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ON THE LOCAL HOLOMORPHIC EXTENSION OF CR FUNCTIONS

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0. Introduction. The history of the holomorphic continuation, across a hypersurface Σ, of functions defined and holomorphic on one side of Σ goes back to the discovery of strong pseudoconvexity — and to the proof in Levi [1] that, in a strongly pseudoconvex domain of $\mathbb{C}^2$ with smooth boundary Σ, there are holomorphic functions that have no holomorphic extension across the boundary, to the concave side. Later, in Lewy [1], it was shown that every holomorphic function on the concave side extends to the convex side. It is now customary to rephrase such results in the language of CR functions on Σ, and of their germs at a point: *on a strongly pseudoconvex hypersurface in* $\mathbb{C}^N$ *there are germs of CR functions that do not extend to the pseudoconcave side* (Levi); *every germ of CR function extends to the pseudoconvex side* (Lewy).

The question of the extension (always, for us, holomorphic extension) of germs is radically different from that of the extension of CR functions defined in the whole boundary Σ. Extension of the latter kind can be viewed as an aspect of the *Hartogs phenomenon.* Let us recall how. Let $\Omega \subset \mathbb{C}^N (N \geq 2)$ be an open and bounded set, whose complement consists of a single, unbounded, connected component, and whose boundary Σ is fairly smooth, say of class $\mathcal{C}^2$. Let h be a function defined and $\mathcal{C}^2$ in the whole of Σ. Provided h satisfies the tangential Cauchy-Riemann equations, one can find a $\mathcal{C}^2$ extension $\tilde{h}$ to Ω such that $\bar{\partial}\tilde{h}$, as well as its first partial derivatives, vanish on Σ. Let $g = \bar{\partial}\tilde{h}$ in Ω, $g \equiv 0$ in $\mathbb{C}^N \backslash \Omega$; we have $g \in \mathcal{C}^1(\mathbb{C}^N)$ and $\bar{\partial} g = 0$. We can then solve $\bar{\partial} u = g$ in $\mathbb{C}^N$, with $u \in \mathcal{C}^1(\mathbb{C}^N)$ and $u \equiv 0$ in $\mathbb{C}^N \backslash \Omega$ (since $N \geq 2$). Clearly, $\tilde{h} - u$ is holomorphic in Ω and is equal to h on Σ. The extension of the globally defined CR function h depends on the *topology* of Σ; its *geometry*, *e.g.*, whether Σ has convex or concave parts, is irrelevant.

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The situation is quite different when one tries to extend the germs of CR functions.

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1. The Hypersurface Case.

First of all, we recall the definitions and fix the notation. We consider a real hypersurface Σ in $\mathbb{C}^{n+1}(n \geq 1)$, of class $\mathcal{C}^2$ (this will always be our smoothness hypothesis, unless specified otherwise). Call $z_1, ..., z_n$ and w the complex coordinates in $\mathbb{C}^{n+1}$. We shall assume that $0 \in \Sigma$ and that the tangent hyperplane to Σ at 0 is the hyperplane $\Im w = 0$. In other words, we are going to assume that, in an open ball $\Omega \subset \mathbb{C}^{n+1}$ centered at the origin, Σ is defined by an equation

$$\Im w = \phi(z, \Re e w), \tag{1.1}$$

with ϕ real-valued and of class $\mathcal{C}^2$ in the closure of Ω, and

$$\phi|_0 = 0, d\phi|_0 = 0. \tag{1.2}$$

The pullbacks to Σ of the differentials $dz_i (1 \leq i \leq n)$, dw span a vector subbundle, here denoted by $T'^{1,0}\Sigma$, of the complexified tangent bundle $\mathbb{C}T^*\Sigma$; the rank of $T'^{1,0}\Sigma$ is equal to $n+1$. Its orthogonal in $\mathbb{C}T\Sigma$ is the vector subbundle of rank n, $T^{0,1}\Sigma$, spanned by the vector fields tangent to Σ that are linear combinations of $\partial/\partial\bar{z}_i (1 \leq i \leq n)$ and $\partial/\partial\bar{w}$. By a CR function h on Σ we shall mean a continuous function in Σ such that $Lh = 0$ whatever the $\mathcal{C}^1$ section L of $T^{0,1}\Sigma$. The equation $Lh = 0$ can, and must, be understood in the distribution sense.

As our standpoint is strictly local, we may as well assume that $\Sigma = \Sigma \cap \Omega$, and that the hypersurface Σ subdivides Ω into two *sides*: Ω^+, in which $\Im w > \phi(z, \Re e w)$; Ω^-, where $\Im w < \phi(z, \Re e w)$. The boundary value of any continuous function in $\Omega^+ \cup \Sigma$ (i.e., "continuous up to the boundary") that is holomorphic in Ω^+ is a CR function on Σ. The problem we wish to discuss is the localized converse: to seek properties of Σ which ensure that every (continuous) CR function h in a neighborhood of 0 in Σ is the boundary value $bv\tilde{h}$, say in the distribution sense, and possibly in a smaller neighborhood, of a holomorphic function $\tilde{h}$ in a set $\tilde{V} \cap \Omega^+$ with $\tilde{V}$ some open neighborhood of 0 in $\mathbb{C}^{n+1}$. If the latter is true, we say that the germ of h *extends to* Ω^+. We recall that the fact that $\tilde{h}$ has a distribution boundary value, is equivalent to the property that, in $\tilde{V}, \tilde{h}$ grows slowly at the *edge* $\tilde{V} \cap \Sigma$, i.e., to each compact subset K of $\tilde{V}$ there is an integer $k > 0$ and a constant $C > 0$ such that

$$|\tilde{h}(z)| \leq C \operatorname{dist}(z, \Sigma)^{-k}, \ \forall z \in K \cap \Omega^+.$$

(Here the variable is denoted by z rather than (z, w).)

We now recall a number of basic facts, all fairly elementary, and most well known; and first of all, the property of "unique continuation" across a hypersurface:

Proposition 1.1. *If the germ of a CR function at 0, on Σ, extends to both sides Ω^+ and Ω^-, then it is the restriction to Σ of the germ of a holomorphic function in $\mathbb{C}^{n+1}$.*

The next observation is a particular case of a more general statement (see e.g., Baouendi-Jacobowitz-Treves [1], Lemma 2.4).

Proposition 1.2. *Suppose a holomorphic function $\tilde{h}$ in $\tilde{V} \cap \Omega^+$, with $\tilde{V}$ an open neighborhood of 0 in $\mathbb{C}^{n+1}$, has a boundary value h on $V = \tilde{V} \cap \Sigma$ in the distribution sense. If h is a continuous function in V, then $\tilde{h}$ is continuous in $V \cup \Omega^+$.*

The next statement is a direct consequence of the Baire category theorem (cf. Lemma III.5.1 in Treves [1]):

Proposition 1.3. *Let U be an open neighborhood of 0 in Σ with the property that to each CR function $h \in C^0(U)$ there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that h extends holomorphically to $\tilde{V} \cap \Omega^+$. Then $\tilde{V}$ can be chosen independently of h.*

Extension of (germs at 0 of) CR functions to one side, say Ω^+, is the same as extension to a full neighborhood of 0 in $\mathbb{C}^{n+1}$ (to the germ $(\mathbb{C}^{n+1}, 0)$) of (germs at 0 of) holomorphic functions in Ω^-. This is a consequence of the classical decomposition of CR functions (see Andreotti and Hill [1]):

Proposition 1.4. *If h is a continuous CR function in an open neighborhood U of 0 in Σ, then there are an open neighborhood $\tilde{V} \subset \Omega$ of 0 in $\mathbb{C}^{n+1}$, and holomorphic functions $\tilde{h}^+$ and $\tilde{h}^-$ in $\tilde{V} \cap \Omega^+$ and $\tilde{V} \cap \Omega^-$ respectively, such that $h = bv\, \tilde{h}^+ - bv\, \tilde{h}^-$ in $\tilde{V} \cap \Sigma$.*

Below we shall also make use of the following approximation results:

Proposition 1.5. *To each open neighborhood $\tilde{V} \subset \Omega$ of 0 in $\mathbb{C}^{n+1}$ there is another open neighborhood $\tilde{V}_0 \subset \tilde{V}$ such that every continuous function in $\tilde{V} \cap (\Omega^+ \cup \Sigma)$ which is holomorphic in $\tilde{V} \cap \Omega^+$ is the uniform limit, in $\tilde{V}_0 \cap (\Omega^+ \cup \Sigma)$, of a sequence of holomorphic polynomials.*

Remark: By a simple translation argument the following variant of Proposition 1.5 is also true. To each open neighborhood $\widetilde{V} \subset \Omega$ of 0 in $\mathbb{C}^{n+1}$ there is another open neighborhood $\widetilde{V}_0 \subset \widetilde{V}$ such that every function which is holomorphic in $\widetilde{V} \cap \Omega^+$ is the uniform limit, on compact subsets of $\widetilde{V}_0 \cap \Omega^+$, of a sequence of holomorphic polynomials.

Proposition 1.6. *To each open neighborhood U of O in Σ there is another open neighborhood $U_0 \subset U$ of O such that every continuous CR function in U is the uniform limit, in U_0, of a sequence of holomorphic polynomials.*

Prop.1.5 is stated, and proved, as Th. V.7.2 in Treves [1]. Prop.1.6 is a direct consequence of the approximation formula in Baouendi-Treves [1] (also Th. II.2.1 in Treves [1]). [The authors would like to thank J. P. Rosay for pointing out an embarassing mistake in an earlier version of the article, specifically in an attempt to derive directly Prop.1.6 from Prop.1.5.]

Propositions 1.3 and 1.6 have the following consequence:

Proposition 1.7. *The following properties are equivalent:*

(1.3) *To each open neighborhood U of 0 in Σ there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that every CR function $h \in \mathcal{C}^0(U)$ extends holomorphically to $\tilde{V} \cap \Omega^+$.*

(1.4) *To each open neighborhood U of 0 in Σ there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that any holomorphic polynomial that vanishes in $\tilde{V} \cap \Omega^+$ also vanishes in U.*

Proof. Let U and $\tilde{V}$ be as in (1.3). If the polynomial P does not vanish in U, $1/P$ extends holomorphically to $\tilde{V} \cap \Omega^+$, and therefore P cannot vanish there.

Let now U and U_0 be related as in Prop.1.6. There is an open (and bounded) neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that, given any holomorphic polynomial P and any $(z,w) \in \tilde{V} \cap \Omega^+$, then $P(z_0,w_0) = P(z,w)$ for some $(z_0,w_0) \in U_0$. This entails that if a sequence of holomorphic polynomials P_ν converges in $\mathcal{C}^0(U_0)$ to a CR function h then P_ν converges uniformly in $\tilde{V} \cap \Omega^+$, to a holomorphic function h whose boundary value on U_0 is equal to h. $\square$

Prop. 1.5 has also the following consequence, which will be of use later.

Proposition 1.8. *The following properties are equivalent:*

(1.5) *Given any open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ there is a compact set $K \subset \tilde{V} \cap \Omega^+$ such that the origin belongs to the polynomial hull of K,*

$$\hat{K} = \{z_0 \in \mathbb{C}^{n+1}; \forall P \in \mathbb{C}[z], |P(z_0)| \le \max_K |P|\}.$$

(1.6) *To each open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ there is an open subneighborhood $\tilde{V}_1 \subset \tilde{V}$ such that every holomorphic function in $\tilde{V} \cap \Omega^+$ extends as a holomorphic function in $\tilde{V}_1$.*

The reader will notice that, in (1.6), the holomorphic functions in $\tilde{V} \cap \Omega^+$ are not required to grow slowly at the edge $\tilde{V} \cap \Sigma$.

Proof. (1.5) $\Rightarrow$ (1.6). Let $\tilde{V}$ be an arbitrary open neighborhood of 0 in $\mathbb{C}^{n+1}$. Let $\tilde{V}_0$ be related to $\tilde{V}$ as in Prop. 1.5, and assume that $0 \in \hat{K}$, with $K \subset\subset \tilde{V}_0 \cap \Omega^+$. Denote by $\bar{\mathcal{B}}$ the closed unit ball in $\mathbb{C}^n$; if $\epsilon > 0$ is small enough $K + \epsilon\bar{\mathcal{B}}$ is a compact subset of $\Omega^+ \cap \tilde{V}_0$. The polynomial hull of $K + \epsilon\bar{\mathcal{B}}$ contains that of $K + \epsilon u$, i.e., the set $\hat{K} + \epsilon u$, whatever $u \in \bar{\mathcal{B}}$. This shows that the polynomial hull of $K + \epsilon\bar{\mathcal{B}}$ contains $\hat{K} + \epsilon\bar{\mathcal{B}}$. Since $0 \in \hat{K}$, $\epsilon\bar{\mathcal{B}}$ is contained in the polynomial hull of $K + \epsilon\bar{B}$. Let $\{P_\nu\}$ be a sequence of polynomials that converge uniformly in $K + \epsilon\bar{B}$ to the function h, defined and holomorphic in $\tilde{V} \cap \Omega^+$, the same will be true in $\epsilon\mathcal{B}$, to a holomorphic function which must be equal to h in $\Omega^+ \cap (\epsilon\mathcal{B})$.

(1.6) $\Rightarrow$ (1.5), by a standard argument. Suppose (1.5) does not hold, i.e., there is an open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ such that $0 \notin \hat{K}$, whatever the compact set $K \subset \tilde{V} \cap \Omega^+$. Let $\{K_\nu\}$ be a sequence of compact subsets of $\Omega^+ \cap \tilde{V}$ such that $K_\nu \subset K_{\nu+1}$ and that $\Omega^+ \cap \tilde{V}$ is the union of the sets K_ν. By hypothesis $0 \notin \hat{K}_\nu$ for all ν. As a consequence, for each ν there is a point $z_\nu \in \Omega^+ \backslash K_\nu$, $|z_\nu| \le 1/\nu$, and a polynomial P_ν such that $\max_{K_\nu} |P_\nu| < 1, P_\nu(z_\nu) = 1$. Raising P_ν to a suitable power allows us to assume $\max_{K_\nu} |P_\nu| < 2^{-\nu}$. It follows that the infinite product $\prod_{\nu=1}^{+\infty} [1 - P_\nu(z)]^\nu$ converges uniformly on every compact subset of $\Omega^+ \cap \tilde{V}$, to a holomorphic function h in $\Omega^+ \cap \tilde{V}$ which vanishes to infinite order at 0, and therefore cannot be extended holomorphically to a full neighborhood of the origin. □

The *characteristic set* of the CR structure of Σ is, by definition, the intersection of $T'^{1,0}\Sigma$ with the real cotangent bundle $T^*\Sigma$; it is the set of common zeros, in $T^*\Sigma$, of the symbols of all the sections of $T^{0,1}$. At each point of Σ it is spanned by the one-form $\theta = ds + \imath(Bd\bar{z} - \bar{B}dz)$, where $B = [1 + \imath\phi_s(z,s)]^{-1}\phi_{\bar{z}}(z,s)$ (we have used the notation $s = \Re ew$). Then, if L is any $\mathcal{C}^1$ section of $T^{0,1}\Sigma$ over Σ, we may regard the real function $(2\imath)^{-1}\langle\theta, [L, \bar{L}]\rangle$ as the *Levi form* $\mathcal{Q}$ of Σ evaluated at the section L. If $\mathcal{Q}(L) \geq 0$ for every $L \in \mathcal{C}^1(\Sigma; T^{0,1}\Sigma)$, Σ is said to be *weakly pseudoconvex.* If Σ is weakly pseudoconvex, then the domain Ω^+ is pseudoconvex. Keep in mind that Ω is a ball. The test case is $\phi(z,s) = |z|^2$; then $\theta = ds + \imath(zd\bar{z} - \bar{z}dz)$, L can be taken to be $\partial/\partial\bar{z} - \imath z\partial/\partial s$, and $[L, \bar{L}] = 2\imath\partial/\partial s$. Of course, the pseudoconvex side is the convex side, $\Im w > |z|^2$.

Theorem 1.9. *If Σ is weakly pseudoconvex, then not every germ at 0 of a CR function in Σ extends to Ω^-.*

Proof. Let $\mathcal{B} \subset \Omega$ denote an open ball in $\mathbb{C}^{n+1}$ centered at 0. Given $\epsilon > 0$ very small denote by $\mathcal{B}_\epsilon$ the set of points $(z,w) \in \mathcal{B}$ such that $\Im w > \phi(z, \Re ew) - \epsilon$. If Σ is weakly pseudoconvex the domain $\mathcal{B}_\epsilon$ is pseudoconvex, and therefore (this is where the depth of the result lies) it is a domain of holomorphy. There is a holomorphic function $\tilde{h}$ in $\mathcal{B}_\epsilon$ which does not extend holomorphically across the boundary of $\mathcal{B}_\epsilon$ at the point $z = 0$, $w = -\imath\epsilon$. The trace of $\tilde{h}$ on $\mathcal{B} \cap \Sigma$ does not extend holomorphically to the intersection $\tilde{V} \cap \mathcal{B}$ if the open neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$ contains the point $z = 0$, $w = -\imath\epsilon$. The sought conclusion ensues by Prop. 1.3. $\square$

Remark 1.10. According to Nirenberg-Kohn [1] there exist weakly pseudoconvex hypersurfaces $\Sigma \ni 0$ in $\mathbb{C}^2$ such that any holomorphic curve through 0 must cross into both sides of Σ. But combining Prop. 1.7 and Th. 1.9 shows that there is an open neighborhood U of 0 in Σ such that, given any neighborhood $\tilde{V}$ of 0 in $\mathbb{C}^{n+1}$, there are one-dimensional holomorphic varieties (and, by proximity, submanifolds) which intersect $\tilde{V} \cap \Omega^-$ but do not intersect U. Thus holomorphic curves in the pseudoconcave side Ω^- may get arbitrarily close to 0 without crossing Σ.

We now come to the most important result, obtained so far, in the question of the extension of CR functions on a hypersurface. It is Trepreau's theorem:

Theorem 1.11. *For at least some germ at 0 of a CR function on Σ not to extend to Ω^+ and at least some germ not to extend to Ω^-, it is necessary and sufficient that Σ contain the germ at 0 of a holomorphic variety of complex codimension one (the zero set of the germ at 0 of a holomorphic function).*

A Proof of Trepreau's Theorem

The sufficiency of the condition is easy to show. Let h be a holomorphic function in some open ball $\mathcal{B} \subset \Omega$ centered at 0, whose zero set contains 0 and is contained in $\Sigma \cap \mathcal{B}$. Whatever the real number $a \neq 0$, if $(z, w - \imath a) \in \mathcal{B}$ and if $h(z, w - \imath a) = 0$, then $\Im w - \phi(z, \Re e w) = a$ and therefore $(z, w) \notin \Sigma \cap \mathcal{B}$. If $\mathcal{B}_1$ is a strictly smaller ball, also centered at 0, and if $|a|$ is sufficiently small, the function $1/h(z, w - \imath a)$ is holomorphic in a full neighborhood of $\Sigma \cap \mathcal{B}_1$ in $\mathcal{B}_1$, but does not extend to any neighborhood of 0 in $\mathbb{C}^{n+1}$ that contains the point $z = 0$, $w = \imath a$. The conclusion follows then from Prop. 1.3, by letting a go to zero.

The original proof of the necessity, in Trepreau [1], exploited the hypothesis that no extension either to Ω^+ or to Ω^- occurs, to build a holomorphic hypersurface as a union of *analytic disks* that lie entirely in Σ. Instead, we shall apply Prop. 1.8, to negate the existence of any analytic disk whose boundary lies inside Ω^+ or Ω^- (and arbitrarily near 0) and whose interior contains the origin. (The proof in Tumanov [19] uses a different type of deformation of analytic disks.)

Let $\mathbf{S}^1$ denote the unit circle, and $\dot{H}^1(\mathbf{S}^1; \mathbb{R})$ the first Sobolev space of real-valued functions in $\mathbf{S}^1$ such that $\int_0^{2\pi} u(\theta) d\theta = 0$. We shall denote by $\|u\|_0$ the norm of u in $L^2(\mathbf{S}^1)$, and by $\|u\|_1 = \{\|u\|_0^2 + \|u_\theta\|_0^2\}^{\frac{1}{2}}$ the norm in $\dot{H}^1(\mathbf{S}^1; \mathbb{R})$.

Let J denote the restriction of the *Hilbert transform* to $\dot{H}^1(\mathbf{S}^1; \mathbb{R})$, i.e., the linear isometry of which transforms $\Im \sum_{p=1}^{\infty} c_p e^{\imath p\theta}$ into $\Re e \sum_{p=1}^{+\infty} c_p e^{\imath p\theta}$. We have $J^2 = -I$. We recall rapidly the solution of the *Bishop equation*, Eq. (1.8) below.

Lemma 1.12. *Let $\psi(\theta, s)$ be a real-valued C^2 function in $\mathbf{S}^1 \times \mathbb{R}$, such that*

$$|\psi_s| \leq \frac{1}{4}, \; |\psi_{ss}| + |\psi_{s\theta}| \leq \frac{1}{4}. \tag{1.7}$$

Then there is a unique function $v \in \dot{H}^1(\mathbf{S}^1; \mathbb{R})$ such that, if $0 \leq \theta \leq 2\pi$,

$$v(\theta) = \psi(\theta, \, \mathbf{J}v(\theta)) - \int_0^{2\pi} \psi(\omega, \, \mathbf{J}v(\omega)) d\omega / 2\pi. \tag{1.8}$$

Proof. Call $\mathcal{K}(\mathbf{J}v)(\theta)$ the right-hand side in (1.8). If $f \in \dot{H}^1(\mathbf{S}^1; \, \mathbb{R})$, $\mathbf{J}f$ is a continuous function. Direct differentiation with respect to θ shows that $\mathcal{K}(\mathbf{J}f) \in \dot{H}^1(\mathbf{S}^1; \, \mathbb{R})$. And the mean value theorem entails that, under Hypothesis (1.7), we have, for all f, $g \in \dot{H}^1(\mathbf{S}^1; \, \mathbb{R})$,

$$\|\mathcal{K}(\mathbf{J}f) - \mathcal{K}(\mathbf{J}g)\|_0 \leq \frac{1}{2} \|f - g\|_0 \, ,$$

$$\|\mathcal{K}(\mathbf{J}f) - \mathcal{K}(\mathbf{J}g)\|_1 \le \frac{1}{2}\|f-g\|_1 + \frac{1}{4}\|f-g\|_0 \cdot \|g\|_1.$$

We define

$$v_0 = \mathcal{K}(0);\ v_\nu = \mathcal{K}(\mathbf{J}v_{\nu-1}) \text{ if } \nu \ge 1.$$

Induction on $\nu \ge 1$ and the above estimates entail

$$\|v_\nu - v_{\nu-1}\|_0 \le 2^{-\nu}\|\mathcal{K}(0)\|_0,\ \|v_\nu - v_{\nu-1}\|_1 \le \nu 2^{-\nu}\|\mathcal{K}(0)\|_1 \,. \tag{1.9}$$

From (1.9) we conclude that v_ν converges in $\dot{H}^1(\mathbf{S}^1;\mathbb{R})$ to $v = \mathcal{K}(\mathbf{J}v)$. Uniqueness of the solution follows directly from the inequality $\|\mathcal{K}(\mathbf{J}f) - \mathcal{K}(\mathbf{J}g)\|_0 \le \frac{1}{2}\|f-g\|_0$. $\square$

We return to our function $\phi(z,s)$. After extending ϕ to the whole of $\mathbb{R}^{2n}\times\mathbb{R}$ and multiplying it by a cutoff function, we may assume that $\phi \in C^2(\mathbb{R}^{2n}\times\mathbb{R};\mathbb{R})$ and that $\phi \equiv 0$ outside a compact set. We apply Lemma 1.12 with

$$\psi(\theta,s) = \phi(\zeta z + \lambda\zeta^m z',\ s),$$

where $\zeta = \rho e^{\imath\theta}$, $\lambda \in \mathbb{C}$, $m \in \mathbb{Z}_+$ and $z,\ z' \in \mathbb{C}^n$. The unique solution in Lemma 1.12 will be denoted by $v_m(z,\ z',\ \zeta,\ \lambda)$. We have

$$\begin{aligned} v_m(z,z',\ \zeta,\ \lambda) &= \phi\left(\zeta z + \lambda\zeta^m z',\ \mathbf{J}v_m(z,z',\ \zeta,\ \lambda)\right) \\ &\quad - (2\pi\imath)^{-1} \oint_{|\tau|=\rho} \phi\left(\tau z + \lambda\tau^m z',\ \mathbf{J}v_m(z,z',\tau,\lambda)\right) d\tau/\tau. \end{aligned} \tag{1.10}$$

The following property will be of pivotal importance:

(1.11) $\exists r_0$, $\rho_0 > 0$, *and for each integer* $m = 1,2,\ldots$, *a number* $\delta_m > 0$, *such that, for all* z , $z' \in \mathbb{C}^n$, $|z|+|z'| < r_0$, *all* $\rho < \rho_0$, *all* $\lambda \in \mathbb{C}$, $|\lambda| < \delta_m$,

$$\oint_{|\tau|=\rho} \phi(\tau z + \lambda\tau^m z',\ \mathbf{J}v_m(z,\ z',\tau,\lambda))d\tau/\tau = 0. \tag{1.12}$$

Lemma 1.13. *If Property (1.11) does not hold true then, either every germ at* 0 *of a CR function on* Σ *extends to* Ω^+ *or else every germ extends to* Ω^-.

Proof. Suppose there are sequences of points z, z' converging to 0 in $\mathbb{C}^n$, and of numbers $\lambda \in \mathbb{C}$, $\rho > 0$ converging to zero, such that, for some integer $m \ge 0$ (possibly depending on z, z', λ, ρ), the integral at the left in (1.12) is, say, strictly negative. Let $F_m(\zeta) = F_m(\zeta; z, z', \lambda, \rho)$ be the holomorphic function in the disk $\Delta_\rho = \{\zeta \in \mathbb{C};\ |\zeta| < \rho\}$ equal to

$$\mathbf{J}v_m(z, z', \rho e^{\imath\theta}, \lambda) + \imath v_m(z, z', \rho e^{\imath\theta}, \lambda)$$

on the boundary $|\zeta| = \rho$. It follows at once from (1.9) that, as z, z', ρ, $\lambda \to 0$, $F_m(\partial\Delta_\rho)$ will be contained in arbitrarily small neighborhoods of 0; $F_m(0) = 0$, since $v_m \in \dot{H}^1(\mathbf{S}^1; \mathbb{R})$. The origin belongs to the polynomial hull of the image of $\partial\Delta_\rho$ under the map $\bar{\Delta}_\rho \ni \zeta \to (\zeta z + \lambda\zeta^m z', F_m(\zeta)) \in \mathbb{C}^n \times \mathbb{C})$. Thus Prop. 1.8 yields the result. □

Lemma 1.13 allows us to hypothesize that (1.11) holds. By (1.12) we have

$$v_m(z, z', \zeta, \lambda) = \phi(\zeta z + \lambda\zeta^m z', \mathbf{J}v_m(z, z', \zeta, \lambda)). \tag{1.13}$$

It is important to note that, when $\lambda = 0$, v_m is independent of z' as well as of m. Let us therefore write $v(z, \zeta) = v_m(z, z', \zeta, 0)$. As a matter of fact, $v(z, \zeta)$ is a function of the product ζz alone. Putting $\lambda = 0$ in (1.13) yields

$$v(z, \zeta) = \phi(\zeta z, \mathbf{J}v(z, \zeta)). \tag{1.14}$$

Keep in mind that $\mathbf{J}$ acts with respect to ζ; define $u(z, \zeta) = \mathbf{J}v(z, \zeta)$, $f(z) = u(z, 1) + \imath v(z, 1)$. Then $(z, f(z)) \in \Sigma$ provided $z \in \mathbb{C}^n$ is sufficiently close to 0. Moreover, if we note that zero is the solution of (1.14) when $z = 0$, we conclude that $f(0) = 0$. In view of this, Trepreau's theorem will be an immediate consequence of the following

Lemma 1.14. *If Property (1.11) holds, there is an open neighborhood of the origin in $\mathbb{C}^n$ in which f is holomorphic.*

Proof. We get, by differentiating (1.13) with respect to λ:

$$\partial_\lambda v_m = \zeta^m z' \cdot \phi_z(\zeta z + \lambda\zeta^m z', \mathbf{J}v_m) + \phi_s(\zeta z + \lambda\zeta^m z', \mathbf{J}v_m)\mathbf{J}\partial_\lambda v_m. \tag{1.15}$$

We extend the action of $\mathbf{J}$ to complex-valued functions by linearity and we set, for any (complex-valued) function $f \in \mathcal{C}^0(\mathbf{S}^1)$,

$$\mathcal{T}_\zeta f(z, \zeta) = \phi_s(\zeta z, \mathbf{J}v(z, \zeta))(\mathbf{J}f)(\theta),$$

recalling that $\zeta = |\zeta|e^{\imath\theta}$. We derive from (1.15) where we put $\lambda = 0$:

$$(\partial_\lambda v_m)(z, z', \zeta, 0) = (I - \mathcal{T}_\zeta)^{-1}[\zeta^m z' \cdot \phi_z(\zeta z, \mathbf{J}v(z, \zeta))]. \tag{1.16}$$

By differentiating both sides of (1.14) with respect to $z_j (j = 1, ..., n)$ one gets

$$z' \cdot v_z = \zeta z' \cdot \phi_z(\zeta z, \mathbf{J}v) + \phi_s(\zeta z, \mathbf{J}v) z' \cdot \partial_z \mathbf{J}v. \tag{1.17}$$

The operator $\mathbf{J}$ acts in the variable ζ and therefore commutes with $\partial/\partial z_j$, whence

$$\zeta z' \cdot \phi_z(\zeta z, \mathbf{J}v(z,\zeta)) = (I - \mathcal{T}_\zeta)(z' \cdot v_z(z,\zeta)) .$$

Substituting in (1.16) leads to

$$(\partial_\lambda v_m)(z, z', \zeta, 0) = (I - \mathcal{T}_\zeta)^{-1} \left(\zeta^{m-1}(I - \mathcal{T}_\zeta)(z' \cdot v_z(z,\zeta))\right) . \tag{1.18}$$

By differentiating the integral at the left in (1.12) with respect to λ [and taking (1.13) into account], putting $\lambda = 0$ and thanks to (1.18), we get

$$\oint_{|\tau|=\rho} (I - \mathcal{T}_\tau)^{-1} \left(\tau^{m-1}(I - \mathcal{T}_\tau)[z' \cdot v_z(z,\tau)]\right) d\tau/\tau = 0. \tag{1.19}$$

As we are going to show, a consequence of (1.19) is that, for fixed z and z', $z' \cdot v_z(z,\zeta)$ is a holomorphic function of ζ in some disk $|\zeta| < \rho_1$. If this is so then, perforce, $\mathbf{J}$ acts on $z' \cdot v_z(z,\zeta)$ as multiplication by $\imath$ and thus

$$\zeta^{m-1}\mathbf{J}(z' \cdot v_z(z,\zeta)) = \mathbf{J}\left(\zeta^{m-1} z' \cdot v_z(z,\zeta)\right) \tag{1.20}$$

and, as a consequence of (1.18) & (1.20),

$$(\partial_\lambda v_m)(z, z', \zeta, 0) = \zeta^{m-1} z' \cdot v_z(z,\zeta). \tag{1.21}$$

Next we let the operator $\bar\partial_\lambda$ act on both sides of Eq. (1.15) and put $\lambda = 0$:

$$(I - \mathcal{T}_\zeta)\partial_\lambda\bar\partial_\lambda v_m\big|_{\lambda=0} = |\zeta^m|^2[(z' \cdot \partial_z)(\bar z' \cdot \bar\partial_z)\phi](\zeta z, \mathbf{J}v) +$$

$$2\Re e\left\{\zeta^m (z' \cdot \partial_z \phi_s)(\zeta z, \mathbf{J}v)(\mathbf{J}\bar\partial_\lambda v_m\big|_{\lambda=0})\right\} + \phi_{ss}(\zeta z, \mathbf{J}v)\left|\mathbf{J}\bar\partial_\lambda v_m\big|_{\lambda=0}\right|^2 .$$

Taking (1.20) and (1.21) into account yields

$$(I - \mathcal{T}_\zeta)\partial_\lambda\bar\partial_\lambda v_m\big|_{\lambda=0} = |\zeta^m|^2[(z' \cdot \partial_z)(\bar z' \cdot \bar\partial_z)\phi](\zeta z, \mathbf{J}v) + \tag{1.22}$$

$$2\Re e\left\{\zeta^m (z' \cdot \partial_z \phi_s)(\zeta z, \mathbf{J}v)(\zeta^{m-1} z' \cdot \mathbf{J}v_z)\right\} + \phi_{ss}(\zeta z, \mathbf{J}v)|\zeta^{m-1} z' \cdot \mathbf{J}v_z|^2,$$

where $\mathbf{J}$ acts in the ζ variable. From (1.17) we derive

$$(I - \mathcal{T}_\zeta)(\bar z' \cdot \bar\partial_z)(z' \cdot \partial_z v) = |\zeta|^2(\bar z' \cdot \bar\partial_z)(z' \cdot \partial_z \phi)(\zeta z, \mathbf{J}v) + \tag{1.23}$$

$$2\Re e\left(\zeta(z'\cdot\partial_z\phi_s)(\zeta z,\mathbf{J}v)\bar{z}'\cdot\bar{\partial}_z\mathbf{J}v\right)+\phi_{ss}(\zeta z,\mathbf{J}v)|\mathbf{J}z'\cdot v_z|^2.$$

If we compare (1.22) & (1.23) we obtain:

$$\partial_\lambda\bar{\partial}_\lambda v_m\big|_{\lambda=0}=(I-\mathcal{T}_\zeta)^{-1}\left(|\zeta^{m-1}|^2(I-\mathcal{T}_\zeta)(\bar{z}'\cdot\bar{\partial}_z)(z'\cdot\partial_z v)\right), \tag{1.24}$$

whence, by (1.12) and (1.13),

$$\oint_{|\tau|=r}[(\bar{z}'\cdot\bar{\partial}_z)(z'\cdot\partial_z v)](z,\tau)d\tau/\tau=0. \tag{1.25}$$

We are assuming that $z'\cdot v_z(z,\zeta)$ is holomorphic with respect to ζ ; then the same must be true of $(\bar{z}'\cdot\bar{\partial}_z)(z'\cdot\partial_z v(z,\zeta))$. But by the same token, $\bar{z}'\cdot\bar{\partial}_z v(z,\zeta)$ is antiholomorphic with respect to ζ (recall that v is real) and the same must be true of $(z'\cdot\partial_z)(\bar{z}'\cdot\bar{\partial}_z v(z,\zeta))$. We conclude that $(z'\cdot\partial_z)(\bar{z}'\cdot\bar{\partial}_z v(z,\zeta))$ must be constant with respect to ζ. Thanks to (1.25) we conclude that

$$(\bar{z}'\cdot\bar{\partial}_z)(z'\cdot\partial_z v(z,\zeta))=0 \tag{1.26}$$

if $|\zeta|=r$ and for all z, z' sufficietly close to 0. But as v is a function of ζz alone, it is permitted to put $\zeta=1$, provided z stays in an appropriately small neighborhood of 0. We reach the conclusion that $v(z,1)$ is pluriharmonic, as we wanted.

It remains to show that the validity of (1.19), for all integers $m\geq 1$, all sufficiently small $\rho>0$ and all z, $z'\in\mathbb{C}^n$ sufficiently close to 0, implies that $z'\cdot v_z(z,\zeta)$ is a holomorphic function of ζ, $|\zeta|<\rho_1$. Putting $z'=z$ in (1.19) yields

$$\oint_{|\tau|=\rho}(I-\mathcal{T}_\tau)^{-1}\left(\tau^{m-1}(I-\mathcal{T}_\tau)(\tau\partial/\partial\tau)v(z,\tau)\right)d\tau/\tau=0\ . \tag{1.27}$$

In the remainder of the proof we may omit mention of z; we write $g(\zeta)=v(z,\zeta)$ and $g(\rho e^{\imath\theta})=\sum_{\nu=1}^{+\infty}[c_\nu(\rho)e^{\imath\nu\theta}+\bar{c}_\nu(\rho)e^{-\imath\nu\theta}]$. We must show that g is harmonic in a neighborhood of zero. If $\zeta=\rho e^{\imath\theta}$, $2\zeta g_\zeta(\zeta)=\sum_{\nu=1}^{+\infty}[p_\nu(\rho)e^{\imath\nu\theta}+q_\nu(\rho)e^{-\imath\nu\theta}]$ with $p_\nu=\rho c_\nu'+\nu c_\nu$, $q_\nu=\rho\bar{c}_\nu'-\nu\bar{c}_\nu$. We are going to prove that $q_\nu\equiv 0$, i.e., $c_\nu(\rho)=c_\nu(0)\rho^\nu$, for all $\nu\geq 1$, which indeed means that g is harmonic.

We write $\mathcal{T}_\zeta f=\mathcal{T}f=\varphi_s(\zeta\mathbf{J}g)\mathbf{J}f$ and we note that

$$(I-\mathcal{T})^{-1}[\zeta^{m-1}(I-\mathcal{T})(\zeta g_\zeta)]=\zeta^m g_\zeta+(I-\mathcal{T})^{-1}[\mathcal{T},\zeta^{m-1}](\zeta g_\zeta). \tag{1.28}$$

But $[\mathcal{T}, \zeta^{m-1}](\zeta g_\zeta) = \varphi_s(\zeta, \mathbf{J}g)[\mathbf{J}, \zeta^{m-1}](\zeta g_\zeta)$. We continue to write $\zeta = \rho e^{\imath\theta}$; then

$$2\imath\rho^{1-m}\zeta^{m-1}\mathbf{J}[\zeta g_\zeta(\zeta)] = \sum_{\nu=1}^{+\infty}[p_\nu(\rho)e^{\imath(\nu+m-1)\theta} - q_\nu(\rho)e^{-\imath(\nu-m+1)\theta}],$$

$$2\imath\rho^{1-m}\mathbf{J}(\zeta^m\partial_\zeta g)(\zeta) =$$

$$\sum_{\nu=1}^{+\infty} p_\nu(\rho)e^{\imath(\nu+m-1)\theta} + \sum_{\nu=1}^{m-1} q_\nu(\rho)e^{-\imath(\nu-m+1)\theta} - \sum_{\nu=m}^{+\infty} q_\nu(\rho)e^{-\imath(\nu-m+1)\theta}.$$

We reach the following conclusion

$$[\mathbf{J}, \zeta^{m-1}](\zeta g_\zeta) = -\imath\rho^{m-1}\sum_{\nu=1}^{m-1} q_\nu(\rho)e^{-\imath(\nu-m+1)\theta}.$$

Suppose we have proved $q_\nu \equiv 0$ for all $\nu < m-1$ (trivially true if $m \leq 2$). We derive

$$[\mathbf{J}, \zeta^{m-1}](\zeta g_\zeta) = -\imath\rho^{m-1}q_{m-1}(\rho). \tag{1.29}$$

This in turn implies

$$[\mathcal{T}, \zeta^{m-1}](\zeta g_\zeta) = \varphi_s(\zeta, \mathbf{J}g)[\mathbf{J}, \zeta^{m-1}](\zeta g_\zeta) = -\imath\rho^{m-1}\varphi_s(\zeta, \mathbf{J}g)q_{m-1}(\rho).$$

Since, on the other hand,

$$(2\imath\pi)^{-1}\oint_{|\zeta|=\rho} \zeta^m g_\zeta(\zeta)d\zeta/\zeta = \rho^{m-1}q_{m-1}(\rho)\ ,$$

we conclude that, after division by $2\imath\pi\rho^{m-1}$ of the integral at the left in (1.27) we get

$$\left\{1 - \oint_{|\zeta|=\rho} (I-\mathcal{T})^{-1}\varphi_s(\zeta, \mathbf{J}g(\zeta))d\zeta/2\pi\zeta\right\} q_{m-1}(\rho) = 0. \tag{1.30}$$

Since the L^∞ norm of φ_s is $\leq \frac{1}{4}$ [by (1.7)] the norm of the linear operator $\mathcal{T}$ acting on the space $L^2(\mathbf{S}^1)$ is also $\leq \frac{1}{4}$ (recall that $\mathbf{J}$ is an isometry). We conclude that

$$\left|\oint_{|\zeta|=\rho} (I-\mathcal{T})^{-1}f(\zeta)d\zeta/2\pi\zeta\right| \leq \frac{4}{3}\left\{\int_0^{2\pi} |f(re^{\imath\theta})|^2 d\theta/2\pi\right\}^{\frac{1}{2}}.$$

We apply this with $f(\zeta) = \varphi_s(\zeta, \mathbf{J}g(\zeta))$ and take once again advantage of (1.7). We reach the following conclusion:

$$\left| \oint_{|\zeta|=\rho} (I - \mathcal{T})^{-1} \varphi_s(\zeta, \mathbf{J}g(\zeta)) d\zeta / 2\pi\zeta \right| \leq \frac{1}{3}.$$

Taking this estimate into account in (1.30) yields $q_{m-1} \equiv 0$, which completes the proof of Trepreau's theorem. □

Remark 1.15. According to the preceding proof, the property that Σ contains the germ at 0 of a holomorphic variety of codimension one (which, a priori, might have singularities) is equivalent to (1.11).

As shown, (1.11) ensures the existence of a holomorphic function f in an open neighborhood of 0 in $\mathbb{C}^n$, such that $\Im f(z) = \phi(z, \Re e f(z))$ and $f(0) = 0$. In other words, if (1.11) holds, the hypersurface Σ contains the germ at 0 of an n-dimensional holomorphic submanifold (without singularities) of $\mathbb{C}^{n+1}$, namely the zero set of $w - f(z)$. □

Beyond Trepreau's Theorem

The key question left unanswered by Th.1.11 is that of the side to which extension occurs, when the hypersurface Σ does not contain the germ of any holomorphic hypersurface. In this connection only partial results are known.

The first result is a consequence of Theorems 1.9 and 1.11 combined:

Proposition 1.16. *If Σ is weakly pseudoconvex and does not contain the germ at 0 of a holomorphic hypersurface, then every germ of a CR function on Σ extends to the pseudoconvex side.*

Prop. 1.16 was proved in Bedford and Fornaess [1] under the additional hypothesis that the hypersurface Σ be smooth and have *finite type* at the origin. This is the same as saying that, given any holomorphic map $\mathbb{C}^n \ni \zeta \to (z(\zeta), w(\zeta)) \in \mathbb{C}^{n+1}$ with $(z(0), w(0)) = 0$, the Taylor expansion of the $n \times n$ matrix $\partial_\zeta \bar{\partial}_\zeta [\phi(z(\zeta), \Re e w(\zeta))]$ does not vanish identically (assuming $\phi \in \mathcal{C}^\infty$). By Remark 1.15 this precludes that Σ contain a holomorphic hypersurface passing through the origin.

Still under the hypothesis that Σ is of class $\mathcal{C}^\infty$ and of finite type at the origin, added precision is provided by the *sector property* of Baouendi and Treves [2], and by the more general *rays condition* of Fornaess and Rea [1]. These conditions are best described by looking at holomorphic *curves* (rather than hypersurfaces). Let Δ denote the open unit disk in $\mathbb{C}$. If Σ is of finite type at 0, there is a holomorphic map $\gamma : \Delta \ni \zeta \to (z(\zeta), w(\zeta)) \in \mathbb{C}^{n+1}$, with $z(0) = 0$, $w(0) = 0$, such that the Taylor expansion of $\phi(z(\zeta), \Re e w(\zeta))$ at $\zeta = 0$ contains a homogeneous term $p_m(\zeta, \bar{\zeta})$ (a real polynomial, homogeneous of degree m) that is not harmonic. Let m denote the smallest integer (perforce ≥ 2) for which this occurs:

$$\partial_\zeta \bar{\partial}_\zeta [\phi(z(\zeta), \Re e w(\zeta)) - p_m(\zeta, \bar{\zeta})] = O(|\zeta|^{m-1}).$$

The idea is to look at the arcs of the unit circle $\mathbf{S}^1$ in which $p_m(\cos\,\theta, \sin\,\theta) > 0$. If one such arc has length $> \pi/m$ (i.e., if the sector property holds) extension to Ω^+ does occur. Because p_m is not harmonic, there always is an arc of length $> \pi/m$ in which p_m is either > 0 or < 0, and thus extension always occurs to one side at least, also a consequence of Trepreau's theorem, or to both sides, as when m is odd (Th. I.2 in Baouendi and Treves [2]) or sometimes when m is even. Results similar to the last mentioned can be found in Boggess and Pitts [1].

The rays condition is a refinement of the sector property. Let $\Gamma \subset \mathbf{S}^1$ be the set that is left after every arc of length $\geq \pi/m$, in which $p_m(\cos\,\theta, \sin\,\theta) < 0$, has been deleted. The rays property holds if $p_m(\cos\,\theta_0, \sin\,\theta_0) > 0$ for some θ_0 in a connected component of Γ whose length is $> \pi/m$. In that case, extension to Ω^+ takes place. There is a partial converse to this, provided by the *grouped sectors property*, when all the connected components of Γ have length $< \pi/m$. Under this hypothesis, there are germs at 0 of CR functions on Σ that do not extend to Ω^+ (Th. 1 in Fornaess and Rea [1]).

When $m = 2, 4$, the sector property is necessary and sufficient for extension of every germ of CR function to Ω^+. The situation in the case $m = 4$ can be fully described. For simplicity, suppose $\Sigma \subset \mathbb{C}^2$ is defined by the equation $\Im w = p_4(x, y)$. There is no loss of generality in assuming $p_4(\cos\,\theta, \sin\,\theta) = c(1 - a\cos 2\theta)$ with $c \neq 0$. Extension of every germ of CR function on Σ to both sides occurs if $|a| > \sqrt{2}$, and only to one side (determined by the sign of c) if $|a| \leq \sqrt{2}$. Observe that weak pseudoconvexity or weak pseudoconcavity correspond to $|\alpha| \leq 4/3 (< \sqrt{2})$.

Obviously, the rays condition entails the sector property. For $m \geq 6$, there are examples when the rays property holds while the sector property does not: this is the case of the homogeneous polynomial $p_6(x, y) = -(y^2 - \epsilon x^2)^2 (y^2 - \alpha x^2)$ with $\alpha = tan^2(\pi/9)$ and $\epsilon > 0$ sufficiently small (Fornaess and Rea, *loc. cit.*, Example 2).

In certain borderline cases, between the rays condition and the grouped sector condition, extendability might well be determined by higher order terms in the Taylor expansion of $\phi(z(\zeta), \Re e w(\zeta))$.

All this suggests that we do not have, presently, at our disposal the conceptual tools needed to characterize the extendability to a given side. The place where to test such tools would be a *rigid* hypersurface in $\mathbb{C}^2$, i.e., a three-dimensional submanifold Σ of $\mathbb{C}^2$ defined by an equation $\Im w = \phi(z)(z \in \mathbb{C})$.

2. The Higher Codimension Case.

Here we consider a *generic* submanifold $\mathcal{M}$ of $\mathbb{C}^N(N \geq 2)$, of class $\mathcal{C}^\infty$. "Generic" means that the pullbacks to $\mathcal{M}$ of the differentials $dz_1, \ldots, dz_N$ are linearly independent and therefore span a vector subbundle $T'^{1,0}\mathcal{M}$ of $\mathbb{C}T^*\mathcal{M}$ whose rank (over $\mathbb{C}$) is equal to N. Its orthogonal in $\mathbb{C}T\mathcal{M}$, which we shall denote by $T^{0,1}\mathcal{M}$, is spanned by the linear combinations of $\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_N$ that are tangent to $\mathcal{M}$. Its complex rank is equal to $n = \dim_{\mathbb{R}} \mathcal{M} - N(\geq 0)$. The vector bundles $T'^{1,0}\mathcal{M}$ and $T^{0,1}\mathcal{M}$ define a CR structure on $\mathcal{M}$ whose characteristic set has rank $d = codim_{\mathbb{R}}\mathcal{M} = N - n$.

The local extension of CR functions on a generic manifold $\mathcal{M}$ most commonly considered is the holomorphic extension to *wedges* (with edges on $\mathcal{M}$). There are various definitions of such a wedge. We are going to use a rather concrete one, well suited to the our discussion. We reason in an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^N$. Let $z_1, \ldots, z_n, w_1, \ldots, w_d$ denote the coordinates in $\mathbb{C}^N$; we write
$z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_d)$. We assume $0 \in \mathcal{M}$ and that the tangent space to $\mathcal{M}$ at 0 is the real vector subspace $\Im w = 0$. Equations of $\mathcal{M}$ in $\mathcal{O}$ will be

$$\Im w_k = \phi_k(z, \Re e w_k),\ k = 1, \ldots d, \tag{2.1}$$

with $\phi = (\phi_1, \ldots, \phi_d) \in \mathcal{C}^2(\mathcal{O}; \mathbb{R}^d)$, $\phi_k(0,0) = 0$, $d\phi_k(0,0) = 0$ for all k.

Let $U \subset \mathcal{O}$ be an open neighborhood of 0 in $\mathcal{M}$ and Γ an open and convex cone in $\mathbb{R}^d$, with vertex at the origin. We define what for us will be a typical wedge with *edge* U,

$$\mathcal{W}_\delta(U, \Gamma) = \{(z, w + \imath t) \in \mathbb{C}^N \ ; \ (z, w) \in U, \, t \in \Gamma, \, |t| < \delta\} \tag{2.2}$$

$(\delta > 0)$. It is a routine matter to define the germ of a wedge in $\mathbb{C}^N$, at the point 0 and in the direction $\dot{v} \in \mathbf{S}^{d-1}$ (by letting U range over all open neighborhoods of 0 in $\mathcal{M}$, Γ over all open cones in $\mathbb{R}^d$ containing $\dot{v}$ and by letting δ tend to zero).

The distribution boundary value (on the edge U) of a holomorphic function h in $\mathcal{W}_\delta(U, \Gamma)$ is defined in the obvious manner: if $u \in \mathcal{C}_c^\infty(U)$, then

$$\langle bv \ h, u \rangle = \lim_{\lambda \to +0} (2\imath)^{-n} \int h(z, w + \imath\lambda v) u(z, w) dz \wedge d\bar{z} \wedge dw,$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, $d\bar{z} = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, $dw = dw_1 \wedge \cdots \wedge dw_d$, v is an arbitrary (but fixed) unit vector in Γ and the integration is carried out over U. The distribution $bv\, h$ on U is well defined (and independent of $v \in \Gamma$) if the absolute value of h *grows slowly at the edge*, i.e., if $|h(z, w + \imath\lambda v)|$ grows slower than some power of $1/\lambda$, as $\lambda \to +0$.

We say that there is *wedge extendability in* $\mathcal{M}$ *at* 0 when,

(2.3) *to each open neighborhood U of 0 in $\mathcal{M}$, there is an open subneighborhood V, an open cone $\Gamma \subset \mathbb{R}^d$ and a number $\delta > 0$ such that each CR distribution in U is the boundary value, in V, of a holomorphic function in the wedge $\mathcal{W}_\delta(V, \Gamma)$ whose absolute value grows slowly at the edge.*

One could as well look at CR hyperfunctions (in which case the extension would be an arbitrary holomorphic function in the wedge).

Next we turn to another, less familiar, feature of a CR structure — its *orbits*. By definition, an orbit of the CR structure of $\mathcal{M}$ in Ω, is an equivalence class of points in an arbitrary open subset Ω of $\mathcal{M}$, for the equivalence relation: $p' \approx p''$ defined as follows: there is a finite sequence of points in Ω, $p' = p_0$, $p_1, \ldots, p_r = p''$ such that, for each i, p_i and p_{i+1} lie on one and the same integral curve of some vector field $\Re e L = \frac{1}{2}(L + \bar{L})$, $L \in \mathcal{C}^\infty(\Omega; T^{0,1}\mathcal{M})$ (i.e., L a smooth section of $T^{0,1}\mathcal{M}$ over Ω). A general theorem in Sussman [1] states that every orbit is an immersed $\mathcal{C}^\infty$ submanifold of $\mathcal{M}$. Let $\mathfrak{g}(\Omega)$ denote the Lie algebra, for the commutation bracket, generated by all the real vector fields $\Re e L$, $L \in \mathcal{C}^\infty(\Omega; T^{0,1}\mathcal{M})$. The tangent space at an arbitrary point p to the orbit $\mathcal{L}_p$ in Ω through p contains the freezing of $\mathfrak{g}(\Omega)$ at p; and the orbits are minimal for this property. But the tangent space might not be equal to $\mathfrak{g}(\Omega)\big|_p$.

Example 2.1. Let $\mathcal{M}$ be a hypersurface, i.e., $d = 1$, and suppose there is a relatively compact subset Ω' of Ω in which the Levi form of $\mathcal{M}$ vanishes identically. Suppose the CR structure of $\mathcal{M}$ is of finite type in $\Omega \backslash \Omega'$. In Ω, there is only one orbit, Ω itself. In Ω' the orbits are the holomorphic submanifolds (of complex codimension 1) that are the leaves of the natural foliation of Ω'. $\square$

The significance of leaves is apparent in the following result (see Treves [1], Th. II.3.3):

Theorem 2.2. *The support of any CR distribution in an open subset Ω of $\mathcal{M}$ is a union of orbits in Ω of the CR structure of $\mathcal{M}$.*

In other words, if h is a CR distribution in Ω and if an orbit in Ω intersects supp h, it is entirely contained in supp h. Or, to rephrase this property: *the orbits of the CR structure propagate the zeros of the CR distributions.*

There is a local converse to Th. 2.2, proved in Baouendi and Rothschild [1]:

Theorem 2.3. *Let $\mathcal{L}$ be an orbit in Ω of the CR structure of $\mathcal{M}$, 0 a point of $\mathcal{L}$. If the open neighborhood U of 0 in $\mathcal{M}$ is sufficiently small, then there is a CR distribution u in U whose support is equal to $\mathcal{L} \cap U$.*

Proof. There is an open neighborhood $\mathcal{N}$ of 0 in $\mathcal{L}$ diffeomorphic to an embedded (but possibly not closed) submanifold of Ω, also denoted by $\mathcal{N}$. Since $\mathcal{L}$ is an orbit the restriction of $T^{0,1}\mathcal{M}$ to $\mathcal{N}$ is contained in $\mathbb{C}T\mathcal{N}$. It follows that the pullbacks to $\mathcal{N}$ of $dz_1, \dots, dz_n, dw_1, \dots, dw_d$ span a vector subbundle $T'_{\mathcal{N}}$ of $\mathbb{C}T^*\mathcal{N}$. Among these differentials we may find a basis $\omega_1, \dots, \omega_r$ of $T'_{\mathcal{N}}$ over $U \cap \mathcal{N}$ — provided the open neighborhood U of 0 in $\mathcal{M}$ is suitably small. Consider then the linear functional u on $C_c^\infty(U)$ defined by

$$\langle u, v\rangle = \int_{U \cap \mathcal{N}} v\, \omega_1 \wedge \cdots \wedge \omega_r \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

(Notice that $\dim_{\mathbb{R}} \mathcal{N} = rank\ T^{0,1}\mathcal{M} + rank\ T'_{\mathcal{N}} = n + r$. The cases $r = 0$ or $r = n + d$ are not excluded.) It is evident that supp $u \subset \mathcal{N} \cap U$ and it is checked at once, if U is sufficiently small, that u is not identically equal to zero. Finally, if L_j is the section of $T^{0,1}\mathcal{M}$ over U such that $L_j \bar{z}_k = \delta_{jk} (1 \le j, k \le n)$ then

$$\begin{aligned} &L_j v\, \omega_1 \wedge \cdots \wedge \omega_r \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \\ &\qquad = \pm d(v\, \omega_1 \wedge \cdots \wedge \omega_r \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n) \end{aligned}$$

and since supp v is compact, Stokes' theorem entails

$$\begin{aligned} \langle L_j u, v\rangle = \pm \int_{U \cap \mathcal{N}} & d(v\, \omega_1 \wedge \cdots \wedge \omega_r \wedge d\bar{z}_1 \wedge \cdots \\ & \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge \cdots \wedge d\bar{z}_n) = 0. \end{aligned}$$

This proves that $L_j u = 0$, i.e., u is a CR distribution. $\square$

We say, following Tumanov [1], that $\mathcal{M}$ is *not minimal at* 0 if there is an open neighborhood Ω of 0 in $\mathcal{M}$ in which the orbit $\mathcal{L}$ through 0 has dimension $<$ dim $\mathcal{M}$. Then supp u (with u as in Th. 2.3) is contained in a proper submanifold of U. It is impossible for (2.3) to hold: any holomorphic function h in a wedge $\mathcal{W}_\delta(V, \Gamma) (V \subset U)$ such that $bv\ h = u$ in V would vanish in an open and dense subset of V and therefore in the whole wedge $\mathcal{W}_\delta(V, \Gamma)$. We may restate this result as follows (Baouendi and Rothschild [1]):

Theorem 2.4. *For wedge extendability in the CR manifold $\mathcal{M}$ to hold at 0 it is necessary that $\mathcal{M}$ be minimal at 0 .*

The converse is Tumanov's theorem:

Theorem 2.5. *For wedge extendability in the CR manifold $\mathcal{M}$ to hold at 0 it is sufficient that $\mathcal{M}$ be minimal at 0.*

The proof, in Tumanov [1], is based on the construction of analytic disks with boundaries on the manifold $\mathcal{M}$. See also Baouendi-Rothschild [2].

References

1. Andreotti, A. and Hill, D. C., *[1] Complex characteristic coordinates and tangential Cauchy-Riemann equations*, Scuola Norm. Sup. Pisa Sci. Fis. Mat. **26** (1972), 299–324.
2. Baouendi, M. S., Jacobowitz, H. and Treves, F., *[1] On the analyticity of CR mappings*, Ann. of Math. **122** (1985), 365–400.
3. Baouendi, M. S. and Rothschild, L. P., *[1] Cauchy-Riemann functions on manifolds of higher codimension in complex space*, Invent. Math. **101** (1990), 45–56.
4. ———, *[2] Minimality and the extension of functions from generic manifolds*, preprint.
5. ———, *[3] A generalized complex Hopf lemma and its applications to CR mappings*, preprint.
6. Baouendi, M. S. and Treves, F., *[1] A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Annals of Math. **113** (1981), 387–421.
7. ———, *[2] About the holomorphic extension of CR functions on real hypersurfaces in complex space*, Duke Math. J. **51** (1984), 77–107.
8. Bedford, E. and Fornaess, J. E., *[1] Local extension of CR-functions from weakly pseudoconvex boundaries*, Michigan Math. J. **25** (1978), 259–262.
9. Boggess, A. and Pitts, J., *[1] CR extension near a point of higher type*, Duke Math. J. **52** (1985), 67–102.
10. Fornaess, J. E. and Rea, Cl., *[1] Local holomorphic extendability and non-extendability of CR-functions on smooth boundaries*, Ann. Scuola Norm. Sup. Pisa Sci. **XII** (1985), 491–502.
11. Helms, L. L., *[1] Introduction to Potential Theory, Wiley-Interscience 1969.*
12. Kohn, J. J. and Nirenberg, L., *[1] A pseudoconvex domain not admitting a holomorphic support function*, Math. Ann. **201** (1973), 265–268.
13. Levi, E. E., *[1] Sulle ipersuperfici dello spazio a 4 dimensioni che possono essere frontiera del campo di esistenza di une funzione analitica di due variabili complesse*, Ann. Mat. Pura Appl. **18** (1911), 69–79.
14. Lewy, H., *[1] On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann of Math. **74** (1956), 514–522.
15. Stensones, B., *Ph. D. Thesis*, Princeton University (1985).

16. Sussmann, H. J., *[1] Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc. **180** (1973), 171–188.
17. Trépreau, J. M., *[1] Sur le prolongement holomorphe des fonctions CR définies sur une hypersurface réelle de classe C^2 dans $\mathbb{C}^n$*, Invent. Math. **83** (1986), 583–592.
18. Treves, F., *[1] Hypo-analytic structures. Local theory*, Princeton University Press, Princeton, N.J. (1992).
19. Tumanov, A. E., *Extension of CR functions into a wedge from a manifold of finite type*, Mat. Sbornik **136** (1988), 128–139 (Russian); English translation in USSR Sbornik **64** (1989), 129–140.

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