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SCATTERING OF WAVES IN A MEDIUM DEPENDING PERIODICALLY ON TIME

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I. INTRODUCTION

We obtain the asymptotic behaviour as $t \to \infty, |x| \leq a < \infty$ of solutions of exterior mixed problems for hyperbolic equations and systems when the boundary of a domain and coefficients of the equations depend periodically on time. Our method can be regarded as an alternative one to the Lax-Phillips scattering theory. Using the Lax-Phillips method we have to construct at first waves operators and a scattering matrix. Then we study some analytic properties of the scattering matrix and some properties of a special Lax-Phillips semigroup $Z(t)$ and then we derive asymptotic behavior of solutions of the exterior mixed problem as $t \to \infty$. In our direct method at first we find the asymptotic behavior of the solution of the exterior mixed problem. Unlike Lax-Phillips we do it without using any abstract result on spectral representation, outgoing and ingoing subspaces and so on. Then we obtain existence of the wave operators and the scattering operator. In fact, it is not a difficult problem if you know asymptotic behavior of the solutions.

Both of these methods were constructed earlier in the stationary case, when the domain and coefficients of the equations did not depend on time (there are references in [6]). Recently a few papers by J. Cooper and W. Strauss appeared which contain some results of Lax-Phillips theory for scattering of waves by a body moving periodically in t ([1],[2],[3]). Another method of research of this problem is based on the theorem of RAGE type and is suggested by V. Petkov [4]. These authors proved the existence of a scattering operator for wave equation in exterior of a body which depends periodically on t if $n \geq 3$ and obtained asymptotic behavior of solutions of this problem for odd n. They also studied hyperbolic systems of first order when dimension n is odd. Our method gives the possibility to study general time periodic systems of any order and moreover the dimension of the space can be arbitrary and the energy of solution can be unbounded with respect to time. Some of the proofs given below are very concise. The omitted details can be reconstructed with the help of [7], [8], [9].

II. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Let $x \in I\!R^n$, $\partial_t = \partial/\partial_t$, $\partial_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $\Omega \in I\!R^{n+1}_{(t,x)}$ be the exterior of the cylinder with a curvelinear boundary which depends periodically on t. Let $u = (u^{(1)}, ..., u^{(\ell)})$,
$L = L(t, x, \partial_t, \partial_x) = \{L_{i,j}\}$ be a hyperbolic $\ell \times \ell$ matrix. We consider the exterior mixed problem

$$(1) \qquad \begin{cases} Lu = 0, \quad (t,x) \in \Omega, \quad t > \tau; \qquad Bu|_{\partial\Omega} = 0, \quad t > \tau; \\ \partial_t^j u|_{t=\tau} = f_j, \quad 0 \leq j \leq m-1, \quad x \in \Omega_\tau = \Omega \bigcap \{t = \tau\}. \end{cases}$$

Here $B = B(t, x, \partial_x)$ is a boundary operator of general type, $m = \max_{i,j}$ ord $L_{i,j}$.

The main problem of this part of the article is the following. Let $f = (f_0, ..., f_{m-1})$ be a function with a compact support. The asymptotic behavior of solution u is to be found when $t \to \infty$ and x is bounded, that is the initial data are localized in space and the solution at large t is of interest only in the limited part of the space.

We fix an arbitrary constant a for which $\partial\Omega \subset \{(t,x) : |x| < a - 1\}$, condition H_1 is satisfied and $f = 0$ when $|x| > a$.

Conditions.

H_1. The medium is homogeneous in the neighborhood of infinity, that is $L = L_0(\partial_t, \partial_x)$ when $|x| > a$, where L_0 is a homogeneous matrix with constant coefficients.

H_2. The problem (1) is time periodic, that is $\Omega_{t+T} = \Omega_t$ and coefficients of the operators L and B are periodic functions with respect to t with the same period T.

H_3. The problem (1) is correct and Duhamel principle is valid.

Let $C_a^\infty(\bar{\Omega}_\tau), C_a^\infty(\bar{\Omega})$ be spaces of infinitely smooth functions in $\bar{\Omega}_\tau$ or $\bar{\Omega}$ which are equal to zero when $|x| > a$; $H^s(D)$ be a Sobolev space of functions in domain D, $H^s_{loc}(\bar{D})$ be a space of functions in the domain D belonging to $H^s(V)$ for any bounded domain $V \subset D$;

$\psi \in H^{s,A}$ if $\exp(At)\psi \in H^s(\Omega)$;
$\psi \in H^{s,A}_{a,0}$ if $\psi \in H^{s,A}$ and $\psi = 0$ when $|x| \geq a$ *or* $t < 0$.

If $\nu \Rightarrow (\nu_0, ..., \nu_{m-1})$, then we denote $H^\nu(\Omega_\tau) = \sum_{0 \leq j \leq m-1} H^{\nu_j}(\Omega_\tau)$. Let $f = (f_0, \dots, f_{m-1}) \in F_\tau$ if $f_j \in C^\infty_a(\bar{\Omega}_\tau)$ and compatibility conditions are satisfied, that is there exists $w \in H^m_{loc}(\bar{\Omega})$ for which boundary and initial data of problem (1) are valid.

We shall use the same notation for the space of functions and vector-functions if the latter is a direct product of n copies of the space of functions. At last let H(ν) be the closure of the space F_τ with respect to the norm of the space $H^\nu(\Omega_\tau)$.

The correctness of the problem (1) means that it has the unique solution $u \in H^m_{loc}(\bar{\Omega} \bigcap \{t \geq \tau\})$ for any $f \in F_\tau$ and there are $\nu_j, q \in I\!R$ such that the operator

$$U_\tau : f \to \begin{cases} u, & t > \tau \\ 0, & t < \tau, \end{cases} \quad f \in F_\tau$$

has the following continuous extension: $U_\tau : H(\nu) \to H^q_{loc}(\bar{\Omega})$.

According to Duhamel principle there exist $A_0(s)$ such that the problem

$$Lw = g, \quad (t, x) \in \Omega; \quad Bw|_{\partial\Omega} = 0; \quad w = 0 \quad \text{when} \quad t < 0$$

is uniquely solvable in the space $H^{s,A}$ for any $g \in H^{s,A}_{a,0}$ if $s \geq m, A \geq A_0(s)$. Besides the operator

(2) $$V : H^{s,A}_{a,0} \to H^{s,A}, \quad Vg = w, \quad s \geq m, \quad A \geq A_0(s)$$

is bounded and

$$w(t, x) = \int_0^t u(t, \tau, x)\, d\tau .$$

Here u is the solution of the problem (1) with $f = Pg(\tau, \cdot)$, where $Pg = (0, ..., 0, g(\tau, x))$. It is implied that $Pg \in H(\nu)$ if $g \in H^{m,A}_{a,0}$.

The condition H_3 means that the boundary of the body must not move too quickly. For example, for the wave equation the velocity of the moving boundary must be lower than the velocity of propagation of waves in the medium. In this case the condition H_3 is satisfied for all the basic problems for wave equation.

In the case of general hyperbolic equations and systems we change the variables $(t, x) \to (t, y)$, $y = y(t, x)$ so that Ω could take the form of the straight cylinder. The velocity of the moving boundary must be such that the system in the new variables remains hyperbolic at t. Then the condition H_3 is satisfied if boundary operators satisfy uniform Shapiro-Lopatinsky condition.

H_4. Non-trapping condition. It means the following.

Let $E = E(t, \tau, x, x^0)$ be the Schwartz kernel of the operator U_τ, that is E is Green matrix of the problem (1). It is supposed that there exists such a function $T(\rho)$, that E is infinitely smooth when $|x|, |x^0| < \rho, \quad t - \tau > T(\rho)$. This condition is equivalent to the following: all the bicharacteristics are outgoing to infinity when t tends to infinity.

H_5. The operator L_0 has no waves with zero propagation velocity, that is $det L_0(0, \sigma) \neq 0$ when $\sigma \neq 0$. One can give up this condition in the same way as it was done in the stationary case in |5|.

Let (1^0) denote the problem (1), when $\tau = 0$.

THEOREM 1. *Let the conditions $H_1 - H_5$ be satisfied, $f \in H(\nu)$. Then there exists a sequence of complex points k_j which are called the scattering frequencies and integers p, q, p_j and periodic on t functions $u_0(t, x), u_{j,l}(t, x) \in C^\infty$ with period T such that*

1) $-\pi/\mathrm{T} \leq \mathrm{Re} k_j < \pi/\mathrm{T}, \quad \mathrm{Im} k_{j+1} \leq \mathrm{Im}\, k_j, \quad \mathrm{Im}\, k_j \to -\infty \quad as \quad j \to \infty$

2) *If n is odd then the solution of the problem (1^0) has the following expansion*

$$u = \sum_{j=1}^{N} \sum_{t=0}^{p_j} C_{j,l} u_{j,l}(t, x) t^l \exp(-i k_j t) + u_N, \tag{3}$$

where there exist λ and $C = C(a, N, j, \alpha)$ such that

$$|\partial_t^j \partial_x^\alpha u_N| \leq C t^\lambda \exp(\mathrm{Im} k_{N+1} t) \|f\|_{H(\nu)}, \quad |x| \leq a, \quad t \to \infty. \tag{4}$$

3) If *n is even then*

$$u = \sum_{\mathrm{Im} k_j \geq 0} \sum_{l=0}^{p_j} C_{j,l} u_{j,l}(t, x) t^l \exp(-i k_j t) + C_0 u_0(t, x) t^p ln^q t + w, \tag{5}$$

where $C_{j,l} = C_{j,l}(f), C_0 = C_0(f)$ and

$$|\partial_t^j \partial_x^\alpha w| \leq C |\partial_t^j (t^p ln^{q-1} t)| \; \|f\|_{H(\nu)}, \quad |x| \leq a, \quad t \to \infty. \tag{6}$$

Remark. The scattering frequencies k_j belonging to the upper half plane correspond to the exponentially growing terms. They are finite in number. The scattering frequencies k_j belonging to the real axis correspond to the terms, which are the product of the oscillating exponent with periodic function $u_{j,l}$. Such points are finite in number too. The points on the lower half-plane correspond to the exponentially decreasing terms. The less $\operatorname{Im} k_j$ are, the faster they decrease.

PROOF. We change the variables $(t,x) \to (t,y)$, $y = y(t,x)$ so that Ω takes the form of a straight cylinder and $y = x$ when $|x| > a$. The condition that L is a hyperbolic operator isn't used in the proof of the theorem, and the condition that L_0 is a hyperbolic operator is used only in the proof of lemma 6. The matrix L_0, conditions $H_1 - H_5$ and the assertions of the theorem don't change when the variables are changed. We'll use the same notations (t,x) for new variables. Thus we can suppose that there exists a domain $\omega \subset \mathbb{R}^n$ such that $\Omega = \mathbb{R} \times \omega \subset \mathbb{R}^{n+1}_{(t,x)}$.

A special parametrix W_τ of the problem (1) plays an important role in the proof of the theorem. Let us construct this parametrix. We can choose the function $T = T(\rho)$ defined in the condition H_4 in such a way that $T \subset C^\infty(\mathbb{R})$ and $T(\rho) = T(a)$ if $\rho \le a$. Let $T_1 \in C^\infty(\mathbb{R})$, $T_1(\rho) > T(\rho)$ for any $\rho \ge 0$. Let $\zeta = \zeta(t-\tau, x)$ be such a function, that $\zeta \in C^\infty(\mathbb{R}^{n+1})$, $\zeta = 1$ when $t - \tau < T(|x|)$, $\zeta = 0$ when $t - \tau > T_1(x)$. Let $\psi \in C^\infty(\mathbb{R}^n_x)$, $\psi = 1$ when $|x| > a - 1$, $\psi = 0$ when $|x| < a - 2/3$. We define:

$$W_\tau = \zeta U_\tau - \psi N_\tau, \qquad N_\tau = V^0 \psi [L, \zeta] U_\tau$$

Here $[L,\zeta]$ is a commutator of L and the multiplication operator on the function ζ; V^0 is the operator (2) for case $L = L_0$, $\Omega = \mathbb{R}^{n+1}$; the function $\psi[L,\zeta]U_\tau$ is not defined in the domain $\mathbb{R}^{n+1} \backslash \Omega$ and we continue it by zero in this domain (here $\psi = 0$).

It is easy to see that for any $f \in F_\tau$

$$\text{(7)} \qquad \begin{cases} LW_\tau f = G\tau f, & t > \tau, \quad x \in \omega; \\ BW_\tau f|_{\partial\Omega} = 0, & t > \tau; \quad \partial_t^j W_\tau f|_{t=\tau} = f_j, \quad 0 \le j \le m-1. \end{cases}$$

where

$$\text{(8)} \qquad G_\tau = (1 - \psi^2)[L,\zeta]U_\tau - [L_0, \psi] V^0 \psi [L,\zeta] U_\tau .$$

Let P be the operator defined in condition $H_3, l_t h = h(t,\cdot)$ for any $h = h(t,x)$, $G(t,\tau) = l_t G_\tau$.

LEMMA 1. A. *The Schwartz kernel* $g = g(t,\tau,x,x^0)$ *of the operator* G_τ *has* *the following properties when* $|x^0| \le a$:

$$\begin{aligned} &g \in C^\infty, \\ &g = 0 \quad \text{if} \quad |x| \ge a \quad \text{or} \quad t - \tau \le T(a), \\ &g(t+T, \tau+T, x, x^0) = g(t, \tau, x, x^0). \end{aligned}$$

B. *The equation*

$$\phi(t,\cdot) + \int_0^t G(t,\tau)P\phi(\tau,\cdot)d\tau = -G(t,0)f, \quad \phi \in I\!R^l, \tag{9}$$

is uniquely solvable in the space $C_a^\infty(\bar{\Omega})$ *for any* $f \in F_\tau$. *Also* $\phi = 0$ *when* $t < 0$.

C. *If* $f \in F_\tau$ *then the solution* $u = U_0 f$ *of problem* (1^0) *is equal to*

$$u = W_0 f + \int_0^t W_\tau P\phi(\tau,\cdot)d\tau, \tag{10}$$

where ϕ *is the solution of equation (9).*

PROOF. Assertion A follows from (8) and conditions $H_1 - H_4$. Assertion B is the consequence of assertion A and the fact that equation (9) is the equation of Volterra type. Assertion C follows from (7).

Formula (8) when $t - \tau > T_1(a)$ can be transformed in the following way. Since $(1-\psi^2)[L,\zeta] = 0$ for $t - \tau > T_1(a)$ we have

$$G_\tau = -[L_0,\psi]V^0\psi[L,\zeta]U_\tau, \quad t - \tau > T_1(a).$$

Since $LU_\tau f = \delta(t-\tau)f$ where δ is the delta function, we have

$$G_\tau f = -[L_0,\psi]V^0\psi L\zeta U_\tau f + [L_0,\psi]V^0\psi(\delta(t-\tau)f), \quad t - \tau > T_1(a).$$

We transform the first summand with the help of the relations:

$$\psi L = \psi L_0 = -[L_0,\psi] + L_0\psi, \quad V^0 L_0 \psi\zeta U_\tau = \psi\zeta U_\tau.$$

Since $[L_0,\psi]\psi\zeta = 0$ for $t - \tau > T_1(a)$, we have

$$G_\tau f = [L_0,\psi]V^0 Qf + [L_0,\psi]V^0\psi(\delta(t-\tau)f), \quad t - \tau > T_1(a) \tag{11}$$

where $Qf = [L_0,\psi]\zeta U_\tau f$ is zero when $|x| \ge a$ or $t - \tau > T_1(a)$. From (11) it follows that asymptotic behaviors of the functions $G(t,\tau)f$ and $V^0(\delta(t-\tau)f)$ as $t - \tau \to \infty$ are alike. From this and assertion A of lemma 1 the following lemma can be received.

LEMMA 2. 1) *For any* s *there exists* $A_0(s)$ *such that the operators*

$$\begin{aligned} &G : H_{a,0}^{s,A} \to H_{a,0}^{s+1,A}, \quad (G\phi)(t) = \textstyle\int_0^t G(t,\tau)P\phi(\tau)d\tau, \\ &G_0 : H(\nu) \to H_{a,0}^{s,A}, \quad (G_0 f)(t) = G(t,0)f \end{aligned} \tag{12}$$

are bounded.

2) *The solution of equation* (9) *belongs to the space* $H^{s,A}_{a,0}$, *for any* s *and* $A \geq A_0(s)$.

We research equation (9) with the help of the transformation

$F' = \exp(i\theta t/T)F$, where F is the transformation of Fourier-Bloch-Gelfand:

$$\phi(t) \to (F\phi)(\theta, t) = \sum_{k=-\infty}^{\infty} \phi(kT + t) \exp(ik\theta).$$

Let $C^{\infty}_{a,per} = \{\phi : \phi \in C^{\infty}_{a}(\bar{\Omega}), \quad \phi(t + T, x) = \phi(t, x)\}$, where T is defined in condition H_2. Let $H^s_{a,per}$ be the closure of the space $C^{\infty}_{a,per}$ with respect to the norm of the space $H^s(\Omega \cap \{0 < t < T\})$. The next lemma easily follows from the definition of the operator F'.

LEMMA 3. 1) *The operator* $F' : H^{s,A}_{a,0} \to H^s_{a,per}$ *is bounded and analytically depends on* θ *when* $\operatorname{Im}\theta > AT$.

2) If $\phi \in H^{s,A}_{a,0}$ then

$$(13) \qquad \begin{aligned} (F\phi)(\theta + 2\pi, t) &= (F\phi)(\theta, t), \quad \operatorname{Im}\theta > AT, \\ (F'\phi)(\theta + 2\pi, t) &= (F'\phi)(\theta, t), \quad \operatorname{Im}\theta > AT, \end{aligned}$$

$$(14) \qquad \phi(t) = \tfrac{1}{2\pi} \int_{d_\alpha} (F\phi)(\theta, t) d\theta, \quad d_\alpha = [\alpha i - \pi, \alpha i + \pi], \quad \alpha > AT.$$

Let us fix s and $A > A_0(s)$. Let us apply the transformation F' with $\operatorname{Im}\theta > AT$ to the equation (9). Since $G(t, \tau) = 0$ when $t < \tau$ and $\phi(\tau) = 0$ when $\tau < 0$, we can replace the interval of integration in (12) by $\mathbb{R}$. Then the operators G and F become commutative. Therefore $F'G\phi = G(\theta)F'\phi$ where for any $h \in H^s_{a,per}$ we have

$$\begin{aligned} G(\theta)h &= \textstyle\int_{\infty}^{\infty} G(t, \tau) \exp(i\theta(t - \tau)/T) Ph(\tau) d\tau = \\ &= \textstyle\sum_{k=-\infty}^{\infty} \int_{-kT}^{(-k+1)T} G(t, \tau) \exp(i\theta(t - \tau)/T) Ph(\tau) d\tau = \\ &= \textstyle\int_0^T (F'G)(\theta, t, \tau) \exp(-i\theta\tau/T) Ph(\tau) d\tau. \end{aligned}$$

So the function $\psi = F'\phi$ is a solution of the equation

$$(15) \qquad \psi + G(\theta)\psi = -G_0(\theta)f, \qquad (G_o(\theta)f)(t) = F'G(t, 0)f,$$

where $\operatorname{Im}\theta > AT$ and the operators

$$G(\theta) : H^s_{a,per} \to H^s_{a,per}, \qquad G_0(\theta) : H(\nu) \to H^s_{a,per}$$

are compact and analytically depend on θ when $\operatorname{Im}\theta > AT$.

The important property of the parametrix W_τ is the existence of a meromorphic extension of the operators $G(\theta)$, $G_0(\theta)$ in the domain $\operatorname{Im}\theta < AT$.

Let H be a Hilbert space. A family of the operators $A(\theta) : H \to H$ is called finitely-meromorphic if 1) the operators $A(\theta)$ depend meromorphically on the parameter θ, 2) for any pole $\theta = \theta_0$ of the family $A(\theta)$ the coefficients of negative powers of $(\theta - \theta_0)$ in the Laurent-series expansions of $A(\theta)$ are finite-dimensional operators (i.e. they take the whole space H into a finite-dimensional subspace of H). We denote by $\mathbb{C}'$ the complex θ-plane $\mathbb{C}$ with cuts:

$$l_k = \{\theta : \theta = 2k\pi - i\rho, \rho > 0\}, k = 0, \pm 1, \pm 2, \cdots$$

We'll say that $A(\theta)$ possesses property $S(S')$ if either n is odd and $A(\theta), \theta \in \mathbb{C}$, is a finitely-meromorphic family or n is even, $A(\theta), \theta \in \mathbb{C}'$, is a finitely-meromorphic family and $A(\theta)$ has the following asymptotic behavior as $\theta \to 0$

a) for property S:

$$A(\theta) = B(\theta)\ln\theta + \sum_{0\leq j\leq m} B_j\theta^{-j} + C(\theta), \quad m < \infty,$$

where the operators $B(\theta), C(\theta)$ analytically depend on θ when $|\theta| << 1$, and operators $B_j, \partial_\theta^j B|_{\theta=0}, j \geq 0$, are finite-dimensional

b) for property S':

$$A(\theta) = A_0(\theta) + \theta^{-m} \sum_{j\geq 0} \left(\frac{\theta}{P(\ln\theta)}\right)^j P_j(\ln\theta), \tag{16}$$

where $A_0(\theta)$ analytically depends on θ when $|\theta| << 1$, P is a polynomial with constant coefficients, P_j are polynomials of orders less or equal to jl, constants $m, l \geq 0$ are integers, coefficients of P_j are finite-dimensional operators.

LEMMA 4. ([6]). *If a family of compact operators $G(\theta) : H \to H$ possesses property* S *and there exists $\theta = \theta_0$ such that the operator $1 + G(\theta_0)$ is reversible then the family of operators $(1 + G(\theta))^{-1}$ possesses property S'.*

LEMMA 5. *Let $Q : H^{s,A}_{a,0} \to H^{s,A}_{a,0}$ be a bounded operator and its kernel $q(t, \tau, x, x^0)$ possesses the following properties*

$$q(t+T, \tau+T, x, x^0) = q(t, \tau, x, x^0),$$
$$q(t, \tau, x, x^0) = 0 \quad if \quad 0 \leq t - \tau \leq T_0 < \infty.$$

Then there exists an analytical operator-valued function $Q(\theta) : H^s_{a,per} \to H^s_{a,per}$, $\theta \in \mathbb{C}$, such that $F'Q = Q(\theta)F'$.

Let $\chi \in C^\infty(\mathbb{R}^n_x)$, $\chi = 1$ when $|x| < a - 1$, $\chi = 0$ when $|x| > a$. Let $\alpha \in C^\infty(\mathbb{R})$, $\alpha(t) = 1$ when $t > 1$, $\alpha(t) = 0$ when $t < 0$. Let $V^{0,\alpha}$ be the operator with the kernel equal to $\alpha(t-t_0)E^0(t-\tau, x-x^0)$, where E^0 is kernel of the operator V^0.

LEMMA 6. *There exist such t_0 and such a family of compact operators $P(\theta)$: $H^s_{a,per} \to H^s_{a,per}$, that $P(\theta)$ possesses property S and $F'\chi V^{0,\alpha} = P(\theta)F'$.*

Lemma 5 is rather simple, lemma 6 is a consequence of Herglotz-Petrovskii formulas. From Lemmas 5,6 and formula (11) we obtain the following lemma.

LEMMA 7. *The operators $G(\theta)$ and $G_0(\theta)$ admit meromorphic extensions, which possesses property S.*

From this, (15) and lemma 4 it follows that

(17) $\quad F'\phi = L(\theta)f \quad$ where $\quad L(\theta) = -(1+G(\theta))^{-1}G_0(\theta) : H(\nu) \to H^s_{a,per}$

is an operator which possesses property S'. The operator $L(\theta)$ has no poles when $\operatorname{Im}\theta > AT$ since $F'\phi$ exists for any f if $\operatorname{Im}\theta > AT$. From (17) and (10) it follows that there exist t_0 and an operator-function $R(\theta) : H(\nu) \to H^s_{a,per}$, such that $R(\theta)$ possesses property S' and has no poles when $\operatorname{Im}\theta > AT$ and

$$F'(\chi\alpha(t-t_0)u) = R(\theta)f \tag{18}$$

We denote by θ_j the poles of the operator $R(\theta)$ lying in the stripe $-\pi \leq \operatorname{Re}\theta < \pi$. We number them so that $\operatorname{Im}\theta_{j+1} \leq \operatorname{Im}\theta_j$. Let $k_j = \theta_j/T$. From (14) and (18) it follows that

$$\chi u = (2\pi)^{-1}) \int_{d_\alpha} R(\theta)f\exp(-i\theta t/T)d\theta, \qquad t > t_0+1. \tag{19}$$

Let n be odd. From (19), (13) and lemma 7 it follows that

(20)

$$\chi u = i \sum_{\operatorname{Im}\theta_j \geq \operatorname{Im}\theta_{N+1}} res_{\theta=\theta_j} R(\theta)f\exp(-i\theta t/T) + \frac{1}{2\pi}\int_{d_\beta} R(\theta)f\exp(-i\theta t/T)d\theta,$$

where $t > t_0+1$ and $\beta = \operatorname{Im}\theta_{N+1} - \epsilon, 0 < \epsilon << 1$. The estimate (4) is true for the second term in right side of (20). Thus (3), (4) follow from (20). Let n be even. Then

$$\chi u = i \sum_{\operatorname{Im}\theta_j \geq \theta} res_{\theta=\theta_j} R(\theta)f\exp(-i\theta t/T) + \frac{1}{2\pi}\int_{d} R(\theta)f\exp(-i\theta t/T)d\theta, \tag{21}$$

where $t > t_0+1$ and $d = d_+ \cup \lambda_\epsilon \cup d_-$, $\quad d_\pm = d_{-\epsilon} \cap \{\pm\operatorname{Re}\theta > 0\}, 0 < \epsilon << 1, \lambda_\epsilon$ is the circle $|\theta| = \epsilon$ with the beginning in the point $-i\epsilon - 0$ and the end in the point $-i\epsilon + 0$. The integral along $d_{-\epsilon}$ can be estimated. The asymptotic behavior of the integral along λ_ϵ at $t \to \infty$ can be found with the help of expansion (16) for $R(\theta)$. So if n is even the assertion of the theorem 1 follows from (21). The theorem 1 is proved.

III. SYSTEMS WITH BOUNDED ENERGY, SCATTERING OPERATOR

In this part of the article we consider only systems of the first order ($m = 1$) for the sake of simplicity. In this case there are no difficulties with the description of energy space. Energy of a solution is its norm in the space $L_2(\Omega_t)$. We suppose that the following additional condition is satisfied:

H_6. There exists such constant $M < \infty$ that the norm of operator

$$U(t,\tau): \quad L_2(\Omega_\tau) \to L_2(\Omega_t)$$

is not greater than M if $t \geq \tau$. Here $U(t,\tau) = l_t U_\tau$, where U_τ was defined in condition H_3, l_t is the operator of restriction on hyperplane $t = const$.

Let us denote the monodromy operator $U(T,0)$ of the problem (1^0) by M and its eigenvalues belonging to the unit circle S^1 by $\exp(i\lambda_j T)$, $\lambda_j \in [0, 2\pi/T]$. Let $L_{2,b}$ be the space of functions belonging to $L_2(\Omega_0)$ and equal to zero when $|x| > b$.

THEOREM 2. *Let conditions* $H_1 - H_6$ *be satisfied. Then the operator M does not have more than the finite number of the eigenvalues* $\exp(i\lambda_j T), 1 \leq j \leq N$ *(taking their multiplicity into account) belonging to the unit circle* S^1. *Let* f_j, $1 \leq j \leq N$, *be the corresponding system of linearly independent eigenfunctions.*

There exist such eigenfunctions h_j *of the operator* M^* *with the eigenvalues* $\exp(-i\lambda_j T)$, $1 \leq j \leq N$ *that the solutions* $u = U(t,0)f$ *of the problem* (1^0) *with* $f \in L_{2,a}$ *have the following asymptotic behavior as* $t \to \infty$

$$u = \sum_{j=1}^{M} C_j U(t,0) f_j + w, \qquad C_j = (f, h_j), \tag{22}$$

where the following estimates are valid:

1) if n is odd then there exists $\epsilon > 0$ *such that for arbitrary* $j, \alpha = (\alpha_1, \cdots, \alpha_n)$ *and some* $C = C(j,\alpha,a)$ *we have*

$$|\partial_t^j \partial_x^\alpha w| \leq C \exp(-\epsilon t) ||f||_{L_{2,a}}, \quad t \to \infty, \quad |x| \leq a; \tag{23}$$

2) if n is even then for the same j, α we have

(24) $$|\partial_t^j \partial_x^\alpha w| \le C|\partial_t^j \ln^{-1} t| \|f\|_{L_{2,a}}, \qquad t \to \infty, \quad |x| \le \alpha.$$

Remark. The functions $U(t,0)f_j$ in (22) have form $u_j(t,x)\exp(i\lambda_j t)$ where u_j are time periodic functions.

PROOF. From condition H_6 it follows that expansions (3), (5) have no terms increasing as $t \to \infty$. Thus for any $b > a$, $|x| < b$ and $f \in L_{2,b}$ these expansions can be rewritten as following

(25) $$u = \sum_{j=1}^{N} C_j u_j(t,x)\exp(i\lambda_j t) + o(1), \qquad t \to \infty, \quad |x| < b,$$

where $\mathrm{Im}\lambda_j = 0$, $u_j(t+T,x) = u_j(t,x)$, the functions u_j with the same λ_j are linearly independent and the estimates (23), (24) with b instead of a are valid for remainder $o(1)$. It follows from (1^0) and (25) that the functions u_j exp $(i\lambda_j t), 1 \le j \le N$, satisfy the equation and boundary conditions of problem (1^0).

If $u_j = 0$ when $|x| < a$ then

$$L_0(\partial_t, \partial_x)u_j \exp(i\lambda_j t) = 0, \qquad |x| < b.$$

Let $u_j = \sum a_n(x)\exp(i2\pi nt/T)$ be the Fourier-series expansion of the function u_j. Then $a_n(x) = 0$ when $|x| < a$ and

$$L_0(i(\lambda_j + 2\pi n/T), \partial_x)a_n = 0, \qquad |x| < b.$$

Thus $a_n = 0$ and $u_j = 0$ when $|x| < b$. From here it follows that the numbers λ_j, N in (25) don't depend on b and we can choose the same functions u_j for all b.

From H_6 and (1) it follows that

(26) $$||C_j u_j||_{L_2(\Omega_t)} \le C\|f\|_{L_{2,a}}$$

where the constant C does not depend on a. In particular $u_j(t,\cdot) \in L_2(\Omega_t)$. And as $u_j \exp(i\lambda_j t)$ satisfies the equation and the boundary condition of problem (1) the functions $u_j(0,x)$ are eigenfunctions of the monodromy operator M with the eigenvalues $\exp(i\lambda_j T)$.

Further, linear functionals $f \to C_j = C_j(f)$ are defined on the dense set S of $L_2(\Omega_0)$ (on functions with compact supports) and they are bounded according to (26). Thus by Riesz theorem there exists h_j such that

(27) $$C_j = (f, h_j), \qquad h_j \in L_2(\Omega_0).$$

If f has a compact support then Mf also has a compact support, as the propagation velocity is finite for the solutions of equation $Lu = 0$. Using successively the equality $U(t,T) = U(t-T,0)$, expansion (25), (27) for the solution of the problem (1^0) at the time $t-T$ with the initial data Mf and the periodicity of the functions u_j we obtain:

$$U(t,T)Mf = U(t-T,0)Mf =$$
$$= \sum_{j=1}^N C_j' u_j(t-T,x) \exp(i\lambda_j(t-T)) + o(1) =$$
$$= \sum_{j=1}^N C_j' u_j(t,x) \exp(i\lambda_j(t-T)) + o(1), \quad C_j' = (Mf, h_j)$$

when $t \to \infty$. On the other hand according to (25), (27)

$$U(t,0)f = \sum_{j=1}^N C_j u_j(t,x) \exp(i\lambda_j t) + o(1), \qquad C_j = (f, h_j).$$

Left parts of the last two equalities coincide. Thus,

$$(Mf, h_j) = (f, h_j \exp(-i\lambda_j T))$$

and h_j are eigenfunctions of M^* with the eigenvalues $\exp(-i\lambda_j T)$.

In order to complete the proof of theorem 2 it remains to prove the following assertion (A): if f is an eigenfunction of operator M with the eigenvalue $\exp(i\lambda T)$, $\lambda \in [0, 2\pi/T]$, then one or several numbers λ_j in formula (25) are equal to λ, and f is a linear combination of the corresponding functions $u_j(0,x)$.

Let us consider a sequence of functions f_ϵ with compact supports such that

$$\|f_\epsilon - f\|_{L_2(\Omega_0)} \to 0 \quad \text{when} \quad \epsilon \to 0. \tag{28}$$

If u_ϵ is the solution of problem (1^0) with initial data f_ϵ then by virtue of (25)

$$u_\epsilon = \sum_{j=1}^N C_j^\epsilon u_j(t,x) \exp(i\lambda_j t) + o(1), \qquad t \to \infty. \tag{29}$$

it follows from (28) and condition H_6 that

$$\|u_\epsilon - u(t,x)\exp(i\lambda t)\|_{L_2(\Omega_0)} \to 0 \qquad \text{when} \qquad \epsilon \to 0. \tag{30}$$

The assertion (A) follows from (29), (30). Theorem 2 is proved.

Let us denote the space of functions belonging to $L_2(\Omega_0)$ and ortohqgonal to h_j, $1 \le j \le N$, by H. Let $U_0 = U_0(t-\tau)$ denote the opeartor $U(t,\tau)$ for case $L = L_0$, $\Omega = I\!R^{n+1}$.

THEOREM 3. *Let conditions $H_1 - H_6$ be satisfied. Then the wave operators*

$$W_-(U, U_0) : L_2(\mathbb{R}^n) \to H,$$
$$W_+(U_0, U) : H \to L_2(\mathbb{R}^n)$$

and scattering operator $S = W_+(U_0, U)W_-(U, U_0)$ exist and are bounded.

This theorem easily follows from the theorem 2. The only difficulty is the following: the remainder in (22) is not integrable with respect to time in the neighborhood of infinity if n is even. But we can show that the remainder can be expressed as a finite sum of an integrable summand and of summands which have the form $\alpha(t)h(t,x)$. Here $|\partial_t \alpha| \leq Ct^{-1}ln^{-2}t$ as $t \to \infty$ and function h is smooth, periodic with respect to time and equal to zero when $|x| \geq a$. Then we use the following assertion: if functions α and w have the above mentioned properties then the integral: $\int_0^t U_0(-s)\alpha(s)h(s,x)ds$ converges in the space $L_2(\mathbb{R}^n)$ when $t \to \infty$.

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