n \geq 2$ admits a holomorphic projective connection, then all q-th projective Weyl forms with $2q \geq n$ vanish. If, further, it is of Kähler, then all q-th projective Weyl forms with $q \geq 1$ vanish.*

Theorem 2.2 and the corollary above give a refinement of [KO, Theorem 3.1].

Remark. Let D be a reduced effective divisor on X with only normal crossing singularities. With respect to the *logarithmic pair* (X, D) [I], we can consider the *logarithmic* projective connection and its associated *logarithmic* projective Weyl curvature tensor and get a logarithmic analogue of Theorem 2.2. See [K3] for the detail.

3 Conformal structures

In this section, we assume that $n = \dim X \geq 3$. We use the notation of sections 1. The symbols $\sigma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ in this section are different from those in section 2 by a constant multiple $\dfrac{n+1}{n}$. On $U_\alpha \cap U_\beta$, we define a scalar-valued holomorphic 1-form

$$\sigma_{\alpha\beta} = n^{-1} d\log(\det \tau_{\alpha\beta}) = n^{-1}\mathrm{Trace}(\tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}).$$

and a $n \times n$ matrix-valued holomorphic 1-form $\rho_{\alpha\beta}$ by

$$(\rho_{\alpha\beta})^j_k = \sigma_{\alpha\beta k} dz^j_\beta$$

where

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta j} dz^j_\beta.$$

Definition 3.1 *We say that a complex manifold X of dimension $n \geq 1$ admits a holomorphic conformal structure if the structure group of the tangent bundle reduces as a holomorphic bundle to the conformal group $CO(n, \mathbf{C})$.*

Let $S \subset GL(n, \mathbf{C})/\mathbf{C}^*$ be the set of non-singular symmetric matrices factored by the non-zero scalar matrices. We form a holomorphic fibre bundle

$$Z = \left(\bigcup_\alpha U_\alpha \times S \right) / \sim$$

on X with the typical fibre S by identifying $(z_\alpha, s_\alpha) \in U_\alpha \times S$ with $(z_\beta, s_\beta) \in U_\beta \times S$ if and only if $z_\alpha = z_\beta$ and $s_\beta = {}^t\tau_{\alpha\beta} s_\alpha \tau_{\alpha\beta}$. Let $\pi : Z \to X$ be the natural projection. That X admits a holomorphic conformal structure is equivalent to saying that π admits a holomorphic section. A holomorphic section g of π is also called a holomorphic conformal structure of X.

Suppose that X admits a holomorphic conformal structure g. On each U_α, g is represented by a holomorphic symmetric $(2,0)$-form

$$g_\alpha = g_{\alpha ij} dz^i_\alpha dz^j_\alpha$$

such that

$$g_\beta = f_{\beta\alpha} g_\alpha \text{ on } U_\alpha \cap U_\beta, \tag{6}$$

where $f_{\beta\alpha}$ is a nowhere vanishing holomorphic function defined on $U_\alpha \cap U_\beta$ and

$$\det(g_{\alpha ij}(x)) \neq 0 \text{ for all } x \in U_\alpha.$$

Let F be the holomorphic line bundle on X formed by the 1-cocycle $\{f_{\alpha\beta}\}$. Then $\{g_\alpha\}$ can be regarded as an element of $\Gamma(X, \mathcal{S}^2(\Omega^1) \otimes F)$, where $\mathcal{S}^2(\Omega^1)$ indicates the 2nd symmetric power of Ω^1. Note that two sections $\{g_\alpha\}$ and $\{h_\alpha\}$ in $\Gamma(X, \mathcal{S}^2(\Omega^1) \otimes F)$ represent the same conformal structure if and only if on each U_α there is a nowhere vanishing holomorphic function f_α such that $g_\alpha = f_\alpha h_\alpha$. Put

$$G_\alpha = (g_{\alpha ij}),$$

where the (i,j)-component is given by $g_{\alpha ij}$. By (6), we have

$$g_{\beta rs} = f_{\beta\alpha} g_{\alpha ij} (\tau_{\alpha\beta})^i_r (\tau_{\alpha\beta})^j_s \text{ on } U_\alpha \cap U_\beta.$$

We have easily the following

Lemma 3.1 [KO] $nc_1[F]+2c_1(X)=0.$

Now we shall define *the conformal Weyl curvature tensor* associated with the holomorphic conformal structure $g=\{g_\alpha\}$. We put

$$\rho^*_{\alpha\beta}=-G_\beta^{-1t}\rho_{\alpha\beta}G_\beta$$

or

$$(\rho^*_{\alpha\beta})^j_k=-g^{jr}_\beta(\rho_{\alpha\beta})^s_r g_{\beta sk},$$

where g^{jk}_β is the (j,k)-component of G_β^{-1}. Then, on $U_\alpha\cap U_\beta\cap U_\gamma$, we have

$$\rho^*_{\alpha\gamma}=\tau^{-1}_{\beta\gamma}\rho^*_{\alpha\beta}\tau_{\beta\gamma}+\rho^*_{\beta\gamma}.$$

This implies that the set $\{\rho^*_{\alpha\beta}\}$ defines a 1-cocycle in $Z^1(X,\Omega^1\otimes \mathrm{End}\Theta)$. We define a 1-cocycle $\{c_{\alpha\beta}\}$ by

$$c_{\alpha\beta}=a_{\alpha\beta}-\rho_{\alpha\beta}-\rho^*_{\alpha\beta}-\sigma_{\alpha\beta}I,$$

which is also an element of $Z^1(X,\Omega^1\otimes \mathrm{End}\Theta)$. As we see by the following argument that *the cohomology class represented by $\{c_{\alpha\beta}\}$ turns out to be zero.* By means of the representative $\{g_\alpha\}$ of g, we can construct explicitly a 0-cochain $\{c_\alpha\}\in C^0(\mathcal{U},\Omega^1\otimes\mathrm{End}\Theta)$ whose coboundary coincides with the 1-cocycle $\{c_{\alpha\beta}\}$, i.e.,

$$c_\beta=c_{\alpha\beta}+\tau^{-1}_{\alpha\beta}c_\alpha\tau_{\alpha\beta}.$$

The 0-cochain $\{c_\alpha\}$ is called a *holomorphic conformal connection* of X. We define the Christoffel symbols $\left\{{l \atop ij}\right\}_\alpha$ associated with the symmetric tensor g_α on U_α by

$$\left\{{l \atop ij}\right\}_\alpha=\frac{1}{2}g^{lk}_\alpha\left(\frac{\partial g_{\alpha ik}}{\partial z^j_\alpha}+\frac{\partial g_{\alpha jk}}{\partial z^i_\alpha}-\frac{\partial g_{\alpha ij}}{\partial z^k_\alpha}\right).$$

The conformal connection $\{c_\alpha\}$ associated with the conformal structure $\{g_\alpha\}$ is defined by

$$\begin{aligned}
c_\alpha &= \left(c^l_{\alpha i}\right),\\
c^l_{\alpha i} &= c^l_{\alpha ij}dz^j_\alpha,\\
c^l_{\alpha ij} &= \left\{{l \atop ij}\right\}_\alpha-\frac{\delta^l_i}{n}\left\{{a \atop aj}\right\}_\alpha-\frac{\delta^l_j}{n}\left\{{a \atop ai}\right\}_\alpha+\frac{1}{n}g^{lb}_\alpha g_{\alpha ij}\left\{{a \atop ab}\right\}_\alpha.
\end{aligned}$$

The following lemma can be proved by calculations.

Lemma 3.2

$$\begin{aligned}
c^j_{\alpha ij} &= 0\\
c^l_{\alpha ij} &= c^l_{\alpha ji}\\
c_\beta &= c_{\alpha\beta}+\tau^{-1}_{\alpha\beta}c_\alpha\tau_{\alpha\beta}.
\end{aligned}\tag{7}$$

By (7), we see that the cochain $\{c_\alpha\}$ is determined by g and independent of the choice of the representative $\{g_\alpha\}$. Put

$$F_\alpha = dc_\alpha + c_\alpha \wedge c_\alpha.$$

Write the (j,k)-component of F_α as

$$(F_\alpha)^j_k = F^j_{\alpha krs} dz^r_\alpha \wedge dz^s_\alpha.$$

We set

$$\begin{aligned} F_{\alpha kl} &= F^m_{\alpha klm}, \\ \Phi_\alpha &= g^{kl}_\alpha F_{\alpha kl}. \end{aligned}$$

Then

$$\begin{aligned} W^j_{\alpha klm} &= F^j_{\alpha klm} + \frac{1}{n-2}\left(\delta^j_l F_{\alpha km} - \delta^j_m F_{\alpha kl}\right) \\ &\quad + \frac{1}{n-2} g^{jr}_\alpha \left(g_{\alpha km} F_{\alpha rl} - g_{\alpha kl} F_{\alpha rm}\right) \\ &\quad + \frac{1}{(n-1)(n-2)}\left(\delta^j_m g_{\alpha kl} - \delta^j_l g_{\alpha km}\right)\Phi_\alpha \end{aligned}$$

satisfy

$$W^j_{\beta klm} = (\tau^{-1}_{\alpha\beta})^j_i W^i_{\alpha rst} (\tau_{\alpha\beta})^r_k (\tau_{\alpha\beta})^s_l (\tau_{\alpha\beta})^t_m. \tag{8}$$

Define an $n \times n$-matrix valued holomorphic 2-form W_α by

$$(W_\alpha)^j_k = W^j_{\alpha klm} dz^l_\alpha \wedge dz^m_\alpha. \tag{9}$$

Then (8) is written as

$$W_\beta = \tau^{-1}_{\alpha\beta} W_\alpha \tau_{\alpha\beta}, \tag{10}$$

i.e., $\{W_\alpha\}$ is an element of $\Gamma(X, \Omega^2 \otimes \mathrm{End}\Theta)$.

Definition 3.2 *The holomorphic tensor field $\{W_\alpha\}$ is called the conformal Weyl curvature tensor associated with the holomorphic conformal structure g.*

We remark that the conformal Weyl curvature tensor is defined independently of the choice of $\{g_\alpha\}$ which represents g. We shall define holomorphic $2q$-forms, $\mathcal{C}_q(g)$, $q = 0, 1, \ldots, n$, associated with the holomorphic conformal structure g by

$$\mathcal{C}_q(g) = \varphi_q(W_\alpha).$$

In view of (10), the $\mathcal{C}_q(g)$ are indeed defined on the whole X.

Theorem 3.1 [K2] *i) $\mathcal{C}_q(g)$ is a d-closed holomorphic 2q-form.*
ii) The de Rham cohomology class $[\mathcal{C}_q(g)]$ is a real cohomology class and is independent of the choice of the holomorphic conformal structure g.

Definition 3.3 [K2] *The d-closed holomorphic 2q-form $\mathcal{C}_q(g)$ is called the q-th conformal Weyl form associated with the holomorphic conformal structure g.*

Theorem 3.1 follows immediately from the property of the Chern forms and the following formula.

Theorem 3.2 [K2] *Let X be a complex manifold of dimension $n \geq 3$ which admits a holomorphic conformal structure g on X. Then there exists a C^∞ affine connection θ on X which satisfies the equality*

$$\sum_{q=0}^{n} \mathcal{C}_q(g)t^q = (1 - b^2t^2)\sum_{q=0}^{n}(1 - bt)^{n-q}t^q c_q(\theta),$$

or equivalently,

$$\sum_{q=0}^{n} c_q(\theta)t^q = \frac{(1 + bt)^{n+2}}{1 + 2bt}\sum_{q=0}^{n}\left(\frac{t}{1 + bt}\right)^q \mathcal{C}_q(g),$$

where

$$\begin{aligned} b &= \frac{1}{n}c_1(\theta), \\ c_q(\theta) &= \text{the } q\text{-th Chern forms associated with } \theta, \\ \mathcal{C}_q(g) &= \text{the } q\text{-th conformal Weyl form associated with } g. \end{aligned}$$

In the course of the proof of Theorem 3.2, we obtain the following as a corollary.

Theorem 3.3 [K2] *Let X be a complex manifold of dimension $n \geq 3$ with a holomorphic conformal structure g. Then*

$$\mathcal{C}_{2q+1}(g) = 0, \quad q = 0, 1, \ldots, [\frac{n}{2}].$$

By Theorems 3.2 and 3.3, the conformal Weyl forms are, for example,

$$\begin{aligned} \mathcal{C}_0(g) &= 1, \\ \mathcal{C}_1(g) &= 0, \\ \mathcal{C}_2(g) &= \frac{-n^2 + n - 2}{2n^2}c_1^2(\theta) + c_2(\theta), \end{aligned}$$

$$\begin{aligned}
\mathcal{C}_3(g) &= \frac{(n-1)(n-2)}{3n^2}c_1^3(\theta) - \frac{n-2}{n}c_1(\theta)c_2(\theta) + c_3(\theta) = 0, \\
\mathcal{C}_4(g) &= -\frac{(n-1)(n^2-5n+2)}{8n^3}c_1^4(\theta) + \frac{(n-1)(n-4)}{2n^2}c_1^2(\theta)c_2(\theta) \\
&\quad -\frac{n-3}{n}c_1(\theta)c_3(\theta) + c_4(\theta).
\end{aligned}$$

We obtain the following from Theorem 3.2 by the same reasoning as the proof of Corollary 1.1.

Corollary 3.1 [K2] *If a compact complex manifold with dimension $n \geq 3$ admits a holomorphic conformal structure, then all q-th conformal Weyl forms with $2q \geq n$ vanish. If, further, the manifold is of Kähler then all q-th, $q \geq 1$, conformal Weyl classes are zero.*

Theorem 3.2 and the corollary above give a refinement of [KO, Theorem 3.20] for $n \geq 3$.

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