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Appendix: Coxeter groups and unipotent representations

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APPENDIX: COXETER GROUPS AND UNIPOTENT REPRESENTATIONS

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1. INTRODUCTION

1.1. Let (W, S) be a finite Coxeter group. We shall regard W as a group generated by reflections in a real $|S|$–dimensional vector space E defined as in [2, Ch.4, 4.3]; we shall write $\det(q-w) \in \mathbb{R}[q]$ for the characteristic polynomial of $w \in W$ on E (q is an indeterminate).

Our objective is to attach to W a finite set $\mathcal{U}(W)$ together with a function Deg which associates to each $\rho \in \mathcal{U}(W)$ a polynomial $\mathrm{Deg}(\rho) \in \mathbb{R}[q]$ which, in the case where W is a Weyl group, should be the set of unipotent representations over $\mathbb{C}$ of a corresponding Chevalley group over a finite field, together with the dimension of such representations, regarded as a polynomial in the cardinal of the finite field.

The fact that our objective is reasonable, is suggested by the following considerations:

(a) in the case where (W, S) is a Weyl group, $\mathcal{U}(W)$ and Deg are provided by the results of [12];

(b) the classification and dimensions of unipotent representations of Chevalley groups of type B_n and C_n given in [12] is the same, so that it depends only on the underlying Coxeter group and not on the root datum; thus, one may hope that this makes sense even when there is no root system and no Chevalley group around, that is, in the non-crystallographic case;

(c) a crucial ingredient in the description of $\mathcal{U}(W)$ for Weyl groups is its partition into families; these are indexed by the two-sided cells of W and the complexity of a family is related to the pattern of intersections of left cells and right cells inside a fixed two-sided cell; now cells are well defined even in the non-crystallographic case (see [7]).

We will give a description of $\mathcal{U}(W)$ and Deg in the general case, based on heuristic arguments: namely, we will postulate various properties for them

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that are known in the crystallographic case and make sense in the general case.

It should be noted that the "results" below (which were obtained in 1980–1982, except for those in 4.2) are not theorems in the accepted sense; rather, they are an indication of a not yet discovered theory. A rigorous treatment of these matters and an explanation of their meaning remain to be found.

It would be interesting to understand the non-abelian Fourier transform (see [10]) for families in the non-crystallographic case.

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1.2. We set $\theta = e^{2\sqrt{-1}\pi/3}, \zeta = e^{2\sqrt{-1}\pi/5}$. Let $\lambda = -(\zeta^2+\zeta^{-2}) = (1+\sqrt{5})/2$ (the golden ratio) and let $\lambda' = -(\zeta+\zeta^{-1}) = (1-\sqrt{5})/2$. Let $\Phi_d \in \mathbb{Z}[q]$ be the d–th cyclotomic polynomial; thus, $(\Phi_1, \Phi_2, \Phi_3, \dots) = (q-1, q+1, q^2+q+1, \dots)$. For $d \in \{5, 10, 15, 20, 30\}$ we write $\Phi_d = \Phi'_d \Phi''_d$ where

$$\Phi'_5 = q^2+\lambda q+1\,, \Phi''_5 = q^2+\lambda' q+1$$

$$\Phi'_{10} = q^2-\lambda q+1\,, \Phi''_{10} = q^2-\lambda' q+1$$

$$\Phi'_{15} = q^4-\lambda q^3+\lambda q^2-\lambda q+1\,, \Phi''_{15} = q^4-\lambda' q^3+\lambda' q^2-\lambda' q+1$$

$$\Phi'_{20} = q^4-\lambda q^2+1\,, \Phi''_{20} = q^4-\lambda' q^2+1$$

$$\Phi'_{30} = q^4+\lambda q^3+\lambda q^2+\lambda q+1\,, \Phi''_{30} = q^4+\lambda' q^3+\lambda' q^2+\lambda' q+1\,.$$

1.3. We set $P(W) = \sum_{w\in W} q^{l(w)} \in \mathbb{Z}[q]$ where $l(w)$ is the length of w.

Let S' be a subset of S. Let $\langle S'\rangle$ be the subgroup of W generated by S'. Then $(\langle S'\rangle, S')$ is a Coxeter group. Let $n(S')$ be the number of reflections in $\langle S'\rangle$. Let $W(S')$ be the Coxeter group on the generators $S-S'$ such that for $s \neq t$ in $S-S'$, the order of st in $W(S')$ is

$$2(n(S'\cup\{s,t\}) - n(S'))/(n(S'\cup\{s\}) + n(S'\cup\{t\}) - 2n(S'))\,.$$

Let $f : S-S' \to \{1, 2, \dots\}$ be a function such that $f(s) = f(t)$ whenever $s \neq t$ in $S-S'$ are such that st has odd order in $W(S')$ and let $\mathcal{H}(W(S'); f)$ be the Iwahori–Hecke algebra over $\mathbb{C}(\sqrt{q})$ attached to the Coxeter group $W(S')$ and to f; thus, the quadratic relation satisfied by the generator T_s is $(T_s+1)(T_s - q^{f(s)}) = 0$. If χ is a simple $\mathcal{H}(W(S'); f)$–module, we denote by $D(\chi) \in \mathbb{C}(q)$ the formal degree of χ. We extend f to a function $W(S') \to \{0, 1, 2, \dots\}$ by

$$f(s_1 s_2 \cdots s_p) = f(s_1) f(s_2) \cdots f(s_p)$$

where $s_1, s_2, \cdots, s_p$ is any sequence in $S-S'$ such that $s_1 s_2 \dots s_p$ is a reduced expression in $W(S')$.

We set $P(W(S'); f) = \sum_{w\in W(S')} q^{f(w)} \in \mathbb{Z}[q]$. Taking $S' = \emptyset$ and $f = 1$, this specializes to $P(W) \in \mathbb{Z}[q]$. Replacing W by $\langle S'\rangle$, we obtain $P(\langle S'\rangle) \in \mathbb{Z}[q]$.

2. Postulates

In this section we postulate a number of properties of $\mathcal{U}(W)$ and Deg.

2.1. $\mathrm{Deg}(\rho)$ divides the polynomial $q^{n(S)}P(W)(q-1)^{|S|}$ and the coefficient of the highest power of q appearing in $\mathrm{Deg}(\rho)$ is a real algebraic number > 0. (Hence all coefficients of $\mathrm{Deg}(\rho)$ are real algebraic numbers.)

2.2. Let σ be an automorphism of the field of real algebraic numbers. There exists a bijection $\tilde{\sigma} : \mathcal{U}(W) \to \mathcal{U}(W)$ such that, for any ρ, the polynomial $\mathrm{Deg}(\tilde{\sigma}(\rho))$ is equal up to sign to the polynomial obtained by applying σ to each coefficient of $\mathrm{Deg}(\rho)$.

2.3. If W' is another finite Coxeter group, then
(a) $\mathcal{U}(W \times W') = \mathcal{U}(W) \times \mathcal{U}(W')$ and $\mathrm{Deg}(\rho, \rho') = \mathrm{Deg}(\rho)\,\mathrm{Deg}(\rho')$ for $\rho \in \mathcal{U}(W), \rho' \in \mathcal{U}(W')$.

2.4. Let $\mathcal{U}^0(W)$ be the subset of $\mathcal{U}(W)$ consisting of those ρ such that the highest power of $(q-1)$ dividing $\mathrm{Deg}(\rho)$ is $(q-1)^{|S|}$. The elements in this subset are said to be *cuspidal.* Then $\mathcal{U}(W)$ is naturally partitioned into subsets $\mathcal{U}(W; S'; \rho')$ indexed by pairs (S', ρ'), where $S' \subset S$ and $\rho' \in \mathcal{U}^0(\langle S' \rangle)$ such that

(a) $\mathcal{U}(W; W; \rho) = \{\rho\}$ for any $\rho \in \mathcal{U}^0(W)$ and

(b) $\mathcal{U}(W; S'; \rho')$ is naturally in bijection with the set of isomorphism classes of simple modules of the algebra $\mathcal{H}(W(S'); f)$ for a well defined function f as in 1.3.

Let χ be a simple $\mathcal{H}(W(S'); f)$-module and let $S'[\rho', \chi]$ be the corresponding element of $\mathcal{U}(W; S'; \rho')$. Then

(c) $\mathrm{Deg}(S'[\rho', \chi]) = \mathrm{Deg}(\rho')D(\chi)\frac{P(W)}{P(\langle S' \rangle)P(W(S'); f)}$.

(It follows that, if $\rho \in \mathcal{U}(W; S'; \rho')$, then the highest power of $(q-1)$ dividing $\mathrm{Deg}(\rho)$ is $(q-1)^{|S'|}$.) Further,

(d) if $s \in S - S'$, then the function $f : S - S' \to \{1, 2, \dots\}$ takes the same value at s as the analogous function $\{s\} \to \{1, 2, \dots\}$, defined like f, in terms of $S' \cup \{s\}, S'$ instead of S, S'.

2.5. In the case where the longest element of W is central in W, there exists an involution $\rho \mapsto \tilde{\rho}$ of $\mathcal{U}(W)$ such that $\mathrm{Deg}(\tilde{\rho})$ is $\pm$ the polynomial obtained from $\mathrm{Deg}(\rho)$ by the change of variable $q \mapsto -q$.

2.6. Let $w \in W$ be a regular element in the sense of Springer [13]. Let $d \geq 1$ be the order of w and let $\omega \in \mathbb{C}^*$ be a primitive root of 1 of order d which is a root of $\det(q - w)$. Let $Z(w)$ be the centralizer of w in W. Let $\mathcal{U}(W)_\omega = \{\rho \in \mathcal{U}(W) |\, \mathrm{Deg}(\rho)|_{q=\omega} \neq 0\}$. Then

(a) there exists a $1-1$ correspondence $\phi \leftrightarrow \rho_\phi$ between the set of irreducible characters of $Z(w)$ and the set $\mathcal{U}(W)_\omega$ such that $\mathrm{Deg}(\rho_\phi)|_{q=\omega} = \pm\phi(1)$ for all ϕ.

Moreover, we have

(b) $\sum_\rho \mathrm{Deg}(\rho_\phi)|_{q=\omega} \mathrm{Deg}(\rho_\phi) = (-1)^{l(w)} P(W) q^{n(S)} (q-1)^{|S|} \det(q-w)^{-1}$.

2.7. In the setup of 2.6, assume in addition that the centralizer of w in W consists precisely of the powers of w. Then

(a) there exist distinct elements $\lambda_1, \lambda_2, \ldots, \lambda_d$ of $\mathbb{C}[\sqrt{q}]$ (each λ_j is a root of 1 times a power of $\sqrt{q}$) and distinct elements $\rho_1, \rho_2, \ldots, \rho_d$ of $\mathcal{U}(W)$ such that

$\mathrm{Deg}(\rho_j) = \pm P(W) q^{n(S)} (q-1)^{|S|} \det(q-w)^{-1} \lambda_j^{-1} \prod_{j'; j' \neq j} (\lambda_j - \lambda_{j'})^{-1}$ for $j = 1, 2, \ldots, d$;

(b) we have $\lambda_1 = 1$ and $\mathrm{Deg}(\rho_1) = q^{n(S)}$;

(c) $\mathcal{U}(W)_\omega = \{\rho_1, \rho_2, \ldots, \rho_d\}$.

If W is irreducible or 1 and w is a Coxeter element, then the corresponding set $\mathcal{T}(W) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ (d=Coxeter number) can be determined inductively as follows. There is a subset $\mathcal{T}^0(W)$ of $\mathcal{T}(W)$ such that

(d) $\mathcal{T}(W) = \sqcup_{S'} \{\lambda q^m | \lambda \in \mathcal{T}^0(\langle S' \rangle), 0 \leq m \leq |S - S'|\}$

(here S' runs over the subsets of S such that $\langle S' \rangle$ is irreducible or 1;

(e) if $\lambda \in \mathcal{T}^0(W)$ then λ is $(\sqrt{q})^{|S|}$ times a root of 1;

(f) if $W = 1$ then $\mathcal{T}(W) = \{1\}$;

(g) $\prod_{\lambda \in \mathcal{T}(W) - \{1\}} (1 - \lambda) = (-1)^{|S|} P(W) (q-1)^{|S|} \det(q-w)^{-1}$.

2.8. We have

(a) $\sum_\rho \mathrm{Deg}(\rho)^2 = |W|^{-1} P(W)^2 (q-1)^{2|S|} \sum_{w \in W} \det(q-w)^{-2}$.

(ρ runs over $\mathcal{U}(W)$.)

2.9. We now comment on the postulates above assuming that W is a Weyl group. Let G be the group of rational points of a split reductive algebraic group over a finite field with Weyl group W.

The property in 2.1 comes from the fact that the dimension of an irreducible representation of G divides $|G|$. The bijection in 2.2 is the identity map. The bijection in 2.3(a) is given by external tensor product of unipotent representations.

For the properties in 2.4, see [9, 3.25, 3.26]. The property in 2.5 is stated in [10, no.8].

For $w \in W$, let R_w be the virtual representation of G attached in [5] to the maximal torus of type w and the unit character.

It is reasonable to expect that the non-zero numbers (ρ, R_w) (multiplicities of unipotent representations ρ) are the same up to sign as the degrees of the irreducible characters of $Z(w)$ for good enough w (both sets of numbers have the sum of squares equal to $|Z(w)|$). (Whether such a property holds was a question that C. W. Curtis and J. Bernstein separately asked the author in the early 1980's.) This had been checked in some examples, but not in general; it was reasonable to use it as a heuristic tool (this has been now checked for all regular w in [5], thm. 3.2).

In the setup of 2.6, the multiplicity (ρ, R_w) is given by setting $q = \omega$ in the polynomial in q which gives the dimension of ρ (see [9, 3.30]). This justifies 2.6(a).

The properties in 2.7 appeared for w a Coxeter element in [8, 6.1], where the λ_j were the eigenvalues of Frobenius. The analogous properties for w as in 2.7, were known in exceptional groups (the example of the element of order 24 in the Weyl group of type E_8 is discussed in [12, 11.6]) (these have been now checked in all cases in [6], §2).

The property 2.8(a) appeared in [9, 3.13].

2.10. Since the case where W is a Weyl group is understood, we see using 2.3 that it is enough to define $\mathcal{U}(W)$ and Deg in the case where W is irreducible and non-crystallographic, hence a dihedral group $I_2(p)$ of order $2p$ ($p = 5$ or $p \geq 7$) or a Coxeter group of type H_3 or H_4.

3. Types H_2, H_3, H_4

3.1. We consider the case where W is a dihedral group of order 10; we say that W has type H_2. By 2.4, the subset $\mathcal{U}(W; \emptyset; 1)$ may be identified with the set of irreducible representations (up to isomorphism) of the Hecke algebra $\mathcal{H}(W(\emptyset); 1)$ (1 is the function with constant value 1 on S); since the corresponding formal degrees are known from [6], we see that $\mathcal{U}(W; \emptyset; 1)$ consists of 4 elements, with Deg() given by

$$1\,; \frac{5-\sqrt{5}}{10} q(q+1)\Phi_5'\,; \frac{5+\sqrt{5}}{10} q(q+1)\Phi_5''; q^5\,,$$

respectively.

With the notation of 2.7, we have $\mathcal{T}(W) = \{1, q, q^2, \zeta^2 q, \zeta^{-2} q\}$; the two elements of $\mathcal{U}(W)$ corresponding to $\zeta^2 q, \zeta^{-2} q$ have Deg() given by the same polynomial $\frac{1}{\sqrt{5}} q(q-1)(q^2-1)$, hence are cuspidal. The six objects of $\mathcal{U}(W)$ constructed above exhaust $\mathcal{U}(W)$ by 2.8(a).

3.2. We consider the case where W is of type H_3.

By 2.4, the subset $\mathcal{U}(W; \emptyset; 1)$ may be identified with the set of irreducible representations (up to isomorphism) of the Hecke algebra $\mathcal{H}(W(\emptyset); 1)$ (1 is the function with constant value 1 on S); since the corresponding formal degrees are known from [11], we see that $\mathcal{U}(W;\emptyset;1)$ consists of 10 elements, with the Deg–function given by

$$1;\ \frac{5+\sqrt{5}}{10}q\Phi_3\Phi_6\Phi_5'\Phi_{10}';\ \frac{5-\sqrt{5}}{10}q\Phi_3\Phi_6\Phi_5''\Phi_{10}'';\ q^2\Phi_5\Phi_{10};\ \frac{1}{2}q^3\Phi_2^3\Phi_6\Phi_{10};$$
$$\frac{1}{2}q^3\Phi_2^3\Phi_6\Phi_{10};\ q^5\Phi_5\Phi_{10};\ \frac{5+\sqrt{5}}{10}q^6\Phi_3\Phi_6\Phi_5'\Phi_{10}';\ \frac{5-\sqrt{5}}{10}q^6\Phi_3\Phi_6\Phi_5''\Phi_{10}'';\ q^{15}$$

respectively. With the notation of 2.7, we have

$$\mathcal{T}(W)=\{1,q,q^2,q^3,\zeta^2q,\zeta^2q^2,\zeta^{-2}q,\zeta^{-2}q^2,\sqrt{-1}q^{3/2},-\sqrt{-1}q^{3/2}\};$$

the four elements of $\mathcal{U}(W)$ corresponding to $\zeta^2q,\zeta^2q^2,\zeta^{-2}q,\zeta^{-2}q^2$ have Deg() given by

$$\frac{1}{\sqrt{5}}q\Phi_1^2\Phi_2^2\Phi_3\Phi_6;\ \frac{1}{\sqrt{5}}q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6;\ \frac{1}{\sqrt{5}}q\Phi_1^2\Phi_2^2\Phi_3\Phi_6;\ \frac{1}{\sqrt{5}}q^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$$

respectively; the two elements of $\mathcal{U}(W)$ corresponding to $\sqrt{-1}q^{3/2},-\sqrt{-1}q^{3/2}$ have Deg() given by

$$\frac{1}{2}q^3\Phi_1^3\Phi_3\Phi_5;\ \frac{1}{2}q^3\Phi_1^3\Phi_3\Phi_5$$

respectively (the last two objects are cuspidal). The 16 objects of $\mathcal{U}(W)$ constructed above exhaust $\mathcal{U}(W)$ by 2.8(a).

3.3. We consider the case where W is of type H_4. By 2.4, the subset $\mathcal{U}(W;\emptyset;1)$ may be identified with the set of irreducible representations (up to isomorphism) of the Hecke algebra $\mathcal{H}(W(\emptyset);1)$ (1 is the function with constant value 1 on S); since the corresponding formal degrees are known from [1], we see that $\mathcal{U}(W;\emptyset;1)$ consists of 34 elements, with explicitly known Deg().

Let $S'\subset S$ be such that $\langle S'\rangle$ is of type H_2. The corresponding Coxeter group $W(S')$ (see 1.3) is a dihedral group of order 20, and the function $f:S-S'\to\{1,2,\dots\}$ attached to one of the two cuspidal objects ρ',ρ'' of $\mathcal{U}(\langle S'\rangle)$ takes the values 1 and 5 on the two elements of $S-S'$ (this can be deduced from the analysis of the case H_3). The formal degrees of the simple modules of the corresponding Iwahori–Hecke algebra $\mathcal{H}(W(S');f)$ are known from [6]. Using 2.4, we obtain the classification of objects of $\mathcal{U}(W;S';\rho')\sqcup\mathcal{U}(W;S';\rho'')$. We thus obtain 16 new objects of $\mathcal{U}(W)$ with explicitly known Deg().

With the notation of 2.7, we have

$$\begin{aligned}\mathcal{T}(W)=\{&1,q,q^2,q^3,q^4,\zeta^2q,\zeta^2q^2,\zeta^2q^3,\zeta^{-2}q,\zeta^{-2}q^2,\zeta^{-2}q^3,\\&\sqrt{-1}q^{3/2},\sqrt{-1}q^{5/2},-\sqrt{-1}q^{3/2},-\sqrt{-1}q^{5/2},-\zeta q^2,-\zeta^2q^2,-\zeta^3q^2,-\zeta^4q^2,\\&\theta\zeta q^2,\theta\zeta^{-1}q^2,\theta^2\zeta q^2,\theta^2\zeta^{-1}q^2,\zeta q^2,\zeta^{-1}q^2,\\&-q^2,\theta q^2,\theta^2q^2,-\theta q^2,-\theta^2q^2\};\end{aligned}$$

the four elements of $\mathcal{U}(W)$ corresponding to $\sqrt{-1}q^{3/2}, \sqrt{-1}q^{5/2}, -\sqrt{-1}q^{3/2}, -\sqrt{-1}q^{5/2}$ have Deg() computable as in 2.4 and are seen to be new (non-cuspidal) objects; the 15 elements of $\mathcal{U}(W)$ corresponding to the last 15 elements of $\mathcal{T}(W)$ have Deg() computable as in 2.4 and are seen to be new (cuspidal) objects.

Applying 2.2 (with σ such that $\sigma(\sqrt{5}) = -\sqrt{5}$) to the objects known so far, we obtain 6 new cuspidal objects with explicitly known value of Deg().

Applying property 2.5 ($q \mapsto -q$) to the objects known so far, we obtain further 26 cuspidal objects of $\mathcal{T}(W)$ with explicitly known values of Deg().

We now use 2.7 for an element of order 20 in W with an eigenvalue ω on E. Using the definition and the known values of Deg(), we can check that 18 of the objects of $\mathcal{U}(W)$ known so far belong to $\mathcal{U}(W)_\omega$. Using 2.7(a) and the known values of Deg()–function for these 18 objects, we can determine the set $\lambda_1, \lambda_2, \ldots, \lambda_{20}$ in our case. From this we can compute the value of Deg() for the two missing objects of $\mathcal{U}(W)_\omega$ and we find two new cuspidal objects of $\mathcal{U}(W)$ with explicitly known value of Deg().

We now use 2.6 for a regular element w of order 5 in W. Let ω be an eigenvalue of w on E. The centralizer of w in W is isomorphic to the binary icosahedral group times a cyclic group of order 5; this has 45 irreducible characters. Using the definitions we see that from the objects of $\mathcal{U}(W)$ known so far, 44 belong to $\mathcal{U}(W)_\omega$. By 2.6(a), there is one new object ρ of $\mathcal{U}(W)$ which appears in $\mathcal{U}(W)_\omega$ and we have $\text{Deg}(\rho)|_{q=\omega} = \pm 2$. Using 2.6(b) we can determine $\text{Deg}(\rho)$ and we see that we have found a new cuspidal object of $\mathcal{U}(W)$.

Applying 2.5 ($q \mapsto -q$) to this cuspidal object gives us yet another (cuspidal) object of $\mathcal{U}(W)$ with explicitly known value of Deg() (which differs from that of the previous cuspidal object).

The 104 objects of $\mathcal{U}(W)$ constructed above exhaust $\mathcal{U}(W)$ by 28(a). They are listed (by specifying the values of Deg() on them) in the Table. Note that 50 out of the 104 objects are cuspidal. They all belong to one family consisting of the 74 objects ρ such that the highest power of q dividing $\text{Deg}(\rho)$ is q^6.

4. DIHEDRAL GROUPS

4.1. We now assume that W is a dihedral group of order $2p$ where $p \geq 3$. Let $\xi = e^{2\sqrt{-1}\pi/p}$. We define an integer $k \geq 1$ by $p = 2k+1$ if p is odd and by $p = 2k+2$ if p is even.

By methods similar to those in §3 we see that, if $p = 2k+1$, then $\mathcal{U}(W)$ consists of two objects $1, S$ and of k^2 other objects denoted ρ_i $(1 \leq i \leq k)$ and $\rho_{i,j}, \rho'_{i,j}$ $(1 \leq i < j \leq k)$; if $p = 2k+2$, then $\mathcal{U}(W)$ consists of two objects $1, S$ and of k^2+k+2 other objects denoted ρ_i, ρ'_i $(1 \le i \le k)$, $\rho_{i,j}, \rho'_{i,j}$ $(1 \le i < j \le k)$ and ϵ', ϵ''.

The values of Deg() are given as follows.

$\mathrm{Deg}(1) = 1, \mathrm{Deg}(S) = q^p$,

$$\mathrm{Deg}(\rho_i) = \frac{(1-\xi^i)(1-\xi^{-i})}{p} \frac{q(1-q^2)(1-q^p)}{(1-q)^2(q-\xi^i)(q-\xi^{-i})},$$

$$\mathrm{Deg}(\rho'_i) = \frac{(1+\xi^i)(1+\xi^{-i})}{p} \frac{q(1-q^2)(1-q^p)}{(1+q)^2(q-\xi^i)(q-\xi^{-i})},$$

$$\mathrm{Deg}(\rho_{i,j}) = \mathrm{Deg}(\rho'_{i,j}) = \frac{\xi^i+\xi^{-i}-\xi^j-\xi^{-j}}{p} \frac{q(1-q^2)(1-q^p)}{(q-\xi^i)(q-\xi^{-i})(q-\xi^j)(q-\xi^{-j})},$$

$$\mathrm{Deg}(\epsilon') = \mathrm{Deg}(\epsilon'') = \frac{2}{p} \frac{q(1-q^2)(1-q^p)}{(1-q^2)^2}.$$

4.2. The objectives stated in 1.1 can be extended to include Coxeter groups (W, S) with a given automorphism ψ. We can again seek to define a set $\mathcal{U}(W, \psi)$ with a function Deg on it which should reduce to the case discussed earlier for $\psi = 1$ and which, for $\psi \ne 1$, should mimic as closely as possible the theory of unipotent representations of non-split reductive groups over a finite field. We can again assume that W is irreducible. The only case which is not covered by the known theory is the dihedral group of order $2p$ with a non-trivial ψ.

Let $\xi = e^{2\sqrt{-1}\pi/p}, \tilde{\xi} = e^{\sqrt{-1}\pi/p}$. With the notation of 4.1, we see that, if $p = 2k+1$, then $\mathcal{U}(W, \psi)$ consists of two non-cuspidal objects $1, S$ and of k^2 cuspidal objects denoted ρ_i $(1 \le i \le k)$ and $\rho_{i,j}, \rho'_{i,j}$ $(1 \le i < j \le k)$; if $p = 2k+2$, then $\mathcal{U}(W, \psi)$ consists of two non-cuspidal objects $1, S$ and of k^2+k cuspidal objects denoted $\tilde{\rho}_{i,j}, \tilde{\rho}'_{i,j}$ ($i < j$ odd in the interval $[1, p-1]$).

The values of the Deg–function are given as follows.

$\mathrm{Deg}(1) = 1, \mathrm{Deg}(S) = q^p$,

$$\mathrm{Deg}(\rho_i) = \frac{(1-\xi^i)(1-\xi^{-i})}{p} \frac{q(q^2-1)(q^p+1)}{(q+1)^2(q+\xi^i)(q+\xi^{-i})},$$

$$\mathrm{Deg}(\rho_{i,j}) = \mathrm{Deg}(\rho'_{i,j}) = \frac{\xi^i+\xi^{-i}-\xi^j-\xi^{-j}}{p} \frac{q(q^2-1)(q^p+1)}{(q+\xi^i)(q+\xi^{-i})(q+\xi^j)(q+\xi^{-j})},$$

$$\mathrm{Deg}(\tilde{\rho}_{i,j}) = \mathrm{Deg}(\tilde{\rho}'_{i,j}) = \frac{\tilde{\xi}^i+\tilde{\xi}^{-i}-\tilde{\xi}^j-\tilde{\xi}^{-j}}{p} \frac{q(q^2-1)(q^p+1)}{(q-\tilde{\xi}^i)(q-\tilde{\xi}^{-i})(q-\tilde{\xi}^j)(q-\tilde{\xi}^{-j})}.$$

For $p = 3, 4, 6$ we recover the classification and degrees of unipotent representations of the unitary group in three variables, the Suzuki groups of type ${}^2B_2$ and the Ree groups of type ${}^2G_2$.

Table for H_4.

$$1$$
$$q^4\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$
$$q^5\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$
$$q^{15}\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$
$$q^{16}\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$
$$q^{60}$$

$$\tfrac{5-\sqrt5}{10}\,q\,\Phi_4^2\,\Phi_5'\,\Phi_{10}'\,\Phi_{12}\,\Phi_{15}'\,\Phi_{20}\,\Phi_{30}'$$
$$\tfrac{5+\sqrt5}{10}\,q\,\Phi_4^2\,\Phi_5''\,\Phi_{10}''\,\Phi_{12}\,\Phi_{15}''\,\Phi_{20}\,\Phi_{30}''$$
$$\tfrac{1}{\sqrt5}\,q\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{20}$$
$$\tfrac{1}{\sqrt5}\,q\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{20}$$

$$\tfrac{5-\sqrt5}{10}\,q^2\,\Phi_3^2\,\Phi_5'\,\Phi_6^2\,\Phi_{10}'\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}''\,\Phi_{30}$$
$$\tfrac{5+\sqrt5}{10}\,q^2\,\Phi_3^2\,\Phi_5''\,\Phi_6^2\,\Phi_{10}''\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}'\,\Phi_{30}$$
$$\tfrac{1}{\sqrt5}\,q^2\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{30}$$
$$\tfrac{1}{\sqrt5}\,q^2\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{30}$$

$$\tfrac12\,q^3\,\Phi_2^3\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}\,\Phi_{30}$$
$$\tfrac12\,q^3\,\Phi_2^3\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}\,\Phi_{30}$$
$$\tfrac12\,q^3\,\Phi_1^3\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}$$
$$\tfrac12\,q^3\,\Phi_1^3\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}$$

$$\tfrac12\,q^{18}\,\Phi_2^3\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}\,\Phi_{30}$$
$$\tfrac12\,q^{18}\,\Phi_2^3\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}\,\Phi_{30}$$
$$\tfrac12\,q^{18}\,\Phi_1^3\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}$$
$$\tfrac12\,q^{18}\,\Phi_1^3\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}$$

$$\tfrac{5+\sqrt5}{10}\,q^{22}\,\Phi_3^2\,\Phi_5''\,\Phi_6^2\,\Phi_{10}''\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}'\,\Phi_{30}$$
$$\tfrac{5-\sqrt5}{10}\,q^{22}\,\Phi_3^2\,\Phi_5'\,\Phi_6^2\,\Phi_{10}'\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}''\,\Phi_{30}$$
$$\tfrac{1}{\sqrt5}\,q^{22}\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{30}$$
$$\tfrac{1}{\sqrt5}\,q^{22}\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{30}$$

$$\tfrac{5+\sqrt5}{10}\,q^{31}\,\Phi_4^2\,\Phi_5''\,\Phi_{10}''\,\Phi_{12}\,\Phi_{15}''\,\Phi_{20}\,\Phi_{30}''$$
$$\tfrac{5-\sqrt5}{10}\,q^{31}\,\Phi_4^2\,\Phi_5'\,\Phi_{10}'\,\Phi_{12}\,\Phi_{15}'\,\Phi_{20}\,\Phi_{30}'$$
$$\tfrac{1}{\sqrt5}\,q^{31}\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{20}$$
$$\tfrac{1}{\sqrt5}\,q^{31}\,\Phi_1^2\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{20}$$

$$\frac{1}{10} q^6\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, \Phi_{10}\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{8} q^6\, \Phi_2^4\, \Phi_4^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{10} q^6\, \Phi_2^4\, \Phi_5\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{2-\lambda}{30} q^6\, \Phi_3^2\, \Phi_4^2\, {\Phi''_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi'_{30}$$
$$\frac{1+\lambda}{30} q^6\, \Phi_3^2\, \Phi_4^2\, {\Phi'_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi''_{30}$$
$$\frac{1}{10} q^6\, \Phi_3^2\, \Phi_4^2\, \Phi_5\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{20} q^6\, \Phi_2^4\, \Phi_4^2\, {\Phi''_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi''_{20}\, \Phi_{30}$$
$$\frac{1}{20} q^6\, \Phi_2^4\, \Phi_4^2\, {\Phi'_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi'_{20}\, \Phi_{30}$$
$$\frac{13-8\lambda}{120} q^6\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, {\Phi'_5}^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{5+8\lambda}{120} q^6\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, {\Phi''_5}^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{5-3\lambda}{30} q^6\, \Phi_2^4\, \Phi_3^2\, {\Phi'_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi''_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{2+3\lambda}{30} q^6\, \Phi_2^4\, \Phi_3^2\, {\Phi''_5}^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi'_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1+\lambda}{30} q^6\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi''_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{2-\lambda}{30} q^6\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi'_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{40} q^6\, \Phi_2^4\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{12} q^6\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi'_5\, \Phi_6^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi'_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi'_5\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi'_{15}\, \Phi''_{20}\, \Phi_{30}$$
$$\frac{\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi''_5\, \Phi_6^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi''_{30}$$
$$\frac{\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi''_5\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi''_{15}\, \Phi'_{20}\, \Phi_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi'_5\, \Phi_6^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi'_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi'_5\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi'_{15}\, \Phi''_{20}\, \Phi_{30}$$
$$\frac{\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi''_5\, \Phi_6^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi''_{30}$$
$$\frac{\lambda}{10} q^6\, \Phi_1^2\, \Phi_2^4\, \Phi_3^2\, \Phi_4^2\, \Phi''_5\, \Phi_6^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi''_{15}\, \Phi'_{20}\, \Phi_{30}$$
$$\frac{1}{8} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{10} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_5^2\, \Phi_{10}\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{20} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi''_{20}\, \Phi_{30}$$
$$\frac{1}{20} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi'_{20}\, \Phi_{30}$$
$$\frac{13-8\lambda}{120} q^6\, \Phi_1^4\, \Phi_4^2\, {\Phi''_5}^2\, \Phi_6^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{5+8\lambda}{120} q^6\, \Phi_1^4\, \Phi_4^2\, {\Phi'_5}^2\, \Phi_6^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{5-3\lambda}{30} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_5^2\, \Phi_6^2\, {\Phi'_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi''_{30}$$
$$\frac{2+3\lambda}{30} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_5^2\, \Phi_6^2\, {\Phi''_{10}}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi'_{30}$$
$$\frac{1}{40} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_4^2\, \Phi_{10}^2\, \Phi_{12}\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1}{12} q^6\, \Phi_1^4\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, \Phi_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^4\, \Phi_2^2\, \Phi_3^2\, \Phi_4^2\, {\Phi'_5}^2\, \Phi_6^2\, \Phi'_{10}\, \Phi_{12}\, \Phi'_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^4\, \Phi_2^2\, \Phi_3^2\, \Phi_4^2\, {\Phi'_5}^2\, \Phi_6^2\, \Phi'_{10}\, \Phi_{12}\, \Phi'_{15}\, \Phi_{20}\, \Phi_{30}$$
$$\frac{1-\lambda}{10} q^6\, \Phi_1^4\, \Phi_2^2\, \Phi_3^2\, \Phi_4^2\, \Phi_5^2\, \Phi_6^2\, \Phi'_{10}\, \Phi_{12}\, \Phi_{15}\, \Phi''_{20}\, \Phi'_{30}$$

$$\frac{1-\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi'_{10}\,\Phi_{12}\,\Phi_{15}\,\Phi''_{20}\,\Phi'_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_5''^2\,\Phi_6^2\,\Phi''_{10}\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_5''^2\,\Phi_6^2\,\Phi''_{10}\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi''_{10}\,\Phi_{12}\,\Phi_{15}\,\Phi'_{20}\,\Phi''_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^2\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi''_{10}\,\Phi_{12}\,\Phi_{15}\,\Phi'_{20}\,\Phi''_{30}$$
$$\frac{1}{4}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{30}$$
$$\frac{1}{4}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{30}$$
$$\frac{1}{6}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{20}\,\Phi_{30}$$
$$\frac{1}{6}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{20}\,\Phi_{30}$$
$$\frac{1}{6}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{20}$$
$$\frac{1}{6}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{15}\,\Phi_{20}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi''_{20}\,\Phi''_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi''_{20}\,\Phi''_{30}$$
$$\frac{1-\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi'_{20}\,\Phi'_{30}$$
$$\frac{1-\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{12}\,\Phi_{15}\,\Phi'_{20}\,\Phi'_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi''_{15}\,\Phi''_{20}\,\Phi_{30}$$
$$\frac{\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi''_{15}\,\Phi''_{20}\,\Phi_{30}$$
$$\frac{1-\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi'_{15}\,\Phi'_{20}\,\Phi_{30}$$
$$\frac{1-\lambda}{10}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi'_{15}\,\Phi'_{20}\,\Phi_{30}$$
$$\frac{\sqrt{5}}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi'_{15}\,\Phi_{20}\,\Phi''_{30}$$
$$\frac{\sqrt{5}}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi'_{15}\,\Phi_{20}\,\Phi''_{30}$$
$$\frac{\sqrt{5}}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi'_{30}$$
$$\frac{\sqrt{5}}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_4^2\,\Phi_5^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi'_{30}$$
$$\frac{2-\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5'^2\,\Phi_6^2\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{2-\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5'^2\,\Phi_6^2\,\Phi_{12}\,\Phi''_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{1+\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5''^2\,\Phi_6^2\,\Phi_{12}\,\Phi'_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{1+\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5''^2\,\Phi_6^2\,\Phi_{12}\,\Phi'_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{2-\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}'^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi''_{30}$$
$$\frac{2-\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}'^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi''_{30}$$
$$\frac{1+\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}''^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi'_{30}$$
$$\frac{1+\lambda}{30}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_6^2\,\Phi_{10}''^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi'_{30}$$
$$\frac{\sqrt{5}}{15}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}$$
$$\frac{\sqrt{5}}{15}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}$$
$$\frac{\sqrt{5}}{15}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}$$
$$\frac{\sqrt{5}}{15}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_4^2\,\Phi_5^2\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{20}$$
$$\frac{1}{60}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_3^2\,\Phi_5^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$
$$\frac{1}{60}q^6\,\Phi_1^4\,\Phi_2^4\,\Phi_6^2\,\Phi_{10}^2\,\Phi_{12}\,\Phi_{15}\,\Phi_{20}\,\Phi_{30}$$

References

[1] D. Alvis and G. Lusztig, *The representations and generic degrees of the Hecke algebras of type H_4*, J. für reine und angew. Math. **336** (1982), 201–212.

[2] N. Bourbaki, *Goupes et algèbres de Lie, Ch. 4,5 et 6*, Hermann, 1968.

[3] M. Broué, G. Malle and J. Michel, *Generic blocks of finite reductive groups*, This issue (1993), 7–91.

[4] M. Broué and G. Malle, *Zyklotomische Heckealgebren*, This issue (1993), 119–191.

[5] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. **103** (1976), 103–161.

[6] W. Feit and G. Higman, *The non-existence of generalized polygons*, J. Algebra **1** (1964), 114–131.

[7] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[8] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **38** (1976), 101–159.

[9] ———, *Representations of finite Chevalley groups*, Regional Conf. Series in Math. 39, Amer. Math. Soc., 1978.

[10] ———, *Unipotent representations of a finite Chevalley group of type E_8*, Quart. J. Math. Oxford **30** (1979), 315–338.

[11] ———, *A class of irreducible representations of a Weyl group II*, Proc. Kon. Nederl. Akad. (A) **85** (1982), 219–226.

[12] ———, *Characters of reductive groups over a finite field*, Ann. Math. Studies 107, Princeton U. Press, 1984.

[13] T. A. Springer, *Regular elements of finite reflection groups*, Invent. Math. **25** (1974), 159–193.

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