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**ON NONLINEAR SCATTERING OF STATES
WHICH ARE CLOSE TO A SOLITON**

V.S.Buslaev and G.S.Perelman

1 Solitons

Consider the nonlinear Schroedinger equation

$$(1.1) \quad i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi, \psi = \psi(x, t) \in \mathbf{C},$$

$x, t \in \mathbf{R}$. Assume that

- i) F is a given smooth ($\in C^\infty$) real function bounded from below,
- ii) the point $\xi = 0$ is a (sufficiently strong) root of the function F :

$$(1.2) \quad F(\xi) = F_1 \xi^p (1 + O(\xi)), p > 0.$$

Further consider the function

$$(1.3) \quad U(\phi, \alpha) = -\frac{1}{8}\alpha^2\phi^2 - \frac{1}{2}\int_0^{\phi^2} F(\xi)d\xi.$$

If $\alpha \neq 0$ this function is negative for sufficiently small ϕ . The next assumption on F will be given in a slightly implicit, but absolutely elementary form:

- iii) for α from some interval, $\alpha \in A \subset \mathbf{R}_+$, the function $\phi \rightarrow U(\phi, \alpha)$ has a positive root; if $\phi_0 (= \phi_0(\alpha))$ is the smallest positive root then $U_\phi(\phi_0, \alpha) > 0$.

Under all these assumptions there exists the unique even positive solution $y \rightarrow \phi(y)$ of the equation

$$(1.4) \quad \phi_{yy} = -U_\phi = \frac{1}{4}\alpha^2\phi + F(\phi^2)\phi$$

vanishing at infinity. More precisely

$$(1.5) \quad \phi = \phi(y|\alpha) \sim \phi_\infty \exp(-\frac{1}{2}\alpha|y|), y \rightarrow \infty.$$

The following functions of x can be called the *soliton states*:

$$(1.6) \quad w(x|\sigma) = \exp(-i\beta + i\frac{1}{2}vx)\phi(x - b|\alpha),$$

here

$$(1.7) \quad \sigma = (\beta, \omega, b, v), \omega = \frac{1}{4}(v^2 - \alpha^2),$$

$\beta, \omega, b, v \in \mathbf{R}, \alpha \in A$. The set of the allowable σ will be denoted by Σ . If σ is a solution of the Hamiltonian system:

$$(1.8) \quad \beta' = \omega, \omega' = 0, b' = v, v' = 0.$$

the function $w(x|\sigma(t))$ is a solution of the equation (1.1) called the *soliton*.

2 The linearization of equation (1.1)

Consider the linearization of the equation (1.1) on the soliton $w(x|\sigma(t))$:

$$(2.1) \quad i\chi_t = -\chi_{xx} + F(|w|^2)\chi + F'(|w|^2)w(\bar{w}\chi + w\bar{\chi}).$$

Instead of χ introduce the function f :

$$(2.2) \quad \chi(x, t) = \exp(i\Phi)f(y, t), \Phi = -\beta(t) + \frac{1}{2}vx, y = x - b(t).$$

The function f obeys the following equation:

$$(2.3) \quad if_t = L(\alpha)f,$$

where

$$(2.4) \quad L(\alpha)f = -f_{yy} + \frac{1}{4}\alpha^2f + F(\phi^2)f + F'(\phi^2)\phi^2(f + \bar{f}),$$

$\phi = \phi(y|\alpha)$. Equation (2.3) is only a real-linear equation. Introduce its complexification:

$$(2.5) \quad if_t = H(\alpha)\vec{f}, \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix},$$

$$(2.6) \quad H(\alpha) = H_0(\alpha) + V(\alpha), H_0(\alpha) = (-\partial_y^2 + \frac{1}{4}\alpha^2)\sigma_3,$$

$$(2.7) \quad V(\alpha) = [F(\phi^2) + F'(\phi^2)\phi^2]\sigma_3 + iF'(\phi^2)\phi^2\sigma_2,$$

σ_2, σ_3 are the standard Pauli matrices:

$$(2.8) \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3 Properties of the operator $H(\alpha)$

The operator $H(\alpha)$ can be treated as a linear operator in $\mathbf{L}_2(\mathbf{R} \rightarrow \mathbf{C}^2)$. Define it on the domain where $H_0(\alpha)$ is self-adjoint. It possesses the properties:

$$(3.1) \quad \sigma_3 H = H^* \sigma_3, \sigma_2 H = -H^* \sigma_2, \sigma_1 H = -H \sigma_1.$$

As a result the spectrum of H is invariant with respect to the following transformations: $E \rightarrow \bar{E}, E \rightarrow -E$.

The continuous spectrum consists of two half-axis $[E_0, \infty)$ and $(-\infty, -E_0]$, $E_0 = \frac{1}{4}\alpha^2$. Its multiplicity is equal to 2.

Owing to the exponential decay of the potential term $V(\alpha)$ at infinity the discrete spectrum of $H(\alpha)$ contains only a finite number of eigenvalues and the corresponding *root subspaces* have only finite dimension.

The point $E = 0$ is always a point of the discrete spectrum. One can indicate two *eigenfunctions*

$$(3.2) \quad \vec{\xi}_1 = \begin{pmatrix} u_1 \\ \bar{u}_1 \end{pmatrix}, \vec{\xi}_3 = \begin{pmatrix} u_3 \\ \bar{u}_3 \end{pmatrix},$$

where

$$(3.3) \quad u_1 = -i\phi(y|\alpha), u_3 = -\phi_y,$$

and two *adjoint functions*:

$$(3.4) \quad \vec{\xi}_2 = \begin{pmatrix} u_2 \\ \bar{u}_2 \end{pmatrix}, \vec{\xi}_4 = \begin{pmatrix} u_4 \\ \bar{u}_4 \end{pmatrix},$$

where

$$(3.5) \quad u_2 = -\frac{2}{\alpha}\phi_\alpha, u_4 = \frac{i}{2}y\phi.$$

They obey the relations:

$$(3.6) \quad H\vec{\xi}_1 = H\vec{\xi}_3 = 0, H\vec{\xi}_2 = i\vec{\xi}_1, H\vec{\xi}_4 = i\vec{\xi}_3.$$

Actually, the spectrum of $H(\alpha)$ can lie only in the real axis and in the imaginary axis of the E -plane, see [We2], for example. It is known also that the spectrum of $H(\alpha)$ is real and the root subspace corresponding to the point $E = 0$ is generated by the vectors $\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3, \vec{\xi}_4$ if and only if

$$(3.7) \quad \partial_\alpha \|\phi\|^2 > 0.$$

Consider the resolvent $R(E) = (H - E)^{-1}$. Its kernel $R(y, y'|E)$ is an analytic function in the *extended* E -plane: it admits an analytic continuation through the continuous spectrum as a meromorphic function. The resolvent kernel goes to infinity when E tends to the branch points $\mp E_0$ if the equation $H(\alpha)\psi = \mp E_0\psi$, treated as a differential equation, has nontrivial solutions bounded at infinity. In this case the points $\mp E_0$ will be called *resonances*.

4 Nonlinear equation

Consider the Cauchy problem for equation (1.1) with the initial data

$$(4.1) \quad \psi(x, 0) = \psi_0(x),$$

where $\psi_0 \in H^1, H^1$ is the standard Sobolev space with the norm:

$$(4.2) \quad \|f\|_{H^1}^2 = \|f\|_2^2 + \|f'\|_2^2.$$

The problem has a solution $\psi = \psi(x, t)$ which belongs to H^1 with respect to x for each t , moreover $\psi \in C(\mathbf{R} \rightarrow H^1)$. Any such solution ψ obeys two *conservation laws*:

$$(4.3) \quad \int |\psi(x, t)|^2 dx = \text{const}, \int [|\psi_x(x, t)|^2 + U(|\psi(x, t)|)] dx = \text{const},$$

where U is the function (1.3). The second formula (4.3) leads to the following estimate:

$$(4.4) \quad \|\psi(\cdot, t)\|_{H^1} \leq c(\|\psi_0\|_{H^1}) \|\psi_0\|_{H^1},$$

here $c = \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is a smooth function. If in addition ψ_0 has the finite norm: $\|(1 + |x|)\psi_0\|_2 < \infty$, the solution ψ also has the finite, but growing in time, similar norm:

$$(4.5) \quad \|(1 + |x|)\psi(x, t)\|_2 \leq c(\|\psi_0\|_{H^1}) [\|(1 + |x|)\psi\|_2 + t\|\psi_0\|_{H^1}].$$

5 Theorem

Let $\sigma_0 = (\beta_0, \omega_0, b_0, v_0) \in \Sigma, \omega_0 = \frac{1}{4}(v_0^2 - \alpha_0^2)$. Consider the Cauchy problem for equation (1.1) with the initial data:

$$(5.1) \quad \psi_0(x) = w(x|\sigma_0) + \chi_0(x).$$

Our aim is to describe the asymptotic behavior of the solution ψ as $t \rightarrow \infty$. Assume that:

T_1) the norm

$$(5.2) \quad N = \|(1 + x^2)\chi_0\|_2 + \|\chi_0'\|_2$$

is sufficiently small;

T_2) $E = 0$ is the only point of the discrete spectrum of $H(\alpha_0)$ and the dimension of the corresponding root subspace is equal to 4;

T_3)the points $\mp E_0$ are not resonances;

T_4)the function F is a polynomial¹ and $p \geq 4$.

Then there exist $\sigma_+ \in \Sigma$ and $f_+ \in \mathbf{L}_2 \cap \mathbf{L}_\infty$ such that

$$(5.3) \quad \psi = w(\cdot, \sigma_+(t)) + \exp(-il_0 t) + o(1)$$

as $t \rightarrow \infty$. In this formula: $\sigma_+(t)$ is the trajectory of the system (1.8) with the initial data $\sigma_+(0) = \sigma_+$; $l_0 = -\partial_x^2$; $o(1)$ is meant \mathbf{L}_2 -norm. Moreover σ_+ in (5-3) is sufficiently close to σ_0 and f_+ is sufficiently small.

It is worth to note that the operator $H(\alpha)$ possesses theses two properties T_2) and T_3) if α is sufficiently close to α_0 and the operator $H(\alpha_+)$ possesses both these properties naturally.

6 Literature

Of course, simple formulas (1.6-8) for the soliton are well known. But in many dimensional x -space the situation is quite different, see, for example, [Str2, Be-Li]. Properties of the spectrum of the operator $H(\alpha)$ were considered in [We2]. The Cauchy problem for equation (1.1) was considered in the space H^1 in [G-V, K] and in some other works. The Cauchy problem with the initial data of the form (5.1) was treated in [Sh-Str, Ca-Li, We1, We2]. The main result states that for the Cauchy data (5.1) the solution always remains in a small H^1 -vicinity of the orbit generated by the trajectory $\sigma_0(t)$, $\sigma_0(0) = \sigma_0$. As for the scattering behavior of the solution when $t \rightarrow \infty$, some series of works devoted to the scattering in the absence of bound states should be mentioned [Str1, G-V]. The only result which is close to formula (5.3) is contained in [Sof-We]. The authors of the work have considered the equation

$$i\psi_t = -\Delta \psi + [V(x) + \lambda|\psi|^{m-1}]\psi,$$

$$\psi = \psi(x, t), x \in \mathbf{R}^n, 1 < m < \frac{n+2}{n-2}, n = 2, 3.$$

In this situation the soliton appears as the perturbation of an eigenfunction of the operator $\psi \rightarrow -\Delta \psi + V(x)\psi$, which is supposed to be unique and simple. The main difference between the theorem of [Sof-We] and our theorem is generated by the fact that in the first case the center of the soliton is *stable*. As a result our work contains some number of technical details which differ it from [Sof-We] although the main line is the same. However, it is worth to emphasize that this common mane line is also similar to the corresponding one in the investigation of asymptotic regimes for nonlinear parabolic equations, see [He].

¹This assumption is not crucial, it is accepted only for the simplicity

7 Separation of motions

Consider a trajectory $\sigma(t) = (\beta(t), \omega(t), b(t), v(t)) \in \Sigma$ which generally is not a solution of system (1.8). Consider the corresponding trajectory in the set of soliton states

$$(7.1) \quad w(x|\sigma(t)) = \exp(i\Phi)\phi(y|\alpha),$$

$$\Phi = -\beta(t) + \frac{1}{2}v(t)x, y = x - b(t), \alpha = \alpha(t).$$

Write the solution ψ of the Cauchy problem (1.1),(5.1) as the sum

$$(7.2) \quad \psi(x, t) = w(x|\sigma(t)) + \chi(x, t).$$

Instead of the equation (1.1) one can get a system for σ and χ introducing some conditions on the splitting (7.2). Let w_σ be the derivative of $w(x|\sigma)$ with respect to the parameter σ and $w_\sigma(t) = w_\sigma(\cdot|\sigma(t))$. One will use the following condition to fix the splitting:

$$(7.3) \quad \langle \vec{\chi}(t), \sigma_3 \vec{w}_\sigma(t) \rangle = 0,$$

here $\langle \cdot, \cdot \rangle$ is the scalar product in the space $\mathbf{L}_2(\mathbf{R} \rightarrow \mathbf{C}^2)$. Note that

$$(7.4) \quad \vec{w}_\beta = \exp(i\sigma_3\Phi)\vec{\xi}_1, \quad \vec{w}_\omega = \exp(i\sigma_3\Phi)\vec{\xi}_2,$$

$$(7.5) \quad \vec{w}_b = \exp(i\sigma_3\Phi)\vec{\xi}_3, \quad \vec{w}_v = \exp(i\sigma_3\Phi)\left(\vec{\xi}_4 - \frac{1}{2}b\vec{\xi}_1 - \frac{1}{2}v\vec{\xi}_2\right).$$

So conditions (7.3) can be represented in the form:

$$(7.6) \quad \langle \vec{f}(t), \sigma_3 \vec{\xi}_i(t) \rangle = 0,$$

where

$$(7.7) \quad \chi(x, t) = \exp(i\Phi)f(y, t), \quad \vec{\xi}_i(t) = \vec{\xi}_i(y|\alpha(t)).$$

A different but an equivalent form of the splitting condition is

$$(7.8) \quad \text{im}(f(t), u_i(t)) = 0.$$

The geometrical sense of (7.3)=(7.6)=(7.8) is very simple: condition (7.6) implies that $\vec{f}(t)$ belongs to the subspace of the continuous spectrum of the operator $H(\alpha(t))$. Actually the condition leads to decomposition which is in accordance with the asymptotic behavior (5.3).

Write down the system for σ and χ (or f) in more explicit form. Replace the set $\sigma = (\beta, \omega, b, v)$ by some other set of variables (γ, ω, c, v) , where

$$(7.9) \quad b = \int_0^t v(\tau)d\tau + c, \quad \beta = \int_0^t \omega(\tau)dt + \gamma.$$

In terms of new variables system (1.8) acquires the form:

$$(7.10) \quad \gamma' = 0, \omega' = 0, c' = 0, v' = 0.$$

Rewrite equation (1.1) in terms of f :

$$(7.11) \quad if_t = L(\alpha)f + N(\phi, f) + l(\sigma)f + l(\sigma)\phi + i(\omega' - \frac{1}{2}vv')\frac{2}{\alpha}\phi_\alpha,$$

where

$$(7.12) \quad N(\phi, f) = F(|\phi + f|^2)(\phi + f) - F(\phi^2)\phi - F(\phi^2)f - F'(\phi^2)\phi^2(f + \bar{f}),$$

$$(7.13) \quad l(\sigma) = \frac{1}{2}v'y + ic'\partial_y + \left(\frac{1}{2}bv' - \gamma'\right).$$

Consider the derivative of splitting condition in form (7.8) with respect to t and substitute expression (7.11) for f_t in the obtained relation. The result can be written down as follows:

$$(7.14) \quad (A_0 + A_1)\lambda = G,$$

where

$$(7.15) \quad \lambda = (\gamma' - \frac{1}{2}bv', \omega' - \frac{1}{2}vv', c', v'), \quad A_0 = \{im(u_i, u_j)\}_{i,j=1}^4,$$

$$(7.16) \quad (A_1\lambda)_j = -re(l(\sigma)f, u_j) - \frac{2}{\alpha}(\omega' - \frac{1}{2}vv')im(f, u_{j\alpha}), \quad G = re(N, u_j).$$

Obtain the explicit expression for the matrix A_0 :

$$(7.17) \quad A_0 = \begin{pmatrix} 0 & e & 0 & 0 \\ -e & 0 & 0 & 0 \\ 0 & 0 & 0 & n \\ 0 & 0 & -n & 0 \end{pmatrix}, \quad n = -\frac{1}{4}\|\phi\|_2^2, \quad e = \frac{4}{\alpha} \frac{dn}{d\alpha}.$$

Under assumptions T_2, T_3)

$$(7.18) \quad \det A_0(\alpha_0 \neq 0),$$

see (3.7). If $\alpha(t)$ is close to α_0 and $f(t)$ is sufficiently small (actually we are going to prove it), equation (7.14) can be used to estimate λ . Substituting λ from (7.14) to the right side of (7.11) one obtains the system:

$$(7.19) \quad \sigma_t = G_1(\sigma, f), \quad if_t = L(\alpha)f + N_1(\sigma, f).$$

Equation (7.14) is not a complete equivalent of conditions (7.3)=(7.6)=(7.8). To get the equivalence one has to add to equation (7.14) condition (7.3)=(7.6)=(7.8) at the time-moment $t = 0$:

$$(7.20) \quad \langle \vec{\chi}_0, \sigma_3 \vec{w}_\sigma(\cdot | \sigma_0) \rangle = 0.$$

Generally this condition is not satisfied by the given decomposition (5.1) of the initial data ψ_0 . But if χ_0 is sufficiently small it is possible to reconstruct decomposition (5.1) of the initial data ψ_0 in order to satisfy to condition (7.20)

In fact, one has to solve the equation:

$$(7.21) \quad \langle \vec{\psi}_0 - vecw(\cdot|\sigma_1), \sigma_3 \vec{w}_\sigma(\sigma_1) \rangle = 0$$

with respect to σ_1 . Here ψ_0 should be given by (5.1) with sufficiently small χ_0 , see (5.2). The local solvability of (7.21) is guaranteed by the nondegeneration of the corresponding Jacobi matrix:

$$(7.22) \quad - \langle \vec{w}_{\sigma_i}(\cdot|\sigma_0), \sigma_3 \vec{w}_{\sigma_j}(\cdot|\sigma_0) \rangle = -2iA_0(\alpha_0).$$

So one can assume that decomposition (5.1) obeys condition (7.20).

Since $\psi \in C(\mathbf{R} \rightarrow H^1)$ a little more general constructions show that condition (7.3) has to fulfil on some small time-interval. Some estimates which will be given in next sections, will show also that at the end of this time- interval the solution has the structure (5.1) with the small second term. It gives us the possibility to continue the constructions and to solve equation (7.3) for all $t \in [0, t_1]$.

8 Reduction to a spectral problem

Now one can describe the main line of the following constructions.

1) System (7.19) will be investigated on a large finite interval $t \in [0, t_1]$. In the end one will be able to consider the limit $t_1 \rightarrow \infty$.

2) On the interval $[0, t_1]$ one can pick out the leading term of system (7.19) in the form:

$$(8.1) \quad \sigma_t = 0, if_t = L(\alpha)f.$$

The first equation should be completed by more stable final data: $\sigma(t_1) = \sigma_1$, with the undefinite for the moment values:

$$\begin{aligned} \sigma_1 &= \sigma(t_1), \sigma_1 = (\beta_1, \omega_1, b_1, v_1), \\ \omega_1 &= \frac{1}{4}(v_1^2 - \alpha_1^2), b_1 = v_1 t_1 + c_1, \beta_1 = \omega_1 t_1 + \gamma_1. \end{aligned}$$

Naturally now one has to put $L(\alpha) = L(\alpha_1)$. After that the second equation oppositely should be completed by the known initial data.

3) Rewrite full equation (7.11) in order to get the operator $L(\alpha_1)$ as the main term of the of the right side. Introduce the new function g :

$$(8.2) \quad \chi = exp(i\Phi_1)g(z, t), \Phi_1 = -\omega_1 t - \gamma_1 + \frac{1}{2}v_1 x, z = x - v_1 t - c_1.$$

It obeys the equation:

$$(8.3) \quad ig_t = L(\alpha_1)g + D(\sigma, g),$$

and D is given by the formulas

$$(8.4) \quad \begin{aligned} D &= D_0 + D_1 + D_2 + D_3 + D_4, \\ D_0 &= \exp(-i\Delta) \left[l(\sigma)\phi(y|\alpha) + i\frac{2}{\alpha}(\omega' - \frac{1}{2}v\nu')\phi_\alpha(y|\alpha) \right], \\ \Delta &= \Phi_1 - \Phi, \\ D_1 &= [F(\phi^2(y|\alpha) + F'(\phi^2(y|\alpha))\phi^2(y|\alpha)] g - \\ &\quad [F(\phi^2(z|\alpha_1)) + F'(\phi^2(z|\alpha_1))\phi^2(z|\alpha_1)] g, \\ D_2 &= F'(\phi^2(y|\alpha))\phi^2(y|\alpha)[\exp(-2i\Delta) - 1]\bar{g}, \\ D_3 &= [F'(\phi^2(y|\alpha))\phi^2(y|\alpha) - F'(\phi^2(z|\alpha_1))\phi^2(z|\alpha_1)]\bar{g}, \\ D_4 &= \exp(-i\Delta)N(\phi(y|\alpha), \exp(i\Delta)g). \end{aligned}$$

In order to investigate the long-time behavior of the solution of the second equation (8.1) and its full form (8.3) one has to separate the contributions of the discrete spectrum and of the continuous spectrum of the operator $L(\alpha_1)$, more precisely, of the operator $H(\alpha_1)$. Consider the representation

$$(8.5) \quad \vec{g} = \vec{k} + \vec{h},$$

where \vec{k} and \vec{h} are the indicated contributions. One can use condition (7.6) to express the component \vec{k} in terms \vec{h} . Since

$$(8.6) \quad \vec{k} = \sum_i \kappa_i \vec{\xi}_i(z|\alpha_1),$$

condition (7.6) leads to the relation:

$$(8.7) \quad \sum_i \kappa_i \langle \Lambda \vec{\xi}_i(z|\alpha_1), \sigma_3 \vec{\xi}_j(y|\alpha) \rangle + \langle \Lambda \vec{h}, \sigma_3 \vec{\xi}_j(y|\alpha) \rangle = 0,$$

where

$$(8.8) \quad \Lambda = \begin{pmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{pmatrix}.$$

The main term of equation (8.7) is again defined by the matrix A_0 :

$$(8.9) \quad \langle \vec{\xi}_i(z|\alpha_1), \sigma_3 \vec{\xi}_j(y|\alpha_1) \rangle \sim -2iA_0(\alpha_1).$$

At last for \vec{h} one can write down the following integral representation (equation):

$$(8.10) \quad \vec{h} = \exp(-iH_1 t)\vec{h}_0 - i \int_0^t \exp[-iH_1(t-\tau)]P_1 \vec{D} d\tau.$$

Here P_1 is the spectral projection operator on the subspace of the continuous spectrum of H_1 and

$$(8.11) \quad \vec{h}_0 = P_1 \vec{g}_0, g_0(z) = \exp \left[i\gamma_1 - i\frac{1}{2}v_1(z + c_1) \right] \chi_0(z + c_1).$$

The final form of the equations which are used in order to investigate the dynamical system on the interval $t \in [0, t_1]$ is given by relations (7.14), (8.7), (8.10).

9 Linear evolution

Consider the operator $H = H(\alpha)$ with some fixed α and assume that H satisfies conditions $T_2), T_3)$ (with α instead of α_0 in them). Let $U(t) = \exp(-iHt)$ be the corresponding evolution operator and P be the spectral projection operator on the subspace of the continuous spectrum of H . Equation (8.11) shows that one has to have some estimates of the evolution $U(t)P$. Such estimates will be presented in this section. They are enough transparent and can be proved by means of simple (but unfortunately not short) computations which use the spectral resolution of H . So let $\vec{h} = P\vec{h}$, then

$$(9.1) \quad \|U(t)\vec{h}\|_\infty \leq ct^{-1/2}[\|\vec{h}\|_2 + NR(\vec{h})];$$

$$(9.2) \quad \|U(t)\vec{h}\|_\infty \leq c(1+t)^{-1/2}[\|\vec{h}\|_{H^1} + NR(\vec{h})]$$

$$(9.3) \quad \|\varrho U(t)\vec{h}\|_2 \leq c(1+t)^{-3/2}[\|\vec{h}\|_2 + NR(\vec{h})].$$

Here

$$NR(\vec{h}) \text{ can be equal } \|(1+x^2)\vec{h}\|_1 \text{ or } \|(1+x^2)\vec{h}\|_2,$$

$$\varrho(x) = (1+|x|)^{-\kappa}, \kappa > 3, 5.$$

10 Estimates of nonlinear terms

All nonlinear terms of equation (8.10) can be estimated with the following set o:

$$M_0(t) = |\alpha^2 - \alpha_0^2|, M_1(t) = |d(t)|, d = y - z,$$

$$M_2(t) = \|\vec{\kappa}\|, \vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4), M_3(t) = \|\varrho(z)\vec{h}(z, t)\|_2, M_4 = \|\vec{h}(z, t)\|_\infty,$$

$$\mathbf{M}_0(t) = \sup_{\tau \leq t} M_0(\tau), \text{bf } \mathbf{M}_1(t) = \sup_{\tau \leq t} M_1(\tau),$$

$$\mathbf{M}_2(t) = \sup_{\tau \leq t} (1+\tau)^{3/2} M_2(\tau), \mathbf{M}_3(t) = \sup_{\tau \leq t} (1+\tau)^{3/2} M_3(\tau),$$

$$\mathbf{M}_4(t) = \sup_{\tau \leq t} (1 + \tau)^{1/2} M_4(\tau).$$

At last

$$\mathbf{M}_j = \mathbf{M}_j(t_1).$$

These definitions and relation (7.14) lead more or less directly to the inequalities:

$$(10.1) \quad \|\lambda\| \leq W(\mathbf{M})(\mathbf{M}_2 + \mathbf{M}_3)^2(1 + t)^{-3}, t < t_1.$$

Here $W(\mathbf{M})$ is a function of $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$, which is a bounded function on a finite vicinity of the point $\mathbf{M}_j = 0$ and can acquire infinite values outside of some larger vicinity. It is possible to present an explicit expression for W but this expression is useless for our purpose. From (10.1) one can obtain:

$$(10.2) \quad \mathbf{M}_0, \mathbf{M}_1 \leq W(\mathbf{M})(\mathbf{M}_2 + \mathbf{M}_3)^2.$$

Inequalities (10.1) together with the relation (8.7) generate also the estimate

$$(10.3) \quad \mathbf{M}_2 \leq W(\mathbf{M})(\mathbf{M}_2 + \mathbf{M}_3)^3.$$

Now pick out from D_4 all terms containing at least one power of ϕ and denote their sum by D_{II} , the remainder will be denoted by D_{III} . Finally, let $D_I = D_1 + D_2 + D_3$.

Direct computations permit to prove the following estimates:

$$(10.4) \quad \|(1 + z^2)P_1(D_0 + D_I + D_{II})\|_2 \leq W(\mathbf{M})(\mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4)^2(1 + t)^{-3/2}.$$

In order to obtain a similar estimate for D_{III} one has to use additionally some information on solutions of nonlinear equation (1.1), more precisely one has to use conservation law (4.3) and estimate (4.5). As result one has obtain:

$$(10.5) \quad \|P_1 D_{III}\|_2 + \|(1 + z^2)P_1 D_{III}\|_1 \leq W(\mathbf{M})\mathbf{M}_4^{2p-1}(1 + t)^{-3/2}.$$

Just here it is important to assume that $4 \leq p$.

11 Final estimates

Using equation (8.10) and combining estimates (9.1-3),(10.4-5) one can obtain finally:

$$(11.1) \quad \mathbf{M}_3, \mathbf{M}_4 \leq W(\mathbf{M}) [N + (\mathbf{M}_2 + \mathbf{M}_3)^2 + \mathbf{M}_4^2 + \mathbf{M}_4^{2p-1}],$$

where

$$N = \|(1 + x^2)\chi_0\|_2 + \|\chi'_0\|_2.$$

The first term N is originated from the first free term of the right side of equation (8.10). It is controlled by the second variants of estimates (9.2-3). Other terms in brackets are originated from the integral term of equation (8.10). They are controlled by estimates (10.4-5) and by both variants of (9.1),(9.3).

Now one has obtained a closed set of inequalities (10.2-3),(11.1) and can try to solve it. Formulas (10.3),(11.1) give the system:

$$(11.2) \quad \mathbf{M}_2 + \mathbf{M}_3, \mathbf{M}_4 \leq W(\mathbf{M}) [N + (\mathbf{M}_2 + \mathbf{M}_3)^2 + (\mathbf{M}_2 + \mathbf{M}_3)^3 + \mathbf{M}_4^2 + \mathbf{M}_4^{2p-1}].$$

If N is sufficiently small, the system shows that the pair $\mathbf{M}_2 + \mathbf{M}_3, \mathbf{M}_4$ can belong either to a small vicinity of the point $(0,0)$ or to some domain whose distance from $(0,0)$ is limited from below uniformly with respect to N . It is clear that only the first possibility can be realized. Therefore all the functions $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ are sufficiently small:

$$(11.3) \quad \mathbf{M}_j(t) \leq \mu(N)N,$$

here $\mu(N)$ is a bounded function defined for small N . Since all constants in the estimates do not depend on t_1 the same estimates are true for $\mathbf{M}_j(t)$ uniformly in $t \in \mathbf{R}_+$:

$$(11.4) \quad \mathbf{M}_j(t) \leq \mu(N)N.$$

12 The limiting soliton

Return to (10.1) again. Estimates (11.4) show now that

$$(12.1) \quad \|\lambda\| \leq \mu(N)N^2(1+t)^{-3}.$$

It implies that all variables γ, ω, c, v have limits $\gamma_\infty, \omega_\infty, c_\infty, v_\infty$ as $t \rightarrow \infty$. So one can introduce the limiting trajectory $\sigma_+(t)$:

$$(12.2) \quad \beta_+(t) = \omega_+t + \gamma_+, \omega_+ = \omega_\infty, \gamma_+ = \gamma_\infty + \int_0^\infty (\omega(\tau) - \omega_\infty) d\tau,$$

$$(12.3) \quad b_+(t) = v_+t + c_+, v_+ = v_\infty, c_+ = c_\infty + \int_0^\infty (v(\tau) - v_\infty) d\tau.$$

It is clear that

$$(12.4) \quad \sigma(t) - \sigma_+(t) = O(t^{-1}),$$

as $t \rightarrow \infty$. Now the *limiting soliton* $w(x|\sigma_+(t))$ arises naturally and

$$(12.5) \quad w(x|\sigma(t)) - w(x|\sigma_+(t)) = O(t^{-1})$$

in the space $\mathbf{L}_2 \cap \mathbf{L}_\infty$.

13 Dispersion reminder

The second term χ of the total solution $\psi = w(x|\sigma(t)) + \chi$ can be studied asymptotically if one uses the same representation of χ as in section 8, but with ∞ instead of t_1 in transformation (8.2):

$$(13.1) \quad \chi = \exp(i\Phi_\infty)g(z, t),$$

$$\Phi_\infty = -\beta_+(t) - \frac{1}{2}v_+x, z = x - b(t).$$

Now the operators H_1 and P_1 should be replaced by the naturally defined operators H_+ and P_+ and all construction of section 8-11 can be duplicated.

Particularly one again can separate the contributions of the discrete and the continuous spectra of H_+ :

$$(13.2) \quad \vec{g} = \vec{k} + \vec{h}.$$

From

$$(13.3) \quad M_2(t) \leq \mu(N)N(1+t)^{-3/2},$$

see (11.4), one can obtain at once the estimate:

$$(13.4) \quad \vec{k} = O(t^{-3/2}),$$

in the space $\mathbf{L}_2 \cap \mathbf{L}_\infty$.

Representation (8.10) for \vec{h} acquires the form:

$$(13.5) \quad \vec{h} = \exp(-iH_+t)P_+\vec{h}_0 - i \int_0^t \exp[-iH_+(t-\tau)]P_+\vec{D}d\tau,$$

with the respectively transformed D .

Introduce the representation

$$(13.6) \quad \vec{h} = \exp(-iH_+t)\vec{h}_\infty + \vec{R},$$

$$(13.7) \quad \vec{h}_\infty = P_+(\vec{h}_0 + \vec{h}_1), \vec{h}_1 = -i \int_0^\infty \exp(iH_+\tau)\vec{D}d\tau,$$

$$(13.8) \quad \vec{R} = -i \int_t^\infty \exp[-iH_+(t-\tau)]P_+\vec{D}d\tau.$$

Here $\vec{h}_\infty \in \mathbf{L}_2 \cap \mathbf{L}_\infty$; $\vec{h}_0 \in \mathbf{L}_2 \cap \mathbf{L}_\infty$ since $\vec{h}_0 \in H^1$; $\vec{h} \in \mathbf{L}_2 \cap \mathbf{L}_\infty$ in accordance with (9.1), (10.4-5). Inequalities (10.4-5) imply immediately that:

$$(13.9) \quad \vec{R} = O(t^{-1/2})$$

in \mathbf{L}_2 -norm,

$$(13.10) \quad \vec{R} = O(t^{-1})$$

in \mathbf{L}_∞ -norm. So one can formulate the following result:

$$(13.11) \quad \begin{aligned} \psi &= w(x|\sigma_+(t)) + \\ &+ \exp(i\Phi_\infty) \left[\exp(-iH_+t)\vec{h}_\infty \right]_1(z, t) + R, \end{aligned}$$

where R admits estimates (13.9-10). The brackets $[\vec{v}]_1$ are used in order to indicate the first component of \mathbf{C}^2 - vector \vec{v} .

14 Scattering

In the *dispersive term* $\exp(i\Phi_\infty) \exp(-iH_+t)\vec{h}_\infty$ the element \vec{h}_∞ belongs to the subspace of the continuous spectrum of the operator H_∞ . So its behavior as $t \rightarrow \infty$ is *scattering behavior* (in \mathbf{L}_2 - norm):

$$(14.1) \quad \exp(-iH_+t)\vec{h}_\infty = \exp(-iH_0t)\vec{h}_+ + o(1),$$

where $\vec{h}_+ \in \mathbf{L}_2$ and is related to \vec{h}_∞ in terms of the corresponding *wave operator* W_+ :

$$(14.2) \quad \vec{h}_\infty = W_+\vec{h}_+.$$

It is not essential that the operator H_∞ is not self-adjoint in our case since its spectral resolution has the same structure as for a self-adjoint operator. We are not going to discuss here the properties of \vec{h}_+ and the reminder in more detail. Only note that

$$(14.3) \quad \exp(i\Phi_\infty) [\exp(-iH_0t)\vec{h}_+]_1(z) = [\exp(-il_0t)f_+](x),$$

and

$$(14.4) \quad l_0 = -\partial_x^2, f_+(x) = \exp(-i\gamma_+ - i\frac{1}{2}v_+x)h_+(x - c_+).$$

Introduce representation (14.3) in formula (13.11) and write down the *final result*:

$$(14.5) \quad \psi = w(\cdot|\sigma_+(t)) + \exp(-il_0t)f_+ + o(1).$$

From the first author (V.B.):

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