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Shrawan Kumar and Michèle Vergne

Introduction

In the text of this article, definitions, propositions, theorems, lemmas, examples, corollaries are numerated in the same sequential order. Formulas follow an independent sequential order.

Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and let M be a smooth manifold on which G acts. Let $\mathcal{A}(M) = \sum_i \mathcal{A}^i(M)$ be the space of smooth differential forms on M and let $\mathcal{A}_{cpt}(M)$ be the subspace of forms with compact support. Let us recall (Cartan ;[10]) that the G-equivariant de Rham complex of M is by definition the differential $\mathbb{Z}_+$-graded algebra

$$\mathcal{A}_G(M) := (S(\mathfrak{g}') \otimes \mathcal{A}(M))^G$$

endowed with the tensor product graded algebra structure (where elements of $\mathfrak{g}'$ are assigned degree 2) together with the equivariant de Rham differential $d_{\mathfrak{g}}$ of degree 1 (see section 2, Formula 5). Its cohomology denoted $H^*_G(M)$ is called the G- equivariant de Rham cohomology of M. Alternatively, an element $\alpha \in \mathcal{A}_G(M)$ can be thought of as a differential form $\alpha(X)$ on M depending polynomially on $X \in \mathfrak{g}$, such that α is equivariant:

$$\alpha(g \cdot X) = g \cdot \alpha(X),$$

for all $g \in G$.

The complex $\mathcal{A}_G(M)$ admits a subcomplex

$$\mathcal{A}_{cpt,G}(M) := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M))^G,$$

and its cohomology is called the G-equivariant de Rham cohomology with compact support $H^*_{cpt,G}(M)$.

Sometimes, it is natural to consider the space $\mathcal{A}^\infty_G(M)$ of equivariant forms $\alpha(X)$ on M depending smoothly on $X \in \mathfrak{g}$. The differential $d_{\mathfrak{g}}$ extends to this space and the cohomology of the complex $(\mathcal{A}^\infty_G(M), d_{\mathfrak{g}})$ is denoted by $H^\infty_G(M)$.

This cohomology $H^\infty_G(M)$ is studied in the preceding article of this volume (notation differs slightly from the ones used in the preceding article. In particular, the dual vector space of $\mathfrak{g}$ is denoted here by $\mathfrak{g}'$ instead of $\mathfrak{g}^*$, the space denoted here by $\mathcal{A}^\infty_G(M)$ (resp. $H^\infty_G(M)$) was denoted by $\mathcal{A}^\infty_G(\mathfrak{g}, M)$ (resp.by $\mathcal{H}^\infty_G(\mathfrak{g}, M)$)). In some situations, it is also important to consider the space $\mathcal{A}^{-\infty}_G(M)$ of equivariant forms depending in a generalized way on the variable $X \in \mathfrak{g}$ (cf. section 2, Definition 3 for a precise definition). The differential $d_{\mathfrak{g}}$ still has a meaning on $\mathcal{A}^{-\infty}_G(M)$ and the cohomology of the complex $(\mathcal{A}^{-\infty}_G(M), d_{\mathfrak{g}})$ is denoted by $H^{-\infty}_G(M)$. (The space $\mathcal{A}^{-\infty}_G(M)$ and its cohomology $H^{-\infty}_G(M)$ were introduced in [12].) One similarly defines $H^{-\infty}_{cpt,G}(M)$. When M is a point, $H^{-\infty}_G(point)$ is equal to the space $C^{-\infty}(\mathfrak{g})^G$ of G-invariant generalized functions on $\mathfrak{g}$. There is a natural map $H^\infty_G(M) \to H^{-\infty}_G(M)$.

If M is compact and G-oriented, integration over M gives us a map from $H^{-\infty}_G(M)$ to $C^{-\infty}(\mathfrak{g})^G$. More generally if $p : M \to B$ is a G-equivariant fibration, with G-oriented fibers, then there is defined an integration along the fiber map

$$p_* : H^{-\infty}_{cpt,G}(M) \to H^{-\infty}_{cpt,G}(B)$$

(cf. Formula 8).

If M is non compact, and if $\alpha(X)$ is an equivariant form on M depending smoothly on $X \in \mathfrak{g}$, the integral of $\alpha(X)$ over M may sometimes exist in a generalized sense: after integrating $\alpha(X)$ against a test function Φ on the Lie algebra $\mathfrak{g}$, the form $(\alpha, \Phi) := \int_{\mathfrak{g}} \alpha(X)\Phi(X)dX$ may become integrable over M and we can define $\int_M \alpha \in C^{-\infty}(\mathfrak{g})^G$ by

$$(\int_M \alpha, \Phi) = \int_M (\alpha, \Phi).$$

Many important examples of generalized functions on $\mathfrak{g}$ arise this way. For instance, characters of representations of G attached to a generic coadjoint orbit M are given by the integral of an equivariant form over M (see [21], [12]). If G is compact, the formula of [20] for the index of a G-transversally elliptic operator D on a compact G-manifold B is given by the integration (in the generalized sense) over $M = T^*B$ of a G-equivariant form $\alpha(\sigma)(X)$ on M (depending smoothly on $X \in \mathfrak{g}$) attached to the symbol σ of D.

It will thus be useful to understand the space $H^{-\infty}_G(M)$. The aim of this article is to start a systematic study of the cohomology space $H^{-\infty}_G(M)$.

Now we describe some of the results we prove in this article.

We first prove (in section 2) that for a G-equivariant real vector bundle $p : \mathcal{V} \to B$, the canonical pull-back map $p^* : H^{-\infty}_G(B) \to H^{-\infty}_G(\mathcal{V})$ is an isomorphism (cf. Proposition 8). In particular, for a real representation V of G, $H^{-\infty}_G(V) \cong C^{-\infty}(\mathfrak{g})^G$. Similarly, we prove the Thom isomorphism; asserting

that if G is compact and the fibers of p are G-oriented, then the integration along the fiber map

$$p_* : H^{-\infty}_{cpt,G}(\mathcal{V}) \to H^{-\infty}_{cpt,G}(B)$$

is an isomorphism (cf. Proposition 11).

Let $K \subset G$ be a closed subgroup . Let $\chi = \chi_{G/K} : K \to \{\pm 1\}$ be the character of K defined by $\chi(k) = sign \det_{\mathfrak{g}/\mathfrak{k}} k$, for all $k \in K$. Let M be a K-manifold and let $H^{-\infty}_{K,\chi}(M)$ be the cohomology of the complex

$$(\mathcal{A}^{-\infty}_{K,\chi}(M), d_{\mathfrak{k}}) := (C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^{\chi}, d_{\mathfrak{k}})$$

(cf. Definition 49). Fix an orientation o on $\mathfrak{g}/\mathfrak{k}$. Consider the space $G \times_K M$, fibered over G/K with fiber M. In section 5, we define a cochain map

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,\chi}(M) \to \mathcal{A}^{-\infty}_{G}(G \times_K M)$$

(cf. Proposition 50) and prove that if K is compact, $Ind_{G/K,o}$ induces an isomorphism in cohomology (cf. Theorem 52). This is one of the central results of this article. The proof of this result relies on a study of the homology of the perturbed Koszul complexes defined in sections 3 and 4. This technique is already used in [13] for the study of G-equivariant cohomology with smooth coefficients.

Taking $M = point$, we get the isomorphism

$$C^{-\infty}(\mathfrak{k})^{\chi} \cong H^{-\infty}_{G}(G/K)$$

where $C^{-\infty}(\mathfrak{k})^{\chi} := \{f \in C^{-\infty}(\mathfrak{k}); k \cdot f = \chi(k) f, \text{ for all } k \in K\}$.

The explicit description of the isomorphism (cf. Proposition 43) indicates the analogy between $\int_{G/K} Ind_{G/K,o} f$ and characters of induced representations (cf. Proposition 44).

Recall [13] that $H^{\infty}_{G}(G/K)$ is canonically isomorphic to $C^{\infty}(\mathfrak{k})^{K}$. We determine the canonical map

$$C^{\infty}(\mathfrak{k})^{K} \cong H^{\infty}_{G}(G/K) \to C^{-\infty}(\mathfrak{k})^{\chi} \cong H^{-\infty}_{G}(G/K),$$

coming from the natural map $H^{\infty}_{G}(G/K) \to H^{-\infty}_{G}(G/K)$, in Proposition 53.

From now on in the introduction, the notation K will be reserved to denote a compact connected Lie group with maximal torus T and Weyl group W. The Lie algebras of K, T are denoted by $\mathfrak{k}$, $\mathfrak{t}$ respectively.

In section 6, we prove a Künneth theorem: Let D be a compact K-manifold such that $H_K(D)$ is free over $H_K(point)$ (e.g. $D = K/U$, for a closed subgroup $U \subset K$ of the same rank, cf. Lemma 65). Then, for any K-manifold M, the canonical Künneth map

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H^{-\infty}_{K}(M) \to H^{-\infty}_{K}(D \times M)$$

is an isomorphism (cf. Theorem 61). In fact Theorem 61 is true without the assumption on K to be connected, but then we need to add the assumption that the evaluation map $H_K(D) \to H(D)$ is surjective. In particular for any such D, taking M to be a point, we get

$$H_K^{-\infty}(D) \cong C^{-\infty}(\mathfrak{k})^K \otimes_{S(\mathfrak{k}')^K} H_K(D)$$

(cf. Corollary 64).

Using Künneth theorem and the Induction isomorphism, we obtain in Proposition 66 an extension of Chevalley's theorem. Let $C^{-\infty}(\mathfrak{t})^\epsilon$ be the space of all the generalized W-anti-invariant functions on $\mathfrak{t}$. Then the multiplication of generalized functions on $\mathfrak{t}$ by polynomial functions induces an isomorphism

$$C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \cong C^{-\infty}(\mathfrak{t}).$$

By the same technique, we obtain the Reduction Theorem asserting that for any K-manifold M, we have a canonical isomorphism

$$H_T^{-\infty}(M)^W \cong H_K^{-\infty}(M),$$

where $H_T^{-\infty}(M)^W$ refers to the W-invariants under the canonical action of W on $H_T^{-\infty}(M)$ (cf. Theorem 74). The proof of this reduction theorem is inspired by the proof of Theorem 4.2 in Atiyah [1].

Again combining the Künneth theorem and the Induction isomorphism, we obtain an isomorphism of $H_L(point)$-modules

$$H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_{L,\chi_{K/L}}^{-\infty}(M),$$

for any closed subgroup $L \subset K$ of the same rank and any K-manifold M (cf. Theorem 70). In particular, taking $M = K/U$ (for any closed subgroup U of K), we obtain

$$H_{L,\chi_{K/L}}^{-\infty}(K/U) \cong S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} (C^{-\infty}(\mathfrak{u})^{\chi_{K/U}}).$$

If M is a T-manifold, we give a homology spectral sequence (in section 10) with

$$E_p^2 = Tor_p^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$$

converging to the cohomology $H_T^{-\infty}(M)$, where $S(\mathfrak{t}')$ acts by multiplication on $C^{-\infty}(\mathfrak{t})$ (cf. Theorem 102). We show that this spectral sequence degenerates at the E^2-term for any homogeneous space $M = K/U$ (U any closed subgroup of K) (cf. Proposition 106).

In section 9, we study free actions. Let P be a principal G-bundle (for any Lie group G) and let $q : P \to P/G$ be the quotient map. Assume that the fibers

of q admit a G-orientation o. Then we prove (cf. Theorem 89) that $H_G^{-\infty}(P)$ is a free module over $H_G(P) \cong H(P/G)$ with a generator γ_o. We determine this generator explicitly (cf. Proposition 80). Consider, for example, the free action of T on $P = K$ by right translations. Then the space $H_T^{-\infty}(P)$ is a vector space of dimension $|W|$ over $\mathbb{R}$. We use the description of $H_T^{-\infty}(K)$ to conclude that the canonical map $H_T^{\infty}(K) \to H_T^{-\infty}(K)$ is identically 0 (cf. Corollary 96).

More generally, we consider the case of a manifold P with a right action of a Lie group G and where we assume that a normal closed subgroup N of G acts principally on P (cf Definition 75). In addition, assume that the principal N-bundle $q_N : P \to P/N$ admits a G-invariant connection and that the fibers of q_N admit a G-orientation o. We then construct a map

$$m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P),$$

and show that m_o is an isomorphism if G is compact (cf. Theorem 91).

In section 11, we prove a Localization theorem for any compact oriented T-manifold M. We first need to take a T-equivariant embedding of M in a representation space V of T. This gives rise to a certain non-zero polynomial $P \in S(\mathfrak{t}')$. Now we determine

$$P(X) \int_M \alpha(X) \in C^{-\infty}(\mathfrak{t}),$$

for any $\alpha \in H_T^{-\infty}(M)$, in terms of the restriction of α to M^T and of the equivariant Euler class of the normal bundle of the submanifold $M^T \subset M$ (cf. Theorem 107). One striking difference from the smooth case is that it is possible to have $\int_M \alpha(X) \neq 0$, for $\alpha \in H_T^{-\infty}(M)$, even though M^T may be empty. In fact, we prove that $\int_K : H_T^{-\infty}(K) \to C^{-\infty}(\mathfrak{t})$ is injective, where T acts on K by right multiplication (cf. Proposition 95, section 9).

Finally in the appendix, we prove that if M is a paracompact manifold, then the de Rham differential d admits a continuous splitting on the space of exact differential forms on M. This result seems to be new and interesting on its own. In a similar way, we prove that the equivariant de Rham differential $d_{\mathfrak{g}}$ for the action of a compact Lie group G on a paracompact manifold M admits a continuous splitting. We were motivated to prove this result, as this enables us to obtain the spectral sequence of section 10 for any paracompact T-manifold.

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1 Notation

By a manifold M, we shall always mean a paracompact C^∞ real manifold without boundary (unless otherwise stated). We denote by $C^\infty(M)$ the space of C^∞- (real-valued) functions on M. We denote by $C^{-\infty}(M)$ the space of generalized (real-valued) functions on M. By definition, $C^{-\infty}(M)$ is the continuous dual of the space of smooth compactly supported densities on M under the C^∞-topology. The space $C^\infty(M)$ is canonically a subspace of $C^{-\infty}(M)$ and $C^{-\infty}(M)$ is a module over $C^\infty(M)$.

We denote the space of smooth differential forms on M (with real coefficients) by $\mathcal{A}^*(M)$. We denote the subspace of compactly supported differential forms by $\mathcal{A}^*_{cpt}(M)$. The exterior derivative is denoted by d_M or simply by d. If ξ is a vector field on M, we denote by $\iota(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ the contraction by the vector field ξ. We denote by $\mathcal{L}(\xi) : \mathcal{A}^*(M) \to \mathcal{A}^*(M)$ the Lie derivative action of ξ. The operators $d, \iota(\xi), \mathcal{L}(\xi)$ on $\mathcal{A}(M)$ satisfy the Cartan relation :

$$d\iota(\xi) + \iota(\xi)d = \mathcal{L}(\xi). \tag{1}$$

If M is oriented, for $\alpha \in \mathcal{A}_{cpt}(M) = \oplus_{i=0}^{\dim M} \mathcal{A}^i_{cpt}(M)$, we note $\int_M \alpha$ the integral of the component of α in $\mathcal{A}^{\dim M}_{cpt}(M)$.

Let G be a real Lie group. By a G-manifold, we mean a manifold on which G acts smoothly. Let $\mathfrak{g}$ be the Lie algebra of G. If $X \in \mathfrak{g}$, we denote by X_M (or simply X, if no confusion is likely) the vector field on M such that

$$(X_M \cdot \varphi)(x) = \frac{d}{d\varepsilon}\varphi((\exp -\varepsilon X)x)|_{\varepsilon=0}$$

for $\varphi \in C^\infty(M)$, $x \in M$.

Unless otherwise stated, vector spaces are over $\mathbb{R}$, and the dual $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of a vector space V is denoted by V'. If $E^i, 1 \leq i \leq n$ is a basis of a n-dimensional vector space V, E_i denotes the dual basis of V'.

Tensor products without subscripts will mean over $\mathbb{R}$. Unless otherwise indicated, cohomology of a manifold is taken to be the de Rham cohomology (with real coefficients).

In this article, $\mathbb{Z}/2$-graded objects will carry a superscript $\bullet$, while $\mathbb{Z}$-graded objects will carry a superscript $*$. A vector space with a $\mathbb{Z}/2$-grading will often

be called a superspace. In defining actions on the tensor product $V \otimes W$ of two $\mathbb{Z}/2$-graded vector spaces V, W, we will respect usual rules of signs. For example, an odd endomorphism A of the $\mathbb{Z}/2$-graded vector space W is extended to an endomorphism still denoted by A of $V \otimes W$ by defining

$$A(v \otimes w) = v \otimes Aw \qquad \text{if } v \in V^{even}, w \in W. \tag{2}$$

$$A(v \otimes w) = -v \otimes Aw \qquad \text{if } v \in V^{odd}, w \in W. \tag{3}$$

Any $\mathbb{Z}$-graded object C^* can of course be thought of as a $\mathbb{Z}/2$-graded object $C^\bullet$ by defining

$$C^{even} = \sum_{n \in \mathbb{Z}} C^{2n}, \qquad C^{odd} = \sum_{n \in \mathbb{Z}} C^{2n+1}.$$

Lie algebra of any real Lie group will be denoted by the same lower case German letter.

If a group G acts on a set E, we denote by E^G the subset of invariants.

2 G-equivariant cohomology with generalized coefficients - Basic definitions

Let G be a Lie group and let M be a G-manifold. Let us recall the definition of the G-equivariant de Rham cohomology $H^*_G(M)$ of M:

Let $S(\mathfrak{g}')$ be the symmetric algebra of $\mathfrak{g}'$. Consider the $\mathbb{Z}_+$-graded space $S(\mathfrak{g}') \otimes \mathcal{A}(M)$, where the degree of an element $P \otimes \alpha, P \in S^p(\mathfrak{g}'), \alpha \in \mathcal{A}^q(M)$ is defined by

$$\deg(P \otimes \alpha) = 2p + q. \tag{4}$$

We refer to this degree as being the total degree.

Let E^i be a basis of $\mathfrak{g}$ and let $E_i \in \mathfrak{g}'$ be the dual basis. Define the operator $d_{\mathfrak{g}}$ of degree 1 on $S(\mathfrak{g}') \otimes \mathcal{A}(M)$ by:

$$d_{\mathfrak{g}}(P \otimes \alpha) = P \otimes d_M \alpha - \sum_i E_i P \otimes \iota(E^i_M)\alpha$$

for $P \in S(\mathfrak{g}'), \alpha \in \mathcal{A}(M)$. This expression is independent of the choice of the basis E^i, as the element $\sum_i E_i \otimes E^i \in \mathfrak{g}' \otimes \mathfrak{g}$ is the canonical element $I \in \mathrm{End}(\mathfrak{g})$, where I is the identity element of $\mathrm{End}(\mathfrak{g})$.

We often will identify $S(\mathfrak{g}')$ with the space of polynomial functions on $\mathfrak{g}$. Writing $X \in \mathfrak{g}$ as $X = \sum x_i E^i$, we identify E_i with the linear coordinate function x_i. An element α of the space $S(\mathfrak{g}') \otimes \mathcal{A}(M)$ can be viewed as a

polynomial map $X \mapsto \alpha(X)$ from $\mathfrak{g}$ to $\mathcal{A}(M)$ and then the operator $d_{\mathfrak{g}}$ is given by the formula:

$$(d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \iota(X_M)(\alpha(X)) \tag{5}$$

or by

$$(d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \sum_i x_i \iota(E_M^i)(\alpha(X)). \tag{6}$$

Consider the action of G on $S(\mathfrak{g}')$ induced from the adjoint representation of G on $\mathfrak{g}$ and the action of G on $\mathcal{A}(M)$ induced from the action of G on M. Let

$$\mathcal{A}_G(M) := (S(\mathfrak{g}') \otimes \mathcal{A}(M))^G$$

be the space of G-invariants in $S(\mathfrak{g}') \otimes \mathcal{A}(M)$. In other words, an element α of $\mathcal{A}_G(M)$ is an equivariant polynomial map (i.e.$\alpha(g \cdot X) = g \cdot (\alpha(X))$) from $\mathfrak{g}$ to $\mathcal{A}(M)$. The operator $d_{\mathfrak{g}}$ commutes with the tensor product action of G on $S(\mathfrak{g}') \otimes \mathcal{A}(M)$, thus $d_{\mathfrak{g}}$ preserves $\mathcal{A}_G(M)$. The Cartan relation (1)

$$\mathcal{L}(X_M) = d\iota(X_M) + \iota(X_M)d$$

implies $(d_{\mathfrak{g}}^2\alpha)(X) = -\mathcal{L}(X_M)(\alpha(X))$. Thus $(d_{\mathfrak{g}}^2\alpha)(X) = 0$ for $\alpha \in \mathcal{A}_G(M)$ and thus $(\mathcal{A}_G(M), d_{\mathfrak{g}})$ is a complex.

Definition 1 *Define:*

$$Z_G(M) = \{\alpha \in \mathcal{A}_G(M), d_{\mathfrak{g}}\alpha = 0\},$$

$$B_G(M) = \{\alpha \in \mathcal{A}_G(M), \alpha = d_{\mathfrak{g}}\beta, \text{for some} \beta \in \mathcal{A}_G(M)\}$$

and

$$H_G(M) = Z_G(M)/B_G(M).$$

The space $H_G(M)$ is called the *G-equivariant de Rham cohomology of M*. The cohomology $H_G(M)$ inherits the $\mathbb{Z}_+$-grading from $\mathcal{A}_G^*(M)$. The graded algebra $S(\mathfrak{g}')^G$ of invariant polynomial functions on $\mathfrak{g}$ acts by multiplication on $\mathcal{A}_G(M)$. This action commutes with the differential $d_{\mathfrak{g}}$. Thus $H_G^*(M)$ is a $\mathbb{Z}$-graded $S(\mathfrak{g}')^G$-module.

In particular for G reduced to the identity element, the space $Z_G(M) \subset \mathcal{A}(M)$ is the subspace $Z(M)$ of closed differential forms on M, the space $B_G(M) \subset \mathcal{A}(M)$ is the space $B(M)$ of exact differential forms and $H_G^*(M)$ is the usual de Rham cohomology $H^*(M)$ with real coefficients.

If K is a closed subgroup of G, the restriction to $\mathfrak{k}$ of a function defined on $\mathfrak{g}$ induces a map from $H_G^*(M)$ to $H_K^*(M)$. In particular, evaluation at $0 \in \mathfrak{g}$: $\alpha \mapsto \alpha(0)$ induces a map from $H_G^*(M)$ to $H^*(M)$.

The complex $\mathcal{A}_G(M)$ has a subcomplex

$$\mathcal{A}_{cpt,G}(M) := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M))^G.$$

The *compactly supported G-equivariant de Rham cohomology* $H^*_{cpt,G}(M)$ of M is defined as the cohomology of the complex $(\mathcal{A}^*_{cpt,G}, d_{\mathfrak{g}})$.

We may also consider the space $C^\infty(\mathfrak{g}, \mathcal{A}(M))$ of C^∞-maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. The group G acts naturally on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$. The operator $d_{\mathfrak{g}}$ is defined by the same formula (5) on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$. The space $C^\infty(\mathfrak{g}, \mathcal{A}(M))$ has a $\mathbb{Z}/2$-grading given by parity of differential forms. The operator $d_{\mathfrak{g}}$ is an odd operator on this superspace. However it is impossible to define a $\mathbb{Z}_+$-grading on $C^\infty(\mathfrak{g}, \mathcal{A}(M))$, such that $d_{\mathfrak{g}}$ would be of degree 1.

We denote by

$$\mathcal{A}^\infty_G(M) = C^\infty(\mathfrak{g}, \mathcal{A}(M))^G,$$

the space of G-equivariant C^∞ maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. The Cartan relation implies again $d^2_{\mathfrak{g}} = 0$ on $\mathcal{A}^\infty_G(M)$.

Definition 2 *Define:*

$$Z^\infty_G(M) = \{\alpha \in \mathcal{A}^\infty_G(M), d_{\mathfrak{g}}\alpha = 0\},$$

$$B^\infty_G(M) = \{\alpha \in \mathcal{A}^\infty_G(M), \alpha = d_{\mathfrak{g}}\beta \quad \text{for some } \beta \in \mathcal{A}^\infty_G(M)\}$$

and

$$H^\infty_G(M) = Z^\infty_G(M)/B^\infty_G(M).$$

Introduce (as in [12]) the space $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ of generalized functions on $\mathfrak{g}$ with values in the space $\mathcal{A}(M)$. This is, by definition, the space of continuous $\mathbb{R}$-linear maps $\mathrm{Hom}(\mathcal{D}(\mathfrak{g}), \mathcal{A}(M))$ from the space of smooth compactly supported densities $\mathcal{D}(\mathfrak{g})$ on $\mathfrak{g}$ to the space $\mathcal{A}(M)$, where $\mathcal{D}(\mathfrak{g})$ and $\mathcal{A}(M)$ are both endowed with the C^∞-topologies. Thus, if α is an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ and if Φ is a smooth compactly supported density on $\mathfrak{g}$, then (α, Φ) is a differential form on M denoted by $\int_{\mathfrak{g}} \alpha(X) d\Phi(X)$. A compactly supported C^∞ density on $\mathfrak{g}$ will be called a *test density* (on $\mathfrak{g}$). A compactly supported C^∞ function on $\mathfrak{g}$ will be called a *test function.* We write d_M for the operator on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ defined by

$$(d_M\alpha, \Phi) = d_M(\alpha, \Phi), \qquad \text{for } \Phi \text{ a test density,}$$

and ι for the operator defined by

$$(\iota\alpha, \Phi) = \sum_i \iota(E^i_M)(\alpha, x_i\Phi).$$

Then define the operator $d_{\mathfrak{g}}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ by

$$d_{\mathfrak{g}}\alpha = d_M\alpha - \iota\alpha.$$

Observe that for $\alpha \in C^\infty(\mathfrak{g}, \mathcal{A}(M)) \subset C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$, the operator $d_{\mathfrak{g}}$ coincides with the operator $d_{\mathfrak{g}}$ introduced above. We thus will also write informally

$$(d_{\mathfrak{g}}\alpha)(X) = d_M(\alpha(X)) - \sum_i x_i \iota(E_M^i)(\alpha(X))$$

for $\alpha \in C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$.

The group G acts naturally on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$:

$$(g\alpha, \Phi) = g \cdot (\alpha, g^{-1} \cdot \Phi).$$

It can be easily seen that the operators d and ι commute with the action of G. Define

$$\mathcal{A}_G^{-\infty}(M) = C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))^G$$

as the space of G -equivariant $C^{-\infty}$-maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. An element of the space $\mathcal{A}_G^{-\infty}(M)$ will be called a *G-equivariant form with generalized coefficients*, or simply *an equivariant form.* If Φ is a test function on $\mathfrak{g}$, we denote by Φ^g the function $\Phi^g(X) = \Phi(gX)$. Let dX be an Euclidean measure on $\mathfrak{g}$. For $\alpha \in \mathcal{A}_G^{-\infty}(M)$ and $g \in G$, we have

$$|\det{}_{\mathfrak{g}}(g)|\left(\int_{\mathfrak{g}} \alpha(X)\Phi^g(X)dX\right) = g^{-1} \cdot \left(\int_{\mathfrak{g}} \alpha(X)\Phi(X)dX\right). \tag{7}$$

The operator $d_{\mathfrak{g}}$ preserves $\mathcal{A}_G^{-\infty}(M)$ and the Cartan relation (1) implies again $d_{\mathfrak{g}}^2 = 0$ on $\mathcal{A}_G^{-\infty}(M)$.

Definition 3 *Define:*

$$Z_G^{-\infty}(M) = \{\alpha \in \mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}}\alpha = 0\},$$

$$B_G^{-\infty}(M) = \{\alpha \in \mathcal{A}_G^{-\infty}(M), \alpha = d_{\mathfrak{g}}\beta \qquad \text{for some } \beta \in \mathcal{A}_G^{-\infty}(M)\}$$

and

$$H_G^{-\infty}(M) = Z_G^{-\infty}(M)/B_G^{-\infty}(M).$$

An equivariant form in $Z_G^{-\infty}(M)$ (resp. $B_G^{-\infty}(M)$) is said to be *closed* (resp. *exact*).

Observe that the parity of the exterior degree on $\mathcal{A}(M)$ induces a $\mathbb{Z}/2$-degree on the preceding spaces. We denote them by

$$Z_G^{-\infty}(M)^\bullet, B_G^{-\infty}(M)^\bullet, H_G^{-\infty}(M)^\bullet.$$

The ring $S(\mathfrak{g}')^G$ of invariant polynomial functions on $\mathfrak{g}$ acts by multiplication on $\mathcal{A}_G^{-\infty}(M)$. This action commutes with the differential $d_{\mathfrak{g}}$. Thus $H_G^{-\infty}(M)^\bullet$

is a $S(\mathfrak{g}')^G$-module. In fact $H_G^{-\infty}(M)$ is a module for $H_G^{\infty}(M)$ under left multiplication.

If M is a point, then, from the definition, it is clear that $H_G(point) = S(\mathfrak{g}')^G$, $H_G^{\infty}(point) = C^{\infty}(\mathfrak{g})^G$ and $H_G^{-\infty}(point) = C^{-\infty}(\mathfrak{g})^G$.

There is a natural map

$$H_G^{\infty}(M) \to H_G^{-\infty}(M).$$

Define similarly $(\mathcal{A}_{cpt,G}^{-\infty}(M), d_{\mathfrak{g}})$ as the subcomplex of $(\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}})$, consisting of all $\alpha \in \mathcal{A}_G^{-\infty}(M)$ such that $(\alpha, \Phi) \in \mathcal{A}_{cpt}(M)$, for all test densities Φ, and define $H_{cpt,G}^{-\infty}(M)$ as the cohomology of this subcomplex.

If $\phi : N \to M$ is a G- equivariant map between two G-manifolds, then the pull-back of differential forms induces a cochain map

$$\phi^* : (\mathcal{A}_G^{-\infty}(M), d_{\mathfrak{g}}) \to (\mathcal{A}_G^{-\infty}(N), d_{\mathfrak{g}}),$$

in particular a map in cohomology (again denoted by)

$$\phi^* : H_G^{-\infty}(M) \to H_G^{-\infty}(N).$$

Thus the correspondence $M \mapsto H_G^{-\infty}(M)$ is a contravariant functor from the category of G-manifolds and G-equivariant maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g}')^G$-modules.

Similarly the correspondence $M \mapsto H_{cpt,G}^{-\infty}(M)$ is a contravariant functor from the category of G-manifolds and G-equivariant proper maps to the category of $\mathbb{Z}/2$-graded $S(\mathfrak{g}')^G$-modules.

Definition 4 *Let $p : M \to B$ be a G-equivariant fibration of G-manifolds. Then the map p is said to have* G-oriented *fibers if the fibers of p are oriented with an orientation varying continuously and if the G-action on M preserves the orientation of all the fibers.*

If $p : \mathcal{V} \to B$ is a G-equivariant real vector bundle and if p has G-oriented fibers, we will just say that the *vector bundle* $\mathcal{V}$ is *G-oriented.* We say that a *G-manifold* B is *G-oriented* if the tangent bundle of B is G-oriented.

If M is G-oriented, integration over M defines a map $\int_M$ from $\mathcal{A}_{cpt,G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$:

$$(\int_M \alpha, \Phi) := \int_M (\alpha, \Phi), \qquad \text{for any test density } \Phi \text{ on } \mathfrak{g}.$$

This map induces a map from $H_{cpt,G}^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$.

If $p : M \to B$ is a G-equivariant fibration of G-manifolds with G-oriented fibers, then the integration over the fibers gives a cochain map denoted by $\int_{M/B}$ or by p_* from $\mathcal{A}^{-\infty}_{cpt,G}(M)$ to $\mathcal{A}^{-\infty}_{cpt,G}(B)$:

$$(\int_{M/B} \alpha, \Phi) := \int_{M/B} (\alpha, \Phi), \qquad \text{for any test density } \Phi \text{ on } \mathfrak{g}. \tag{8}$$

In particular, we get an induced map in cohomology (again denoted by) p_* or $\int_{M/B}$ (depending on the choice of a G-orientation):

$$p_* : H^{-\infty}_{cpt,G}(M) \to H^{-\infty}_{cpt,G}(B).$$

Similarly if, in addition, p is a proper map, we get the integration map

$$p_* : H^{-\infty}_G(M) \to H^{-\infty}_G(B).$$

Observe that if $\alpha \in H^{\infty}_{cpt,G}(M), \beta \in H^{-\infty}_G(B)$, then $\alpha \wedge p^*\beta \in H^{-\infty}_{cpt,G}(M)$ and we have

$$p_*(\alpha \wedge p^*\beta) = p_*\alpha \wedge \beta. \tag{9}$$

(Our sign convention for p_* is as in [3], chapter 1.)

If $\mathcal{E} \to M$ is a G-equivariant real vector bundle, we introduce the manifold $M_{\mathcal{E}}$: An element of $M_{\mathcal{E}}$ is a couple (m, o), where $m \in M$ and o is an orientation of the fiber $\mathcal{E}_m$. Then $M_{\mathcal{E}}$ has a canonical G-manifold structure. Define a G-equivariant diffeomorphism of $M_{\mathcal{E}}$ by $\epsilon(m, o) = (m, -o)$. As in section 5 of [13], we may also consider the $\mathcal{E}$-twisted cohomology group $H_G(M)_{\mathcal{E}}$, which is by definition the cohomology of the complex

$$\mathcal{A}_G(M)_{\mathcal{E}} := \{\alpha \in \mathcal{A}_G(M_{\mathcal{E}}); \epsilon \cdot \alpha = -\alpha\}.$$

We define similarly the $\mathcal{E}$-twisted groups $H^{\pm\infty}_G(M)_{\mathcal{E}}$ and $H^{\pm\infty}_{cpt,G}(M)_{\mathcal{E}}$. If $p : M \to B$ is a G-equivariant fibration with vertical tangent bundle $\mathcal{V}$, then we get the integration map

$$p_* : H^{\pm\infty}_{cpt,G}(M)_{\mathcal{V}} \to H^{\pm\infty}_{cpt,G}(B).$$

If $TM \to M$ is the tangent bundle, the manifold M_{TM} is denoted by M_t, the TM-twisted group $H_G(M)_{TM}$ is denoted by $H_G(M)_t$ and the compactly supported twisted group by $H_{cpt,G}(M)_t$. If M is G-oriented, the space $H_G(M)_t$ is canonically isomorphic with $H_G(M)$. In general $H_G(M)_t$ is a module over $H_G(M)$. Taking the fibration $p : M \to \textit{point}$, we get

$$p_* = \int_M : H_{cpt,G}(M)_t \to S(\mathfrak{g}')^G$$

even if M is not oriented. We thus can form the bilinear map $(,) : H_G(M) \times H_{cpt,G}(M)_t \to S(\mathfrak{g}')^G$ defined by: for $\alpha \in H_G(M)$ and $\beta \in H_{cpt,G}(M)_t$

$$(\alpha, \beta) = \int_M \alpha\beta.$$

If $G = e$, the bilinear form above (over $\mathbb{R}$) is non-degenerate. For lack of reference we include a proof of the following proposition.

Proposition 5 *Let G be a compact Lie group and let M be a G-manifold. If the evaluation map $H_G(M) \to H(M)$ is surjective, then*

1. *the evaluation map $H_{cpt,G}(M)_t \to H_{cpt}(M)_t$ is surjective*

2. *the spaces $H_G(M)$ and $H_{cpt,G}(M)_t$ are free modules over $S(\mathfrak{g}')^G$*

3. *if M is compact, the bilinear form $(.,.)$ induces an isomorphism of $H_G(M)_t$ with $Hom_{S(\mathfrak{g}')^G}(H_G(M), S(\mathfrak{g}')^G)$.*

Remark 6 *Let G be a compact connected Lie group and let M be a G-manifold such that $H_G(M)$ is a free $H_G(point)$-module. Then the Eilenberg-Moore spectral sequence (see [16], chapter 3, section 1) degenerates at the E^2-term. In particular, the evaluation map $H_G(M) \to H(M)$ is surjective. Observe that the assumption that G is connected can not be dropped here. Consider for example $G = O(3)$ and $M = O(3)/O(2)$.*

Proof: Let $n = \dim M$. For $\alpha \in S(\mathfrak{g}') \otimes \mathcal{A}(M)$ we define the exterior degree of α to be the smallest integer k such that $\alpha \in S(\mathfrak{g}') \otimes \left(\sum_{i=0}^k \mathcal{A}^i(M)\right)$ and we write $\alpha = \sum_{i=0}^k \alpha_{[i]}$ with $\alpha_{[i]} \in S(\mathfrak{g}') \otimes \mathcal{A}^i(M)$.

Let us prove (1). As $H_G(M)$ surjects on $H(M)$ under evaluation at 0, the group G acts trivially on $H(M)$ and (by Poincaré duality) also on $H_{cpt}(M)_t$. Let $[\alpha] \in H_{cpt}(M)_t$ and let α be an element of $Z_{cpt}(M)_t^G$ representing the (G-invariant) cohomology class $[\alpha]$. Assume α to be homogeneous. Let $\tilde{\alpha}$ be a homogeneous (for the total degree given by formula 4) element of $\mathcal{A}_{cpt,G}(M)_t := (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M)_t)^G$ such that $\tilde{\alpha}(0) = \alpha$. Let $\beta = d_{\mathfrak{g}}\tilde{\alpha}$. As $d\alpha = 0$, $\beta(0) = 0$. Let k be the exterior degree of β. Let $\tilde{\nu} \in Z_G(M)$ be homogeneous of total degree $(n-k)$. Since $\tilde{\nu}$ is $d_{\mathfrak{g}}$-closed,

$$(\tilde{\nu}, \beta) = \int_M \tilde{\nu}\beta = \int_M \tilde{\nu}(d_{\mathfrak{g}}\tilde{\alpha}) = 0.$$

As only the terms of exterior degree n of $\tilde{\nu}\beta$ contribute to the integral, we obtain that the polynomial $(\tilde{\nu}(0), \beta_{[k]})$ is 0. If $\beta_{[k]} = \sum P_i\beta_i$, where $P_i \in S(\mathfrak{g}')$

are linearly independent and $\beta_i \in \mathcal{A}^k(M)$, we obtain $\sum_i P_i \int_M \tilde{\nu}(0)\beta_i = 0$. Thus $\int_M \tilde{\nu}(0)\beta_i = 0$ for all homogeneous elements $\tilde{\nu} \in Z_G(M)$ of total degree $n-k$. Since the evaluation map $H_G(M) \to H(M)$ is surjective, this implies that $\beta_i \in B_{cpt}(M)_t$ (by Poincaré duality). Thus $\beta_{[k]} \in (S(\mathfrak{g}') \otimes B_{cpt}(M)_t)^G$. As G is compact, we can choose $\beta_{[k]} = d\omega$ with homogeneous $\omega \in (S(\mathfrak{g}') \otimes \mathcal{A}_{cpt}(M)_t)^G$ of exterior degree $(k-1)$ and such that $\omega(0) = 0$. Thus $\gamma := \tilde{\alpha} - \omega$ is still such that $\gamma(0) = \alpha$ and $d_{\mathfrak{g}}(\gamma)$ is of exterior degree strictly less than k. By induction on k, we can construct a $d_{\mathfrak{g}}$-closed form κ such that $\kappa(0) = \alpha$. Thus the evaluation map at 0 gives a surjective map from $H_{cpt,G}(M)_t$ to $H_{cpt}(M)_t$.

Let us prove (2): Let x_a be a homogeneous basis of $H(M)$ over $\mathbb{R}$ and let α_a be any homogeneous elements of $Z_G(M)$ such that $[\alpha_a(0)] = x_a$. Let us prove that the cohomology classes of the elements α_a form a system of generators for $H_G(M)$ over $S(\mathfrak{g}')^G$. Let $\mathcal{H} \subset Z(M)^G$ be the subspace generated by the elements $\alpha_a(0)$. We have $Z(M) = \mathcal{H} \oplus B(M)$. Take $\alpha \in Z_G(M)$ (say of exterior degree k). Then

$$\alpha_{[k]} \in (S(\mathfrak{g}') \otimes Z(M))^G = S(\mathfrak{g}')^G \otimes \mathcal{H} \oplus (S(\mathfrak{g}') \otimes B(M))^G.$$

Arguing as in the proof of (1), we see that the cohomology class of α is congruent to an element $\beta + \sum_a P_a \alpha_a$ with $P_a \in S(\mathfrak{g}')^G$ and $\beta \in Z_G(M)$ of exterior degree strictly less than k. By induction on k, this proves that the equivariant cohomology classes of the elements α_a form a system of generators for $H_G(M)$. Let us prove that they are independent over $S(\mathfrak{g}')^G$. Let $\beta = \sum_a P_a \alpha_a \in Z_G(M)$ be such that β is 0 in $H_G(M)$. Let k be the maximum of the exterior degrees of those α_a's such that $P_a \neq 0$. Then for every $\nu \in Z_{cpt,G}(M)_t$ homogeneous of total degree $(n-k)$, the polynomial

$$(\beta, \nu) = \sum_a P_a \int_M (\alpha_a(0))_{[k]} \nu(0)$$

is equal to zero. By (1) the evaluation map $H_{cpt,G}(M)_t \to H_{cpt}(M)_t$ is surjective. Thus the polynomial map $\sum_{deg x_a = k} P_a x_a$ from $\mathfrak{g}$ to $H(M)$ is identically 0. But the elements x_a being linearly independent, this implies that the elements P_a for which x_a is of exterior degree k are identically 0. This is in contradiction with the definition of k. This prove that $\{\alpha_a\}$ are linearly independent over $S(\mathfrak{g}')^G$. One can similarly prove that $H_{cpt,G}(M)_t$ is a free module over $S(\mathfrak{g}')^G$.

Proceeding in a similar way and using Poincaré duality we can construct (if M is compact) a basis $\alpha^b \in H_G(M)_t$ such that $(\alpha_a, \alpha^b) = \delta_a^b$. This proves (3). ∎

Definition 7 *If Q is a continuous operator on $\mathcal{A}(M)$, we still denote by Q the operator on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$ defined by*

$$(Q \cdot \alpha, \Phi) = Q \cdot (\alpha, \Phi), \qquad \text{for all test densities } \Phi \text{ on } \mathfrak{g}.$$

We will often write the above as $(Q\alpha)(X) = Q \cdot (\alpha(X))$ *and say that* Q *is the* pointwise extension *of* Q *to* $C^{-\infty}(\mathfrak{g}, \mathcal{A}(M))$.

We start to compute the space $H_G^{-\infty}$ in some elementary situations.

Let $p : \mathcal{V} \to B$ be a G-equivariant real vector bundle over B. Let $i : B \to \mathcal{V}$ be the inclusion from B to $\mathcal{V}$ as the zero section.

Proposition 8 *The canonical maps*

$$i^* : H_G^{-\infty}(\mathcal{V}) \to H_G^{-\infty}(B)$$

and

$$p^* : H_G^{-\infty}(B) \to H_G^{-\infty}(\mathcal{V})$$

are inverses to each other. In particular, both of them are isomorphisms.

Proof: The proof is similar to the Poincaré lemma for the de Rham complex. Let $\mathcal{R}$ be the vertical Euler vector field on $\mathcal{V}$. Extend the operator $\mathcal{L}(\mathcal{R})$ pointwise to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. Extend similarly the operator $\iota(\mathcal{R})$ to $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $\iota(E^i)\iota(\mathcal{R}) + \iota(\mathcal{R})\iota(E^i) = 0$, for all i, Cartan relation implies

$$\mathcal{L}(\mathcal{R}) = d_{\mathfrak{g}}\iota(\mathcal{R}) + \iota(\mathcal{R})d_{\mathfrak{g}}$$

on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(\mathcal{V}))$. As $\mathcal{R}$ commutes with the action of G, the operators $\mathcal{L}(\mathcal{R})$, $\iota(\mathcal{R})$ preserve $\mathcal{A}_G^{-\infty}(\mathcal{V})$. Let $h_t(v) = tv$. Let $\beta \in \mathcal{A}(\mathcal{V})$. Then $h_t^*\beta = \beta$ for $t = 1$, while $h_t^*\beta = p^*i^*\beta$ for $t = 0$. We compute

$$\frac{d}{dt}h_t^*\beta = t^{-1}\mathcal{L}(\mathcal{R})h_t^*\beta.$$

(As $\mathcal{R}$ vanishes at 0, the right hand side depends smoothly on $t \in \mathbb{R}$). Clearly, the same relation persists for differential forms with parameters. Thus for $\beta \in \mathcal{A}_G^{-\infty}(\mathcal{V})$, we have

$$\frac{d}{dt}h_t^*\beta = t^{-1}\mathcal{L}(\mathcal{R})h_t^*\beta = t^{-1}(\iota(\mathcal{R})d_{\mathfrak{g}} + d_{\mathfrak{g}}\iota(\mathcal{R}))h_t^*\beta.$$

Define $H : \mathcal{A}_G^{-\infty}(\mathcal{V}) \to \mathcal{A}_G^{-\infty}(\mathcal{V})$ by

$$H\beta = \int_0^1 h_t^*(\iota(\mathcal{R})\beta)t^{-1}dt.$$

Then we obtain

$$\beta - p^*i^*\beta = \int_0^1 \frac{d}{dt}h_t^*\beta dt = (d_{\mathfrak{g}}H + Hd_{\mathfrak{g}})\beta,$$

for all $\beta \in A_G^{-\infty}(\mathcal{V})$. Thus we see that $p^*i^* = I$ in cohomology, where I is the identity operator. Of course, $pi = I$, in particular $i^*p^* = I$. This proves the proposition. ∎

Considering the case of a vector space V, we have

Corollary 9 *Let G be a Lie group and V be a finite dimensional real representation space for G, then*

$$H_G^{-\infty}(V) = C^{-\infty}(\mathfrak{g})^G.$$

We now prove the Thom isomorphism for compactly supported cohomology of vector bundles.

Definition 10 *Let G be a compact Lie group and let $p : \mathcal{V} \to B$ be a G-equivariant G-oriented vector bundle over a compact base B.*

An element $u \in H_{cpt,G}(\mathcal{V})$ such that $p_* u = 1$ in $H_G(B)$ will be called a Thom class.

Given a G-orientation o on $\mathcal{V}$, recall [18] that there exists a unique Thom class $u_o \in H_{cpt,G}(\mathcal{V})$. Multiplication by u_o induces a map m_o: $m_o(\alpha) = u_o \wedge p^*\alpha$ from $H_G(B)$ to $H_{cpt,G}(\mathcal{V})$ and the map m_o is an isomorphism. Similarly, as $u_o \in H_{cpt,G}(\mathcal{V})$, we can define the map $m_o(\alpha) = u_o \wedge p^*\alpha$ from $H_G^{-\infty}(B)$ to $H_{cpt,G}^{-\infty}(\mathcal{V})$.

Proposition 11 . *Let G be a compact Lie group and let $\mathcal{V} \to B$ be a G-equivariant G-oriented real vector bundle over a compact base B. Then the maps*

$$m_o : H_G^{-\infty}(B) \to H_{cpt,G}^{-\infty}(\mathcal{V})$$

and

$$p_* : H_{cpt,G}^{-\infty}(\mathcal{V}) \to H_G^{-\infty}(B)$$

are inverses to each other. In particular both of them are isomorphisms.

Remark 12 *In the above proposition, we have assumed the base B to be compact, just in order to simplify notation. If B is not necessarily compact, the same proof will lead to isomorphism of $H_{cpt,G}^{-\infty}(B)$ with $H_{cpt,G}^{-\infty}(\mathcal{V})$. It is also clear that a similar Thom isomorphism will hold between $H_G^{-\infty}(B)$ and the cohomology $H_{vcpt,G}^{-\infty}(\mathcal{V})$ of the complex of equivariant forms on $\mathcal{V}$ with compact support along the fibers.*

Proof: The proof is similar to the proof of the Thom isomorphism in equivariant cohomology given in [19]. For $\alpha \in \mathcal{A}(\mathcal{V})$, we denote by $\tilde{\alpha}$ the image of α under the automorphism $x \mapsto -x$ of $\mathcal{V}$. Let us consider the bundle $\mathcal{V} \oplus \mathcal{V}$ over B and let $\sigma(x, y) = (y, -x)$ be the automorphism of $\mathcal{V} \oplus \mathcal{V}$. Denote by $\sigma_t, t \in \mathbb{R}$ the transformation

$$\sigma_t(x, y) = ((\cos t)x + (\sin t)y, -(\sin t)x + (\cos t)y)$$

of the fibers of $\mathcal{V} \oplus \mathcal{V}$. Then σ_0 is the identity, while $\sigma_{\pi/2}$ is equal to σ. Let

$$S = (\frac{d}{dt}\sigma_t)$$

be the vector field on $\mathcal{V} \oplus \mathcal{V}$ induced by the group of transformations σ_t. We have

$$\mathcal{L}(S) = d\iota(S) + \iota(S)d$$

on $\mathcal{A}_{cpt}(\mathcal{V} \oplus \mathcal{V})$.

Extending these transformations pointwise to $C^{-\infty}(\mathfrak{g}, \mathcal{A}_{cpt}(\mathcal{V} \oplus \mathcal{V}))$ (cf. Definition 7) we obtain the relation:

$$\mathcal{L}(S) = d_{\mathfrak{g}}\iota(S) + \iota(S)d_{\mathfrak{g}}$$

on $C^{-\infty}(\mathfrak{g}, \mathcal{A}_{cpt}(\mathcal{V} \oplus \mathcal{V}))$. The transformations σ_t commute with the action of G. Thus $\iota(S)$ and $\mathcal{L}(S)$ preserve $\mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$.

Define $H : \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V}) \to \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$ by

$$H\nu = \int_0^{\pi/2} (\sigma_t^* \iota(S)\nu)dt.$$

We obtain, as in the proof of the preceding proposition (8)

$$\sigma^*\nu - \nu = (d_{\mathfrak{g}}H + Hd_{\mathfrak{g}})\nu$$

for any $\nu \in \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$.

Let $p_i : \mathcal{V} \oplus \mathcal{V} \to \mathcal{V}, i = 1, 2$ be the natural maps obtained by projections on the first or second component respectively. Consider $\alpha \in \mathcal{A}^{\infty}_{cpt,G}(\mathcal{V})$ and $\beta \in \mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V})$. Then $p_1^*\alpha \wedge p_2^*\beta$ is a well defined element of $\mathcal{A}^{-\infty}_{cpt,G}(\mathcal{V} \oplus \mathcal{V})$. If α, β are closed equivariant forms, then $p_1^*\alpha \wedge p_2^*\beta$ is closed and is in the same cohomology class as $\sigma^*(p_1^*\alpha \wedge p_2^*\beta) = p_2^*\alpha \wedge p_1^*\tilde{\beta}$. Let us integrate over the fibers of p_2. It is clear that $(p_2)_*(p_1^*\alpha \wedge p_2^*\beta) = p^*(p_*\alpha)\beta$. Thus the equality in cohomology

$$p_1^*\alpha \wedge p_2^*\beta \cong p_2^*\alpha \wedge p_1^*\tilde{\beta}$$

implies (if N is the rank of $\mathcal{V}$, and $|\alpha| \in \{0, 1\}$ is the parity of α):

$$p^*(p_*\alpha) \wedge \beta \cong (-1)^{N(|\alpha|-1)}\alpha \wedge p^*(p_*\beta)$$

in cohomology. In particular let $\alpha = u_o$ be the equivariant Thom class of $\mathcal{V} \to B$, which is of parity N. We obtain the relation

$$\beta \cong u_o \wedge p^*p_*\beta$$

in $H^{-\infty}_{cpt,G}(\mathcal{V})$. Thus, we see that $m_o p_* = I$ in cohomology. Of course $p_* m_o = I$. This proves the proposition. ∎

We finish this section by giving an example of a non-trivial closed equivariant form with generalized coefficients.

We need a notation.

Let V be a real vector space of dimension n. Let ν' be a non-zero element in $\Lambda^n V'$. We denote by $|\nu'|^{-1}\delta_V$ the element of $C^{-\infty}(V)$ defined by

$$(10) \qquad \int_V |\nu'|^{-1}\delta_V(X)\Phi(X)dX = \Phi(0), \qquad \text{for any test function } \Phi \text{ on } V,$$

where dX is the Euclidean density on V determined by ν'.

We denote by $\delta_{V,o} \in C^{-\infty}(V) \otimes \Lambda^n V'$ the element:

$$(11) \qquad \delta_{V,o} = |\nu'|^{-1}\delta_V \otimes \nu'.$$

The element $\delta_{V,o}$ depends only on the orientation o of V determined by ν'.

Let G be a Lie group. Consider the action of G on itself by left translations. Let $n = \dim G$. Fix an orientation o on $\mathfrak{g}$. Let $\nu' \in \Lambda^n \mathfrak{g}'$ be a positive element. Let dg be the unique left invariant form of maximal degree on G such that $(dg)_e = \nu'$, where e is the identity element of G.

Lemma 13 *The form*

$$\alpha_{G,o}(X) := |\nu'|^{-1}\delta_{\mathfrak{g}}(X) \otimes |\det_{\mathfrak{g}} g| dg$$

is a closed equivariant form on G, which depends only on the choice of o.

Proof: The form $\alpha_{G,o}$ is equivariant, as $g_0 \cdot (|\nu'|^{-1}\delta_{\mathfrak{g}}) = |\det_{\mathfrak{g}} g_0|(|\nu'|^{-1}\delta_{\mathfrak{g}})$ and $|\det_{\mathfrak{g}}(g_0^{-1}g)| = |\det_{\mathfrak{g}} g_0|^{-1}|\det_{\mathfrak{g}} g|$. It is immediate to see that $\alpha_{G,o}$ depends only on o. As $\alpha_{G,o}$ is of maximal degree, $d\alpha_{G,o} = 0$. Also, as the generalized function $|\nu'|^{-1}\delta_{\mathfrak{g}}$ is annihilated by multiplication by all the coordinates functions on $\mathfrak{g}$, we see that $\iota\alpha_{G,o} = 0$. ∎

We will prove in section 5 that $H_G^{-\infty}(G) = \mathbb{R}\alpha_{G,o}$

3 Koszul complexes

Our main aim in sections 4 and 5 will be the study of the cohomology of "perturbed" Koszul complexes. Thus, in this section, we recall some well-known facts on Koszul complexes.

Let V be a finite dimensional real vector space of dimension n with basis $e^i, 1 \leq i \leq n$ and dual basis $e_i \in V'$. Let $S(V')$ be the ring of polynomial functions on V. Let L be a $S(V')$-module. We still denote by e_i the action of $e_i \in V'$ on L. We denote by $\iota(e^i) : \Lambda^* V' \to \Lambda^{*-1} V'$ the contraction by the vector $e^i \in V$.

Consider the space

$$L \otimes \Lambda^* V'$$

which is $\mathbb{Z}_+$-graded by the exterior degree. On this space, the operator

$$j_L = \sum_{i=1}^{n} e_i \otimes \iota(e^i) \tag{12}$$

is an operator of degree -1 and its square is zero. We denote by $H(j_L)$ the homology space of j_L. It is a $\mathbb{Z}_+$-graded vector space. If $L = S(V')$ (considered as a $S(V')$-module under multiplication), we denote the operator j_L by j_V. We denote:

$$A^* = S(V') \otimes \Lambda V'.$$

The following proposition is basic

Proposition 14 *Consider the operator j_V on A^*. We have:*

1. *If $i > 0, H_i(j_V) = 0$.*

2. *If $i = 0$, the map $\phi \mapsto \phi(0)$ from $A^0 = S(V')$ to $\mathbb{R}$ induces an isomorphism from $H_0(j_V)$ with $\mathbb{R}$.*

Even though this proposition is well known, we give a proof as we will use the explicit homotopy given below in the rest of the article.

Proof: We identify the space $A^* = S(V') \otimes \Lambda^* V'$ with the space of differential forms with polynomial coefficients on V. We write an element $x \in V$ as $x = \sum_i x_i e^i$, so that $e_i(x) = x_i$. If $I = (i_1, i_2, \ldots, i_k)$ is a multi-index: $1 \leq i_1 < \cdots < i_k \leq n$, we identify $e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_k} \in \Lambda V'$ with the k-form $dx_I = dx_{i_1} dx_{i_2} \cdots dx_{i_k}$.

Consider the partial derivative ∂^i in the direction of $e^i \in V$. Let E_V be the Euler vector field $E_V = \sum_i x_i \partial^i$. We denote the contraction operator $\iota(e^i)$ on $\Lambda V'$ by ι^i. Thus

$$j_V = \sum_i x_i \iota^i = \iota(E_V).$$

We denote by ϵ_i the multiplication by dx_i. Let $d_V = \sum_i \partial^i \otimes \epsilon_i$ be the de Rham differential on A^*. Let $\mathcal{L}_V$ be the Euler operator on $S(V') \otimes \Lambda V'$ given by the Lie derivative action of E_V.

$$\mathcal{L}_V(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + kf) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

for $f \in S(V')$.

Then, Cartan relation implies $\mathcal{L}_V = d_V j_V + j_V d_V$.

Consider the subcomplex A_0^* of A^* such that $A_0^k = A^k$, if $k > 0$, while $A_0^0 = \{\phi \in S(V'), \phi(0) = 0\}$. The operator $\mathcal{L}_V$ keeps A_0^* stable and induces an invertible operator of degree 0 on A_0^*. We can give an integral formula for the inverse F_V of the Euler operator $\mathcal{L}_V$: Define F_V on A_0^* by

1. If $f dx_I \in A^k$, $k > 0$,

$$F_V(f dx_I) = (\int_0^1 f(tx) t^{k-1} dt) dx_I.$$

2. If $f \in A_0^0$

$$F_V f = \int_0^1 f(tx) t^{-1} dt.$$

It is well defined as $f(tx)$ vanishes for $t = 0$.

It is immediate to see that $F_V \mathcal{L}_V \phi = \phi$ for every $\phi \in A_0^*$. The operator $F_V = \mathcal{L}_V^{-1}$ commutes with j_V and d_V.

Let $h_V := F_V d_V$, then, if $\phi \in A_0^*$,

$$\phi = (h_V j_V + j_V h_V)\phi.$$

This formula clearly implies the proposition. ∎

Let L, N be two $S(V')$-modules. The tensor product space (over $\mathbb{R}$) $L \otimes N$ is given a structure of $S(V')$-module by defining the action of an element $f \in V'$ to be $f \cdot (m \otimes n) = fm \otimes n - m \otimes fn$.

Consider the operator $j_{L \otimes N}$ on $L \otimes N \otimes \Lambda V'$. The homology space $H_0(j_{L \otimes N})$ in degree 0 is the quotient of $L \otimes N$ by the subspace spanned by elements of the form $fm \otimes n - m \otimes fn$, for $f \in V'$. This quotient is by definition $L \otimes_{S(V')} N$:

$$H_0(j_{L \otimes N}) = L \otimes_{S(V')} N.$$

If N is a $S(V')$-module, denote by N^0 the space N with the trivial action of V'.

Lemma 15 *Let N be a $S(V')$-module. The operator $R := \exp \sum_i \partial^i \otimes e_i$ gives an isomorphism of the $S(V')$-module $S(V') \otimes N$ with the $S(V')$-module $S(V') \otimes N^0$.*

Proof: It is sufficient to check this assertion when V is a 1-dimensional vector space, where it is checked easily. ∎

Corollary 16 *If L is a free $S(V')$-module, then $L \otimes N$ is also free; hence $H_i(j_{L \otimes N}) = 0$ if $i > 0$.*

Lemma 17 *For any $S(V')$-modules L and N, the homology space of the operator $j_{L\otimes N}$ on the complex*

$$L \otimes N \otimes \Lambda V'$$

is equal to the torsion group $Tor^{S(V')}(L,N)$, i.e.,

$$H_i(j_{L\otimes N}) = Tor_i^{S(V')}(L,N).$$

Proof: Let us consider the complex

$$0 \to S(V') \otimes N \otimes \Lambda^n V' \overset{j_{S(V')\otimes N}}{\to} \quad \dots \overset{j_{S(V')\otimes N}}{\to} S(V') \otimes N \otimes V' \overset{j_{S(V')\otimes N}}{\to} S(V') \otimes N \overset{m}{\to} N \to 0$$

where the last map is the surjective map $m : S(V') \otimes N \to N : m(\phi \otimes n) = \phi n$. By the preceding corollary, this complex is exact. Furthermore if we endow the space $S(V') \otimes N \otimes \Lambda V'$ with the $S(V')$ module structure $S(V') \otimes (N \otimes \Lambda V')^0$, the homomorphisms $j_{S(V')\otimes N}$ and m are $S(V')$-module morphisms. Thus the complex above is a free resolution of N as a $S(V')$-module. We may calculate the torsion group $Tor^{S(V')}(L,N)$ using this resolution. The space $L \otimes_{S(V')} S(V') \otimes (N \otimes \Lambda^* V')^0$ is isomorphic with $L \otimes N \otimes \Lambda^* V'$. The operator $I \otimes_{S(V')} (j_{S(V')\otimes N})$ under this isomorphism becomes the operator $j_{L\otimes N}$. This proves the lemma. ∎

We now introduce another vector space P considered as a parameter space. Let $W = V \oplus P$. We write an element $w \in W$ as $w = x + y$, with $x \in V$, $y \in P$.

Let us consider the space

$$A^{\infty,*} = C^\infty(W) \otimes \Lambda^* V'.$$

The multiplication by the coordinate function x_i is an operator on $C^\infty(W)$. Thus the operator $j_V^\infty := \sum_{i=1}^n x_i \otimes \iota(e^i)$ is an operator of degree -1 on $A^{\infty,*}$ and $(j_V^\infty)^2 = 0$. Let $r_P : C^\infty(W) \to C^\infty(P)$ be the restriction map.

Proposition 18 *Consider the operator j_V^∞ on the complex $A^{\infty,*}$. We have*

1. *If $i > 0, H_i(j_V^\infty) = 0$.*

2. *If $i = 0$, the map r_P from $A^0 = C^\infty(W)$ to $C^\infty(P)$ induces an isomorphism from $H_0(j_V^\infty)$ with $C^\infty(P)$.*

Proof: The method of proof is identical to the proof of Proposition 14. For simplicity, we denote j_V^∞ by j_V. Let

$$d_V = \sum_i \partial^i \otimes \epsilon_i \tag{13}$$

be the partial de Rham differential in the direction of V on the complex $A^{\infty,*}$. Let $\mathcal{L}_V$ be the Euler operator on $A^{\infty,*}$ with respect to the variables x_i, dx_i:

$$\mathcal{L}_V(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\sum_i x_i \partial^i f + kf) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

if $f \in C^\infty(W)$.

Then, $\mathcal{L}_V = d_V j_V + j_V d_V$.

Let $A_0^{\infty,*}$ be the subcomplex of $A^{\infty,*}$ defined by $A_0^{\infty,k} = A^{\infty,k}$ if $k > 0$ and $A_0^{\infty,0} = \{\phi \in C^\infty(W); r_P \phi = 0\}$.

The Euler operator $\mathcal{L}_V$ induces an operator of degree 0 on $A_0^{\infty,*}$, which is invertible. Its inverse F_V is given explicitly by an integral formula as in the proof of Proposition 14:

Definition 19 *Let us consider the operator F_V of degree 0 on $A_0^{\infty,*}$ defined by*

1. *If $f dx_I \in A_0^{\infty,k}$, $k > 0$,*

$$F_V(f dx_I) = (\int_0^1 f(tx + y) t^{k-1} dt) dx_I.$$

2. *If $f \in A_0^{\infty,0}$,*

$$F_V f = \int_0^1 f(tx + y) t^{-1} dt.$$

It is well defined as $f(tx + y)$ vanishes for $t = 0$.

The operator F_V commutes with d_V, j_V. It is easy to prove

$$F_V \mathcal{L}_V \phi = \phi$$

for every $\phi \in A_0^{\infty,*}$. Thus if

(14) $$h_V = F_V d_V,$$

$$\phi = (h_V j_V + j_V h_V)\phi, \qquad \text{for } \phi \in A_0^{\infty,*}.$$

Thus h_V is a homotopy for the complex $A_0^{\infty,*}$. The existence of h_V implies that the subcomplex $A_0^{\infty,*}$ is exact. This in turn implies the proposition. ∎

Observe that if $f \in C^\infty_{cpt}(W) \otimes \Lambda V'$ is compactly supported, then $F_V f$ is not necessarily compactly supported.

Remark 20 *If a Lie group G acts linearly on V and P, the operator j_V^∞ commutes with the action of G. Thus $(A^{\infty,*})^G$ is a subcomplex of $A^{\infty,*}$. The homotopy h_V, that we have constructed above, commutes with the action of G. It results that the homology of the subcomplex $(A^{\infty,*})^G$ of $A^{\infty,*}$ exists only in degree 0 and in degree 0 is isomorphic to $C^\infty(P)^G$.*

Now, consider the space:

$$A^{-\infty,*} = C^{-\infty}(W) \otimes \Lambda^* V'.$$

The operator $j_V^{-\infty}$ is similarly defined on $A^{-\infty,*}$. The complex $j_V^{-\infty} : A^{-\infty,*} \to A^{-\infty,*-1}$ is called the *Koszul complex with $C^{-\infty}$-coefficients* (with the space P as a parameter space).

Choose an orientation o on V. By Formula (11) of section 2, this determines an element $\delta_{V,o}$ of $C^{-\infty}(V) \hat{\otimes} \Lambda^n V'$. If $f \in C^{-\infty}(P)$ is a generalized function on P, the product $\delta_{V,o}(x)f(y)$ is in $C^{-\infty}(W) \otimes \Lambda^n V'$.

It is easy to identify the homology of $j_V^{-\infty}$ in top degree.

Lemma 21 *The kernel of $j_V^{-\infty}$ on $A^{-\infty,n} = C^{-\infty}(W) \otimes \Lambda^n V'$ is equal to the space $\delta_{V,o} \otimes C^{-\infty}(P)$.*

Proof: As $x_i \delta_{V,o}(x) = 0$ for all i, the subspace $\delta_{V,o}(x) \otimes C^{-\infty}(P)$ of $C^{-\infty}(W) \otimes \Lambda^n V'$ is in $Ker(j_V^{-\infty})$. Reciprocally if $f \otimes \nu' \in C^{-\infty}(W) \otimes \Lambda^n V'$ is such that $j_V^{-\infty}(f \otimes \nu') = 0$, we see that $x_i f(x+y) = 0$ for all i. Thus f is the product of the δ- function on the transverse subspace V with a generalized function on P. ∎

Proposition 22 *We have*

1. *If $i \neq n$, $H_i(j_V^{-\infty}) = 0$.*

2. *$H_n(j_V^{-\infty}) = \delta_{V,o} \hat{\otimes} C^{-\infty}(P)$*

Remark 23 *Let K be a compact group acting linearly on W and preserving the direct sum decomposition $W = V \oplus P$. Thus the group K acts on $A^{-\infty,*}$ and the operator j_V commutes with the action of K. Hence $(A^{-\infty,*})^K$ is a subcomplex of $A^{-\infty,*}$. Let $\chi(k) := det_V(k)$. Then, K being compact, $\chi(k) = \pm 1$ and moreover $k \cdot \delta_{V,o} = \chi(k)\delta_{V,o}$. By averaging over K the equation $\alpha = j_V^{-\infty}\beta$, we see that the homology of the subcomplex $(A^{-\infty,*})^K$ of $A^{-\infty,*}$ is also equal to 0, except in top degree n, while in top degree $H_n((A^{-\infty,*})^K) = \delta_{V,o} \otimes C^{-\infty}(P)^\chi$, where $C^{-\infty}(P)^\chi = \{f \in C^{-\infty}(P); k \cdot f = \chi(k)f, \text{for all } k \in K\}$.*

We now give a proof of Proposition 22.

Proof: Let us consider the subcomplex $L^* = C^{-\infty}_{cpt,V}(W) \otimes \Lambda^* V'$ of $A^{-\infty,*}$, where $C^{-\infty}_{cpt,V}(W)$ denotes the space of generalized functions ϕ on W, with support contained in a set of the form $F \oplus P$ where F is a compact subset of V (F depending upon the generalized function ϕ) .

Lemma 24 *The inclusion $L^* \to A^{-\infty,*}$ induces an isomorphism in homology.*

Proof: Let us denote the operator $j_V^{-\infty}$ by j_V. Choose a scalar product on V and an orthonormal basis e^i of V. Let $e_i \in V'$ be the dual basis and let us denote by ϵ_i the exterior multiplication by e_i on $\Lambda V'$. Let ϵ_V be the operator of degree $+1$ on $A^{-\infty,*}$ defined by $\epsilon_V = \sum_i x_i \epsilon_i$. It is easily verified that $\epsilon_V j_V + j_V \epsilon_V = |x|^2 I$.

Let χ be a smooth function on V such that $\chi(x) = 1$ for $|x| < 1/2$ and $\chi(x) = 0$ for $|x| > 1$. Extend χ to a smooth function on W by setting $\chi(x+y) = \chi(x)$. The multiplication by the function χ on $A^{-\infty}$ commutes with j_V. It sends $A^{-\infty}$ to L. The function $(1 - \chi(x))(|x|^2)^{-1}$ is a C^∞ function on V, thus on W. We write for $\phi \in C^{-\infty}(W) \otimes \Lambda V'$

$$\phi = \phi_0 + \phi_1$$

with $\phi_0 = \chi\phi$ and

$$\phi_1 = \frac{1-\chi(x)}{|x|^2}|x|^2\phi = \frac{1-\chi(x)}{|x|^2}(j_V\epsilon_V + \epsilon_V j_V)\phi.$$

If $j_V\phi = 0$, both elements ϕ_0 and ϕ_1 are annihilated by j_V. Furthermore ϕ_1 is in the image of j_V. Thus each element of $H(A^{-\infty})$ has a representative in the subcomplex L and hence the natural map $H(L) \to H(A^{-\infty})$ is surjective. Now let $\phi \in L$ be such that $\phi = j_V\alpha$ with $\alpha \in A^{-\infty}$. We can find a C^∞ function θ on W such that it is equal to 1 on the support of ϕ and such that $\theta\alpha \in L$. Thus $\phi = \theta\phi = j_V(\theta\alpha)$ and the homology class of ϕ in $H(L)$ is zero. Thus the natural map $H(L) \to H(A^{-\infty})$ is injective. ∎

If $f \otimes \nu' \in L^n$, i.e $f \in C^{-\infty}_{cpt,V}(W)$, we can define $I(f \otimes \nu') \in C^{-\infty}(P)$ by

$$\int_P I(f \otimes \nu')(y)\phi(y)dy = \int_W f(x,y)\phi(y)dxdy$$

where dx is the positive density on V associated to ν'. If $\ell = \delta_{V,o}(x)g(y)$, then $I(\ell) = g$. We denote by L_0^* the subcomplex of L^* such that $L_0^k = L^k$ if $k \neq n$, while $L_0^n = \{\phi \in L^n, I(\phi) = 0\}$. We will construct an explicit homotopy for $j_V^{-\infty}$ on the subcomplex L_0^*.

Consider the complex $R^* = C^\infty_{cpt,P}(W) \otimes \Lambda^* V'$, where $C^\infty_{cpt,P}(W)$ denotes the space of smooth functions on W, with support contained in a set of the form $V \oplus F$ where F is a compact subset of P. Define the subcomplex R_0^* by $R_0^k := R^k$, if $k > 0$, while $R_0^0 := \{\phi \in R^0; r_P \phi = 0\}$, where r_P is the restriction map from $C^\infty(W)$ to $C^\infty(P)$. The operator F_V given in Definition 19 preserves R_0^*. Thus the homotopy $h_V = F_V d_V$ of $A_0^{\infty,*}$ is also a homotopy for R_0^*.

Let us choose a Lebesgue measure dy on P. The pairing $(,)$ between the complexes L^* and R^{n-*} defined by

$$(\phi dx_I, f dx_J) = \int_W \phi(x+y) f(x+y)(dx_I \wedge dx_J) dy$$

is a non degenerate pairing. The space L_0^n is the orthogonal of the subspace $1 \otimes C^\infty_{cpt}(P)$ of functions on W constant in the $x \in V$ variables. Thus $(,)$ induces a non degenerate pairing between L_0^* and R_0^{n-*}.

The operator j_V satisfies

$$(j_V \alpha, \beta) + (-1)^{|\alpha|}(\alpha, j_V \beta) = 0$$

for $\alpha \in L^*, \beta \in R^{n-*}$.

Thus we can transpose the homotopy for R_0^* and obtain a homotopy for L_0^*. More explicitly, define the operator U_V of degre 0 on L_0^* by

$$(U_V \alpha, \beta) = (\alpha, F_V \beta)$$

for $\alpha \in L_0^*, \beta \in R_0^{n-*}$. We extend the partial de Rham differential d_V from $A^{\infty,*}$ to $A^{-\infty,*}$ (again denoted by d_V) by the same formula (13). The operator d_V also satisfies

$$(d_V \alpha, \beta) + (-1)^{|\alpha|}(\alpha, d_V \beta) = 0.$$

Thus $d_V L^{n-1} \subset L_0^n$ and d_V is an operator of degree 1 on L_0^*. The operator U_V commutes with d_V. Let $k_V = U_V d_V = d_V U_V$. Then, for $\alpha \in L_0$, $\beta \in R_0$,

$$(k_V \alpha, \beta) + (-1)^{|\alpha|}(\alpha, h_V \beta) = 0.$$

Thus, for $\alpha \in L_0$,

$$\alpha = (k_V j_V + j_V k_V)\alpha$$

as follows from the transpose relation $\beta = (h_V j_V + j_V h_V)\beta$.

The complex L_0^* is thus exact and this implies the proposition. ∎

Let us now consider a Lie group G with Lie algebra $\mathfrak{g}$. Recall the definition of the Koszul differential c_L on the space $T^* = L \otimes \Lambda^* \mathfrak{g}'$ calculating the cohomology

of a $\mathfrak{g}$-module L: For $\alpha \in L \otimes \Lambda^p \mathfrak{g}'$, $c_L \alpha \in L \otimes \Lambda^{p+1} \mathfrak{g}'$ is defined by

$$(15) \qquad \begin{aligned}(c_L\alpha)(X_1, ..., X_{p+1}) = \\ \sum_i (-1)^{i+1} X_i \cdot \alpha(X_1, ..., \hat{X}_i, ..., X_{p+1}) \\ + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1}),\end{aligned}$$

where X_1,...,X_{p+1} are elements of $\mathfrak{g}$.

Consider $\iota(X) : T^* \to T^{*-1}$, the contraction by an element $X \in \mathfrak{g}$ and $\mathcal{L}(X)$ the action of $\mathfrak{g}$ by tensor product on T. It is not difficult to verify the relation $\mathcal{L}(X) = c_L \iota(X) + \iota(X) c_L$. The space $\Lambda \mathfrak{g}'$ acts by exterior multiplication on T and c_L satisfies the Leibniz's rule: $c_L(\alpha\xi) = c(\alpha)\xi + (-1)^{|\alpha|} \alpha c_L(\xi)$, where c in this formula denotes the Koszul differential of the complex $\Lambda \mathfrak{g}'$ (corresponding to the trivial one dimensional representation L).

Let K be a Lie subgroup of G. Assume that L is a $(\mathfrak{g}, K)$-module. Consider the subspace

$$T_K^* = (L \otimes \Lambda^*(\mathfrak{g}/\mathfrak{k})')^K$$

of $L \otimes \Lambda \mathfrak{g}'$. From the relation $\mathcal{L}(X) = c_L \iota(X) + \iota(X) c_L$, it is easy to see that (T_K^*, c_L) is a subcomplex of $(L \otimes \Lambda^* \mathfrak{g}', c_L)$. The cohomology of the subcomplex (T_K^*, c_L) is by definition the relative Lie algebra cohomology $H^*(\mathfrak{g}, K, L)$ of the $(\mathfrak{g}, K)$-module L.

Consider the algebra $D(\mathfrak{g})$ of differential operators on $\mathfrak{g}$ with polynomial coefficients. Then the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ determines a Lie algebra homomorphism τ from $\mathfrak{g}$ into $D(\mathfrak{g})$. If L is a $D(\mathfrak{g})$-module, then L is a $\mathfrak{g}$-module, via the adjoint action. Furthermore, as $D(\mathfrak{g})$ contains $S(\mathfrak{g}')$, the module L is a $S(\mathfrak{g}')$-module.

Let e^i be a basis of $\mathfrak{g}$, $e_i \in \mathfrak{g}'$ the dual basis. We consider the element $\Omega = \sum_i e_i \tau(e^i)$ of $D(\mathfrak{g})$

Lemma 25 *The element $\Omega \in D(\mathfrak{g})$ is identically 0.*

Proof: For $X \in \mathfrak{g}$, denote by ∂_X the constant coefficient vector field on $\mathfrak{g}$ equal to X. The adjoint vector field $\tau(e^i)$ is given by: $\tau(e^i) = -\sum_j x_j \partial_{[e^i, e^j]}$ and hence

$$\Omega = -\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$$

which is equal to zero, as the vector field $-\sum_{i,j} x_i x_j \partial_{[e^i, e^j]}$ is the vector field equal at the point $X \in \mathfrak{g}$ to $[X, X] = 0$. ∎

Let L be a $D(\mathfrak{g})$-module. Consider the space

$$A^* := L \otimes \Lambda \mathfrak{g}'.$$

As L is a $S(\mathfrak{g}')$-module, we can consider the operator $j = j_L : A^* \to A^{*-1}$ given (as in Formula (12)) by $j = \sum_i e_i \otimes \iota(e^i)$.

On the other hand, the $\mathfrak{g}$-module structure on L gives rise to the Koszul differential $c : L^* \to L^{*+1}$.

Lemma 26 . *Let L be a $D(\mathfrak{g})$-module. The operator $j + c$ satisfies*

$$(j + c)^2 = 0$$

Proof: As $j^2 = 0$, $c^2 = 0$, we have to verify that $jc + cj = 0$. We have, (setting $\iota^i = \iota(e^i)$) and using the Leibniz's rule:

$$jc+cj = \sum_i (e_i\iota^i c+ce_i\iota^i) = \sum_i e_i(\iota^i c+c\iota^i)+\sum_i c(e_i)\iota^i = \sum_i e_i\mathcal{L}(e^i)+\sum_i c(e_i)\iota^i.$$

The action $\mathcal{L}(e^i)$ is by the tensor product action $\tau(e^i) \otimes I + I \otimes \mathcal{L}_\Lambda(e^i)$, where $\mathcal{L}_\Lambda(e^i)$ is the action of $\mathfrak{g}$ on $\Lambda\mathfrak{g}'$ induced from the adjoint representation. Thus, as $\sum_i e_i\tau(e^i) = 0$ from the preceding lemma, it remains to see that $\sum_i e_i\mathcal{L}_\Lambda(e^i)\xi + \sum_i c(e_i)\iota^i\xi = 0$ for $\xi \in \Lambda\mathfrak{g}'$. Writing ξ as a product of elements $\alpha \in \mathfrak{g}'$, it is sufficient to prove this relation for $\xi \in \mathfrak{g}'$ where this is checked easily. ∎

The spaces $L = C^{\pm\infty}(\mathfrak{g})$ have natural $D(\mathfrak{g})$-module structures. Thus on $L \otimes \Lambda\mathfrak{g}'$, the operators j and c are defined and satisfy $(j + c)^2 = 0$. We will see in section 5 that we obtain this example of perturbed Koszul complex when computing the G-equivariant cohomology of a Lie group G provided with the free action of G on itself given by left translation. We compute the cohomology of slightly more complicated complexes in the next section.

4 Induction of equivariant differential complexes

Let (A, d) be a differential complex, i.e. a $\mathbb{Z}/2$-graded vector space over $\mathbb{R}$, with a differential d of odd degree. We will assume that A is a Fréchet space and that d is continuous. (In most of the applications, A will be the space of smooth differential forms on a G-manifold M.) Let G be a Lie group acting on A. We assume that the action of G on A is differentiable. As in H.Cartan [10], we say that (A, d) is a *G-differential complex*, if

1. The action of G preserves the $\mathbb{Z}/2$-grading and commutes with d.

2. There are given continuous contraction operators $\iota(X), X \in \mathfrak{g}$ of odd degree satisfying $\iota(X)\iota(Y) + \iota(Y)\iota(X) = 0$ for all $X, Y \in \mathfrak{g}$ and $g\iota(X)g^{-1} = \iota(gX)$, for all $g \in G$, $X \in \mathfrak{g}$.

3. The Lie derivative action $\mathcal{L}(X)$ of the action of G on A satisfies $\mathcal{L}(X) = d\iota(X) + \iota(X)d$, for all $X \in \mathfrak{g}$.

If A is a G-differential complex, define the space A_{horG} of *G-horizontal* elements to be

$$A_{horG} = \{\alpha \in A; \iota(Y)\alpha = 0, \quad \text{for all } Y \in \mathfrak{g}\}.$$

The space A_{basG} of *G-basic* elements is by definition the space of elements of A which are G-invariant and horizontal:

$$A_{basG} = A^G_{horG}.$$

Remark that the differential d leaves the space A_{basG} stable.

Let (A, d) be a G-differential complex. We call (A, d) a *G-differential algebra* if, in addition, A has a $\mathbb{Z}/2$-graded algebra structure satisfying the following:

1. The action of any $g \in G$ on A is by algebra automorphisms.

2. The operator d and the operators $\iota(X), X \in \mathfrak{g}$, are odd derivations of the algebra A.

If G acts smoothly on a manifold M, then $\mathcal{A}^\bullet(M) = \mathcal{A}^{even}(M) \oplus \mathcal{A}^{odd}(M)$ is a G-differential algebra. If L is a differentiable G-module, the tensor product G-module $L \otimes \Lambda\mathfrak{g}'$ together with the Koszul differential c_L and the contraction operators $I \otimes \iota(X)$ is a G-differential complex. In particular, taking L to be the trivial one dimensional G-module, $\Lambda\mathfrak{g}'$ is a G-differential complex, in fact a G-differential algebra.

The tensor product (over $\mathbb{R}$) of two G-differential complexes is canonically a G-differential complex. Thus, for any G- differential complex A, we can form the G-differential complex $A \otimes \Lambda\mathfrak{g}'$. We write an element $\alpha \in A \otimes \Lambda\mathfrak{g}'$ as $\alpha = \sum_k \alpha_{[k]}$ with $\alpha_{[k]} \in A \otimes \Lambda^k\mathfrak{g}'$. We denote by $r : A \otimes \Lambda\mathfrak{g}' \to A$ the projection of an element $\alpha \in A \otimes \Lambda\mathfrak{g}'$ on its component $\alpha_{[0]}$ of exterior degree 0.

Lemma 27 *Let A be a G-differential complex. The map $r : A \otimes \Lambda\mathfrak{g}' \to A$ induces an isomorphism from $(A \otimes \Lambda\mathfrak{g}')_{horG}$ to A.*

Proof: For $X \in \mathfrak{g}$, we denote by $\iota_t(X)$ the tensor product contraction on $A \otimes \Lambda\mathfrak{g}'$. If $\alpha \in (A \otimes \Lambda\mathfrak{g}')_{horG}$ is such that $\alpha_{[0]} = 0$, it is easy to see by induction on the exterior degree that $\alpha = 0$. Let us prove that r is surjective: Let E^i be a basis of $\mathfrak{g}$ with dual basis $E_i \in \mathfrak{g}'$. Denote by ϵ_i the exterior multiplication by E_i on $A \otimes \Lambda\mathfrak{g}'$ from the left. Let $h_i = I - \epsilon_i\iota_t(E^i)$. It is easy to see that the operators

h_i commute with each other. In particular, the operator $h = \prod_i (I - \epsilon_i \iota_t(E^i))$ acting on $A \otimes \Lambda \mathfrak{g}'$ is well defined, i.e. it does not depend upon the order in which the product is taken. We verify : $\iota_t(E^i)h_i = 0, \iota_t(E^j)h_i = h_i \iota_t(E^j)$ for $i \neq j$. Thus h is a projector from $A \otimes \Lambda \mathfrak{g}'$ to the set $(A \otimes \Lambda \mathfrak{g}')_{hor_G}$. If $a \in A = A \otimes \Lambda^0 \mathfrak{g}'$, the element $h(a) = a - (-1)^{|a|-1} \sum_i \iota(E^i) a \otimes E_i + \cdots$ is a horizontal element of $A \otimes \Lambda \mathfrak{g}'$ with component of exterior degree 0 equal to a. Thus r is surjective. ∎

Definition 28 *The operator* $h := \prod_i (I - \epsilon_i \iota_t(E^i))$ *on* $A \otimes \Lambda \mathfrak{g}'$ *is called the horizontal projection.*

By the preceding lemma, if $\alpha \in A \otimes \Lambda \mathfrak{g}'$, the element $h(\alpha)$ is the unique horizontal element of $A \otimes \Lambda \mathfrak{g}'$ whose component of exterior degree 0 is $\alpha_{[0]}$.

If (A, d_A) is a G-differential complex, we can define the spaces $\mathcal{A}^{\pm\infty} := C^{\pm\infty}(\mathfrak{g}, A)$. We denote both of them by $\mathcal{A}$ when there is no need to indicate precisely the smoothness properties of a function $f : \mathfrak{g} \to A$ that we assume. The space $\mathcal{A}$ inherits a $\mathbb{Z}/2$-graded structure from that of A.

For any $E \in \mathfrak{g}$, the contraction operator $\iota(E)$ is extended to $\mathcal{A}$ pointwise:

$$(\iota(E)f)(X) = \iota(E)(f(X)).$$

Similarly the differential d_A is extended on $\mathcal{A}$ by

$$(d_A f)(X) = d_A(f(X)).$$

Thus we can define on $\mathcal{A}$ the operator

$$\iota_{\mathfrak{g}} = \sum x_i \iota(E^i)$$

i.e. $(\iota_{\mathfrak{g}} f)(X) = \sum_i x_i (\iota(E^i) f(X))$ and the operator

$$d_{\mathfrak{g}} = d_A - \iota_{\mathfrak{g}}.$$

When $\mathfrak{g}$ is understood, we will just write ι for $\iota_{\mathfrak{g}}$. If we take $A = \mathcal{A}(M)$, for a G-manifold M, then $\mathcal{A}^{\pm\infty}$ was introduced in section 2 and $d_{\mathfrak{g}}$ here coincides with the operator $d_{\mathfrak{g}}$ of section 2.

Lemma 29 *The operator* $d_{\mathfrak{g}}$ *is odd and satisfies* $d_{\mathfrak{g}}^2 = 0$ *on*

$$\mathcal{A}_G^{\pm\infty} := C^{\pm\infty}(\mathfrak{g}, A)^G.$$

More generally if χ *is a character of* G *trivial on its connected component, then*

$$(\mathcal{A}_{G,\chi}^{\pm\infty}, d_{\mathfrak{g}}) := (C^{\pm\infty}(\mathfrak{g}, A)^{\chi}, d_{\mathfrak{g}})$$

is a complex, where

$$C^{\pm\infty}(\mathfrak{g}, A)^{\chi} := \{f \in C^{\pm\infty}(\mathfrak{g}, A) : g \cdot f = \chi(g) f, \text{ for all } g \in G\}.$$

Proof: It is easy to see that $\iota^2 = 0$. So to prove that $d_{\mathfrak{g}}^2 f = 0$ for $f \in \mathcal{A}_{G,\chi}^{\pm\infty}$, it suffices to show that $(d_A\iota + \iota d_A)f = 0$. Now $((d_A\iota + \iota d_A)f)(X) = \sum_i x_i \mathcal{L}_A(E^i)(f(X))$, where $\mathcal{L}_A(E^i)$ is the Lie derivative action on A of the element $E^i \in \mathfrak{g}$. The invariance condition on an element $f \in C^{\pm\infty}(\mathfrak{g}, A)^\chi$ implies that $\mathcal{L}_A(E^i)(f(X)) = \frac{d}{d\epsilon} f(X + \epsilon[E^i, X])$. Thus, for $f \in C^{\pm\infty}(\mathfrak{g}, A)^\chi$, we obtain in the notation of Lemma 25 (section 3)

$$((d_A\iota + \iota d_A)f)(X) = \sum_i x_i \frac{d}{d\epsilon} f(X + \epsilon[E^i, X]) = \sum_{i,j} x_i x_j \partial_{[E^i,E^j]} f(X) = 0.$$

∎

Let G be a Lie group and let K be a closed subgroup of G. Let (A, d_A) be a K-differential complex. Let

$$L^{\pm\infty} = C^{\pm\infty}(\mathfrak{g}, A).$$

As for $\mathcal{A}$, we denote both of these by L, when there is no need to indicate the precise smoothness assumption. The $\mathbb{Z}/2$-graded structure on A induces a $\mathbb{Z}/2$-structure on L. Consider on L the structure of G-module, defined by $(g \cdot f)(X) = f(g^{-1} \cdot X)$, for $f \in C^{\pm\infty}(\mathfrak{g}, A)$. This structure of G-module of course induces a structure of $\mathfrak{g}$-module on differentiation. Thus on $L \otimes \Lambda\mathfrak{g}'$, we can define the Koszul differential c (associated to this $\mathfrak{g}$-module structure on L). However, we take in account sign rules in defining c: c coincides with the Koszul differential $c_{L^{even}}$ (see Formula 15 of section 3) on $L^{even} \otimes \Lambda\mathfrak{g}'$, while we define $c = -c_{L^{odd}}$ on $L^{odd} \otimes \Lambda\mathfrak{g}'$.

We extend the operator d_A pointwise on L: $(d_A f)(X) = d_A(f(X))$. We still denote by d_A the operator $d_A \otimes I$ on $L \otimes \Lambda\mathfrak{g}'$.

Let E^i be a basis of $\mathfrak{g}$. We extend the operator $\iota_\Lambda(E^i)$ to $L \otimes \Lambda\mathfrak{g}'$ following the sign rules (2), (3) given in section 1.

Consider the operator

$$j = \sum x_i \iota_\Lambda(E^i)$$

on $L \otimes \Lambda\mathfrak{g}'$.

Lemma 30 *The operator $c_{\mathfrak{g}} := d_A + c + j$ satisfies $c_{\mathfrak{g}}^2 = 0$.*

Proof: The $\mathfrak{g}$-module structure on L is induced from its $D(\mathfrak{g})$-module structure via the adjoint representation. It follows from Lemma 26 of section 3 that we have $(c + j)^2 = 0$. Furthermore, as can be easily seen, we have $(c + j)d_A + d_A(c + j) = 0$. ∎

The space L is equipped with the operators still denoted by $\iota_A(E), E \in \mathfrak{k}$, defined pointwise by their action on A.

Total contraction operators $\iota_t(Y), Y \in \mathfrak{k}$, are defined by the tensor product contraction on $L \otimes \Lambda\mathfrak{g}'$. Let us consider the action of K on L by

$$(k \cdot f)(X) = k \cdot f(k^{-1}X)$$

and the action on $L \otimes \Lambda\mathfrak{g}'$ by tensor product. We denote by $\mathcal{L}_t(Y), Y \in \mathfrak{k}$, the corresponding infinitesimal action of $Y \in \mathfrak{k}$.

We define $(L \otimes \Lambda\mathfrak{g}')_{horK} = \{\alpha \in L \otimes \Lambda\mathfrak{g}'; \iota_t(Y)\alpha = 0, \text{for all } Y \in \mathfrak{k}\}$. The subspace $(L \otimes \Lambda\mathfrak{g}')_{basK}$ of K-basic elements is the subspace of elements of $L \otimes \Lambda\mathfrak{g}'$ which are horizontal and invariant under K:

$$(L \otimes \Lambda\mathfrak{g}')_{basK} = ((L \otimes \Lambda\mathfrak{g}')_{horK})^K.$$

Lemma 31 *The operators $c_A := c + d_A$ and j preserve the subspace of K-basic elements.*

Proof: The operator j clearly preserves the space of horizontal elements. It commutes with the action of K, thus it preserves the space of K basic elements.

Using the relation $\iota_t(Y)c_A + c_A\iota_t(Y) = \mathcal{L}_t(Y)$, we see that the space of K-basic elements is stable under c_A. ∎

Definition 32 *Let A be a K-differential complex. Define the induced complex $Ind_{G/K}^{\pm\infty}A$ from K to G of the K-differential complex A to be the space*

$$Ind_{G/K}^{\pm\infty}A = (C^{\pm\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}$$

with the differential $c_{\mathfrak{g}} = c + d_A + j$.

If $(A, d_A) = (\mathbb{R}, 0)$, then $Ind_{G/K}^{\pm\infty}\mathbb{R} = (C^{\pm\infty}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}/\mathfrak{k})')^K$ with differential $c_{\mathfrak{g}} = c + j$, where c is the Koszul differential of the $(\mathfrak{g}, K)$-module $C^{\pm\infty}(\mathfrak{g})$.

Our aim is to compute the cohomology of the complex $(Ind_{G/K}^{\pm\infty}A, c_{\mathfrak{g}})$ in terms of the cohomology of the complex $(C^{\pm\infty}(\mathfrak{k}, A)^K, d_{\mathfrak{k}})$.

The cohomology of the complex $Ind_{G/K}^{\infty}A$ is determined in [13]. Recall the results:

Theorem 33 *The restriction map*

$$r_{\mathfrak{k}} : C^{\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}' \to C^{\infty}(\mathfrak{k}, A)$$

given by

$$(r_{\mathfrak{k}}\alpha)(Y) = \alpha(Y) \qquad \text{for } \alpha \in C^{\infty}(\mathfrak{g}, A), Y \in \mathfrak{k}$$
$$(r_{\mathfrak{k}}\alpha) = 0 \qquad \text{if } \alpha \in C^{\infty}(\mathfrak{g}, A) \otimes \Lambda^{\geq 1}\mathfrak{g}'$$

defines a cochain map from $(Ind_{G/K}^{\infty}(A), c_{\mathfrak{g}})$ to $(C^{\infty}(\mathfrak{k}, A)^K, d_{\mathfrak{k}})$. Furthermore if the principal bundle $G \to G/K$ possesses a G-invariant connection, then the restriction map $r_{\mathfrak{k}}$ induces an isomorphism in cohomology.

In particular, if $A = \mathbb{R}$, we obtain a map $r_{\mathfrak{k}} : Ind_{G/K}\mathbb{R} \to C^\infty(\mathfrak{k})^K$ and this map is an isomorphism in cohomology when K is compact.

Remark here that the restriction map does not extend to $Ind^{-\infty}_{G/K}(A)$ as generalized functions cannot usually be restricted to a subspace.

The assumption that $G \to G/K$ possesses a G-invariant connection is satisfied for example when K is a reductive Lie group, in particular when K is a compact subgroup.

We now consider $Ind^{-\infty}_{G/K}A$. Let $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$. We identify $\mathfrak{r}'$ with $\mathfrak{k}^\perp = \{f \in \mathfrak{g}', f|\mathfrak{k} = 0\}$. Let $n = \dim \mathfrak{r}$. Consider the character $\chi_{G/K}(k) = sign(det_{\mathfrak{r}}k)$ of K. As G and K are fixed, we denote $\chi_{G/K}$ simply by χ. Let $\nu' \in \Lambda^n\mathfrak{r}'$ be a nonzero element. The element ν' determines an Euclidean measure $|d\nu'|$ and an orientation o on $\mathfrak{r}$. If dX is an Euclidean measure on $\mathfrak{g}$, we denote by dY the Euclidean measure on $\mathfrak{k}$ such that $dX = |d\nu'|dY$.

Remark that, as $n = \dim \mathfrak{g}/\mathfrak{k}$, the space $\Lambda\mathfrak{k}' \otimes \Lambda^n\mathfrak{r}'$ is naturally embedded as a subspace of $\Lambda\mathfrak{g}'$. Consider the horizontal projection operator (see Definition 28) $h_K : A \to (A \otimes \Lambda\mathfrak{k}')_{horK}$. Thus, for $a \in A$, the element $h_K(a) \wedge \nu'$ belongs to $(A \otimes \Lambda\mathfrak{g}')_{horK}$.

Definition 34 *Let $f \in \mathcal{A}^{-\infty}_{K,\chi} := C^{-\infty}(\mathfrak{k}, A)^\chi$. Choose ν' a nonzero element of $\Lambda^n\mathfrak{r}'$. We define $Ind_{G/K,\nu'}f \in Ind^{-\infty}_{G/K}A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}$ by*

$$(Ind_{G/K,\nu'}f, \Phi dX) = h_K(\int_{\mathfrak{k}} f(Y)\Phi(Y)dY) \wedge \nu', \text{ for any test function } \Phi \text{ on } \mathfrak{g}$$

(with $dX = dY|d\nu'|$).

It is easy to see that the map $Ind_{G/K,\nu'}$ depends only on the orientation o of $\mathfrak{g}/\mathfrak{k}$ determined from ν'. Thus we write $Ind_{G/K,o}$ for $Ind_{G/K,\nu'}$. Remark that the map $Ind_{G/K,o}$ is injective.

Proposition 35 *The map $Ind_{G/K,o}$ is a cochain map of parity degree equal to $\dim G/K$ from the $\mathbb{Z}/2$-cochain complex $(\mathcal{A}^{-\infty}_{K,\chi}, d_{\mathfrak{k}})$ to the $\mathbb{Z}/2$-cochain complex $(Ind^{-\infty}_{G/K}A, c_{\mathfrak{g}})$.*

Proof: Let $L = C^{-\infty}(\mathfrak{g}, A)$ and let $L^{\det} \subset L$ be the subset of elements $F \in L$ satisfying $k \cdot F = (\det_{\mathfrak{r}}k)F$. If $F \in L$, define $h_K(F) \in L \otimes \Lambda\mathfrak{k}'$ by $h_K(F)(X) = h_K(F(X))$. The map $F \to h_K(F) \wedge \nu'$ sends L to $(L \otimes \Lambda\mathfrak{k}')_{horK} \wedge \nu'$. It sends $L^{\det}$ to $(L \otimes \Lambda\mathfrak{g}')_{basK} = Ind^{-\infty}_{G/K}A$.

Lemma 36 *For every $F \in L^{\det}$, we have $c_A(h_K F \wedge \nu') = h_K(d_A F) \wedge \nu'$, where c_A is the operator on $Ind^{-\infty}_{G/K}A$ defined in Lemma 31.*

Proof: We compute $c_A(h_K(F) \wedge \nu')$ for $F \in L^{\det}$. Let K^i be a basis of $\mathfrak{k}$ with dual basis K_i. It is easy to see that $c(\nu') = -\sum_i (Tr\,(ad_{\mathfrak{k}} K^i))K_i \wedge \nu'$. Thus, by Leibniz's rule,

$$c_A(h_K(F) \wedge \nu') \in L \otimes (\Lambda \mathfrak{g}' \wedge \nu') = L \otimes (\Lambda \mathfrak{k}' \wedge \nu').$$

As c_A preserves K-basic elements (cf. Lemma 31), $c_A(h_K(F) \wedge \nu')$ is K-basic, in particular is horizontal. The space $(L \otimes (\Lambda \mathfrak{g}' \wedge \nu'))_{horK}$ is isomorphic with L by the map $\alpha \wedge \nu' \to \alpha_{[0]} \in L$ for $\alpha \in L \otimes \Lambda \mathfrak{g}'$, where $\alpha_{[0]}$ is the component of α in the zeroth exterior degree. Thus, if $c_A(h_K(F) \wedge \nu') = G \wedge \nu'$, then $c_A(h_K(F) \wedge \nu') = h_K(G_{[0]}) \wedge \nu'$.

We have $G = d_A(h_K(F)) + c(h_K F) + (-1)^{|F|+1} h_K F \wedge \sum_i (Trad_{\mathfrak{k}} K^i) K_i$. Looking at the term of zeroth-exterior degree, it follows that $G_{[0]} = d_A F$. This proves the lemma. ∎

For $f \in C^{-\infty}(\mathfrak{k}, A)^\chi$, the element $F \in C^{-\infty}(\mathfrak{g}, A)$ defined by $(F, \Phi dX) = \int_{\mathfrak{k}} f(Y)\Phi(Y)dY$ belongs to $L^{\det}$ and $Ind_{G/K,o} f = h_K(F) \wedge \nu'$. The preceding lemma implies $c_A Ind_{G/K,o} f = Ind_{G/K,o} d_A f$. Proposition 35 is now a consequence of the following

Lemma 37

$$j(Ind_{G/K,o} f) = -Ind_{G/K,o}(\iota_{\mathfrak{k}} f).$$

Proof: Let E^i be a basis of $\mathfrak{g}$ such that the first elements form a basis of $\mathfrak{k}$. Let E_i be the dual basis. Then the last n coordinates x_i vanish on $\mathfrak{k}$. We denote by $y_i, 1 \leq i \leq \dim \mathfrak{k}$ the coordinates on $\mathfrak{k}$ corresponding to the basis of $\mathfrak{k}'$ dual to the basis E^i of $\mathfrak{k}$. We have then

$$(j(Ind_{G/K,o} f), \Phi dX) = \sum_{i=1}^{\dim \mathfrak{k}} (\iota_\Lambda(E^i) h_K(\int_{\mathfrak{k}} f(Y) y_i \Phi(Y) dY)) \wedge \nu'.$$

As h_K is a projector on the K-horizontal elements for the tensor product contraction, it satisfies for $E^i \in \mathfrak{k}$ and $a \in A$, $\iota_\Lambda(E^i)(h_K a) + \iota_A(E^i)(h_K a) = 0$. But $\iota_A(E^i) h_K = h_K \iota_A(E^i)$, and we obtain the lemma. ∎

The proof of this lemma completes the proof of Proposition 35. ∎

If A is a K-differential algebra, then $C^{-\infty}(\mathfrak{k}, A)^\chi$ is a module over $C^\infty(\mathfrak{k}, A)^K$. Similarly $Ind^{-\infty}_{G/K} A$ is a module over $Ind^{\infty}_{G/K} A$. Remark the following relation between the maps $r_{\mathfrak{k}}$ and $Ind_{G/K,o}$

Lemma 38 *If $\alpha \in Ind^{\infty}_{G/K} A$ and $s \in C^{-\infty}(\mathfrak{k}, A)^\chi$, then*

$$\alpha Ind_{G/K,o} s = Ind_{G/K,o}((r_{\mathfrak{k}} \alpha) s).$$

The main result of this section is the following

Theorem 39 *Assume that the group K is compact. Then the map $Ind_{G/K,o}$: $(\mathcal{A}^{-\infty}_{K,\chi}, d_{\mathfrak{k}}) \to (Ind^{-\infty}_{G/K}A, c_{\mathfrak{g}})$ induces an isomorphism in cohomology.*

Proof: We can choose a K-invariant decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}.$$

To this direct sum decomposition is associated the tensor product decomposition $\Lambda\mathfrak{g}' = \Lambda\mathfrak{k}' \otimes \Lambda\mathfrak{r}'$.

Let $h_K : L \to L \otimes \Lambda\mathfrak{k}'$ be the projection on K-horizontal vectors for the tensor product contraction (see definition 28). The map

$$W(v \otimes \xi) = h_K(v) \wedge \xi$$

for $v \in L$ and $\xi \in \Lambda\mathfrak{r}'$ is an isomorphism from the space $L \otimes \Lambda\mathfrak{r}'$ to the space $(L \otimes \Lambda\mathfrak{g}')_{horK}$. The map W commutes with the action of K and allows us to identify the space

$$T_K := (L \otimes \Lambda\mathfrak{r}')^K$$

with the space

$$Ind^{-\infty}_{G/K}A = (C^{-\infty}(\mathfrak{g}, A) \otimes \Lambda\mathfrak{g}')_{basK}.$$

On the space T_K, we will use the $\mathbb{Z}_+$-gradation given by the exterior degree $T_K = \oplus_{p=0}^n T_K^p = \oplus_{p=0}^n (L \otimes \Lambda^p\mathfrak{r}')^K$.

Let R^i be a basis of $\mathfrak{r}$ with dual basis R_i and let K^j be a basis of $\mathfrak{k}$ with dual basis K_j. We write $X \in \mathfrak{g}$ as $X = Y + R$, with $R = \sum_i x_i R^i$, $Y = \sum_j y_j K^j$.

The operator $\iota_{\mathfrak{k}} = \sum_j y_j \iota_A(K^j)$ is defined on $L = C^{-\infty}(\mathfrak{g}, A)$. Let $j_{\mathfrak{r}}$: $L \otimes \Lambda^*\mathfrak{r}' \to L \otimes \Lambda^{*-1}\mathfrak{r}'$ be given by $j_{\mathfrak{r}} = \sum_i x_i \iota_\Lambda(R^i)$. It is easy to see that $\iota_{\mathfrak{k}}$ and $j_{\mathfrak{r}}$ commute with the action of K on $L \otimes \Lambda\mathfrak{r}'$.

Lemma 40 *For all $\alpha \in L \otimes \Lambda\mathfrak{r}'$,*

$$jW(\alpha) = W(j_{\mathfrak{r}}\alpha - \iota_{\mathfrak{k}}\alpha).$$

Proof: We have, for $v \in L$ and $\xi \in \Lambda\mathfrak{r}'$,

$$j(h_K v \wedge \xi) = \sum_i x_i \iota_\Lambda(R^i)(h_K v \wedge \xi) + \sum_j y_j (\iota_\Lambda(K^j) h_K(v)) \wedge \xi.$$

Further, for $K^j \in \mathfrak{k}$, $\iota_\Lambda(K^j)h_K(v) + h_K(\iota_A(K^j)v) = 0$, and we obtain the lemma. ∎

Lemma 41 *If $\alpha \in T_K^p$, then $W^{-1}c_A W(\alpha) \in T_K^p \oplus T_K^{p+1} \oplus T_K^{p+2}$.*

Proof: This is obvious since, $\mathfrak{k}$ being a subalgebra, $c(\Lambda^q\mathfrak{k}' \otimes \Lambda^p\mathfrak{r}') \subset (\Lambda^{q+1}\mathfrak{k}' \otimes \Lambda^p\mathfrak{r}') \oplus (\Lambda^q\mathfrak{k}' \otimes \Lambda^{p+1}\mathfrak{r}') \oplus (\Lambda^{q-1}\mathfrak{k}' \otimes \Lambda^{p+2}\mathfrak{r}')$. ∎

Let us now prove Theorem 39. By the map W, we identify $Ind^{-\infty}_{G/K}A$ with T_K and write still $c_{\mathfrak{g}}$ for the operator $W^{-1}c_{\mathfrak{g}}W$ on T_K. As we have seen in Proposition 22 and Remark 23 (of section 3), the homology groups of the operator $j_{\mathfrak{r}} : T^*_K \to T^{*-1}_K$ are equal to zero except in maximal degree $n = \dim \mathfrak{r}$. Furthermore, if $\alpha \in T^n_K$ is such that $j_{\mathfrak{r}}\alpha = 0$, then $\alpha = \delta_{\mathfrak{r},o}(x) \otimes f(y)$ for some $f \in C^{-\infty}(\mathfrak{k}, A)^\chi$ by Lemma 21 and Remark 23 (of section 3). But then, by definition of $Ind_{G/K,o}$, we obtain $\alpha = Ind_{G/K,o}f$.

Now, if $\alpha \in T^n_K$ is such that $c_{\mathfrak{g}}\alpha = 0$, writing this equation componentwise, we see that α satisfies the relation $j_{\mathfrak{r}}\alpha = 0$. Thus α is of the form $Ind_{G/K,o}f$. As $c_{\mathfrak{g}}Ind_{G/K,o}f = Ind_{G/K,o}d_{\mathfrak{k}}f$ (by Proposition 35), we see that $f \in Kerd_{\mathfrak{k}}$, since $Ind_{G/K,o}$ is an injective map. Consider now an element $\alpha = \sum_{k\geq k_0} \alpha_{[k]}$ in the kernel of $c_{\mathfrak{g}}$. From the degree consideration, we see that its component of minimal exterior degree k_0 is annihilated by $j_{\mathfrak{r}}$. If k_0 is less than n, there exists an element $\beta \in T^{k_0+1}_K$ such that $\alpha_{[k_0]} = j_{\mathfrak{r}}\beta$. The element $\alpha - c_{\mathfrak{g}}\beta$ is in the same cohomology class as α and all its non-zero exterior degree components are of degree strictly greater than k_0. By induction, α has a representative in T^n_K and we see that the map $Ind_{G/K,o}$ is surjective in cohomology.

Suppose now that $Ind_{G/K,o}f = c_{\mathfrak{g}}\beta$ with $\beta = \sum_{k\geq k_0} \beta_{[k]}$. Writing this equation component wise, we see that $j_{\mathfrak{r}}\beta_{[k_0]} = 0$. If $k_0 < n$, changing β to $\beta' = \beta - c_{\mathfrak{g}}\gamma$ with $\gamma \in T^{k_0+1}_K$ and $j_{\mathfrak{r}}\gamma = \beta_{[k_0]}$, we still have $Ind_{G/K,o}f = c_{\mathfrak{g}}\beta'$. By choice, $\beta' = \sum_{k>k_0} \beta'_{[k]}$. Hence, by induction, we obtain an element $\tilde{\beta} \in T^n_K$ such that $Ind_{G/K,o}f = c_{\mathfrak{g}}\tilde{\beta}$. From degree consideration, $j_{\mathfrak{r}}\tilde{\beta} = 0$. But then $\tilde{\beta} = Ind_{G/K,o}g$ for some $g \in C^{-\infty}(\mathfrak{k}, A)^\chi$. The equation $c_{\mathfrak{g}}Ind_{G/K,o}g = Ind_{G/K,o}f$ reads as $Ind_{G/K,o}d_{\mathfrak{k}}g = Ind_{G/K,o}f$. But $Ind_{G/K,o}$ being an injective map, we get $d_{\mathfrak{k}}g = f$. This proves that the map $Ind_{G/K,o}$ is injective in cohomology, thereby completing the proof of Theorem 39. ∎

As a particular case, if K is compact and if $(A, d_A) = (\mathbb{R}, 0)$, we obtain that the map $Ind_{G/K,o}$ induces an isomorphism from $C^{-\infty}(\mathfrak{k})^\chi$ to the cohomology of the complex $((C^{-\infty}(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}/\mathfrak{k})')^K, j + c)$.

We apply these calculations in the next section to the calculation of the equivariant cohomology of fiber bundles over homogeneous spaces.

5 Equivariant cohomology of homogeneous spaces

Let G be a Lie group and let K be a closed subgroup of G. Let $D = G/K$. Let $e \in D$ be the base point of D. We identify the tangent space of G at a point $g \in G$ with $\mathfrak{g}$ by sending $X \in \mathfrak{g}$ to the tangent vector to the curve

$g \exp \epsilon X$ in G. Let $n = \dim D$. Let $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$. The tangent space T_eD at $e \in D$ is identified with $\mathfrak{r}$. If $g \in G$, we again denote by L_g the map $T_eD \to T_{g \cdot e}D$ induced from the left multiplication map $L_g : D \to D$. For $\alpha \in C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(D)) := \sum_{p=0}^{\dim D} C^{\pm\infty}(\mathfrak{g}, \mathcal{A}^p(D))$, $\alpha_{[p]}$ is the p-th component of α. If $x \in D$ and $\xi_1, \cdots, \xi_p \in T_xD$, then $(\alpha_{[p]})_x(\xi_1, \cdots, \xi_p)$ is a function (maybe generalized) on $\mathfrak{g}$ given by $X \mapsto (\alpha_{[p]}(X))_x(\xi_1, \cdots, \xi_p)$.

Let $\alpha \in \mathcal{A}_G^{\pm\infty}(D) = C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(D))^G$. Let $R^1, .., R^p \in \mathfrak{r}$, $X \in \mathfrak{g}$ and $g \in G$. As α iz an equivariant form

$$\alpha_{[p]}(Ad(g)(X))_{g \cdot e}(L_g R^1, \ldots, L_g R^p) = \alpha_{[p]}(X)_e(R^1, ..., R^p).$$

Let $\tilde{\alpha}(X) = \alpha(X)_e$. Thus $\tilde{\alpha}$ is a function on $\mathfrak{g}$ with values in $\Lambda \mathfrak{r}'$ and the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism from the space $\mathcal{A}_G^{\pm\infty}(D)$ to the space $T_K = (C^{\pm\infty}(\mathfrak{g}) \otimes \Lambda \mathfrak{r}')^K$, where the action of K on both $C^{\pm\infty}(\mathfrak{g})$ and $\Lambda \mathfrak{r}'$ is induced from the adjoint representation.

Let R^i be a basis of $\mathfrak{r} = \mathfrak{g}/\mathfrak{k}$, with dual basis $R_i \in \mathfrak{r}'$. Let $x_i = R_i(X)$. Let $j_{\mathfrak{r}}$ be the operator on T_K given by

$$j_{\mathfrak{r}} = \sum_i x_i \iota_\Lambda(R^i).$$

Let c be the Koszul differential on T_K (see Formula 15 of section 3). From [13], we have

Lemma 42 *For $\alpha \in \mathcal{A}_G^{\pm\infty}(D)$, we have $\widetilde{(d_{\mathfrak{g}}\alpha)} = (c + j_{\mathfrak{r}})\tilde{\alpha}$.*

For example, the complex $\mathcal{A}_G^{-\infty}(G)$ becomes isomorphic, under evaluation at e, to $C^{-\infty}(\mathfrak{g}) \otimes \Lambda \mathfrak{g}'$ and the differential $d_{\mathfrak{g}}$ becomes the perturbed differential $c + j$.

In the notation of the preceeding section, if $(A, d_A) = (\mathbb{R}, 0)$, we have $\mathcal{A}_G^{\pm\infty}(D) \cong Ind_{G/K}^{\pm\infty}(\mathbb{R})$, as cochain complexes.

Let $\chi_{G/K}(k) := sign(det_{\mathfrak{r}} k)$. As G and K are fixed in this section, we denote $\chi_{G/K}$ by χ. Recall the definition (see Theorem 33 and Definition 34 of section 4) of the maps

$$r_{\mathfrak{k}} : Ind_{G/K}^{\infty}(\mathbb{R}) \to C^{\infty}(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to Ind_{G/K}^{-\infty}(\mathbb{R}).$$

Using the identification $\alpha \mapsto \tilde{\alpha}$, we get maps again denoted by $r_{\mathfrak{k}}$ and $Ind_{G/K,o}$:

$$r_{\mathfrak{k}} : \mathcal{A}_G^{\infty}(D) \to C^{\infty}(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_G^{-\infty}(D).$$

Let us describe explicitly these maps.

The point e is a K-stable submanifold of D. The successive restriction maps $\mathcal{A}_G^\infty(D) \to \mathcal{A}_K^\infty(D) \to \mathcal{A}_K^\infty(e)$ are well defined. The composed map $\mathcal{A}_G^\infty(D) \to \mathcal{A}_K^\infty(e) = C^\infty(\mathfrak{k})^K$ coincides obviously with the map $r_{\mathfrak{k}}$:

$$(r_{\mathfrak{k}}\alpha)(Y) = (\alpha_{[0]})_e(Y)$$

for $Y \in \mathfrak{k}$.

We now describe the map $Ind_{G/K,o}$ from $C^{-\infty}(\mathfrak{k})^\chi$ to $\mathcal{A}_G^{-\infty}(G/K)$. Let $\nu' \in \Lambda^n\mathfrak{r}'$ be a non-zero element. The element ν' determines an orientation o and an Euclidean measure $|d\nu'|$ on $\mathfrak{g}/\mathfrak{k}$. If dX is an Euclidean measure on $\mathfrak{g}$, we denote by dY the Euclidean measure on $\mathfrak{k}$ such that $dX = |d\nu'|dY$.

We identify the space $\mathcal{A}(G/K)$ with $C^\infty(G, \Lambda\mathfrak{r}')^K$.

Proposition 43 *The map $Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^\chi \to \mathcal{A}_G^{-\infty}(G/K)$ is given, for $f \in C^{-\infty}(\mathfrak{k})^\chi$, by*

$$(16) \qquad (\int_{\mathfrak{g}} (Ind_{G/K,o}f)(X)\phi(X)dX)(g) = |det_{\mathfrak{g}}(g)|(\int_{\mathfrak{k}} f(Y)\phi(gY)dY)\nu',$$

where ϕ is any test function on $\mathfrak{g}$ and $g \in G$.

Moreover, for any $f \in C^{-\infty}(\mathfrak{k})^\chi$, the equivariant form $Ind_{G/K,o}(f)$ is $d_{\mathfrak{g}}$-closed.

Proof: It is easy to verify that $Ind_{G/K,o}f$ defined by the Formula 16 is indeed an element of $\mathcal{A}_G^{-\infty}(D)$. It obviously coincides with the map given in Definition 34 of section 4 (denoted also by $Ind_{G/K,o}$) at $g = e$. The fact that $Ind_{G/K,o}f$ is $d_{\mathfrak{g}}$ closed follows from Proposition 35 of section 4. It is also easy to check it directly. ∎

Assume that G/K is compact and G-oriented. Thus $\chi = 1$. We can integrate over G/K an equivariant cohomology class and we obtain then a G-invariant generalized function on $\mathfrak{g}$. The next formula is just the integration over G/K of the formula given in Proposition 43 for $Ind_{G/K,o}f$. However, it indicates the analogy between $\int_{G/K} Ind_{G/K,o}f$ and characters of induced representations.

Proposition 44 *Assume G/K compact and G-oriented. Let $f \in C^{-\infty}(\mathfrak{k})^K$. Then*

$$\int_{G/K,o} (Ind_{G/K,o}f, \Phi dX) = \int_{G/K,o} |\det{}_{\mathfrak{g}}(g)|(\int_{\mathfrak{k}} f(Y)\Phi(gY)dY)dg/dk.$$

for any test function Φ on $\mathfrak{g}$ and compatible choices of the left-invariant Haar measure dg on G, of the G-invariant measure dg/dk on G/K and of the Euclidean measure dY on $\mathfrak{k}$.

We still denote by $r_{\mathfrak{k}}$ the map $H_G^\infty(G/K) \to C^\infty(\mathfrak{k})^K$ induced from the map $r_{\mathfrak{k}}$ at the cohomology level and by $Ind_{G/K,o}$ the map $C^{-\infty}(\mathfrak{k})^\chi \to H_G^{-\infty}(G/K)$ induced from the map $Ind_{G/K,o}$ at the cohomology level.

As a particular case of [13] (see Theorem 33 of section 4), we have the following

Proposition 45 *Assume that K is compact. Then the map $r_{\mathfrak{k}}$ gives an isomorphism from $H_G^\infty(G/K)$ with $C^\infty(\mathfrak{k})^K$.*

Further Theorem 39 of section 4 gives as an immediate corollary the following

Theorem 46 *Assume that K is compact. Then the map $Ind_{G/K,o}$ gives an isomorphism from $C^{-\infty}(\mathfrak{k})^\chi$ with $H_G^{-\infty}(G/K)$. The map $Ind_{G/K,o}$ is of even (resp. odd) degree if $\dim G/K$ is even (resp. odd).*

When K is compact, let us give a formula for the map $Ind_{G/K,o}$ in terms of generalized functions.

Choose a K-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$$

and let $pr_{\mathfrak{k}}$ (resp. $pr_{\mathfrak{r}}$) be the projection of $\mathfrak{g}$ on $\mathfrak{k}$ (resp. on $\mathfrak{r}$) determined by this decomposition.

With the notation of Formula 10 of section 2, we have (by Proposition 43), for $f \in C^{-\infty}(\mathfrak{k})^\chi$,

$$(Ind_{G/K,o} f(X))(e) = |\nu'|^{-1} \delta_{\mathfrak{r}}(pr_{\mathfrak{r}} X) f(pr_{\mathfrak{k}} X) \nu', \tag{17}$$

where we have identified the space $\mathcal{A}(G/K)$ with $C^\infty(G, \Lambda \mathfrak{r}')^K$.

Consider the case where K is the trivial subgroup. Recall that we have defined the element

$$\alpha_{G,o}(X) = |\nu'|^{-1} \delta_{\mathfrak{g}}(X) \otimes |\det_{\mathfrak{g}} g| dg$$

in Lemma 13 of section 2. From Formula 17, we see that it is also equal to the element $Ind_{G,o} 1$. Thus we obtain from Theorem 46:

Lemma 47 *Let G be a Lie group acting on itself by left translations, then*

$$H_G^{-\infty}(G) \cong \mathbb{R} \alpha_{G,o}.$$

A more general result is proved in Theorem 89 in section 9.

Now we are going to generalize Proposition 45 and Theorem 46 as follows. Let M be a K-manifold. Consider the product manifold $G \times M$. The group K acts freely on the right on $G \times M$ by $(g,m)k = (gk, k^{-1}m)$. Consider the fiber space $\mathcal{M} = G \times_K M$ of orbits of the K-action. The group G acts on the left on $\mathcal{M}$. When M is a point, the space $\mathcal{M}$ is $D = G/K$. The projection $(g,m) \mapsto g$ induces a map $\mathcal{M} \to D$. Thus the space $\mathcal{M}$ is a fiber space over D with fiber M.

If $\alpha \in \mathcal{A}(\mathcal{M}) \subset \mathcal{A}(G \times M)$, and $g \in G$, then $\alpha(g)$ is an element of $(\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK}$, where $\mathfrak{g}'$ is identified with left invariant 1-forms on G.

Thus

$$\mathcal{A}(\mathcal{M}) = C^\infty(G, (\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK})^K$$

where K-invariants are taken with respect to the action of K by right multiplication on G, left action on M and adjoint action on $\Lambda\mathfrak{g}'$.

If $\alpha(X) \in \mathcal{A}(\mathcal{M})$, then $\tilde{\alpha}(X) := \alpha(X)_e \in (\Lambda\mathfrak{g}' \otimes \mathcal{A}(M))_{horK}$.

By G-invariance, the space $\mathcal{A}_G^{\pm\infty}(\mathcal{M})$ is thus identified with

$$Ind_{G/K}^{\pm\infty}(\mathcal{A}(M)) := (C^{\pm\infty}(\mathfrak{g}, \mathcal{A}(M)) \otimes \Lambda\mathfrak{g}')_{basK}.$$

Let $A = \mathcal{A}(M)$ be our K-differential complex, then $Ind_{G/K}^{\pm\infty}(A)$ is provided with a differential $c_\mathfrak{g}$. As before, the map $\alpha \mapsto \tilde{\alpha}$ is an isomorphism from $\mathcal{A}_G^{\pm\infty}(G \times_K M)$ to $Ind_{G/K}^{\pm\infty}(A)$ and the following lemma ([13]) iz a generalization of Lemma 42.

Lemma 48 *For any* $\alpha \in \mathcal{A}_G^{-\infty}(\mathcal{M})$,

$$\tilde{d_\mathfrak{g}\alpha} = c_\mathfrak{g}\tilde{\alpha}.$$

Thus $(\mathcal{A}_G^{\pm\infty}(\mathcal{M}), d_\mathfrak{g})$ is identified with the complex $(Ind_{G/K}^{\pm\infty}(\mathcal{A}(M)), c_\mathfrak{g})$ of section 4. The complex $(C^\infty(\mathfrak{k}, A)^K, d_\mathfrak{k})$ defined in section 4 is the complex $(\mathcal{A}_K^\infty(M), d_\mathfrak{k})$ of the K-equivariant cohomology of M. As χ is trivial on the connected component of K, the operator $d_\mathfrak{k}$ on $C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi$ still satisfies $d_\mathfrak{k}^2 = 0$, as follows from Lemma 29 of section 4.

Definition 49 *Let us denote by* $\mathcal{A}_{K,\chi}^{-\infty}(M)$ *the space* $C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi$. *We define* $H_{K,\chi}^{-\infty}(M)$ *to be the cohomology of the complex* $(C^{-\infty}(\mathfrak{k}, \mathcal{A}(M))^\chi, d_\mathfrak{k})$.

Recall the definition of the maps

$$r_\mathfrak{k} : Ind_{G/K}^\infty(\mathcal{A}(M)) \to \mathcal{A}_K^\infty(M)$$

and

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,\chi}(M) \to Ind^{-\infty}_{G/K}(\mathcal{A}(M))$$

from section 4. Using the identification $\alpha \mapsto \tilde{\alpha}$, we get maps again denoted by $r_{\mathfrak{k}}$ and $Ind_{G/K,o}$:

$$r_{\mathfrak{k}} : \mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$$

and

$$Ind_{G/K,o} : \mathcal{A}^{-\infty}_{K,\chi}(M) \to \mathcal{A}^{-\infty}_{G}(\mathcal{M}).$$

Let us describe explicitly these maps.

The fiber of $\mathcal{M} \to D$ over the point $e \in D$ is canonically identified with M and is a K-stable submanifold of $\mathcal{M}$. The successive restriction maps $\mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$ are well defined. Obviously, the composed map $\mathcal{A}^{\infty}_{G}(\mathcal{M}) \to \mathcal{A}^{\infty}_{K}(M)$ coincides with the map $r_{\mathfrak{k}}$.

The following proposition gives an explicit description of the map $Ind_{G/K,o}$ generalizing Proposition 43, whose proof is identical (and hence is omitted).

Proposition 50 *For any $f \in \mathcal{A}^{-\infty}_{K,\chi}(M)$, $Ind_{G/K,o} f \in \mathcal{A}^{-\infty}_{G}(\mathcal{M})$ is given by*

$$(Ind_{G/K,o} f, \Phi dX)(g) = |det_{\mathfrak{g}} g| h_K(\int_{\mathfrak{k}} f(Y)\Phi(gY)dY) \wedge \nu'$$

where Φ is any test function on $\mathfrak{g}$ and $dX = dY|d\nu'|$.

We still denote by $r_{\mathfrak{k}}$ the map from $H^{\infty}_{G}(\mathcal{M})$ to $H^{\infty}_{K}(M)$ induced from the map $r_{\mathfrak{k}}$ at the cohomology level. As a generalization of Proposition 45, we have the following Theorem ([13]), (got from Theorem 33 of section 4):

Theorem 51 *Let K be a compact subgroup of a Lie group G and let M be a K-manifold. Then the map $r_{\mathfrak{k}}$ gives an isomorphism from $H^{\infty}_{G}(\mathcal{M})$ with $H^{\infty}_{K}(M)$.*

We still denote by $Ind_{G/K,o}$ the map from $H^{-\infty}_{K,\chi}(M)$ to $H^{-\infty}_{G}(G \times_K M)$ induced from the map $Ind_{G/K,o}$ at the cohomology level. As a corollary of Theorem 39 of section 4, we get the main result of this section which generalizes Theorem 46.

Theorem 52 *Let K be a compact subgroup of a Lie group G and let M be a K-manifold. Then the map $Ind_{G/K,o}$ gives an isomorphism in cohomology from $H^{-\infty}_{K,\chi}(M)$ to $H^{-\infty}_{G}(G \times_K M)$. This map is of even (resp. odd) degree, if $\dim(G/K)$ is even (resp. odd).*

When K is compact, let us write the map $Ind_{G/K,o}$ in terms of generalized functions. We identify $\mathcal{A}(\mathcal{M})$ with the subspace of K-basic elements of $\mathcal{A}(G \times M)$.

Choose a K-invariant decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$$

and let $pr_{\mathfrak{k}}$ (resp. $pr_{\mathfrak{r}}$) be the projection of $\mathfrak{g}$ on $\mathfrak{k}$ (resp. on $\mathfrak{r}$) determined by this decomposition. Let $\nu' \in \Lambda^n \mathfrak{r}'$ be a positive element. With the help of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$, we consider ν' as an element of $\Lambda^n \mathfrak{g}'$. With the notation of Formula 10 of section 2, we have by Proposition 50, for any $m \in M$ and $f \in \mathcal{A}_{K,\chi}^{-\infty}(M)$,

$$(18) \qquad ((Ind_{G/K,o} f)(X))_{(e,m)} = |\nu'|^{-1} \delta_{\mathfrak{r}}(pr_{\mathfrak{r}} X)(h_K(f(pr_{\mathfrak{k}} X)))_m \wedge \nu'.$$

Assume that K is compact. Choose an orientation o on $\mathfrak{r}$. Then there are canonical isomorphisms

$$r_{\mathfrak{k}} : H_G^{\infty}(G/K) \cong C^{\infty}(\mathfrak{k})^K$$

and

$$Ind_{G/K,o} : C^{-\infty}(\mathfrak{k})^{\chi} \cong H_G^{-\infty}(G/K)$$

guaranteed by Proposition 45 and Theorem 46 respectively.

The natural map $H_G^{\infty}(G/K) \to H_G^{-\infty}(G/K)$ thus gives rise (using the above two identifications) to a map

$$M_o : C^{\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{k})^{\chi}.$$

Comparing the $\mathbb{Z}/2$ degree of the maps, we see that this map is identically zero if G/K is not of even dimension. The orientation o determines a polynomial square root $Y \mapsto det_{\mathfrak{g}/\mathfrak{k},o}^{1/2}(Y)$ of the polynomial function $Y \mapsto det_{\mathfrak{g}/\mathfrak{k}}(Y)$ on $\mathfrak{k}$. The normalization of this square root is as in ([12], Formula 12).

Proposition 53 *If $f \in C^{\infty}(\mathfrak{k})^K$, then*

$$M_o(f)(Y) = (-2\pi)^{\dim(G/K)/2} det_{\mathfrak{g}/\mathfrak{k},o}^{1/2}(Y) f(Y)$$

for $Y \in \mathfrak{k}$.

(If $\dim G/K$ is odd, then the function $det_{\mathfrak{g}/\mathfrak{k}}$ is identically 0.)

Proof: The map M_o is a morphism of $C^{\infty}(\mathfrak{k})^K$-modules by Lemma 38 of section 4. It is thus sufficient to prove that

$$1 \cong Ind_{G/K,o}((-2\pi)^{\dim(G/K)/2} det_{\mathfrak{g}/\mathfrak{k},o}^{1/2})$$

in $H_G^{-\infty}(G/K)$.

We will prove this relation using the method of the proof of Proposition 31 of [12]. As K is compact, we can choose a G-invariant metric $< .,. >$ on $D = G/K$. Let $X \in \mathfrak{g}$. Let $\omega(X)$ be the 1-form on D given by $\omega(X)(\xi) =< X_D, \xi >$ for any vector field ξ on D. We denote by ω the G-equivariant form on D given by $X \mapsto \omega(X)$. We have

$$\frac{d}{dt}(e^{td_{\mathfrak{g}}\omega}) = d_{\mathfrak{g}}(\omega e^{td_{\mathfrak{g}}\omega}).$$

Integrating this relation from $t = 0$ to T, we obtain

$$e^{Td_{\mathfrak{g}}\omega} - 1 = d_{\mathfrak{g}}(\int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt).$$

Let $n = \dim G/K$. Let us show that, when $T \to \infty$, the form $e^{Td_{\mathfrak{g}}\omega}$ tends to $Ind_{G/K,o}((-2\pi)^{n/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})$ in $\mathcal{A}_G^{-\infty}(D)$ while $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega} dt$ also has a limit in the space $\mathcal{A}_G^{-\infty}(D)$.

By G-invariance, it is sufficient to compute at the base point $e \in G/K$. Choose a K-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{r}$ (as in the proof of Theorem 52). The elements $e^{td_{\mathfrak{g}}\omega}$ and $\omega e^{td_{\mathfrak{g}}\omega}$ evaluated at the base point e are elements of $C^\infty(\mathfrak{g}) \otimes \Lambda\mathfrak{r}'$. The G-invariant Riemannian metric on D gives us a K-invariant scalar product $< .,. >$ on $\mathfrak{r}$. We choose an oriented orthonormal basis E^a of $\mathfrak{r}$ with dual basis $E_a \in \mathfrak{r}'$. Let us write any $X \in \mathfrak{g}$ as $X = X_0 + X_1$ with $X_0 \in \mathfrak{k}$ and $X_1 \in \mathfrak{r}$. We write $X_1 \in \mathfrak{r}$ as $X_1 = \sum_a x_a(X) E^a$. The function $X \mapsto x_a(X)$ is a linear function on $\mathfrak{g}$. Let E_I be the homogeneous basis of $\Lambda\mathfrak{r}'$ indexed by subsets I of $\{1, 2, ..., n\}$. In particular, $E_{\{1,2,...,n\}} = E_1 \wedge \cdots \wedge E_n$. Then $\omega_e \in C^\infty(\mathfrak{g}) \otimes \Lambda\mathfrak{r}'$ is given by the formula $\omega_e = -\sum_a x_a \otimes E_a$. We compute $d_{\mathfrak{g}}\omega$ using Lemma 42 of Section 5. For any $X \in \mathfrak{g}$, we have $j_{\mathfrak{r}}(\omega_e)(X) = \|X_1\|^2$ and $c(\omega_e)(X) = \kappa(X)$ where $\kappa(X)$ is the element of $\Lambda^2\mathfrak{r}'$ given by

$$\kappa(X)(v, v') =< [v, X]_1, v' > - < [v', X]_1, v > + < X_1, [v, v'] > .$$

In particular, if $X_0 \in \mathfrak{k}$, $\kappa(X_0)(v, v') = -2 < (ad_{\mathfrak{r}} X_0) \cdot v, v' >$. We write $f(t) = (e^{td_{\mathfrak{g}}\omega})_e$. We have

$$f(t, X_0, X_1) = e^{-t\|X_1\|^2} \sum_I P_I(X_0, X_1) t^{|I|/2} E_I$$

where P_I is a homogeneous polynomial of degree $|I|/2$ on $\mathfrak{g}$. It is easy to see that $P_{\{1,2,...,n\}}(X_0) = (-2)^{n/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(X_0)$. Let us show that $f(t, X_0, X_1)$ has a limit when $t \to \infty$, as a generalised function on $\mathfrak{g}$ with values in $\Lambda\mathfrak{r}'$. For a test function ϕ on $\mathfrak{g}$

$$\int_{\mathfrak{g}} f(t, X)\phi(X) dX = \sum_I \left(\int_{\mathfrak{g}} e^{-t\|X_1\|^2} t^{|I|/2} P_I(X_0, X_1)\phi(X_0, X_1) dX_0 dX_1 \right) E_I.$$

The change of variable $X_1 \mapsto t^{-1/2}X_1$ shows that this is also equal to:

$$\sum_I t^{|I|/2-n/2}\left(\int_{\mathfrak{g}} e^{-\|X_1\|^2} P_I(X_0, t^{-1/2}X_1)\phi(X_0, t^{-1/2}X_1)dX_0dX_1\right) E_I.$$

Thus the limit when $t \to \infty$ of $\int_{\mathfrak{g}} f(t,X)\phi(X)dX \in \Lambda\mathfrak{r}'$ exists. Furthermore, when $t \to \infty$, the coefficient of E_I tends to zero except when $I = \{1,2,...n\}$, and the coefficient of $E_{\{1,2,...,n\}}$ tends to $\int_{\mathfrak{g}} e^{-\|X_1\|^2} P_{\{1,2,...,n\}}(X_0,0)\phi(X_0,0)dX_0dX_1$. Computing the constants, we obtain that the form $e^{td_{\mathfrak{g}}\omega}$ tends to

$$Ind_{G/K,o}((-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})$$

when $t \to \infty$.

It remains to show that $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega}dt$ also has a limit in the space $\mathcal{A}_G^{-\infty}(D)$.

Remark that $\omega(X)$ is homogeneous of degree 1 in the variable X_1. Let $g(t) = (\omega e^{td_{\mathfrak{g}}\omega})_e$. We have

$$g(t,X) = \sum_{a,I}\left(e^{-t\|X_1\|^2} t^{|I|/2} x_a P_I(X_0,X_1)\right) E_a \wedge E_I.$$

Here all the subsets I occuring in this sum are such that $|I| \leq n-2$. Now

$$\int_{\mathfrak{g}} g(t,X)\phi(X)dX = \sum_{a,I}\left(\int_{\mathfrak{g}} e^{-t\|X_1\|^2} t^{|I|/2} x_a P_I(X_0,X_1)\phi(X_0,X_1)dX_0dX_1\right) E_a \wedge E_I.$$

The same change of variable (as earlier) shows that this is equal to

$$\sum_{a,I} t^{|I|/2-1/2-n/2}\left(\int_{\mathfrak{g}} e^{-\|X_1\|^2} x_a P_I(X_0, t^{-1/2}X_1)\phi(X_0, t^{-1/2}X_1)dX_0dX_1\right) E_a \wedge E_I.$$

In particular, as $|I| \leq (n-2)$, this function of t is uniformly bounded when $t \to \infty$ by $O(t^{-3/2})$. Thus the function $t \mapsto \int_{\mathfrak{g}} g(t,X)\phi(X)dX$ is integrable on $[0,\infty]$. This shows that $\alpha(T) = \int_0^T \omega e^{td_{\mathfrak{g}}\omega}dt$ has a limit in $\mathcal{A}_G^{-\infty}(G/K)$ when $T \to \infty$. This concludes the proof. ∎

We now turn to the question of the explicit determination of the inverse of the map $Ind_{G/K,o}$.

Definition 54 *Define the Chern-Weil map*

$$w_{G/K} : C^\infty(\mathfrak{k})^K \to H_G^\infty(G/K)$$

as the inverse of the map $r_{\mathfrak{k}}$.

Define the map

$$S_o : H_G^{-\infty}(G/K) \to C^{-\infty}(\mathfrak{k})^\chi$$

as the inverse of the map $Ind_{G/K,o}$.

We denote $w_{G/K}$ simply by w if G and K are understood. An explicit formula for $w_{G/K}$ can be given in terms of the curvature of the principal bundle $G \to G/K$, see [21]. On the other hand we are not able to give an explicit expression for S_o in general. We are able to do it only under some restrictive assumptions or on some particular subspaces. Lemma 38 of section 4 implies that, if $\alpha \in H_G^{-\infty}(G/K)$ and $p \in C^\infty(\mathfrak{k})^K$,

$$S_o(w(p)\alpha) = pS_o(\alpha). \tag{19}$$

It follows from proposition 53 that $S_o\alpha = (-2\pi)^{\dim(G/K)/2}(det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})\, r_{\mathfrak{k}}\alpha$ for $\alpha \in H_G^\infty(G/K)$.

We can improve this result as follows. Let $\alpha \in \mathcal{A}_G^{-\infty}(G/K)$. Then $\alpha_e \in C^{-\infty}(\mathfrak{g}) \otimes \Lambda\mathfrak{r}'$. We say that α_e admits a restriction to $\mathfrak{k}$ if each component of α_e admits a restriction to $\mathfrak{k}$ (see [12]). We can then define $r_{\mathfrak{k}}\alpha = (\alpha_e)_{[0]}|_{\mathfrak{k}}$. Then $r_{\mathfrak{k}}\alpha \in C^{-\infty}(\mathfrak{k})^K$ is a generalized function on $\mathfrak{k}$. This definition extends the definition of $r_{\mathfrak{k}} : \mathcal{A}_G^\infty(G/K) \to C^\infty(\mathfrak{k})^K$.

Proposition 55 *Let $\alpha \in \mathcal{A}_G^{-\infty}(G/K)$ be a closed equivariant differential form on G/K. Assume that α_e admits a restriction to $\mathfrak{k}$. Then*

$$S_o\alpha = (-2\pi)^{\dim(G/K)/2}(det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})\, r_{\mathfrak{k}}\alpha.$$

Proof: Using notation of the proof of Proposition 53, we have $\alpha \cong \alpha e^{td_{\mathfrak{g}}\omega}$. Then using the same arguments as in the proof of Proposition 53, we obtain the desired result. Indeed, if f is a generalized function on $\mathfrak{g}$ admitting a restriction $r_{\mathfrak{k}}f$ to $\mathfrak{k}$, then for any test function ϕ on $\mathfrak{g}$, the limit when $t \to \infty$ of

$$t^{|I|/2} \int_{\mathfrak{g}} e^{-t\|X_1\|^2} f(X_0, X_1)\phi(X_0, X_1) dX_0 dX_1$$

is 0 if $|I| < n$ while for $|I| = n$ this is $(\pi)^{n/2} \int_{\mathfrak{k}} (r_{\mathfrak{k}}f)(X_0)\phi(X_0)dX_0$. ∎

Remark 56 *Proposition 31 of [12] becomes then a consequence of Proposition 44 and Proposition 55 above.*

Let $q^* : C^{-\infty}(\mathfrak{g})^G \to H_G^{-\infty}(G/K)$ be induced from the map $q : G/K \to$ *point*. Thus we get a map (still denoted by) S_o

$$S_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^\chi$$

taking f to $S_o(q^*f)$. The map S_o exists under the only assumption that K is compact, and extends the map $f \mapsto (-2\pi)^{\dim(G/K)/2}(det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})\, r_{\mathfrak{k}}f$, defined on generalised functions f admitting a restriction $r_{\mathfrak{k}}f$ to $\mathfrak{k}$. On the open set where $\det_{\mathfrak{g}/\mathfrak{k}} Y \neq 0$, the G-orbits are transverse to $\mathfrak{k}$. Thus the restriction to $\mathfrak{k}$ of an

invariant generalized function on $\mathfrak{g}$ has a meaning on this open set. By the same proof as Proposition 36, we see that $S_o(f)$ coincides with

$$(-2\pi)^{\dim(G/K)/2}(det^{1/2}_{\mathfrak{g}/\mathfrak{k},o})\, r_{\mathfrak{k}} f$$

on this open set, for any $f \in C^{-\infty}(\mathfrak{g})^G$. However, we are able to compute the map $S_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^\chi$ on arbitrary invariant generalised functions and on the full space $\mathfrak{k}$ explicitly only when G itself is compact.

Let G be a compact connected Lie group. We also assume that $\chi = 1$ so that G/K is orientable (this is only for convenience). We choose the orientation on G/K given by o.

Definition 57 *If $\Phi \in C^\infty(\mathfrak{k})^K$, define $C_o(\Phi) \in C^\infty(\mathfrak{g})^G$ by*

$$C_o(\Phi)(X) = \int_{G/K,o} (w\Phi)(X)$$

where w is the Chern-Weil homomorphism.

Define

$$F_o : C^{-\infty}(\mathfrak{g})^G \to C^{-\infty}(\mathfrak{k})^K$$

as the transpose of the map C_o:

$$vol(G/K, dg/dk) \int_{\mathfrak{k}} F_o(f)(Y)\Phi(Y)dY = \int_{\mathfrak{g}} f(X) C_o(\Phi)(X)dX$$

for any $\Phi \in C^\infty_{cpt}(\mathfrak{k})^K$, and where the measures dX on $\mathfrak{g}$, dY on $\mathfrak{k}$ and dg/dk on G/K are chosen in a compatible way.

It is easy to see that C_o sends invariant compactly supported functions on $\mathfrak{k}$ to invariant compactly supported functions on $\mathfrak{g}$, hence the map F_o is well defined.

Lemma 58 *If $f \in C^\infty(\mathfrak{g})^G$, then $F_o(f)(Y) = (-2\pi)^{\dim(G/K)/2} det^{1/2}_{\mathfrak{g}/\mathfrak{k},o}(Y) f(Y)$, for $Y \in \mathfrak{k}$.*

Proof: The integral formula ([12], page 43) for equivariant cohomology classes gives the lemma. ∎

Proposition 59 *Assume that G is a compact connected Lie group and K a closed subgroup of G such that G/K is oriented. Then for every $f \in C^{-\infty}(\mathfrak{g})^G$ and $p \in C^\infty(\mathfrak{k})^K$,*

$$fw(p) \sim Ind_{G/K,o}(F_o(f)p)$$

in $H^{-\infty}_G(G/K)$.

Proof: Using Formula 19, it is sufficient to prove the formula of this proposition when $p = 1$.

Let Φ be a G-invariant test function on $\mathfrak{g}$ and let $p' \in C^\infty(\mathfrak{k})^K$. Let us compute

$$\int_{\mathfrak{g}} \int_{G/K,o} \Phi(X) f(X) w(p')(X) dX = \int_{\mathfrak{g}} \Phi(X) f(X) C_o(p')(X) dX.$$

If $q^* f \cong Ind_{G/K,o} u$, for $u \in C^{-\infty}(\mathfrak{k})^K$, then $fw(p') \cong Ind_{G/K,o}(p'u)$ by Lemma 38 of section 4. Thus, using Proposition 44, we have:

$$\begin{aligned} \int_{\mathfrak{g}} \int_{G/K,o} \Phi(X) f(X) w(p')(X) dX &= \int_{\mathfrak{g}} \int_{G/K,o} \Phi(X) (Ind_{G/K,o}(p'u))(X) dX \\ &= \int_{G/K,o} (\int_{\mathfrak{g}} \Phi(X) Ind_{G/K,o}(p'u)(X) dX) \\ &= \int_{G/K,o} (\int_{\mathfrak{k}} \Phi(gY) p'(Y) u(Y) dY) dg/dk \end{aligned}$$

Let p' be compactly supported. Taking Φ with sufficiently large support, we obtain

$$\int_{\mathfrak{g}} f(X) C_o(p')(X) dX = vol(G/K, dg/dk) \int_{\mathfrak{k}} p'(Y) u(Y) dY$$

which is what we needed to prove. ∎

The preceding proposition determines the inverse S_o of the map $Ind_{G/K,o}$ on the subspace of $H_G^{-\infty}(G/K)$ spanned by elements of the form $f\alpha$ where $f \in C^{-\infty}(\mathfrak{g})^G$ and $\alpha \in H_G^\infty(G/K)$. We will see in section 6 that this space is equal to $H_G^{-\infty}(G/K)$ provided that G and K have equal rank.

We compute even more explicitly the map F_o when G is a compact connected Lie group and $K = T$ is a maximal torus of G.

Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{t})$. Let $C^{\pm\infty}(\mathfrak{t})^\epsilon$ be the space of W-anti-invariant smooth functions (resp. anti-invariant generalized functions) on $\mathfrak{t}$. Recall the definitions of the maps C_o and F_o from Definition 57.

Lemma 60 *The restriction of the map C_o to $C^\infty(\mathfrak{t})^\epsilon$ gives an isomorphism between $C^\infty(\mathfrak{t})^\epsilon$ and $C^\infty(\mathfrak{g})^G$.*

Furthermore the image of F_o is contained in $C^{-\infty}(\mathfrak{t})^\epsilon$ and the map F_o gives an isomorphism between $C^{-\infty}(\mathfrak{g})^G$ and $C^{-\infty}(\mathfrak{t})^\epsilon$.

Proof: If $\phi \in C^\infty(\mathfrak{t})^\epsilon$, then ϕ is divisible by $\det_{\mathfrak{g}/\mathfrak{t},o}(Y)^{1/2}$ and the restriction of $C_o\phi$ to $\mathfrak{t}$ is equal to $|W|(-2\pi)^{n/2} \det_{\mathfrak{g}/\mathfrak{t},o}(Y)^{-1/2}\phi(Y)$, where $n := \dim(G/T)$. Thus the first assertion follows from Chevalley's theorem for C^∞-functions, see for example [12]. The second is a consequence of the first, as C_o preserves the subspace of compactly supported functions. ∎

Let us describe $F_o f$, when f is the δ-function: Let $\delta_{\mathfrak{g}}(X)$ (resp. $\delta_{\mathfrak{t}}(Y)$) be the δ function on $\mathfrak{g}$ (resp. on $\mathfrak{t}$), given with respect to the Euclidean measure on $\mathfrak{g}$ (resp. $\mathfrak{t}$) associated to the Killing form. Let $\alpha \in i\mathfrak{t}'$ be a root . Using the identification of $\mathfrak{t}$ with $\mathfrak{t}'$ determined by the Killing form, we can consider the differential operator $\prod_{\alpha>0} \partial_{i\alpha}$ on $\mathfrak{t}$, where the product is taken over all the positive roots of $\mathfrak{g}$ for an order compatible with the orientation o as defined in [12]. Then $F_o \delta_{\mathfrak{g}} = \prod_{\alpha>0} \partial_{i\alpha} \delta_{\mathfrak{t}}$.

6 Künneth formula and applications

Let K be a Lie group. Let D and M be K-manifolds. Consider the complex $\mathcal{A}_K(D)$ of K-equivariant forms on D with polynomial coefficients and its cohomology $H_K(D) = Z_K(D)/B_K(D)$ defined in section 2, Definition 1. Recall that these spaces are $\mathbb{Z}_+$-graded. The evaluation map E at zero taking $\alpha \mapsto \alpha(0)$ gives a map from $H_K(D)$ to the usual De Rham cohomology $H(D)$ of D. Consider the map m from $\mathcal{A}_K(D) \otimes \mathcal{A}_K(M)$ to $\mathcal{A}_K(D \times M)$ given by $m(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$. It induces a map (still denoted by m) from $H_K(D) \otimes H_K(M)$ to $H_K(D \times M)$.

Similarly, we can also consider the map $m^{-\infty}$ from $\mathcal{A}_K(D) \otimes \mathcal{A}_K^{-\infty}(M)$ to $\mathcal{A}_K^{-\infty}(D \times M)$ given by $m^{-\infty}(\alpha \otimes \beta)(X) = \alpha(X) \wedge \beta(X)$: this is well defined as we can multiply a generalized function by a polynomial function. It induces a map (still denoted by $m^{-\infty}$) from $H_K(D) \otimes H_K^{-\infty}(M)$ to $H_K^{-\infty}(D \times M)$.

Theorem 61 *Let K be a compact Lie group. Let D be a compact K-manifold. Assume that the evaluation map $E : H_K(D) \to H(D)$ is surjective. Then, for any K-manifold M, the multiplication map $m^{-\infty}$ induces an isomorphism*

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_K^{-\infty}(D \times M).$$

Remark 62 *Proposition 5 implies that $H_K(D)$ is free over $H_K(point)$. As is well known, from the Künneth spectral sequence (see [16]; Proposition 6.1, page 50), the map m induces an isomorphism*

$$\hat{m} : H_K(D) \otimes_{H_K(point)} H_K(M) \cong H_K(D \times M)$$

under the hypothesis of the theorem. (This also follows from the same argument as that for $\hat{m}^{-\infty}$ given below.)

Proof: We can assume D to be oriented. Indeed let us consider the two-fold cover D_t of D defined in section 2 before Proposition 5. An element of D_t is a couple (m, o), where $m \in D$ and o is an orientation of $T_m D$. Thus the manifold D_t is canonically oriented. Consider the map $\epsilon(m, o) = (m, -o)$ and

its action on $H_K(D_t)$. We have $H_K(D_t) = H_K(D) \oplus H_K(D)_t$, where $H_K(D)$ is isomorphic to the eigenspace with eigenvalue 1 for the action of ϵ, while $H_K(D)_t$ is by definition the eigenspace of eigenvalue -1 for the action of ϵ. Furthermore by Proposition 5, the manifold D_t also satisfies the hypothesis that the evaluation map $H_K(D_t) \to H(D_t)$ is surjective. The map

$$\hat{m}^{-\infty} : H_K(D_t) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D_t \times M)$$

clearly commutes with the action of ϵ. So if we show that $\hat{m}^{-\infty}$ is an isomorphism for the oriented manifold D_t, by taking the eigenspace of eigenvalue 1 for ϵ we will obtain the desired isomorphism

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \cong H_K^{-\infty}(D \times M).$$

Let $\mathcal{A}^{(p)}(D \times M) = \Gamma(D \times M, \Lambda^p T'D \otimes \Lambda T'M)$, where Γ denotes the space of smooth sections and $T'M$ denotes the cotangent bundle of M.

We write $\mathcal{A}(D \times M) = \oplus_{p=0}^{\dim D} \mathcal{A}^{(p)}(D \times M)$. The total exterior differential $d_{D\times M}$ on $\mathcal{A}(D \times M)$ breaks up into the sum $d_D + d_M$ of partial exterior differential d_D along D

$$d_D : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p+1)}(D \times M)$$

and partial exterior differential d_M along M

$$d_M : \mathcal{A}^{(p)}(D \times M) \to \mathcal{A}^{(p)}(D \times M).$$

Let us consider the complex $\mathcal{A}_K^{-\infty}(D \times M) = C^{-\infty}(\mathfrak{k}, \mathcal{A}(D \times M))^K$. We write

$$B^p = C^{-\infty}(\mathfrak{k}, \mathcal{A}^{(p)}(D \times M))^K.$$

The operator $d_{\mathfrak{k}}$ can be written as a sum of homogeneous operators $d_{\mathfrak{k}} = d_1 + r_0 + r_{-1}$, with

$$d_1 : B^p \to B^{p+1}, r_0 : B^p \to B^p, r_{-1} : B^p \to B^{p-1}.$$

We have $d_1 = d_D$, $r_0 = d_M - \sum_i x_i \iota(E_M^i)$, and $r_{-1} = -\sum_i x_i \iota(E_D^i)$, where E^i is a basis of $\mathfrak{k}$. We write $d = d_D$.

Let us choose a K-invariant metric on D and consider D as a Riemannian manifold. This endows the space $\mathcal{A}(D)$ with an inner product. Let $d^* : \mathcal{A}^p(D) \to \mathcal{A}^{p-1}(D)$ be the adjoint operator to $d = d_D$. Let $\mathcal{H}(D) = Ker(d) \cap Ker(d^*)$ be the space of harmonic forms on D. It is a K-invariant finite dimensional space of d-closed forms on D. The map $\mathcal{H}(D) \to H(D)$ is an isomorphism. Recall that we do not necessarily assume K to be connected. However as the evaluation map at 0 is surjective, K acts trivially on $H(D)$ (this would be automatic if K were connected). Thus every element of $\mathcal{H}(D)$ is K-invariant.

Lemma 63 *For any $\alpha_0 \in \mathcal{H}^p(D)$, there exists $\alpha \in Z_K(D)$ such that*

$$\alpha_0 - \alpha(X) \in \sum_{j<p} \mathcal{A}^j(D), \qquad \text{for every } X \in \mathfrak{k}.$$

Proof: Our hypothesis implies that if $\alpha_0 \in \mathcal{A}^p(D)$ is d-closed, we can find $\gamma \in \mathcal{A}^{p-1}(D)$ and $\lambda \in Z_K(D)$ such that $\alpha_0 - d\gamma = \lambda(0)$. If α_0 is K-invariant, we may assume (eventually after averaging this equation by the action of K) that γ is K-invariant. Take $\alpha = \lambda + d_{\mathfrak{k}}\gamma$, then $\alpha \in Z_K(D)$ and $\alpha(0) = \alpha_0$. The complex $\mathcal{A}_K(D)$ is $\mathbb{Z}_+$-graded by its total equivariant degree. We may thus assume that α is of total degree p. Thus $\alpha(X) - \alpha(0) \in \mathcal{A}^{p-2}(D) \oplus \mathcal{A}^{p-4}(D) \oplus \cdots$. This proves the lemma. ∎

We continue with the proof of Theorem 61.

Let P be the orthogonal projection of $\mathcal{A}(D)$ onto $\mathcal{H}(D)$. We have $Pd = dP = 0$. Let $G : \mathcal{A}^p(D) \to \mathcal{A}^{p-1}(D)$ be the Green kernel. It satisfies $Gd + dG = I - P$. We can extend the operator P, by the formula $P(\alpha)(X) = P(\alpha(X))$, to an operator still denoted by P,

$$P : B^p \to \mathcal{H}^p(D) \otimes \mathcal{A}_{\mathfrak{k}}^{-\infty}(M).$$

Similarly we can extend pointwise the operator G

$$G : B^p \to B^{p-1}.$$

Let $r = r_0 + r_{-1}$ and let $N = Gr + rG$. The operator N decreases strictly the exterior degree in D . Let $\nu \in B = \sum_j B^j$. The equation $Gd + dG = I - P$ gives the perturbed equation

$$Gd_{\mathfrak{k}}\nu + d_{\mathfrak{k}}G\nu = \nu - (P - N)\nu.$$

Assume $d_{\mathfrak{k}}\nu = 0$. Let us write $\nu = \sum_{j \le p} \nu_j$, with $\nu_j \in B^j$. We will prove by induction on p that ν has a representative in $Z_K(D) \otimes Z_K^{-\infty}(M)$. The equation above implies that $\nu \cong \nu' := (P - N)\nu$. We have $\nu'_p = P\nu_p \in \mathcal{H}^p(D) \otimes \mathcal{A}_K^{-\infty}(M)$. Let us write the equation $d_{\mathfrak{k}}\nu' = 0$ component by component. We obtain, in particular, the equation $r_0\nu'_p + d\nu'_{p-1} = 0$. Applying P, we get $Pr_0\nu'_p + Pd\nu'_{p-1} = 0$. As $Pd = 0$, this implies that $Pr_0\nu'_p = r_0 P\nu'_p = r_0\nu'_p = 0$. Thus $\nu'_p \in \mathcal{H}^p(D) \otimes Z_K^{-\infty}(M)$. The preceding lemma allows us to find an element $\xi \in Z_K(D) \otimes Z_K^{-\infty}(M)$ such that $\nu' - \xi \in \sum_{j<p} B^j$. Thus the map $m^{-\infty}$ is surjective.

Let us prove that $\hat{m}^{-\infty}$ is an isomorphism. It suffices to show that $\hat{m}^{-\infty}$ is injective: Recall from Proposition 5 that $H_K(D)$ is free over $H_K(point)$. Furthermore we can find a basis P^p for the $H_K(point)$-module $H_K(D)$ and a basis P_q for the $H_K(point)$-module $H_K(D)_t$ such that $\int_M P^p P_q = \delta_p^q$. We can

write an element $\alpha \in H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M)$ uniquely as $\alpha = \sum_a P^a \otimes \nu_a$. Let $\nu \in H_K^{-\infty}(D \times M)$. For any fixed a, consider the map $\nu \mapsto \int_D P_a(X)\nu(X)$. This is a well defined map from $H_K^{-\infty}(D \times M) \to H_K^{-\infty}(M)$. We have

$$\int_D P_a(X)(\hat{m}^{-\infty}\alpha)(X) = \nu_a(X).$$

Thus if $\hat{m}^{-\infty}(\alpha) = 0$, $\nu_a(X) = 0$, for all a, and hence $\hat{m}^{-\infty}$ is injective ∎

Applying Theorem 61 to the case where M is a point, we obtain the following

Corollary 64 *Let K be a compact Lie group. Let D be a compact K-manifold . Assume that $H_K(D)$ surjects on $H_K(point)$. Then*

$$H_K^{-\infty}(D) \cong C^{-\infty}(\mathfrak{k})^K \otimes_{S(\mathfrak{k}')^K} H_K(D).$$

Recall from [14] that if K is a compact connected Lie group acting on a compact symplectic manifold D in an Hamiltonian way, then $H_K(D)$ is free over $H_K(point)$. In particular, the following well-known lemma gives some examples of compact K-manifolds D such that $H_K(D)$ is free over $H_K(point) = S(\mathfrak{k}')^K$.

Lemma 65 *Let K be a compact connected Lie group and let L be a closed subgroup of K. Then $H_K(K/L)$ is free over $H_K(point)$ if and only if K and L have the same rank.*

Proof: Let $D = K/L$. Recall that the equivariant cohomology $H_K(D)$ is isomorphic to $H_L(point) = S(\mathfrak{l}')^L$ by the map $r_\mathfrak{l}$, induced from the inclusion of the base point $e \in K/L$. The restriction of polynomial functions on $\mathfrak{k}$ to polynomial functions on $\mathfrak{l}$ gives a homomorphism from $S(\mathfrak{k}')^K$ to $S(\mathfrak{l}')^L$. If $H_K(K/L)$ is free over $H_K(point)$, then in particular the homomorphism $S(\mathfrak{k}')^K \to S(\mathfrak{l}')^L$ is injective. This implies that K and L have the same rank. Conversely, suppose now that L is a closed subgroup of K with equal rank. Let T be a maximal torus of L (and hence of K. Let $W = N_K(T)/T, W_L = N_L(T)/T$ be the respective Weyl groups of (K,T) and (L,T). By Chevalley's theorem, $S(\mathfrak{l}')^L$ is isomorphic to $S(\mathfrak{t}')^{W_L}$, while the ground ring $S(\mathfrak{k}')^K$ is isomorphic to $S(\mathfrak{t}')^W$. Further $S(\mathfrak{t}')^{W_L}$ is a free module over $H_K(point) \cong S(\mathfrak{t}')^W$. ∎
The case where $D = K/T$ is a particularly important example. Considering this case we will see that we obtain as a consequence of Theorem 61 the following:

Proposition 66 *Let K be a compact connected Lie group with maximal torus T and Weyl group W. Let $C^{-\infty}(\mathfrak{t})^\epsilon$ be the space of W-anti-invariant generalized functions on $\mathfrak{t}$. Then the map from*

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \to C^{-\infty}(\mathfrak{t})$$

given by $P \otimes f \mapsto Pf$, for $P \in S(\mathfrak{t}'), f \in C^{-\infty}(\mathfrak{t})^\epsilon$, is an isomorphism.

Proof: Let us consider $D = K/T$ and $M = point$ in Theorem 61. We thus obtain an isomorphism

$$\hat{m}^{-\infty} : H_K(K/T) \otimes_{H_K(point)} C^{-\infty}(\mathfrak{k})^K \to H_K^{-\infty}(K/T). \tag{20}$$

We have $H_K(point) \cong S(\mathfrak{t}')^W$. Recall the isomorphisms

$$w_{K/T} : S(\mathfrak{t}') \to H_K(K/T) \tag{21}$$

where $w_{K/T}$ is the Chern-Weil homomorphism (see Definition 54 of section 5), the isomorphism F_o proved in Lemma 60

$$F_o : C^{-\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{t})^\epsilon, \tag{22}$$

and the isomorphism (cf. Theorem 46)

$$Ind_{K/T,o} : C^{-\infty}(\mathfrak{t}) \to H_K^{-\infty}(K/T). \tag{23}$$

Using the above identifications, the isomorphism (20) gives an isomorphism, again denoted by

$$\hat{m}^{-\infty} : S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \to C^{-\infty}(\mathfrak{t}). \tag{24}$$

By Proposition 59 of section 5, we have for $a \in S(\mathfrak{t}'), f \in C^{-\infty}(\mathfrak{k})^K$

$$w_{K/T}(a)f = Ind_{G/K,o}(aF_o f).$$

Hence we obtain our proposition. ∎

7 Equivariant cohomology and subgroups

Let K be a compact connected Lie group and let M be a K-manifold. If L is a compact subgroup of K of equal rank, then $H_K(K/L)$ is free over $H_K(point)$. In this section, we will use Theorem 61 to compare $H_K^{-\infty}(M)$ and $H_L^{-\infty}(M)$ (and also $H_K(M)$ and $H_L(M)$).

Let $D = K/L$. The space $H_K(D)$ is isomorphic to $H_L(point) = S(\mathfrak{l}')^L$ by the Chern-Weil isomorphism:

$$w_D : H_L(point) = S(\mathfrak{l}')^L \to H_K(D). \tag{25}$$

Consider the natural restriction map $r_L : H_K(M) \to H_L(M)$ of a K-equivariant form on M to a L-equivariant form on M.

Let us recall the following proposition (see [16]; page 38). We include a proof for completeness.

Proposition 67 *Let K be a compact connected Lie group and let M be a K-manifold. Let L be a compact subgroup of K of equal rank. Then the map*

$$I_M : H_L(point) \otimes_{H_K(point)} H_K(M) \to H_L(M)$$

given by

$$P \otimes \omega \mapsto P(r_L\omega)$$

for $P \in H_L(point)$ and $\omega \in H_K(M)$ is an isomorphism.

In particular, $H_L(M)$ is generated by $r_L H_K(M)$ over $H_L(point) = S(\mathfrak{l}')^L$.

Proof: Consider the manifold $\mathcal{M} = K \times_L M$. By Theorem (51), the restriction map induced from the inclusion $i_{L,M}$ of the fiber M at $e \in D$ in the fibered space $\pi : \mathcal{M} \to D$ induces an isomorphism

$$i^*_{L,M} : H_K(\mathcal{M}) \to H_L(M) \tag{26}$$

Consider the map $\mu : [k, m] \mapsto km$ from $K \times_L M$ to M. The map $t = (\pi \times \mu)$ given by $t([k, m]) = (kL, k \cdot m)$ is a K-equivariant isomorphism from $\mathcal{M}$ to $D \times M$, where K acts on $D \times M$ as the diagonal action. Thus we have an isomorphism

$$t^* : H_K(D \times M) \to H_K(\mathcal{M}). \tag{27}$$

As follows from Theorem 61 and Lemma 65, the multiplication map $\hat{m}$ is an isomorphism:

$$\hat{m} : H_K(D) \otimes_{H_K(point)} H_K(M) \to H_K(D \times M). \tag{28}$$

Using the isomorphisms (25) -(28), we obtain an isomorphism

$$I : H_L(point) \otimes_{H_K(point)} H_K(M) \to H_L(M). \tag{29}$$

It remains to show that I is equal to I_M. For this, we have to compute for $P \in S(\mathfrak{l}')^L$ and $\omega \in H_K(M)$

$$\begin{aligned} i^*_{L,M}(\pi^*(w_D P) \wedge \mu^*\omega) &= i^*_{L,M}\pi^*(w_D P) \wedge i^*_{L,M}\mu^*\omega \\ &= P \wedge r_L\omega. \end{aligned}$$

This proves the proposition. ∎

In particular, consider $M = K/U$ for a closed subgroup U of K. Then the equivariant cohomology space $H_K(M)$ is equal to $H_U(point)$ via the Chern-Weil homomorphism w_M and we obtain :

Proposition 68 *Let K be a compact connected Lie group. Let L be a closed subgroup of K of same rank. Let U be a closed subgroup of K. Then the map $P \otimes A \mapsto P(r_L(w_M A))$ for $P \in S(\mathfrak{l}')^L$, $A \in S(\mathfrak{u}')^U$ determines an isomorphism:*

$$S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} S(\mathfrak{u}')^U \cong H_L(K/U) \tag{30}$$

Remark that by our hypothesis, $H_K(K/L) = S(\mathfrak{l}')^L$ is free over $S(\mathfrak{k}')^K$ with rank equal to $\dim H(K/L)$. Thus as a vector space

$$H_L(K/U) \cong H(K/L) \otimes_{\mathbb{R}} S(\mathfrak{u}')^U.$$

Corollary 69 *If U also has the same rank as K, then $H_L(K/U)$ is a free (finitely generated) $H_L(point)$-module.*

Proof: As $S(\mathfrak{u}')^U$ is free over $S(\mathfrak{k}')^K$, the $S(\mathfrak{l}')^L$ -module $S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} S(\mathfrak{u}')^U$ is free with rank equal to $\dim H(K/U)$. ∎

We turn now to the determination of the equivariant cohomology $H_L^{-\infty}(M)$ in the case where M is a K-manifold. We denote by $\chi_{K/L}$ the character of L with values in ± 1 given by $\gamma \to \det_{\mathfrak{k}/\mathfrak{l}} \gamma$. Recall the definition of $H_{L,\chi_{K/L}}^{-\infty}(M)$ from Definition 49 of section 5.

Theorem 70 *Let K be a compact connected Lie group and let M be a K-manifold. Let L be a compact subgroup of K of equal rank. Choose an orientation o on $\mathfrak{k}/\mathfrak{l}$. Then, there is a natural isomorphism of $H_L(point)$-modules:*

$$I_{M,o}^{-\infty} : H_L(point) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_{L,\chi_{K/L}}^{-\infty}(M).$$

In particular, for $M = K/U$, we obtain

$$H_{L,\chi_{K/L}}^{-\infty}(M) \cong S(\mathfrak{l}')^L \otimes_{S(\mathfrak{k}')^K} C^{-\infty}(\mathfrak{u})^{\chi_{K/U}}.$$

Proof: The proof is almost the same as the proof of Proposition 67. Using the same notation, we use the chain of isomorphisms:

$$Ind_{K/L,o} : H_{L,\chi_{K/L}}^{-\infty}(M) \to H_K^{-\infty}(\mathcal{M}). \tag{31}$$

The isomorphism t of $\mathcal{M}$ with $D \times M$ induces an isomorphism

$$t^* : H_K^{-\infty}(D \times M) \to H_K^{-\infty}(\mathcal{M}). \tag{32}$$

As follows from Theorem 61 and Lemma 65, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D \times M). \tag{33}$$

Using the isomorphisms (25), (31)-(33), we obtain an isomorphism

$$I^{-\infty}_{M,o} : H_L(point) \otimes_{H_K(point)} H^{-\infty}_K(M) \to H^{-\infty}_{L,\chi_{K/L}}(M). \tag{34}$$

The description of $I^{-\infty}_{M,o}$ is as follows: for $P \in H_L(point) = S(\mathfrak{l}')^L$ and $\omega \in H^{-\infty}_K(M)$, the element $\alpha = I^{-\infty}_{M,o}(P \otimes \omega)$ is the unique element in $H^{-\infty}_{L,\chi_{K/L}}(M)$ such that $Ind_{K/L,o}\alpha = \pi^*(w_D P) \wedge \mu^*\omega$. It follows from Lemma 38 of section 4 that

$$\pi^*(w_D Q) Ind_{K/L,o}\alpha = Ind_{K/L,o} Q\alpha$$

for any $Q \in H_L(point)$. Hence $I^{-\infty}_{M,o}$ is an isomorphism of $H_L(point)$-modules. The last statement in Theorem 70 follows easily from Theorem 46. ∎

The isomorphism $I^{-\infty}_M$ is not so easy to determine explicitly as the isomorphism I_M. We determine it as much as we can.

Consider $H^{-\infty}_K(M)$ as a $H_K(M)$-module and $H^{-\infty}_{L,\chi_{K/L}}(M)$ as a $H_L(M)$-module.

Lemma 71 *If $\alpha \in H_K(M)$ and $\beta \in H^{-\infty}_K(M)$, then*

$$I^{-\infty}_{M,o}(1 \otimes \alpha\beta) = I_M(1 \otimes \alpha) I^{-\infty}_{M,o}(1 \otimes \beta).$$

Proof: If $I^{-\infty}_M(1 \otimes \beta) = \omega$, we have $\mu^*\beta = Ind_{K/L,o}\omega$. Then $\mu^*\alpha \wedge \mu^*\beta = Ind_{K/L,o}(r_L\alpha)\omega$ by Lemma 38 of section 4. This proves the lemma. ∎

We assume that K/L is orientable so that $\chi_{K/L} = 1$. Recall the map $F_o : C^{-\infty}(\mathfrak{k})^K \to C^{-\infty}(\mathfrak{l})^L$ given in Definition 57 of section 5. Then

Lemma 72 *Let $f \in C^{-\infty}(\mathfrak{k})^K$, $\alpha \in H_K(M)$, then $I^{-\infty}_{M,o}(1 \otimes f\alpha) = (F_o f) r_L \alpha$.*

Proof: This follows immediately from Proposition 59 of section 5. ∎

Thus we know $I^{-\infty}_M$ on the subspace $S(\mathfrak{l}')^L \otimes_{H_K(point)} C^{-\infty}(\mathfrak{k})^K H_K(M)$ of $S(\mathfrak{l}')^L \otimes_{H_K(point)} H^{-\infty}_K(M)$.

If M is compact and $H_K(M)$ is a free module over $H_K(point)$, then $H^{-\infty}_K(M)$ is equal to $C^{-\infty}(\mathfrak{k})^K \otimes_{H_K(point)} H_K(M)$ (cf. Corollary 64 of section 6). Thus in this case, the isomorphism $I^{-\infty}_{M,o}$ is entirely determined by the knowledge of F.

8 Reduction to the maximal torus

Let K be a compact connected Lie group and let T be its maximal torus. Let $W = N(T)/T$ be the Weyl group.

Let M be a K-manifold. It is well known (cf. [16]; chapter 3, section 1, Proposition 1) that the natural restriction map $\mathcal{A}_K(M) \to \mathcal{A}_T(M)$ induces an isomorphism between $H_K(M)$ and $H_T(M)^W$. In this section, we prove a similar statement for the generalized K-cohomology $H^{-\infty}_K(M)$.

We first need to define a map from $H_T^{-\infty}(M)$ to $H_K^{-\infty}(M)$: Choose a non-zero K-invariant form ν' on K/T of maximal exterior degree. In particular, ν' determines an Euclidean measure $|d\nu'|$ on $\mathfrak{k}/\mathfrak{t}$ and an orientation o. For $f \in C^{-\infty}(\mathfrak{t}, \mathcal{A}(M))$, define $A_o(f) \in C^{-\infty}(\mathfrak{k}, \mathcal{A}(K \times M))$ by: If Φ is a test function on $\mathfrak{k}$

$$(A_o(f), \Phi dX)_{(k,m)} := \nu' \wedge (k \cdot (\int_{\mathfrak{t}} f(Y)\Phi(k \cdot Y)dY))_m$$

where dY is the Euclidean measure on $\mathfrak{t}$ which is quotient of dX by $|d\nu'|$ on $\mathfrak{k}/\mathfrak{t}$. In particular A_o depends only on the orientation o on K/T associated to ν'.

Lemma 73 *Let $D = K/T$. Consider $D \times M$ as a K-manifold under the diagonal action. Then the map A_o defines a cochain map from $(\mathcal{A}_T^{-\infty}(M), d_{\mathfrak{t}})$ to $(\mathcal{A}_K^{-\infty}(D \times M), d_{\mathfrak{k}})$.*

Proof: It is not difficult to check that if f is T -invariant, then $A_o(f)$ is in $\mathcal{A}_K^{-\infty}(D \times M)$. Now, as ν' is of maximal degree, we see that $d_{D\times M}(A_o(f)) = A_o(d_M f)$. It is also easy to prove that $\iota_{\mathfrak{k}} A_o(f) = A_o(\iota_{\mathfrak{t}} f)$. ∎

Consider the projection $\pi : D \times M \to M$ with fiber D. We denote the map $\pi_* : \mathcal{A}_K^{-\infty}(D \times M) \to \mathcal{A}_K^{-\infty}(M)$ by $\int_D$ (cf. Formula 8 of section 2).

Define $B(f) \in \mathcal{A}_K^{-\infty}(M)$ by

$$B(f) = \int_{D,o} A_o(f),$$

where the orientation on D is the orientation o. In particular, we see that B does not depend on o. If we denote by dk/dt the positive density on K/T associated to ν', we have for $f \in \mathcal{A}_T^{-\infty}(M)$ and Φ a test function on $\mathfrak{k}$,

$$\int_{\mathfrak{k}} B(f)(X)\Phi(X)dX = \int_{K/T} k \cdot (\int_{\mathfrak{t}} f(Y)\Phi(k \cdot Y)dY)dk/dt.$$

The map B is a cochain map from $(\mathcal{A}_T^{-\infty}(M), d_{\mathfrak{t}})$ to $(\mathcal{A}_K^{-\infty}(M), d_{\mathfrak{k}})$. The Weyl group W canonically acts on $\mathcal{A}_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$.

Theorem 74 *Let K be a compact connected Lie group and let T be its maximal torus. Let W be the Weyl group of K.*

The restriction of the cochain map B to $\mathcal{A}_T^{-\infty}(M)^W$ induces an isomorphism in cohomology

$$b : H_T^{-\infty}(M)^W \to H_K^{-\infty}(M).$$

Proof: Again, this theorem is an easy consequence of Theorem 61 of section 6. As in the proof of Theorem 70, we consider $\mathcal{M} = K \times_T M$ and we use the isomorphism (cf. Theorem 52 of section 5):

$$Ind_{K/T,o} : H_T^{-\infty}(M) \to H_K^{-\infty}(\mathcal{M}). \tag{35}$$

Composing this with the isomorphism (see Formula 32 of section 7):

$$t^* : H_K^{-\infty}(D \times M) \to H_K^{-\infty}(\mathcal{M}), \tag{36}$$

we obtain an isomorphism

$$(t^*)^{-1} \circ \mathrm{ind}_{K/T,o} : H_T^{-\infty}(M) \to H_K^{-\infty}(D \times M). \tag{37}$$

It is not difficult to see that $(t^*)^{-1} Ind_{K/T,o} f = A_o f$, for $f \in H_T^{-\infty}(M)$,

Let ϵ be the character of W given by $\epsilon(w) = det_{\mathfrak{t}}(w)$. Let r be the action of W on $K/T \times M$ given by $w \cdot (kT, m) = (kw^{-1}T, m)$. This action commutes with the diagonal action of K and hence induces an action still denoted by r on $H_K^{-\infty}(D \times M)$. Under the isomorphism (37), the natural action of W on $H_T^{-\infty}(M)$ becomes the action $r \otimes \epsilon$.

The K-equivariant cohomology $H_K(D) \cong S(\mathfrak{t}')$ of D is free over $H_K(point) \cong S(\mathfrak{t}')^W$. Hence, as follows from Theorem 61, the multiplication map $\hat{m}^{-\infty}$ is an isomorphism:

$$\hat{m}^{-\infty} : H_K(D) \otimes_{H_K(point)} H_K^{-\infty}(M) \to H_K^{-\infty}(D \times M). \tag{38}$$

For the action of the group W on $H_K(D)$, induced by the action of W by right translation on $D = K/T$, the subspace $H_K(D)^\epsilon$ of $H_K(D)$ is a free $H_K(point)$ -module of rank one, in fact $H_K(D)^\epsilon = H_K(point) w_D(\chi)$, where $w_D(\chi) \in H_K(D)^\epsilon$ is the image under the Chern-Weil homomorphism w_D of the W-anti-invariant polynomial function

$$\chi(Y) = (2\pi)^{-\dim(D)/2} |W|^{-1} \det{}^{1/2}_{\mathfrak{k}/\mathfrak{t},o}(Y), Y \in \mathfrak{t}.$$

We have $\int_D w_D(\chi)(X) = 1$, for all $X \in \mathfrak{k}$. The space $H_K(D)^\epsilon$ is isomorphic to $H_K(point)$ under $\alpha \mapsto \int_D \alpha(X)$. Thus, by (38), the map $\alpha \mapsto \int_D \alpha$ induces an isomorphism (depending on the choice of an orientation on D)

$$H_K^{-\infty}(D \times M)^\epsilon \cong H_K^{-\infty}(M).$$

and hence, by (37),

$$H_T^{-\infty}(M)^W \cong H_K^{-\infty}(M).$$

The above isomorphism is given by the restriction to $H_T^{-\infty}(M)^W$ of the map $B = \int_{D,o} A_o$. Thus we obtain the formula of the theorem. ∎

In particular, when $M = point$, the isomorphism given by Theorem 74 is the well known isomorphism $b : C^{-\infty}(\mathfrak{t})^W \to C^{-\infty}(\mathfrak{k})^K$ given by:

$$(b(f), \Phi dX) = vol(K/T, dk/dt)(\int_{\mathfrak{t}} f(Y)\Phi(Y)dY)$$

if Φ is a K-invariant test function on $\mathfrak{k}$.

9 The case of a free action

Let G be a Lie group. Let P be a right G-manifold (i.e. G acts on P from the right). (Of course any left G-manifold can be thought of as a right G-manifold under $x \cdot g := g^{-1} \cdot x$, for $x \in P$ and $g \in G$.)

Definition 75 *Let P be a right G-manifold. We will say that the* action of G on P is principal *(or that G acts* principally *on P) if the orbit space P/G is a smooth manifold such that $P \to P/G$ is a smooth principal G-bundle.*

If G is compact, then a right action is principal if and only if the action is free.

Let P be a right G-manifold. If G acts principally on P, it is known [11] that the G- equivariant de Rham cohomology of P is isomorphic to the de Rham cohomology of the quotient space P/G. In this section, we prove similarly that the space $H_G^{-\infty}(P)$ is isomorphic to $H(P/G)$. We also consider the following more general situation:

(S):*Let G be a Lie group and let N be a closed normal subgroup of G. Let P be a right G-manifold. Assume that the subgroup N acts principally on P.*

We ask the question: Under what hypothesis are $H_G^{-\infty}(P)$ and $H_{G/N}^{-\infty}(P/N)$ isomorphic. In this section, we prove this affirmatively when G is compact connected. On the other hand when $N = G$, we need no compactness hypothesis on G to prove the isomorphism $H_G^{-\infty}(P) \cong H(P/G)$ and the proof for this case is comparatively easy. The reader only interested in the case where $N = G$ can go directly to the proof of Theorem 89.

An important example of this situation (S) is the following:

Example 76 *Let U and K be two Lie groups and let $G := U \times K$ be the direct product. Let L be a $U \times K$- manifold. For convenience, we assume that U acts on the left and K on the right. Assume the right action of K on L is principal. Let M be a K-manifold and let $P := L \times M$. Define the action of an element $g = (u, k) \in G = U \times K$ on P by :$(x, m) \cdot (u, k) = (u^{-1}xk, k^{-1}m)$, for $x \in L$, $m \in M$. Then the action of the (normal) subgroup K of G is principal on P. The quotient manifold $(L \times M)/K$ is the left U-space $\mathcal{M} = L \times_K M$ fibered over L/K with fiber M.*

Consider the quotient map $q : P \to P/N$ under the situation (S). Recall Definition 4 of a G-equivariant fibration with G-oriented fibers. Later in the section, we will need to impose the following conditions (77) and (78)

Condition 77 *There exists a G-orientation o for the fibers of q.*

This condition (77) is satisfied, for example, when $\det {}_n g > 0$, for all $g \in G$. In particular this is satisfied if G is connected.

Condition 78 *There exists a G-invariant connection ω for the principal N-bundle $q : P \to P/N$.*

This condition (78) is always satisfied when G is compact.

It is proved in [13] that the canonical map $q^* : H^\infty_{G/N}(P/N) \to H^\infty_G(P)$ is an isomorphism, when the condition (78) is satisfied. Furthermore, an explicit formula for the inverse of q^* is given in terms of the equivariant curvature of ω. The reader should however be warned that the natural map q^* is sometimes equal to zero when applied to the equivariant cohomology with generalized coefficients.

Whenever the conditions (77) and (78) are satisfied, we will construct a natural map (cf. Proposition 82)

$$m_o : H^{-\infty}_{G/N}(P/N) \to H^{-\infty}_G(P)$$

and will show that m_o is an isomorphism, if either $N = G$ or G is compact.

We begin by constructing a natural element $\gamma_o \in H^{-\infty}_G(P)$ (assuming the validity of conditions (77) and (78)):

Let $B := P/N$ be the space of N-orbits. Consider the projection $q : P \to B$. The vertical tangent bundle V is a G-equivariant real vector bundle over P. By assumption, the bundle V is a G-orientable vector bundle. As the group N acts principally, the bundle V is a trivial bundle over P canonically isomorphic to $P \times \mathfrak{n}$. The isomorphism is obtained by sending $(x, X) \in P \times \mathfrak{n}$ to the vertical tangent vector $(X_P)_x$. The action of an element $g \in G$ on $V = P \times \mathfrak{n}$ is given by $(x, Y) \cdot g = (xg, g^{-1} \cdot Y)$ for $x \in P, Y \in \mathfrak{n}$. (Observe that if $\det_{\mathfrak{n}}(g) > 0$, for all $g \in G$, then any choice of orientation of $\mathfrak{n}$ gives rise to a G-orientation of the vector bundle V, i.e., in this case the condition (77) is satisfied.)

Let us choose a G-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$. Using ω, we obtain a G-invariant decomposition

$$TP = V \oplus H$$

of the tangent bundle as sum of vertical and horizontal subbundles.

Similarly, using ω, we have an isomorphism

$$U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n} \tag{39}$$

given by $U(x, X) = (x, Y, Q)$, where $Q \in \mathfrak{g}/\mathfrak{n}$ is the projection of $X \in \mathfrak{g}$ and where $Y = \omega_x(X_P) \in \mathfrak{n}$.

Consider the dual bundle $V' = P \times \mathfrak{n}'$ to the vertical tangent bundle V. The projection $TP \to V$, given by the connection ω, determines a G-invariant injection s_ω of V' in the cotangent bundle $T'P$. Consider the canonical 1-form

α on the manifold $T'P$. Let $\alpha_\omega := s_\omega^* \alpha \in \mathcal{A}^1(V')$. It is a G-invariant differential form on V'. The form

$$\beta_\omega := e^{id_{\mathfrak{g}}\alpha_\omega} \in \mathcal{A}_G^\infty(V') \otimes_{\mathbb{R}} \mathbb{C}$$

is a closed G-equivariant differential form. Consider the projection $p : V' \to P$. We will prove below that if Φ is a test function on $\mathfrak{g}$, then $\int_{\mathfrak{g}} \beta_\omega(X)\Phi(X)dX$ is a differential form on $V' = P \times \mathfrak{n}'$ rapidly decreasing in the direction $\mathfrak{n}'$. Thus, as the vector bundle V' is G-oriented, we may define $p_*\beta_\omega$ as an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P)) \otimes_{\mathbb{R}} \mathbb{C}$ by setting:

$$\int_{\mathfrak{g}} (p_*\beta_\omega)(X)\Phi(X)dX = p_*\Big(\int_{\mathfrak{g}} \beta_\omega(X)\Phi(X)dX\Big)$$

(the map p_* depends on the choice of o).

A representative of the element γ_o will be defined as the integral of β_ω over the fibers of p, normalized in order that $\gamma_o(X)$ is a differential form on P with real coefficients.

Proposition 79 *Let $p : V' \to P$ be the projection as above. Let us choose a G-orientation o on the vector bundle V'. Let $n = \dim \mathfrak{n}$. Let $c_n = 1$, if n is even, and $c_n = -i$ if n is odd. Define*

$$\gamma_{\omega,o} := c_n(2\pi)^{-n} p_*(e^{id_{\mathfrak{g}}\alpha_\omega}).$$

Then $\gamma_{\omega,o}$ is an element of $\mathcal{A}_G^{-\infty}(P)$ and is $d_{\mathfrak{g}}$-closed.

The cohomology class of $\gamma_{\omega,o}$ in $H_G^{-\infty}(P)$ is independent of the choice of the G-invariant connection ω. It depends only on the G-orientation o. We denote it by γ_o.

Proof: Writing $\beta_\omega = e^{id_{\mathfrak{g}}\alpha_\omega}$, we compute $p_*\beta_\omega$.

Let E^j be a basis of $\mathfrak{n}$ with dual basis E_j. We write an element of $\mathfrak{n}'$ as $y = \sum_j y^j E_j$. Let

$$\omega = \sum_j \omega_j E^j \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$$

be the connection form. By definition $(\omega_k, E_P^j) = \delta_k^j$ and ω of course by definition vanishes on the horizontal vectors. Under the identification $V' \cong P \times \mathfrak{n}'$, the 1-form

$$\alpha_\omega = \sum_j y^j \omega_j.$$

Define, as in ([3], chapter 7), the moment $\mu \in \mathfrak{g}' \otimes C^\infty(P) \otimes \mathfrak{n}$ of the connection ω by setting, for any $X \in \mathfrak{g}$

$$\mu(X) = -\omega(X_P).$$

Thus $\mu(X)_x$ is an element of $\mathfrak{n}$ and $\mu(X+Y)_x = \mu(X)_x - Y$ for all $Y \in \mathfrak{n}$ and $x \in P$. Let us compute

$$d_X\alpha_\omega := (d_{\mathfrak{g}}\alpha_\omega)(X).$$

We obtain

$$d_X\alpha_\omega = (y, \mu(X)) + \sum_j (dy^j\omega_j + y^j d\omega_j).$$

We have $e^{id_X\alpha_\omega} = e^{i(y,\mu(X))}A(y, dy, \omega, d\omega)$, where A is a polynomial expression in $y^j, dy^j, \omega_j, d\omega_j$. If Φ is a test function on $\mathfrak{g}$, the integral

$$\int_{\mathfrak{g}} e^{i(y,\mu(X))}\Phi(X)dX$$

is a function on V' rapidly decreasing over the fiber $\mathfrak{n}'$. This can be seen as follows: Consider the isomorphism (39) of $P \times \mathfrak{g}$ with $P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$.

For $x \in P$, let

$$\mathfrak{q}_x := \{X \in \mathfrak{g}; \omega_x(X_P) = 0\}.$$

Thus $\mathfrak{q}_x$ is isomorphic to $\mathfrak{g}/\mathfrak{n}$ under the natural projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ and

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_x. \tag{40}$$

We fix $x \in P$, and write $\mathfrak{q}_x = \mathfrak{q}$. Let $X \in \mathfrak{g}$. Using the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$, we write $X = Y + Q$, with $Y \in \mathfrak{n}, Q \in \mathfrak{q}$. We have $(y, \mu(X)) = -(y, Y)$. Writing $\Phi(X)dX = \Phi(Y, Q)dQdY$, we get

$$\int_{\mathfrak{g}} e^{i(y,\mu(X))}\Phi(X)dX = \int_{\mathfrak{n}} e^{-i(y,Y)}\Psi(Y)dY,$$

where $\Psi(Y) = \int_{\mathfrak{q}} \Phi(Y, Q)dQ$. Clearly $\Psi(Y)$ is a C^∞-function with compact support on $\mathfrak{n}$. As Fourier transform of test functions are rapidly decreasing, it follows that

$$\int_{\mathfrak{g}} e^{id_X\alpha_\omega}\Phi(X)dX = A(y, dy, \omega, d\omega)\int_{\mathfrak{n}} e^{-i(y,Y)}\Psi(Y)dY$$

is a form on V' rapidly decreasing over the fiber $\mathfrak{n}'$ of the projection $V' \to P$.

This proves that $p_*\beta_\omega$ exists, as an element of $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P)) \otimes_{\mathbb{R}} \mathbb{C}$. It is clearly G-invariant, so that $p_*\beta_\omega \in \mathcal{A}_G^{-\infty}(P) \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore $p_*\beta_\omega$ is $d_{\mathfrak{g}}$-closed:

$$(d_{\mathfrak{g}}p_*\beta_\omega, \Phi dX) = p_*(d_{\mathfrak{g}}\beta_\omega, \Phi dX) = 0.$$

If ω_t is a one-parameter smooth family of G-invariant connections, we denote $\alpha_{\omega_t}, \beta_{\omega_t}$ by α_t, β_t respectively. We have

$$\frac{d}{dt}\beta_t = \frac{d}{dt}(e^{id_{\mathfrak{g}}\alpha_t}) = id_{\mathfrak{g}}((\frac{d}{dt}\alpha_t) \wedge \beta_t).$$

The integral of $(\frac{d}{dt}\alpha_t \wedge \beta_t)$ over the fiber $\mathfrak{n}'$ exists in the sense of generalized functions and

$$\frac{d}{dt}p_*\beta_t = id_{\mathfrak{g}}p_*(\frac{d}{dt}\alpha_t \wedge \beta_t). \tag{41}$$

Thus the cohomology class of $p_*\beta_t$ is independent of the choice of the connection ω_t.

We now compute explicitly the element $\gamma_{\omega,o}$ defined in Proposition 79 and show, in particular, that $\gamma_{\omega,o}(X)$ is a differential form with real coefficients.

Let us fix $x \in P$. The orientation o on the vector bundle V gives rise to an orientation o_x on $\mathfrak{n}$ (which may depend on the connected component of $x \in P$). Let E^j be an ordered basis of $\mathfrak{n}$. We will say that this order is o-compatible, if this basis is of orientation o_x. The exterior product $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ of the components ω_i of the connection ω is a vertical form on P of maximum dimension.

Let $\nu' \in \Lambda^n\mathfrak{n}'$ be such that $(\nu', E^1 \wedge \cdots \wedge E^n) = 1$. The element ν' also determines an Euclidean measure dY on $\mathfrak{n}$ and a δ- function $|\nu'|^{-1}\delta_{\mathfrak{n}}(Y) \in C^{-\infty}(\mathfrak{n})$ (cf Section 2, Formula 10).

We can also write

$$|\nu'|^{-1}\delta_{\mathfrak{n}}(Y) = (2\pi)^{-\dim \mathfrak{n}} \int_{\mathfrak{n}'} e^{i(y,Y)}dy,$$

where dy is the measure on $\mathfrak{n}'$ dual to the Euclidean measure dY on $\mathfrak{n}$. Let $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ be the curvature of the connection ω. Thus $\Omega \in (\mathcal{A}^2(P) \otimes \mathfrak{n})^G$. We write

$$\Omega = \sum_j \Omega_j E^j.$$

Define the equivariant curvature of ω (as in [3], chap 7) by

$$\Omega(X) = \mu(X) + \Omega.$$

To simplify notation, we will use ν' to identify generalized functions and distributions and write $\delta_{\mathfrak{n}}$ instead of $|\nu'|^{-1}\delta_{\mathfrak{n}}$.

Let us show that the generalized function $\delta_{\mathfrak{n}}(\Omega(X)) \in C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ is well defined. We describe, at each point $x \in P$, $\delta_{\mathfrak{n}}(\Omega(X))_x$ as a generalized function on $\mathfrak{g}$ with values in the vector space $\Lambda T'_x P$. In the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_x$ given by formula 40, we write $X = Y + Q$. Then $\Omega(Y + Q)_x = -Y + \Omega_x$ and we define $\delta_{\mathfrak{n}}(\Omega(X))_x$ by its Taylor expansion:

$$\delta_{\mathfrak{n}}(\Omega(X))_x = \delta_{\mathfrak{n}}(-Y) + \sum_j \Omega_j(\partial_{E^j}\delta_{\mathfrak{n}})(-Y) + \cdots$$

Proposition 80 *Let $x \in P$ and let E^j be an ordered basis of $\mathfrak{n}$, with an order compatible with the orientation o_x. Then*

$$\gamma_{\omega,o}(X)_x = |\nu'|^{-1}\delta_{\mathfrak{n}}(\Omega(X))_x \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x.$$

Proof: The highest degree component of $e^{id\alpha_\omega} = e^{i(dy,\omega)+i(y,d\omega)}$ in dy_j's is equal to

$$c_n^{-1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y,d\omega)}.$$

The curvature Ω is equal to $d\omega$ modulo terms in ω_j. Thus

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n e^{i(y,d\omega)} = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \wedge e^{i(y,\Omega)}$$

and

$$\begin{aligned} p_*\beta_\omega(X)_x &= c_n^{-1}\Big(\int_{\mathfrak{n}'} e^{i(y,\mu(X)+\Omega)}dy\Big) \wedge (\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\ &= c_n^{-1}(2\pi)^n|\nu'|^{-1}\delta_{\mathfrak{n}}(\Omega(X))_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x. \end{aligned}$$

This proves the proposition. ∎
In particular, we see that $\gamma_{\omega,o}$ indeed belongs to $\mathcal{A}_G^{-\infty}(P)$ and we obtain Proposition 79. ∎

Remark 81 *It is easy to check directly from the formula of the above proposition that $\gamma_{\omega,o}$ is $d_{\mathfrak{g}}$ closed.*

Fix $x \in P$ and consider the decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}$ given by (40). The space $\mathfrak{q}$ is isomorphic to $\mathfrak{g}/\mathfrak{n}$. The generalized function $\gamma_{\omega,o}(X)_x$ iz constant in the direction $\mathfrak{q}$: $\gamma_{\omega,o}(Y+Q)_x = \gamma_{\omega,o}(Y)_x$. Thus, if $f(Q)$ is a generalized function on $\mathfrak{g}/\mathfrak{n}$, we can multiply $\gamma_{\omega,o}(Y)_x$ by $f(Q)$ and we obtain a generalized function on $\mathfrak{g}$ with values in $\Lambda T'_x P$.

Define the map

$$m_{\omega,o} : \mathcal{A}_{G/N}^{-\infty}(P/N) \to \mathcal{A}_G^{-\infty}(P)$$

by setting

$$m_{\omega,o}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega,o},$$

for $\alpha \in \mathcal{A}_{G/N}^{-\infty}(P/N)$.

The above discussion shows that $m_{\omega,o}$ is well defined. It is a cochain map of differential complexes

$$m_{\omega,o} : (\mathcal{A}_{G/N}^{-\infty}(P/N), d_{\mathfrak{g}/\mathfrak{n}}) \to (\mathcal{A}_G^{-\infty}(P), d_{\mathfrak{g}}).$$

Proposition 82 *The induced map in cohomology*

$$H^{-\infty}_{G/N}(P/N) \to H^{-\infty}_{G}(P),$$

from the cochain map

$$m_{\omega,o}(\alpha) = q^*(\alpha) \wedge \gamma_{\omega,o},$$

does not depend upon the choice of the G-invariant connection ω on the principal N-bundle $q : P \to P/N$. We will denote it by m_o.

Proof: This follows easily from Formula 41 on the variation of $p_*\beta_\omega$. ∎

We describe now the properties of $m_{\omega,o}$ in relation to the $\mathcal{A}^\infty_G(P)$-module structure on $\mathcal{A}^{-\infty}_G(P)$.

If $\alpha \in \mathcal{A}^\infty_G(P)$, we can define $\beta \in \mathcal{A}^\infty_G(P)$, by setting

$$\beta(X) := \alpha(X + \Omega(X)).$$

Explicitly, for $x \in P$, $Y \in \mathfrak{n}$, $Q \in \mathfrak{q}$, $X = Y + Q$, then $X + \Omega(X) = Y + Q + \Omega_x - Y = Q + \Omega_x$, thus $\beta(X)_x = \alpha(Q + \Omega)_x$ is defined by its Taylor expansion

$$\alpha(\Omega + Q)_x = \alpha(Q)_x + \sum_j \Omega_j (\partial_{E^j}\alpha)(Q)_x + \cdots$$

Thus $\beta(X + Y)_x = \beta(X)_x$, for $X \in \mathfrak{g}, Y \in \mathfrak{n}$. Hence $\beta \in C^\infty(\mathfrak{g}/\mathfrak{n}, \mathcal{A}(P))$.

Let $\Gamma \subset \mathcal{A}(P)$ be the subspace of horizontal forms. The group G acts on Γ. The connection ω defines a horizontal projector $h : \mathcal{A}(P) \to \Gamma$ which commutes with the action of G. Define, for $\alpha \in \mathcal{A}^\infty_G(P)$,

$$W_\omega(\alpha)(X) := h(\alpha(X + \Omega(X))).$$

As $W_\omega\alpha = h(\beta)$, $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma)$. The G-invariance implies that $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma)^G$. As N is a normal subgroup of G, N acts trivially on $\mathfrak{g}/\mathfrak{n}$, in particular $W_\omega(\alpha) \in C^\infty(\mathfrak{g}/\mathfrak{n}, \Gamma^N)$. The space Γ^N is the space of forms on $B := P/N$, and we think of $W_\omega(\alpha)$ as an element of $C^\infty(\mathfrak{g}/\mathfrak{n}, \mathcal{A}(B))$. The G-invariance implies that $W_\omega(\alpha) \in \mathcal{A}^\infty_{G/N}(P/N)$. Thus we have obtained a map

$$W_\omega : \mathcal{A}^\infty_G(P) \to \mathcal{A}^\infty_{G/N}(P/N).$$

Remark 83 . *The map W_ω is a generalization of the Chern-Weil map: If $N = G$, let $\phi \in C^\infty(\mathfrak{g})^G$ and consider $\phi(X)1 \in \mathcal{A}^\infty_G(P)$. Then $W_\omega(\phi(X)1) = \phi(\Omega)$ is the characteristic form on P/G associated to ϕ by the classical Chern-Weil homomorphism.*

Proposition 84 *If $\alpha \in \mathcal{A}_G^\infty(P)$, then*

$$\alpha \wedge \gamma_{\omega,o} = q^*(W_\omega \alpha) \wedge \gamma_{\omega,o} = m_{\omega,o}(W_\omega(\alpha)).$$

In particular, for $\beta \in \mathcal{A}_{G/N}^{-\infty}(P/N)$,

$$\alpha \wedge m_{\omega,o}(\beta) = m_{\omega,o}(W_\omega \alpha \wedge \beta).$$

Proof: The proof follows easily from our formula given in Proposition 80 for $\gamma_{\omega,o}$. We have, for $x \in P$, $X = Y + Q$, $Y \in \mathfrak{n}, Q \in \mathfrak{q}$

$$\begin{aligned} \alpha(X)_x \wedge \gamma_{\omega,o}(X)_x &= \alpha(Y+Q)_x \delta_{\mathfrak{n}}(\Omega - Y)_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\ &= \alpha(\Omega + Q)_x \delta_{\mathfrak{n}}(\Omega - Y)_x(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n)_x \\ &= \alpha(X + \Omega(X))_x \wedge \gamma_{\omega,o}(X)_x. \end{aligned}$$

As $\gamma_{\omega,o}$ is already of top degree in the vertical directions, we see that

$$\alpha(X + \Omega(X))_x \wedge \gamma_{\omega,o}(X)_x = h(\alpha(X + \Omega(X)))_x \wedge \gamma_{\omega,o}(X)_x$$

which is the formula we want. ∎

The following proposition is proved in [13]. We give another proof, which is an easy application of Proposition 84.

Proposition 85 *Let G be a Lie group and let $N \subset G$ be a closed normal subgroup of G. Let P be a right G-manifold such that the action of N is principal. Assume that the principal N-bundle $q : P \to P/N$ admits a G-invariant connection ω. Then the map*

$$W_\omega : \mathcal{A}_G^\infty(P) \to \mathcal{A}_{G/N}^\infty(P/N)$$

defined above is a cochain map:

$$W_\omega d_{\mathfrak{g}} = d_{\mathfrak{g}/\mathfrak{n}} W_\omega.$$

Furthermore, if $\beta \in \mathcal{A}_{G/N}^\infty(P/N)$ then

$$W_\omega q^* \beta = \beta.$$

Proof: The last equation follows from the definition of W_ω, as $\alpha(Y+Q)_x = (q^*\beta)(Q)$ is independent of the variable $Y \in \mathfrak{n}$, thus $\alpha(\Omega + Q)_x = \alpha_x(Q)$ and is horizontal.

Let us prove that W_ω commutes with differentials. Let $\alpha \in \mathcal{A}_G^\infty(P)$ and write γ instead of $\gamma_{\omega,o}$ and W instead of W_ω. Let us compute $d_{\mathfrak{g}}(\alpha \wedge \gamma)$:

As γ is $d_{\mathfrak{g}}$-closed, we obtain from Proposition 84,

$$d_{\mathfrak{g}}(\alpha \wedge \gamma) = d_{\mathfrak{g}}\alpha \wedge \gamma = q^*(W(d_{\mathfrak{g}}\alpha)) \wedge \gamma.$$

We also have

$$\begin{aligned} d_{\mathfrak{g}}(\alpha \wedge \gamma) &= d_{\mathfrak{g}}(q^*W(\alpha) \wedge \gamma) \\ &= d_{\mathfrak{g}}(q^*W(\alpha)) \wedge \gamma \\ &= q^*(d_{\mathfrak{g}/\mathfrak{n}}W(\alpha)) \wedge \gamma. \end{aligned}$$

Thus

$$m_{\omega,o}(W(d_{\mathfrak{g}}\alpha)) = m_{\omega,o}(d_{\mathfrak{g}/\mathfrak{n}}W(\alpha)).$$

But the map $m_{\omega,o}$ is easily seen to be injective and hence we obtain

$$W(d_{\mathfrak{g}}\alpha) = d_{\mathfrak{g}/\mathfrak{n}}W(\alpha).$$

∎

We also have ([13])

Theorem 86 *Let the notation and assumptions be as in the above proposition (85). Then the cochain map*

$$q^* : \mathcal{A}^{\infty}_{G/N}(P/N) \to \mathcal{A}^{\infty}_{G}(P)$$

induces an isomorphism in cohomology.

As $W_\omega q^* = I$, where I is the identity operator, the map W_ω provides an explicit inverse for q^* in cohomology.

Let us consider example 76 for the following special case: The manifold L is equal to the Lie group U and K is a closed subgroup of U. The manifold $L = U$ is a $U \times K$ manifold, where the action of $(u, k) \in U \times K$ on $x \in L$ is given by $x \cdot (u, k) = u^{-1}xk$. Let $G = U \times K$ and $P = U \times M$, for a K-manifold M. The action of both of the (normal) subgroups U and K of G on P are principal.

Consider first the action of U. The space P/U is our K-manifold M we started with. Consider the canonical Maurer-Cartan connection $\omega_U \in (\mathcal{A}^1(U) \otimes \mathfrak{u})^U$ defined by $\omega_U(X_U) = X$ for every $X \in \mathfrak{u}$. Here X_U is the vector field associated to the action of U on L by left translation, i.e $(X_U)_x$ is the tangent vector to the curve $\exp(-\epsilon X)x$. We have $\omega_U(Y_L)_x = -x \cdot Y$, for $x \in L$, $Y \in \mathfrak{k}$, as K acts by right translations on L. The connection ω_U extends trivially to a $(U \times K)$-invariant connection $\omega_U \otimes 1$ for the principal U-bundle $q_U : U \times M \to M$.

As before, trivialize the vertical bundle $V_U \cong P \times \mathfrak{u}$ (for q_U) by the map defined by sending the vertical tangent vector $(X_{U\times M})_{x,m}$ to (x, m, X), for $x \in$

$U, m \in M, X \in \mathfrak{u}$. Choose an orientation $o_{\mathfrak{u}}$ on $\mathfrak{u}$. We obtain a U-invariant orientation o_U on V_U, by setting $(o_U)_{x,m} = sign(\det_{\mathfrak{u}} x) o_{\mathfrak{u}}$. This orientation is $U \times K$ invariant, if and only if $\det_{\mathfrak{u}} k > 0$, for every $k \in K$. Thus, under this condition, we get a map

$$m_{o_U} : H_K^{-\infty}(M) \to H_{U\times K}^{-\infty}(U \times M).$$

Consider now the action of K. The quotient space $U \times M$ by the action of K is the induced space

$$\mathcal{M} = U \times_K M$$

with left action of U, that we considered in section 5. Assume there exists a U-invariant connection $\omega \in (\mathcal{A}^1(U)^U \otimes \mathfrak{k})^K$ for the principal K-bundle $U \to U/K$. Then the form $\omega \otimes 1$ on $U \times M$ is a U-invariant connection form for the principal K-bundle $q_K : U \times M \to U \times_K M$. Assume that $\det_{\mathfrak{k}} k > 0$ for every $k \in K$. Choose an orientation $o_{\mathfrak{k}}$ on $\mathfrak{k}$, then the fibration q_K has a unique $U \times K$-invariant orientation o_K given by $o_{e,m} = o_{\mathfrak{k}}$ for each point $m \in M$, where e is the identity of U. Thus, under these conditions, there exists a map

$$m_{o_K} : H_U^{-\infty}(U \times_K M) \to H_{U\times K}^{-\infty}(U \times M).$$

Given orientations $o_{\mathfrak{u}}, o_{\mathfrak{k}}$ of $\mathfrak{u}, \mathfrak{k}$ respectively, they determine an orientation o on $\mathfrak{u}/\mathfrak{k}$. We have $\det_{\mathfrak{u}/\mathfrak{k}} k > 0$ for all $k \in K$, as both numbers $\det_{\mathfrak{u}} k$ and $\det_{\mathfrak{k}} k$ are > 0, by assumption. Recall the map

$$Ind_{U/K,o} : H_K^{-\infty}(M) \to H_U^{-\infty}(U \times_K M)$$

from Section 5, Proposition 50

Lemma 87 *Let U be a Lie group and let K be a closed subgroup of U, such that the principal K-bundle $U \to U/K$ admits a U-invariant connection. Assume $\det_{\mathfrak{k}} k > 0$, $\det_{\mathfrak{u}} k > 0$ for all $k \in K$. Let $o_{\mathfrak{u}}, o_{\mathfrak{k}}, o$ be compatible orientations on $\mathfrak{u}, \mathfrak{k}, \mathfrak{u}/\mathfrak{k}$, then*

$$m_{o_U} = m_{o_K} Ind_{U/K,o}.$$

Proof: First, we explicitly compute the map m_{o_U}. Consider the canonical connection ω_U. Its curvature is 0. The equivariant curvature of ω_U is given, for the identity element e of U, $m \in M$, $X \in \mathfrak{u}, Y \in \mathfrak{k}$, by

$$\Omega_U(X, Y)_{(e,m)} = -X + Y.$$

Let $\ell = \dim U$. Let $\nu'_U \in \Lambda^\ell \mathfrak{u}'$ be a positive element (with respect to the orientation $o_{\mathfrak{u}}$). Let dx be the unique left U-invariant form on U, such that

$(dx)_e = \nu'$. Then ν'_U determines a δ-function $|\nu'_U|^{-1}\delta_{\mathfrak{u}}$ on $\mathfrak{u}$, and for $X \in \mathfrak{u}$, $Y \in \mathfrak{k}$,

$$\gamma_{\omega_U,o_U}(X,Y)_{(e,m)} = |\nu'_U|^{-1}\delta_{\mathfrak{u}}(Y-X)(dx)_e.$$

Thus, for $\alpha \in \mathcal{A}_K^{-\infty}(M)$,

$$(m_{\omega_U,o_U}\alpha)(X,Y)_{(e,m)} = |\nu'_U|^{-1}\delta_{\mathfrak{u}}(Y-X)(\alpha(Y)\wedge dx)_{(e,m)}.$$

More explicitly, for Φ_1 a test function on $\mathfrak{u}$, Φ_2 a test function on $\mathfrak{k}$,

$$(\int_{\mathfrak{u}\times\mathfrak{k}}(m_{\omega_U,o_U}\alpha)(X,Y)\Phi_1(X)\Phi_2(Y)dXdY)_{(e,m)} = ((\int_{\mathfrak{k}}\alpha(Y)\Phi_1(Y)\Phi_2(Y)dY)\wedge dx)_{(e,m)}$$

where dX is the Euclidean density on $\mathfrak{u}$ determined by ν'.

Now let us analyse the action of K on $U \times M$. Let $k = \dim K$, $n = \dim(U/K)$. Let E^i be an oriented basis of $\mathfrak{k}$ with dual basis $E_i \in \mathfrak{k}'$. Let $\nu'_K = E_1 \wedge \cdots \wedge E_k$. The connection ω determines a K-invariant decomposition $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{r}$. Thus E_i can be thought of as an element of $\mathfrak{u}'$ vanishing on $\mathfrak{r}$. If $\omega = \sum_i \omega_i E^i$, the form ω_i is the unique left U-invariant 1-form on U such that $(\omega_i)_e = E_i$. Let $pr_{\mathfrak{k}}$ (resp. $\mathrm{pr}_{\mathfrak{r}}$) be the projection from $\mathfrak{u}$ to $\mathfrak{k}$ (resp. $\mathfrak{r}$) determined by ω.

Consider the connection $\tilde{\omega} := \omega \otimes 1$ for the principal K-bundle $q_K : U \times M \to U \times_K M$. Let $\Omega \in \mathcal{A}^2(U) \otimes \mathfrak{k}$ be the curvature of ω. The equivariant curvature of $\tilde{\omega}$ at the point $(e,m) \in U \times M$ is given, for $X \in \mathfrak{u}, Y \in \mathfrak{k}$ by

$$\Omega(X,Y)_{(e,m)} = (pr_{\mathfrak{k}}X - Y) + \Omega_e.$$

Thus the element $\gamma_{\tilde{\omega},o_K}$ is given by

$$(\gamma_{\tilde{\omega},o_K})(X,Y)_{(e,m)} = |\nu'_K|^{-1}\delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X - Y) + \Omega_e)(\omega_1 \wedge \ldots \wedge \omega_k)_e.$$

Let $\mu' \in \Lambda^n\mathfrak{r}'$ be such that $\mu' \wedge \nu'_K = \nu'_U$. Let dr be the unique left U-invariant n-form on U such that $(dr)_e = \mu'$. The element μ' determines also a δ-function $|\mu'|^{-1}\delta_{\mathfrak{r}}$ on the vector space $\mathfrak{r}$. Let $\alpha \in \mathcal{A}_K^{-\infty}(M)$. By definition, for $X \in \mathfrak{u}$, $(Ind_{U/K,o}\alpha)(X)_{(e,m)}$ is the projection (on the horizontal elements for the diagonal K-action) of $|\mu'|^{-1}\delta_{\mathfrak{r}}(pr_{\mathfrak{r}}X)(\alpha(pr_{\mathfrak{k}}X)\wedge dr)_{(e,m)}$. As $\gamma_{\tilde{\omega},o_K}$ is already of top vertical dimension in the direction K, we have

$$((m_{\tilde{\omega},o_K}Ind_{U/K,o}\alpha)(X,Y))_{(e,m)} = |\mu'|^{-1}\delta_{\mathfrak{r}}(pr_{\mathfrak{r}}X)(\alpha(\mathrm{pr}_{\mathfrak{k}}X)\wedge dr \wedge |\nu'_K|^{-1}\delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X - Y) + \Omega)\wedge\omega_1\wedge\ldots\wedge\omega_k)_{(e,m)}.$$

Now $dr \wedge \omega_1 \ldots \wedge \omega_k$ is the form dx of top degree on U and Ω is a form on U, so that $dx \wedge \delta_{\mathfrak{k}}((pr_{\mathfrak{k}}X - Y) + \Omega) = dx \wedge \delta_{\mathfrak{k}}(pr_{\mathfrak{k}}X - Y)$. Thus, if Φ_1 is a test function on $\mathfrak{u}$ and Φ_2 a test function on $\mathfrak{k}$, we obtain

$$(\int_{\mathfrak{u}\times\mathfrak{k}} (m_{\tilde{\omega},o_K} Ind_{U/K,o}\alpha)(X,Y)\Phi_1(X)\Phi_2(Y)dXdY)_{e,m} = \\ ((\int_{\mathfrak{k}} \alpha(Y)\Phi_1(Y)\Phi_2(Y)dY) \wedge dx)_{(e,m)}.$$

Comparing with the preceding calculation, we obtain the equality, for $X \in \mathfrak{u}, Y \in \mathfrak{k}, m \in M$,

$$(m_{\omega_U,o_U}\alpha)(X,Y)_{(e,m)} = (m_{\tilde{\omega},o_K} Ind_{U/K,o}\alpha)(X,Y)_{(e,m)}.$$

By U-invariance we obtain the equality at each point $(x,m) \in U \times M$ and the lemma is proved. ∎

Proposition 88 *Let U be a Lie group. Let K be a compact subgroup of U such that $\det_{\mathfrak{u}} k = 1$, $\det_{\mathfrak{k}} k = 1$ for all $k \in K$. Then for any K-manifold M, the maps*

$$m_{o_U} : H_K^{-\infty}(M) \to H_{U\times K}^{-\infty}(U \times M)$$

and

$$m_{o_K} : H_U^{-\infty}(U \times_K M) \to H_{U\times K}^{-\infty}(U \times M),$$

(defined earlier) are both isomorphisms.

Proof: Let $G = U \times K$. The space $P = U \times M$ is also the induced space $G \times_{\Delta(K)} M$, where K is embedded in $G = U \times K$ by the diagonal map Δ. It is easy to see from the explicit calculation above that the map m_{o_U} coincides with the map $Ind_{G/K,o}$. As K is compact, Theorem 52 of section 5 implies that m_{o_U} is an isomorphism. As $Ind_{U/K,o}$ is also an isomorphism, Lemma 87 gives us the proposition. ∎

Let us return to the general situation (S) of a right G-manifold P, with principal action of a normal subgroup N of G, satisfying Conditions (77) and (78). Then the map $m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P)$ is defined. Although it would be desirable to know that m_o is always an isomorphism, we are able to prove it only under additional hypotheses.

First consider the case where $G = N$. Thus $H_{G/N}^{-\infty}(P/N)$ is simply equal to $H(P/G) = H(B)$. As $P \to B = P/G$ is a principal bundle with group G, we can find a (G-invariant) connection ω for the bundle $P \to B$. We assume that this fibration has G-oriented-fibers. Thus we can construct a canonical element (up to the G-orientation o) $\gamma_o \in H_G^{-\infty}(P)$ and the map m_o.

Let Ω be the curvature of the connection ω for $P \to P/G$. If $\phi(X) \in C^\infty(\mathfrak{g})^G$, then $\phi(\Omega) \in H(B)$ and is independent of the choice of ω. We thus define a structure of $C^\infty(\mathfrak{g})^G$-module on $H(B)$ via the Chern -Weil homomorphism: $\phi \cdot \beta = \phi(\Omega) \wedge \beta$, for $\phi \in C^\infty(\mathfrak{g})^G$ and $\beta \in H(B)$.

Theorem 89 *Let G be a Lie group acting principally (from the right) on a manifold P. Assume further that the quotient map $q : P \to P/G$ has G-oriented fibers under an orientation o. Then the map*

$$m_o : H(P/G) \to H_G^{-\infty}(P)$$

given by $m_o(\alpha) = q^*\alpha \wedge \gamma_o$ is an isomorphism of $C^\infty(\mathfrak{g})^G$-modules.

Proof: The fact that m_o is a morphism of $C^\infty(\mathfrak{g})^G$-modules, follows readily from Proposition 84.

It remains to see that m_o is an isomorphism of vector spaces:

If $P = G \times B$ is the direct product of G and B, with the action of an element $g_0 \in G$ given by $(g, m) \cdot g_0 = (g_0^{-1}g, m)$, for $g \in G, m \in B$, then by Proposition 88 (for $U = G$, $K = e$, $M = B$), the equivariant cohomology $H_G^{-\infty}(P)$ is isomorphic with $H(B)$ under the map m_o. Thus our theorem is true when the fibration $P \to B$ is trivial. (Remark: a trivial principal G-bundle is usually trivialized as $G \times B$, where the action of G is on the right $(g, m) \cdot g_0 = (gg_0, m)$. We can use the isomorphism $(g, m) \to (g^{-1}, m)$ to change this usual trivialization to the trivialization used above.)

Let us now return to the general situation . Choose a (G-invariant) connection form ω for the principal G-bundle q. Consider the element $\gamma_o \in H_G^{-\infty}(P)$ given in Proposition 79 (with respect to the given G-orientation o). Let U be an open subset of B. Denote by γ_U the restriction of γ_o to $q^{-1}(U)$. We denote by $m_U : H(U) \to H_G^{-\infty}(q^{-1}(U))$ the map m_o restricted to $q^{-1}(U)$: $m_U(\alpha) = q^*\alpha \wedge \gamma_U$.

Lemma 90 *Let U and V be two open subsets of B. Assume that the maps m_U, m_V, $m_{U\cap V}$ are isomorphisms, then $m_{U\cup V}$ is an isomorphism.*

Proof: This lemma is proved by a standard Mayer-Vietoris argument: Both the sequences

$$0 \to \mathcal{A}(U \cup V)) \to \mathcal{A}(U) \oplus \mathcal{A}(V) \to \mathcal{A}(U \cap V) \to 0,$$

$$0 \to \mathcal{A}_G^{-\infty}(q^{-1}(U \cup V)) \to \mathcal{A}_G^{-\infty}(q^{-1}(U)) \oplus \mathcal{A}_G^{-\infty}(q^{-1}(V)) \to \mathcal{A}_G^{-\infty}(q^{-1}(U \cap V)) \to 0$$

are exact. The surjectivity of the last map can be seen as follows: Choose a partition of unity f_U, f_V for $U \cup V$ subordinate to U, V. Then the functions $q^* f_U, q^* f_V$ are G-invariant functions. If $\beta \in \mathcal{A}_G^{-\infty}(q^{-1}(U \cap V))$, then β is the image of $(-(q^* f_V)\beta, (q^* f_U)\beta) \in \mathcal{A}_G^{-\infty}(q^{-1}(U)) \oplus \mathcal{A}_G^{-\infty}(q^{-1}(V)))$.

Thus the above Mayer-Vietoris sequences induce the long exact sequences in cohomology. The lemma follows from the five lemma. ∎

Now Theorem 89 follows by recalling that there is a finite open cover U_i of the base B such that the bundle $q^{-1}(U_i) \to U_i$ is trivial. ∎

We now return to the general situation (S) . Assume now that G is compact. Then the condition (78) on the existence of a G-invariant connection for the map $q : P \to P/N$ is always satisfied. We prove the following theorem

Theorem 91 *Let G be a compact Lie group and let P be a right G-manifold. G. Let N be a closed normal subgroup of G acting freely on P. We assume furthermore that the fibers of $q : P \to P/N$ admit a G-orientation o. Then the map*

$$m_o : H_{G/N}^{-\infty}(P/N) \to H_G^{-\infty}(P)$$

is an isomorphism.

Proof: We choose a G-invariant connection form $\omega \in (\mathcal{A}^1(P) \otimes \mathfrak{n})^G$ and use notation of the proof of Proposition 79. Let $\Gamma \subset \mathcal{A}(P)$ be the space of horizontal forms. The action of G on $\mathcal{A}(P)$ preserves the subspace Γ. Let us consider the algebra homomorphism $C : \Lambda \mathfrak{n}' \to \mathcal{A}(P)$, determined by sending $E_j \in \mathfrak{n}'$ to ω_j. We still denote by

$$C : \Gamma \otimes \Lambda \mathfrak{n}' \to \mathcal{A}(P)$$

the map given by $C(\alpha \otimes \xi) = \alpha \wedge C(\xi)$, for $\alpha \in \Gamma, \xi \in \Lambda \mathfrak{n}'$. The map C is an isomorphism. Furthermore C commutes with the action of G.

Recall the isomorphism (11) $U : P \times \mathfrak{g} \to P \times \mathfrak{n} \times \mathfrak{g}/\mathfrak{n}$. Let us explicitly write U, using coordinates. Let $n = \dim \mathfrak{n}$. Choose a basis Q^a, $n < a \leq \dim \mathfrak{g}$, of $\mathfrak{g}/\mathfrak{n}$. We choose a basis G^j of $\mathfrak{g}$ such that the first n-vectors are the vectors E^j and the last ones are representatives of Q^a. Let $Q = \sum_{i>n} x_i Q^i$ be an element of $\mathfrak{g}/\mathfrak{n}$. Define, for $p \in P$,

$$k_j(p, Q) = (\omega_p(\sum_{i>n} x_i G^i)_P, E_j).$$

Then $Q \mapsto k_j(p, Q)$ is a linear function in Q varying smoothly in p. Let $X = \sum_i x_i G^i$. In these coordinates $x = (x_i)$, we have

$$U(p, X) = (p, Y(x), Q(x))$$

with

$$Q(x) = \sum_{i>n} x_i Q^i, Y(x) = \sum_{j=1}^{n} (x_j + k_j(p, Q(x))E^j.$$

We denote by U^* the isomorphism

$$U^* : C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P)) \to C^{-\infty}(\mathfrak{g}, C^\infty(P))$$

given by $(U^*s)(X,p) = s(U(p,X))$, for $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$.

Formula above shows that U^*s is indeed smooth in p and generalized in X.

Let

$$A = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda\mathfrak{n}'.$$

With the help of U and C, we can define an isomorphism

$$T : A \to C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$$

by the following formula: For $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$, $\alpha \in \Gamma \otimes \Lambda\mathfrak{n}'$,

$$T(s\alpha) = (U^*s)C(\alpha).$$

The group G acts on A by the action induced by the adjoint representation of G on $\mathfrak{n}, \mathfrak{g}/\mathfrak{n}$, and its natural action on Γ. We denote by A_G the space of G-invariants in A. Then T commutes with the action of G and induces an isomorphism still denoted by T between A_G and $\mathcal{A}_G^{-\infty}(P)$.

Consider the $\mathbb{Z}_+$-gradation on A

$$A^* = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda^*\mathfrak{n}'.$$

We still denote by $d_\mathfrak{g}$ the operator on the space A obtained from the operator $d_\mathfrak{g}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ under the isomorphism T.

We write $Y \in \mathfrak{n}$ as $Y = \sum y_j E^j$. Let $\iota(E^j)$ be the contraction on $\Lambda\mathfrak{n}'$ by the vector E^j. Let $j_\mathfrak{n}$ be the operator of degree -1 on A given by

$$j_\mathfrak{n} = \sum_j y_j \iota(E^j).$$

The components Ω_j of the curvature Ω are horizontal forms. Thus exterior multiplication by Ω_j is an operator on Γ. We can consider the operator f acting on $C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda\mathfrak{n}'$ given by

$$f = \sum_j \Omega_j \otimes \iota(E^j).$$

The operator f is homogeneous of degree -1.

Let us write, using the $\mathbb{Z}_+$-grading of A, the operator $d_\mathfrak{g}$ on A as a sum of homogeneous operators d_i of degree i.

Lemma 92 *We have*

$$d_{\mathfrak{g}} = d_{-1} + d_0 + d_1$$

with $d_{-1} = -j_{\mathfrak{n}} + f$.

Proof: Let Q^a be a basis of $\mathfrak{g}/\mathfrak{n}$. Let $p \in P$. At the point $p \in P$, consider the decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{q}_p$$

given in (40). We write $Q_p^a \in \mathfrak{q}_p$ for the unique element of $\mathfrak{q}_p$ above $Q^a \in \mathfrak{g}/\mathfrak{n}$. We have $\omega_p(Q_P^a) = 0$. The contraction by Q_P^a produces an operator ι^a on the space of horizontal forms Γ. Let $Q = \sum_a q_a Q^a \in \mathfrak{g}/\mathfrak{n}$. The coordinate function q_a acts on $C^{-\infty}(\mathfrak{g}/\mathfrak{n})$ by multiplication. The operator $\iota_{\mathfrak{g}/\mathfrak{n}} = \sum q_a \iota^a$ is an operator of degree 0 on $C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma) \otimes \Lambda \mathfrak{n}'$.

It is easy to see that the operator $\iota = \iota_{\mathfrak{g}}$ on $C^{-\infty}(\mathfrak{g}, \mathcal{A}(P))$ becomes the operator $\iota_{\mathfrak{g}/\mathfrak{n}} \oplus j_{\mathfrak{n}}$ on A under the isomorphism T.

Let us now analyse the differential d_P under the isomorphism T. If $I = \{1 \leq i_1 < i_2 \ldots < i_k \leq n\}$ is an ordered multiindex, we write $\omega_I = \omega_{i_1} \wedge \ldots \wedge \omega_{i_k}$. Let $s \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P)), \alpha \in \Gamma$. We compute $d_P((U^* s)\alpha \wedge \omega_I)$. As d_P is a derivation, we analyse the exterior differential of each term of this product.

As $d\omega + \frac{1}{2}[\omega, \omega] = \Omega$, where Ω is horizontal, we see that the differential $d\omega_j$ of the component ω_j of the connection ω is the sum of an element of $C(\Lambda^2 \mathfrak{n}')$ and of $\Omega_j \in \Gamma$.

The differential d_P does not necessarily keep the space Γ of horizontal forms stable, but $d_P(\alpha) \in \Gamma \oplus \Gamma \otimes \mathfrak{n}'$, for $\alpha \in \Gamma$.

Finally, for $s(Y, Q, p) \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, C^\infty(P))$, we have, with $Y = \sum_j y_j E^j$,

$$T^{-1} d_P T s = d_P s + \sum_j \partial_{y_j} s(Y, Q, p) d_P k_j(p, Q).$$

Combining all these observations, we see that d_P becomes a sum of the homogeneous operators $f_{-1} + f_0 + f_1$ under the isomorphism T. Furthermore the term f_{-1} is the operator f. Hence we obtain the lemma. ∎

We now prove Theorem 91 by an induction argument similar to the argument of the proof of Theorem 39. Actually we will make use of the bigrading

$$A^{k,q} = C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}, \Gamma^q) \otimes \Lambda^k \mathfrak{n}',$$

where Γ^q refers to the $\mathbb{Z}^+$-grading on Γ given by the exterior degree.

Each of the spaces $A^{k,q}$ is stable by G, since N is a normal subgroup of G. As the group G is compact, the proof of Proposition 22 and Remark 23 (of section 3) implies that the homology groups of the operator $j_{\mathfrak{n}} : A_G^{*,q} \to A_G^{*-1,q}$ are equal to zero, except in maximal degree $n = \dim \mathfrak{n}$.

Let $\alpha \in \mathcal{A}_G^{-\infty}(P)$ be such that $d_{\mathfrak{g}}\alpha = 0$. We first show that α is homologous to an element β divisible by $\gamma_{\omega,o}$. We work with A_G and write again α for the element $T^{-1}(\alpha) \in A_G$. Let $\alpha = \sum_{k \geq k_0} \alpha_k$ with $\alpha_k \in A_G^k$. From the degree consideration (cf. Lemma 92), we see that $(j_{\mathfrak{n}} - f)\alpha_{k_0} = 0$.

Write now $\alpha_{k_0} = \sum_{q \geq q_0} \alpha_{(k_0,q)}$ with $\alpha_{(k_0,q)} \in A^{k_0,q}$. The operator f sends $A^{k_0,q}$ to $A^{k_0-1,q+2}$. Thus, we see again from degree considerations in q that $j_{\mathfrak{n}}(\alpha_{k_0,q_0}) = 0$. So if $k_0 < n$, there exists $\beta \in A_G^{k_0+1,q_0}$ such that $\alpha_{k_0,q_0} = j_{\mathfrak{n}}\beta$. The element $\alpha + d_{\mathfrak{g}}\beta$ is homologous to α and its term of degree k_0 is in $\sum_{q>q_0} A^{k_0,q}$. (Of course $\alpha + d_{\mathfrak{g}}\beta$ has no term of degree strictly less than k_0.) By successive approximations, we thus see that we can construct a representative of α in A_G^n. Now, let $\alpha \in A_G^n$ be such that $d_{\mathfrak{g}}\alpha = 0$. In particular, $(j_{\mathfrak{n}} - f)(\alpha) = 0$. We can write at the point $p \in P$ $\alpha(Y,Q)_p = \lambda(Y,Q,p)(\omega_1 \wedge \ldots \wedge \omega_n)_p$ where $\lambda(Y,Q,p) \in C^{-\infty}(\mathfrak{n} \oplus \mathfrak{g}/\mathfrak{n}) \otimes \Lambda H_p^*$, where H_p is the space of horizontal vectors. Let us write $Y = \sum_j y_j E^j$. For every $j, 1 \leq j \leq n$, the equation $(j_{\mathfrak{n}} - f)\alpha = 0$ implies

$$(y_j - \Omega_j)\lambda(Y,Q,p) = 0.$$

It is not difficult to see (using for example the translation $\lambda(Y,Q,p) \mapsto \lambda(Y + \Omega,Q,p)$) that $\lambda(Y,Q,p) = \delta_{\mathfrak{n}}(-Y + \Omega)\beta(Q,p)$ where $\beta(Q,p)$ is a generalized function on $\mathfrak{g}/\mathfrak{n}$ with values in ΛH_p^*. This way, we construct an element $\beta \in C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma)$. As $\alpha \in A_G$ is G-invariant, $\beta \in (C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma))^G$. As N is normal, the group N acts trivially on $\mathfrak{g}/\mathfrak{n}$. Thus, we see that $\beta \in (C^{-\infty}(\mathfrak{g}/\mathfrak{n},\Gamma^N))^{G/N} = \mathcal{A}_{G/N}^{-\infty}(P/N)$ and $\alpha = q^*\beta \wedge \gamma_{\omega,o}$.

The equation $d_{\mathfrak{g}}\alpha = 0$ and the injectivity of the map $m_{\omega,o}$ at the cochain level (cf. Proof of Proposition 85) implies that $d_{\mathfrak{g}/\mathfrak{n}}\beta = 0$. Thus β is closed and the map m_o is surjective. The injectivity is proved by a similar argument. ∎

Let K and U be compact subgroups of a Lie group L. Then L can be thought of as a $U \times K$-manifold under $x \cdot (u,k) = u^{-1}xk$, for $x \in L$, $u \in U$ and $k \in K$. If M is a K-manifold, we consider the $U \times K$ manifold $P = L \times M$, with twisted action as in Example 76.

Thus, specializing Theorem 91 to this example, we obtain

Proposition 93 *Let K, U be compact subgroups of a Lie group L. Assume that there exists a $U \times K$-invariant orientation o_K for the principal K-bundle $q_K : L \times M \to L \times_K M$. Then the map*

$$m_{o_K} : H_U^{-\infty}(L \times_K M) \to H_{U \times K}^{-\infty}(L \times M)$$

is an isomorphism. In particular, when $M =$ point,

$$H_U^{-\infty}(L/K) \cong H_{U \times K}^{-\infty}(L).$$

In the case of a free action, we have seen that $H_G^{-\infty}(M)$ is isomorphic to $H_G(M)$ under the multiplication by γ_o. However it may happen that the natural inclusion $H_G(M) \to H_G^{-\infty}(M)$ is identically 0. This is for example the case for the action of G on itself, at least when G is compact: The element $1 \in H_G(G) = \mathbb{R}$ has integral zero over G, while the integral of $\gamma_o \in H_G^{-\infty}(G)$ is equal to $vol(G, dg)\delta_{\mathfrak{g}}(X)$, as follows from the explicit formula for γ_o given above.

Assume that M is compact and oriented. Thus $\int_M$ defines a map from $H_G^{-\infty}(M)$ to $C^{-\infty}(\mathfrak{g})^G$. It is clear from the formula, given in Proposition 80 for the generator γ_o that if $\alpha = \beta \wedge \gamma_o$ with $\beta \in H_G(M)$, then $\int_M \alpha$ is a derivative $P(\partial)\delta_{\mathfrak{g}}$ of the $\delta_{\mathfrak{g}}$- function on $\mathfrak{g}$. Moreover, the order of the derivative is less or equal that of $\dim(B)/2$. We will determine explicitly this map in a special case.

Let K be a compact connected semi-simple Lie group and let T be its maximal torus. Let W be the Weyl group of (K, T). Let $S(\mathfrak{t})^W$ be the subalgebra of W-invariants in $S(\mathfrak{t})$. Let I be the ideal in $S(\mathfrak{t})$ generated by all the invariants of positive degree. Similarly, let $S(\mathfrak{t}')^W$ be the subalgebra of W-invariants in $S(\mathfrak{t}')^W$ and let J be the ideal in $S(\mathfrak{t}')$ generated by all the invariants of positive degree. Let $\delta_{\mathfrak{t}}$ be the δ function on $\mathfrak{t}$ determined by the Euclidean measure on $\mathfrak{t}$ associated to the Killing form.

If $f \in C^{-\infty}(\mathfrak{t})$ and $Q \in S(\mathfrak{t})$, then the derivative $Q(\partial)f$ of f by the constant coefficient differential operator $Q(\partial)$ is well defined.

Similarly, if $P \in S(\mathfrak{t})$ is a polynomial function on $\mathfrak{t}'$ and $Q \in S(\mathfrak{t}')$, we can define $Q(\delta)P$. An element $P \in S(\mathfrak{t})$ is called *harmonic*, if $Q(\delta)P = 0$, for all $Q \in J$. We denote by $\mathcal{H}$ the set of harmonic elements of $S(\mathfrak{t})$.

Lemma 94 *Let*

$$\mathcal{J} := \{f \in C^{-\infty}(\mathfrak{t}), Pf = 0, \text{ for all } P \in J\}$$

be the set of generalized functions on $\mathfrak{t}$ *annihilated by all the* W*-invariant functions* P *without constant terms under multiplications. Then* $\mathcal{J}$ *is equal to*

$$\mathcal{J} = \{Q(\partial)\delta_{\mathfrak{t}}; Q \in \mathcal{H}\}.$$

Proof: Choose a W-invariant norm $|x|$ on $\mathfrak{t}$. If $f \in \mathcal{J}$, then f is annihilated by the invariant polynomial function $|x|^2$. Hence f is supported at the origin and there exists a $Q \in S(\mathfrak{t})$ such that $f = Q(\partial) \cdot \delta_{\mathfrak{t}}$. The equation $J \cdot Q(\partial) \cdot \delta_{\mathfrak{t}} = 0$ implies, by Fourier transform, that Q is harmonic. ∎

Let $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{r}$ be the T-invariant decomposition of $\mathfrak{k}$, and let $n = \dim \mathfrak{r}$. Choose compatible orientations $o_{\mathfrak{k}}, o_{\mathfrak{r}}, o_{\mathfrak{t}}$ on $\mathfrak{k}, \mathfrak{r}, \mathfrak{t}$. Let κ', μ', ν' be the forms of maximal degree on $\mathfrak{k}, \mathfrak{r}, \mathfrak{t}$ respectively, associated to the Killing form $(,)$ and our choice of orientations . We denote also by κ' the left K-invariant form on K

coinciding with κ' at the identity e of K. Similarly, we extend ν' (resp. μ') as a left K-invariant $\dim \mathfrak{t}$-form (resp. $\dim \mathfrak{r}$-form) on K coinciding with ν' (resp μ') at e.

Let $\lambda \in \mathfrak{t}'$. The bilinear form on $\mathfrak{r}$ given by $B_\lambda(X,Y) = (\lambda, [X,Y])$ is an element of $\Lambda^2\mathfrak{r}'$. Let $\Delta = \{\alpha \in i\mathfrak{t}'\}$ be the set of roots of $(\mathfrak{k}, \mathfrak{t})$. Choose an order on Δ compatible with the orientation $o_{\mathfrak{r}}$, as in ([12], page 40). Let U be the polynomial function $U(\lambda) = \prod_{\alpha>0}(\lambda, i\alpha)$. Then $U \in \mathcal{H}$ and the map $P \in S(\mathfrak{t}') \mapsto P(\partial)U$ induces an isomorphism from $S(\mathfrak{t}')/J$ to $\mathcal{H}$. Furthermore it is easy to see that $B_\lambda^{n/2} = ((n/2)!)U(\lambda)\mu'$.

Consider the free action of T on $M = K$ by $k \cdot t = kt$. The space $H_T(K)$ is isomorphic to $H(K/T)$. The Chern-Weil map $W : S(\mathfrak{t}') \to H(K/T)$ is surjective, with kernel J. Thus we identify $H_T(K)$ with the $S(\mathfrak{t}')$-module $S(\mathfrak{t}')/J$.

Proposition 95 *The map*

$$\int_K : H_T^{-\infty}(K) \to C^{-\infty}(\mathfrak{t})$$

is an isomorphism from $H_T^{-\infty}(K)$ *to* $\mathcal{J}$. *Furthermore, we have*

$$\int_K \gamma_o = (-1)^{n/2} vol(K) U(\partial) \cdot \delta_{\mathfrak{t}},$$

where $n = \dim K$.

Proof: Consider the curvature Ω of $K \to K/T$, determined by the T-invariant decomposition $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{r}$. It is an element of $\Lambda^2\mathfrak{r}' \otimes \mathfrak{t}$. Let us compute $\exp \Omega$ in the algebra $\Lambda\mathfrak{r}' \otimes S(\mathfrak{t})$. The component of $\exp \Omega$ of exterior degree n is given by the formula $(\exp \Omega)_{[n]} = \mu' \otimes U$. The term of exterior degree n of $\delta_{\mathfrak{t}}(\Omega - X)$ is thus equal to $(-1)^{n/2}\mu' \otimes U(\partial)\delta_{\mathfrak{t}}$. Formula for $\gamma_o \in \mathcal{A}_T^{-\infty}(K)$ given in Proposition 80 shows that the term of maximal exterior degree of γ_o is

$$(\gamma_o)_{[\dim K]}(X) = (-1)^{n/2}(U(\partial) \cdot \delta_{\mathfrak{t}})(X)\kappa'.$$

Integrating over K, we obtain the formula for $\int_K \gamma_o$ given in the proposition.

As $H_T(K)$ is generated by 1 over $S(\mathfrak{t}')$, we obtain the equality $H_T^{-\infty}(K) = (S(\mathfrak{t}')/J)\gamma_o$. Furthermore, as seen by Fourier transform, the map $P \mapsto P(U(\partial) \cdot \delta_{\mathfrak{t}})$ induces an isomorphism from $S(\mathfrak{t}')/J$ to $\mathcal{J}$ and we obtain our proposition. ∎

Corollary 96 *The natural map* $H_T(K) \to H_T^{-\infty}(K)$ *is identically* 0.

Proof: Elements of $H_T(K) = H(K/T)$ come from the base, thus have integral zero on K. ∎

Remark 97 *If a torus acts on a compact oriented manifold M without fixed points, every (equivariant) cohomology class in $H_T(M)$ is of integral equal to zero, as follows from the localization formula (see [3], chap 7). The preceding example (i.e T acting on K by right translations) gives a striking case of an action of T without fixed points, where any non-zero equivariant cohomology class with generalized coefficients has a non-zero integral.*

10 A spectral sequence for T-equivariant cohomology

Let K be a compact connected Lie group and M a K- manifold. Let T be a maximal torus of K. In section 8, we have seen that the K-equivariant cohomology $H_K^{-\infty}(M)$ of M can be computed in terms of the T-equivariant cohomology of M. In this section, we will establish a spectral sequence relating the $S(\mathfrak{t}')$- modules $H_T(M)$ and $H_T^{-\infty}(M)$.

Let T be an abelian Lie group (not necessarily compact) and let M be a T-manifold Let $\mathfrak{t}$ be the Lie algebra of T. Then, as T is abelian,

$$\mathcal{A}_T(M) = S(\mathfrak{t}') \otimes \mathcal{A}(M)^T.$$

Similarly

$$\mathcal{A}_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T).s$$

We can then consider $\mathcal{A}_T^{-\infty}(M)$ as obtained from the space $\mathcal{A}_T(M)$ by "extension" of coefficients.

Let us consider the space:

$$\Omega = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'$$

$\mathbb{Z}$-graded by its exterior degree with respect to $\Lambda\mathfrak{t}'$, i.e.

$$\Omega^p = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}') \otimes \Lambda^p\mathfrak{t}'.$$

Let E^i be a basis of $\mathfrak{t}$ with dual basis E_i of $\mathfrak{t}'$. An element $X \in \mathfrak{t}$ is written as $X = \sum_i x_i E^i$. We can consider an element of

$$V = C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes S(\mathfrak{t}')$$

as a form $\alpha(X, Y) \in \mathcal{A}(M)^T$ depending in a generalized way on the first variable $X \in \mathfrak{t}$ and in a polynomial way on the second variable $Y \in \mathfrak{t}$.

We consider on V the $\mathbb{Z}/2$-grading given by the parity of an element in $\mathcal{A}(M)^T$. Consider the $S(\mathfrak{t}')$-module structure on the space V defined by $(E_i \cdot$

$\alpha)(X,Y) = (x_i - y_i)\alpha(X,Y)$, i.e. $E_i(\theta \otimes P) = x_i\theta \otimes P - \theta \otimes y_i P$, for $\theta \in C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$ and $P \in S(\mathfrak{t}')$.

Let j be the Koszul differential of degree -1 on $\Omega = V \otimes \Lambda\mathfrak{t}'$, got from the $S(\mathfrak{t}')$-module V (cf. Formula 12 of section 3), i.e.

$$j = \sum_i (x_i - y_i) \otimes \iota_\Lambda(E^i).$$

(As usual, in extending $\iota_\Lambda(E^i)$ to the tensor product of the two superspaces V and $\Lambda\mathfrak{t}'$, we respect the sign rules (2) and (3) of section 1).

If $\alpha(X,Y) = \theta(X) \otimes P(Y) \in V$, the restriction $\alpha(X,X) = \theta(X)P(X)$ of α to the diagonal is well defined. Thus, for any $\beta \in \Omega$, the restriction $\beta(X,X)$ of β to the diagonal is an element of $C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T) \otimes \Lambda\mathfrak{t}'$. Let us denote by $r(\beta)$ the component of exterior degree zero of $\beta(X,X)$. Thus the map r is a map from Ω to $\mathcal{A}_T^{-\infty}(M)$.

We can also write

$$\Omega = C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) \otimes \Lambda\mathfrak{t}',$$

where by definition

$$C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) = \sum_{p \geq 0} C^{-\infty}(\mathfrak{t}, \mathcal{A}_T^p(M))$$

and $\mathcal{A}_T^*(M)$ refers to the $\mathbb{Z}_+$-grading of $\mathcal{A}_T(M)$ defined in section 2.

We extend pointwise the differential $d_\mathfrak{t}$ of $\mathcal{A}_T(M)$ to $C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M))$ by defining $(\overline{d_\mathfrak{t}} f)(X) = d_\mathfrak{t}(f(X))$. Consider the operator

$$d_0 = \overline{d_\mathfrak{t}} \otimes I$$

of degree 0 (with respect to the $\mathbb{Z}$-grading of $\Lambda\mathfrak{t}'$) on Ω.

The operators j and d_0 satisfy $j^2 = 0$, $d_0^2 = 0$, $jd_0 + d_0 j = 0$.

Consider the $\mathbb{Z}/2$ grading on Ω given by the parity of forms on $\mathcal{A}(M)^T$ together with the $\mathbb{Z}$-grading of $\Lambda\mathfrak{t}'$. Then d_0 and j are odd operators. Define the operator

$$D = j + d_0.$$

The operator D is an odd operator on Ω of square equal to 0.

Let $H(\Omega, D)$ be the cohomology space of D. It is a $\mathbb{Z}/2$-graded space.

Proposition 98 *The map $r : \Omega \to \mathcal{A}_T^{-\infty}(M)$ satisfies $rD = d_\mathfrak{t} r$. Moreover r induces an isomorphism in cohomology. Thus the cohomology of the complex (Ω, D) is isomorphic to $H_T^{-\infty}(M)$.*

Proof: Since $rj = 0$, and $r(\theta(X) \otimes y_i \iota(E_M^i) P(Y)) = x_i \iota(E_M^i)(\theta(X)P(X))$ if $\theta(X) \in C^{-\infty}(\mathfrak{t})$ and $P(Y) \in \mathcal{A}_T(M)$, the first assertion is immediate,. Let $n = \dim \mathfrak{t}$. As V is a tensor product of the free module $S(\mathfrak{t}')$ by $C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$, the space V is a free $S(\mathfrak{t}')$-module (cf. Corollary 16 of section 3). Thus by Proposition 14 the Koszul complex

$$0 \to \Omega^n \xrightarrow{j} \quad \cdots \xrightarrow{j} \Omega^1 \xrightarrow{j} \Omega^0 \xrightarrow{r} \mathcal{A}_T^{-\infty}(M) \to 0$$

is exact at all the levels Ω^i, for all $i > 0$. Exactness at Ω^0 is easy to check.

Let Ω' be the exact complex for j defined by $\Omega'^i = \Omega^i$ if $i > 0$ and $\Omega'^0 = Kerr$. Choose any homotopy h of Ω' of degree 1 i.e. $hj + jh = I_{\Omega'}$. Consider $N = hd_0 + d_0 h$. Then N is an operator of degree 1 on Ω'. We have $hD + Dh = I + N$ on Ω', and N is a nilpotent operator commuting with D. Let us prove that r is surjective in cohomology: Let $\theta \in \mathcal{A}_T^{-\infty}(M)$ be such that $d_{\mathfrak{t}}\theta = 0$. We lift θ as a form in two variables $\Theta(X, Y) = \theta(X)$ constant in Y, i.e $\Theta = \theta \otimes 1$. Then $r(D\Theta) = 0$ i.e $D\Theta = d_0\Theta \in \Omega'$. Thus $(I + N)D\Theta = (hD + Dh)D\Theta = DhD\Theta$ i.e. $D(\Theta - (I + N)^{-1} hD\Theta) = 0$. The element $w(\Theta) := \Theta - (I + N)^{-1} hD\Theta$ still satisfies $r(w(\Theta)) = \theta$, and is a cocycle for D.

Similarly, we prove that r is injective: Let $\alpha \in \Omega$ be such that $D\alpha = 0$ and $r(\alpha) = d_{\mathfrak{t}}\theta$. Then $\alpha' := \alpha - D\Theta$ satisfies $D\alpha' = 0$ and $r\alpha' = 0$. Then $\alpha' = D(I + N)^{-1} h\alpha'$ is a boundary. This proves the proposition. ∎

We give below a more explicit way to construct a representative in $H(\Omega, D)$ of an element in $H_T^{-\infty}(M)$.

If $Y \in \mathfrak{t}$, we define as in section 4 the tensor product contraction $\iota_t(Y) = \iota(Y_M) + \iota_\Lambda(Y)$ on Ω. The horizontal space Ω_{hor} is then defined as

$$\Omega_{hor} = \{\alpha \in \Omega, \iota_t(Y)\alpha = 0 \text{ for all } Y \in \mathfrak{t}\}.$$

The space Ω_{hor} is stable by D. There is a canonical projection map (see Definition 28 of section 4) from Ω to Ω_{hor} given by

$$h = \prod_i (I - \epsilon_i \iota_t(E^i))$$

where ϵ_i denotes the multiplication by E_i on $\Lambda\mathfrak{t}'$.

We denote by $w : \mathcal{A}_T^{-\infty}(M) \to \Omega$ the map $w(\theta) = h(\Theta)$, where Θ is the lift of θ constant in Y.

We have

$$w(\theta) = \Theta + (-1)^{|\theta|} \sum_i \iota(E_M^i)\Theta \otimes E_i - \sum_{i<j} \iota(E_M^i)\iota(E_M^j)\Theta \otimes E_i \wedge E_j + \cdots$$

Lemma 99 *The map w satisfies: $wd_{\mathfrak{t}}\theta = Dw\theta$. Further w induces an isomorphism in cohomology, inverse to the map in cohomology induced by r.*

Proof: Both terms of this equation belong to Ω_{hor}. Thus, to prove that they are equal, we need only to compute their terms of zeroth exterior degree. The element $wd_{\mathfrak{t}}\theta$ has zero-exterior degree term equal to the lift of $(d_{\mathfrak{t}}\theta)(X) = d_M\theta(X) - \sum_i x_i\iota(E_M^i)\theta(X)$, constant in Y.

The element $Dw\theta$ has zero-exterior degree term

$$d_M\theta(X) - \sum_i y_i\iota(E_M^i)\theta(X) + \sum_{i,j}(y_i - x_i)\iota_\Lambda(E^i)(\epsilon_j(\iota(E_M^j)\theta(X)))$$

which is equal to

$$d_M\theta(X) - \sum_i x_i\iota(E_M^i)\theta(X).$$

It is clear that $rw = 1$. But since the map r induces an isomorphism in cohomology, we get that w also induces isomorphism in cohomology inverse to that of r. ∎

The complex (Ω, D) admits an increasing filtration $\mathcal{F} = \{\mathcal{F}_x\}_{0\leq p\leq \dim \mathfrak{t}}$ by the exterior degree in $\Lambda\mathfrak{t}'$ i.e . $\mathcal{F}_p = \otimes_{k\leq p}\Omega^k$. This canonically gives rise to a convergent homology spectral sequence E^r converging to $H(\Omega, D)$.

Lemma 100 *Assume that T is compact* abelian *and M is a paracompact T-manifold, such that $H_T^*(M)$ is finite dimensional in each degree. Then*

$$E_p^1 = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda^p\mathfrak{t}'.$$

Proof: By definition

$$\begin{aligned} E_p^1 &= H(\mathcal{F}_p/\mathcal{F}_{p-1}, D) \\ &= H(\mathcal{F}_p/\mathcal{F}_{p-1}, d_0) \qquad \text{since } j(\mathcal{F}_p) \subset \mathcal{F}_{p-1}, \\ &\cong H(C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)) \otimes \Lambda^p\mathfrak{t}', d_0) \\ &\cong H(C^{-\infty}(\mathfrak{t}, \mathcal{A}_T(M)), \overline{d}_{\mathfrak{t}}) \otimes \Lambda^p\mathfrak{t}'. \end{aligned}$$

It is easy to see that $Ker(\overline{d}_{\mathfrak{t}}) = C^{-\infty}(\mathfrak{t}, Z_T(M))$ and moreover $Im(\overline{d}_{\mathfrak{t}}) \subset C^{-\infty}(\mathfrak{t}, B_T(M))$. Further, by Theorem 117 (of the Appendix), we get a continuous splitting of the map

$$\mathcal{A}_T^{n-1}(M) \xrightarrow{d_{\mathfrak{t}}} B_T^n(M),$$

and hence $Im(\overline{d}_{\mathfrak{t}}) = C^{-\infty}(\mathfrak{t}, B_T(M))$. Also $H_T^n(M)$ being finite dimensional, the projection $Z_T^n(M) \to H_T^n(M)$ admits a continuous splitting. From this we easily conclude that

$$H(C^{-\infty}(\mathfrak{t}, \mathcal{A}_T^*(M)), \overline{d}_{\mathfrak{t}}) \cong C^{-\infty}(\mathfrak{t}, H_T^*(M)).$$

This proves the lemma. ∎

Remark 101 *If T is compact and $H^*(M)$ is finite dimensional in each degree then so is $H_T^*(M)$. This follows from the Serre spectral sequence for the fibration $M \to E(T) \times_T M \to B(T)$. In particular, for compact M, $H_T^*(M)$ is finite dimensional in each degree.*

The differential $d^1 : E_p^1 \to E_{p-1}^1$ of degree -1 induced by $D = d_0 + j$ on $E^1 = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda\mathfrak{t}'$ is the Koszul differential j associated to the canonical $S(\mathfrak{t}')$-module stuctures on $C^{-\infty}(\mathfrak{t})$ and $H_T(M)$. Hence, combining Proposition 98, Lemma 100 and Lemma 17 we obtain the main result of this section.

Theorem 102 *Let T be a compact abelian Lie group and let M be a manifold such that $H_T^*(M)$ is finite dimensional in each degree. Then the cohomology group $H_T^{-\infty}(M)$ has an increasing $\mathbb{Z}_+$-filtration H_p, and a convergent homology spectral sequence with*

$$E_p^2 = Tor_p^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$$

and

$$E_p^\infty = H_p/H_{p-1},$$

where $C^{-\infty}(\mathfrak{t})$ and $H_T(M)$ have their canonical $S(\mathfrak{t}')$-module structures.

This spectral sequence is functorial with respect to the T-equivariant smooth maps. Further the total $\mathbb{Z}/2$-grading given by the standard $\mathbb{Z}_+$ degree on $H_T^*(M)$ together with the p index in Tor is compatible with the $\mathbb{Z}/2$-grading of $H_T^{-\infty}(M)$.

We obtain a number of corollaries:

Corollary 103 *Let M, N be T-manifolds such that $H_T^*(M)$ is finite dimensional in each degree, with a T- equivariant smooth map $f : M \to N$. Assume that the induced map $f^* : H_T^*(N) \to H_T^*(M)$ is an isomorphism in T- equivariant cohomology . Then the induced map*

$$f^* : H_T^{-\infty}(N) \to H_T^{-\infty}(M)$$

is also an isomorphism.

Proof: It follows immediately from the above spectral sequence. ∎

The following corollary was obtained in section 6 for compact T- manifolds as a consequence of Theorem 61 (cf. Corollary 64).

Corollary 104 *For any T-manifold M such that $H_T^*(M)$ is a projective finitely generated $S(\mathfrak{t}')$- module, the canonical map*

$$\beta_{T,M} : C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} H_T(M) \to H_T^{-\infty}(M)$$

is an isomorphism.

Proof: Since $H_T(M)$ is $S(\mathfrak{t}')$-projective, the spectral sequence of Theorem 102 has $E_p^2 = 0$, unless $p = 0$. In particular the spectral sequence degenerates at the E^2-term itself. Also

$$E_0^2 = Tor_0^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \approx C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} H_T(M).$$

This proves the corollary. ∎

Let T be an abelian Lie group and let M be a T-manifold. The T-equivariant de Rham complex with generalized coefficients admits a graded subcomplex obtained by forming the algebraic tensor product

$$\tilde{\mathcal{A}}_T^{-\infty}(M) := C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T.$$

This subcomplex is stable by the action of $S(\mathfrak{t}')$. We denote the cohomology of this subcomplex by $\tilde{H}_T^{-\infty}(M)$. We have the following comparison:

Proposition 105 *Let T be a compact abelian Lie group and let M be a T-manifold such that $H_T^*(M)$ is finite dimensional in each degree. Then the canonical map $\tilde{H}_T^{-\infty}(M) \to H_T^{-\infty}(M)$, induced from the inclusion $\tilde{\mathcal{A}}_T^{-\infty}(M) \to \mathcal{A}_T^{-\infty}(M)$, is an isomorphism.*

Proof: Recall the definition of the complex (Ω, D) and define a subcomplex

$$\tilde{\Omega} := C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T \otimes S(\mathfrak{t}') \otimes \Lambda\mathfrak{t}'.$$

The cochain map $r : \Omega \to C^{-\infty}(\mathfrak{t}, \mathcal{A}(M)^T)$ restricts to a cochain map (denoted by) $\tilde{r} : \tilde{\Omega} \to C^{-\infty}(\mathfrak{t}) \otimes \mathcal{A}(M)^T$. By the same proof as that of Proposition 98, we can easily see that $\tilde{r}$ induces isomorphism in cohomology.

Thus the augmented complex

$$\tilde{\Omega} \xrightarrow{\tilde{r}} \tilde{\mathcal{A}}_T^{-\infty}(M)$$

maps by the natural inclusion i into the augmented complex

$$\Omega \xrightarrow{r} \mathcal{A}_T^{-\infty}(M).$$

The filtration $\{\mathcal{F}_p\}$ of Ω gives rise to the filtration $\{\tilde{\mathcal{F}}_p := \mathcal{F}_p \cap \tilde{\Omega}\}$ of $\tilde{\Omega}$. In particular, we get the induced map $\tilde{E}_p^r \to E_p^r$, where $\tilde{E}_p^r$ is the spectral sequence corresponding to the filtration $\{\tilde{\mathcal{F}}_p\}$. We have $\tilde{E}_p^1 = C^{-\infty}(\mathfrak{t}) \otimes H_T(M) \otimes \Lambda^p\mathfrak{t}'$. In particular, $\tilde{E}_p^1 \to E_p^1$ is an isomorphism, and hence the inclusion $\tilde{\Omega} \to \Omega$ induces isomorphism in cohomology. But then the map $i : \tilde{\mathcal{A}}_T^{-\infty}(M) \to \mathcal{A}_T^{-\infty}(M)$ also induces an isomorphism in cohomology. ∎

The spectral sequence obtained in Theorem 102 may sometimes be used to determine the torsion groups $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$. For example, if K is a

compact connected Lie group with maximal torus T and if M is a K-manifold such that $Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M))$ is equal to zero except for $i = i_0$ (for some i_0), then from the degenerate spectral sequence of Theorem 102 and Theorem 70,

$$Tor_{i_0}^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{k}')^K} H_K^{-\infty}(M).$$

As we next show, this hypothesis is valid when M is homogeneous under K. Let U be a closed subgroup of K. Let us choose a maximal torus T_U of U and let T be a maximal torus of K containing T_U.

Proposition 106 *Let $M = K/U$, where K is a compact connected Lie group and U a closed subgroup. Let $\chi : U \to \pm 1$ be the character $\chi(u) := \det_{\mathfrak{t}/\mathfrak{u}} u$. Then the group*

$$Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) = 0 \qquad \text{for } i \neq d := \dim(T/T_U)$$

and

$$Tor_d^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{k}')^K} C^{-\infty}(\mathfrak{u})^\chi.$$

Proof: Let $\mathfrak{t}_U$ be the Lie algebra of T_U. Let $W_U \subset GL(\mathfrak{t}_U)$ be the Weyl group of the pair (U, T_U) (i.e.$W_U = N_U(T_U)/T_U$). If $P \in S(\mathfrak{t}')^W$ is a W-invariant function, its restriction to $\mathfrak{t}_U$ is W_U invariant (to see this, use Chevalley's theorem to conclude that P is the restriction to $\mathfrak{t}$ of a K-invariant polynomial on $\mathfrak{k}$) . From Proposition 68 of section 7, we have $H_T(M) \cong S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}_U')^{W_U}$. Thus

$$Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}_U')^{W_U}).$$

Consider $N := S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}_U')$ as a $(S(\mathfrak{t}'), W_U)$-module by the action of $S(\mathfrak{t}')$ on the left and the action of W_U on the right factor. Then $N^{W_U} = S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}_U')^{W_U})$. The space $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)$ thus carries a canonical structure of $(S(\mathfrak{t}'), W_U)$-module and moreover by the standard averaging process

$$Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), H_T(M)) \cong Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)^{W_U}.$$

Thus, to prove the vanishing part, we prove that $Tor_i^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N) = 0$ except for $i = d$.

Let $\mathfrak{t}_1 = \mathfrak{t}/\mathfrak{t}_U$, so that $\mathfrak{t}_1' \subset \mathfrak{t}'$. Consider the partial Koszul complex $j_1 = j_{\mathfrak{t}_1} : S(\mathfrak{t}') \otimes \Lambda \mathfrak{t}_1' \to S(t_U')$. This gives a $S(\mathfrak{t}')$-free resolution of $S(\mathfrak{t}_U')$. As $S(\mathfrak{t}')$ is free over $S(\mathfrak{t}')^W$, the complex $S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda \mathfrak{t}_1')$ with differential $I \otimes j_1$ gives a $S(\mathfrak{t}')$-free resolution of $S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}_U')$. Thus $Tor^{S(\mathfrak{t}')}(C^{-\infty}(\mathfrak{t}), N)$ is the homology of the complex

$$C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')} (S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda \mathfrak{t}_1')) \cong C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda \mathfrak{t}_1')$$

i.e. of the complex
(42)

$$0 \to C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda^d \mathfrak{t}_1') \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \Lambda^{d-1} \mathfrak{t}_1')$$
$$\cdots \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} (S(\mathfrak{t}') \otimes \mathfrak{t}_1') \xrightarrow{j_1} C^{-\infty}(\mathfrak{t}) \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \to 0.$$

Consider the isomorphism obtained in Proposition 66 of section 6

$$C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}') \cong C^{-\infty}(\mathfrak{t})$$

induced from the multiplication map. Hence, the map $P_1 \otimes F \otimes P_2 \mapsto P_1 \otimes FP_2$ gives an isomorphism of

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t})^\epsilon \otimes_{S(\mathfrak{t}')^W} S(\mathfrak{t}')$$

with

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} C^{-\infty}(\mathfrak{t}).$$

Thus we obtain an isomorphism of the complex (42) with the complex

$$S(\mathfrak{t}') \otimes_{S(\mathfrak{t}')^W} (C^{-\infty}(\mathfrak{t}) \otimes \Lambda \mathfrak{t}_1')$$

under the differential $I \otimes j_1^{-\infty}$. By Proposition 22 of section 3, the homology of the complex $(C^{-\infty}(\mathfrak{t}) \otimes \Lambda \mathfrak{t}_1', j_1^{-\infty})$ is non-zero only in degree d. As $S(\mathfrak{t}')$ is free over $S(\mathfrak{t}')^W$, we obtain the vanishing part of the proposition. Furthermore, by the remark just before this proposition and Theorem 46 of section 5, we obtain the assertion regarding Tor_d. ∎

11 Localization formula

Let T be a torus, i.e. a compact connected abelian Lie group, acting on a compact oriented manifold M. Let $\alpha \in H_T^{-\infty}(M)$. The integral $\Theta(X) := \int_M \alpha(X)$ of α is a generalized function on $\mathfrak{t}$. When $\alpha \in H_T^\infty(M)$, the localization formula (see [3], chapter 7) gives $\Theta(X)$ in terms of the restriction of α to the fixed submanifold M^T of M. As shown by Proposition 95 of section 9 (where M^T is empty but the map $\int_M$ is not zero), it is not possible to determine $\int_M \alpha$ in terms of $\alpha | M^T$ in the generalized case. The main reason for the difference between the $C^{\pm\infty}$-cases is that the space $H_T^{-\infty}(point) = C^{-\infty}(\mathfrak{t})$ is not torsion free over $S(\mathfrak{t}')$. Indeed, for $\alpha \in H_T^{-\infty}(M)$, we will find a non-zero polynomial $P \in S(\mathfrak{t}')$ and determine $P(X) \int_M \alpha(X)$ in terms of $\alpha | M^T$, as in the C^∞-case.

The localization formula, we are going to give in the generalized case, involves choosing a T-equivariant embedding of M in a real representation space

V of T. This is always possible, see ([9], Chap 6, Theorem 4.1). Let V_0 be the subspace of T-fixed vectors and let

$$V = V_0 \oplus V_1$$

be the T-invariant decomposition. Thus $\det_{V_1}(X)$ is a non zero polynomial on $\mathfrak{t}$. The fixed submanifold $M_0 = M^T$ of M is given by $M \cap V_0$. The space V_1 is even dimensional. Let us choose an orientation on V_1. This orientation determines a polynomial square root of $\det_{V_1}(X)$. Using a T-invariant metric on V, we view the normal bundle $\mathcal{N}$ of M_0 in M as a T-equivariant subbundle of the trivial bundle $M_0 \times V_1$. The bundle $\mathcal{N}$ is T-orientable and is of even rank. Let us denote by $\mathcal{Q}$ the supplementary bundle:

$$M_0 \times V_1 = \mathcal{N} \oplus \mathcal{Q}.$$

The bundle $\mathcal{Q}$ is a T-equivariant bundle over M_0. We choose orientations of $V_1, \mathcal{N}, \mathcal{Q}$ in a compatible way. Let $u_{\mathcal{N}} \in H_{cpt,T}(\mathcal{N})$, $u_{\mathcal{Q}} \in H_{cpt,T}(\mathcal{Q})$ be the T-equivariant Thom classes (see Definition 10 section 2) of $\mathcal{N}$, $\mathcal{Q}$ respectively. Let $\chi(\mathcal{N}) \in H_T(M_0)$ (resp.$\chi(\mathcal{Q})$) be the equivariant Euler class of the bundle $\mathcal{N} \to M_0$ (resp. $\mathcal{Q}$). By definition (we differ here from the definition of [3], chapter 7), the restriction of $u_{\mathcal{N}}$ (resp. $u_{\mathcal{Q}}$) to M_0 via the zero section is equal to $\chi(\mathcal{N})$ (resp. $\chi(\mathcal{Q})$). We have the following equality in $H_T(M_0)$:

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \cong \chi(\mathcal{N})(X)\chi(\mathcal{Q})(X). \tag{43}$$

Let us fix an orientation of M and consider the compatible orientation of M_0. Following is the localization formula in generalized cohomology.

Theorem 107 *Let T be a torus acting on a compact oriented manifold M. For $\alpha \in H_T^{-\infty}(M)$, we have the equality*

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \int_M \alpha(X) = \int_{M^T} \alpha(X)\chi(\mathcal{Q})(X)$$

as elements of $C^{-\infty}(\mathfrak{t})$.

Proof: The proof is obtained by imitating the proof in the C^∞-case given in [2]. Consider the Thom class $u_1(X) \in H_{cpt,T}(V_1)$ of the T-vector space V_1, thought of as a T-equivariant vector bundle $q : V_1 \to \textit{point}$. We have

$$(-2\pi)^{-\dim V_1/2} q^*(\det_{V_1}^{1/2}(X)) \sim u_1(X)$$

as elements of $H_T(V_1)$. Consider the map $p : M \to V_1$ induced by the projection of $V = V_0 \oplus V_1$ to V_1. Thus

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \int_M \alpha(X) = \int_M \alpha(X) p^* u_1(X).$$

We can take a representative of u_1 (as a cohomology class in $H_{cpt,T}(V_1)$) supported in a sufficiently small neighborhood of 0 in V_1. Thus we may assume that p^*u_1 is compactly supported in a T-stable tubular neighborhood U of $M_0 = p^{-1}(0)$. Let π be the T-equivariant projection of $U \to M_0$. Let i be the inclusion of M_0 in M. We have seen in section 2, Proposition 8, that the restriction $\alpha|U$ of α to U is equivalent to $\pi^*i^*\alpha$ in $H_T^{-\infty}(U)$. As p^*u_1 is compactly supported in U, we obtain

$$\begin{aligned}\int_M \alpha(X)p^*u_1(X) &= \int_U \alpha(X)p^*u_1(X) \\ &= \int_U \pi^*i^*\alpha(X)p^*u_1(X) \\ &= \int_{M_0} \alpha(X)\pi_*p^*u_1(X).\end{aligned}$$

Let $\beta = p^*u_1 \in H_{cpt,T}(U)$. It remains to show that $\pi_*\beta(X) = \chi(\mathcal{Q})(X)$ in $H_T(M_0)$.

The restriction of $\beta(X)$ to M_0 is equal to $(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X)$. The tubular neighborhood U of M_0 in M is T- equivariantly diffeomorphic to the normal bundle $\mathcal{N} \to M_0$. Let $u_{\mathcal{N}}$ be the equivariant Thom class of $\mathcal{N}$. Then, it is well known (see the proof of Proposition 11 of section 2) that $\beta = (\pi^*\pi_*\beta)u_{\mathcal{N}}$ in $H_{cpt,T}(U)$. By restricting this equality to M_0, we obtain

$$(-2\pi)^{-\dim V_1/2} \det_{V_1}^{1/2}(X) \cong \pi_*\beta(X)\chi(\mathcal{N})(X)$$

in $H_T(M_0)$. As $\det_{V_1}(X)$ is a non zero polynomial, $\chi(\mathcal{N})(X)$ is invertible on the open set $\det_{V_1}(X) \neq 0$. By Formula (43), we obtain the equality $\pi_*\beta(X) = \chi(\mathcal{Q})(X)$. This proves the theorem. ∎

Let us illustrate the localization formula in the simple example of $M = P_1(\mathbb{C})$.

Let p_1 be the point at infinity of M. Then $U = M - \{p_1\}$ is isomorphic to $\mathbb{C}$. We consider the action of $T = \{e^{i\theta}\}$ on $P_1(\mathbb{C})$ given by $z \mapsto e^{i\theta}z$. This action has two fixed points $p_0 = 0$ and $p_1 = \infty$. We write still p_0, p_1 for the injections of p_0 and p_1 in M.

We write an element of $\mathfrak{t}$ as $X = \theta J$, with $\exp 2\pi J = 1$. Let us first describe the T-equivariant cohomology of M. It is a free $S(\mathfrak{t}')$-module with two generators α, β. We can normalize these two generators, by requiring

$$p_0^*(\alpha) = 1, \qquad \int_M \alpha = 0$$

while

$$p_0^*(\beta) = 0, \qquad \int_M \beta = 1.$$

Identifying $X = \theta J$ with θ, some specific representatives of α and β are

$$\alpha = 1$$

$$\beta(\theta) = (2\pi)^{-1}(\theta|z|^2(1+|z|^2)^{-1} + i(1+|z|^2)^{-2}dz \wedge d\bar{z}).$$

The restriction maps $p_0^*, p_1^* : H_T(M) \to S(\mathfrak{t}')$ satisfy

$$2\pi(p_1^* - p_0^*) = \theta \int_M .$$

Consider now $H_T^{-\infty}(M)$. Let $\delta(\theta)$ be the δ function at 0. The element

$$v(\theta) = \delta(\theta)\beta(\theta)$$

is in $H_T^{-\infty}(M)$.

As $\theta\delta(\theta) = 0$, we have

$$v(\theta) = (-2i\pi)^{-1}\delta(\theta)(1+|z|^2)^{-2}dz \wedge d\bar{z}.$$

Thus, the element v does not have component in zero exterior degree, in particular its restriction to $M^T = \{p_0\} \cup \{p_1\}$ is zero. The integral $\int_M v(\theta)$ is equal to $\delta(\theta)$ and is supported at 0. This is compatible with the localization theorem which asserts that $\theta \int_M v(\theta) = 0$.

Let $P = p_0^* \oplus p_1^*$ be the map:

$$P : H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t}) \oplus C^{-\infty}(\mathfrak{t}).$$

Thus v is in the kernel of P.

In fact, we have the exact sequence

$$0 \to \mathbb{C}v \to H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t}) \oplus C^{-\infty}(\mathfrak{t}) \to 0.$$

The exactness of this sequence can be seen as follows: As $H_T(M)$ is free over $S(\mathfrak{t}')$, we have

$$H_T^{-\infty}(M) = C^{-\infty}(\mathfrak{t})\alpha + C^{-\infty}(\mathfrak{t})\beta.$$

Writing $\nu = f\alpha + g\beta$, we see that if $P\nu = 0$, then $f = 0$ and $\theta g = 0$. Thus ν is proportional to v. Let us see that P is surjective. The restriction maps $p_0^*, p_1^* : H_T^{-\infty}(M) \to C^{-\infty}(\mathfrak{t})$ still satisfy

$$(2\pi)(p_1^* - p_0^*) = \theta \int_M .$$

Thus we have $p_0^*\nu = f$ and $(2\pi)p_1^*\nu = \theta g + (2\pi)f$. As it is always possible to divide by θ in the space $C^{-\infty}(\mathfrak{t})$, we see that $P = p_0^* \oplus p_1^*$ is surjective.

12 Appendix – A splitting for $d_{\mathfrak{g}}$

Recall from section 1 that, for a manifold M, $\mathcal{A}^*(M)$ denotes the $\mathbb{Z}_+$-graded space of smooth forms on M and $d = d_M$ denotes the exterior derivative. Let $B^*(M)$ (resp. $Z^*(M)$) denote the space of exact (resp. closed) forms. As in the earlier part of this article, we assume M to be paracompact and equip $\mathcal{A}(M)$ with the C^∞-topology and $B(M), Z(M)$ are given the subspace topology. We make the following

Definition 108 *The exterior derivative d is said to admit a* continuous splitting on M, *if there exists a continuous graded linear map $s : B^*(M) \to \mathcal{A}^{*-1}(M)$ (of degree -1) such that $d \circ s = I$.*

Observe that, by virtue of Hodge theorem, d admits a continuous splitting on any compact manifold M. In fact, we will prove in an elementary way the following

Theorem 109 *Let M be any (paracompact) manifold. Then d admits a continuous splitting on M.*

Proof: The proof of the theorem will be broken into the following lemmas.

Lemma 110 *For any manifold M, $Z(M)$ and $B(M)$ are closed subspaces of $\mathcal{A}(M)$.*

Proof: Being the kernel of d, $Z(M) \subset \mathcal{A}(M)$ is clearly a closed subspace. By Poincaré duality, we have

$$B(M) = \{\alpha \in Z(M); \int_M \alpha\gamma = 0\}$$

for all $\gamma \in Z_{cpt}(M)_t$. Thus $B(M)$ is a closed subspace of $\mathcal{A}(M)$. ∎

Lemma 111 *Let M be any contractible manifold. Then d admits a continuous splitting on M.*

Proof: Choose a C^∞-contraction $\phi : \mathbb{R} \times M \to M$, i.e. $\phi|_{\{0\}\times M} = I_M$ and $\phi|_{\{1\}\times M} = m_0$, for some fixed point $m_0 \in M$. Define the map $H : \mathcal{A}^*(M) \to \mathcal{A}^{*-1}(M)$ by

$$H(\omega) = \int_0^1 \phi^*\omega$$

for $\omega \in \mathcal{A}^*(M)$. Then H is a homotopy operator, i.e.

$$dH\omega + Hd\omega = \omega - \phi_1^*\omega$$

for $\omega \in \mathcal{A}^*(M)$, where $\phi_1 : M \to M$ is defined by $\phi_1 := \phi|_{\{1\}\times M}$. Now define $s : B^*(M) \to \mathcal{A}^{*-1}(M)$, by $s = H|_{B^*(M)}$. Then s gives a splitting for d. ∎

Lemma 112 *Let M be a manifold, and $W \subset U$ be two open subsets satisfying $\bar{W} \subset U$, where $\bar{W}$ is the closure of W. Let $[\omega] \in H(M)$ be a cohomology class such that $[\omega]_{|U} = 0$, as an element of $H(U)$. Then there exists a form $\tilde{\omega} \in Z(M)$ which satisfies:*

1. *$\tilde{\omega}|_W$ is identically zero, and*

2. *$[\tilde{\omega}] = [\omega]$ as elements of $H(M)$.*

Proof: Choose an open subset V of M such that $\bar{W} \subset V \subset \bar{V} \subset U$. Write $\omega|_U = d\theta$, for some $\theta \in \mathcal{A}^*(U)$. Choose a C^∞-function f on M such that $f \equiv 1$ on $\bar{W}$ and $f \equiv 0$ on $M \backslash V$. Then $f\theta$ is a smooth form on the whole of M . Now set $\tilde{\omega} = \omega - d(f\theta)$. Then $\tilde{\omega}$ satisfies the requirements of the lemma. ∎

Lemma 113 *Let U and V be two open subsets of a manifold M, such that the exterior derivative d admits a continuous splitting on U, V and $U \cap V$. Assume further that $H(U \cap V)$ is finite dimensional. Then d admits a continuous splitting on the union $W := U \cup V$.*

Proof: Choose a continuous splitting s_1 (resp. s_2) of d on U (resp. V), and define a splitting $s : B^*(U) \oplus B^*(V) \to \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V)$ by $s = s_1 \oplus s_2$ (i.e., $s(\omega_1 + \omega_2) := s_1(\omega_1) + s_2(\omega_2)$, for $\omega_1 \in B^*(U)$ and $\omega_2 \in B^*(V)$).

Consider the commutative diagram (where the upper horizontal sequence is exact):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}^{*-1}(W) & \xrightarrow{\gamma_1} & \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V) & \xrightarrow{\gamma_2} & \mathcal{A}^{*-1}(U \cap V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_W} & & \downarrow{\scriptstyle d_U \quad d_V} & & \downarrow{\scriptstyle d_{U\cap V}} & & \\ 0 & \longrightarrow & B^*(W) & \xrightarrow{\hat{\gamma}_1} & B^*(U) \oplus B^*(V) & \xrightarrow{\hat{\gamma}_2} & B^*(U \cap V) & & \end{array}$$

where $\gamma_1, \gamma_2, \hat{\gamma}_1$ and $\hat{\gamma}_2$ are the canonical maps. From the commutativity of the above diagram, $\gamma_2 s \hat{\gamma}_1(\omega) \in Z^{*-1}(U \cap V)$, for any $\omega \in B^*(W)$; and moreover, from the definition of the coboundary map $\delta : H^{*-1}(U \cap V) \to H^*(W)$, we get $\delta[\gamma_2 s \hat{\gamma}_1(\omega)] = [\omega] = 0$. In particular, the cohomology class $[\gamma_2 s \hat{\gamma}_1(\omega)]$ lies in the image $F \subset H(U \cap V)$ of $H(U) \oplus H(V)$ under γ_2.

Now choose any linear map $\beta : F^{*-1} \to Z^{*-1}(U) \oplus Z^{*-1}(V)$ such that $[\gamma_2 \beta(x)] = x$, for all $x \in F^{*-1}$. Since F is finite dimensional ($H(U \cap V)$ being finite dimensional by assumption), any such β is automatically continuous. With the help of β, we define the continuous linear map

$$s_\beta : B^*(W) \longrightarrow \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V),$$

by $s_\beta(\omega) = s\hat{\gamma}_1(\omega) - \beta[\gamma_2 s\hat{\gamma}_1(\omega)]$, for $\omega \in B^*(W)$. It can be easily seen that $\gamma_2 s_\beta(\omega) \in B^{*-1}(U \cap V)$. Choose a partition of unity $\{f_U, f_V\}$ subordinate to the cover $\{U, V\}$ of W and choose a continuous splitting s_3 of d on the intersection $U \cap V$. Then $f_V \cdot (s_3\gamma_2 s_\beta(\omega)) \in \mathcal{A}^{*-2}(U)$ and $f_U \cdot (s_3\gamma_2 s_\beta(\omega)) \in \mathcal{A}^{*-2}(V)$, for any $\omega \in B^*(W)$. Finally, define the continuous map $\theta : B^*(W) \to \mathcal{A}^{*-1}(U) \oplus \mathcal{A}^{*-1}(V)$ by

$$\theta(\omega) = s_\beta(\omega) - \left(d_U(f_V \cdot (s_3\gamma_2 s_\beta(\omega))) \oplus d_V(-f_U \cdot (s_3\gamma_2 s_\beta(\omega)))\right) ,$$

for $\omega \in B^*(W)$.

It is easy to see that $\gamma_2 \circ \theta = 0$, in particular, the map θ lifts to a (continuous) map $\bar{\theta} : B^*(W) \to \mathcal{A}^{*-1}(W)$ and moreover the map $\bar{\theta}$ provides a continuous splitting for d_W, i.e., $d_W \circ \bar{\theta} = I$. This completes the proof of the lemma. ∎

Let us now prove a stabilization lemma for splittings.

Proposition 114 *Let $\{V_i\}_{i=1,2,\ldots}$ be an open and locally finite cover of M. Set, for any $k = 1, 2, \ldots$,*

$$U_k = \bigcup_{i=1}^{k} V_i .$$

Also set $U_0 = \emptyset$. Assume that d admits a continuous splitting on V_k and $U_k \cap V_{k+1}$ and assume that $H(V_k)$ and $H(U_k \cap V_{k+1})$ are finite dimensional for all $k \geq 1$. Then d_M admits a continuous splitting.

Proof: We proceed as in the above lemma with the pair $U = U_k$ and $V = V_{k+1}$, but with a special choice of the map β. Mayer-Vietoris long exact sequence implies that $H(U_k)$ is finite dimensional. Let $F_k \subset H(U_k \cap V_{k+1})$ be the image of $H(U_k) \oplus H(V_{k+1})$ under the map $\gamma_2 := \gamma_2^k$. For $i \leq k$, let $H(U_k)_i$ be the subspace of elements of $H(U_k)$ which restrict to 0 in $H(U_i)$. We have the following

Lemma 115 *For any i, there exists $k(i) \geq i$ such that for all $k \geq k(i)$,*

$$\gamma_2 \left(H(U_k)_i \oplus H(V_{k+1})\right) = F_k.$$

Proof: For any $k \geq i$, let $R_k \subset H(U_i)$ be the image of $H(U_k)$ under the natural restriction. As $H(U_i)$ is finite dimensional, the decreasing sequence of subspaces R_k of $H(U_i)$ is stationary. Thus there exists an index $k(i)$ such that for $k \geq k(i)$, $R_k = R_{k(i)}$. Let us show that for $k \geq k(i)$, $\gamma_2 \left(H(U_k)_i \oplus H(V_{k+1})\right) = F_k$. Indeed let $\alpha = \nu - \kappa \in F_k$ where ν (resp. κ) is the restriction to $U_k \cap V_{k+1}$ of an element still denoted by $\nu \in H(U_k)$ (resp. $\kappa \in H(V_{k+1})$). As $R_k = R_{k+1}$, we may write $\nu = \nu_0 + \nu'|_{U_k}$, with $\nu_0 \in H(U_k)_i$ and $\nu' \in H(U_{k+1})$. Thus $\alpha = \nu_0 - (\kappa - \nu'|_{V_{k+1}})$ is in $\gamma_2(H(U_k)_i + H(V_{k+1}))$. ∎

We continue with the proof of Proposition 114. We fix an open refinement $\{W_i\}$ of $\{V_i\}$, i.e., $\bar{W}_i \subset V_i$ and $\cup_i W_i = M$. Let us choose more carefully the map $\beta_k : F_k \to Z(U_k) \oplus Z(V_{k+1})$. Let $b(k)$ be the largest integer $i \geq 0$ such that $\gamma_2(H(U_k)_i \oplus H(V_{k+1})) = F_k$ and choose $\beta_k = \beta_k^0 \oplus \beta_k'$ valued in $Z(U_k)_{b(k)} \oplus Z(V_{k+1})$, where $Z(U_k)_i$ is the space of closed differential forms θ on U_k such that $[\theta]|_{U_i} = 0$ as an element of $H(U_i)$. Furthermore with the help of Lemma 112, we can assume that for any $\omega \in F_k$ the component $\beta_k^0(\omega)$ vanishes identically on the open subset $\bigcup_{i=1}^{b(k)} W_i$.

With this choice of β_k, we get a continuous splitting s_{k+1} of d on the manifold U_{k+1}. This completes the inductive procedure to construct a splitting s_k of d on the manifold U_k, for all values of k. Now define a map $s : B^*(M) \to \mathcal{A}^{*-1}(M)$ by

$$s(\omega) = \lim_{k \to \infty} s_k(\omega_{|U_k}), \qquad \text{for } \omega \in B(M).$$

Observe that for any relatively compact open subset V of M, there exists a large enough k_0 (depending only upon V) such that $(s_k(\omega_{|U_k}))_{|V} = (s_{k_0}(\omega_{|U_{k_0}}))_{|V}$ for all $k \geq k_0$. In particular, the map s is well defined and continuous. It is clear that s provides a splitting of d on the whole of M. ∎

To prove Theorem 109, it is then sufficient to prove the existence of a covering of M satisfying the conditions of Proposition 114.

Lemma 116 *Consider a locally finite covering of M by geodesically convex open subsets with respect to a fixed Riemannian metric on M, then this covering satisfies the conditions of Proposition 114.*

Proof: For any open subset U of M, let $n(U)$ be the smallest number of geodesically convex open subsets of U required to cover U (if no such finite cover exists, we decree $n(U) = \infty$). We first prove by induction on $n(U)$ that d_U admits a continuous splitting on any open subset $U \subset M$ with $n(U) < \infty$: The case $n(U) = 1$ is taken care of by Lemma 111. Observe that for any two convex open subsets of M, their intersection is also convex, so the general case follows by induction on $n(U)$ and Lemma 113 (together with [8], Chapter 1, Proposition 5.3.1). Thus a locally finite open cover $M = \bigcup V_i$, $i \geq 1$ by geodesically convex open subsets of M satisfies the hypothesis of Proposition 114. ∎

This completes the proof of Theorem 109. ∎

Let us generalize Theorem 109 to the equivariant case.

Theorem 117 *Let G be a compact Lie group acting on a paracompact manifold M. Then, for all $p \in \mathbb{Z}_+$, the subspaces $Z_G^p(M)$ and $B_G^p(M)$ are closed subspaces of $\mathcal{A}_G^p(M)$. Furthermore the equivariant de Rham differential $d_{\mathfrak{g}} : \mathcal{A}_G^p(M) \to B_G^{p+1}(M)$, where $\mathfrak{g} :=$ Lie G, admits a continuous splitting for all $p \geq 0$.*

Proof: We start with the following preliminary lemmas.

Lemma 118 *Let G be a compact group acting on a manifold M. If $H(M)$ is finite dimensional, then $H_G^p(M)$ is finite dimensional, for all $p \in \mathbb{Z}_+$.*

Proof: Consider the filtration of $\mathcal{A}_G(M)$ by the subspaces

$$\mathcal{A}_G(M)_k = \left(S(\mathfrak{g}') \otimes \sum_{i=0}^{k} \mathcal{A}^i(M) \right)^G$$

of $\mathcal{A}_G(M)$ consisting of equivariant differential forms of exterior degree less or equal to k. Consider the space $Z_G(M)_k$ of closed equivariant differential forms with exterior degree less or equal to k. It is easy to see (as in the proof of Proposition 5) that the map $\alpha \mapsto \alpha_{[k]}$ induces an injective map from $V_k = Z_G(M)_k / \left(Z_G(M)_{k-1} + (d_{\mathfrak{g}} \mathcal{A}_G(M)_{k-1}) \right)$ to $(S(\mathfrak{g}') \otimes H^k(M))^G$. Clearly V_k surjects on $H_G(M)_k / H_G(M)_{k-1}$, where $H_G(M)_k$ is the subspace of $H_G(M)$ consisting of those cohomology classes with a representative in $Z_G(M)_k$. This, in particular gives that $H_G(M)_k$ is finite dimensional in each $\mathbb{Z}$-graded degree and for all k. This proves the lemma.

Lemma 119 *Let $f : V \to W$ be a continuous linear map between Fréchet spaces such that $\operatorname{Im} f$ is of finite codimension in W. Then $\operatorname{Im} f$ is a closed subspace of W.*

Proof: Let us take a vector space complement U of $\operatorname{Im} f$ in W, which is finite dimensional by assumption. Also let K be the kernel of f. Consider the direct sum $V \oplus U$ and define a continuous linear map $\hat{f} : V \oplus U \to W$ by $\hat{f}|_V = f$ and $\hat{f}|_U$ is the inclusion. It is a surjective linear map between Fréchet spaces, so it is an open map (by the open map theorem). In particular, the map $\hat{f}$ gives a (linear) homeomorphism $(V/K) \oplus U \to W$. But V/K is closed in the direct sum $(V/K) \oplus U$, and hence its image $\operatorname{Im} f$ is closed in W. ∎

Thus we obtain

Lemma 120 *If $H_G^p(M)$ is finite dimensional, the space $B_G^p(M) \subset \mathcal{A}_G^p(M)$ is a closed subspace, for all $p \in \mathbb{Z}_+$.*

Lemma 121 *Let $p : \mathcal{V} \to M$ be a G-equivariant real vector bundle over a G-manifold M. If $d_{\mathfrak{g}}$ has a continuous splitting on M, then $d_{\mathfrak{g}}$ admits a continuous splitting on $\mathcal{V}$.*

Proof: Consider the map $\phi(t, v) = tv$ from $\mathbb{R} \times \mathcal{V}$ to $\mathcal{V}$. This map commutes with the action of G. Keeping the same notation for the operator H as in

Lemma 32, and denoting by i the inclusion of M as the zero section of $\mathcal{V}$, we obtain

$$\omega - p^* i^* \omega = (d_{\mathfrak{g}} H + d_{\mathfrak{g}} H)\omega$$

for all $\omega \in \mathcal{A}_G(\mathcal{V})$. Thus if s is a continuous splitting for $d_{\mathfrak{g}}$ on M, then $s_{\mathcal{V}} = (H + p^* s i^*)|_{B_G(\mathcal{V})}$ is a continuous splitting for $d_{\mathfrak{g}}$ on $\mathcal{V}$. ∎

Lemma 122 *Let N be a contractible manifold. Let $K \subset G$ be a closed subgroup of G. Consider the G-manifold $M = (G/K) \times N$, where G acts by left action on the first factor and trivially on the second factor. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the G-manifold M.*

Proof: Note first that if $N = point$, i.e., $M = G/K$ is homogeneous, then (see section 5) $\mathcal{A}_G(M) \cong (S(\mathfrak{g}') \otimes \Lambda(\mathfrak{g}/\mathfrak{k})')^K$ is finite-dimensional in each degree, thus $d_{\mathfrak{g}}$ admits a continuous splitting on M. Proceeding as in Lemma 121, we obtain a continuous splitting for the product $(G/K) \times N$. ∎

We will also use the following equivariant analogue of Lemma 113, which follows by the same proof (using a G-invariant partition of unity, which exists since G is compact).

Lemma 123 *Let G be a compact Lie group and let M be a G-manifold with G-stable open subsets U and V. Assume that $d_{\mathfrak{g}}$ admits a continuous splitting on U, V and $U \cap V$. Assume further that $H_G^p(U \cap V)$ is finite dimensional in each degree p. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the union $W := U \cup V$.*

Proposition 124 *Let G be a compact Lie group and let M be a G-manifold with a locally finite covering by G-stable open subsets V_i, $i \geq 1$. Let $U_k = \bigcup_{i=1}^k V_i$. Assume that $d_{\mathfrak{g}}$ admits a continuous splitting on V_k and $U_k \cap V_{k+1}$, for all $k \geq 1$. Assume further that $H_G^p(U_k \cap V_{k+1})$ and $H_G^p(V_k)$ are finite dimensional in each degree p. Then*

(1) $d_{\mathfrak{g}}$ admits a continuous splitting on M,

(2) $B_G^p(M)$ is a closed subspace of $\mathcal{A}_G^p(M)$.

Proof: Mayer-Vietoris long exact sequence implies that $H_G^p(U_i)$ is finite-dimensional. We construct a continuous splitting s_i for $d_{\mathfrak{g}}$ on U_i as in the proof of Proposition 114. Then the splittings s_i stabilize to give rise to a continuous splitting s on M. Let us prove the assertion (2). Let $\alpha \in \overline{B_G^*(M)}$. Since $B_G(U_i)$ is a closed subspace of $\mathcal{A}_G(U_i)$ by Lemma 120, $\alpha|_{U_i} = d_{\mathfrak{g}} s_i(\alpha|_{U_i})$. By construction of s we see that $s_i(\alpha|_{U_i}) \in \mathcal{A}_G^{*-1}(U_i)$ stabilizes in an element $\beta \in \mathcal{A}_G^{*-1}(M)$ such that $d_{\mathfrak{g}}\beta = \alpha$. ∎

Lemma 125 *Let X be a G-manifold with exactly one orbit type. Then $d_{\mathfrak{g}}$ admits a continuous splitting on X.*

Proof: Since G is compact, our assumption implies that the space of orbits $M := X/G$ is a manifold and that the quotient map $q : X \to X/G$ is a locally trivial fibration (see [9], Chapter 2, Theorem 5.8). Take a locally finite open covering of M by convex open subsets V_i (in particular V_i being contractible, q is trivial over V_i) and set $\tilde{V}_i := q^{-1}(V_i)$. Now take any $x_0 \in \tilde{V}_i$, then $\tilde{V}_i$ is G-diffeomorphic to $G \cdot x_0 \times V_i$, where G acts trivially on V_i and G acts canonically on the orbit $G \cdot x_0$. Thus $d_{\mathfrak{g}}$ admits a continuous splitting on $\tilde{V}_i$ by Lemma 122. Arguing as in the proof of Lemma 116, we see that the covering $X = \bigcup_i \tilde{V}_i$ satisfies the hypothesis of Proposition 124. Thus we obtain our lemma. ∎

Let M be a compact G-manifold with boundary $\delta(M)$. Observe that $\delta(M)$ is automatically G-stable. Let $M^o = M \backslash \delta(M)$. We call M^o the interior of M.

Lemma 126 *Let M be a compact G-manifold possibly with boundary. Then $d_{\mathfrak{g}}$ admits a continuous splitting on the interior M^o of M.*

Proof: By [9], chapter 8, Theorem 3.13, any compact manifold has finitely many orbit types. Let us assume by induction on the number of orbit types $\eta(M^o)$ of M^o that the lemma is true for all compact manifolds M (possibly with boundary) with $\eta(M^o) \leq n$ and take a compact manifold M with $\eta(M^o) = n+1$. The case where $\eta(M^o) = 1$ is taken care of by Lemma 125. Let G_1 be an isotropy subgroup such that G_1 is not properly contained in any other isotropy subgroup and let N be the set of all the elements of M^o with isotropy group G_1. Then $N = (M^o)^{G_1}$ by the maximality property of G_1. Thus N is a closed submanifold of M^o. Let K be the normalizer of G_1 in G. The map $(g, x) \mapsto g \cdot x$ induces an isomorphism of $G \times_K N$ on its image F. Thus F is a G-invariant closed submanifold of M^o. Cover M^o as $M^o = U \cup V$ where U is a G-stable open tubular neighborhood of F and $V = M^o \backslash F$. Clearly $\eta(V) = n$ and $\eta(F) = 1$. Also U, $U \cap V$ and V are interiors of compact G-manifolds with boundary. Further the existence of a continuous splitting for the operator $d_{\mathfrak{g}}$ on F (guaranteed by Lemma 125) gives rise to a splitting of $d_{\mathfrak{g}}$ on U, by Lemma 121. By induction hypothesis, $d_{\mathfrak{g}}$ admits a continuous splitting on V and $U \cap V$. Further $U \cap V$ being the interior of a compact manifold with boundary, $H(U \cap V)$ is finite dimensional and so is $H^p_G(U \cap V)$ for every p. Now the lemma follows by applying Lemma 123 to the open cover $M^o = U \cup V$. ∎

Let us now prove Theorem 117. Choose a G-invariant Morse function

$$f : M \to [0, \infty)$$

for the G-manifold M in the sense of ([22], par. 4). We further assume that f is a proper map. Let $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ be the complete list (possibly infinite) of critical values of f. Choose real numbers $\mu_i < \overline{\mu}_i$ such that $\lambda_i < \mu_i < \overline{\mu_i} < \lambda_{i+1}$. Define $V_i := f^{-1}(\mu_{i-1}, \overline{\mu}_i)$ (μ_0 is set as $-\infty$) and $U_k = \bigcup_{i=1}^{k} V_i$. Then $V_k, U_k, U_k \cap V_{k+1}$ are interiors of compact G-manifolds with boundaries

and the open cover V_i of M satisfies the hypothesis of Proposition 124. This completes the proof of Theorem 117. ∎

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