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On the Spectrum of Gauge-Periodic Elliptic Operators

Jochen Brüning and Toshikazu Sunada

1. Introduction

This note presents an extension of the results in [1] concerning the spectrum of symmetric elliptic operators on complete noncompact Riemannian manifolds. Thus consider a complete Riemannian manifold, M, of dimension m, with a properly discontinuous action of a discrete group, Γ, of isometries; we assume that the orbit space is compact. Moreover, let $E \to M$ be a hermitian vector bundle with a unitary representation

$$U : \Gamma \to L^2(E)\,. \tag{1.1a}$$

More precisely, we assume that Γ acts unitarily on E, via γ_*, and put

$$U_\gamma f(p) := \gamma_* f(\gamma^{-1}(p))\,. \tag{1.1b}$$

Thus each U_γ maps $C_0^\infty(E)$ to itself. Finally, let D be a symmetric elliptic differential operator on $C_0^\infty(E)$. In [1] we have assumed that D is, in addition, periodic in the sense that it commutes with all U_γ on $C_0^\infty(E)$. Now we bring in a second unitary representation, the *gauge*,

$$\begin{aligned} &V : \Gamma \to C^\infty(\operatorname{End} E)\,,\\ &V_\gamma \mid E_p \text{ is unitary for all } \gamma \in \Gamma,\, p \in M, \end{aligned} \tag{1.2}$$

which induces a unitary representation on $L^2(E)$. This representation will also be denoted by V. In general,

$$W_\gamma := V_\gamma U_\gamma \tag{1.3}$$

will not define a representation any more, since $[V_{\gamma_1}, U_{\gamma_2}]$ maybe nonzero. But frequently we have a good substitute namely

$$U_{\gamma_1} V_{\gamma_2} = X(\gamma_1, \gamma_2) V_{\gamma_2} U_{\gamma_1}\,, \tag{1.4a}$$

where

$$X(\gamma_1, \gamma_2) \text{ is in } C^\infty(\operatorname{End} E), \text{ unitary on each fiber, and} \\ \text{a character of } \Gamma \text{ in each variable separately.} \tag{1.4b}$$

Moreover, we want that

$$X(\gamma, \gamma) = 1 \quad \text{for all } \gamma \in \Gamma\,. \tag{1.4c}$$

The operator D is called *gauge-periodic* if

$$[W_\gamma, D] = 0 \quad \text{on } C_0^\infty(E)\,. \tag{1.5}$$

The periodic case is obviously contained with V, X trivial. An interesting example with nontrivial gauge is provided by the Schrödinger operator with constant magnetic field in $\mathbb{R}^2$. This will be our main application which we deal with in greater detail below.

Assuming (1.5) we associate a C^*-algebra with D as follows. Fix a fundamental domain, $\mathcal{D}$, for Γ and introduce the isometry

$$\begin{aligned} &\Phi : L^2(E) \to L^2(\Gamma, L^2(E \mid \mathcal{D}))\,, \\ &\Phi f(\gamma) := r_{\mathcal{D}} \circ W_\gamma(f)\,, \end{aligned} \tag{1.6}$$

where $r_{\mathcal{D}}$ denotes restriction $L^2(E) \to L^2(E \mid \mathcal{D}) =: H$. Let R_γ, L_γ be right translation by γ and left translation by γ^{-1} in $L^2(\Gamma)$, respectively, and define the unitary operator X_γ in $L^2(\Gamma)$ for $\gamma \in \Gamma$ by

$$X_\gamma \sigma(\delta) := X(\delta, \gamma)\sigma(\delta)\,. \tag{1.7}$$

Then it is easy to compute that

$$\tilde{R}_\gamma := \Phi W_\gamma \Phi^{-1} = X_\gamma R_\gamma \otimes I\,. \tag{1.8}$$

Since X is a bicharacter, it is also readily seen that

$$[X_{\gamma_1} L_{\gamma_1} \otimes I, \tilde{R}_{\gamma_2}] = 0 \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma\,. \tag{1.9}$$

We will see that this is satisfied in our main example (and probably in many other cases). Then we abbreviate $\tilde{L}_\gamma =: X_\gamma L_\gamma$ and introduce the C^*-algebra $\mathcal{C}_W(\Gamma)$ which is generated by $(\tilde{L}_\gamma)_{\gamma \in \Gamma}$ in $\mathcal{L}(L^2(\Gamma))$. With $\mathcal{K} = \mathcal{K}(H)$, the ideal of compact operators on $H = L^2(E \mid \mathcal{D})$, we introduce, as in [1],

$$\mathcal{C}_W(\Gamma, \mathcal{K}) := \mathcal{C}_W(\Gamma) \otimes \mathcal{K}\,. \tag{1.10}$$

On this algebra we can again define a natural trace tr_Γ (to be described in Sec. 3), such that all spectral projections of D have a finite trace. We say that $\mathcal{C}_W(\Gamma, \mathcal{K})$ has the *Kadison property* if there is a constant $C > 0$ such that

$$\mathrm{tr}_\Gamma P \geq C\,, \tag{1.11}$$

for all nonzero orthogonal projections $P \in \mathcal{C}_W(\Gamma, \mathcal{K})$. The largest constant in (1.11) will be called the *Kadison constant* of $\mathcal{C}_W(\Gamma, \mathcal{K})$, to be denoted $C_W(\Gamma)$.

We can show that D has a unique self-adjoint extension, $\bar{D}$, with spectral resolution

$$\bar{D} = \int_{-\infty}^{+\infty} \lambda dE_\lambda\,.$$

Quite analogously to [1] we then obtain

Theorem 1 *If $\lambda_1 > \lambda_2 \in \mathbb{R}\backslash \mathrm{spec}\, \bar{D}$ then $E_{\lambda_1} - E_{\lambda_2} \in \mathcal{C}_W(\Gamma, \mathcal{K})$. If $\mathcal{C}_W(\Gamma)$ has the Kadison property then the spectrum of $\bar{D}$ has band structure in the sense that the intersection of the resolvent set with any compact interval of real numbers has finitely many components.*

As noted in [1], the proof of Theorem 1 gives some quantitive information which we exploit in connection with the magnetic Schrödinger operator in $\mathbb{R}^2$. Recall that this operator is defined on $C_0^\infty(\mathbb{R}^2)$ by

$$D_A := \sum_{i=1}^{2} \Big(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_i} + a_i\Big)^2 + v\,, \tag{1.12}$$

where $a_i, v \in C^\infty(\mathbb{R}^2)$. The magnetic field is assumed to be constant,

$$b(x) := \Big(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\Big)(x) \equiv b(0) =: b\,,$$

and we assume moreover that v is $\mathbb{Z}^2$-periodic. b is also equal to the magnetic flux over a unit cell,

$$b = \int_{0 \leq x_1, x_2 \leq 1} b(x_1, x_2) dx_1 dx_2 =: 2\pi\theta\,. \tag{1.13}$$

This operator fits into our framework as follows. Since the magnetic field is constant we may assume that

$$a_1(x) = bx_2/2\,, \quad a_2(x) = -bx_1/2\,.$$

With ω the standard symplectic form in $\mathbb{R}^2$, we define for $z \in \mathbb{Z}^2$

$$\begin{aligned} U_z f(x) &:= f(x - z)\,, \\ V_z f(x) &:= e^{\sqrt{-1}\, b/2\, \omega(x,z)} f(x)\,. \end{aligned} \tag{1.14}$$

Then it follows that

$$X(z_1, z_2) = e^{\sqrt{-1}\, b/2\, \omega(z_1,z_2)} \tag{1.15}$$

and

$$\tilde{L}_{z_1}\tilde{L}_{z_2} = e^{\sqrt{-1}\, b\, \omega(z_2,z_1)}\tilde{L}_{z_2}\tilde{L}_{z_1}\,. \tag{1.16}$$

Now we regard the quantity b as a parameter restricted by $|b| \leq C_1$, say. It is known that the precise band structure of spec $\bar{D}_A$ in a given interval $[\lambda_1, \lambda_2]$ depends rather subtly on the arithmetic nature of θ in (1.13). We will prove

Theorem 2 *Assume that $\theta = p/q \in \mathbb{Q}$ with $(p,q) = 1$, and that $\lambda_1 > \lambda_2 \in \mathbb{R} \setminus$ spec $\bar{D}_A$. There is a constant C depending only on C_1, λ_1, λ_2, and v such that*

$$G(D_A, \lambda_1, \lambda_2) := \sharp\{\ \textit{gaps in}\ \operatorname{spec} \bar{D}_A \cap [\lambda_2, \lambda_1]\}$$

satisfies the estimate

$$G(D_A, \lambda_1, \lambda_2) \leq C(C_1, \lambda_1, \lambda_2, v)\, q\,. \tag{1.17}$$

The proof of this result uses the fact that the Kadison constant of $\mathcal{C}_W(\Gamma)$ satisfies $C_W(\Gamma) \geq q^{-1}$. This degeneration then allows the possible development of Cantor structures if θ approaches irrational numbers. It has been shown in [3] that, for suitable v, G also has a similar lower bound. Crucial for this result was a thorough study of Harper's equation, a discrete approximation to D_A. Using only the structure of the rotation algebras (which are brought in by (1.16)) it has been shown in [2] that the maximum number of gaps is realized by Harper's operator. One might thus hope that our approach, which links all gauge-periodic operators with the rotation algebra, opens a way to bypass the discrete approximation and to establish directly that "sufficiently complicated" operators in the rational rotation algebra will indeed have the maximum number of gaps. Of course, this need not be so for every operator as illustrated by the case $v = 0$. Since $C_W(\Gamma) = 0$ for irrational θ, we also see that for a vanishing Kadison constant no general conclusion concerning the structure of the spectrum is possible.

We are indebted to Victor Guillemin and Johannes Sjöstrand for some enlightening discussions.

2. Parametrix construction

We follow essentially the outline of [1, Sec. 2]. Since Γ acts properly discontinuously the sets

$$\Gamma(K) := \{\gamma \in \Gamma \mid K \cap \gamma K \neq \emptyset\} \tag{2.1}$$

are finite for all compact $K \subset M$. It follows that for any sequence $(u_\gamma)_{\gamma\in\Gamma} \subset L^2(E)$ with $\operatorname{supp} u_\gamma \subset \gamma K$ for some compact K and all γ, $u := \sum_{\gamma\in\Gamma} u_\gamma$ is well defined. Moreover, we find the norm estimate

$$\|u\|^2_{L^2(E)} \le \sharp\Gamma(K) \sum_{\gamma\in\Gamma} \|u_\gamma\|^2_{L^2(E)} \,. \tag{2.2}$$

In particular, the convergence of the right hand side implies $u \in L^2(E)$.

On the other hand, if $\psi_1, \psi_2 \in C_0^\infty(M)$ and $B \in \mathcal{L}(L^2(E))$ and if we put, for $u \in L^2(E))$,

$$B_\gamma u := W_\gamma \psi_1 B \psi_2 W_\gamma^* u =: \psi_{1\gamma} W_\gamma B W_\gamma^* \psi_{2\gamma} u \,, \tag{2.3}$$

then we can easily prove the estimate

$$\sum_\gamma \|B_\gamma u\|^2_{L^2(E)} \le (\sup_M \psi_1^2)(\sup_M \psi_2^2)\sharp\Gamma(\bar{\mathcal{D}} \cup \operatorname{supp}\, \psi_1 \cup \operatorname{supp}\, \psi_2)\|B\|^2\|u\|^2_{L^2(E)} \,. \tag{2.4}$$

Now consider a gauge-periodic operator D with domain $C_0^\infty(E)$ in $L^2(E)$. To show that D is essentially self-adjoint we consider $u \in L^2(E)$ with

$$D^* u = \sqrt{-1} u \,.$$

Since $u \in H^\rho_{\mathrm{loc}}(E)$, $\rho := \operatorname{ord} D$, by elliptic regularity we have $\psi_\gamma u \in H^\rho_0(E)$ for $\psi \in C_0^\infty(M)$. Now pick ψ such that

$$\sum_{\gamma\in\Gamma} \psi_\gamma = 1 \,. \tag{2.5}$$

Then we compute

$$\begin{aligned} 0 &= (u, (D^* - \sqrt{-1})u) \\ &= \sum_{\gamma,\gamma'} (\psi_\gamma u, D\psi_{\gamma'} u) + \sqrt{-1}\|u\|^2 \\ &= \overline{\sum_{\gamma,\gamma'} (\psi_\gamma u, D\psi_{\gamma'} u)} + \sqrt{-1}\|u\|^2 \,, \end{aligned}$$

which implies $u = 0$, as desired.

We identify D with its closure in $L^2(E)$. Assuming next $D \geq 0$ and $\rho > m$ we construct a local paramatrix for the heat operator $\partial_t + D$ as in [1, Lemma 1]. Using the same notation, we define the global fundamental solution by

$$\mathcal{F}_t u := \sum_{\gamma \in \Gamma} W_\gamma \varphi F_t \psi W_\gamma^* u \,, \tag{2.6}$$

and the remainder term by

$$\mathcal{R}_t u := \sum_{\gamma \in \Gamma} W_\gamma (\partial_t + D) \varphi F_t \psi W_\gamma^* u \,, \tag{2.7}$$

where ψ satisfies (2.5) and $\varphi \in C_0^\infty(M)$ equals 1 near supp ψ. Going through the proof of [1, Lemma 2] we obtain the analogous result (using (2.2) and (2.4)) i.e.

Lemma 1 *Fix $T > 0$.*
1) Uniformly in $t \in (0, T]$, we have

$$\|\mathcal{F}_t\|_{L^2(E)} + \|\mathcal{R}_t\|_{L^2(E)} \leq C_2 \,.$$

2) For $u \in L^2(E)$, the functions $\mathcal{F}_t u$ and $\mathcal{R}_t u$ are continuous in $(0, T]$ with

$$\lim_{t \to 0} \mathcal{F}_t u = u \,.$$

3) $\mathcal{F} u$ is differentiable in $(0, T]$, has values in $\mathcal{D}(D)$, and satisfies the equation

$$(\partial_t + D) \mathcal{F}_t u = \mathcal{R}_t u \,.$$

As in loc. cit. we can now derive the Neumann series

$$\exp(-tD) = \sum_{j=0}^{\infty} (-1)^j (\mathcal{F} *^j \mathcal{R})_t \,, \tag{2.8}$$

where

$$(\mathcal{F} *^0 \mathcal{R})_t := \mathcal{F}_t \,, \quad (\mathcal{F} *^{j+1} \mathcal{R})_t = \int_0^t (\mathcal{F} *^j \mathcal{R})_{t-u} \mathcal{R}_u du \,. \tag{2.9}$$

The kernel estimates in [1, Lemma 1] then lead, as before, to the following result.

Lemma 2 *$\exp(-tD)$ has a smooth kernel (with respect to the given L^2-structures),*

$$K_t(p,q) \in E_p \otimes E_q^*, \quad t > 0, \; (p,q) \in M \times M .$$

This kernel satisfies the estimate

$$|K_t(p,q)|_{E_p \otimes E_q^*} \leq C_3 t^{-m/\rho} \exp(-C_4 d_M(p,q)^{\rho/(\rho-1)} t^{-1/(\rho-1)}), \tag{2.10}$$

uniformly in $(0,T] \times M \times M$; here d_M denotes the Riemannian distance.

Moreover, as $t \searrow 0$ we have the asymptotic relation

$$\mathrm{tr}_{E_p} K_t(p,p) \sim t^{-m/\rho} A(p), \tag{2.11}$$

with an explicitly computable function $A(p)$ (cf. [1, (0.1)]).

3. C^*-algebras

Following the outline of [1] further, we have to introduce the trace tr_Γ on $\mathcal{C}_W(\Gamma, \mathcal{K})$, defined in (1.10). To do so we introduce the commutant of $(\tilde{\mathcal{R}}_\gamma)_{\gamma \in \Gamma}$,

$$\mathcal{M}_W(\Gamma) := \{A \in \mathcal{L}(L^2(\Gamma, H)) \mid [A, \tilde{\mathcal{R}}_\gamma] = 0 \text{ for } \gamma \in \Gamma\}. \tag{3.1}$$

Then we define the Fourier coefficients of $A \in \mathcal{M}_W(\Gamma)$ by

$$\hat{A}(\gamma)(v) := \tilde{\mathcal{R}}_\gamma A(\delta_1^v)(1), \tag{3.2}$$

where $\gamma \in \Gamma$, $v \in H$ (such that $\hat{A}(\gamma) \in \mathcal{L}(H)$), and $\delta_1^v \in L^2(\Gamma, H)$ is given by

$$\delta_1^v(\gamma) = \begin{cases} v, & \gamma = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

The following properties are easily checked.

Lemma 3 *1) For $\gamma \in \Gamma$, $K \in \mathcal{K}(H)$ we have*

$$\widehat{\tilde{L}_\gamma \otimes K}(\gamma') = \begin{cases} K, & \gamma = \gamma'. \\ 0 & \textit{otherwise.} \end{cases}$$

2) For $A \in \mathcal{M}_W(\Gamma)$ and $\tau \in L^2(\Gamma, H)$, $\gamma \in \Gamma$,

$$A\tau(\gamma) = \sum_{\gamma' \in \Gamma} X(\gamma', \gamma) \hat{A}(\gamma\gamma'^{-1})(\tau(\gamma')) .$$

3) For $A \in \mathcal{M}_W(\Gamma)$, $\gamma \in \Gamma$

$$\widehat{A^*}(\gamma) = (\hat{A}(\gamma^{-1}))^* .$$

4) For $A, B \in \mathcal{M}_W(\Gamma)$, $\gamma \in \Gamma$

$$\widehat{AB}(\gamma) = \sum_{\gamma'} X(\gamma', \gamma)\hat{A}(\gamma\gamma'^{-1})\hat{B}(\gamma') ,$$

in particular

$$\widehat{A^*A}(1) = \sum_{\gamma} \hat{A}(\gamma)^* \hat{A}(\gamma) .$$

5) For $A \in \mathcal{M}_W(\Gamma)$ we have

$$\|A\| \le \sum_{\gamma\in\Gamma} \|\hat{A}(\gamma)\| .$$

Proof Properties 1) through 4) follow by straightforward computations.
5) follows from the arguments in [1, Lemma 3]. □
Thus we arrive at the crucial

Lemma 4 *If D is a gauge-periodic symmetric elliptic differential operator in $L^2(E)$, of even order $\rho > m$, then*

$$e^{-D} \in \mathcal{C}_W(\Gamma, \mathcal{K}) .$$

Proof We have $e^{-D} \in \mathcal{M}_W(\Gamma)$ by assumption, and it is easily computed that for $v \in H = L^2(E \mid \mathcal{D})$, $p \in \mathcal{D}$, and $A := \Phi e^{-D} \Phi^{-1}$ one has

$$\hat{A}(\gamma)(v)(p) = \int_{\mathcal{D}} e^{-D}(p, \gamma q)\gamma_* v(q) d\,\mathrm{vol}_M(q) . \tag{3.4}$$

Thus all Fourier coefficients are compact.

In Γ we introduce the minimal word length with respect to a fixed finite set of generators; this defines a translation invariant metric, d_Γ, on $\Gamma \times \Gamma$. Then there is a constant, C_5, such that

$$d_\Gamma(\gamma_1, \gamma_2) \le C_5 (\inf_{p,q\in\mathcal{D}} d_M(\gamma_1 p, \gamma_2 q) + 1) . \tag{3.5}$$

Now we put $r(\gamma) := d_\Gamma(\gamma, 1)$ and observe that

$$\sharp\{\gamma \in \Gamma \mid r(\gamma) \le R\} \le C_6 e^{C_7 R} .$$

Then the estimate (2.10) implies that

$$\sum_{\gamma\in\Gamma} \|\hat{A}(\gamma)\| < \infty .$$

Combining this with Lemma 3, 1) and 5), we reach the desired conclusion. □

It remains to define tr_Γ. We put, for $A \in \mathcal{M}_W(\Gamma)^+ :=$ the positive part of $\mathcal{M}_W(\Gamma)$,

$$\mathrm{tr}_\Gamma A := \mathrm{tr}_H \hat{A}(1) . \tag{3.6}$$

It follows from Lemma 3, 4) that tr_Γ is a faithful trace on $\mathcal{M}_W(\Gamma)$, hence on $\mathcal{C}_W(\Gamma, \mathcal{K})$. Now if $\lambda_1 > \lambda_2$ are real numbers, and D is a gauge-periodic symmetric elliptic operator with spectral resolution $D = \int_{-\infty}^{+\infty} \lambda dE_\lambda$, then $E_{\lambda_1} - E_{\lambda_2}$ is an integral operator with smooth kernel [4]. It follows as in (3.4) that

$$\mathrm{tr}_\Gamma(E_{\lambda_1} - E_{\lambda_2}) = \int_{\mathcal{D}} \mathrm{tr}_{E_p}(E_{\lambda_1} - E_{\lambda_2})(p,p) d\,\mathrm{vol}_M(p) . \tag{3.7}$$

Thus we arrive at

Lemma 5 *For any gauge-periodic symmetric elliptic operator D and real numbers $\lambda_1 > \lambda_2$ we have an estimate*

$$0 \le \mathrm{tr}_\Gamma(E_{\lambda_1} - E_{\lambda_2}) \le C(\lambda_1, \lambda_2, D) . \tag{3.8}$$

The dependence on D is only through the coefficients and their derivatives in an arbitrary neighborhood of $\bar{\mathcal{D}}$.

4. Proof of Theorem 1 and Theorem 2

The proof of Theorem 1 now follows from Lemmas 4 and 5, precisely as the proof of [1, Theorem 1].

Proof of Theorem 2 Fix $\lambda_1 > \lambda_2$ not in spec D_A, and restrict the magnetic field to $|b| \le C_1$. Then we obtain from Lemma 5

$$\mathrm{tr}_\Gamma(E_{\lambda_1} - E_{\lambda_2}) \le C(\lambda_1, \lambda_2, C_1) . \tag{3.9}$$

The theorem will thus be proved if we can show that

$$C_W(\Gamma) \ge 1/q , \tag{3.10}$$

if $b = 2\pi\theta$ and $\theta = p/q$, $(p,q) = 1$. To prove (3.10) we introduce the (universal) rotation algebra $\mathcal{A}_\theta$, with generators u, v satisfying

$$vu = e^{2\pi\sqrt{-1}\,\theta} uv =: \mu\, uv . \tag{3.11}$$

Recall that $\mathcal{A}_\theta$ admits a canonical action of $T^2 = S^1 \times S^1 \ni (w_1, w_2) =: w \mapsto \alpha_w \in \operatorname{Aut} \mathcal{A}_\theta$ such that $\alpha_w u = w_1 u$, $\alpha_w v = w_2 v$. We will also need a distinguished irreducible representation,

$$\begin{aligned} &\pi : \mathcal{A}_\theta \to M(q, \mathbf{C})\,, \\ &\pi(u) = \operatorname{diag}(1, \mu, \ldots, \mu^{q-1}),\ \pi(v) \text{ cyclic permutation of} \\ &\text{the standard basis.} \end{aligned} \tag{3.12}$$

Finally, denote by $\varphi : \mathcal{A}_\theta \times \mathcal{K} \to \mathcal{C}_W(\Gamma)$ the representation sending $u \otimes K$ to $\tilde{L}_{e_1} \otimes K$ and $v \otimes K$ to $\tilde{L}_{e_2} \otimes K$. Then we claim that for all $A \in \mathcal{A}_\theta \otimes K$ with $\operatorname{tr}_\Gamma \varphi(A)$ finite we have

$$\operatorname{tr}_\Gamma \varphi(A) = q^{-1} \int_{T^2} \operatorname{tr}_{\mathbf{C}^q \otimes H} (\pi \otimes I \circ \alpha_w \otimes I)(A) dw\,, \tag{3.13}$$

where dw is normalized Haar measure on T^2. To prove (3.13) we only have to observe that

$$\widehat{\varphi(A)}(0,0) = \int_{T^2} (\alpha_w \otimes I)(A) dw\,.$$

Since φ is an isomorphism, (3.13) implies (3.10) and the proof is complete. □

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